Fusion: Applying Equational Transforms to Simplify Programs

`github.com/ryanorendorff/lc-2017-fusion`

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Outline

Motivation: Simple Programs versus Performance

A brief introduction to GHC

List fusion with $foldr/build$

Stream Fusion

Applications of Fusion

Common way to process a list: map and fold!

As an example, say we want to square all the elements in a list and then sum the result. [1–4]

$$process :: [Int] \rightarrow Int$$
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$$sum = foldr\ (+)\ 0$$

How fast is $process$?

So now that we have our process function, how fast does it run?

$$process :: [Int] \rightarrow Int$$

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Let's try to process a million elements with our $process$ and $process'$, which uses the standard Prelude $foldr$ and map.

$$process\ [0 \mathinner{..} 1,000,000]; process'\ [0 \mathinner{..} 1,000,000]$$

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How does the Prelude do better with the same functions?

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It seems we have matched GHC's performance!

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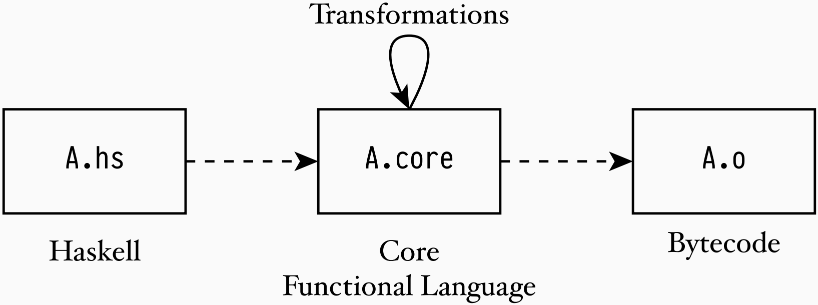
When GHC compiles a Haskell program, it converts the code into an intermediate language called "Core", which is then (eventually) turned into byte code. [5]



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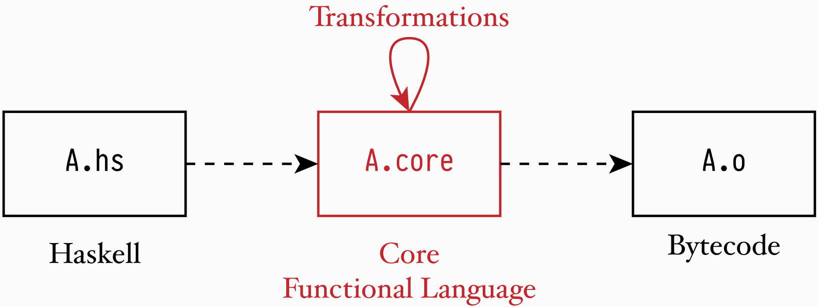

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Rewrite Rules allow us to say two expressions are equivalent

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- After the period are the equivalent statements.

Rules have some restrictions

Rewrite rules have some gotchas. [8]

- Rules doesn't prevent you from doing something silly

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{-# RULES "id5" forall x. id x = 5 #-}
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```

- If multiple rules are possible, GHC arbitrarily chooses one.

We can combine maps to traverse a list once

Let us introduce the following rule about maps. [4]

```
{-# RULES "map/map" forall f g xs.
  map_fuse f (map_fuse g xs) =  map_fuse (f.g) xs #-}
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```

$$mapTest :: [Int] \to [Int]$$
$$mapTest\ xs = map\ (+1)\ (map\ (*2)\ xs)$$

$$mapTest_{fuse} :: [Int] \to [Int]$$
$$mapTest_{fuse}\ xs = map_{fuse}\ (+1)\ (map_{fuse}\ (*2)\ xs)$$

Our map fusion performs (a bit) better!

We can test our functions on a million elements

$$mapTest\ xs = map\ (+1)\ (map\ (*2)\ xs)$$
$$mapTest_{fuse}\ xs = map_{fuse}\ (+1)\ (map_{fuse}\ (*2)\ xs)$$

and find we get a bit better time and space performance.

Function	Time (ms)	Memory (MB)
$mapTest$	26.4	256.00
$mapTest_{fuse}$	17.6	184.00

Through rules, GHC performs fusion

Rules allow us to perform *fusion*, where we remove intermediate data structures from the computation.

In our $process$ function, we create an intermediate list

$$process :: [Int] \to Int$$

$$process\ xs = sum \circ map\ sq\ \$\ xs$$

whereas our "fused" form did not make any intermediate structure, and used an accumulator instead.

$$process_{hand} :: [Int] \to Int$$

$$\begin{aligned} &process_{hand}\ [\,] = 0 \\ &process_{hand}\ (x : xs) = x * x + process_{hand}\ xs \end{aligned}$$

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GHC accomplishes fusion with two functions: foldr and build. [3, 9]

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foldr combines the elements of a list

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&foldr :: (a \rightarrow b \rightarrow b) \rightarrow b \rightarrow [a] \rightarrow b \\
&foldr\ f\ \ z\ [\,] = z \\
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while $build$ builds up a list from a generating function.

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$$

$$
\begin{aligned}
&\mathit{build1}\ l \equiv [1, 2, 3] \\
&\quad \textbf{where} \\
&\qquad l\ \mathit{cons}\ \mathit{nil} = 1\ `\mathit{cons}`\ (2\ `\mathit{cons}`\ (3\ `\mathit{cons}`\ \mathit{nil}))
\end{aligned}
$$

The $foldr/build$ rule removes intermediate fold/build pairs

To remove intermediate data structures (those created by $build$), we eliminate $foldr/build$ pairs with a rule.

```
{-# RULES
"foldr/build"
∀ f z (g :: ∀ b. (a -> b -> b) -> b -> b).
foldr f z (build g) = g f z #-}
```

$$
\begin{aligned}
&foldr\ (+)\ 0\ (build\ l) \equiv l\ (+)\ 0 \equiv 1 + (2 + (3 + 0)) \\
&\quad \textbf{where} \\
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$$

We need a few extra rules to convert maps into fold/builds

To convert our definition of maps into a fold/build pair, we need the following helper function. [8, 9]

$$
\begin{aligned}
&mapFB :: (elt \to lst \to lst) \to (a \to elt) \to a \to lst \to lst \\
&mapFB\ c\ f = \lambda x\ ys \to c\ (f\ x)\ ys
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As an example, lets apply the list cons $c = (:)$ and $f = sq$

$$\lambda x\ ys \to sq\ x : ys$$

We need a few extra rules to convert maps into fold/builds

With that, we have all we need to convert map into build/fold.

```
{-# RULES "map" [~1] ∀ f xs. map f xs =
  build (\c n -> foldr (mapFB c f) n xs) #-}
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We also provide a way to cancel failed fusion by converting back to a map.

```
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$$
\begin{aligned}
& build\ (\lambda c\ n \rightarrow foldr\ mapFB\ c\ f)\ n\ xs) \\
\equiv\ & \{\text{-inline def of build -}\} \\
& (\lambda c\ n \rightarrow foldr\ mapFB\ c\ f)\ n\ xs)\ (:)\ [\,] \\
\equiv\ & \{\text{-remove lambda -}\} \\
& foldr\ (mapFB\ (:)\ f)\ [\,]\ xs
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$$

Manual rewrite rule application

Let's try applying the rewrite rules manually.

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\equiv\quad & \{\ \text{apply lambda}\ \} \\
& foldr\ (\lambda x\ ys \rightarrow sq\ x + ys)\ 0\ xs
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Applying foldr: the empty case

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$$(\lambda x\ ys \to sq\ x + ys)\ x\ (process_{fuse}\ xs)$$

$\equiv$ {-apply lambda -}

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$$process\ (x : xs) = foldr\ (\lambda x\ ys \rightarrow sq\ x + ys)\ 0\ (x : xs)$$

$$\equiv\ \{\text{-expand } foldr \text{ case: } foldr\ f\ z\ (x : xs) = f\ x\ (foldr\ f\ z\ xs)\ \text{-}\}$$

$$(\lambda x\ ys \rightarrow sq\ x + ys)\ x\ (foldr\ (\lambda x\ ys \rightarrow sq\ x + ys)\ z\ xs)$$

$$\equiv\ \{\text{-use def of } process_{fuse}\text{: } foldr\ f\ 0\ xs = process_{fuse}\ xs\ \text{-}\}$$

$$(\lambda x\ ys \rightarrow sq\ x + ys)\ x\ (process_{fuse}\ xs)$$

$$\equiv\ \{\text{-apply lambda -}\}$$

$$sq\ x + process\ xs$$

Applying foldr: the list case

Now let's do the $(x : xs)$ case.

$$process\ (x : xs) = foldr\ (\lambda x\ ys \rightarrow sq\ x + ys)\ 0\ (x : xs)$$

$$\equiv\ \{\text{-expand } foldr \text{ case: } foldr\ f\ z\ (x : xs) = f\ x\ (foldr\ f\ z\ xs) \text{ -}\}$$

$$(\lambda x\ ys \rightarrow sq\ x + ys)\ x\ (foldr\ (\lambda x\ ys \rightarrow sq\ x + ys)\ z\ xs)$$

$$\equiv\ \{\text{-use def of } process_{fuse}\text{: } foldr\ f\ 0\ xs = process_{fuse}\ xs \text{ -}\}$$

$$(\lambda x\ ys \rightarrow sq\ x + ys)\ x\ (process_{fuse}\ xs)$$

$$\equiv\ \{\text{-apply lambda -}\}$$

$$sq\ x + process\ xs$$

$$\equiv\ \{\text{-inline } sq \text{ -}\}$$

$$x * x + process_{fuse}\ xs$$

Bringing both cases back together

If we now combine our two cases, we have the following

$$\begin{aligned} process_{fuse}\ [\,] &= 0 \\ process_{fuse}\ (x : xs) &= x * x + process_{fuse}\ xs \end{aligned}$$

This is the same as what we had originally written manually!

$$\begin{aligned} &process_{hand}\ [\,] = 0 \\ &process_{hand}\ (x : xs) = x * x + process_{hand}\ xs \end{aligned}$$

We achieved list fusion using $foldr$ / $build$ with rewrite rules

We managed to fuse $process$ using our rewrite rules. We can look at the output of the compiler and it confirms what we expected.

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We achieved list fusion using $foldr$ / $build$ with rewrite rules

We managed to fuse $process$ using our rewrite rules. We can look at the output of the compiler and it confirms what we expected.

$$
\begin{aligned}
process_{fuse}\ [\,] &= 0 \\
process_{fuse}\ (x : xs) &= x * x + process_{fuse}\ xs
\end{aligned}
$$

As expected, we get the same performance after performing the fusion rules.

Function	Time (ms)	Memory (MB)
$process$	41.86	265.26
$process'$	26.60	96.65
$process_{hand}$	28.80	96.65
$process_{fuse}$	27.08	96.65

There are many types of fusion concepts out there

While $foldr$ / $build$ works well, it can have problems fusing zip and $foldl$.

There are a few other systems out there. [2, 3]

- $unbuild$ / $unfoldr$, where $unfoldr$ builds a list and $unbuild$ consumes a list. It can have problems fusing $filter$.

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While $foldr$ / $build$ works well, it can have problems fusing zip and $foldl$.

There are a few other systems out there. [2, 3]

- $unbuild$ / $unfoldr$, where $unfoldr$ builds a list and $unbuild$ consumes a list. It can have problems fusing $filter$.
- stream fusion, which works by defining a $Stream$ data type that acts like an iterator.

Table of Contents

Introduction to Stream

The Stream fusion system attempts to do something similar, by defining a list as an iterator. [2, 3]

$$\begin{aligned}&\mathbf{data}\ \mathit{Stream}\ a\ \mathbf{where}\\&\quad \mathit{Stream} :: (s \to \mathit{Step}\ a\ s) \to s \to \mathit{Stream}\ a\end{aligned}$$

Introduction to Stream

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$$\begin{aligned}
&\mathbf{data}\ \mathit{Stream}\ a\ \mathbf{where} \\
&\quad \mathit{Stream} :: (s \to \mathit{Step}\ a\ s) \to s \to \mathit{Stream}\ a
\end{aligned}$$

where $\mathit{Step}\ a\ s$ informs us how to keep processing the stream.

$$\begin{aligned}
\mathbf{data}\ \mathit{Step}\ a\ s = {} & \mathit{Done} \\
\mid {} & \mathit{Skip}\quad\ \ s \\
\mid {} & \mathit{Yield}\ a\ s
\end{aligned}$$

Streams have little helpers to make lists: stream

To work on standard lists, we introduce the following two functions to convert between lists and streams.

$$
\begin{aligned}
&\mathit{stream} :: [\,a\,] \rightarrow \mathit{Stream}\ a\\
&\mathit{stream}\ \mathit{xs} = \mathit{Stream}\ \mathit{uncons}\ \mathit{xs}\\
&\quad \textbf{where}\\
&\qquad \mathit{uncons}\ [\,] \qquad\quad = \mathit{Done}\\
&\qquad \mathit{uncons}\ (y : \mathit{ys}) = \mathit{Yield}\ y\ \mathit{ys}
\end{aligned}
$$

Streams have little helpers to make lists: unstream

To work on standard lists, we introduce the following two functions to convert between lists and streams.

$$
\begin{aligned}
& \mathit{unstream} :: \mathit{Stream}\ a \to [\,a\,] \\
& \mathit{unstream}\ (\mathit{Stream}\ \mathit{next}\ \mathit{s0}) = \mathit{unfold}\ \mathit{next}\ \mathit{s0} \\
& \quad \textbf{where} \\
& \quad\quad \mathit{unfold}\ \mathit{next}\ s = \textbf{case}\ \mathit{next}\ s\ \textbf{of} \\
& \quad\quad\quad \mathit{Done} \qquad\quad \to [\,] \\
& \quad\quad\quad \mathit{Skip}\ s' \qquad \to \qquad \mathit{unfold}\ \mathit{next}\ s' \\
& \quad\quad\quad \mathit{Yield}\ x\ s' \to x : \mathit{unfold}\ \mathit{next}\ s'
\end{aligned}
$$

Let's define map for $Streams$

We can define some standard list processing functions on $Streams$.

Let's try map.

$$
\begin{aligned}
&map_s :: (a \to b) \to Stream\ a \to Stream\ b \\
&map_s\ f\ (Stream\ next0\ s0) = Stream\ next\ s0 \\
&\quad \mathbf{where} \\
&\qquad next\ s = \mathbf{case}\ next0\ s\ \mathbf{of} \\
&\qquad\quad \begin{array}{lll}
Done & \to Done & \\
Skip\ s' & \to Skip & s' \\
Yield\ x\ s' & \to Yield\ (f\ x) & s'
\end{array}
\end{aligned}
$$

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&\qquad next\ s = \mathbf{case}\ next0\ s\ \mathbf{of} \\
&\qquad\quad Done \qquad\ \ \to Done \\
&\qquad\quad Skip\ s' \qquad \to Skip \qquad\quad\ s' \\
&\qquad\quad Yield\ x\ s' \to Yield\ (f\ x)\ s'
\end{aligned}
$$

$$
\begin{aligned}
&map_{[a]} :: (a \to b) \to [a] \to [b] \\
&map_{[a]}\ f = unstream \circ map_s\ f \circ stream
\end{aligned}
$$

Fusion on Streams

Fusion on streams only has one rewrite rule, and it is pretty simple.

```
{-# RULES "stream" ∀ (s :: Stream a).
    stream (unstream s) = s #-}
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$$mapTestStream :: [Int] \to [Int]$$
$$mapTestStream\ xs = map_{[a]}\ (+1) \circ map_{[a]}\ (*2)\ \$\ xs$$

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    stream (unstream s) = s #-}
```

$$mapTestStream :: [Int] \rightarrow [Int]$$
$$mapTestStream\ xs = map_{[a]}\ (+1) \circ map_{[a]}\ (*2)\ \$\ xs$$

$$map_{[a]}\ (+1) \circ map_{[a]}\ (*2)$$
$\equiv$ {-expand mapl -}
$$unstream \circ map_s\ (+1) \circ stream \circ unstream \circ map_s\ (*2) \circ stream$$
$\equiv$ {-apply "stream/unstream" -}
$$unstream \circ map_s\ (+1) \circ map_s\ (*2) \circ stream$$

Map fused by Stream Fusion

Our map example

$$mapTestStream :: [Int] \to [Int]$$
$$mapTestStream\ xs = map_{[a]}\ (+1) \circ map_{[a]}\ (*2)\ \$\ xs$$

gets fused into this result.

$$mapTestStreamCompiled :: [Int] \to [Int]$$
$$mapTestStreamCompiled\ [\,] = [\,]$$
$$mapTestStreamCompiled\ (x : xs) =$$
$$1 + (x * 2) : mapTestStreamCompiled\ xs$$

Table of Contents

We can make $process$ even faster with $Data.Vector$

The $Data.Vector$ package uses stream fusion and many other rewrite rules behind the scenes in order to optimize array based computations.

$$process\ xs = sum0 \circ map\ sq\ \$\ xs$$

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The $Data.Vector$ package uses stream fusion and many other rewrite rules behind the scenes in order to optimize array based computations.

$$process\ xs = sum0 \circ map\ sq\ \$\ xs$$

The vector version looks very similar.

$$\textbf{import}\ qualified\ Data.Vector\ as\ V$$

$$processVec\ n = V.sum\ \$\ V.map\ sq\ \$\ V.enumFromTo\ 1\ (n :: Int)$$

We can make $process$ even faster with $Data.Vector$

$$processVec\ n = V.sum \ \$ \ V.map\ sq \ \$ \ V.enumFromTo\ 1\ (n :: Int)$$

But has awesome performance!

Function	Time (ms)	Memory (MB)
$process$	41.86	265.26
$process_{fuse}$	25.31	96.65
$processVec$	0.7	16×10^{-5}

What code does $Data.Vector$ generate?

The $processVec$ function is pretty simple in Haskell itself.

$$processVec\ n = V.sum\ \$\ V.map\ sq\ \$\ V.enumFromTo\ 1\ (n :: Int)$$

When compiling, GHC fires *202 rules!*

What code does $Data.Vector$ generate?

The $processVec$ function is pretty simple in Haskell itself.

$$processVec\ n = V.sum\ \$\ V.map\ sq\ \$\ V.enumFromTo\ 1\ (n :: Int)$$

When compiling, GHC fires *202 rules!*

Specifically, this appears when using the debug flag `-ddump-rule-firings`. [10]

```
...
Rule fired: stream/unstream [Vector]
Rule fired: stream/unstream [Vector]
...
```

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$$processVec\ n = V.sum\ \$\ V.map\ sq\ \$\ V.enumFromTo\ 1\ (n :: Int)$$

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```
...
Rule fired: stream/unstream [Vector]
Rule fired: stream/unstream [Vector]
...
```

What code does $Data.Vector$ generate?

The $processVec$ function is pretty simple in Haskell itself.

$$processVec\ n = V.sum \$ V.map\ sq \$ V.enumFromTo\ 1\ (n :: Int)$$

And the final code generated is the following.

$$\begin{aligned}
&processVecGHC\ n = loop\ 1\ 0\\
&\quad \mathbf{where}\\
&\qquad loop\ count\ acc = \mathbf{case}\ count \leqslant n\ \mathbf{of}\\
&\qquad\quad False \rightarrow acc\\
&\qquad\quad True \rightarrow loop\ (count + 1)\ (acc + (count * count))
\end{aligned}$$

Other use cases for fusion

Besides vector, fusion is used in a few other places.

- Repa, a parallel list processing library [11]

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Besides vector, fusion is used in a few other places.

- Repa, a parallel list processing library [11]
- Vector instructions by SIMD [12]
- Pipes, a stream processing library [13]

Wrap up

What did we talk about today?

- Goal: simple code that performed as well as a optimized version.
- A brief introduction to compilation in GHC and rewrite rules.
- $foldr$ / $build$ fusion.
- Showed a second type of fusion: stream fusion.
- Went through some libraries using fusion.

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