Lecture 08

- advection diffusion example
- impulsively started object

Also posted

Below are the current office hours. There are 2 set one-half-hour slots. I will be glad to expand these if (1) this is not enough time and/or (2) these don't work for people who wish to talk with me.

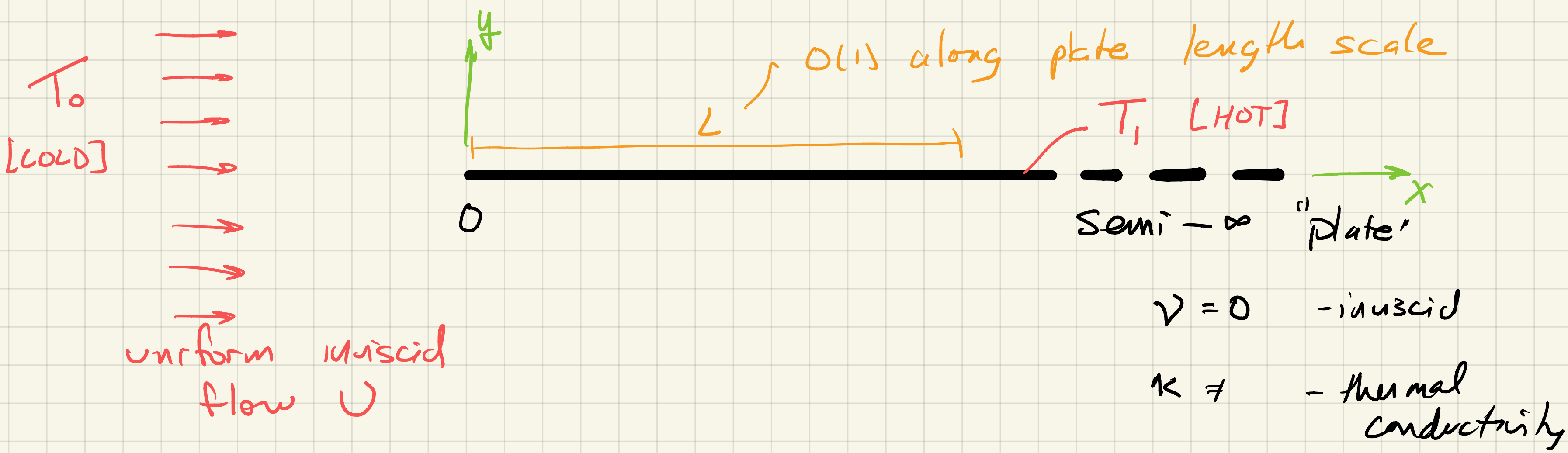
I list "Zoom" for in-person times too... please ping me to remind me to launch it!

I can easily be available evenings for Zoom office hour. Please just reach out. (I work most M-Th evenings regardless, so this is not an imposition at all.)

In general, do not hesitate to set something up. I would be delighted to talk about the material any time we can make work.

Who	When	Where
Freund	W 11:00-11:30	306F Talbot / Zoom
Freund	Th 2:00-2:30	306F Talbot / Zoom
Freund	Evening if needed	Zoom
Freund	By appointment	Zoom/Talbot/NCSA...

Next Example : Advection - Diffusion Eq.



gov. eq

steady

$$\rho c \underbrace{\left(\cancel{\frac{\partial T^*}{\partial t^*}} + U \frac{\partial T^*}{\partial x^*} \right)}_{\frac{D^* T^*}{D t^*}} = \kappa \nabla^{*2} T^*$$

same if $u = U$ const

recall (2D)

steady

$$\cancel{\frac{\partial \omega}{\partial t}} + u \frac{\partial \omega}{\partial x} + \cancel{v} \frac{\partial \omega}{\partial y} = \nu \nabla^2 \omega$$

$v = 0$

$\Rightarrow$ very similar

nondimensionalize

$$T^* = T_0 + (T_1 - T_0)\,T$$

dim

$$T^* \begin{cases} = T_0 & \text{when } T = 0 \\ = T_1 & \text{when } T = 1 \end{cases}$$

$$x^* = L x$$

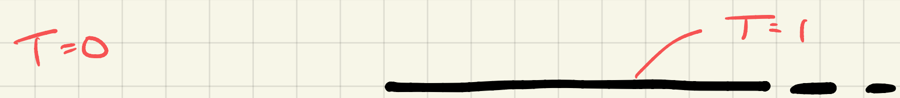
some length along plate

$$\frac{1}{\varepsilon} = \frac{\rho c L U}{k} = Pe\,\#$$

Peclet

$$\frac{\partial T}{\partial x} = \varepsilon \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

like Re # for heat transfer

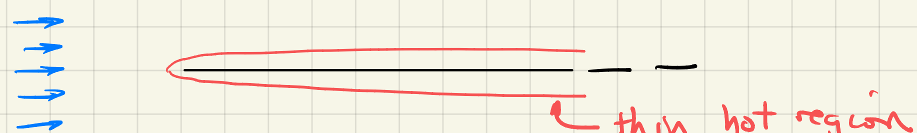


$T(x>0, y=0) = 1$

$T(x \to -\infty, y \to \pm\infty) = 0$

cold far away from plate....

ε v. small

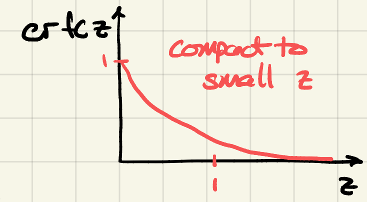


ε not quite as small

⋮

exact solution $\quad T(x,y) = \operatorname{erfc}\eta$

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-\zeta^2}\,d\zeta$$



$|\zeta|e^{i2\phi}$ $\qquad |z|e^{i\theta}$

$$(\eta + i\xi)^2 = \frac{1}{\varepsilon}(x+iy)$$

$$\eta^2-\xi^2+2i\eta\xi = \frac{1}{\varepsilon}x + \frac{i}{\varepsilon}y$$

$$x = \varepsilon(\eta^2-\xi^2)$$

$$y = 2\varepsilon\eta\xi$$

hot region — need $\eta \lesssim 1$

where in (x,y)?

$$x = \varepsilon(\xi^2-\eta^2)$$

$O(1)$ to be hot

hot for $x = O(1)$? — L-scale

$$\xi^2 \sim \left(x/\varepsilon\right) \quad \text{or} \quad \boxed{\xi = O\left(\sqrt{x/\varepsilon}\right)}$$

ξ Large ...

want to switch $\xi \to y$

$$\eta = \frac{y}{2\varepsilon\xi}$$

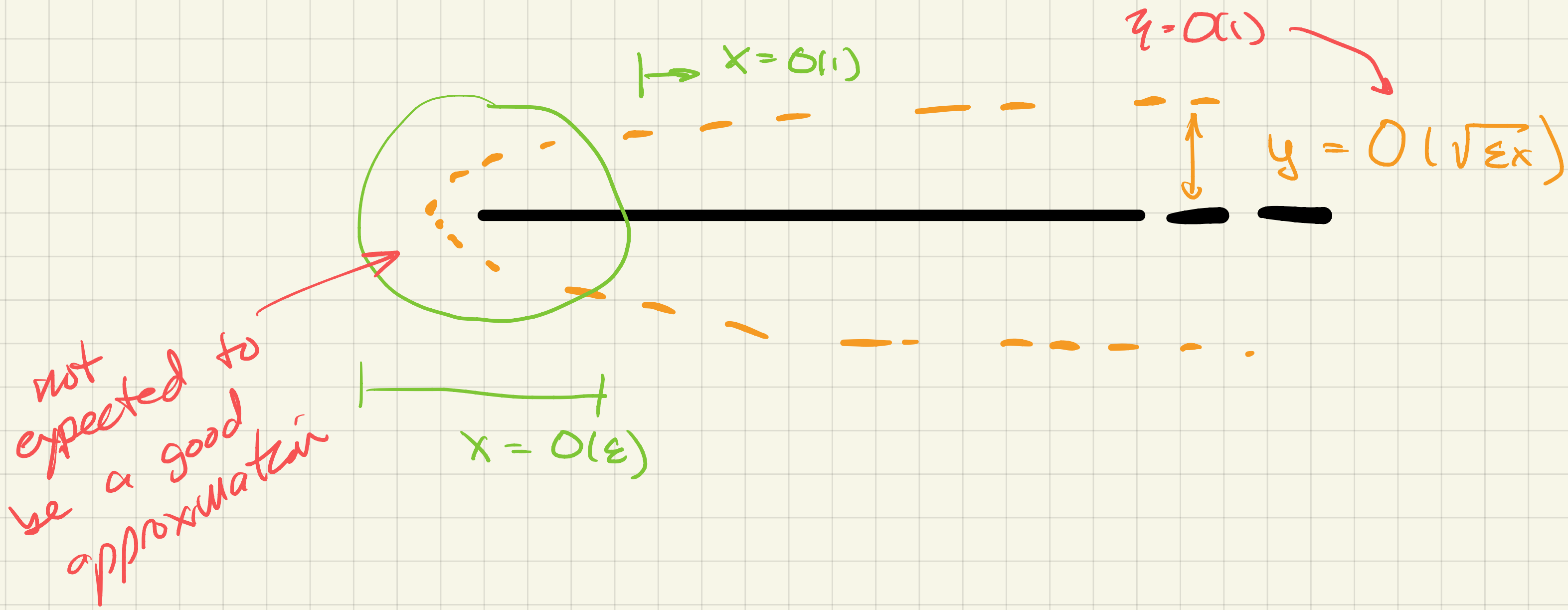
exact

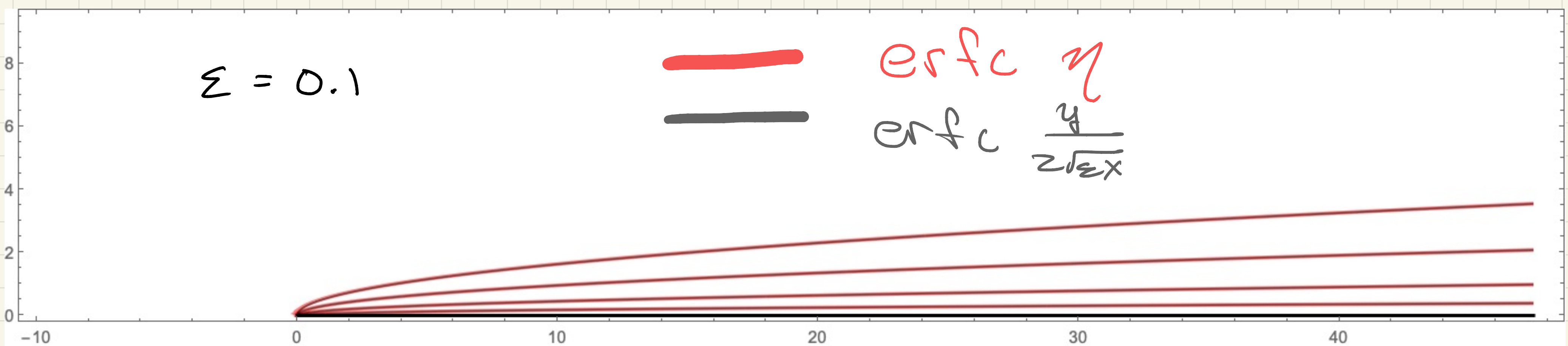
$$\eta = \frac{y}{2\sqrt{\varepsilon x}}$$

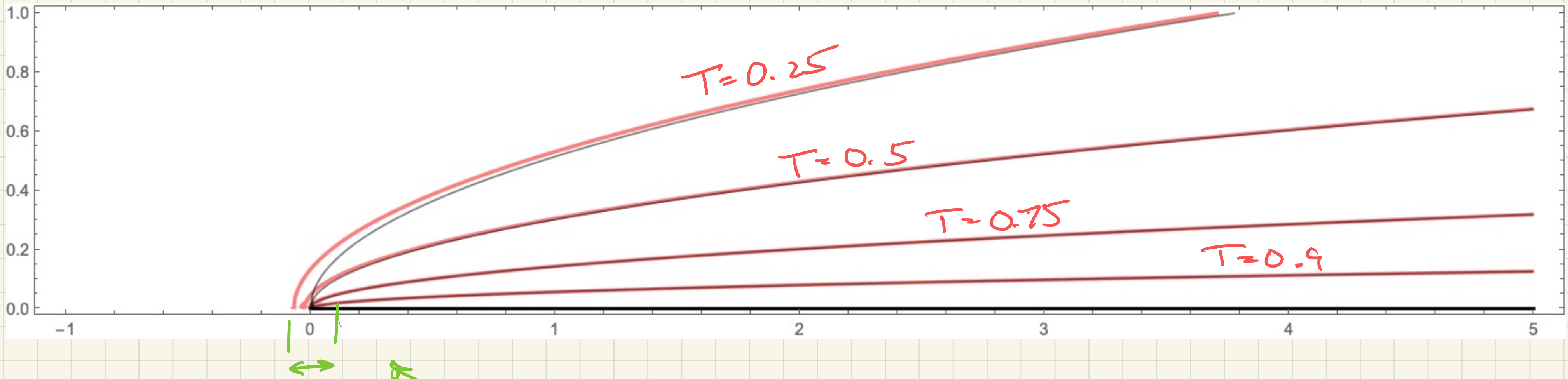
$x = O(1)$ — away from tip

$\eta = O(1)$

$T \neq 0$







$\sim 2\varepsilon$

denotes where expected

Can we arrive we arrive at this w/o exact?

$$\frac{\partial T}{\partial x} = \varepsilon \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

clearly $\varepsilon = 0$ fails

physical insight :

1) $O(1)$ advection : $1 \frac{\partial T}{\partial x}$

2) $O(\varepsilon)$ diffusion in y : $\boxed{\varepsilon} \frac{\partial^2 T}{\partial y^2}$

motivate a rescaling

- this is "made up"
- it only "works" if it "works"

ZOOM in on y $\qquad y = \sqrt{\varepsilon}\, Y$ $\qquad$ ansatz

sub in ...

$$\frac{\partial T}{\partial x} = \varepsilon \left(\underbrace{\frac{\partial^2 T}{\partial x^2}}_{O(\varepsilon)} + \underbrace{\frac{1}{\varepsilon} \frac{\partial^2 T}{\partial Y^2}}_{O(1)} \right)$$

$$\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial Y^2} + O(\varepsilon)$$

$$T(y \to \infty) = 0$$

$$T(y = 0) = 1$$

$$T(x, y) = f(z) \qquad z = \frac{y}{\sqrt{x}}$$

$$f'' + \tfrac{z}{2} f' = 0$$

$\vdots$

$$T(x, Y) = \operatorname{erfc}\left(\frac{Y}{2\sqrt{x}}\right)$$

$$T(x,y) = \operatorname{erfc}\left(\frac{y}{2\sqrt{\varepsilon x}}\right)$$

matches analysis of full solution.

Q: can we "fix" $x, \varepsilon = O(\varepsilon)$ region ?

$$\frac{\partial T}{\partial x} = \varepsilon\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

ansatz $\quad x = \varepsilon X \quad y = \varepsilon Y \quad \rightarrow$ zoom in <u>both</u> x and y ....

$$\frac{1}{\varepsilon}\frac{\partial T}{\partial X} = \varepsilon\left(\frac{1}{\varepsilon^2}\frac{\partial^2 T}{\partial X^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 T}{\partial Y^2}\right)$$

$$\frac{\partial T}{\partial X} = \left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2}\right) \rightarrow \text{all terms stay} \ldots$$

$\rightarrow$ no simplification $\rightarrow$ not more solvable, all physics matter with this approximation

Ockendon + Ockendon

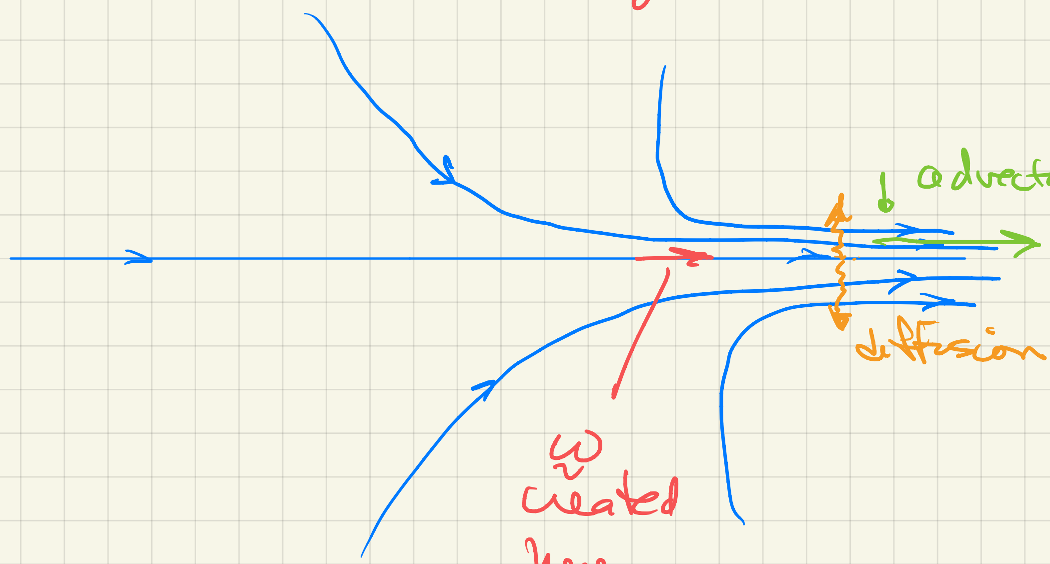
Reminders ... then thought experiment

↓

Bachelor

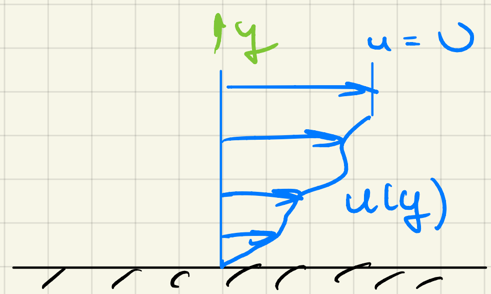
advection
diffusion } of ω → balance → led to exact

ω generation on wall



advection (small)
advection
diffusion } spreads slowly (high Re)

$\underset{\sim}{\omega}$ created here



net BL vorticity

$$\Omega = \int_0^\infty \omega(y)\,dy$$

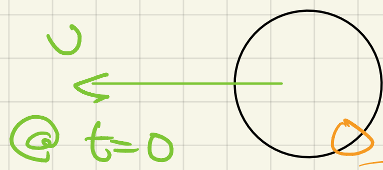
$$= \int_0^\infty \left(\cancel{\frac{\partial v}{\partial x}}^{\,0} - \frac{\partial u}{\partial y}\right) dy$$

$$= -U$$

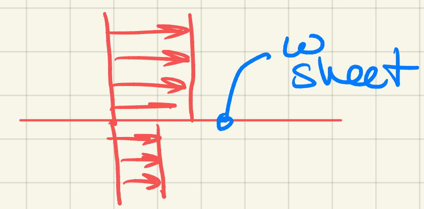
whatever the shape u(y)

→ detail don't matter too much for phenomenology

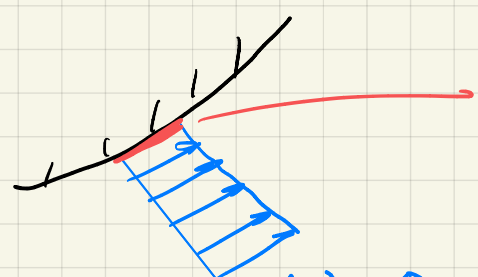
Impulsively started body



$t=0$ $\boxed{\text{I}}$ all vorticity must be on the body



Body from



$\underline{u}$ discontinuity / near discontinuity → vortex sheet

potential flow

II rapid diffusion at first

$\frac{y}{\sqrt{\nu t}}$ — exact solutions

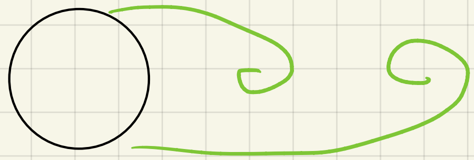
- rapid ω diffusion away from wall
- (also advection along wall — little change to advect $\parallel$ wall)

- ω stays close to wall in a thin layer — δ

III) long time ... eventually $(\nu t)^{1/2}$ not small

— can diffuse far as $t \to \infty$

— time for action to pull ω away from boundary



ω leaving body by advection

→ enabled by stage II S-scale physics