Lecture 12

- Displacement
- Matched asymptotics

Meeting 1 : T - F next week

- sign up for 30 min time slot
- 1st question
- reach out with concerns / questions
- no lecture W
- all in Talbot 306 F
- feedback ... immediate verbal + paragraph

Faulkner - Skan Flows $\quad U = Cx^m$

wall shear stress

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_0$$

F-S $\quad \psi = (U\nu x)^{1/2} f(\eta)$

$$\eta = \left(\frac{U}{\nu x}\right)^{1/2} y$$

$$u = \frac{\partial \psi}{\partial y} = (U\nu x)^{1/2} \frac{df}{d\eta}\frac{d\eta}{dy}$$

$$\tau_w = \mu (U\nu x)^{1/2}\left(\frac{U}{\nu x}\right) f''(0)$$

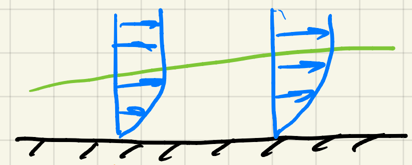
$$\tau_w = \mu \underbrace{\left(\frac{U^3}{\nu x}\right)^{1/2}}_{\text{all downstream } x \text{ dependence}} \underbrace{f''(0)}_{\text{just \#}}$$

$U = Cx^m$

$$\tau_w \propto x^{3m/2}\, x^{-1/2} = x^{\frac{1}{2}(3m-1)}$$

$m = 1/3 \rightarrow$ const τ_w downstream

$m=0$ Blasius



$\rightarrow$ δ thickening downstream by diffusion + some $v>0$ advection

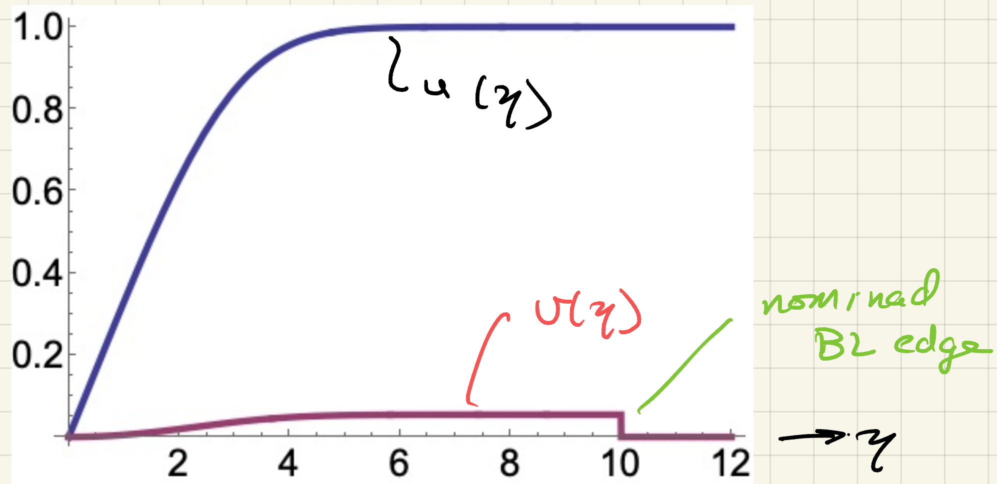
same profiles, same U, but thicker $\Rightarrow$ τ_w decreasing

δ = const

$\rightarrow$ same δ, larger U

same profile $\Rightarrow$ τ_w increasing

Displacement — related to v "jump" at BL edge



OUTER flow (U, V)

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

recast in INNER flow variable

$$\bar{V} = V\sqrt{Re}$$

$$\bar{y} = y\sqrt{Re}$$

$$* \quad \frac{\partial U}{\partial x} + \frac{\partial \bar{V}}{\partial \bar{y}} = 0$$

now in BL scaling

note $U, \bar{V}$ are viewed as fixed at BL edge

↳ i.e. the y dependence does not matter → take values at $y/\delta \to \infty$

INNER flow (BL eq) * $\dfrac{\partial u}{\partial x} + \dfrac{\partial \bar{v}}{\partial \bar{y}} = 0$ just BL eq.

Q : $\bar{V} - \bar{v} = ?$

* − * subtract

$$\frac{\partial}{\partial \bar{y}}(\bar{V} - \bar{v}) = -\frac{\partial}{\partial x}(U - u)$$

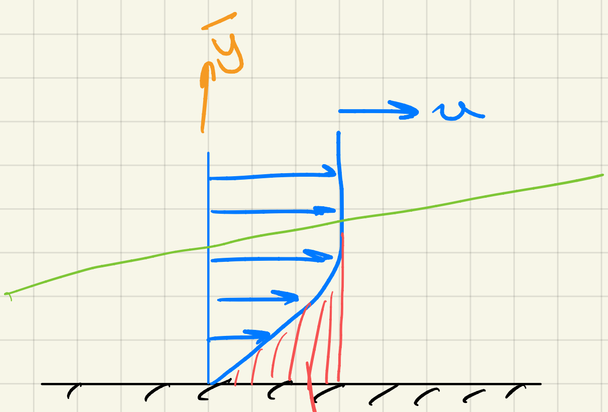
integrate in $\bar{y}$

$$\int_0^\infty \frac{\partial}{\partial \bar{y}}(\bar{V} - \bar{v})\,d\bar{y} = -\int_0^\infty \frac{\partial}{\partial x}(U-u)\,d\bar{y}$$

$$\lim_{y/\delta \to \infty}(\bar{V} - \bar{v}) = -\frac{\partial}{\partial x}\Big[U \underbrace{\int_0^\infty \left(1 - u/U\right) d\bar{y}}_{\delta_1}\Big]$$

The v jump

δ_1 Displacement thickness



the "missing" downstream flow due to BL

$$\lim_{y/\delta \to \infty} \bar{V} - \bar{v} = -\frac{\partial}{\partial x}\big(\underbrace{U\delta_1}\big)$$

└ change $\left(\frac{\partial}{\partial x}\right)$ of how much is "missing"

→ creates flow w/in boundary layer

$m = 0$ $U = \text{const}$

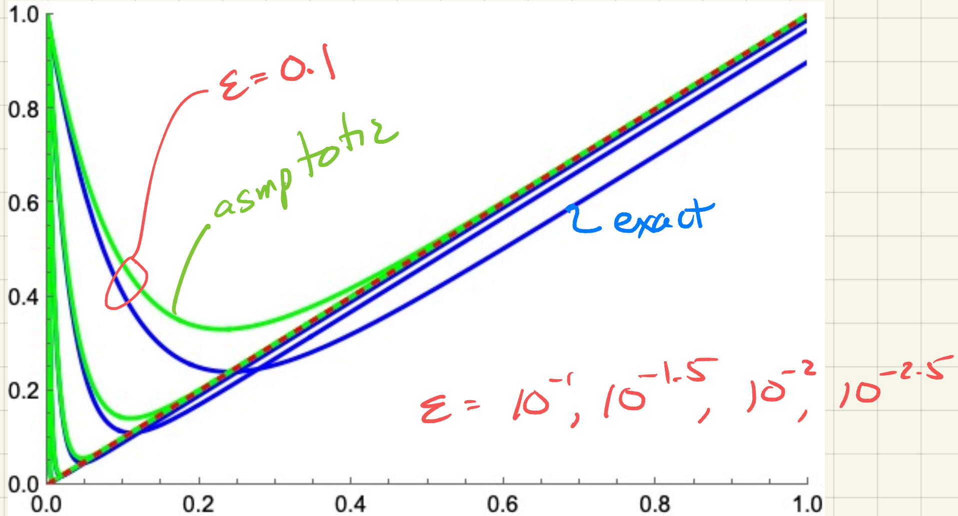
$\delta_1 \sim \sqrt{x}$ Blasius result

→ gives us $\bar{V} - \bar{v} < 0$

End material for Meeting 1

Where does this "jump" come from mathematically?

Recall our model ODE $\varepsilon \frac{du}{dx} + u = x$ $u(0) = 1$



exact $u(x) = (1+\varepsilon)e^{-x/\varepsilon} + x - \varepsilon$

INNER $u(x) = e^{-x/\varepsilon}$

OUTER $u(x) = x$

added $u(x) = e^{-x/\varepsilon} + x$

no jump ...

just works

$$u_{ex} - u_{asmp} = \varepsilon e^{-x/\varepsilon} - \varepsilon$$

→ off $O(\varepsilon)$, but smooth

NOTE: we did <u>not</u> attempt to "just add" the BL and OUTER flow solutions

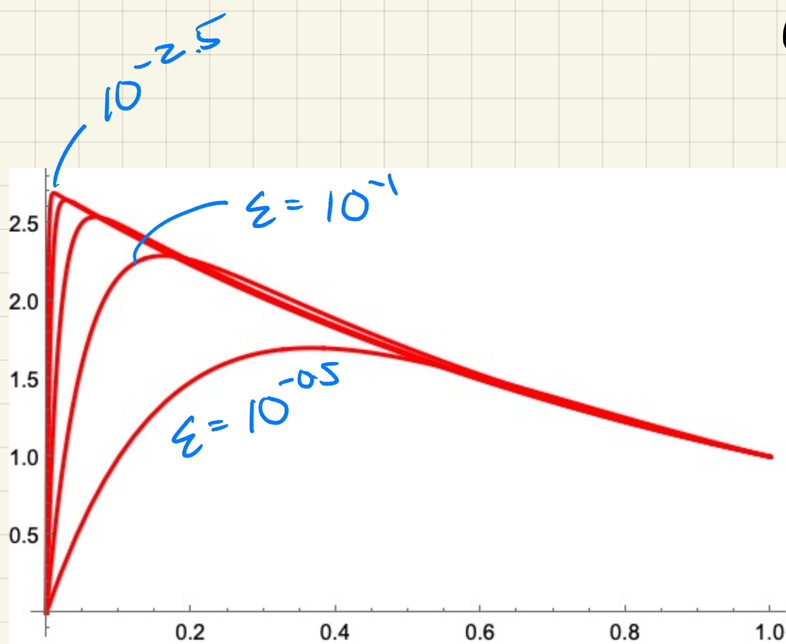
Note: for ODE both $O(\varepsilon)$ where they "intersect"

A better model ... more like flow equations ... has BC's at both "ends"

$$\varepsilon \frac{d^2u}{dy^2} + 2\frac{du}{dy} + 2u = 0 \qquad u(0)=0 \qquad u(1)=1$$

$$\beta_{\pm} = \frac{-2 \pm \sqrt{4 - 4\cdot 2\cdot \varepsilon}}{2\varepsilon} = -\frac{1}{\varepsilon} \pm \frac{\sqrt{1-2\varepsilon}}{\varepsilon}$$

$$u(y) = \frac{e^{\beta_+ y} - e^{\beta_- y}}{e^{\beta_+} - e^{\beta_-}}$$



Drop $\varepsilon \frac{d^2u}{dy^2}$ term (inviscid)

$$\frac{du}{dy} + u = 0$$

$$u(y) = A e^{-y}$$

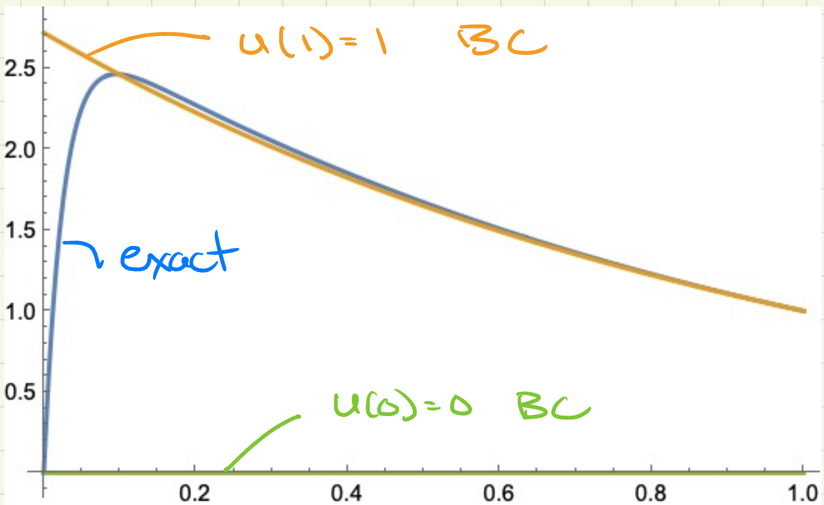
$\boxed{\text{OUTER}}$

can only use 1 of the 2 BC's

$u(0)=0 \rightarrow A=0 \qquad u(y)=0$ ← like $\mu=0$ solution for Stokes 1st

OR

$u(1)=1 \rightarrow A=e^1 \qquad u(y)=e^{1-y}$



INNER

zoom in on "BL" near $y=0$

$Y = y/\varepsilon \Rightarrow \varepsilon \frac{1}{\varepsilon^2}\frac{d^2u}{dY^2} + 2\frac{1}{\varepsilon}\frac{du}{dY} + 2u = 0$

INNER variable

$$\frac{d^2u}{dY^2} + 2\frac{du}{dY} + 2\varepsilon u = 0$$

$$\varepsilon u \to 0$$

BCs: $u(y=0) = u(Y=0) = 0$

$u(y=1) = u(Y=\tfrac{1}{\varepsilon}) = 1$

$$u(Y) = Be^{-2Y} + C$$

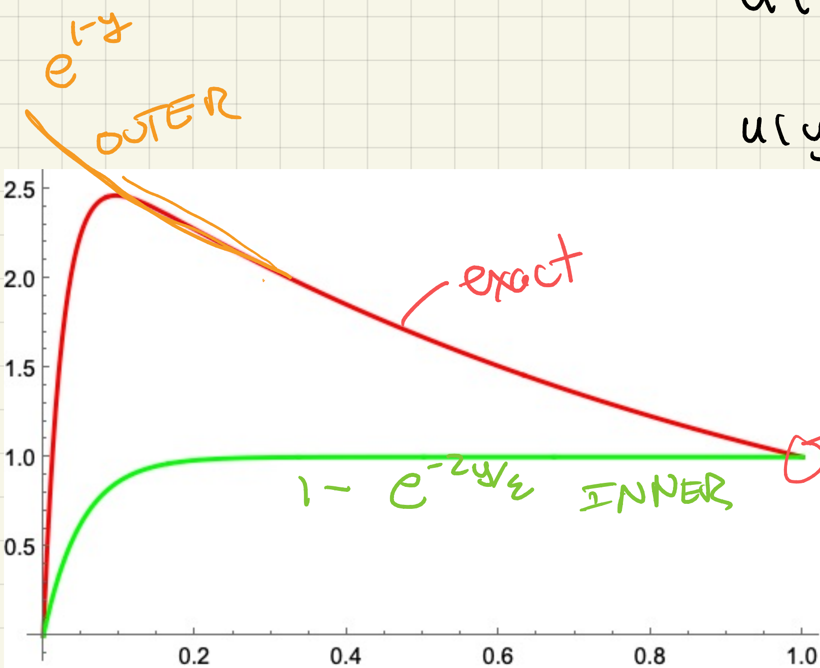
$u(0) = 0 = B + C$

$u(Y\to\infty) = 1 = C$

$B = -1$

$$u(Y) = 1 - e^{-2Y}$$

$$u(y) = 1 - e^{-2y/\varepsilon}$$



can't just add