Lecture 09

- Asymtotic analysis of Jeffrey-Hamel ODE
- N-S BL

Office hours today → Zoom

↳ link canvas and Box

Recall Jeffrey - Hammel ODE

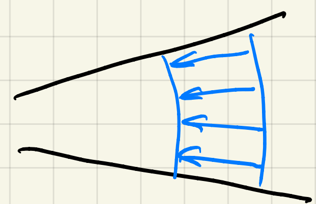
$$f''' + 2\alpha Re\, ff' + 4\alpha^2 f' = 0 \qquad f(\theta) = \frac{u(\theta)}{u_{max}}$$

$\eta = \theta/\alpha$

$f(\eta)$

recall α - small

$\alpha |Re|$ - large "inviscid"



• great but no BL

Schichtung

Asymptotic → zoom in on region of interest

make $\xi = O(1)$ in region of interest

$$\xi = \underbrace{(1-\eta)}_{\xi = 0 \text{ on wall}} \sqrt{-\alpha Re} \equiv A$$

$1/\sqrt{\nu}$

we "think" this is the needed zoom factor change variable

$$f''' + \underbrace{2\alpha Re}_{\to\; -\left(\sqrt{-\alpha Re}\right)^2 \;=\; -A^2} ff' + 4\alpha^2 f' = 0$$

$$f(\eta) = f(\xi)$$

$$f'(\eta) = \frac{df}{d\eta} = \frac{df}{d\xi}\underbrace{\frac{d\xi}{d\eta}}_{-A} = -\dot{f}A$$

$$f''(\eta) = \frac{d^2f}{d\eta^2} = \frac{d^2f}{d\xi^2}\underbrace{\left(\frac{d\xi}{d\eta}\right)^2}_{A^2} + \frac{df}{d\xi}\underbrace{\frac{d^2\xi}{d\eta^2}}_{0} = A^2\ddot{f}$$

$$f''' = \qquad = -A^3\dddot{f}$$

$$-A^3\dddot{f} + 2A^2A\dot{f}f - 4\alpha^2A\dot{f} = 0$$

A large

$$\dddot{f} - 2\dot{f}f = 0$$

A, Re gone

$A \to \infty$ "model"

$f(0) = 0$

$f(\xi \to \infty) = 1$

$f'(\xi \to \infty) = 0$

matching the "uniform flow"

⋮ time + just trying it

$$f(\xi) = 3\tanh^2\left[\frac{\xi}{\sqrt{2}} - \tanh^{-1}\sqrt{\tfrac{2}{3}}\right] - 2$$

$$f(\eta) = 3\tanh^2\left[\left(-\tfrac{1}{2}\alpha Re\right)^{1/2}(1-\eta) - \tanh^{-1}\sqrt{\tfrac{2}{3}}\right] - 2$$

Same, by a procedure on gov. eq.

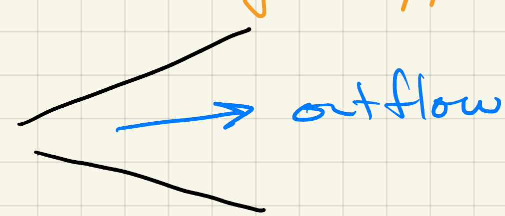
Bachelor

Navier - Stoke BL Hypothesis

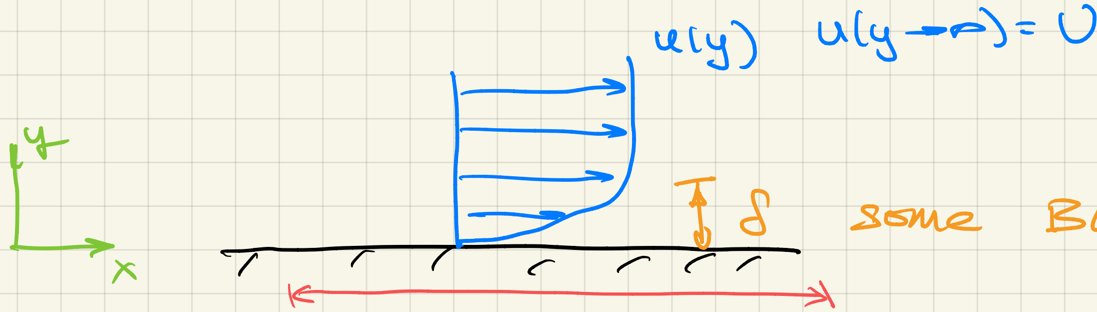
Boundary Layer – a thin layer in which viscosity is important no matter how high Re#

- self-consistent
- seems to work in cases

- no proof
- not always applicable



- are guidelines, no specific class of applicability



some BL scale $\sim \sqrt{\nu}$ $\sim \sqrt{1/Re}$

some (vague) flow scale

expect: $\left|\frac{\partial u}{\partial x}\right| \ll \left|\frac{\partial u}{\partial y}\right|$ and $\left|\frac{\partial^2 u}{\partial x^2}\right| \ll \left|\frac{\partial^2 u}{\partial y^2}\right|$

→ can reason in advance

small?

<u>x-mom</u> $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

large 0

small?

<u>mass</u> $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

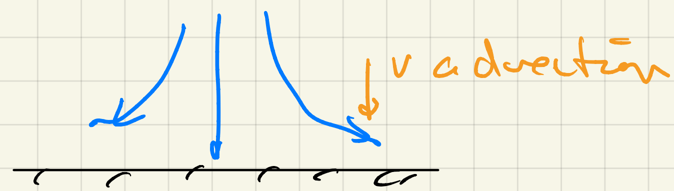
large

choose to preserve exactly

- "leaking" some small mass might be ok ... but maybe not for $t \to \infty$

- pointwise condition - also "strong"
- tells us v small but needs to be retained

so $\frac{\partial}{\partial y} \sim \frac{1}{\delta} \Rightarrow v \sim \delta$

- v important on 

- keep $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$ in x-momentum

- NOTE:

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}}$$

have we
- dropped $\nu\frac{\partial^2 u}{\partial x^2}$ from N-S ?
- added $\nu\frac{\partial^2 u}{\partial y^2}$ to Euler ?

- example inertia/advection versus wall ⊥ diffusion

$$u\frac{\partial u}{\partial x} \quad \text{v.} \quad \nu\frac{\partial^2 u}{\partial y^2}$$

→ must balance

$$O\left[\frac{u\frac{\partial u}{\partial x}}{\nu\frac{\partial^2 u}{\partial y^2}}\right] = 1$$

$u \sim U$

$x \sim L$

$y \sim \delta$

$$\frac{U^2/L}{\nu\, U/\delta^2} = \frac{\delta^2 U}{\nu L} = \frac{\delta^2}{L^2}\underbrace{\frac{UL}{\nu}}_{Re\#} = O(1)$$

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{Re}} \iff \delta \sim \sqrt{\nu}$$

• mass

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

$$\sim \frac{U}{L} \qquad \sim \frac{V}{\delta} \qquad \Longrightarrow \qquad V \sim \frac{\delta U}{L} \approx U / Re^{1/2}$$

$y \sim \delta \qquad v \sim \delta \qquad \nu \sim \delta^2$

• y-momentum

$$\underset{\sim \delta}{\frac{\partial v}{\partial t}} + \underset{\sim \delta}{u \frac{\partial v}{\partial x}} + \underset{\sim \delta^2/\delta}{v \frac{\partial v}{\partial y}} = \underbrace{\underset{\sim \frac{1}{\delta}}{-\frac{1}{\rho}\frac{\partial p}{\partial y}}}_{\text{leading order}} + \underset{\delta^2}{\nu} \left(\underset{\sim \delta}{\frac{\partial^2 v}{\partial x^2}} + \underset{\sim \delta/\delta^2}{\frac{\partial^2 v}{\partial y^2}} \right)$$

$$-\frac{1}{\delta}\frac{\partial p}{\partial y} = O(\delta)$$

$$\boxed{\frac{\partial p}{\partial y} = 0}$$

y momentum to $O(\delta)$

BC: $\quad u = v = 0 \quad$ at $\quad y = 0$

$u \to U(x,t) \quad$ for $\quad y/\delta \to \infty \qquad (p \to p_\infty)$

NOTES

- only have 1 v BC
  - whole system can be one 3rd order equation ψ [Ockendon + Ockendon]
  - v is "asymptotically" small
    → some presumption of $v = 0$ where its really just tiny
- what is δ? → exact value doesn't matter
  $\delta \sim \sqrt{v}$ and smooth transition to OUTER flow
- what is the "OUTER" flow
  → assume effectively inviscid

x-mom $\quad \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$

no viscous

$V = 0$ – no V advection

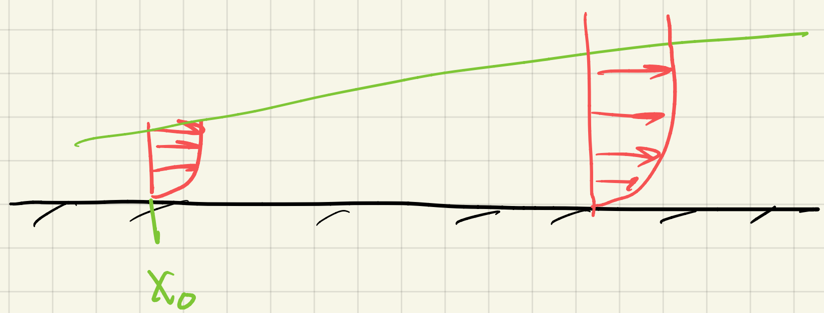
→ stream line on edge of BL

steady $\quad \frac{\partial}{\partial x}\left(\frac{1}{2}U^2 + \frac{1}{\rho}P\right) = 0$

$P + \frac{1}{2}\rho U^2 = \text{const}$

persist to wall $\quad \frac{\partial P}{\partial y} = 0$ in BL

- also need x BC : $u(x_0, y)$



need info at some x_0

- unsteady → also need an IC

SYSTEMATIC single ansatz → result

start $$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p + \frac{1}{Re}\nabla^2\underline{u} \qquad \nabla\cdot\underline{u} = 0$$

$$y' = Re^{1/2}\, y \qquad v' = Re^{1/2}\, v$$

(drop primes)

$\therefore$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}$$

$$0 = -\frac{\partial p}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

A $Re \to \infty$ model for high Re