1 Set Theory

union intercetion composition

$$\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} S_n = \{x : \text{x appears in infinitely many } S_n\}$$

Can be view as dec seq. of sets.

$$\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} S_n = \{x : \text{x appears in all } S_n \text{ except finitely}\}$$

Can be view as inc seq. of sets.

De Morgan's Laws

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

2 Probability Axioms

Axioms

1. $P(A) \geq 0$
2. $P(\Omega) = 1$
3. $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$
   (If A_n are mutually exclusive)

Union Bound

$P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$

3 Continuity of Probability func

Def: Increasing sequence of events

$E_1 \subseteq E_2 \subseteq ... \subseteq E_n \subseteq E_{n+1} ...$

Def: Decreasing sequence of events

$E_1 \supseteq E_2 \supseteq ... \supseteq E_n \supseteq E_{n+1} ...$

Theorem: Interchange of limiting operations

If the events are des/inc, then

$\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$

Borel-Cantlli Lemma

If

$$\sum_{n=1}^{\infty} P(E_n) < \infty$$

then

$$\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_n = 0$$

4 Conditional Probability

Def: Conditional Probability

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

Multiplication Rule

$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)P(A_2|A_1)$
$P(A_3|A_1 \cap A_2)...P(A_n|A_1 \cap A_2 \cap ... \cap A_{n-1})$

Total Probability Theorem

$A_1, A_2, ..., A_n$ are disjoint and
$\bigcup_{i=1}^{n} A_i = \Omega$ (form partition) then
$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$

Bayes' Rule

$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

$P(A_i|B) = \frac{A_i P(B|A_i)}{P(B)}$

$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + ... + P(A_n)P(B|A_n)$

5 Indepenence

Def: Indepenence

$P(A \cap B) = P(A)P(B)$
$P(A|B) = P(A)$

Indepenence of multiple events

$P(\bigcap i \in SA_i) = \Pi i \in SA_i$
"for every(not pairwise)"$S \subseteq 1, 2, 3, ..., n$
example:
$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$
$P(A_1|A_2 \cap A_3) = A_1$

6 CDF and PMF

Def: R.V

Map events to real numbers.

Def: CDF

For any R.V X, CDF of X ist
$F_x(t) = P_x(X \leq t)$

Use CDF to find the probability of an event(I)

1. $P(X \leq a) = F_x(a)$
2. $P(X > a) = 1 - F_x(a)$
3. $P(X < a) = F_x(a^-)$
4. $P(X \geq a) = 1 - F_x(a^-)$
5. $P(X = a) = F_x(a) - F_x(a^-)$

Use CDF to find the probability of an event(II)

1. $P(a < X \leq b) = F_x(b) - F_x(a)$
2. $P(a < X < b) = F_x(b^-) - F_x(a)$
3. $P(a \leq X < b) = F_x(b^-) - F_x(a^-)$
4. $P(a \leq X \leq b) = F_x(b) - F_x(a^-)$

Properties of CDF

1. $F_x(t)$ is non-decreasing
2. $\lim_{t\to\infty} F_x(t) = 1$
3. $\lim_{t\to-\infty} F_x(t) = 0$
4. $F_x(t)$ is right-continuous
   (i.e. $F_x(t^+) = F_x(t)$)

Def: PMF

For any discrete R.V X, PMF of X is

1. $p_x(x) = P_x(X = x_i)$
2. $p_x(x) = 0$, if $x \notin X$
3. $\sum_{i=1}^{\infty} p(x_i) = 1$

7 Entropy

Def

$H(X) = -\sum p(x_i) * log(p(x_i) = E[-log\ p(X)])$
Average randomnesss(higher more random) of a random variable.
0 no randomness, 1 max randomness

8 Expected value

Def

$E[X] = \sum x_i * p(x_i)$

E.V by CDF

$$E[X] = \sum_{i=1}^{\infty} (x_i - x_{i-1}) * (1 - F_X(x_{i-1}))$$

LOTUS

$E[g(X)] = \sum g(x_i) * p(x_i)$

Linear

$E[g(x) + h(x)] = E[g(x)] + E[h(x)]$

9 Moment

n-th moment

$E[X^n]$

n-th central moment

$E[(X - E[X])^n]$

exsistence

$E[|x|] < \infty$

Riemann Rearangement Theorem

Let a_n be sequence of numbers satisfy that:

1. $$\sum_{n=1}^{\infty} a_n\ converges$$
2. $$\sum_{n=1}^{\infty} |a_n| = \infty$$

Then for any real number x, there exists a rearrangement of a_n such that the sum of the rearrangement is x.

10 Variance

Def

$Var(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$

Properties

1. $Var(X) \geq 0$
2. $Var(X) = 0 \iff P(X = E[X]) = 1$
3. $Var(X + c) = Var(X)$
4. $Var(aX) = a^2 Var(X)$
5. $Var(X + Y) = Var(X) + Var(Y)$
   (X, Y are independent)

11 PDF

Def

$f(x)$ is a PDF of random variable X if:

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

CDF

$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$

Expected value

$E[X] = \int_{-\infty}^{\infty} xf(x)dx$

From CDF to PDF

$f(x) = \frac{d}{dx}F(x)$
$f(x)$ must be continuous

12 Inverse Transform Sampleing

Def

X is a random variable with CDF $F(x)$, then $F^{-1}(U)$ is a random variable with CDF $F(x)$, where U is a random variable with uniform distribution on $[0, 1]$

Proof

$$\begin{aligned} F(t) &= P(x \leq t) \\ &= P(T(u) \leq t) \\ &= P(u \leq T^{-1}(t)) \\ &= T^{-1}(t) \\ &= F^{-1}(t) \end{aligned}$$

Algorithm

1. Generate U with uniform distribution on $[0, 1]$
2. Compute $x = F^{-1}(U)$

13 Convolution

Discrete

X and Y are independent random variables, then $Z = X + Y$ is a random variable with PMF:

$$P(Z = z) = \sum_{x=-\infty}^{\infty} P(X = x)P(Y = z - x)$$

Continuous

X and Y are independent random variables, then $Z = X + Y$ is a random variable with PDF:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)dx$$

14 Gaussian Integral

1. $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$
2. $\int_{-\infty}^{\infty} xe^{-a(x-b)^2} dx = b\sqrt{\frac{\pi}{a}}$
3. $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{2a^{3/2}}$

15 Bernoulli R.V

PMF

parameter: p

$$P(X=k)=\begin{cases} p & if\, k=1 \\ 1-p & if\, k=0 \\ 0 & otherwise \end{cases}$$

CDF

$$F(x)=\begin{cases} 0 & if\, x<0 \\ 1-p & if\, 0\le x<1 \\ 1 & if\, x\ge 1 \end{cases}$$

Expected value

$E[X]=p$

Variance

$Var(X)=p(1-p)$

16 Binoial R.V

PMF

parameter: n, p

$$P(X=k)=\begin{cases} \binom{n}{k}p^k(1-p)^{n-k} & if\, k=0,1...,n \\ 0 & otherwise \end{cases}$$

CDF

$$F(x)=\begin{cases} 0 & if\, x<0 \\ \sum_{k=0}^{x}\binom{n}{k}p^k(1-p)^{n-k} & if\, 0\le x<n \\ 1 & if\, x\ge n \end{cases}$$

Expected value

$E[X]=np$

Variance

$Var(X)=np(1-p)$

17 Poisson R.V

PMF

parameter: λ, T

$$P(X=n)=\begin{cases} \frac{(\lambda T)^n}{n!}e^{-\lambda T} & if\, n=0,1,2... \\ 0 & otherwise \end{cases}$$

CDF

$$F(x)=\begin{cases} 0 & if\, x<0 \\ \sum_{k=0}^{x}\frac{(\lambda T)^k}{k!}e^{-\lambda T} & if\, 0\le x<n \\ 1 & if\, x\ge n \end{cases}$$

Expected value

$E[X]=\lambda T$

Variance

$Var(X)=\lambda T$

Sum of independent Poisson R.V

If X_1 and X_2 are independent Poisson R.V with parameters λ_1, T and λ_2, T respectively, then X_1+X_2 is a Poisson R.V with parameter $\lambda_1+\lambda_2, T$.

18 Geometric R.V

PMF

parameter: p

$$P(X=n)=\begin{cases} p(1-p)^{n-1} & if\, n=1,2,3... \\ 0 & otherwise \end{cases}$$

CDF

$$F(x)=\begin{cases} 0 & if\, x<0 \\ 1-(1-p)^{\lfloor x\rfloor} & if\, 0\le x<n \\ 1 & if\, x\ge n \end{cases}$$

Expected value

$E[X]=\frac{1}{p}$

Variance

$Var(X)=\frac{1-p}{p^2}$

19 Discrete Uniform R.V

PMF

parameter: a, b

$$P(X=k)=\begin{cases} \frac{1}{b-a+1} & if\, k=a,a+1,...,b \\ 0 & otherwise \end{cases}$$

CDF

$$F(x)=\begin{cases} 0 & if\, x<a \\ \frac{\lfloor x\rfloor-a+1}{b-a+1} & if\, a\le x<b \\ 1 & if\, x\ge b \end{cases}$$

Expected value

$E[X]=\frac{a+b}{2}$

Variance

$Var(X)=\frac{(b-a+1)^2-1}{12}$

20 Continuous Uniform R.V

PDF

parameter: a, b

$$f(x)=\begin{cases} \frac{1}{b-a} & if\, a\le x\le b \\ 0 & otherwise \end{cases}$$

CDF

$$F(x)=\begin{cases} 0 & if\, x<a \\ \frac{x-a}{b-a} & if\, a\le x<b \\ 1 & if\, x\ge b \end{cases}$$

Expected value

$E[X]=\frac{a+b}{2}$

Variance

$Var(X)=\frac{(b-a)^2}{12}$

21 Exponential R.V

PDF

parameter: λ

$$f(x)=\begin{cases} \lambda e^{-\lambda x} & if\, x\ge 0 \\ 0 & otherwise \end{cases}$$

CDF

$$F(x)=\begin{cases} 0 & if\, x<0 \\ 1-e^{-\lambda x} & if\, x\ge 0 \end{cases}$$

Expected value

$E[X]=\frac{1}{\lambda}$

Variance

$Var(X)=\frac{1}{\lambda^2}$

22 Normal R.V

PDF

parameter: μ, σ^2

$$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

CDF

$$F(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{x}e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt$$

Expected value

$E[X]=\mu$

Variance

$Var(X)=\sigma^2$