Lecture 15

- 2nd order BL

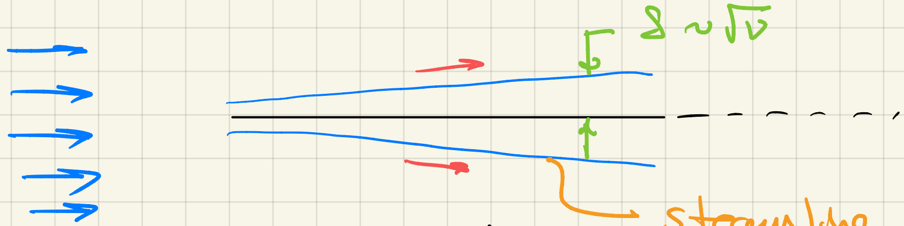
Last time

$Re \to \infty$ — no effect on outer flow

$\delta \to 0$, no displacement

Re large

$\delta \sim \sqrt{\nu}$



streamline at "edge" of BL → it's displaced by δ

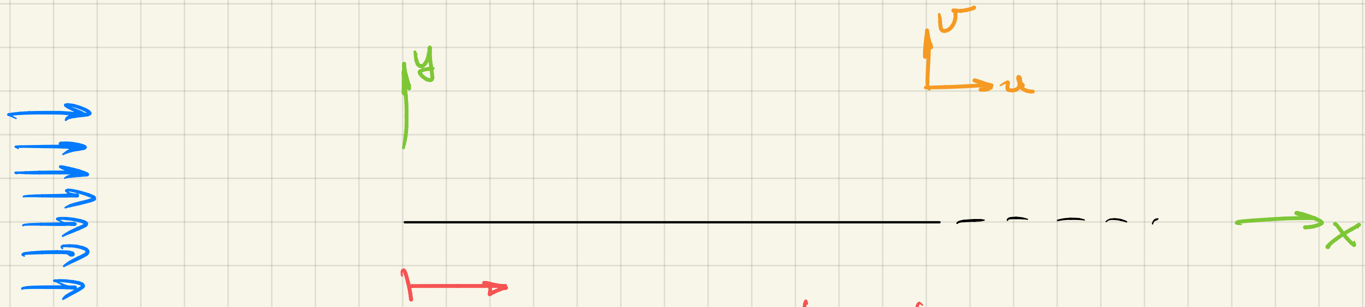
effects missing in first-order BL theory

2nd Order BL theory $\Rightarrow$ how to include these rigorously

- BL approximation due to some U, V — INNER
- U, V response to the BL — OUTER

MATCH, each responding to the other

New



some scale along plate

follows van Dyke + JBF notes

$$Re = \frac{LU}{\nu}$$

assume steady

$$u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = \frac{1}{Re}\nabla^2\omega$$

N-S in 2D, no approximation

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x} \qquad \omega = -\nabla^2\psi$$

Exact NS still

$$\left(\psi_y\frac{\partial}{\partial x} - \psi_x\frac{\partial}{\partial y} - \frac{1}{Re}\nabla^2\right)\nabla^2\psi = 0$$

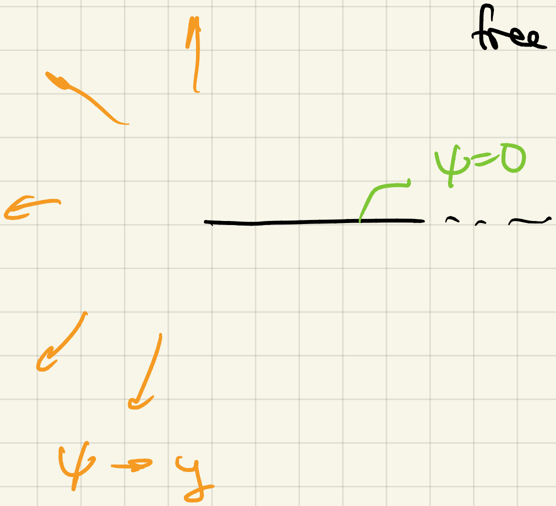
Fun fact: all 2D potential flows satisfy full N-S equation

<u>BC</u> no penetration $\psi(x>0, y=0) = 0$

value is arbitrary, but no flow across a streamline

$v = -\frac{\partial\psi}{\partial x} = 0$ on plate no slip $\quad u(x>0, y=0) = \left.\dfrac{\partial\psi}{\partial y}\right|_{x>0,\, y=0} = 0$

free stream: $\quad$ upstream, $y \to \pm\infty$



$u = U$ $\quad$ [dimensional]

$u = 1$ $\quad$ [non dimensional]

$\psi_y \equiv \dfrac{\partial\psi}{\partial y}$

$\psi = y$ $\quad$ so $\quad u = \psi_y = 1$

$\boxed{\text{OUTER}}$ expansion

$$\psi(x,y) = \delta_1(Re)\,\psi_1(x,y) + \delta_2(Re)\,\psi_2(x,y) + \cdots$$

leave vague for now,
will set as informed

$\delta_1 = 1$

$\delta_2 = 1/Re^{1/2}$

will confirm

appeal to $\psi(x \to -\infty, y)$ BC $\quad \psi = y$
for insight into $\delta_1(Re)$

lowest order

$\psi_1 = \psi / \delta_1$

$$\psi_1 = \lim_{Re \to \infty} \frac{y}{\delta_1(Re)} \sim y$$

so $\delta_1(Re) = 1$

if not, we'd "lose" the solution

first approximation (OUTER)

$$\Big(\underbrace{\psi_{1y} \frac{\partial}{\partial x} - \psi_{1x} \frac{\partial}{\partial y}}_{\frac{D\omega}{Dt}} - \underbrace{\frac{1}{Re}}_{*} \nabla^2 \Big) \overbrace{\nabla^2 \psi_1}^{-\omega} = \underbrace{o(\delta_1)}$$

$\frac{D\omega}{Dt}$ – pure advection ... from upstream

* don't know the precise degree of smallness ... "little oh" but Re large

thing that $\to 0$ faster than δ_1

$\to$ probably $O(\delta_2)$ "big oh"

$\omega \to 0$ upstream, pure advection,

so $\nabla^2 \psi_1 = 0$ ... the OUTER $O(\delta_1)$ flow is potential

OUTER BC $\psi_1(x, 0) = 0$ (stream line, no pen)

$\psi_1 \to y$ for $x \to -\infty$, $y \to \pm\infty$

skip INNER no slip ... → could not apply anyhow...

solution $\psi_1 = y$ – parallel stream lines (of course)

$\boxed{\text{INNER}}$ expansion

zoom in $Y = \dfrac{y}{\Delta_1(Re)}$ some dependence on Re (will be $1/Re^{1/2}$)

goal $Y = O(1)$

→ Δ_1 will come from a consistency condition for INNER expansion of ψ

we know $u = O(1)$ both INNER / OUTER

$$u = \psi_y = \frac{\partial \psi}{\partial y} = O(1) \quad \text{[both]}$$

$y = O(\Delta_1)$ then $\psi = O(\Delta_1)$

INNER expansion $\psi(x,y) = \Delta_1(Re)\underbrace{\Psi_1(x,y)}_{O(1)} + \Delta_2 \Psi_2(x,y) + \dots$

sub into N-S : $\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y} - \frac{1}{Re} \nabla^2 \right) \nabla^2 \psi = 0$

keep Δ_1 terms ... for this bit $\Psi_1 = \Psi$
(for now)

$$\Delta_1^2 \left(\frac{1}{\Delta_1} \Psi_Y \frac{\partial}{\partial X} - \Psi_X \frac{1}{\Delta_1} \frac{\partial}{\partial Y} \right) \overbrace{\left(\cancel{\Psi_{XX}} + \frac{1}{\Delta_1^2} \Psi_{YY} \right)}^{\nabla^2 \Psi}$$

$$- \frac{\Delta_1}{Re} \left(\cancel{\frac{\partial^2}{\partial X^2}} + \frac{1}{\Delta_1^2} \frac{\partial^2}{\partial Y^2} \right) \underbrace{\left(\cancel{\Psi_{XX}} + \frac{1}{\Delta_1^2} \Psi_{YY} \right)}_{\Delta^2 \Psi} = 0$$

left out Δ_2's

crossed out higher order Δ_1 terms

keep $O\left(\frac{1}{\Delta_1}\right)$
and Re term...

$$\left(\frac{1}{\Delta_1} \Psi_Y \frac{\partial}{\partial X} - \Psi_y \frac{1}{\Delta_1} \frac{\partial}{\partial Y} \right) \Psi_{YY} - \frac{\Delta_1}{Re \Delta_1^4} \Psi_{YYYY} = 0$$

useful choice ...

$\Delta_1^2 = 1/Re$

$\Delta_1 = 1/Re^{1/2}$

keeps all terms

→ it work because it work

$$\left(\frac{\partial^2}{\partial Y^2} - \Psi_Y^{B} \frac{\partial}{\partial x} + \Psi_x \overset{A}{\frac{\partial}{\partial Y}}\right) \Psi_{YY} = 0$$

$$\frac{\partial}{\partial Y}\left(\Psi_{YYY} + \Psi_x \Psi_{YY} - \Psi_Y \Psi_{xY}\right) = 0$$

same

$$\cancel{\Psi_{xY} \Psi_{YY}} + \Psi_x \overset{A}{\Psi_{YYY}} - \cancel{\Psi_{YY} \Psi_{xY}} - \Psi_Y^{B} \Psi_{xYY}$$

integrate $(\Psi \xrightarrow{\text{notation}} \Psi_1)$

"const" dep on x

$$\Psi_{1YYY} + \Psi_{1x} \Psi_{YY} - \Psi_{1Y} \Psi_{xY} = g(x)$$

$u = \frac{\partial \Psi}{\partial Y}$

$v = -\frac{\partial \Psi}{\partial x}$

to be confirmed

$$\frac{\partial^2 u}{\partial Y^2} - v \frac{\partial u}{\partial y} - u \frac{\partial u}{\partial x} = \left(\frac{\partial p}{\partial x}\right)$$

x-momentum BL equation

NEXT → MATCH