Formalizing Linear Algebra Algorithms

using Dependently Typed Functional Programming

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For this talk, we will mostly be using Agda syntax (“Haskell-like”).

First step: define a type for a matrix

A matrix can be seen as a table of numbers, which we could encode as

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```

which is equivalent to the Haskell

```
data MatrixOfNumbers a = ConstructMatrixOfNumbers [[a]]
```

and in Python

```
A = TypeVar['A']

@dataclass
class MatrixOfNumbers(Generic[A])
    matrix : List[List[A]]
```

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Mₙ : MatrixOfNumbers ℕ -- Natural numbers
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Conventions used in this talk : `A` is a type, M_i is a matrix, `m n p q` are natural numbers, and `u v x y` are vectors.

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A matrix can be used in a few different cases:

1. Multiply a matrix with a vector (matrix-vector multiply): Mx
2. Transform a matrix to get a new matrix (transpose): $M^T x$
3. Combine matrices (matrix-matrix multiply): $M_1 * M_2$

What is matrix-vector multiplication?

Matrix-vector multiply transforms one vector into another through multiplication and addition.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{M} \underbrace{*}_{*} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{x} \underbrace{=}_{=} \begin{bmatrix} (1*1)+(2*2)+(3*3) \\ (4*1)+(5*2)+(6*3) \end{bmatrix} = \underbrace{\begin{bmatrix} 14 \\ 32 \end{bmatrix}}_{y}$$

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$$M \quad * \quad x \quad = \quad y$$

Another way to think of matrix-vector multiplication is

M is a *function* from vectors of size 3 to vectors of size 2. This function is sometimes called a *linear map*.

Example of a matrix as a function: identity

The identity matrix converts a vector into the same vector.

$$I * v = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} * v = v$$

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If we were to write out the identity matrix as a function, it would be the same as the identity function.

```
list-identity : List A → List A
list-identity l = l
```

Example of a matrix as a function: diag

The diagonal matrix point-wise multiplies one vector with another (written as $*^V$).

$$diag(u) * v = \begin{bmatrix} u_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_n \end{bmatrix} * v = u *^V v$$

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written as a function, this would look like

```
diag : List A → (List A → List A)
diag u = λ v → u *ⱽ v
```

or alternatively as

```
diag u = λ v → zipWith _ (_*_) u v
```

Let’s define a matrix as a function!

We can define a matrix as just a function then that takes a vector and returns a new one.

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data FunctionalMatrix (A : Set) : Set where
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This addresses the matrix-vector ability of a matrix, what else can we tackle functionally?

We have matrix-vector multiply down, can we do more?

With our functional definition of a matrix, we can do other operations like matrix-matrix multiply.

$$(M_1 * M_2)v = M_1(M_2(v))$$

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_·ᶠ_ : FunctionalMatrix A → List A → List A
ConstructFunctionalMatrix f ·ᶠ l = f l

apply_two_matrices : FunctionalMatrix A → FunctionalMatrix A
                   → List A → List A
apply_two_matrices M₁ M₂ v = M₁ ·ᶠ M₂ ·ᶠ v
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```

Hmm that looks a lot like composition:

```
_∘ᶠ_ : FunctionalMatrix A → FunctionalMatrix A → FunctionalMatrix A
M₁ ∘ᶠ M₂ = ConstructFunctionalMatrix (apply_two_matrices M₁ M₂)
```

We often need the transpose matrix at the same time

This type encapsulates the function nature of a matrix, but we often need the transpose as well.

```
data FunctionalMatrixWithTranpose (A : Set) : Set where
    ConstructFMT :  (List A → List A) -- Forward function
                  → (List A → List A) -- Transpose function
                  → FunctionalMatrixWithTranpose A
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We can now define the identity matrix with the transpose matrix function, which is also the identity.

```
Mᵢ,ₜ : FunctionalMatrixWithTranpose A
Mᵢ,ₜ = ConstructFMT (list-identity) (list-identity)
```

What are the benefits of this approach?

We gain a few benefits from using functions directly.

- Write out the model for a process in a more direct manner.

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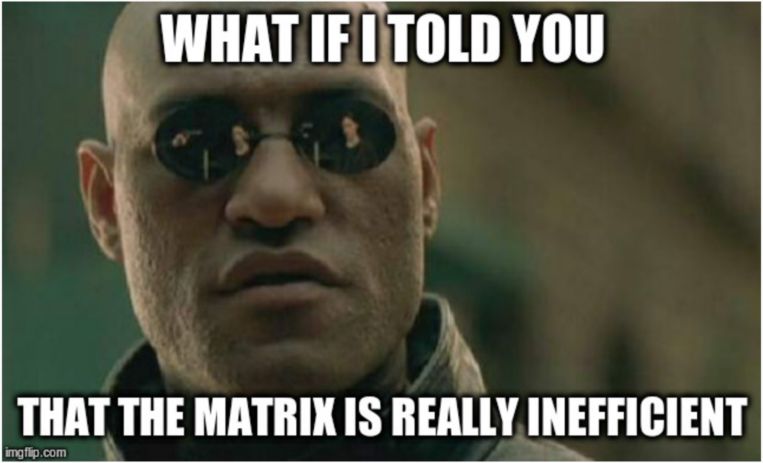
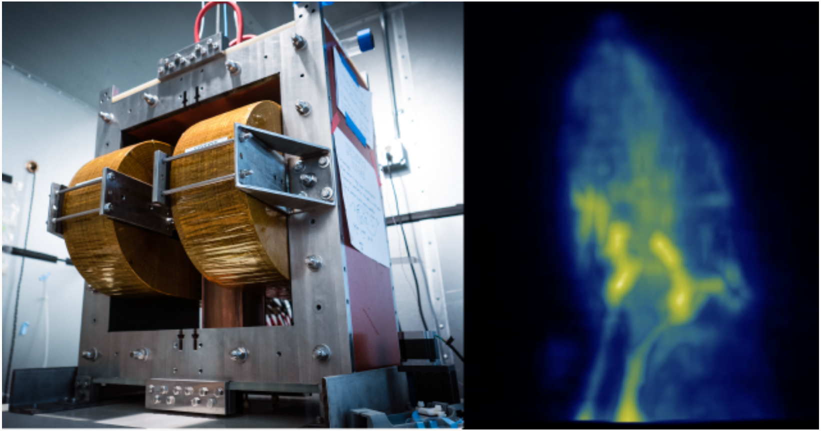
- Write out the model for a process in a more direct manner.
- Speed and time benefits.



Magnetic Particle Imaging reconstructs images using functional matrices

A model of how the device (left) generates signals from the sample (rat, right) is encoded as “matrix-free” functions in Python using PyOp, the python implementation of this idea.



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Metric	Matrix	Matrix-Free	Improvement
Space	150 GB	bytes	10^9x
Time	60 min	2 min	$30x$
Use of functional concepts	No	Yes	Priceless

Intuition check : convert a functional matrix into a number matrix

For the rest of the rules of linear algebra to apply, we should always be able to define an equivalent functions using only multiplication and addition.

For example our original identity function

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identity' : List A → List A
identity' v = v
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identity' v = v
```

could be written as

```
identity' : List A → List A
identity' v = replicate (len v) 1 *ⱽ v
```

where `replicate` creates a list of 1s and `*ⱽ` multiplies each element in two vectors together.

Is `FunctionalMatrixWithTranspose` “correct by construction”?

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Correct by construction linear algebra

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```
f₁ : List ℕ → List ℕ
f₁ v = randomlySizedNewList v


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Mᵣ = ConstructFMT f₁ f₁
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$$M_r = \begin{bmatrix} \text{⚀} & \text{⚁} \\ \text{⚂} & \text{⚃} \end{bmatrix}$$

If we could convert a random number generator to a number, sure! `:-(`

Encoding the length of the vector in the type

Agda allows us to specify what the length of a vector as part of the type.[1]

```
v : Vec ℕ 3
v = [ 1 , 2 , 3 ]ᵛ
```

[1]This is a bit of a misnomer; the difference between term and type is muddled in most dependently typed languages.

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If we try to create a vector of the wrong length, Agda will tell us.

```
v₂ : Vec ℕ 2
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Vec is a *dependent type* because its type *depends on a value*.

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Use functions on Vec to ensure that the shapes match

We can define a matrix type where the shapes are preserved.

```
data SizedMatrix (A : Set) (m n : ℕ) : Set where
    ConstructSizedMatrix :  (Vec A n → Vec A m) -- Forward function
                         → (Vec A m → Vec A n) -- Transpose function
                         → SizedMatrix A m n
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In Haskell, we would write this as

```
data SizedMatrix (A :: *) (m :: Nat) (n :: Nat) where
    ConstructSizedMatrix :: (KnownNat m, KnownNat n)
                         ⇒ (Vec A n → Vec A m) -- Forward function
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We can now define our identity matrix again.

```
id : (A : Set) → A → A

Mᵢ,ₛ : SizedMatrix A n n
Mᵢ,ₛ = ConstructSizedMatrix id id -- id : Vec A n → Vec A n
```

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If we wanted to convert this to multiplication and addition only....

$$M\spadesuit = \begin{bmatrix} \clubsuit & \heartsuit \\ \spadesuit & \diamondsuit \end{bmatrix}$$

Matrices cannot contain just anything! The elements have to be able to be added/multiplied.

Matrices are defined over fields

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    -_  : A → A -- + inverse, - 4
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    0ᶠ  : A -- Identity of _+_, 4 + 0ᶠ = 4
    1ᶠ  : A -- Identity of _*_, 4 * 1ᶠ = 4
```

We can define matrices that operate on Fields only

Now we can restrict our A type to having a defined version of + and *.

```
data SizedFieldMatrix (A : Set) ⦃ F : Field A ⦄ (m n : ℕ) : Set where
    ConstructSFM :  (Vec A n → Vec A m) -- Forward function
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in Haskell this would be written as

```
data SizedFieldMatrix A (m :: Nat) (n :: Nat) where
    ConstructSFM :: (KnownNat m, KnownNat n, Field A)
                 => (Vec A n → Vec A m) -- Forward function
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The card example can no longer be constructed, but the identity matrix still can be constructed.

```
-- + and * must be defined on A
Mₛᶠᵢ : ⦃ F : Field A ⦄ → SizedFieldMatrix A n n
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Are we missing anything else to be “correct by construction”?

Matrices are linear functions

Matrices have the following properties that we'd like to preserve:

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Currently we could define a matrix like so, which has neither property.

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- Linearity : $f(u +^V v) = 1 \;\neq\; f(u) +^V f(v) = 1 +^V 1 = 2$

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How do we ensure that our functions are linear?

For our matrices to make sense, we need the functions that are used for the forward and transpose functions to be linear functions.

```
-- A linear function (aka a linear map)
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    f : (Vec A m → Vec A n)
    f[u+v]≡f[u]+f[v] : (u v : Vec A m) → f (u +ⱽ v) ≡ f u +ⱽ f v
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    f[u+v]≡f[u]+f[v] : (u v : Vec A m) → f (u +ⱽ v) ≡ f u +ⱽ f v
    f[c*v]≡c*f[v] : (c : A) → (v : Vec A m) → f (c ∘ⱽ v) ≡ c ∘ⱽ (f v)
```

How do we ensure that our functions are linear?

For our matrices to make sense, we need the functions that are used for the forward and transpose functions to be linear functions.

```
-- A linear function (aka a linear map)
record _⊸_ {A : Set} ⦃ F : Field A ⦄ (m n : ℕ) : Set where
  field
    f : (Vec A m → Vec A n)
    f[u+v]≡f[u]+f[v] : (u v : Vec A m) → f (u +ⱽ v) ≡ f u +ⱽ f v
    f[c*v]≡c*f[v] : (c : A) → (v : Vec A m) → f (c ∘ⱽ v) ≡ c ∘ⱽ (f v)
```

with this we could define our matrices using linear functions.

```
data LinearMatrix {A : Set} ⦃ F : Field A ⦄ (m n : ℕ) : Set where
  ConstructLinearMatrix : (n ⊸ m) → (m ⊸ n) → LinearMatrix m n
```

What is this ≡ thing?

The ≡ sign means that two things are equal[2] in the sense that the left and the right side can be written with the same order of constructors[3].

[2] Homogenously

[3] Their normal forms are equivalent

What is this ≡ thing?

The ≡ sign means that two things are equal[2] in the sense that the left and the right side can be written with the same order of constructors[3].

The definition of ≡ is

```
data _≡_ (x : A) : A → Set where
  refl : x ≡ x
```

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What is this ≡ thing?

The ≡ sign means that two things are equal[2] in the sense that the left and the right side can be written with the same order of constructors[3].

The definition of ≡ is

```
data _≡_ (x : A) : A → Set where
  refl : x ≡ x
```

we can note two things

- The only way to construct an instance of ≡ is the `refl` constructor
- The `refl` constructor can only be constructed from two pieces that are the same `A`.

[2] Homogenously

[3] Their normal forms are equivalent

Demonstrating equality on natural numbers

For example, if we have the data type for natural numbers

```
data ℕ where
  zero : ℕ      -- 0
  suc  : ℕ → ℕ -- 1 + n
```

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we can demonstrate that two numbers are equivalent by making sure they are the same series of `suc` and `zero`.

```
two = suc (suc zero)

a = suc (suc (suc zero)) -- 3
b = suc (two)            -- 3 as well
```

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```
two = suc (suc zero)

a = suc (suc (suc zero)) -- 3
b = suc (two)            -- 3 as well

_ : a ≡ b
_ = refl -- suc (suc (suc zero)) ≡ suc (suc (suc zero))
```

Fields must follow some properties on top of defining + and *

Fields define more than just + and *; a field must also adhere to some properties.

```
+-assoc   : (a b c : A) → a + (b + c) ≡ (a + b) + c
+-comm    : (a b : A)   → a + b ≡ b + a
+-0       : (a : A)     → a + 0ᶠ ≡ a
+-inv     : (a : A)     → (- a) + a ≡ 0ᶠ
```

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+-inv     : (a : A)     → (- a) + a ≡ 0ᶠ

*-assoc   : (a b c : A) → a * (b * c) ≡ (a * b) * c
*-comm    : (a b : A)   → a * b ≡ b * a
*-1       : (a : A)     → a * 1ᶠ ≡ a
*-inv     : (a : A)     → (a ≢ 0ᶠ) → (a ⁻¹) * a ≡ 1ᶠ
```

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*-assoc   : (a b c : A) → a * (b * c) ≡ (a * b) * c
*-comm    : (a b : A)   → a * b ≡ b * a
*-1       : (a : A)     → a * 1ᶠ ≡ a
*-inv     : (a : A)     → (a ≢ 0ᶠ) → (a ⁻¹) * a ≡ 1ᶠ

*-distr-+ : (a b c : A) → a * (b + c) ≡ (a * b) + (a * c)
```

Proofs can be used to rewrite terms

Let's use the `Field` proofs we have to construct a new proof.

```
new_proof : (b : A) → (b + 0ᶠ) * 1ᶠ ≡ b
new_proof b = begin
  (b + 0ᶠ) * 1ᶠ
```

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  b + 0ᶠ
  ≡⟨ +-0 b ⟩ -- +-1 : (a : A) → a + 0ᶠ ≡ a
  b ∎
```

Proving that the identity function is linear

The linear identity function is simple

```
id₁ : ⦃ F : Field A ⦄ → n ⊸ n
id₁ = record
  { f = id -- Vec A n → Vec A n
```

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  ; f[u+v]≡f[u]+f[v] = λ u v → refl -- id (u +ⱽ v) ≡ id u +ⱽ id v
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```

Now let's try to define the **diag** function as a linear function

```
diag₁ : ⦃ F : Field A ⦄ → Vec A n → n ⊸ n
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  -- *ⱽ∘ⱽ≡∘ⱽ*ⱽ : d *ⱽ (c ∘ⱽ v) ≡ c ∘ⱽ (d *ⱽ v)
  ; f[c*v]≡c*f[v] = λ c v → *ⱽ∘ⱽ≡∘ⱽ*ⱽ c d v
  }
```

Let's go through the linearity proof for **diag**

To show how one proves linearity for `diag`, let's step through the proof.

```
*ⱽ-distr-+ⱽ' : (d u v : Vec A n)
             → d *ⱽ (u +ⱽ v) ≡ d *ⱽ u +ⱽ d *ⱽ v
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*ᵛ-distr-+ᵛ' []ᵛ []ᵛ []ᵛ = refl
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*ⱽ-distr-+ⱽ' (d₀ ∷ⱽ dᵣ) (u₀ ∷ⱽ uᵣ) (v₀ ∷ⱽ vᵣ) = begin
    (d₀ ∷ⱽ dᵣ) *ⱽ ((u₀ ∷ⱽ uᵣ) +ⱽ (v₀ ∷ⱽ vᵣ)) ≡⟨⟩
    (d₀ * (u₀ + v₀)) ∷ⱽ (dᵣ *ⱽ (uᵣ +ⱽ vᵣ))
```

Let's go through the linearity proof for **diag**

To show how one proves linearity for **diag**, let's step through the proof.

```
*V-distr-+V' : (d u v : Vec A n)
             → d *V (u +V v) ≡ d *V u +V d *V v

*V-distr-+V' []V []V []V = refl

*V-distr-+V' (d₀ ::V dᵣ) (u₀ ::V uᵣ) (v₀ ::V vᵣ) = begin
    (d₀ ::V dᵣ) *V ((u₀ ::V uᵣ) +V (v₀ ::V vᵣ)) ≡⟨⟩
    (d₀ * (u₀ + v₀)) ::V (dᵣ *V (uᵣ +V vᵣ))

  ≡⟨ cong ((d₀ * (u₀ + v₀)) ::V_) (*V-distr-+V' dᵣ uᵣ vᵣ) ⟩
    (d₀ * (u₀ + v₀)) ::V (dᵣ *V uᵣ +V dᵣ *V vᵣ)
```

Let's go through the linearity proof for **diag**

To show how one proves linearity for **diag**, let's step through the proof.

```
*ⱽ-distr-+ⱽ' : (d u v : Vec A n)
             → d *ⱽ (u +ⱽ v) ≡ d *ⱽ u +ⱽ d *ⱽ v

*ⱽ-distr-+ⱽ' []ⱽ []ⱽ []ⱽ = refl

*ⱽ-distr-+ⱽ' (d₀ ∷ⱽ dᵣ) (u₀ ∷ⱽ uᵣ) (v₀ ∷ⱽ vᵣ) = begin
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    (d₀ * (u₀ + v₀)) ∷ⱽ (dᵣ *ⱽ (uᵣ +ⱽ vᵣ))

  ≡⟨ cong ((d₀ * (u₀ + v₀)) ∷ⱽ_) (*ⱽ-distr-+ⱽ' dᵣ uᵣ vᵣ) ⟩
    (d₀ * (u₀ + v₀)) ∷ⱽ (dᵣ *ⱽ uᵣ +ⱽ dᵣ *ⱽ vᵣ)

  ≡⟨ cong (_∷ⱽ (dᵣ *ⱽ uᵣ +ⱽ dᵣ *ⱽ vᵣ)) (*-distr-+ d₀ u₀ v₀) ⟩
    (d₀ * u₀ + d₀ * v₀) ∷ⱽ (dᵣ *ⱽ uᵣ +ⱽ dᵣ *ⱽ vᵣ) ≡⟨⟩
    (d₀ ∷ⱽ dᵣ) *ⱽ (u₀ ∷ⱽ uᵣ) +ⱽ (d₀ ∷ⱽ dᵣ) *ⱽ (v₀ ∷ⱽ vᵣ) ∎
```

How we can finally define our LinearMatrix!

We can finally define a linear function.

```
data LinearMatrix {A : Set} ⦃ F : Field A ⦄ (m n : ℕ) : Set where
  ConstructLinearMatrix : (n ⊸ m) → (m ⊸ n) → LinearMatrix m n

id-linear : ⦃ F : Field A ⦄ → LinearMatrix n n
id-linear = ConstructLinearMatrix idₗ idₗ
```

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id-linear = ConstructLinearMatrix id₁ id₁
```

Have we reached “Correct by construction linear algebra”?

Does the transpose match?

Say we defined a matrix as so

```
Mₙₒ : ⦃ F : Field A ⦄ → LinearMatrix n n
Mₙₒ = ConstructLinearMatrix (id₁) (diag₁ (replicate 1ᶠ))
```

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We have mixed up the forward/transpose pairing between our two linear functions.

$$I = I^T$$

$$diag(v) = diag(v)^T$$

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```

We have mixed up the forward/transpose pairing between our two linear functions.

$$I = I^T$$

$$diag(v) = diag(v)^T$$

To solve this problem, we can show that for forward function `M` and transpose function `Mᵀ` that the following property holds.

$$\forall xy. \langle x, My \rangle = \langle y, M^T x \rangle$$

$$\langle a, b \rangle = \text{sum}(a *^V b) = \sum_i^n a_i * b_i$$

Finally we reach our goal!

If we require the user to prove the inner product property, we can *finally* create a “correct by construction” functional matrix.

```
data Mat_×_ {A : Set} ⦃ F : Field A ⦄ (m n : ℕ) : Set where
  ⟦_,_,_⟧ :  (M  : n ⊸ m )
          → (Mᵀ : m ⊸ n )
          → (p : (x : Vec A m) → (y : Vec A n)
                → ⟨ x , M ·ˡᵐ y ⟩ ≡ ⟨ y , Mᵀ ·ˡᵐ x ⟩ )
          → Mat m × n
```

where the inner product (⟨⟩) is defined as

```
⟨_,_⟩ : ⦃ F : Field A ⦄ → Vec A n → Vec A n → A
⟨ x , y ⟩ = sum (x *ⱽ y)
```

The final identity functional matrix

With this, we can finally define the identity matrix in a way that is not possible to make an error.

```
Mᴵ : ⦃ F : Field A ⦄ → Mat n × n
Mᴵ = ⟦ id₁ , id₁ , id-transpose  ⟧
  where

    id-transpose : ⦃ F : Field A ⦄ (x y : Vec A n)
                 → ⟨ x , id y ⟩ ≡ ⟨ y , id x ⟩
```

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```
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Mᴵ = ⟦ id₁ , id₁ , id-transpose  ⟧
  where

    id-transpose : ⦃ F : Field A ⦄ (x y : Vec A n)
                 → ⟨ x , id y ⟩ ≡ ⟨ y , id x ⟩

    id-transpose x y = begin
      ⟨ x , id y ⟩ ≡⟨⟩
      ⟨ x , y ⟩    ≡⟨ ⟨⟩-comm x y ⟩
      ⟨ y , x ⟩    ≡⟨⟩
      ⟨ y , id x ⟩ ∎
```

What can we do with a matrix

We can do a few things with a matrix:

1. Multiply the matrix with a vector (matrix-vector multiply): Mx
2. Transform the matrix to get a new matrix (transpose): $M^T x$
3. Combine matrices (matrix-matrix multiply): $M_1 * M_2$

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We have not done matrix-matrix multiplication, can we implement it with our new definition?

Implementing matrix-matrix multiply on functional matrices

With our first iteration, we were able to define matrix-matrix multiplication

$$M_1 * M_2$$

using function composition

```
apply_two_matrices : FunctionalMatrix A → FunctionalMatrix A
                   → List A → List A
apply_two_matrices F G v = F ·ᶠ G ·ᶠ v

_∘ᶠ_ : FunctionalMatrix A → FunctionalMatrix A → FunctionalMatrix A
F ∘ᶠ G = ConstructFunctionalMatrix (apply_two_matrices F G)
```

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F ∘ᶠ G = ConstructFunctionalMatrix (apply_two_matrices F G)
```

We can do the same with our new definition, by performing composition of the linear functions.

We are going to need a few functions to get there. One to extract the linear functions.

```
Mat-to-⊸ : ⦃ F : Field A ⦄ → Mat m × n → n ⊸ m
Mat-to-⊸ ⟦ f , t , p ⟧ = f

_ᵀ : ⦃ F : Field A ⦄ → Mat m × n → Mat n × m
⟦ f , a , p ⟧ ᵀ = ⟦ a , f , (λ x y → sym (p y x)) ⟧
```

Matrix-matrix multiply toolbox

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⟦ f , a , p ⟧ ᵀ = ⟦ a , f , (λ x y → sym (p y x)) ⟧
```

and a way to compose linear functions

```
_∘ˡ_ : ⦃ F : Field A ⦄ → n ⊸ p → m ⊸ n → m ⊸ p
g ∘ˡ h = record {
    f = λ v → g ·ˡ (h ·ˡ v)
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Mat-to-⊸ : ⦃ F : Field A ⦄ → Mat m × n → n ⊸ m
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and a way to compose linear functions

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_∘ˡ_ : ⦃ F : Field A ⦄ → n ⊸ p → m ⊸ n → m ⊸ p
g ∘ˡ h = record {
    f = λ v → g ·ˡ (h ·ˡ v)
  ; f[u+v]≡f[u]+f[v] = TrustMe!
  ; f[c*v]≡c*f[v] = TrustMe! }
```

Defining matrix-matrix multiply

We can now define matrix-matrix multiply. If we remember that we need to

$$\begin{aligned} \text{forward} &: \; M_1 * M_2 \\ \text{transpose} &: \; (M_1 * M_2)^T = M_2^T * M_1^T \end{aligned}$$

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Which we can directly encode in Agda.

```
_*ᴹ_ : ⦃ F : Field A ⦄ → Mat m × n → Mat n × p → Mat m × p
M₁ *ᴹ M₂ =
  ⟦ (Mat-to-⊸ M₁) ∘ˡ (Mat-to-⊸ M₂)
  , (Mat-to-⊸ (M₂ ᵀ)) ∘ˡ (Mat-to-⊸ (M₁ ᵀ))
```

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  , (Mat-to-⊸ (M₂ ᵀ)) ∘ˡ (Mat-to-⊸ (M₁ ᵀ))

  , TrustMe!
  ⟧
```

What do we have so far with this encoding?

We have gained some nice benefits by moving to a proven type constructor.

- We can define a performant, functional version of matrix algebra.

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- We can guarantee that our implementation is correct.
- We can use equational reasoning to prove two implementations are equivalent.

But there is a cost. For one file in the library that implements this idea, out of 213 lines of code:

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- We can define a performant, functional version of matrix algebra.
- We can guarantee that our implementation is correct.
- We can use equational reasoning to prove two implementations are equivalent.

But there is a cost. For one file in the library that implements this idea, out of 213 lines of code:

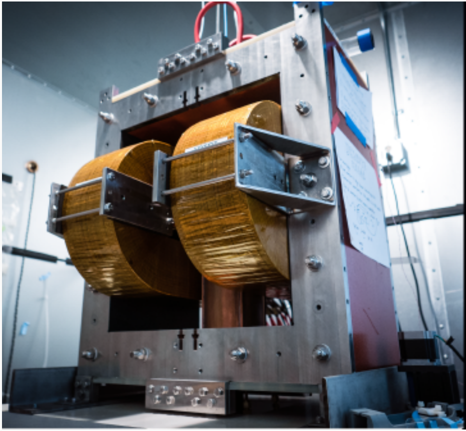
24 lines are function definitions (11.3% of the code). *Everything else is either a proof, a type signature, or an import/control statement*

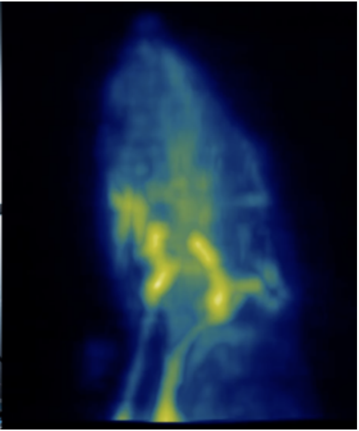
Algorithms using Linear Algebra

Magnetic Particle Imaging can be modeled using linear algebra

In MPI, we are attempting to detect where iron is within a sample.

- v : the voltages coming off of the device.
- f_e : the distribution of iron.
- M : a *function* that converts iron distributions into voltages.





Solving linear equations finds what input produces an observed result

We can write this process of converting iron distributions to voltages as

$$Mf_e = v$$

If we have the voltages v coming off the device, we want to find the iron distribution f_e that produced that signal.

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$$Mf_e = v$$

If we have the voltages v coming off the device, we want to find the iron distribution f_e that produced that signal.

To solve this problem, we want to compare how good our estimate of the input x is at producing the observed output y using the following function.

$$J(f_e) = f_e^T M^T M f_e - 2 f_e M^T v$$

Taking a **step** in the right direction

One simple way to find a better x than some initial guess is to update x *in the direction of steepest descent* ∇J.

$$f_{e,i+1} = f_{e,i} - \alpha \nabla J(f_{e,i})$$
$$f_{e,i+1} = f_{e,i} - \alpha (M^T (M f_{e,i} - v))$$

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We can implement this in Agda as

```
step :  ⦃ F : Field A ⦄
     → (α : A) → (M : Mat m × n)
     → (v : Vec A m) → (fₑ : Vec A n) → Vec A n
step α M v = λ fₑ → fₑ -ⱽ α ∘ⱽ (M ᵀ · (M · fₑ -ⱽ v))
```

Gradient descent is just running **step** a bunch of times

From there, we can find the value of x that best matches y by iterating.

```
gradient-descent : ⦃ F : Field A ⦄
                 → (n : ℕ)         -- Number of iterations to run
                 → (α : A)         -- Scale factor
                 → (M : Mat m × n) -- Model of system
                 → (v : Vec A m)   -- Data
                 → (fₑ : Vec A n)  -- Initial estimate
                 → List (Vec A n)  -- Results (farther is better)
gradient-descent n α M v fₑ = iterate n fₑ (step α M v)

-- iterate _ x f = [x, f x, f (f x), ... ]
```

We cna define equivalent forms of a linear equation

We had defined our step function as

```
step α M v fₑ = fₑ -ⱽ α ∘ⱽ (M ᵀ · (M · fₑ -ⱽ v))
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is there another way to write this function?

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step α M v fₑ = fₑ -ⱽ α ∘ⱽ (M ᵀ · (M · fₑ -ⱽ v))
```

is there another way to write this function?

yes!

```
step' :  ⦃ F : Field A ⦄
      → (α : A) → (M : Mat m × n)
      → (v : Vec A m) → (fₑ : Vec A n) → Vec A n
step' α M v fₑ = fₑ -ⱽ α ∘ⱽ (M ᵀ · M · fₑ -ⱽ M ᵀ · v)
```

Proving the two steps are in lock step

We can prove that **step** and **step'** are the same by saying that when we apply the same inputs to **step** and **step'**, we get the same result.[4]

```
proof :  ⦃ F : Field A ⦄ → (α : A)
      → (M : Mat m × n) → (v : Vec A m) → (fₑ : Vec A n)
      → step α M v fₑ ≡ step' α M v fₑ
proof α M v fₑ = begin
  fₑ -ⱽ α ∘ⱽ (M ᵀ · (M · fₑ -ⱽ v))
```

[4] Proving that **step** and **step'** are the same is an extensional statement, and requires function extensionality.

Proving the two **steps** are in lock step

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proof α M v fₑ = begin
  fₑ -ⱽ α ∘ⱽ (M ᵀ · (M · fₑ -ⱽ v))

  -- M-distr--ⱽ : M (fₑ -ⱽ v) ≡ M fₑ -ⱽ M v
  ≡⟨ cong (λ z → fₑ -ⱽ α ∘ⱽ z) (M-distr--ⱽ (M ᵀ) (M · fₑ) v) ⟩
  fₑ -ⱽ α ∘ⱽ (M ᵀ · M · fₑ -ⱽ M ᵀ · v) ∎
```

[4] Proving that **step** and **step'** are the same is an extensional statement, and requires function extensionality.

Have we accomplished our goal?

Our original goal was

Correct by construction linear algebra

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- Eliminated incorrect function pairing bugs.

Comparing the steps we went through

Through this process, we went through three different implementations of matrices as functions.

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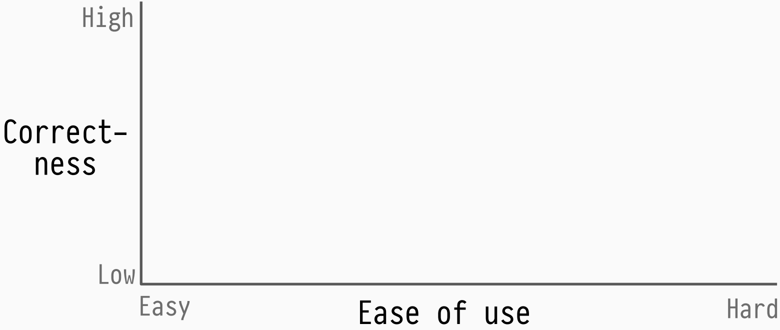
Through this process, we went through three different implementations of matrices as functions.

- Regular functions (Python: `PyOp` library)
- Size-typed functions (Haskell: `convex` library)
- Linear functions (Agda: `FLA` library)

API comparison: how likely am I to use this?

Not every library is a blast to use. How do these three functional approaches stack up?



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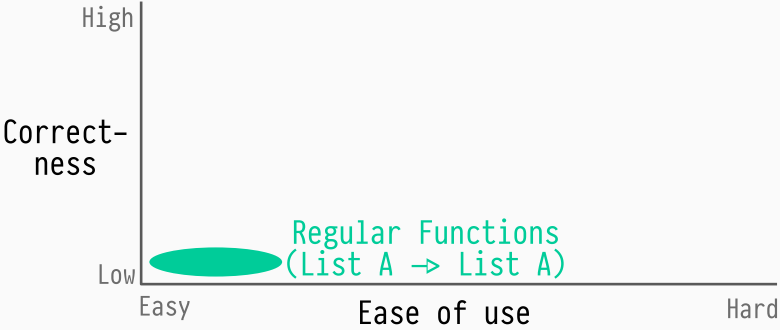

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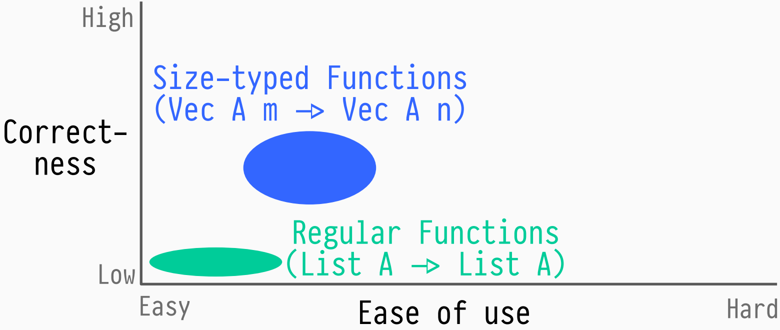

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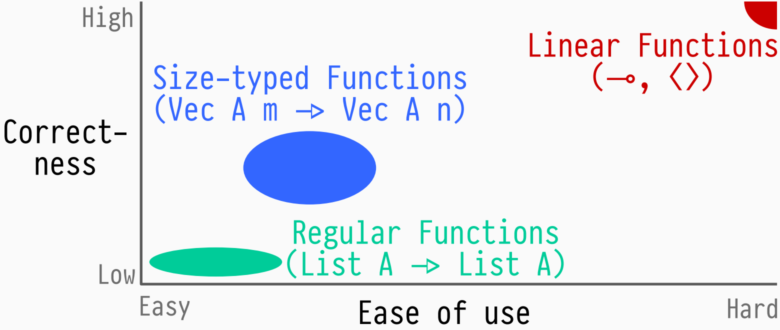

This presentation is a program!

This presentation is an Agda program! Instructions for how to load the presentation in Agda can be found at

github.com/ryanorendorff/lc-2020-linear-algebra-agda

The full library that implements this style (without `TrustMe!`) can be found at

github.com/ryanorendorff/functional-linear-algebra

Questions?

Thanks for listening to my talk!

github.com/ryanorendorff/lc-2020-linear-algebra-agda

Appendix

Instructions for how to run this presentation in Agda

If you have the Nix package manager installed, you can run

```
nix-shell
```

at the root of this presentation's repo and then launch emacs

```
emacs src/FunctionalPresentation.lagda.md
```

More information on the Agda emacs mode can be found https://agda.readthedocs.io/en/v2.6.1.1/tools/emacs-mode.html. If you use Spacemacs, the documentation for its Agda mode is https://www.spacemacs.org/layers/+lang/agda/README.html.