The Algebra and Dissection of Types

`github.com/ryanorendorff/algebra_lambdaconf2016`

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The Reals

A familiar set of symbols is the reals ($\mathbb{R}$). With the reals we can define an operation $+$ that combines two reals together.

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We denote this algebra with the following notation for some set $\mathcal{S}$and some operation Op.

$$(\mathcal{S}, Op)$$

Algebraic Structures

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$$a + (b + c) = (a + b) + c$$
$$a \vee (b \vee c) = (a \vee b) \vee c$$

- There exists an $e \in \mathcal{S}$ such that $e \times a = a \times e = a$.

$$a + 0 \quad = 0 \quad + a = a$$
$$a \vee \mathit{False} = \mathit{False} \vee a = a$$

These types of algebraic structures are *monoids*.

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Algebraic Data Types

In this talk we are going to focus on only algebraic data types (ADTs)[1]. For example, we can define a type that holds multiple values for one constructor

$$\textbf{data}\ Doubled\ b = Doubled\ b\ b$$

and we can use multiple constructors for a type.

$$\textbf{data}\ Cards = Hearts \mid Diamonds \mid Clubs \mid Spades$$

Let's look at each of these cases in turn.

[1] we are actually going to further restrict to regular data types

Products

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We can write the same thing in a slightly different way.

$$\mathbf{data}\ a \times b = a \hat{\times} b$$

The One

Now let's do a little trick. Let's define the following data type.

$$\mathbf{data}\ 1 = \hat{1}$$

We can only construct this value in one way. How many ways are there to construct this?

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Well 1 only has one value ($\hat{1}$), so we can think of the following relation to hold[2].

$$1 \times a \cong a \cong a \times 1$$

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Product is a monoid!

Remember that a monoid has to follow the following rules.

[3]The star $*$ represents a kind.

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$$a \times 1 \cong 1 \times a \cong a$$

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In which case, we expect the following.

$$a + 0 \cong 0 + a \cong a$$

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An identity type.

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ADTs form a semiring!

Star (or Closed) Semiring

A star semiring or closed semiring has the additional unary operator $-^*$, defined as

$$a^* = 1 + aa^* = 1 + a^*a$$

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For lists, we can have lists of no length or one length or ...

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ADTs can be described as a closed semiring.

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Zippers are used in extensively in XMonad.

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Derivatives can also lead to a one hole context

The derivative of a type is often known as a "one hole context".

We can think of this as marking where we are in a data structure.

Derivative of a Constant

We know from calculus a few derivative rules. The basics are that the derivative of a constant is zero.

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$$\partial K = 0$$

How does this look for a value of type K?

$$\partial Nil = 0$$

A value has no context; it is the only thing in town!

Derivative of a variable

The derivative of variable by itself is 1.

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How does this look for a value of type Id?

$$\partial(\hat{Id}\ x) = ()$$

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This represents that we can either be in the left side of a product or the right hand side.

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Take for example a $List$.

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But for dissection, we note those values that have already been processed differently from those yet to be seen.

Dissection follows the same rules

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$$\begin{aligned}
\triangle K &= 0 \\
\triangle Id &= 1 \\
\triangle (p+q) &= \triangle p + \triangle q \\
\triangle (p \times q) &= \triangle p \times \diagdown q + \angle p \times \triangle q
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What is $\diagup$ and $\diagdown$?

$\diagup$ labels the values in the type as having been processed,

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Dissection of a List

What if we want to dissect a $List$? Remember the derivative.

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It is the same as $Zipper$ but where we annotate the parts of the list we have already seen ($\angle\,(List\ a)$) and those we have yet to see ($\rangle\,(List\ a)$).

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$$\triangle(List\ a) = (\angle(List\ a), \llcorner\!\!\!\diagdown(List\ a))$$

If we had used the same label in the dissection, we get the derivative.

$$\triangle(List\ a) = (\angle(List\ a), \angle(List\ a)) \cong ((List\ a), List\ a)$$

Summary

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- How to describe an algebraic structure $(\mathcal{S}, Op)$ and monoids.
- How $\cdot + \cdot$ and $\cdot \times \cdot$ each form monoids, and together form a closed semiring.
- How to take the derivative of a type, giving a one hole context.
- How to take the dissection of a type, giving a one hole context parameterized with where you have been before.

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Identity Data Type

We can define an identity data type that only holds a single value.[7].

$$\textbf{data}\ Id_1\ x = Id_1\ x \quad \text{-- element}$$

We note that this can be a functor with the following definition.

$$\begin{aligned}&\textbf{instance}\ Functor\ Id_1\ \textbf{where}\\ &\quad fmap\ f\ (Id_1\ x) = Id_1\ (f\ x)\end{aligned}$$

[7] All definitions are originally from [**?**]

Constant Data Type

$$\textbf{data}\ K_1\ a\ x = K_1\ a \quad \text{-- constant}$$

This is the same as the $const$ function. $const\ x :: b \to a$ is a function that ignores its input.

We note that this can also be a functor.

$$\begin{aligned}&\textbf{instance}\ Functor\ (K_1\ a)\ \textbf{where}\\ &\quad fmap\ f\ (K_1\ a) = K_1\ a\end{aligned}$$

Pairs

Data types can be defined by the product of two other types. Often this appears as the following.

$$\mathbf{data}\ Pair\ a\ b = Pair\ a\ b$$

We can write this generically as follows.

$$\mathbf{data}\ p \times_1 q\ x = (p\ x\ , q\ x)_1 \quad \text{-- pairing}$$

And again we note that this can be a functor.

$$\mathbf{instance}\ (Functor\ p, Functor\ q) \Rightarrow Functor\ p \times_1 q\ \mathbf{where}$$
$$fmap\ f\ (p\ , q)_1 = (fmap\ f\ p\ , fmap\ f\ q)_1$$

Sums

We can also construct a data type that chooses between one of two alternates.

$$\mathbf{data}\ Either\ a\ b = Left\ a \mid Right\ b$$

We can write this generically as follows.

$$\mathbf{data}\ p +_1 q\ x = L_1\ (p\ x) \mid R_1\ (q\ x) \quad \text{-- choice, aka Either}$$

And again we note that this can be a functor.

$$\begin{aligned}&\mathbf{instance}\ (Functor\ p, Functor\ q) \Rightarrow Functor\ p +_1 q\ \mathbf{where}\\ &\quad fmap\ f\ (L_1\ p) = L_1\ (fmap\ f\ p)\\ &\quad fmap\ f\ (R_1\ q) = R_1\ (fmap\ f\ q)\end{aligned}$$

One

If we think hard, we can think of a type that can only be constructed in one manner.[8]

$$\mathbf{type}\ Unit = ()$$

This can be encoded in our generic patterns using K_1.

$$\mathbf{type}\ 1_1 = K_1\ ()$$
$$1_v = K_1\ ()$$

[8] Remember that the only inhabitant of $Unit$ is $Unit$. We ignore $\perp$.

Zero?

What if I want to define a type that cannot be inhabited at all? I would want something like the following.

$$\mathbf{data}\ 0 = ???$$

Well we can make an uninhabited type in the following way.

$$\mathbf{data}\ 0_1$$

This means we can never call the following function, which is probably a good thing. :-)

Recreating Maybe

Using this construction, we can recreate $Maybe$

$$\begin{aligned}
&\textbf{type } Maybe\ a = (1_1 +_1 Id_1)\ a \\
&nothing = L_1\ (K_1\ ()) \\
&just\ x \quad = R_1\ (Id_1\ x)
\end{aligned}$$

Constructing a value is pretty simple.

$$val = just\ 5 :: Maybe\ Integer$$

And note that $Maybe$ already has a definition of $fmap$!

$$newVal = fmap\ (4+)\ val \quad \text{-- } \equiv just\ 9$$

Back to Monoids

Remember that a monoid $(\mathcal{S}, Op)$ has to have a closed, associative binary operator with an identity element.

A *semiring* is when two monoids are defined over the same set, have two different identity elements, and one of the monoids commutes $(+)$. We write this as follows.

$$(\mathcal{S}, +, 1, \cdot, 0)$$

Product Types

Remember that the $\cdot \times_1 \cdot$ type is defined as follows.

$$\mathbf{data}\ p \times_1 q\ x = (p\ x\ , q\ x)_1 \quad \text{-- pairing}$$

If we define the set of all Haskell types $\mathcal{H}$, then

List

If we attempt to make a list with this formulation, then we could write the following.

$$\mathbf{type}\ ListF\ a = 1_1 +_1 K_1\ a \times_1 Id_1$$

We would like a list that contained $ListF$, but if we try

$$\mathbf{type}\ List = ListF\ List$$

we get an infinite type. To fix this, we need to "tie the knot".

$$\mathbf{data}\ \mu\, p = \hat{\mu}\, p\ (\mu\, p)$$

And now we can define our list as follows.

$$\begin{aligned}
&\mathbf{type}\ List\ \ a\ \ &&= \mu\,(ListF\ a)\\
&[\,] &&= \hat{\mu}\,(L_1\ 1_v)\\
&(:)\ a\ as &&= \hat{\mu}\,(R_1\ (K_1\ a\,, Id_1\ as)_1)
\end{aligned}$$