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Mathematical definition for case (iv) is $s = 0$ unless $r = \alpha \pm i\beta$ are roots of the characteristic equations. in which case s is the multiplicity of this root. Same applies for the other cases (ii) & (iii) with $r = \alpha$ ($\beta = 0$ in these cases) & with case (i) with $r = 0$ (since $\alpha = \beta = 0$ in case (i))

ie. For case (i) $s = 0$ unless $r = 0$ is a root of char. eqn in which case s is the multiplicity of this root

3) Method can also be applied to $g(x)$ which is a sum of suitable terms

E.g. $g(x) = 1 + 2x + \sin\sqrt{3}x$

let $g_1(x) = 1+2x$ & let $g_2(x) = \sin\sqrt{3}x$

solve $L[y_{p_1}] = g_1$ to find $y_{p_1}(x)$ & $L[y_{p_2}] = g_2$ to find $y_{p_2}(x)$. Then let $y_p(x) = y_{p_1}(x) + y_{p_2}(x)$ then

$$L[y_p] = L[y_{p_1} + y_{p_2}] = L[y_{p_1}] + L[y_{p_2}] = g_1(x) + g_2(x) = g$$

4) The method fails miserably in many cases:

- if you make an algebraic error
- if you guess the wrong form of $y_p(x)$
- if $g(x)$ is not a suitable function
- usually fails for variable coefficient problems.

When it fails, you get silmultaneous equations for the coefficients that have no solution.

Examples (only use this method for constant coeff questions)

Ex $L[y] = y'' + 4y' - 2y = 2x^2 - 3x + 6$

- Find complementary solution y_c that solves $L[y_c] = 0$

$r^2 + 4r - 2 = 0 \Rightarrow r = -2 \pm \sqrt{6}$

$y_c = k_1 e^{(-2-\sqrt{6})x} + k_2 e^{(-2+\sqrt{6})x}$

To find $y_p(x)$ guess $y_p(x) = Ax^2 + Bx + C$

$\Rightarrow y_p' = 2Ax + B \qquad y_p'' = 2A$

so $2x^2 - 3x + 6 = L[y_p] = 2A + 4[2Ax + B] - 2[Ax^2 + Bx + C]$
$= -2Ax^2 + (8A - 2B)x + 2A + 4B - 2C$

$[x^2]$ $2 = -2A$ $A = -1$
$[x]$ $-3 = 8A - 2B = -8 - 2B$ $B = -5/2$
$[x^0]$ $6 = 2A + 4B - 2C$ $C = -9$

Thus $y_p(x) = -x^2 - 5/2\, x - 9$
general solution:
$y_c(x) + y_p(x) = k_1 e^{(-2-\sqrt{6})x} + k_2 e^{(-2+\sqrt{6})x} - x^2 - 5/2\, x - 9$

$\underline{Ex}$ $y'' - y' + y = 2\sin 3x$

$r^2 - r + 1 = 0$ $\qquad$ $r = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$y_c(x) = e^{x/2}\left(K_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + K_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

$y_p(x) \Rightarrow$ guess $\quad A\sin(3x) + B\cos(3x)$ $\Leftarrow$ always have both cos & sin together.

$y_p' = 3A\cos(3x) - 3B\sin(3x)$
$y_p'' = -9A\sin(3x) - 9B\cos(3x)$

$$\begin{aligned}
2\sin 3x &= y_p'' - y_p' + y_p \\
&= -9A\sin(3x) - 9B\cos(3x) - 3A\cos(3x) + 3B\sin(3x) \\
&\quad + A\sin 3x + B\cos 3x \\
&= (3B - 8A)\sin 3x + (-8B - 3A)\cos(3x)
\end{aligned}$$

$2 = 3B - 8A$
$0 = -8B - 3A$ $\qquad$ $3A = -8B$ $\qquad$ $B = -3/8 A$ $\qquad$ $A = \frac{-16}{73}$ $\quad$ $B = \frac{6}{73}$

$y_p(x) = \frac{-16}{73}\sin 3x + \frac{6}{73}\cos 3x$

general solution: $y_c(x) + y_p(x) = \text{-----}$

Ex $y'' + 3y' + 2y = e^{2x}$

$r^2 + 3r + 2 = 0 \Rightarrow (r+1)(r+2) = 0$

$\Rightarrow y_c(x) = c_1 e^{-x} + c_2 e^{-2x}$

Guess $y_p(x) = A e^{2x}$ $\quad y'_p(x) = 2Ae^{2x}$ $\quad y''_p(x) = 4A e^{2x}$

$e^{2x} = 4Ae^{2x} + 6Ae^{2x} + 2Ae^{2x}$
$e^{2x} = 12Ae^{2x}$ $\quad A = 1/12$

general solution $y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{12} e^{2x}$

Ex $L[y] = y'' - 5y' + 4y = \delta e^{x}$

Work out y_p first: $y_p(x) = Ae^x$ $\quad y'_p = y''_p = y_p(x)$

$L[y_p] = Ae^x - 5Ae^x + 4Ae^x = 0$ ← wrong guess y_p never solves $L[y_p] = \delta e^x$

Instead, find $y_c(x)$ first:
$r^2 - 5r + 4 = 0 \Rightarrow (r-4)(r-1) = 0$ $\quad e^x = e^{1x}$
so $y_c(x) = c_1 e^{4x} + c_2 e^x$

was actually part of the complimentary function

For $y_p(x)$ now guess $Axe^x \Rightarrow x[Ae^x]$ $\quad \boxed{s=1}$

notice also $r=1$ & $r=4$ are roots of the characteristic equation. so if $g(x) = \delta e^{\alpha x}$ then $s=0$ unless $\alpha = 1$ or $\alpha = 4$ when $s=1$

$y_p(x) = Axe^x \quad y'_p(x) = A(x+1)e^x \quad y''_p(x) = A(x+2)e^x$

so $y''_p - 5y_p' + 4y_p = A(x+2)e^x - 5A(x+1)e^x + 4Axe^x$
$= -3Ae^x$ (x term will always get canceled)

$A = \frac{-8}{3}$

general solution $y(x) = c_1 e^{4x} + (-8/3\, x + c_2)e^x$

<u>Ex</u> $y'' - 2y' + y = e^x$

$r^2 - 2r + 1 = 0 \quad (r-1)^2 = 0 \quad y_c(x) = (k_1 x + k_2)e^x$

Guess $y_p(x) = Ax^2e^x$ $\quad y'_p(x) = A(x^2+2x)e^x$
$s = 2$ $\quad y''_p(x) = A(x^2+4x+2)e^x$

because $g(x) = e^{\alpha x}$ with $\alpha = 1$
& $r = 1$ is a root of mult 2 of
char. equation.

so $e^x = A(x^2+4x+2)e^x - 2A(x^2+2x)e^x + Ax^2e^x$
$e^x = 2Ae^x \quad A = 1/2$

$y(x) = (1/2\, x^2 + c_1 x + c_2)e^x$

$\underline{Ex}$ $L[y] = y'' + 4y = 4x + 10\sin x \qquad y(\pi) = 0 \qquad y'(\pi) = 2$

Warning for nonhomogeneous IVP, compute the general solution $y = y_p + y_c$ including the particular solution y_p before using the initial condition to find the constants.

$g_1(x) = 4x \qquad g_2(x) = 10\sin x$

$r^2 + 4 = 0 \Rightarrow r^2 = -4 \qquad r = \pm 2i$

$y_c(x) = c_1\cos(2x) + c_2\sin(2x)$

$y_{p_1}(x) = Ax + B \qquad y_{p_1}'(x) = A \qquad y_{p_1}''(x) = 0$

$L[y_{p_1}] = 0 + 4(Ax + B)$

$g_1(x) = 4x = 4Ax + 4B \qquad B = 0 \qquad A = 1$

$y_{p_1}(x) = x$

$y_{p_2}(x) = A\sin x + B\cos x$

$y'_{p_2}(x) = A\cos x - B\sin x \qquad y''_{p_2}(x) = -A\sin x - B\cos(x)$

$g_2(x) = 10\sin x = -A\sin x - B\cos x + 4A\sin x + 4B\cos x$

$10\sin x = 3A\sin x + 3B\cos x$

$A = 10/3$

$y_{p_2}(x) = 10/3 \sin x$

$y(x) = c_1\cos(2x) + c_2\sin(2x) + 10/3\sin x + x$

$y'(x) = -2c_1\sin(2x) + 2c_2\cos(2x) + 10/3\cos x + 1$

$y(\pi) = 0 = c_1\cos(2\pi) + c_2\sin(2\pi) + 10/3\sin\pi + \pi$

$0 = c_1 + \pi \qquad c_1 = -\pi$

$y'(\pi) = 2 = 2c_2 + 1 - 10/3 \qquad c_2 = 13/6$

$y(x) = -\pi\cos(2x) + 13/6\sin(2x) + 10/3\sin x + x$