No lecture Wednesday!

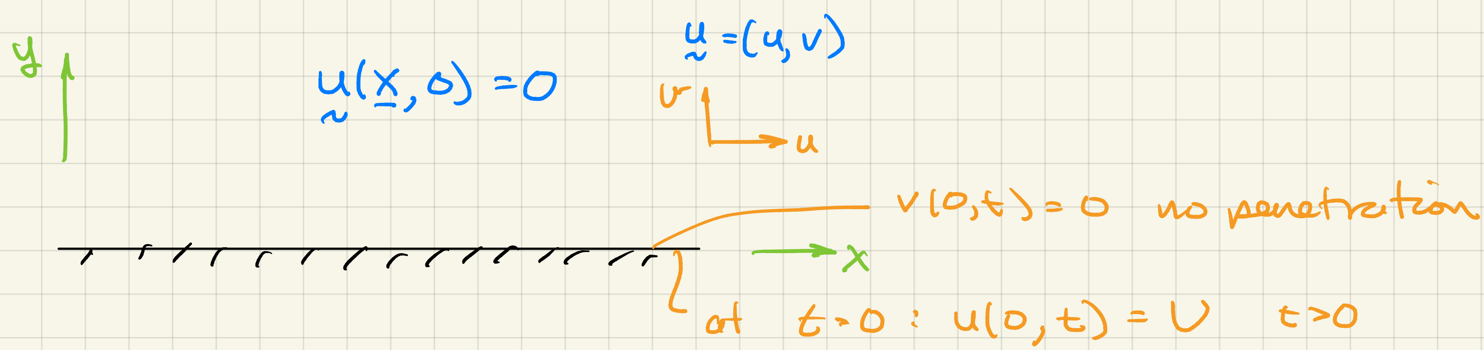
mix Schlichting, Bachelor reference

Exact N-S Solutions w/ BL-like behavior

Stokes 1st Problem (1851) — as a math problem

Rayleigh Problem (1911) — flow

→ we'll do "the math" for lecture 1



assume: $\mu = \text{const}$, $\rho = \text{const}$, $\nabla \cdot \underline{u} = 0$ [incompressible]

no x dependence (∞ plate) $\Rightarrow \frac{\partial}{\partial x} \to 0$

Simplify + Solve NS

- $\nabla \cdot \underline{u} = 0$

$$\cancel{\frac{\partial u}{\partial x}}^{\,0} + \frac{\partial v}{\partial y} = 0$$

$\longrightarrow v = \text{const} \rightarrow$ apply BC $v(0,t) = 0$

$\boxed{v = 0 \text{ everywhere}}$

$V = v$ same

- y-mom

$$\cancel{\frac{\partial v}{\partial t}}^{\,0} + u \cancel{\frac{\partial v}{\partial x}}^{\,0} + \cancel{v}^{\,0} \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu \left(\cancel{\frac{\partial^2 v}{\partial x^2}}^{\,0} + \cancel{\frac{\partial^2 v}{\partial y^2}}^{\,0} \right)$$

$$\frac{\partial p}{\partial y} = 0 \qquad \boxed{p = \text{const}}$$

- x-mom

$$\frac{\partial u}{\partial t} + u \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\partial_x \to 0} + \underbrace{\cancel{v}}_{v=0} \frac{\partial u}{\partial y} = -\underbrace{\cancel{\frac{\partial p}{\partial x}}}_{\partial_x(\text{cons})=0} + \nu \left(\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\partial_x \to 0} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

- boundary / initial condition

$$u(y,0) = 0$$
$$u(0,t) = U \qquad t > 0$$
$$u(y \to \infty, t) = 0$$

2 the same rhs

$\rightarrow$ halmark of self similarity

First lecture ... reasoned $\frac{y}{\sqrt{\nu t}}$ as dimensionally important

(Semi) Automatic way to seek similarity solution

→ often means that 2 independent variable collapse into 1 variable

const

similarity variable

assume $u(y,t) = A\, t^{\alpha} f(\eta)$

unknown function

pick α to help wipe out other t powers

$\to \eta$

$$\xi = \eta^2 = B^2 y^2 t^{\underbrace{2\beta}_{\gamma}}$$

$$\xi = B y^2 t^{\gamma}$$

$$\eta = B y t^{\beta}$$

const

β - affords way to form unknown similarity combination

sub into $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$

$$\frac{\partial u}{\partial t} = A \alpha t^{\alpha-1} f + A t^{\alpha} f' \underbrace{\frac{d\eta}{dt}}_{B y \beta t^{\beta-1}}$$

$\eta = B y t^{\beta}$

$y = \frac{\eta t^{-\beta}}{B}$

$By = \eta t^{-\beta}$

$$\eta t^{-\beta} \beta t^{\beta-1} = \eta \beta t^{-1}$$

$$\frac{\partial u}{\partial y} = A t^\alpha f' \frac{d\eta}{\partial y} = A t^\alpha f' B t^\beta$$

$$\frac{\partial^2 u}{\partial y^2} = A t^\alpha f'' B^2 t^{2\beta}$$

simplest if

$\nu B^2 \rightarrow B^2 = \frac{1}{\nu}$

$\boxed{B = \frac{1}{\sqrt{\nu}}}$

$$\cancel{A}\, \alpha\, \cancel{t^{\alpha-1}}\, f + \cancel{A}\, \cancel{t^\alpha}\, f'\, \eta\, \beta\, \cancel{t^{-1}} = \nu \cancel{A}\, \cancel{t^\alpha}\, f''\, B^2\, \cancel{t^{2\beta}}$$

$t^{\alpha-1}$ $\qquad$ $t^{\alpha-1}$ $\qquad$ $t^{\alpha+2\beta}$

$\boxed{\beta = -\tfrac{1}{2}}$ gets rid of all t's

$$\alpha f - \tfrac{1}{2}\eta f' = f''$$

note: $\eta = \frac{y}{\sqrt{\nu t}}$

as expected

$u(y,t) = A t^\alpha f(\eta)$

<u>BC's</u>

$u(y,0) = 0 \xrightarrow{t=0} \eta \to \infty \Rightarrow f(\infty) = 0$ (BC) if $\alpha = 0$

$u(0,t) = U \xrightarrow{y=0} \eta = 0 \Rightarrow A t^\alpha f(0) = U$ ← t-independent

Must take $\alpha = 0$

can take $A = U \rightarrow f(0) = 1$

$u(y\to\infty, t) = 0 \xrightarrow{y\to\infty} \eta \to \infty \Rightarrow f(\infty) = 0$

$$\boxed{f'' + \tfrac{1}{2}\eta f' = 0 \qquad f(0) = 1,\ f(\infty) = 0}$$

$$g = f'$$

$$g' = -\tfrac{1}{2}\eta g$$

$$\int \frac{1}{g}\, dg = \int -\tfrac{1}{2}\eta \; d\eta + c$$

$$\log g = -\tfrac{1}{4}\eta^2 + c$$

$$f' = \frac{df}{d\eta} = g = C' e^{-\frac{1}{4}\eta^2}$$

$$f(\eta) = \int_0^\eta c' e^{-\eta^2/4}\, d\eta + c''$$

$$f(\eta) = \tilde{c}\ \mathrm{erf}\ \eta/2 + c''$$

"error fct"

Apply BC: $f(\infty) = \tilde{c} + c'' = 0$

$\mathrm{erf}\ \infty = 1$ $\quad f(0) = c'' = 1$ $\quad$ so $\tilde{c} = -1$

$\mathrm{erf}\ 0 = 0$

$$f(\eta) = 1 - \mathrm{erf}\ \eta/2 = \mathrm{erfc}\ \eta/2$$

complementary error function