Fusion: Applying Equational Transforms to Simplify Programs

`github.com/ryanorendorff/lc-2017-fusion`

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May 2017

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Outline

Motivation: Simple Programs versus Performance

A brief introduction to GHC

List fusion with $foldr/build$

Stream Fusion

Applications of Fusion

Motivation: Simple Programs versus Performance

Common way to process a list: map and fold!

As an example, say we want to square all the elements in a list and then sum the result.

$$process :: [Int] \to Int$$

$$process\ xs = sum \circ map\ sq\ \$\ xs$$

Where we have defined the functions as follows.

$$\begin{aligned} map\ \_\ [\,] &= [\,] \\ map\ f\ (x : xs) &= f\ x : map\ f\ xs \end{aligned}$$

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$$foldr :: (a \to b \to b) \to b \to [a] \to b$$
$$\begin{aligned} foldr\ \_\ z\ [\,] &= z \\ foldr\ f\ z\ (x : xs) &= f\ x\ (foldr\ f\ z\ xs) \end{aligned}$$

$$sum = foldr\ (+)\ 0$$

How fast is $process$?

So now that we have our process function, how fast does it run?

$$process :: [Int] \rightarrow Int$$

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Let's try to process a million elements with our $process$ and $process'$, which uses the standard Prelude sum and map.

$$process\ [0 \mathinner{..} 1,000,000]; process'\ [0 \mathinner{..} 1,000,000]$$

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How does the Prelude do better with the same functions?

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It seems we have matched GHC's performance!

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Our manual version $process_{hand}$.

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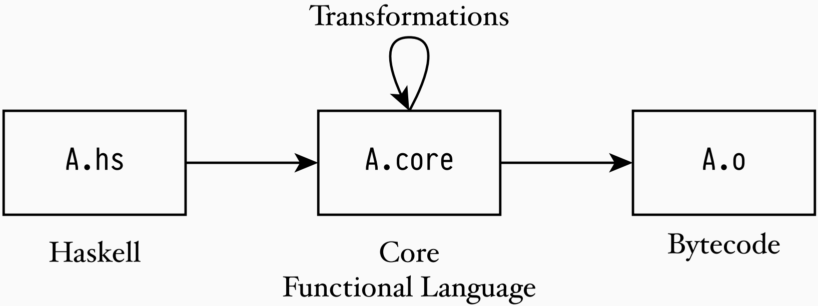
How can we leverage the compiler to write simple code that is fast?

Table of Contents

The GHC Compilation Pipeline converts Haskell into an intermediate language and then bytecode

When GHC compiles a Haskell program, it converts the code into an intermediate language called "Core", which is then (eventually) turned into byte code.



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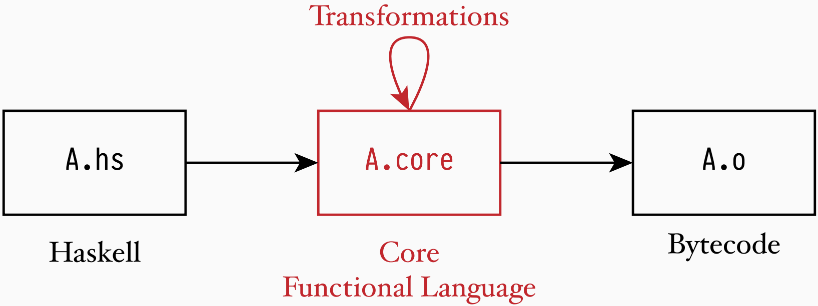

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- After the period is the what we are saying are equivalent statements.

Rules have some restrictions

Rewrite rules have some gotchas.

- Rules doesn't prevent you from doing something silly

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- You can make the compiler go into an infinite loop.

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{-# RULES "fxy" forall x y. f x y = f y x #-}
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  ```

- If multiple rules are possible, GHC will randomly choose one.

We can combine maps to traverse a list once

Let us introduce the following rule about maps.

```
{-# RULES "map/map" forall f g xs.
  map_fuse f (map_fuse g xs) =  map_fuse (f.g) xs #-}
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$$mapTest :: [Int] \to [Int]$$

$$mapTest\ xs = map\ (+1)\ (map\ (*2)\ xs)$$

$$mapTest_{fuse} :: [Int] \to [Int]$$

$$mapTest_{fuse}\ xs = mapfuse\ (+1)\ (mapfuse\ (*2)\ xs)$$

Our map fusion performs (a bit) better!

We can test our functions on a million elements

$$mapTest\ xs = map\ (+1)\ (map\ (*2)\ xs)$$

$$mapTest_{fuse}\ xs = mapfuse\ (+1)\ (mapfuse\ (*2)\ xs)$$

and find we get a bit better time and space performance.

Function	Time (ms)	Memory (MB)
$mapTest$	26.4	256.00
$mapTest_{fuse}$	17.6	184.00

Through rules, GHC performs fusion

Some of the rules work together to perform *fusion*: to combine terms in such a way as to pass over a data structure once.

In our $process$ function, we create an intermediate list

$$process :: [Int] \rightarrow Int$$

$$process\ xs = sum \circ map\ sq\ \$\ xs$$

whereas our "fused" form did not make any intermediate structure, and used an accumulator instead.

$$process_{hand} :: [Int] \rightarrow Int$$

$$process_{hand}\ [\,] = 0$$
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foldr / *build* fusion is used to simplify list computations

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$foldr$ combines the elements of a list

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$$foldr\ f\ \ z\ [\,] = z$$
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$$
\begin{aligned}
&build1\ l \equiv [1, 2, 3] \\
&\quad \textbf{where} \\
&\qquad l\ cons\ nil = 1\ `cons`\ (2\ `cons`\ (3\ `cons`\ nil))
\end{aligned}
$$

The *foldr*/*build* rule removes intermediate fold/build pairs

To remove intermediate data structures (those created by *build*), we eliminate *foldr*/*build* pairs with a rule.

```
{-# RULES
"foldr/build"
∀ f z (g :: ∀ b. (a -> b -> b) -> b -> b).
foldr f z (build g) = g f z #-}
```

$$
\begin{aligned}
&\mathit{foldr}\ (+)\ 0\ (\mathit{build}\ l) \equiv l\ (+)\ 0 \equiv 1 + (2 + (3 + 0))\\
&\quad \textbf{where}\\
&\qquad l\ \mathit{cons}\ \mathit{nil} = 1\ \text{`}\mathit{cons}\text{`}\ (2\ \text{`}\mathit{cons}\text{`}\ (3\ \text{`}\mathit{cons}\text{`}\ \mathit{nil}))
\end{aligned}
$$

We need a few extra rules to convert maps into fold/builds

To convert our definition of maps into a fold/build pair, we need the following helper function.

$$
\begin{aligned}
&mapFB :: (elt \to lst \to lst) \to (a \to elt) \to a \to lst \to lst \\
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With that, we have all we need to convert map into build/fold.

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{-# RULES "map" ∀ f xs. map f xs =
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$$sq\ x + process\ xs$$

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Now let's do the $(x : xs)$ case.

$$process\ (x : xs) = foldfuse\ (\lambda x\ ys \rightarrow sq\ x + ys)\ 0\ (x : xs)$$

$\equiv$ {-expand $foldr$ case: $foldr\ f\ z\ (x : xs) = f\ x\ (foldr\ f\ z\ xs)$ -}

$$(\lambda x\ ys \rightarrow sq\ x + ys)\ x\ (foldr\ (\lambda x\ ys \rightarrow sq\ x + ys)\ z\ xs)$$

$\equiv$ {-use definition of $processFuse$: $foldr\ f\ 0\ xs = processFuse\ xs$ -

$$(\lambda x\ ys \rightarrow sq\ x + ys)\ x\ (processFuse\ xs)$$

$\equiv$ {-apply lambda -}

$$sq\ x + process\ xs$$

$\equiv$ {-inline sq -}

$$x * x + processFuse\ xs$$

Bringing both cases back together

If we now combine our two cases, we have the following

$$process_{hand}\ [\,] = 0$$
$$process_{hand}\ (x : xs) = x * x + process_{hand}\ xs$$

This is the same as what we had originally written manually!

We achieved list fusion using $foldr$ / $build$ with rewrite rules

We managed to fuse $process$ using our rewrite rules. We can look at the output of the compiler and it confirms what we expected.

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Function	Time (ms)	Memory (MB)
$process$	41.86	265.26
$process'$	25.31	96.65
$process_{hand}$	25.31	96.65
$processFuse$	25.31	96.65

Table of Contents

Introduction to Stream

The Stream fusion system attempts to do something similar, by defining a list as a state machine.

$$
\begin{array}{l}
\textbf{data}\ \mathit{Stream}\ a\ \textbf{where} \\
\quad \mathit{Stream} :: (s \rightarrow \mathit{Step}\ a\ s) \rightarrow s \rightarrow \mathit{Stream}\ a
\end{array}
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$$
\begin{array}{lcll}
\textbf{data}\ \mathit{Step}\ a\ s & = & \mathit{Done} & \\
 & \mid & \mathit{Skip} & s \\
 & \mid & \mathit{Yield}\ a & s
\end{array}
$$

Streams have little helpers to make lists

To work on standard lists, we introduce the following two functions to convert between lists and streams.

$$
\begin{aligned}
&steam &&:: [\,a\,] \rightarrow Stream\ a \\
&unstream &&:: Stream\ a \rightarrow [\,a\,]
\end{aligned}
$$

Note that these functions are inverses.

$$
\begin{aligned}
&stream \circ unstream \equiv id_{stream} \\
&unstream \circ stream \equiv id_{[a]}
\end{aligned}
$$

Maps on Streams!

$$
\begin{array}{l}
\mathit{maps} :: (a \to b) \to \mathit{Stream}\ a \to \mathit{Stream}\ b \\
\mathit{maps}\ f\ (\mathit{Stream}\ \mathit{next0}\ \mathit{s0}) = \mathit{Stream}\ \mathit{next}\ \mathit{s0} \\
\quad \mathbf{where} \\
\qquad \mathit{next}\ s = \mathbf{case}\ \mathit{next0}\ s\ \mathbf{of} \\
\qquad\quad \mathit{Done} \to \mathit{Done} \\
\qquad\quad \mathit{Skip}\ s' \to \mathit{Skip}\ s' \\
\qquad\quad \mathit{Yield}\ x\ s' \to \mathit{Yield}\ (f\ x)\ s'
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\qquad\quad \mathit{Yield}\ x\ s' \to \mathit{Yield}\ (f\ x)\ s'
\end{array}
$$

$$
\begin{array}{l}
\mathit{mapl} :: (a \to b) \to [a] \to [b] \\
\mathit{mapl}\ f = \mathit{unstream} \circ \mathit{maps}\ f \circ \mathit{stream}
\end{array}
$$

Stream Fusion!

Fusion on streams only has one rewrite rule, and it is pretty simple.

```
{-# RULES "stream" ∀ (s :: Stream a).
    stream (unstream s) = s #-}
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\begin{aligned}
&\mathit{mapTestStream} :: [\mathit{Int}] \to [\mathit{Int}] \\
&\mathit{mapTestStream}\ \mathit{xs} = \mathit{mapl}\ (+1)\ (\mathit{mapl}\ (*2)\ \mathit{xs})
\end{aligned}
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$$

$$
\begin{aligned}
&\mathit{mapTestStreamCompiled} :: [\mathit{Int}] \to [\mathit{Int}] \\
&\mathit{mapTestStreamCompiled}\ [\,] = [\,] \\
&\mathit{mapTestStreamCompiled}\ (x : \mathit{xs}) = \\
&\quad 1 + (x * 2) : \mathit{mapTestStreamCompiled}\ \mathit{xs}
\end{aligned}
$$

Table of Contents

We can make $process$ even faster with $Data.Vector$

The $Data.Vector$ package uses stream fusion and many other rewrite rules behind the scenes in order to optimize array based computations.

$$process\ xs = sum0 \circ map\ sq\ \$\ xs$$

We can make $process$ even faster with $Data.Vector$

The $Data.Vector$ package uses stream fusion and many other rewrite rules behind the scenes in order to optimize array based computations.

$$process\ xs = sum0 \circ map\ sq\ \$\ xs$$

The vector version looks very similar.

$$\textbf{import}\ qualified\ Data.Vector\ as\ V$$

$$processVec\ n = V.sum\ \$\ V.map\ sq\ \$\ V.enumFromTo\ 1\ (n :: Int)$$

We can make *process* even faster with *Data.Vector*

But has incredible performance!

Function	Time (ms)	Memory (MB)
process	41.86	265.26
processFuse	25.31	96.65
processVec	0.7	16×10^{-5}

What code does $Data.Vector$ generate?

While we wrote this in our program

$$processVec\ n = V.sum\ \$\ V.map\ sq\ \$\ V.enumFromTo\ 1\ (n :: Int)$$

GHC then generates the following code (simplified back to Haskell).

What code does $Data.Vector$ generate?

While we wrote this in our program

$$processVec\ n = V.sum\ \$\ V.map\ sq\ \$\ V.enumFromTo\ 1\ (n :: Int)$$

GHC then generates the following code (simplified back to Haskell).

$$\begin{aligned}
&processVecGHC\ n = loop\ 1\ 0\\
&\quad \textbf{where}\\
&\qquad loop\ count\ acc = \textbf{case}\ count \leqslant n\ \textbf{of}\\
&\qquad\quad False \rightarrow acc\\
&\qquad\quad True \rightarrow loop\ (count + 1)\ (acc + (count * count))
\end{aligned}$$

Repa: A numerical Haskell Library using Fusion

Repa also uses fusion in order to handle parallel array operations.

$$\textbf{import}\ \mathit{qualified}\ \mathit{Data.Array.Repa}\ \mathit{as}\ R$$

$$\mathit{processRepa}\ n = R.\mathit{foldP}\ (+)\ 0 \circ R.\mathit{map}\ \mathit{sq}\ \$\ \mathit{array}$$
$$\quad \textbf{where}$$
$$\qquad \mathit{array} = R.\mathit{fromListUnboxed}\ (R.Z\ R.:.\ (n :: \mathit{Int}))\ [1\mathinner{..}n]$$