Fusion: Applying Equational Transforms to Simplify Programs

github.com/ryanorendorff/lc-2017-fusion

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Outline

Motivation: Simple Programs versus Performance

A brief introduction to GHC

List fusion with $foldr/build$

Stream Fusion

Applications of Fusion

Motivation: Simple Programs versus Performance

Common way to process a list: map and fold!

As an example, say we want to square all the elements in a list and then sum the result.

$$process :: [Int] \to Int$$
$$process\ xs = sum \circ map\ sq\ \$\ xs$$

Where we have defined the functions as follows.

$$\begin{aligned} &map\ \_\ [\,] &&= [\,] \\ &map\ f\ (x : xs) &&= f\ x : map\ f\ xs \end{aligned}$$

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$$foldr :: (a \to b \to b) \to b \to [a] \to b$$

$$\begin{aligned} foldr\ \_\ z\ [\,] &= z \\ foldr\ f\ z\ (x : xs) &= f\ x\ (foldr\ f\ z\ xs) \end{aligned}$$

$$sum :: [Int] \to Int$$

How fast is $process$?

So now that we have our process function, how fast does it run?

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Let's try to process a million elements with our $process$ and $process'$, which uses the standard Prelude sum and map.

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How does the Prelude do so much better with the same functions?

We can get good performance with manual code

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function	time (ms)
$process$	220.0
$process'$	25.31
$process_{hand}$	26.8

It seems we have matched GHC's performance!

GHC generated the simplified version automatically

Our manual version $process_{hand}$.

$$process_{hand} :: [Int] \to Int$$

$$\begin{aligned}
&process_{hand}\ [\,] = 0\\
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\end{aligned}$$

and when we compile the Prelude defined $process'$, GHC produces

$$\begin{aligned}
&processGHC :: [Int] \to Int\\
&processGHC\ [\,] \qquad\quad = 0\\
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\end{aligned}$$

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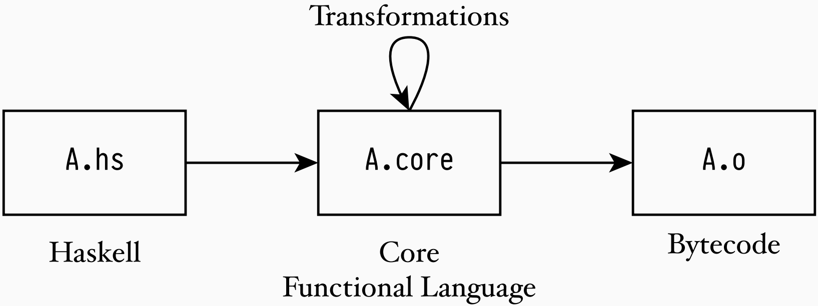
How can we leverage the compiler to write simple code that is fast?

Table of Contents

The GHC Compilation Pipeline converts Haskell into an intermediate language and then bytecode

When GHC compiles a Haskell program, it converts the code into an intermediate language called "Core", which is then (eventually) turned into byte code.



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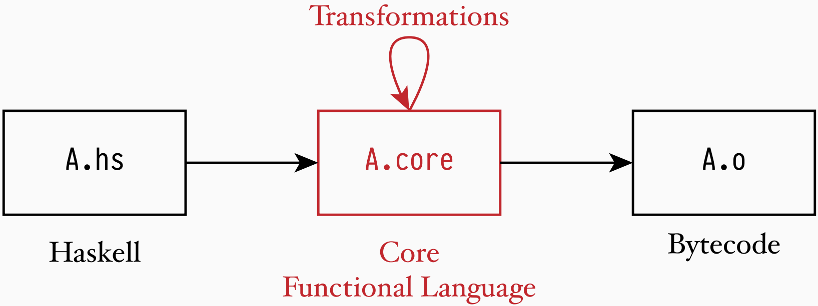

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- *Applying rewrite rules*
- ...

Rewrite Rules allow us to say two expressions are equivalent

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```
{-# RULES "name" forall x. id x = x #-}
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Rewrite rules allow us to replace terms in the program with equivalent terms.

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{-# RULES "name" forall x. id x = x #-}
```

"Any time we see the term $id\ x$, replace it with x".

Rules have some restrictions

Rewrite rules have some gotchas.

- Rules doesn't prevent you from doing something silly

```
{-# RULES "id5" forall x. id x = 5 #-}
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{-# RULES "fxy" forall x y. f x y = f y x #-}
```

- If multiple rules are possible, GHC will randomly choose one.

We can combine maps to traverse a list once

Let us introduce the following rule about maps.

$\texttt{\{-\# RULES "map/map" forall f g xs.}$
$\quad \texttt{map}_{\mathrm{fuse}}\ \texttt{f (map}_{\mathrm{fuse}}\ \texttt{g xs) =}\ \texttt{map}_{\mathrm{fuse}}\ \texttt{(f.g) xs \#-\}}$

We can combine maps to traverse a list once

Let us introduce the following rule about maps.

```
{-# RULES "map/map" forall f g xs.
  map_fuse f (map_fuse g xs) =  map_fuse (f.g) xs #-}
```

$$mapTestUnfused :: [Int] \rightarrow [Int]$$

$$mapTestUnfused\ xs = map\ (+1)\ (map\ (*2)\ xs)$$

$$mapTestFused :: [Int] \rightarrow [Int]$$

$$mapTestFused\ xs = mapfuse\ (+1)\ (mapfuse\ (*2)\ xs)$$

Our map fusion performs (a bit) better!

We can test our functions on a million elements

$$mapTestUnfused\ xs = map\ (+1)\ (map\ (*2)\ xs)$$

$$mapTestFused\ xs = mapfuse\ (+1)\ (mapfuse\ (*2)\ xs)$$

and find we get a bit better time and space performance.

Function	Time (ms)	Memory (MB)
$mapTestUnfused$	26.4	256.00
$mapTestFused$	17.6	184.00

Through rules, GHC performs fusion

Some of the rules work together to perform *fusion*: to combine terms in such a way as to pass over a data structure once.

In our $process$ function, we create an intermediate list

$$process :: [Int] \to Int$$

$$process\ xs = sum \circ map\ sq\ \$\ xs$$

whereas our "fused" form did not make any intermediate structure, and used an accumulator instead.

$$process_{hand} :: [Int] \to Int$$

$$process_{hand}\ [\,] = 0$$
$$process_{hand}\ (x : xs) = x * x + process_{hand}\ xs$$

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foldr / *build* fusion is used to simplify list computations

GHC accomplishes fusion with two functions: foldr and build.

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$foldr$ combines the elements of a list

$$foldrfuse :: (a \rightarrow b \rightarrow b) \rightarrow b \rightarrow [a] \rightarrow b$$

$$foldrfuse\ f\ z\ [\,] = z$$

$$foldrfuse\ f\ z\ (x : xs) = f\ x\ (foldr\ f\ z\ xs)$$

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while $build$ builds up a list from a generating function.

$$buildfuse :: \forall a.(\forall b.(a \to b \to b) \to b \to b) \to [a]$$

$$buildfuse\ g = g\ (:)\ [\,]$$

foldr / *build* fusion is used to simplify list computations

GHC accomplishes fusion with two functions: foldr and build.

foldr combines the elements of a list

$$\mathit{foldrfuse} :: (a \to b \to b) \to b \to [a] \to b$$

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while *build* builds up a list from a generating function.

$$\mathit{buildfuse} :: \forall a.(\forall b.(a \to b \to b) \to b \to b) \to [a]$$

$$\mathit{buildfuse}\ g = g\ (:)\ [\,]$$

$$\mathit{build1}\ \ l \equiv [1, 2, 3]$$
$$\quad \textbf{where}$$
$$\qquad l\ \mathit{cons}\ \mathit{nil} = 1\ \text{‘}\mathit{cons}\text{‘}\ (2\ \text{‘}\mathit{cons}\text{‘}\ (3\ \text{‘}\mathit{cons}\text{‘}\ \mathit{nil}))$$

The *foldr3* / *build1* rule removes intermediate fold/build pairs

To remove intermediate data structures (those created by *build*), we eliminate *foldr* / *build* pairs with a rule.

```
{-# RULES
"foldr/build"
∀ f z (g :: ∀ b. (a -> b -> b) -> b -> b).
foldr f z (build g) = g f z #-}
```

$$
\begin{array}{l}
\mathit{foldr}\ (+)\ 0\ (\mathit{build}\ l) \equiv l\ (+)\ 0 \equiv 1 + (2 + (3 + 0)) \\
\quad \textbf{where} \\
\quad\quad l\ \mathit{cons}\ \mathit{nil} = 1\ \text{`}\mathit{cons}\text{`}\ (2\ \text{`}\mathit{cons}\text{`}\ (3\ \text{`}\mathit{cons}\text{`}\ \mathit{nil}))
\end{array}
$$

We need a few extra rules to convert maps into fold/builds

To convert our definition of maps into a fold/build pair, we need the following helper function.

$$mapFBfuse :: (elt \rightarrow lst \rightarrow lst) \rightarrow (a \rightarrow elt) \rightarrow a \rightarrow lst \rightarrow lst$$

$$mapFBfuse\ c\ f = \lambda x\ ys \rightarrow c\ (f\ x)\ ys$$

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With that, we have all we to convert map into build/fold.

```
{-# RULES "map" ∀ f xs. map f xs =
  build (\c n -> foldr (map_fb c f) n xs) #-}
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  build (\c n -> foldr (map_fb c f) n xs) #-}
```

We also provide a way to combine sequential $mapFB$ functions.

```
{-# RULES "map_fb" ∀ c f g. map_fb (map_fb c f) g =
                                  map_fb c (f . g) #-}
```

Manual rewrite rule application

Let's try applying the rewrite rules manually.

$$sum\ (map\ sq\ xs)$$

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& foldfuse\ (\lambda x\ ys \to sq\ x + ys)\ 0\ xs
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Applying foldr: the empty case

We now look at empty case

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$$0$$

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$$(\lambda x\ ys \rightarrow sq\ x + ys)\ x\ (processfused\ xs)$$

$\equiv$ {-apply lambda -}

$$sq\ x + process\ xs$$

$\equiv$ {-inline sq -}

$$x * x + processfused\ xs$$

Bringing both cases back together

If we now combine our two cases, we have the following

$$
\begin{aligned}
&process_{hand}\ [\,] = 0\\
&process_{hand}\ (x : xs) = x * x + process_{hand}\ xs
\end{aligned}
$$

This is the same as what we had originally written manually!

We achieved list fusion using $foldr$ / $build$ with rewrite rules

We managed to fuse $process$ using our rewrite rules. We can look at the output of the compiler and it confirms what we expected.

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$$process_{hand}\ (x : xs) = x * x + process_{hand}\ xs$$

We achieved list fusion using $foldr$ / $build$ with rewrite rules

We managed to fuse $process$ using our rewrite rules. We can look at the output of the compiler and it confirms what we expected.

$$process_{hand}\ [\,] = 0$$
$$process_{hand}\ (x : xs) = x * x + process_{hand}\ xs$$

Function	Time (ms)	Memory (MB)
$process$	220.0	265.26
$process'$	25.31	96.65
$process_{hand}$	25.31	96.65
$processfused$	25.31	96.65

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Introduction to Stream

The Stream fusion system attempts to do something similar, by defining a list as a state machine.

$$
\begin{array}{l}
\textbf{data}\ \mathit{Stream}\ a\ \textbf{where} \\
\quad \mathit{Stream} :: (s \to \mathit{Step}\ a\ s) \to s \to \mathit{Stream}\ a
\end{array}
$$

Introduction to Stream

The Stream fusion system attempts to do something similar, by defining a list as a state machine.

$$\begin{aligned}&\mathbf{data}\ \mathit{Stream}\ a\ \mathbf{where}\\&\quad \mathit{Stream} :: (s \to \mathit{Step}\ a\ s) \to s \to \mathit{Stream}\ a\end{aligned}$$

$$\begin{aligned}\mathbf{data}\ \mathit{Step}\ a\ s &= \mathit{Done}\\&\mid \mathit{Skip}\quad\ \ s\\&\mid \mathit{Yield}\ a\ s\end{aligned}$$

Streams have little helpers to make lists

$$stream :: [a] \to Stream\ a$$

$$\begin{aligned}
&stream\ xs = Stream\ uncons\ xs \\
&\quad \mathbf{where} \\
&\qquad uncons\ [\,] \qquad\quad = Done \\
&\qquad uncons\ (x : xs) = Yield\ x\ xs
\end{aligned}$$

$$unstream :: Stream\ a \to [a]$$

$$\begin{aligned}
&unstream\ (Stream\ next\ s0) = unfold\ next\ s0 \\
&\quad \mathbf{where} \\
&\qquad unfold\ next\ s = \mathbf{case}\ next\ s\ \mathbf{of} \\
&\qquad\quad Done \to [\,] \\
&\qquad\quad Skip\ s' \to unfold\ next\ s' \\
&\qquad\quad Yield\ x\ s' \to x : unfold\ next\ s'
\end{aligned}$$

Maps on Streams!

$$maps :: (a \rightarrow b) \rightarrow \mathit{Stream}\ a \rightarrow \mathit{Stream}\ b$$

$$
\begin{array}{l}
maps\ f\ (\mathit{Stream}\ next0\ s0) = \mathit{Stream}\ next\ s0 \\
\quad \textbf{where} \\
\qquad next\ s = \textbf{case}\ next0\ s\ \textbf{of} \\
\qquad\quad \mathit{Done} \rightarrow \mathit{Done} \\
\qquad\quad \mathit{Skip}\ s' \rightarrow \mathit{Skip}\ s' \\
\qquad\quad \mathit{Yield}\ x\ s' \rightarrow \mathit{Yield}\ (f\ x)\ s'
\end{array}
$$

$$mapl :: (a \rightarrow b) \rightarrow [\,a\,] \rightarrow [\,b\,]$$

$$mapl\ f = unstream \circ maps\ f \circ stream$$

Stream Fusion!

Fusion on streams only has one rewrite rule, and it is pretty simple.

```
{-# RULES "stream" ∀ (s :: Stream a).
    stream (unstream s) = s #-}
```

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```
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```

$$mapTestStream :: [Int] \to [Int]$$

$$mapTestStream\ xs = mapl\ (+1)\ (mapl\ (*2)\ xs)$$

$$mapTestStreamCompiled :: [Int] \to [Int]$$

$$mapTestStreamCompiled\ [\,] = [\,]$$
$$mapTestStreamCompiled\ (x : xs) =$$
$$\quad 1 + (x * 2) : mapTestStreamCompiled\ xs$$

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We can make $process$ even faster with $Data.Vector$

The $Data.Vector$ package uses stream fusion and many other rewrite rules behind the scenes in order to optimize array based computations.

$$process\ xs = sum0 \circ map\ sq\ \$\ xs$$

We can make $process$ even faster with $Data.Vector$

The $Data.Vector$ package uses stream fusion and many other rewrite rules behind the scenes in order to optimize array based computations.

$$process\ xs = sum0 \circ map\ sq\ \$\ xs$$

The vector version looks very similar.

$$\textbf{import}\ qualified\ Data.Vector\ as\ V$$

$$processVec\ n = V.sum\ \$\ V.map\ sq\ \$\ V.enumFromTo\ 1\ (n :: Int)$$

We can make *process* even faster with *Data.Vector*

But has incredible performance!

Function	Time (ms)	Memory (MB)
$process$	220.0	265.26
$process'$	25.31	96.65
$processmanualfused'$	4.7	96.65
processVec	0.7	16×10^{-5}

What code does $Data.Vector$ generate?

While we wrote this in our program

$$processVec\ n = V.sum\ \$\ V.map\ sq\ \$\ V.enumFromTo\ 1\ (n :: Int)$$

GHC ended up generating the following Core code.

$$\begin{aligned}&loop\ counter\ acc = \mathbf{case}\ counter \leqslant 100000000\\&\quad False \rightarrow acc;\\&\quad True \rightarrow loop\ (counter + 1)\ (acc + counter * counter)\end{aligned}$$

simplify this core, ignore unboxing

Repa: A numerical Haskell Library using Fusion

Repa also uses fusion in order to handle array operations.

Data Parallel Haskell: Nested Data Parallelism made easy

$$processDPH :: [:Int:] \rightarrow Int$$
$$processDPH = sumDPH \circ mapDPH\ sq\ \$\ xs$$

Does dispatch by MPI.