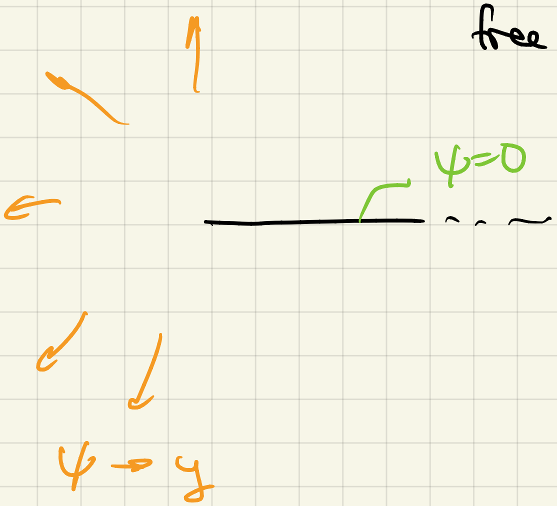
Lecture 16

- 2nd order BL (continued)

no slip $\quad u(x>0, y=0) = \left.\dfrac{\partial \psi}{\partial y}\right|_{x>0,\, y=0} = 0$

free stream: $\quad$ upstream, $y \to \pm\infty$



$u = U \quad$ [dimensional]

$u = 1 \quad$ [non dimensional]

$\psi_y \equiv \dfrac{\partial \psi}{\partial y}$

$\psi = y \quad$ so $\quad u = \psi_y = 1$

$\boxed{\text{OUTER}}$ expansion

$$\boxed{\psi(x,y) = \delta_1(Re)\,\psi_1(x,y) + \delta_2(Re)\,\psi_2(x,y) + \cdots}$$

leave vague for now,
will set as informed

$\delta_1 = 1$

$\delta_2 = 1/Re^{1/2}$

will confirm

appeal to $\psi(x \to -\infty, y)$ BC $\quad \psi = y$
for insight into $\delta_1(Re)$

lowest order

$\psi_1 = \psi / \delta_1$

$$\psi_1 = \lim_{Re\to\infty} \frac{y}{\delta_1(Re)} \sim y$$

so $\delta_1(Re) = 1$

if not, we'd "lose" the solution

first approximation (OUTER)

* don't know the precise degree of smallness ... "little oh"

note Re large

$$\underbrace{\Big(\psi_{1y}\frac{\partial}{\partial x} - \psi_{1x}\frac{\partial}{\partial y}}_{\frac{D\omega}{Dt}} - \underbrace{\frac{1}{Re}\nabla^2}_{*} \Big) \overbrace{\nabla^2\psi_1}^{-\omega} = \underbrace{o(\delta_1)}_{\text{thing that} \to 0 \text{ faster than } \delta_1}$$

$\frac{D\omega}{Dt}$ – pure advection ... from upstream

→ probably $O(\delta_2)$ "big oh"

$\omega \to 0$ upstream, pure advection,

so $\nabla^2\psi_1 = 0$ ... the OUTER $O(\delta_1)$ flow is potential

OUTER BC $\quad \psi_1(x,0) = 0 \quad$ (stream line, no pen)

$\psi_1 \to y \quad$ for $x \to -\infty$, $y \to \pm\infty$

skip INNER no slip ... $\rightarrow$ could not apply anyhow...

solution $\boxed{\psi_1 = y}$ – parallel stream lines (of course)

$\boxed{\boxed{\text{INNER}}}$ expansion

zoom in $Y = \dfrac{y}{\Delta_1(Re)}$ some dependence on Re (will be $1/Re^{1/2}$)

goal $Y = O(1)$

$\rightarrow$ Δ_1 will come from a consistency condition for INNER expansion of ψ

we know $u = O(1)$ both INNER / OUTER

$u = \psi_y = \dfrac{\partial \psi}{\partial y} = O(1)$ [both]

$y = O(\Delta_1)$ then $\psi = O(\Delta_1)$

INNER expansion $\boxed{\psi(x,y) = \Delta_1(Re)\,\underbrace{\Psi_1(x,y)}_{O(1)} + \Delta_2 \Psi_2(x,y) + \dots}$

Sub into N-S : $\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y} - \frac{1}{Re}\nabla^2\right)\nabla^2\psi = 0$

keep Δ_1 terms ... for this bit $\Psi_1 = \Psi$
(for now)

$$\Delta_1^2\left(\frac{1}{\Delta_1}\Psi_Y\frac{\partial}{\partial X} - \Psi_X\frac{1}{\Delta_1}\frac{\partial}{\partial Y}\right)\overbrace{\left(\cancel{\Psi_{XX}} + \frac{1}{\Delta_1^2}\Psi_{YY}\right)}^{\nabla^2\Psi}$$

$$-\frac{\Delta_1}{Re}\left(\cancel{\frac{\partial^2}{\partial X^2}} + \frac{1}{\Delta_1^2}\frac{\partial^2}{\partial Y^2}\right)\underbrace{\left(\cancel{\Psi_{XX}} + \frac{1}{\Delta_1^2}\Psi_{YY}\right)}_{\Delta^2\Psi} = 0$$

left out Δ_2's

crossed out
higher order Δ_1 terms

keep $O\left(\frac{1}{\Delta_1}\right)$
and Re term...

$$\left(\frac{1}{\Delta_1}\Psi_Y\frac{\partial}{\partial X} - \Psi_y\frac{1}{\Delta_1}\frac{\partial}{\partial Y}\right)\Psi_{YY} - \frac{\Delta_1}{Re\,\Delta_1^4}\Psi_{YYYY} = 0$$

useful choice ...

$\Delta_1^2 = 1/Re$

$\boxed{\Delta_1 = (1/Re)^{1/2}}$

keeps all terms

→ it work because it work

$$\left(\frac{\partial^2}{\partial y^2} - \overset{B}{\Psi_y \frac{\partial}{\partial x}} + \overset{A}{\Psi_x \frac{\partial}{\partial y}} \right) \Psi_{yy} = 0$$

$$\frac{\partial}{\partial y} \left(\Psi_{yyy} + \Psi_x \Psi_{yy} - \Psi_y \Psi_{xy} \right) = 0$$

same

$$\cancel{\Psi_{xy} \Psi_{yy}} + \overset{A}{\Psi_x \Psi_{yyy}} - \cancel{\Psi_{yy} \Psi_{xy}} - \overset{B}{\Psi_y \Psi_{xyy}}$$

integrate $(\Psi \xrightarrow{\text{notation}} \Psi_1)$

"const" dep on x

$$\boxed{\Psi_{1yyy} + \Psi_{1x} \Psi_{xy} - \Psi_{1y} \Psi_{xy} = g(x)}$$

$u = \frac{\partial \Psi}{\partial y}$

$v = -\frac{\partial \Psi}{\partial x}$

to be confirmed

$$\frac{\partial^2 u}{\partial y^2} - v \frac{\partial u}{\partial y} - u \frac{\partial u}{\partial x} = \left(\frac{\partial p}{\partial x} \right)$$

x-momentum BL equation

NEW

MATCH INNER/OUTER

... work in terms of ψ_y $(= u)$

makes sense from original path

1 term OUTER : $\psi_y = 1\,\psi_{1y}(x, y)$

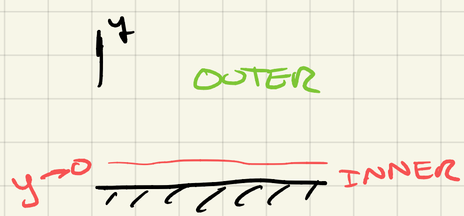
$Y = y/\Delta$, $\Delta \sim 1/Re^{1/2}$, Y is $O(1)$

represent w INNER variables $= \psi_{1y}(x, Y/Re^{1/2})$

$y = Y/Re^{1/2}$

$Re \to \infty$ expand $= \psi_{1y}(x,0) + \frac{Y}{Re^{1/2}}\psi_{1yy}(x,0) + \dots$

just Taylor $f(\varepsilon) = f(0) + \varepsilon f'(\varepsilon)$



from OUTER perspective, match is at $y = 0$

exactly zero $Re \to \infty$ as model for nearly zero

keep 1st term $O(1)$ $= \psi_{1y}(x,0)$

1 term INNER $\psi_y = \Psi_{1Y}(x, Y)$

represent w OUTER variables $= \Psi_{1Y}(x, Re^{1/2} y)$

O(1) in INNER region

$Re \to \infty$ expand $= \Psi_{1Y}(x, \infty)$

Keep 1^{st} term $= \Psi_{1Y}(x, \infty)$

MATCH $\psi_{1y}(x,0) = \Psi_{1Y}(x, \infty)$ *

more precise version of $u \to U$ at some δ-scale

integrate in Y

$Y\psi_{1y}(x,0) = \Psi_1(x,\infty) + C$

$o(Y)$ smaller than Y as $Y \to \infty$

Allows specification of $g(x)$

⋮

↳ plug into INNER flow eq $\circledast$

$$\Psi_{1YYY} + \Psi_{1X}\,\Psi_{1YY} - \Psi_{1Y}\,\Psi_{1XY} = g(x)$$

$\Psi_1 \sim Y$

$$0 + 0 - \psi_{1y}\,\psi_{1xy} = g$$

$$\left. -u\frac{\partial u}{\partial x} = \frac{\partial p}{\partial x} \right\} \rightarrow \text{usual Bernoulli}$$

$u = \psi_y$

$$\frac{\partial}{\partial x}\left(p + \tfrac{1}{2}u^2\right) = 0$$

$g(x) = \dfrac{\partial p}{\partial x}$ is a pressure gradient "source" in x-momentum inner solution

(same as y-momentum conclusion)

INNER BC :

wall $\left| \begin{array}{ll} \Psi_1(x,0) = 0 & \sim \Psi_1 \text{ const} \rightarrow \text{streamline (no penetration)} \\ \Psi_{1Y}(x,0) = 0 & = u(x,0) \rightarrow \text{no slip} \end{array} \right.$

free stream matching $\left| \; \Psi_{1Y}(x,\infty) = \psi_{1y}(x,0) = 1 \right.$ $\quad (\underbrace{\psi_1 = y}_{\text{Blasius}})$

solution ... same, Blasius

$$\Psi_1 = \sqrt{x}\, f(\eta) \qquad \eta = Y/\sqrt{x}$$

$$\vdots$$

$$f''' + \tfrac{1}{2} f f'' = 0$$

$$f(0) = f'(0) = 0$$

$$f'(\infty) = 1$$

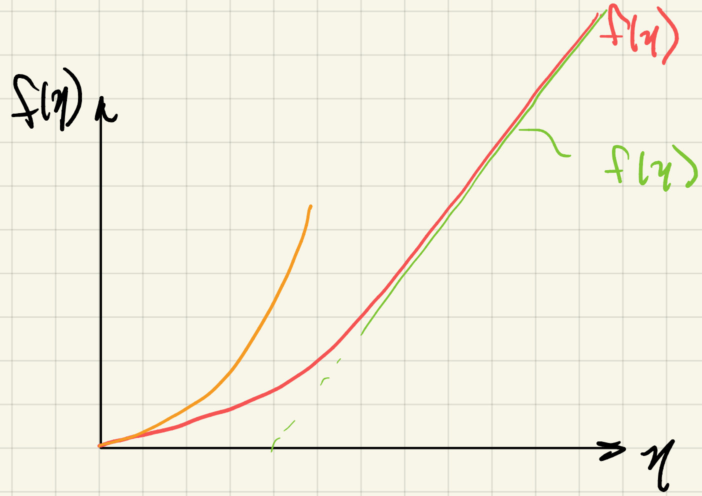
YES, same, but in "larger" matching framework...

recall 2nd order ODE matching

$$B_1 = -e^1 \qquad B_2 = \tfrac{1}{2} e^1$$

$\rightarrow$ just "the right" couple numbers

Numerical solution needed, but only small "parts" will be needed for matching



"offset"

$$f(\eta) = \eta - B_1 + (\text{exponentially small terms}) \quad **$$

diffusion

$f'(\eta) \to 1$

$f(20) = 18.2792$

$$\boxed{B_1 = 1.72079\ldots}$$

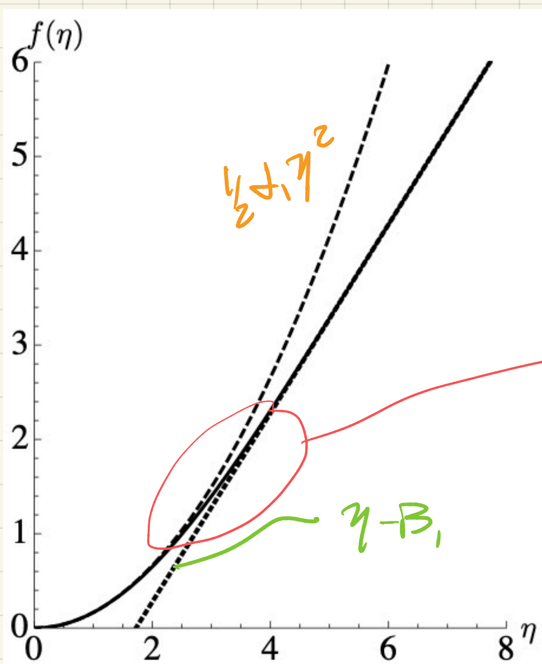
near wall $f(0) = f'(0) = 0$

so small η: $f(\eta) = 0 + 0\eta + \underbrace{\tfrac{1}{2}\alpha_1}\eta^2 + O(\eta^3)$

$f''(0) \neq 0 \to$ finite shear stress

$$\boxed{\alpha_1 = 0.332057}$$

will be used later



note small region of obvious deviation

2nd Order Matching

recall OUTER $\psi(x,y) = \psi_1 + \delta_2(Re)\,\psi_2 + \cdots$

will need to figure this out

INNER: $\psi(x,y) = \Delta_1 \sqrt{x}\, f(\eta)$ $\quad \eta = \dfrac{Y}{\sqrt{x}}$

$1/Re^{1/2}$

** f in OUTER

represent in OUTER variables $= \dfrac{1}{Re^{1/2}} \sqrt{x}\, f_1\left(\dfrac{Re^{1/2} y}{\sqrt{x}}\right)$

exp small

$Re \to \infty$ $= \dfrac{1}{Re^{1/2}} \sqrt{x} \left[\dfrac{Re^{1/2} y}{\sqrt{x}} - B_1 + \exp \right]$

above

$= y - \dfrac{1}{Re^{1/2}} B_1 \sqrt{x}$

Put in Y for convenience $= Y/Re^{1/2} - \dfrac{1}{Re^{1/2}} B_1 \sqrt{x}$

OUTER 2-term $\quad \psi = \underbrace{y}_{\psi_1} + \delta_2(Re)\, \psi_2(x,y)$

represent in terms of INNER variable $\quad = \dfrac{Y}{Re^{1/2}} + \delta_2(Re)\, \psi_2(x, Y/Re^{1/2})$

$Re \to \infty \quad = \dfrac{Y}{Re^{1/2}} + \delta_2(Re)\left[\psi_2(x,0) + \dfrac{Y}{Re^{1/2}}\, \psi_{2y}(x,0) + \cdots\right]$

all smaller

$\delta_2 \sim 1/Re^{1/2}$

to match

take $\quad \delta_2(Re) = \dfrac{1}{Re^{1/2}}$

match $\quad \psi_2 = -B_1 \sqrt{x}$

so $\quad \psi = y - B_1 \frac{1}{Re^{1/2}} \sqrt{x}$

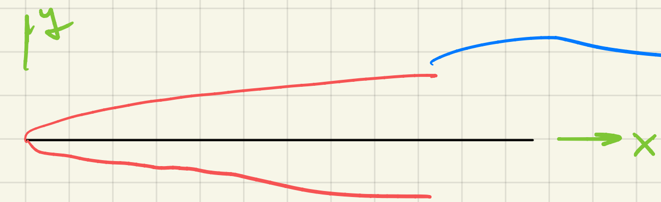
$O(1) \qquad O(1/Re^{1/2})$

Interpretation #1 :

ψ – stream function

$\psi = \text{const} \rightarrow$ streamlines

$\psi = 0$ a streamline $\qquad y = B_1 \dfrac{\sqrt{x}}{Re^{1/2}} \qquad x = y^2$



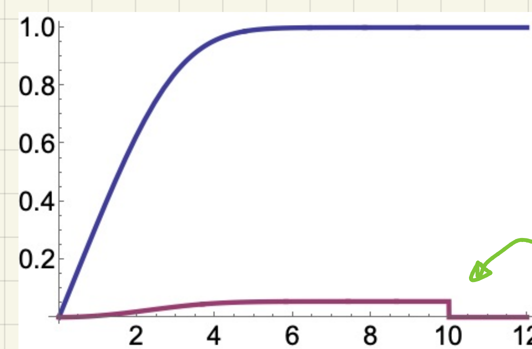
parabolic streamline

$\downarrow$

parabolic body shape due to finite BL thickness...

Interpretation #2 :

$$v = -\psi_x = \frac{1}{2}\frac{B_1}{\sqrt{Re\,x}}$$



this <u>should</u> be the right v to remove the jump we saw at BL "edge"