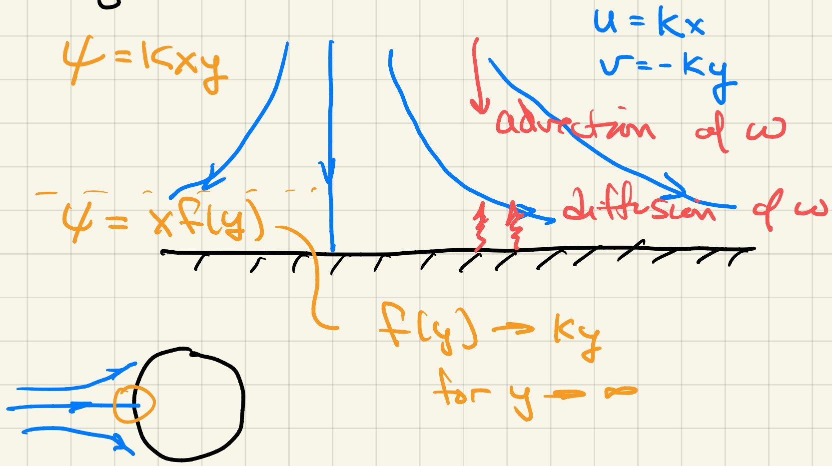
Lecture 05

exact NS

- finish stagnation
- Jeffrey-Hamel

(including Bakca effect)

Stagnation flow



$$-f'f'' + ff''' = -\nu f''''$$

$f'(0) = 0 \quad (u=0)$

$f(0) = 0 \quad (v=0)$

$f(y \to \infty) = ky$

$[f''(y\to\infty) = 0]$

$$y = \left(\frac{\nu}{k}\right)^{1/2} \eta \qquad f(y) = (\nu k)^{1/2} F(\eta)$$

simple substitution $\quad k^{5/2}\, \nu^{-1/2}$ in front of all terms

$$-F'F'' + FF''' + F'''' = 0$$

$$\frac{d}{d\eta}\left(FF'' - F'^2 + F'''\right) = 0$$

integrate $\quad FF'' - F'^2 + F''' = C$

PDE for η large

$F'' \to 0 \qquad F''' \to 0$

BC

$F(0) = 0$

$F'(0) = 0$

$F(\eta \to \infty) = \eta$

$F'(\eta \to \infty) = 1$

$$-F'^2 = C = -1$$

$$F'^2 - FF'' - F''' = 1$$

$F(0) = 0$

$F'(0) = 0$

$F'(\infty) = 1$

→ numerics

```
In[48]:= s = NDSolve[{
        f'''[y] - f'[y] × f'[y] + f[y] × f''[y] == -1,
        f[0] == 0,
        f'[0] == 0,
        f'[4] == 1.0
      },
      f, {y, 0, 4}]
```

ODE

$y \approx \infty$

```
Out[48]= {{f → InterpolatingFunction[ Domain: {{0., 4.}} Output: scalar ]}}
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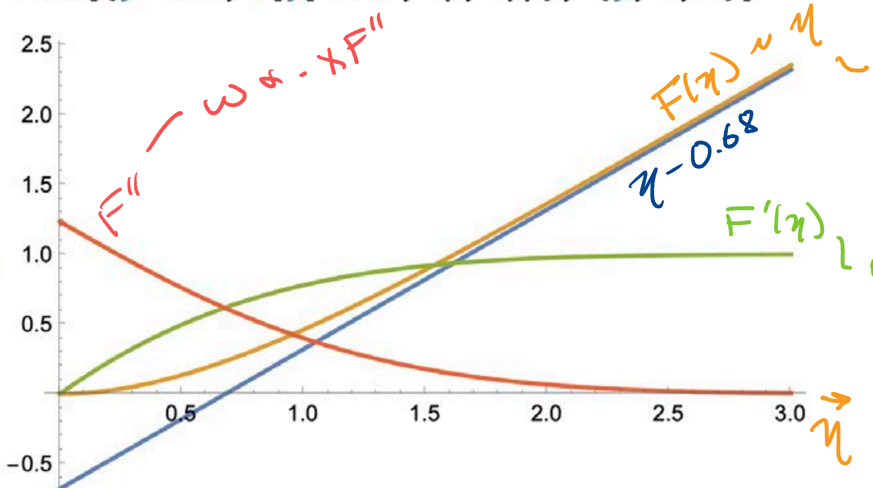
```
In[70]:= fp = D[f[y] /. s, y];
         fpp = D[f[y] /. s, {y, 2}];
         Plot[{y - .68, f[y] /. s , fp, fpp}, {y, 0, 3}]
```

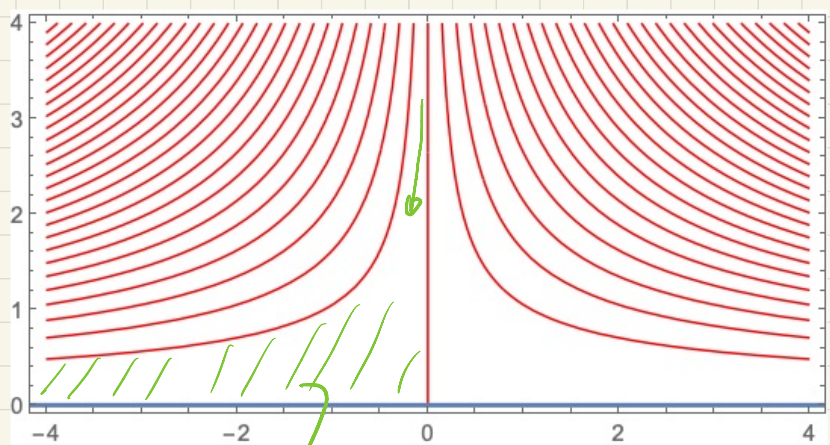
Out[72]=



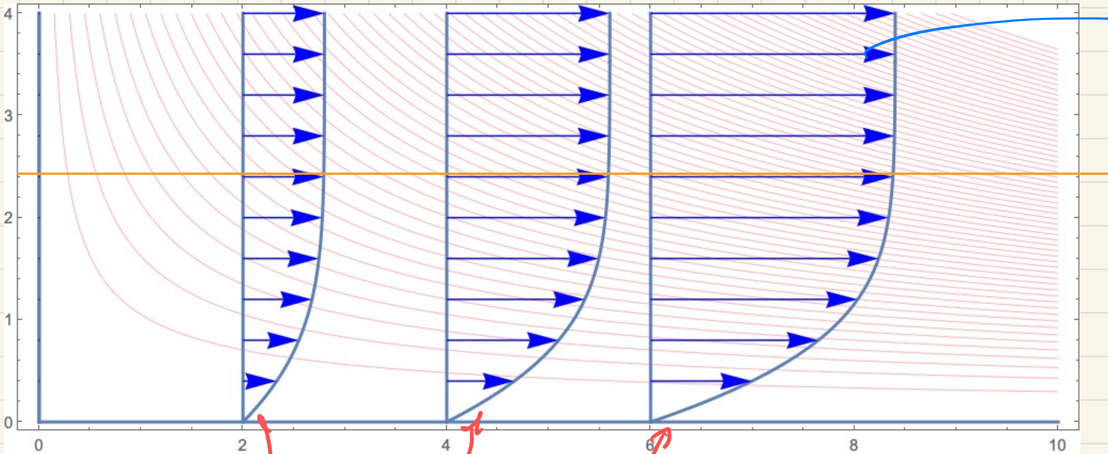
Note: x-dependence $\propto x$

→ Boundary Layer has constant thickness

$$\delta_{99} = 2.4\left(\nu/k\right)^{1/2}$$



slows / is retarded by ν on wall



just x-velocity

δ_{99}

wall gradients getting steeper

$\tau_{wall} = \mu \frac{\partial u}{\partial y}\Big|_{y=0}$ getting higher

Q: pressure? (at stagnation pt.)

y-momentum $$u\cancel{\frac{\partial v}{\partial x}} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \frac{\partial^2 v}{\partial y^2}\right)$$

$x = 0$, stagnation streamline

$\frac{\partial}{\partial x} \to 0$ symmetry

$$ff' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \nu f''$$

$\nu = \mu/\rho$

recall

$v = -f$

Goal: how different from inviscid (Bernoulli) stagnation pressure

$$\frac{\partial p}{\partial y} = -\rho f f' - \mu f''$$

$$\int_\infty^0 \frac{dp}{dy}dy = \int_\infty^0 (-\rho f f' - \mu f'')dy$$

$$P_0 - P_\infty = \left(-\tfrac{1}{2}\rho f^2 - \mu f'\right)\Big|_\infty^0$$

$V = -f \qquad f(0) = 0 \quad \text{BC}$

$f^2(\infty) = V_\infty^2 \qquad f'(0) = 0$

$f^2(0) = 0$

$f(y \to \infty) = ky$

$f'(y \to \infty) = k$

$$P_0 - P_\infty = +\tfrac{1}{2}\rho V_\infty^2 + \mu k$$

$$\underbrace{P_0}_{\text{actual stagnation pressure}} = \underbrace{P_\infty + \tfrac{1}{2}\rho V_\infty^2}_{\text{inviscid stagnation pressure}} + \underbrace{\mu k}_{\text{"error" due to viscosity}}$$

Barker Effect (1922)

(1931 Lotz Method)

P is "a bit" too high

$V_\infty = k h_\infty$ how big an error?

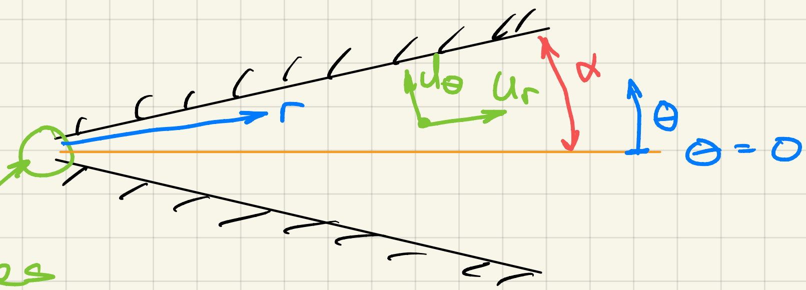
$$\frac{\frac{1}{2}\rho V_\infty^2}{\mu k} = \frac{1}{2}\underbrace{\frac{\rho V_\infty h_\infty}{\mu}}_{\text{a Re\#}}$$

might matter at low Re #

(1915) (1917)

Jeffrey-Hammel Flow – flow in a wedge shaped region

α - half angle



presumes source/sink/singularity

$\underline{u} = (u_r, u_\theta)$

- assume $u_\theta = 0$, pure radial flow

$\Longrightarrow$ brash assumption, but leads to illumination solutions

- steady, ρ const, μ const, incompressible

- **mass** $\nabla \cdot \underline{u} = 0$

$$\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{1}{r}\cancel{\frac{\partial u_\theta}{\partial \theta}}^{\,0} = 0$$

$$r u_r = \hat{f}(\theta)$$

"const" in r, but in general θ dependent

notation $\eta = \dfrac{\theta}{\alpha}$ $\qquad \dfrac{u_r}{u_{max}} = f(\eta)$

$$\begin{cases} r u_{max} = \hat{f}(0) \\ r u_r = \hat{f}(\theta) \end{cases}$$

u_{max} – velocity at $\theta = 0$

no slip BC : $f(\pm 1) = 0 \quad (\theta = \pm\alpha)$

- r - momentum

$$u \frac{\partial u_r}{\partial r} + \underbrace{\cancel{\frac{u_\theta}{r}}}_{0} \frac{\partial u_r}{\partial \theta} - \underbrace{\cancel{\frac{u_\theta^2}{r}}}_{0} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r} - \frac{2}{r^2}\underbrace{\cancel{\frac{\partial u_\theta}{\partial \theta}}}_{0}\right)$$

- θ - momentum

$$0 = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{2\nu}{r^2}\frac{\partial u_r}{\partial \theta} \qquad (\text{non } u_\theta \text{ terms})$$

$\frac{\partial}{\partial \theta}$ (r-mom)

$\frac{\partial}{\partial r}$ (r x θ-mom)

both have $\frac{1}{\rho}\frac{\partial^2 p}{\partial r \partial \theta}$

SUBTRACT $\rightarrow$ u_r only dependent variable

- Sub in $\frac{u_r}{U_{max}} = f(\eta)$

⋮

$$f''' + 2\,Re\,\alpha\, f f' + 4\alpha^2 f' = 0$$

$$Re = \frac{u_{max}\, r\alpha}{\nu}$$

NOTE: Re positive or negative

outflow (u_{max})

inflow (u_{max})

$f(0) = 1$ (peak velocity $u_r/u_{max} = 1$)

$f'(0) = 0$ (symmetry)

$f(\pm 1) = 0$ no slip

$\vdots$ integrate twice

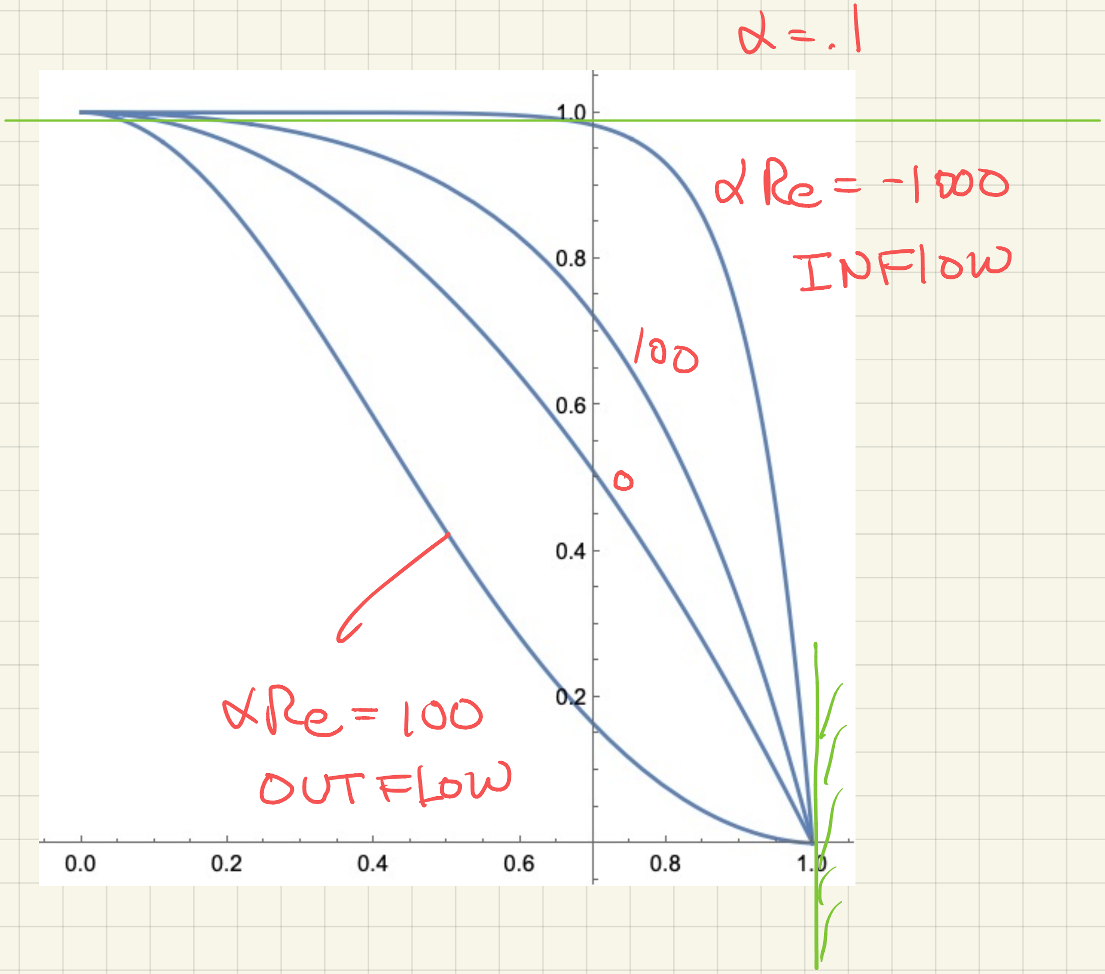
3rd integral is the solution

$$\eta = \int_{f}^{1} \frac{d\phi}{\sqrt{(1-\phi)\left(\frac{2}{3} Re\,\alpha(\phi^2+\phi) + 4\alpha^2\phi + C\right)}}$$

$f(\eta)$

integration const

$f(1) = 0$ $$1 = \int_0^1 \frac{d\phi}{\sqrt{\ \cdots\ C}}$$ $\Leftarrow$ implicit relation for C



α = .1
αRe = -1000
INFlow
100
0
αRe = 100
OUTFLOW
1.0
0.8
0.6
0.4
0.2
0.0
0.2
0.4
0.6
0.8
1.0