Lecture 17

- 2nd order BL
  (continued #2)

Lec 15

$Re \to \infty$ — no effect on outer flow

$\delta \to 0$, no displacement

Re large

$\delta \sim \sqrt{\nu}$

streamline at "edge" of BL → it's displaced by δ

effects missing in first-order BL theory

Note: asymptotically don't distort "boundary" but account for it...

2nd Order BL theory ⟹ how to include these rigorously

- BL approximation due to some U, V — INNER
- U, V response to the BL — OUTER

MATCH, each responding to the other

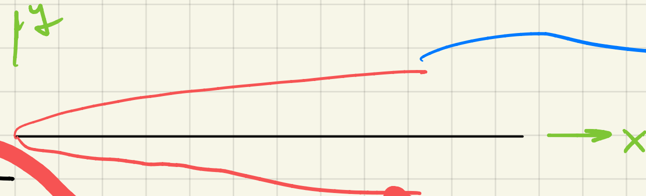
LAST TIME – Inner-to-outer match

Interpretation #1 :

ψ – stream function

ψ = const $\rightarrow$ streamlines

$\psi = 0$ a streamline $\qquad y = B_1 \dfrac{\sqrt{x}}{Re^{1/2}} \qquad x = y^2$



parabolic streamline

$\downarrow$

parabolic body shape due to finite BL thickness...

2nd Order Match

match $\quad \psi_2 = -B_1\sqrt{x}$

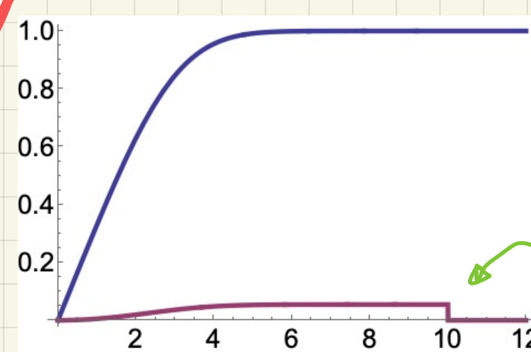
so $\quad \psi = \underbrace{y}_{\psi_1} - \underbrace{B_1 \tfrac{1}{Re^{1/2}} \sqrt{x}}_{\delta_2 \psi_2}$

Note: looks like parabolic streamline, but applied @ $y=0$.

$\dfrac{y}{Re^{1/2}} + \delta_2(Re)\left[\psi_2(x,0) + \right.$

Interpretation #2 :

$$v = -\psi_x = \frac{1}{2}\frac{B_1}{\sqrt{Re\,x}}$$



This should be the right v to remove the jump we saw at BL "edge"

LATEX notes → uploaded

M?

Meeting feedback ...

Still need to solve (Match "above" was just "BC")

$\rightarrow$ sub $\psi = \overset{1}{\delta_1}\psi_1 + \delta_2\psi_2$ into full N-S

full $$\left(\overset{u}{\psi_y}\frac{\partial}{\partial x} - \overset{v}{\psi_x}\frac{\partial}{\partial y} - \frac{1}{Re}\nabla^2\right)\overset{-\omega}{\nabla^2\psi} = 0$$

just substitution

$$\left(\overset{\mathrm{I}}{\psi_{1y}} + \delta_2 \overset{A}{\psi_{2y}}\right)\frac{\partial}{\partial x}\left(\overset{C}{\nabla^2\psi_1} + \delta_2 \overset{\mathrm{III}}{\nabla^2\psi_2}\right)$$

$$-\left(\overset{\mathrm{II}}{\psi_{1x}} + \delta_2 \overset{B}{\psi_{2x}}\right)\frac{\partial}{\partial y}\left(\overset{C}{\nabla^2\psi_1} + \delta_2 \overset{\mathrm{III}}{\nabla^2\psi_2}\right)$$

$$-\underbrace{\frac{1}{Re}}_{O(\delta_2^2)}\nabla^2\nabla^2\psi_1 - \underbrace{\frac{1}{Re}}_{O(\delta_2^2)}\delta_2\nabla^2\nabla^2\psi_2 = 0$$

$O(1)$

$O(1)$

$O(\delta_2) = O(1/Re^{1/2})$

$O(1)$: $$\left(\psi_{1y}\frac{\partial}{\partial x} - \psi_{1x}\frac{\partial}{\partial y}\right)\nabla^2\psi_1 = 0 \longrightarrow \text{already solved } \psi_1 = y$$

$O(\delta_2)$: $$\delta_2\left(\underset{u_2}{\overset{A}{\psi_{2y}}}\frac{\partial}{\partial x} - \underset{-v_2}{\overset{B}{\psi_{2x}}}\frac{\partial}{\partial y}\right)\overset{C\quad \omega_1}{\nabla^2\psi_1}$$

$$+\,\delta_2\left(\underset{u_1}{\overset{\mathrm{I}}{\psi_{1y}}}\frac{\partial}{\partial x} - \underset{-v_1}{\overset{\mathrm{II}}{\psi_{1x}}}\frac{\partial}{\partial y}\right)\overset{\mathrm{III}\quad \omega_2}{\nabla^2\psi_2} = O(1/Re) = O(\delta_2^2)$$

note: still no viscosity

$$\underbrace{u_2 \frac{\partial \omega_1}{\partial x} + v_2 \frac{\partial \omega_1}{\partial y}}_{\text{2nd order advection of 1st order vorticity}} + \underbrace{u_1 \frac{\partial \omega_2}{\partial x} + v_1 \frac{\partial \omega_2}{\partial y}}_{\text{1st order advection of 2nd order vorticity}} = 0$$

$$\frac{D_2 \omega_1}{Dt} \qquad \qquad \frac{D_1 \omega_2}{Dt}$$

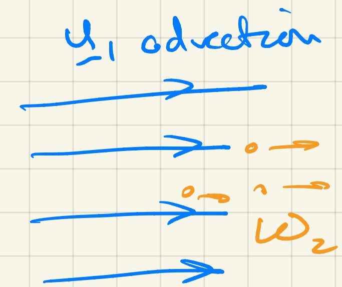
just advection

$\uparrow$

$\psi_1 = y$

already have $\nabla^2 \psi_1 = 0$

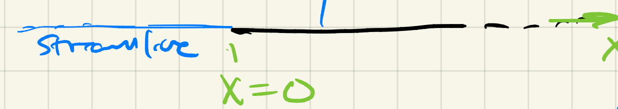
$\omega_1 = 0$



coming from far upstream

$\rightarrow$ no upstream ω to advect $\rightarrow$ $\omega_2 = 0$ everywhere so $\nabla^2 \psi_2 = -\omega_2 = 0$

matching



solve 2nd order OUTER $\nabla^2 \psi_2 = 0$

$y = 0$ BC $\quad \psi_2(x,0) = \begin{cases} 0 & x < 0 \\ -\beta_1 \sqrt{x} & x > 0 \end{cases}$

$y \to \pm\infty,$
$x \to -\infty$ $\quad \psi_2(x,y) = o(y)$ — a degree smaller than $\psi_1 = y$

like our erfc η for Tad-diff

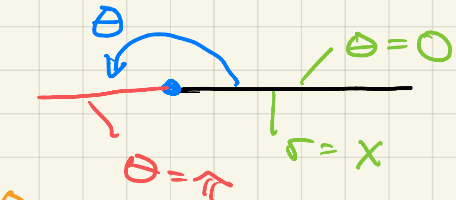
solution $\quad \psi_2 = -\beta_1 \, \mathrm{Real}\left\{ \sqrt{x+iy} \right\}$

can check $\dfrac{\partial^2 \psi_2}{\partial x^2} + \dfrac{\partial^2 \psi_2}{\partial y^2} = 0$

$x + iy = r e^{i\theta}$

$\nabla^2 \psi_2 \rightarrow \quad \dfrac{1}{r} \dfrac{\partial}{\partial r}\left(r \dfrac{\partial \psi_2}{\partial r} \right) + \dfrac{1}{r^2} \dfrac{\partial^2 \psi_2}{\partial \theta^2} = 0$

BC $\psi_2(r,0) = -\beta_1 r^{1/2}$



$e^{i\frac{\theta}{2}} = 1$

$\psi_2(r,\pi) = 0$ $\quad Re\{e^{i\theta/2}\} = 0$

by inspection

$$\psi_2(r,\theta) = -\beta r^{1/2} e^{i\theta/2}$$

if satisfies $\nabla^2 \psi_2 = 0$

sub in ...

sub in

$$-\frac{\beta_1}{4} r^{-3/2} e^{i\theta/2} + \frac{\beta_1}{4} r^{-3/2} e^{i\theta/2} = 0$$

2nd order OUTER

$$\psi = \psi_1 + \delta_2 \Psi_2$$

$$\psi = y - \frac{\beta_1}{Re^{1/2}} Real\{\sqrt{x+iy}\}$$

$$u = \psi_{,y} = 1 - \frac{\beta_1}{Re^{1/2}} \frac{1}{2} (x+iy)^{-1/2} i$$

drop "Real" part restriction

MATCH back to INNER

represent in INNER variables $= 1 - \frac{B_1}{Re^{1/2}} \frac{i}{2} \left(x + i \, Y/Re^{1/2} \right)^{-1/2}$

$Re \to \infty$ expand $= 1 - \underbrace{\frac{B_1}{Re^{1/2}} \frac{i}{2} x^{-1/2}}_{\text{pure imaginary}} + O\left(1/Re \right)$

to $O\left(1/Re^{1/2} \right)$ $= 1 + \frac{\phi}{Re^{1/2}}$

zero to $O\left(Re^{-1/2} \right)$

$\Longrightarrow$

Blasius

INNER $\quad \psi = \Delta_1 \Psi_1(x, Y) + \Delta_2 \Psi_2(x, Y)$

$1/Re^{1/2}$

still not set

represent in OUTER variables

$Y = y \, Re^{1/2}$

$= \Delta_1 \Psi_1\left(x, y \, Re^{1/2}\right) + \Delta_2 \Psi_2\left(x, y \, Re^{1/2}\right) + \ldots$

MATCH $\quad u = \psi_y \quad = \dfrac{1}{\cancel{Re^{1/2}}} \cancel{Re^{1/2}} \Psi_{1y}(x, yRe^{1/2})$

$$+ \Delta_2 Re^{1/2} \Psi_{2y}(x, y Re^{1/2}) + \dots$$

Blasius

$$\Psi_1 = \sqrt{x}\, f_1(\eta) \qquad \eta = \frac{Y}{\sqrt{x}}$$

OUTER variables

$$\Psi_{1y} = \cancel{\sqrt{x}}\, f_1'(\eta) \frac{1}{\cancel{\sqrt{x}}} = f_1'(\eta) = f_1'\left(\frac{y Re^{1/2}}{\sqrt{x}}\right)$$

take $\quad Re \to \infty \quad = f_1'\Big(\underbrace{\frac{y Re^{1/2}}{\sqrt{x}}}_{\infty}\Big) + Re^{1/2} \Delta_2 \Psi_{2y}(x, \infty)$

$f_1'(\infty) = 1$

$1/Re$ to match OUTER term

$$= 1 + \frac{1}{Re^{1/2}} \Psi_{2y}(x, \infty)$$

$= \phi$ to match OUTER

like "BC"

SOLVE 2nd order INNER

$$\psi = \Delta_1 \Psi_1 + \Delta_2 \Psi_2 \Rightarrow \text{full NS}$$

$\vdots$

$O(\Delta_2)$:

$$\frac{\partial}{\partial Y}\Big(\Psi_{2YYY} + \Psi_{1X}\Psi_{2YY} - \Psi_{1Y}\Psi_{2XY} \underbrace{+ \Psi_{2X}\Psi_{1YY} - \Psi_{2Y}\Psi_{1XY}}_{\text{nonlinear } 1^{st}/2^{nd} \text{ mix terms}} \Big) = 0$$

keep a viscous term

Ψ_1 - know (Blasius)

Ψ_2 - linear equation for Ψ_2

BC $\qquad \Psi_2(x,0)=0 \qquad$ no penetration — wall streamline

$\Psi_{2y}(x,0)=0 \qquad$ no slip

$\Psi_{2y}(x,\infty)=0 \qquad$ match

solution is $\Psi_2 = 0$ (unique) ✓

or eigenfunctions (non unique)

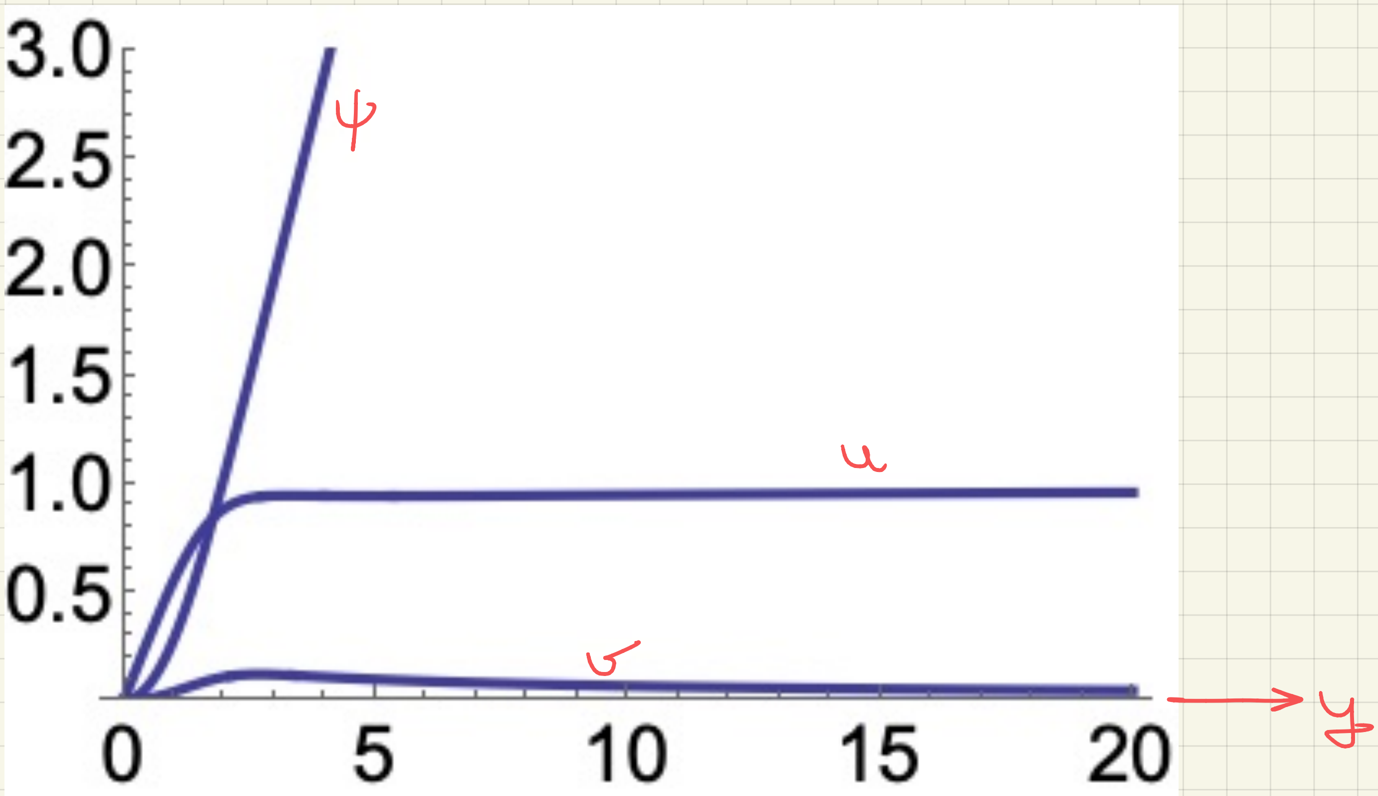
OUTER: $\quad \cancel{y} - \frac{1}{Re^{1/2}} B_1 \, \mathrm{Real}\{\sqrt{x+iy}\}$

INNER $\quad \frac{\sqrt{x}}{Re} f_1\left(\frac{y Re^{1/2}}{\sqrt{x}}\right) + \frac{O}{Re^{1/2}}$

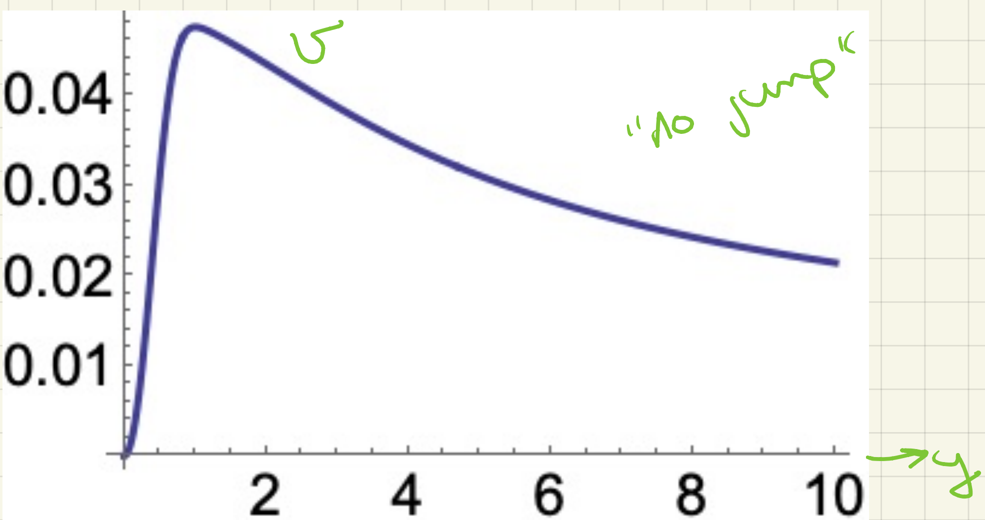
COMMON $\quad \cancel{y} - \frac{B_1}{Re}\sqrt{x}$

COMPOSITE: $\quad \frac{\sqrt{x}}{Re^{1/2}} f_1\left(\frac{y Re^{1/2}}{\sqrt{x}}\right) - \frac{B_1}{Re^{1/2}} \mathrm{Real}\{\sqrt{x+iy}\} + \frac{B_1}{Re^{1/2}}\sqrt{x}$

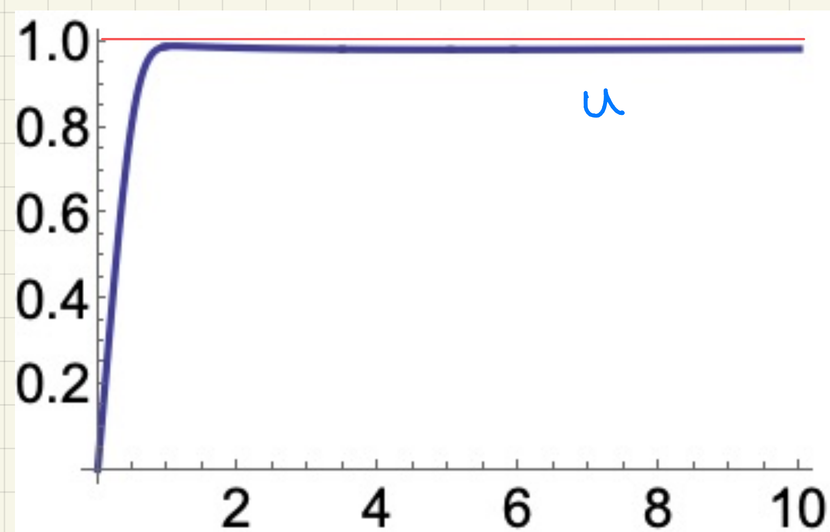
$Re = 100$



3.0
2.5
2.0
1.5
1.0
0.5
0
5
10
15
20
ψ
u
v
y



0.04
0.03
0.02
0.01
2
4
6
8
10
v
"no jump"
y



1.0
0.8
0.6
0.4
0.2
2
4
6
8
10
u