Lecture 23

- Compressible BL

Last times added an energy eq

→ one way coupled   u affected T but not vice versa

missing: change in thermodynamic variable in mass/momentum

now add: state equation (← idea gas)

state dependent properties $\mu = \mu(T)$, $k = k(T)$

WHITE

Compressible NS

linked to p, T by state

$\rho = \text{const} \Rightarrow \nabla \cdot \underline{u} = 0$

mass $\dfrac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{u} = 0$

$\mu = \mu(T)$

mom $\rho \dfrac{D\underline{u}}{Dt} = -\nabla p + \nabla \cdot \underline{\underline{\tau}}$

$$\tau_{ij} = \underbrace{\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)} + \underbrace{\lambda \frac{\partial u_k}{\partial x_k} \delta_{ij}}$$

$$\underline{\underline{\tau}} = \mu(\nabla \underline{u} + \nabla \underline{u}^T) + \lambda(\nabla \cdot \underline{u}) \underline{\underline{\delta}}$$

$k = k(T)$

energy $\rho \dfrac{Dh}{Dt} = \dfrac{Dp}{Dt} + \underbrace{\nabla \cdot (k \nabla T)}_{-\underline{q}} + \underbrace{\underline{\underline{\tau}} : \nabla \underline{u}}_{\Phi}$ = ... NOT very different

$h = e_i + p/\rho$

already Φ affecting T via (Ec)

advection of T in BL

diffusion (Pr)

nondimensionalize

$$\underline{x} = \frac{\underline{x}^*}{L} \qquad t = \frac{t^* U}{L} \qquad \underline{u} = \frac{\underline{u}^*}{U}$$

$$p = \frac{p^* - p_\infty}{\rho_\infty U^2} \qquad T = \frac{T^* - T_\infty}{T_w - T_\infty} \qquad \rho = \frac{\rho^*}{\rho_\infty}$$

$$\Phi = \frac{L^2}{\mu_\infty U^2} \qquad \mu = \frac{\mu^*}{\mu_\infty} \qquad k = \frac{k^*}{k_\infty} \qquad \lambda = \frac{\lambda^*}{\mu_\infty}$$

⋮

non-universal gas constand

Perfect gas $\qquad p = \rho R T \qquad e_i = c_v T$

$$\gamma = c_p/c_v \qquad R = c_p - c_v \qquad h = c_p T$$

⋮

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{u} = 0 \qquad [\text{same}]$$

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \frac{1}{Re_\infty} \nabla \left[\mu \left(\nabla \underline{u} + \nabla \underline{u}^T \right) + \lambda (\nabla \cdot \underline{u}) \underline{\underline{\delta}} \right]$$

$$\frac{\rho^*}{\rho_\infty} \quad \div \quad \Longrightarrow \quad \frac{\rho_\infty}{\rho^*} \; \frac{\mu_\infty}{\rho_\infty L U} \; \frac{\mu^*}{\mu_\infty} = \frac{\mu^*}{\rho^* L U} = Re \quad \text{local Reynold's \#}$$

does not actually form becaus $\mu \neq$ const

$$\rho \frac{DT}{Dt} = Ec_\infty \frac{Dp}{Dt} + \frac{1}{Re_\infty Pr_\infty} \nabla \cdot (k \nabla T) + \frac{Ec_\infty}{Re_\infty} \Phi$$

$$Ec_\infty = \frac{U^2}{c_p T_\infty}$$

$$\underbrace{\frac{\mu_\infty}{\rho_\infty L U} \quad \frac{k_\infty}{c_p \mu_\infty} \quad \frac{k^*}{k_\infty}}_{\text{could collapse}} \times \frac{\rho_\infty}{\rho}$$

$$\frac{1}{Re} \nabla \cdot \frac{1}{Pr} \nabla T$$

Parameters :-

$$Re_\infty = \frac{\rho_\infty U C}{\mu_\infty}$$

$$Pr_\infty = \frac{\mu_\infty c_p}{k_\infty}$$

$$Ec_\infty = \frac{U^2}{c_p T_\infty}$$

— perfect gas

Mach # sound speed

$U = Ma\, a$ → ref state $U = Ma_\infty a_\infty$

$$U^2 = Ma^2 \underbrace{(\gamma R T)}_{a^2}$$

$$= \sqrt{\frac{\gamma p}{\rho}}$$

$$Ec_\infty = \frac{Ma_\infty^2 \gamma R \cancel{T_\infty}}{c_p \cancel{T_\infty}}$$

$$= Ma_\infty^2 \gamma \frac{(c_p - c_v)}{c_p}$$

$(1 - 1/\gamma)$

$= (\gamma-1)\, Ma_\infty^2$ $\rightarrow$ Ma_∞ large, dissipation is important + thermodynamically

BL, 2D $\quad \hat{v} = \sqrt{Re_\infty}\, v \quad \hat{y} = \sqrt{Re}\, y$

$\vdots$

mass $\quad \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho u}{\partial x} + \dfrac{\partial \rho v}{\partial y} = 0$

$\dfrac{\partial \mu}{\partial y}$ – potentially important

x-mom $\quad \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) = -\dfrac{\partial p}{\partial x} + \dfrac{\partial}{\partial y}\left(\mu \dfrac{\partial u}{\partial y}\right)$

applies to k too

y-mom $\quad \dfrac{\partial p}{\partial y} = 0$

energy/enthalpy $\quad \rho\left(\dfrac{\partial h}{\partial t} + u\dfrac{\partial h}{\partial x} + v\dfrac{\partial h}{\partial y}\right) = \dfrac{\partial p}{\partial t} + u\dfrac{\partial p}{\partial x} + \dfrac{\partial}{\partial y}\left(k\dfrac{\partial T}{\partial y}\right) + \underbrace{\mu\left(\dfrac{\partial u}{\partial y}\right)^2}_{\Phi \,\geq\, 0}$

synopsis $\quad$ 3 eq (not counting y-mom)

5 unknowns (u, v, ρ, h, T) [p set outside BL]

2 state eq $\quad T = T(\rho, p) \qquad h = h(T, \rho)$

$\downarrow \qquad\qquad \downarrow$

$p = \rho R T$ $\qquad h = c_p T$

$+$ properties $\quad \mu = \mu(T) \quad k = k(T)$

$\rightarrow$ closed system (given p)

Free stream :

$$\underbrace{\frac{\partial p}{\partial x} = -\rho_\infty \left(\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \right)}_{x\text{-mom}} \Rightarrow \frac{\rho_\infty}{U} \frac{\partial h_\infty}{\partial t} + \rho_\infty \frac{\partial h_\infty}{\partial x} - \frac{1}{U} \frac{\partial p}{\partial t}$$

enthalpy equation

total enthalpy $\rightarrow$ useful for BL

$$H \equiv h + \tfrac{1}{2} u^2 \qquad (\text{BL } v \text{ negligible})$$

H gov eq $\qquad u \times (x\text{-mom}) \qquad u \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial x} \right) = -u \frac{\partial p}{\partial x} + u \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \right)$

$$\rho \frac{D u^2/2}{Dt} = -u \frac{\partial p}{\partial x} + u \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \right)$$

enthalpy $\quad \rho \frac{Dh}{Dt} = \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2$

add $\quad \rho \frac{DH}{Dt} = \cancel{\frac{\partial p}{\partial t}} + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y} + \mu u \frac{\partial u}{\partial y} \right)$

$\mu \left(\frac{\partial u}{\partial y}\right)^2 + u \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$

$Pr \equiv \frac{\mu c_p}{k}$ $\qquad k = \frac{\mu c_p}{Pr}$ $\qquad h = c_p T$

take $c_p \simeq$ const

steady

$\frac{\partial (u^2/2)}{\partial y} = u \frac{\partial u}{\partial y}$

$h = H - u^2/2$ $\qquad$ same

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr} \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial y}\left(\mu u \frac{\partial u}{\partial y} \right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\mu}{Pr} \frac{\partial H}{\partial y} \right) + \frac{\partial}{\partial y}\left[\left(1 - \frac{1}{Pr}\right) \mu u \frac{\partial u}{\partial y} \right]$$

$Pr \approx 1$ for gases

if $Pr = 1 \rightarrow H =$ const is a solution

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr} \frac{\partial H}{\partial y} \right)$$

very helpful, e.g. $s =$ const isentropic flow

$\Rightarrow \quad H = \text{const} \quad \Rightarrow \quad \frac{\partial H}{\partial y} = 0$ $\quad H = h + u^2/2$

$$\Rightarrow \quad \frac{\partial h}{\partial y} + u \frac{\partial u}{\partial y} = 0$$

on wall, $u = 0 \quad \rightarrow \quad \left.\frac{\partial h}{\partial y}\right|_w = 0$

$$\Rightarrow \quad \left.\frac{\partial c_p T}{\partial y}\right|_w = 0 \quad \Rightarrow \quad \left.\frac{\partial T}{\partial y}\right|_w = 0$$

(c_p const)

$\rightarrow$ no wall heat flux

A balance forms: viscous heating Φ v. conduction out of BL κ

does not need const μ, κ

1st Crocco - Busemann relation

a weak substitute for isentropic

$$\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} = \frac{1}{\rho T} \frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial t}\right) + \frac{1}{\rho T} \Phi$$

$$\frac{Ds}{Dt} \neq 0$$

both fundamental violations of isentropic