Lecture 22

- One way coupling $\rightarrow$ T

- Meeting #2 – sign ups ...

TOTAL ENERGY

$$\rho \frac{De_t}{Dt} = -\nabla \cdot \underline{q} + \nabla \cdot \underline{\underline{\tau}} \cdot \underline{u}$$

advection heat work

$=$

KINETIC

$\underline{u} \cdot (\text{mom})$

$$\rho \frac{De_k}{Dt} = \overbrace{\nabla \cdot \underline{\underline{\tau}} \cdot \underline{u} - \underline{\underline{\tau}} : \nabla \underline{u}}^{\underline{u} \cdot \nabla \cdot \underline{\underline{\tau}}}$$

$p \nabla \cdot \underline{u} - \underline{\Phi}$

$+$

$$\rho \frac{De_i}{Dt} = -\nabla \cdot \underline{q} + \underline{\underline{\tau}} : \nabla \underline{u}$$

$-p \nabla \cdot \underline{u} + \underline{\Phi}$

viscous dissipation ≥ 0

INTERNAL

change thermo dynamic variable

$e_i \rightarrow h \rightarrow s \rightarrow T$

$\vdots$

LAST TIME

$$T\frac{Ds}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}$$

$\rho = 1/v$

source of entropy

$$S \qquad \boxed{\rho\frac{Ds}{Dt} = -\frac{1}{T}\nabla\cdot\underline{q} + \frac{1}{T}\Phi}$$

if no thermal conductivity or μ $\rightarrow$ $\frac{Ds}{Dt} = 0$ just advects

specific heat

$h = c_p T$ take c_p const

$$\textcircled{*} \quad \rho c_p \overset{①}{\frac{DT}{Dt}} = \overset{②}{-\nabla\cdot\underset{\sim}{q}} + \overset{③}{\cancel{\frac{Dp}{Dt}}} + \Phi$$

assume isobaric

$p \approx$ const

$\frac{\Delta p}{p_\infty} \ll 1$

small Δp sufficient preserve flow w/o thermodynamic effects

(low Ma #)

$\underset{\sim}{q} = -k\nabla T$ Fourier

$-\nabla\cdot\underline{q} = \nabla\cdot k\nabla T$

ΔP's don't change T

BL limit ?

Assume 1-way coupled u affect T only

NEW

apply BL approximation

non dimensionalize + BL scale (μ, k const)

*dimensional

$x = \frac{x^*}{L}$ $\hat{y} = \frac{y^*}{L}$ $u = \frac{u^*}{U}$ $\hat{v} = \frac{v^*}{U}$ $\theta = \frac{T^* - T_\infty}{\Delta T}$

$y = \sqrt{Re}\,\hat{y}$ $v = \sqrt{Re}\,\hat{v}$

$x^* = Lx$ $y^* = \frac{Ly}{\sqrt{Re}}$ $u^* = Uu$ $v^* = \frac{Uv}{\sqrt{Re}}$ $T^* = \Delta T\theta + T_\infty$

sub into ✪ + assume 2D / steady

① $$\rho c_p \frac{DT^*}{Dt} = \rho c_p \left(u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} \right) = \rho c_p \left(\frac{U u \Delta T}{L} \frac{\partial \theta}{\partial x} + \frac{U \cancel{\sqrt{Re}} \Delta T v}{\cancel{\sqrt{Re}}\, L} \frac{\partial \theta}{\partial y} \right)$$

$$= \frac{\rho c_p U \Delta T}{L} \left(u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} \right)$$

② $$k \nabla^2 T = k \left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} \right) = \frac{k \Delta T}{L^2} \left(\frac{\partial^2 \theta}{\partial x^2} + Re \frac{\partial^2 \theta}{\partial y^2} \right)$$

dominant

$$= \frac{\kappa \Delta T Re}{L^2} \frac{\partial^2 \theta}{\partial y^2}$$

(3) $$\Phi = \frac{1}{2}\mu\left(\nabla^* \underline{u}^* + \nabla^* \underline{u}^{*T}\right) : \left(\nabla^* \underline{u}^* + \nabla^* \underline{u}^*\right) + \lambda\left(\nabla^* \cdot \underline{u}^*\right)^2$$

$$\begin{pmatrix} 0 & x \\ x & x \end{pmatrix}\begin{pmatrix} 0 & x \\ x & x \end{pmatrix}$$

$$\overset{11}{+} \frac{1}{2}\mu\left(2\frac{\partial u^*}{\partial x^*}\right)\left(2\frac{\partial u^*}{\partial x^*}\right) + \overset{21}{\frac{1}{2}\mu}\left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right)\left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right)$$

$$\overset{12}{+} \frac{1}{2}\mu\left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right)\left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right) + \overset{22}{\frac{1}{2}\mu}\left(2\frac{\partial v^*}{\partial y^*}\right)\left(2\frac{\partial v^*}{\partial y^*}\right)$$

$$+ \lambda\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right)^2$$

$$= \mu\frac{U^2}{L^2}\left\{ 2\left[\overset{11}{\underset{O(1)}{\left(\frac{\partial u}{\partial x}\right)^2}} + \overset{22}{\underset{O(1)}{\frac{Re}{Re}\left(\frac{\partial v}{\partial y}\right)^2}}\right] + \overset{12+21}{\left(\underset{O(Re)}{\sqrt{Re}\frac{\partial u}{\partial y}} + \underset{O(1/Re)}{\frac{1}{\sqrt{Re}}\frac{\partial v}{\partial x}}\right)^2} \right\}$$

$$+ \lambda\frac{U^2}{L^2}\left(\underset{O(1)}{\frac{\partial u}{\partial x}} + \underset{O(1)}{\frac{\sqrt{Re}}{\sqrt{Re}}\frac{\partial v}{\partial y}}\right)^2$$

$$= \mu\frac{U^2}{L^2} Re\left(\frac{\partial u}{\partial y}\right)^2$$

① ② ③

$$\frac{\rho C_p U \Delta T}{L}\left(u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}\right) = \frac{k\,\Delta T Re}{L^2}\frac{\partial^2\theta}{\partial y^2} + \mu\frac{U^2 Re}{L^2}\left(\frac{\partial u}{\partial y}\right)^2$$

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y} = \underbrace{\frac{k\cancel{\Delta T} Re\, L}{\rho C_p U \cancel{\Delta T} L^2}}\frac{\partial^2\theta}{\partial y^2} + \frac{\mu U^2 Re\, L}{\rho C_p U \Delta T L^2}\left(\frac{\partial u}{\partial y}\right)^2$$

$$\frac{k\,\cancel{\rho}\cancel{U}\cancel{L}\cancel{L}}{\cancel{\rho}C_p\cancel{U}\cancel{L^2}\mu} \qquad \frac{\cancel{\mu}U^2\cancel{\rho}\cancel{U}\cancel{L}\cancel{L}}{\cancel{\rho}C_p\cancel{U}\Delta T\cancel{\mu}\cancel{L^2}}$$

$$\frac{k}{C_p\mu} = \frac{1}{Pr\#} \qquad \frac{U^2}{C_p\Delta T} = Ec$$

a fluid property → Prandtl # – $\frac{\text{viscous}}{\text{thermal conductivity}}$

Eckert #

"source" – internal heating

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y} = \frac{1}{Pr}\frac{\partial^2\theta}{\partial y^2} + Ec\left(\frac{\partial u}{\partial y}\right)^2$$

[dimensional version $\rho C_p\left(u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}\right) = k\frac{\partial^2\theta}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2$]

Ec# – $\frac{\text{KE}}{\text{dissipation}}$

+ potential to change flow temp relative to ΔT by viscous heating

$\sim Ma^2$

Linear — break into two parts

$$\theta(x, y, Pr, Ec) = \theta_1(x, y, Pr) + Ec\,\theta_2(x, y, Pr)$$

$$u\frac{\partial \theta_1}{\partial x} + v\frac{\partial \theta_1}{\partial y} = \frac{1}{Pr}\frac{\partial^2 \theta_1}{\partial y^2}$$

$$u\frac{\partial \theta_2}{\partial x} + v\frac{\partial \theta_2}{\partial y} = \frac{1}{Pr}\frac{\partial^2 \theta_2}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2$$

$\Delta T = T_w - T_\infty$

$\theta = \dfrac{T - T_\infty}{\Delta T}$

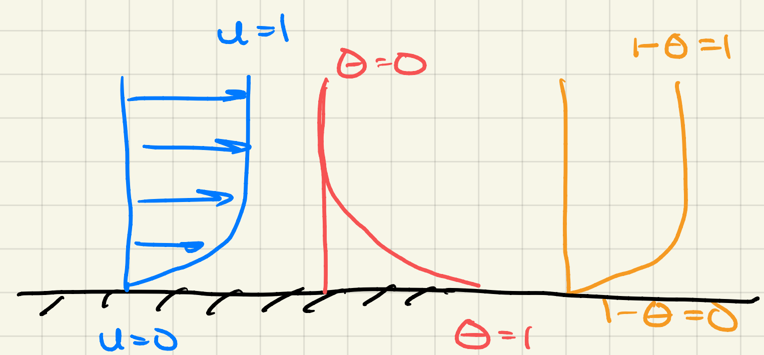
focus on this first (Ec small)

$(\theta_1 \to \theta)$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2}$$

x-mom

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}$$



u and $1-\theta$ have same BC's

$Pr = 1 \longrightarrow$ identical

gases $Pr \approx 1$ air 0.7

$\longrightarrow$ $\delta_{th} \approx \delta_{mom}$ same thickness

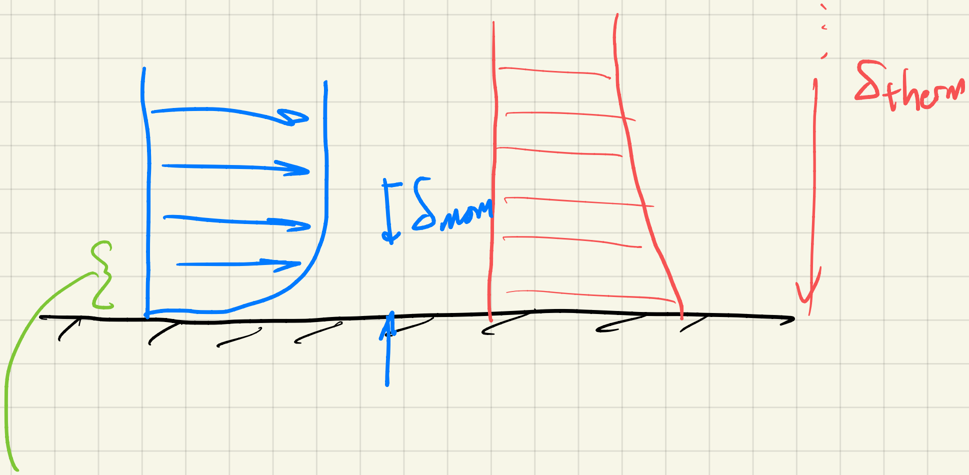
$Pr \to 0$ [liquid metal $\frac{\mu}{c_p K} = \frac{\nu}{\alpha} = Pr$]

α thermal diffusivity

$\delta_{mom} \sim \sqrt{\nu}$

$\delta_{therm} \sim \sqrt{\alpha} \sim \sqrt{\nu / Pr}$

$\delta_{therm} \gg \delta_{mom}$



velocity change only covers a small fraction of T change

schlichting

⋮

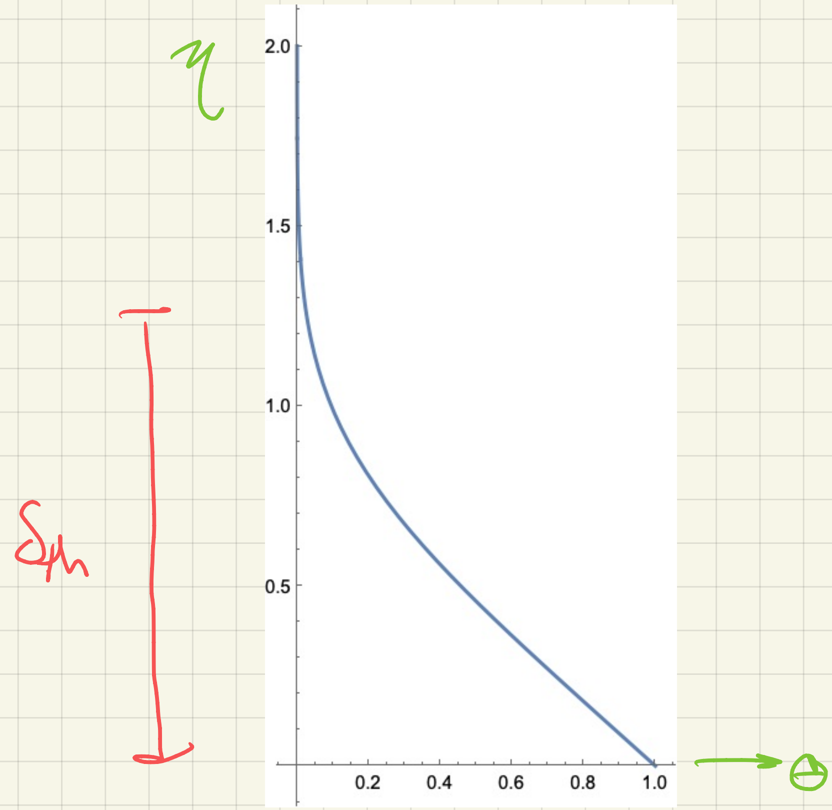
$$\Theta = \operatorname{erf} \eta \qquad \eta = y \frac{U(x)}{2\sqrt{\alpha \int_{x_0}^{x} U(x)\, dx}}$$

$Pr \to \infty$ $\rightarrow$ all T change right on wall where $u \propto y$

$$\ldots \quad \eta = y\sqrt{\frac{\tau_w}{\mu}} \Big/ \left(9\alpha \int_{x_0}^{x} \sqrt{\frac{\tau_w(x)}{\mu}}\, dx \right)$$

$\vdots$

$$\frac{d^2\theta}{d\eta^2} + 3\eta^2 \frac{d\theta}{d\eta} = 0 \quad \ldots\ldots \quad \theta = \frac{\Gamma\left(\frac{1}{3}, \eta^3\right)}{\Gamma(1/3)}$$



Θ_2 ?

→ v. important for very speed flows....