Lecture 27

- Streaming II

U_1 – driving velocity at edge of BL
u_1 – velocity in BL

} determined by linear theory

U_1 - input

$$U_1(x,t) = Re\{\hat{U}(x)e^{int}\}$$

u_1 is like a local Stoke#2 (oscillating plate solution)

$$u_1(x,y,t) = Re\{\hat{U}(x)(1 - e^{-\underbrace{\frac{i+1}{\delta}}_{=\alpha}y})e^{int}\}$$

zero time average — no streaming

v_1 – wall normal due to u_1 via $\nabla \cdot \underline{u} = 0$

$$v_1(x,y,t) = Re\{-e^{int}\frac{d\hat{U}}{dx}(y - \underbrace{\frac{\delta}{i+1}}_{1/\alpha} + \underbrace{\frac{\delta}{i+1}}_{1/\alpha}e^{-\overbrace{\frac{i+1}{\delta}}^{\alpha}y}\}$$

take $u_1 + u_2$ and $U_1 + U_2$ and $v_1 + v_2$ to satisfy full equation (BL equations)

$u_2 \rightarrow U_2$ at edge of BL

outer streaming flow, an order smaller add on to U_1

from Before $\frac{\partial u_1}{\partial t} - \frac{\partial U_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial y^2}$ (*) - solved for u_1, v_1

full $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}} + \nu\frac{\partial^2 u}{\partial y^2}$

(**) $$\frac{\partial(u_1+u_2)}{\partial t} + (u_1+u_2)\frac{\partial(u_1+u_2)}{\partial x} + (v_1+v_2)\frac{\partial(u_1+u_2)}{\partial y} = \left[\frac{\partial(U_1+U_2)}{\partial t} + (U_1+U_2)\frac{\partial(U_1+U_2)}{\partial x}\right] + \nu\frac{\partial^2(u_1+u_2)}{\partial y^2}$$

take $u_1, U_1, v_1 = O(\varepsilon)$

and $u_2, U_2, v_2 = O(\varepsilon^2)$

asymtotic analysis of the $Re \to \infty$ BL theory ...

on top of that

subtract (*) from (**)

$$\overset{\varepsilon^2}{\frac{\partial u_2}{\partial t}} + \left[\overset{\varepsilon^2}{u_1\frac{\partial u_1}{\partial x}} + \overset{\varepsilon^3}{u_2\frac{\partial u_1}{\partial x}} + \overset{\varepsilon^3}{u_1\frac{\partial u_2}{\partial x}} + \overset{\varepsilon^4}{u_2\frac{\partial u_2}{\partial x}}\right] + \left[\overset{\varepsilon^2}{v_1\frac{\partial u_1}{\partial y}} + \overset{\varepsilon^3}{v_2\frac{\partial u_1}{\partial y}} + \overset{\varepsilon^3}{v_1\frac{\partial u_2}{\partial y}} + \overset{\varepsilon^4}{v_2\frac{\partial u_2}{\partial y}}\right]$$

$$= \overset{\varepsilon^2}{\frac{\partial U_2}{\partial t}} + \left[\overset{\varepsilon^2}{U_1\frac{\partial U_1}{\partial x}} + \overset{\varepsilon^3}{U_2\frac{\partial U_1}{\partial x}} + \overset{\varepsilon^3}{U_1\frac{\partial U_2}{\partial x}} + \overset{\varepsilon^4}{U_2\frac{\partial U_2}{\partial x}}\right] + \overset{\varepsilon^2}{\nu\frac{\partial^2 u_2}{\partial y^2}}$$

$O(\varepsilon^2)$:
$$\underbrace{\frac{\partial u_2}{\partial t} - \nu\frac{\partial^2 u_2}{\partial y^2} + \frac{\partial U_2}{\partial t}}_{\text{unknowns}} = \underbrace{U_1\frac{\partial U_1}{\partial x} - u_1\frac{\partial u_1}{\partial x} - v_1\frac{\partial u_1}{\partial y}}_{\text{Knowns}}$$

NOTE: the $a = Re\{\hat{a}e^{int}\}$ "trick" only works cleanly for linear equation

$$U_1 = Re\{\underbrace{\hat{U}e^{int}}_{U}\}$$

$$U_1 = \frac{1}{2}\{U + U^*\}$$ ← also yields real part

$$U_1 = \frac{1}{2}\{\hat{U}e^{int} + \hat{U}^*e^{-int}\}$$

$$\alpha = \frac{i+1}{\delta}$$

complex $$u = \underbrace{\hat{U}(x)(1-e^{-\alpha y})}_{\hat{u}}e^{int}$$

real $$u_1 = \frac{1}{2}\{u + u^*\}$$

$$u_1 = \{\hat{u}e^{int} + \hat{u}^*e^{-int}\}$$

$$v_1 = \frac{1}{2}\{\hat{\beta}e^{int} + \hat{\beta}^*e^{-int}\}$$

all non linear real valued products have form

$$ab = (\hat{a}e^{i\omega t} + \hat{a}^*e^{-i\omega t})(\hat{b}e^{i\omega t} + \hat{b}^*e^{-i\omega t})$$

$$= \underbrace{\hat{a}\hat{b}e^{2i\omega t} + \hat{a}^*\hat{b}^*e^{-2i\omega t}}_{\text{time dependent}} + \underbrace{(\hat{a}^*\hat{b} + \hat{b}^*\hat{a})}_{\text{time independent}}$$

time dependent

2x frequency

$\Downarrow$

zero mean

$\rightarrow$ not our concern

time independent

$\downarrow$

do not time average to zero

$\downarrow$

this is the streaming part

look like these product

time average

$$\overline{\cancel{\frac{\partial u_2}{\partial t}}} - \overline{\cancel{\frac{\partial U_2}{\partial t}}} - \overline{\nu\frac{\partial^2 u_2}{\partial y^2}} = \overline{U_1\frac{\partial U_1}{\partial x}} - \overline{u_1\frac{\partial u_1}{\partial x}} - \overline{v_1\frac{\partial u_1}{\partial y}}$$

$0 \qquad 0$

A B C

for steady streaming flow

Ⓐ

$$u_1 \frac{\partial u_1}{\partial x} = \frac{1}{2}\frac{\partial u_1^2}{\partial x}$$

$$= \frac{1}{8}\frac{\partial}{\partial x}\overline{\left[(\hat{u}e^{int} + \hat{u}^* e^{-int})(\hat{u}e^{int} + \hat{u}^* e^{-int})\right]}$$

$$= \underbrace{\hat{u}^2 e^{2int}}_{\emptyset} + 2\hat{u}\hat{u}^* + \underbrace{\hat{u}^{*2} e^{-2int}}_{\emptyset}$$

$\emptyset \leftarrow$ time average $\rightarrow \emptyset$

$$= \frac{1}{4}\frac{d\hat{u}\hat{u}^*}{dx}$$

Ⓑ

$$\overline{u_1 \frac{\partial u_1}{\partial x}} = \overline{\frac{1}{2}\frac{d u_1^2}{dx}} \qquad u_1 = \frac{1}{2}\left\{\hat{u}(1-e^{-\alpha y})e^{int} + \hat{u}^*(1-e^{-\alpha^* y})e^{-int}\right\}$$

$$= \frac{1}{8}\frac{\partial}{\partial x}\overline{\left[2\hat{u}\hat{u}^*(1-e^{-\alpha y})(1-e^{-\alpha^* y}) + [e^{\pm 2int} \text{ terms}]\right]}$$

average to $\emptyset$

$$= \frac{1}{4}\frac{d\hat{u}\hat{u}^*}{dx}(1-e^{-\alpha y})(1-e^{-\alpha^* y})$$

Ⓒ

$$\overline{v_1 \frac{\partial u_1}{\partial y}} =$$

from above

complex $v = -\frac{d\hat{u}}{dx}\left(y - 1/\alpha + 1/\alpha\, e^{-\alpha y}\right)e^{int}$

real $\frac{\partial u_1}{\partial y} = \frac{1}{2}\left\{\hat{u}\alpha e^{-\alpha y}e^{int} + \hat{u}^*\alpha^* e^{-\alpha^* y}e^{-int}\right\}$

(complex * real "works")

$$= \mathrm{Re}\left\{ \overline{v \frac{\partial u_1}{\partial y}} \right\} = -\frac{1}{2}\mathrm{Re}\left\{ \hat{U}^* \alpha^* \frac{d\hat{U}}{dx}\left(y - 1/\alpha + 1/\alpha\, e^{-\alpha y}\right) e^{-\alpha^* y} \right\}$$

sub in

$$-\nu \frac{\partial^2 \bar{u}_2}{\partial y^2} = \frac{1}{4}\frac{d\hat{U}\hat{U}^*}{dx}\left[1 - (1 - e^{-\alpha y})(1 - e^{-\alpha^* y})\right]$$

$$+ \frac{1}{2}\mathrm{Re}\left\{ \hat{U}^* \frac{\alpha^*}{\alpha}\frac{d\hat{U}}{dx}\left(y\alpha - 1 + e^{-\alpha y}\right) e^{-\alpha^* y} \right\}$$

$\equiv G(x, y)$ like a body force

$\propto e^{-\alpha y}$ decay outside BL

$e^{-(i+1)y/\delta}$

BL's

(1) $\bar{u}_2 = 0$ at $y = 0$ no slip

(2) expect $\bar{u}_2 \to$ const (from BL perspective)

for $y/\delta \to \infty$

since $G \to 0$ outside BL

bc: $\frac{\partial \bar{u}_2}{\partial y} = 0 \quad y/\delta \to \infty$

NOTE: don't need $\frac{\bar{u}_2 L}{\nu} = Re_2$ to be large

$\rightarrow$ BL already exists

solve $-\nu \frac{\partial^2 \bar{u}_2}{\partial y^2} = G(x, y)$

integrate

$$-\nu \int_y^\infty \frac{\partial^2 \bar{u}_2}{\partial y'^2} dy' = \int_y^\infty G(x, y') dy'$$

$$-\nu \left[\cancel{\left.\frac{\partial \bar{u}_2}{\partial y'}\right|_\infty} - \left.\frac{\partial \bar{u}_2}{\partial y'}\right|_y \right] =$$

$\emptyset$ BL

integrate again

$$\int_0^y \nu \frac{\partial \bar{u}_2}{\partial y''} dy'' = \int_0^y \int_{y''}^\infty G(x, y') dy' dy''$$

$$\nu \left[\bar{u}_2 |_y - \cancel{\bar{u}_2 |_0} \right] =$$

0 no slip

$$\bar{U}_2 = \bar{u}_2(y \to \infty) = \frac{1}{\nu} \int_0^\infty \int_{y''}^\infty G(y') dy' dy''$$

$\vdots$ by part $\bar{U}_2 = \frac{1}{\nu} \int_0^\infty y \, G(y) dy$

can integrate full solution

$$\boxed{\bar{U}_2 = \frac{3}{8n}\left(-\frac{d\hat{U}\hat{U}^*}{dx} + i\left(\hat{U}^*\frac{d\hat{U}}{dx} - \hat{U}\frac{d\hat{U}^*}{dx}\right)\right)}$$

$$\bar{U}_2 \propto \frac{U_0^2}{nL}$$

$$\frac{\bar{U}_2}{U_0} \propto \frac{U_0}{nL}$$

$nL \gg U_0$

original assumption

small, but persistent

$$\hat{U}(x) = A(x)\, e^{i\phi(x)}$$ — phenomenology next time ....