Lecture 24

- Reynolds Analogy
- Compressible BL behavior w/ Ma

Meeting # 2 : Covers thru compressible [not streaming]

→ probably Mandy's material

→ will target material since last meeting cut off

Last time → 1st Crocco (1932) - Buseman (1931) Relation

$Pr = 1 \Rightarrow H = h + u^2/2 = \text{const}$ solved BL eq

→ spoke to balance Φ and q

Reynolds Analogy

add $\frac{\partial p}{\partial x} = 0$

x-mom $\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$

enthalpy $\underset{Ⓐ}{\rho u \frac{\partial h}{\partial x}} + \underset{Ⓑ}{\rho v \frac{\partial h}{\partial y}} = \underset{Ⓒ}{\frac{\partial}{\partial y}\left(\frac{\mu}{Pr} \frac{\partial h}{\partial y}\right)} + \overset{Ⓓ}{\mu \left(\frac{\partial u}{\partial y}\right)^2}$

→ v. similar, more so if Pr = 1

Pr = 1
→ only difference is a u-dependent "source" in h equation

assume $\quad h = h(u)$

reminiscent of hodograph plane

$h \to u$

in the sense of it being a useful transforms

↳ not restrictive

compressible potential flow ...

$u(x,y) \qquad v(x,y)$

$x(u,v) \qquad y(u,v)$

(A) $$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial x}$$

(B) $$\frac{\partial h}{\partial y} = \frac{\partial h}{\partial u}\frac{\partial u}{\partial y}$$

(C) $$\frac{\partial}{\partial y}\left(\mu\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial h}{\partial u}\frac{\partial u}{\partial y}\right) = \frac{\partial h}{\partial u}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \boxed{\mu\frac{\partial u}{\partial y}}\,\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial u}\right)$$

$$\frac{\partial}{\partial u}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial h}{\partial u}\frac{\partial u}{\partial y}\right) = \boxed{\frac{\partial^2 h}{\partial u^2}\frac{\partial u}{\partial y}} + \frac{\partial h}{\partial u}\cancel{\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial u}\right)}$$

$\emptyset$

$$\frac{\partial h}{\partial u}\Bigg[\underbrace{\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)}_{=\emptyset}\Bigg]$$

(A) (B)

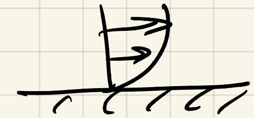
$=\emptyset$ momentum equation

(D) $$= \mu\frac{\partial u}{\partial y}\frac{\partial^2 h}{\partial u^2}\frac{\partial u}{\partial y} + \mu\left(\frac{\partial u}{\partial y}\right)^2$$

$$= \mu\left[1 + \frac{d^2 h}{du^2}\right]\left(\frac{\partial u}{\partial y}\right)^2$$

complete derivatives $h(u)$

$$\left[1+\frac{d^2h}{du^2}\right]\underbrace{\left(\frac{\partial u}{\partial y}\right)^2}_{\neq 0} = 0$$



$$\frac{d^2h}{du^2} = -1$$

$$\frac{dh}{du} = -u + C_1$$

$$h = -\tfrac{1}{2}u^2 + C_1 u + C_2$$

<u>BC</u>

take $h(y=0) = h_w$

no slip $u = 0$

$$C_2 = h_w$$

$h(y\to\infty) = h_\infty$

$u = U$

$$h_\infty = -\tfrac{1}{2}U^2 + C_1 U + h_w$$

$$C_1 = \big(\underbrace{h_\infty + \tfrac{1}{2}U^2}_{H = h_\infty + \frac{1}{2}U^2} - h_w\big)/U$$

H const 1^{st} C-B need

$H_\infty = h_w$ BC

$$\underbrace{\overset{A}{h + \tfrac{1}{2}u^2}}_{H} = (H_\infty - h_w)\,u/U + \overset{C}{h_w}$$

did not go into details

$\Phi \Leftrightarrow q$ balance

H is linear in u in BL

take $c_p = \text{const}$

$$h = c_p T$$

$H_\infty = c_p T_{\text{stagnation}} = c_p T_{aw}$ adiabatic wall

$$\overset{A}{c_p T + u^2/2} = \overset{C}{c_p T_w} + (\overbrace{\underbrace{c_p T_\infty}_{h_\infty} + U^2/2}^{H_\infty} - c_p T_w)\,u/U$$

solve for T

$(\div c_p)$

$$T = T_w + (\overbrace{T_\infty + U^2/c_p 2}^{T_{aw}} - T_w)\frac{u}{U} - u^2/2c_p$$

$$q_w = k_w \frac{\partial T}{\partial y}\Big|_w$$

$$= \frac{(T_{aw} - T_\infty)}{U}\, k_w \frac{\partial u}{\partial y}\Big|_w - k_w \frac{\cancel{u_w}}{c_p}\frac{\partial u}{\partial y}\Big|_w$$

$u_w = 0$

q_w, τ_w are 2 main interests

$\uparrow$ related to τ_w

$$\tau_w = \mu_w \frac{\partial u}{\partial y}\Big|_w$$

$$\frac{\partial u}{\partial y}\Big|_w = \frac{\tau_w}{\mu_w}$$

$$k_w \frac{\partial u}{\partial y}\Big|_w = \frac{k_w \tau_w}{\mu_w} = \frac{c_p \tau_w}{Pr}$$

$$Pr = \frac{c_p \mu}{k}$$

related

$$q_w = \frac{T_{aw} - T_w}{U} \frac{\tau_w c_p}{Pr}$$

1

$$\frac{q_w}{c_p (T_{aw} - T_w)} = \frac{\tau_w}{U}$$

$\div \rho_\infty U$

$$\frac{q_w}{\rho_\infty U c_p (T_{aw} - T_w)} = \frac{\tau_w}{\rho_\infty U^2}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$C_h = C_f/2$$

heat transfer coefficient

friction coefficient

Reynolds Analogy

thermal flux = drag

$\Rightarrow$ viscous heating balanced by conduction cooling

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was a huge "industry" looking for transforms that lead to self similarity . . .

→ cool they exist

→ not illuminating

→ just a hint

Illingworth Transform

$$\rho u = \frac{\partial \psi}{\partial y} \qquad \rho v = -\frac{\partial \psi}{\partial x} \qquad \Rightarrow \nabla \cdot \rho \underline{u} = 0$$

(steady mass)

split form

$$\psi(\xi, \eta) = \int \rho u \, dy = G(\xi) f(\eta)$$

$$u(\xi, \eta) = U(\xi) f'(\eta)$$

$\rightarrow$ viscous effects $\longrightarrow \xi$

$\rightarrow$ density change $\longrightarrow \eta$

$$\xi = \int_0^x \rho_\infty(x)\, U(x)\, \mu_\infty(x)\, dx = \xi(x)$$

$$\eta = \frac{U}{\sqrt{2\xi}} \int_0^y \rho\, dy = \eta(x,y)$$

$\vdots$

$$(C f'')' + (f f'') + \frac{2\xi}{U} \frac{dU}{d\xi} \left(\frac{\rho_\infty}{\rho} - f'^2 \right) = 0$$

$$C = \frac{\rho\mu}{\rho_\infty \mu_\infty} = C(\eta)$$

$\vdots$

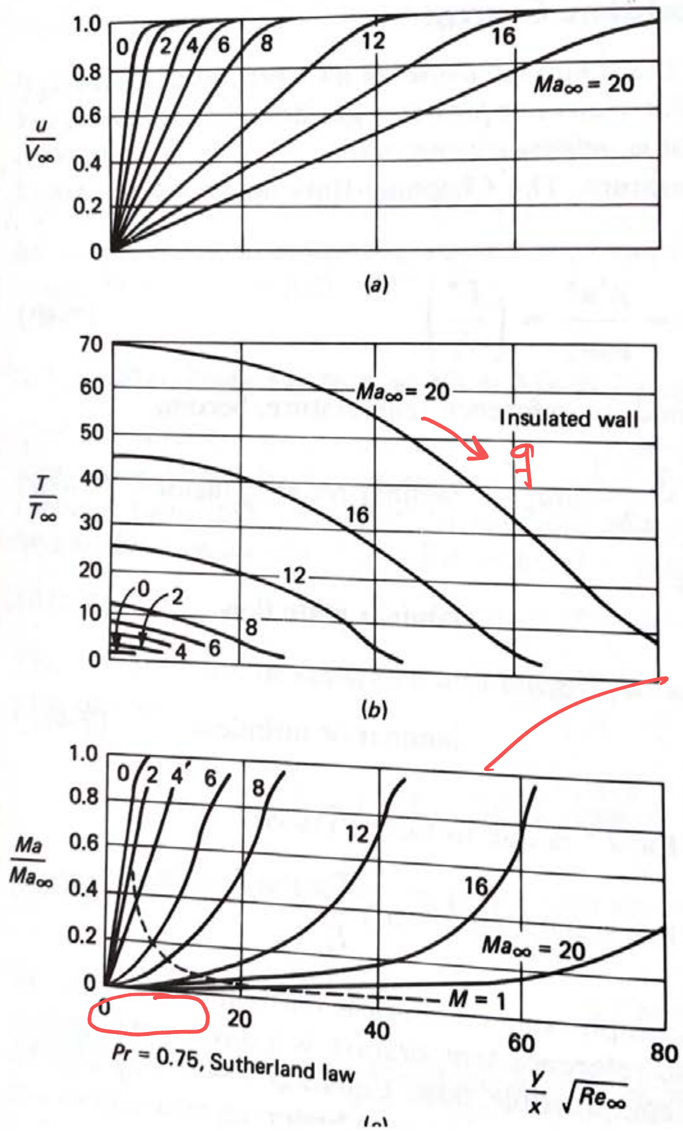
U, ρ, μ cons $\qquad f''' + f f'' = 0$

$\vdots$ cando $\qquad U = K x^m$

Blasius

much thicker at higher Ma

u/U

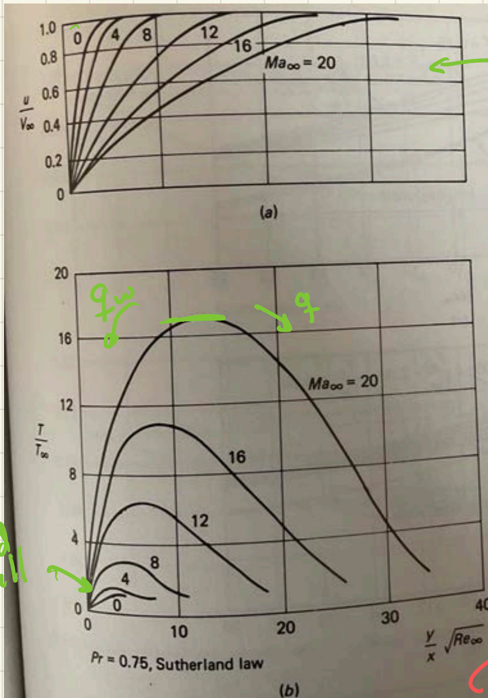


q

mostly subsonic in BL because $a = \sqrt{\gamma R T}$ → v. hot

$q_w = 0$

FIGURE 7-2
Calculations of the laminar ~~compressible~~ boundary layer on an insulated flat plate by van Driest (1952*a*): (*a*) velocity profiles; (*b*) temperature profiles; (*c*) Mach



not quite so thick w/ Ma, but still thicker (thermal expansion)

q_w

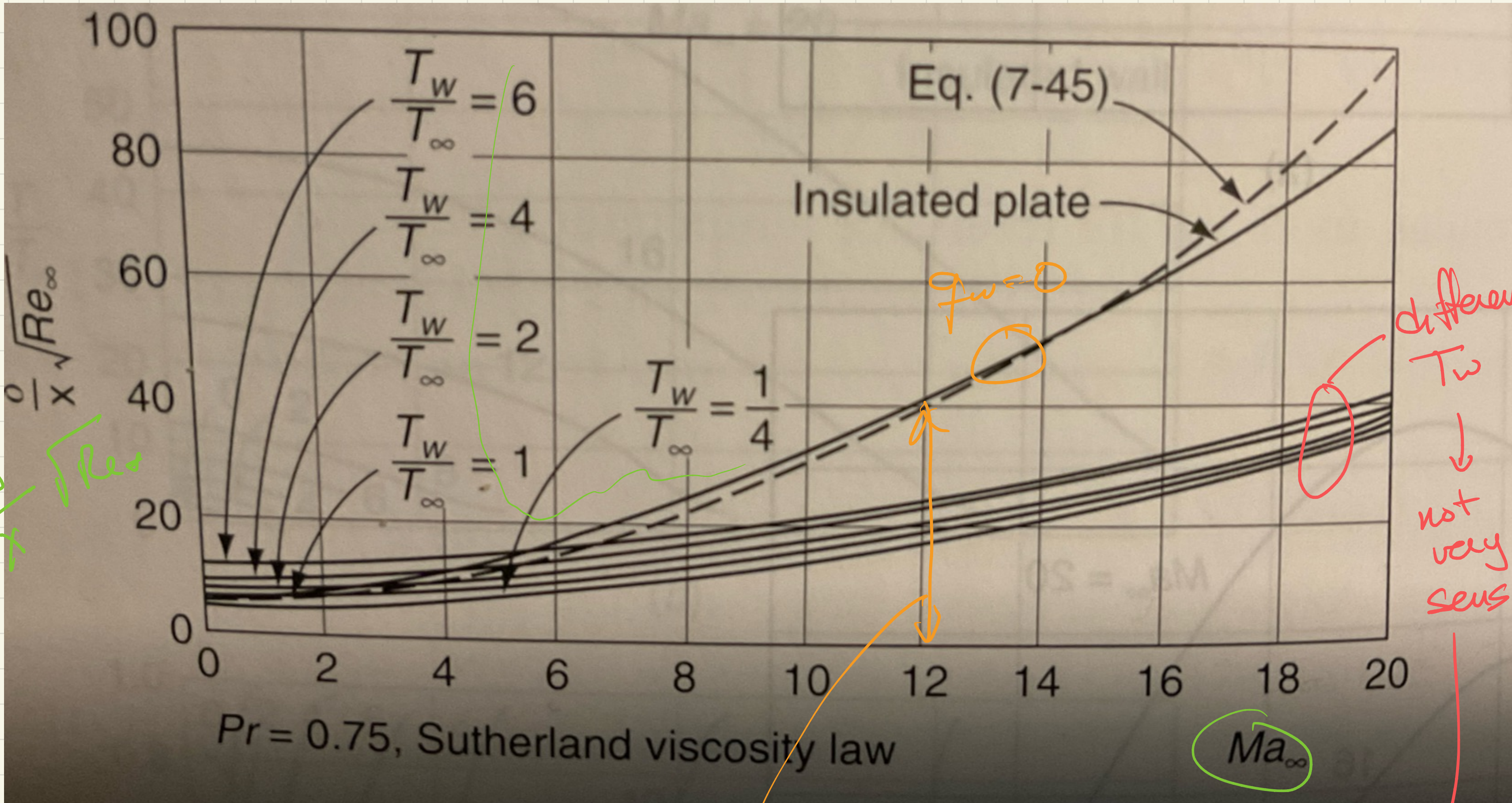
q

cold wall

FIGURE 7-3
Calculations by van Driest (1952*a*) of the compressible laminar boundary layer on a cold flat plate ($T_w = T_e/4$): (*a*) velocity profiles; (*b*) temperature profiles.

cold wall

profiles are well fit



different T_w → not very sensitive

because peak T in BL much higher

about 2x as thick

→ hotter

q only 1-way to free stream