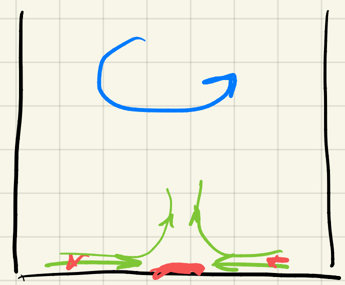
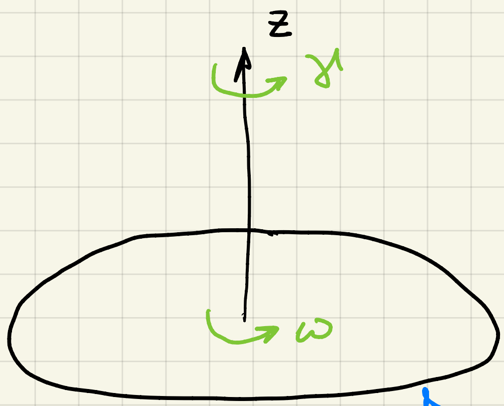
Lecture 20

3D BL.

Recall demo





$\gamma, \omega \rightarrow$ angular rotation rates

γ 2 cases: γ_1 @ $z \rightarrow \infty$

or

γ_2 @ $z \rightarrow d$

∞ flat, rotating "disk"

governing equation

$\frac{\partial}{\partial \theta} \rightarrow 0$ axisymmetric

$\frac{\partial}{\partial t} \rightarrow 0$ steady

r-mom
$$v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 v_r^2 - \frac{v_r}{r^2}\right)$$

θ-mom
$$v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} = \nu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2}\right)$$

z-mom
$$v_z \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 v_z$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

mass (ρ=const)
$$\frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{\partial v_z}{\partial z} = 0$$

more like demo

BC

$z=0$ on "wall"		$z=d$ ← OR →	$z \to \infty$
$v_z = 0$	no pen.	$v_z = 0$	$v_z = ?$
$v_r = 0$	no slip	$v_r = 0$	$v_r = 0$
$v_\theta = r\omega$	solid body rot + no slip	$v_\theta = \gamma_2 r$	$v_\theta = \gamma_1 r$

approach → impose artificial constraints (function form restriction) so the flow is similar enough to the "real" case ...

$\Rightarrow$ $v_z = v_z(z)$ only

not our container

& maybe enough like it away from top and side walls

← not too different from stag flow

$\psi = kxy$

$v = -ky$

fct of y only

<u>mass</u> ... get v_r in terms of v_z ...

$$\frac{1}{r}\frac{\partial r v_r}{\partial r} = -\frac{\partial v_z}{\partial z}$$

no r dependence

$$\frac{\partial r v_r}{\partial r} = -r\frac{\partial v_z}{\partial z}$$

integrate $\quad r v_r = -\frac{1}{2} r^2 \frac{d v_z}{dz} + c$

$$\div r \qquad v_r = -\tfrac{1}{2} r \frac{dv_z}{dz} + \cancel{c/r}$$

$\leftarrow$ bc @ $z=0$

$C \to 0$

only term $v_z(z)$

z-mom

$$\cancel{v_r \frac{\partial v_z}{\partial r}}^{\,0} + \underbrace{v_z \frac{\partial v_z}{\partial z}}_{\frac{\partial \frac{1}{2} v_z^2}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \frac{d^2 v_z}{dz^2}$$

integrate in z :

$$p/\rho = \underbrace{\nu \frac{dv_z}{dz} - \tfrac{1}{2} v_z^2}_{\text{no } r \text{ dependence}} + \Pi(r)$$

integration "const" $\to$ has r dependence

$v_r = -\tfrac{1}{2} r \frac{dv_z}{dz}$ from mass

r-mom

$$v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\cancel{\frac{\partial^2 v_r}{\partial r^2}}^{\,\phi} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\right)$$

$\frac{\partial^2 r}{\partial r^2}$

$$\left(-\tfrac{1}{2} r \frac{dv_z}{dz}\right)\left(-\tfrac{1}{2}\frac{dv_z}{dz}\right) + v_z\left(-\frac{1}{2} r \frac{d^2 v_z}{dz^2}\right) - \frac{v_\theta^2}{r} = -\frac{d\Pi}{dr} - \cancel{\frac{\nu}{2r}\frac{dv_z}{dz} + \frac{\nu}{2}\frac{1}{r}\frac{dv_z}{dz}}$$

$$-\frac{\nu}{2} r \frac{d^3 v_z}{dz^3}$$

$\div r$

$$\frac{1}{4}\left(\frac{dv_z}{dz}\right)^2 - \frac{1}{2} v_z \frac{d^2 v_z}{dz^2} - \underbrace{\frac{v_\theta^2}{r^2} = -\frac{1}{r}\frac{d\Pi}{dr}}_{\text{all } r \text{ dependence}} - \frac{\nu}{2}\frac{d^3 v_z}{dz^3}$$

$$\frac{1}{r}\frac{d\Pi}{dr} - \frac{v_\theta^2}{r^2} = G(z)$$

$$\frac{d\Pi}{dr} = \frac{v_\theta^2}{r} + r\,G(z)$$

$\frac{1}{r}\frac{\partial}{\partial r}(r^2 \ldots)$

BC: at $z=0$ wall $v_\theta = r\omega$ (solid body, no slip)

$$\frac{d\Pi}{dr} = r\omega^2 + r\,G(0)$$

a #

$$\Pi(r) \overset{*}{=} \frac{1}{2}r^2(\omega^2 + G(0))$$

$$C - \frac{v_\theta^2}{r^2} = G(z)$$

$$\Downarrow \quad \frac{v_\theta}{r} \overset{*}{=} H(z) = \sqrt{-G(z)+C}$$

θ-mom $\cdots$ $\frac{d v_z}{dz}\left(\frac{v_\theta}{r}\right) + v_z \frac{\partial}{\partial z}(v_\theta/r) = \nu \frac{\partial^2 v_\theta/r}{\partial z^2}$

v_θ only appears as v_θ/r

$\vdots$ similarity transform $r = \left(\nu/\omega\right)^{1/2}\eta$ $\quad z = \left(\nu/\omega\right)^{1/2}\zeta$

$$v_\theta = (\nu\omega)^{1/2}\,\eta\, f(\zeta) \qquad v_z = (\nu\omega)^{1/2}\,h(\zeta)$$

νr

$\vdots$

r-mom $\quad \frac{1}{4}h'^2 - \frac{1}{2}hh'' - g^2 = -\left(\frac{\omega^2+c}{\omega}\right) - \frac{1}{2}h'''$

θ-mom $\quad -gh' + hg' = g'''$

BC :

$\zeta = 0 \qquad h = h' = 0 \qquad g = 1$

$\zeta \to \infty \qquad h' \to 0 \qquad g \to \gamma_1/\omega$

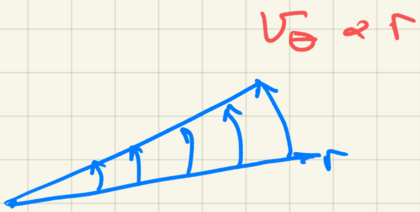
$\delta = \left(\frac{d^2\omega}{\nu}\right)^{1/2} \qquad h = h' = 0 \qquad g = \gamma_2/\omega$

numerical solution

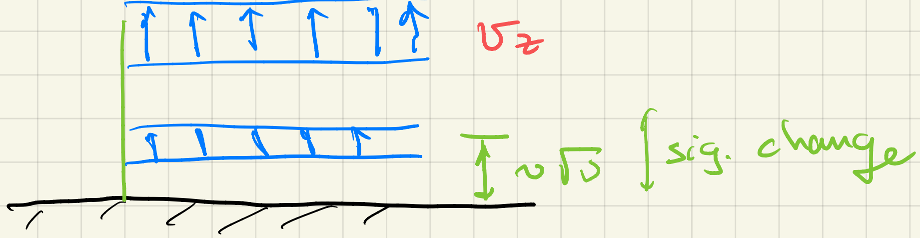
→ ROTATING OUTER FLOW

SIDE

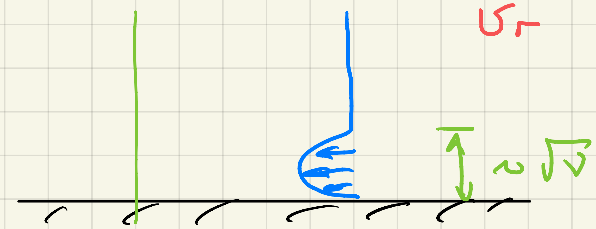
TOP



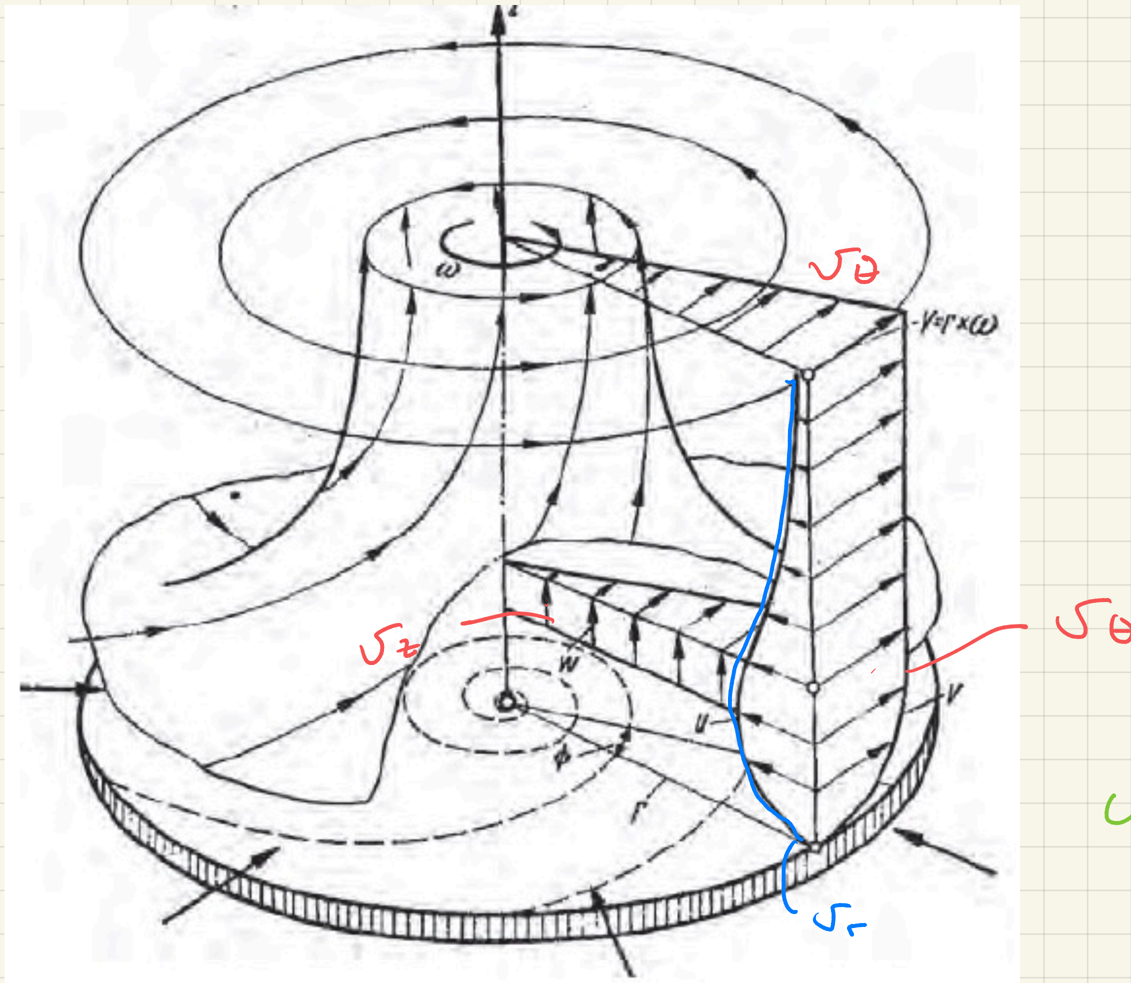
at any z



v_z
$\sim\sqrt{\nu}$
sig. change

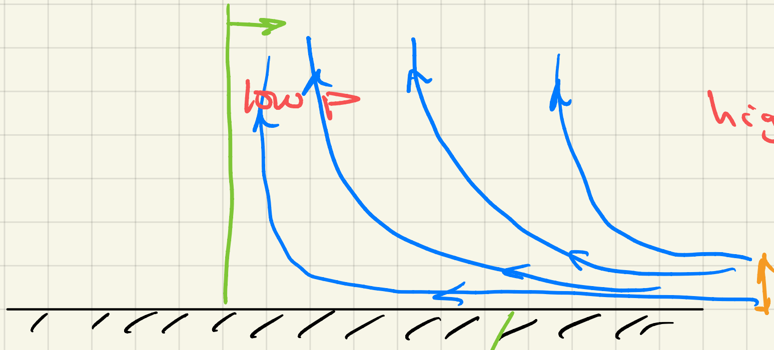


v_r
$\sim\sqrt{\nu}$



v_θ
$V=r\times\omega$
ω
v_z
v_θ
v_r

$\omega = 0$ case



high P → pressure balancing centripital acceleration

P should not change thru BL

at wall, no v_θ → centripidal acceleration cannot balance ∇P

$\omega = 0$ case

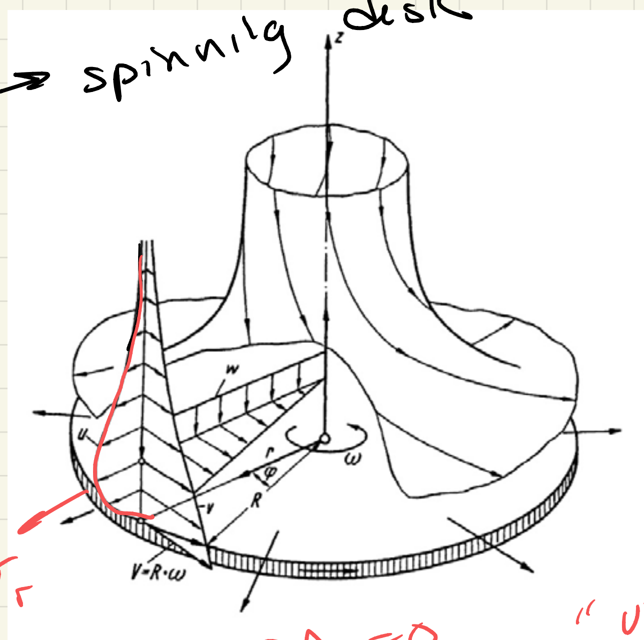
↳ grad P from outside BL must balance something else

→ radial momentum, viscosity

→ "jet" BL along wall toward center...

not our demo

→ spinning disk



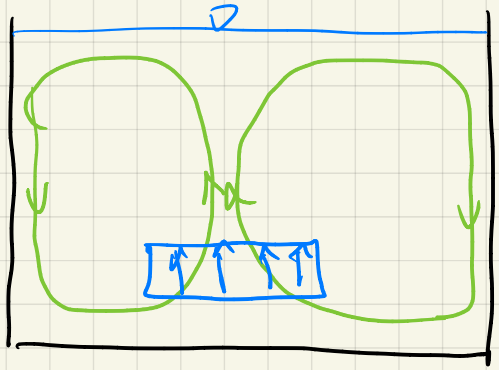
v_r

$\nu = 0$

$\omega \neq 0$

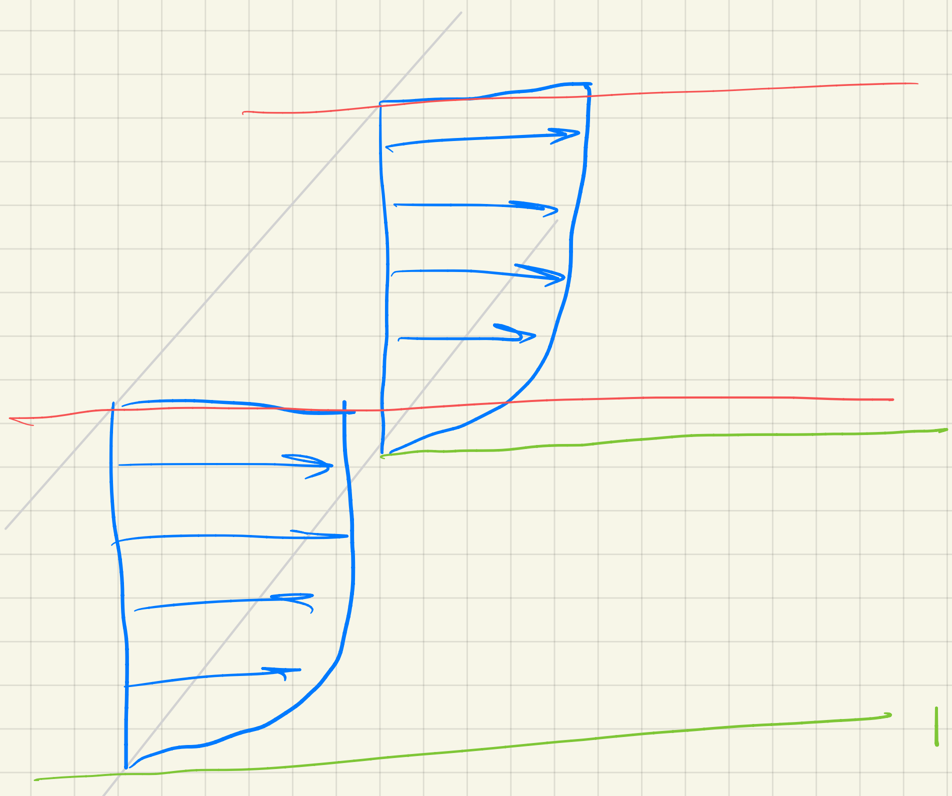
" viscous pump "

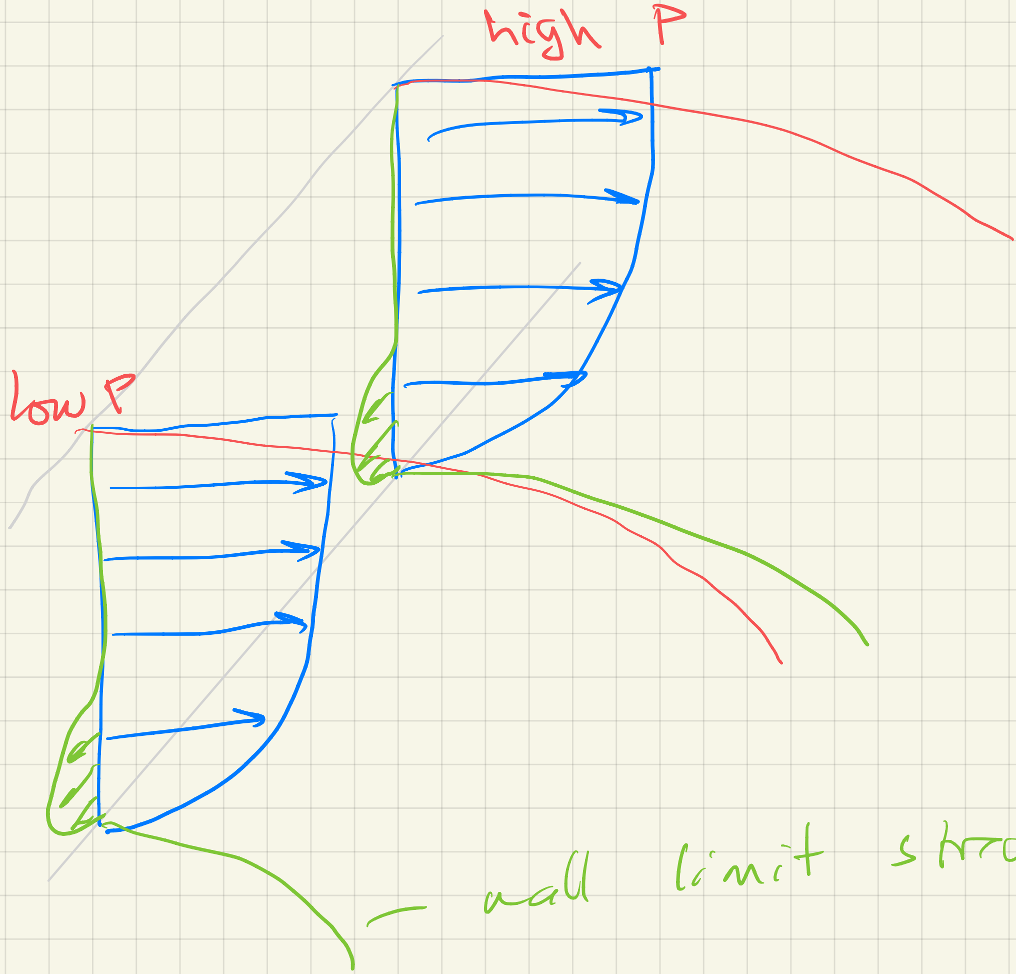
DEMO

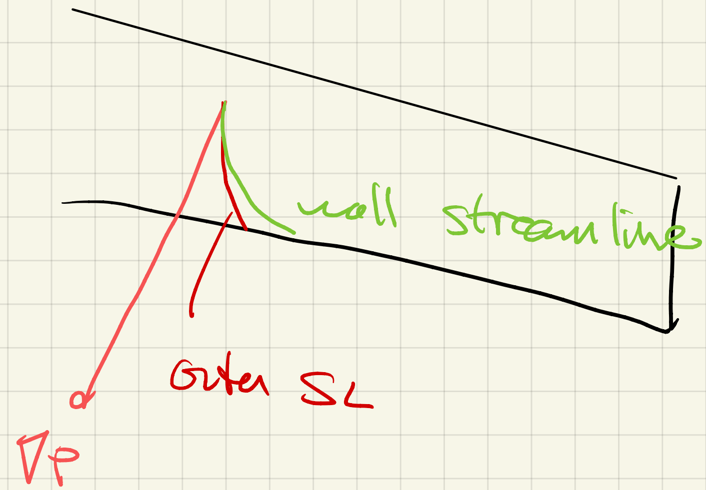


similar enough

Shows up any time streamlines bend above BL







– wall limit streamline bend more