Lecture 14

- 2nd order matching ~~ODE~~
- 2nd order BL

Reproduced from last lecture:

Same model $\varepsilon \dfrac{d^2u}{dy^2} + 2\dfrac{du}{dy} + 2u = 0$ $\quad u(0)=0$, $u(1)=1$

assume $u(y) = u_0(y) + \varepsilon u_1(y) + \varepsilon^2 u_2(y) \cdots\cdots$

"works if it works"

Sub in

$$\varepsilon \frac{d^2u_0}{dy^2} + \varepsilon^2 \frac{d^2u_1}{dy^2} + \cdots + 2\frac{du_0}{dy} + 2\varepsilon\frac{du_1}{dy} + \cdots + 2u_0 + 2\varepsilon u_1 + \cdots = 0$$

INNER $\quad Y = y/\varepsilon$

$$\frac{\varepsilon}{\varepsilon^2}\frac{d^2u_0}{dY^2} + \frac{\varepsilon^2}{\varepsilon^2}\frac{d^2u_1}{dY^2} + \cdots + 2\frac{1}{\varepsilon}\frac{du_0}{dY} + 2\frac{\varepsilon}{\varepsilon}\frac{du_1}{dY} + \cdots + 2u_0 + 2\varepsilon u_1 + \cdots = 0$$

$\times\varepsilon$ rearrange

$$\underbrace{\frac{d^2u_0}{dY^2} + 2\frac{du_0}{dY}}_{O(1)} = \underbrace{-\varepsilon\frac{d^2u_1}{dY^2} - 2\varepsilon\frac{du_1}{dY} - 2\varepsilon u_0}_{O(\varepsilon)} + O(\varepsilon^2)$$

$O(1)$ — SAME as before → $u_0(y) = B_0\left(e^{-2y/\varepsilon} - 1\right)$ left ?

for $u_0(0) = 0$? INNER BC only

OUTER $\div 2$

$$\underbrace{\frac{\partial u_0}{\partial y} + u_0}_{O(1)} = \underbrace{-\frac{\varepsilon}{2}\frac{d^2u_0}{dy^2} - \varepsilon\frac{du_1}{dy} - \varepsilon u_1}_{O(\varepsilon)} + O(\varepsilon^2)$$

$O(1)$ — SAME $\qquad u_0(y) = e^{(1-y)}$

used $u_0(1) = 1$ OUTER BC ONLY

only change so far $u \to u_0$ $\Rightarrow$ $\underbrace{B_0 = -e^1}_{\text{from } O(1) \text{ Matching}}$

next order...

O(ε) INNER :

$$\varepsilon \frac{d^2u_1}{dY^2} + 2\varepsilon \frac{du_1}{dY} = -2\varepsilon u_0$$

"Source" from lower order

$u(0)=0 = u_0(0) + \varepsilon u_1(0)$ $=0$

O(ε) $u_1(0)=0$

$\div\varepsilon$

$$\frac{d^2u_1}{dY^2} + 2\frac{du_1}{dY} = -2e^1(1-e^{-2Y})$$

sub in u_0

can solve

$$\boxed{u_1(Y) = B_1(1-e^{-2Y}) - Ye^1(1+e^{-2Y})}$$

only applied inner BC... → B_1 const....

? from O(ε) outer match

New

O(ε) OUTER :

$$\frac{du_1}{dy} + u_1 = -\frac{1}{2}\frac{d^2u_0}{dy^2} = -\frac{1}{2}e^{1-y}$$

⋮ solve

$$\boxed{u_1(y) = \frac{1}{2}(1-y)e^{1-y}}$$

Match

Match to find B_1 $\rightarrow$ $Y = y/\varepsilon$

OUTER: $$u(y) = \underbrace{e^{1-y}}_{O(1)} + \underbrace{\tfrac{\varepsilon}{2}(1-y)e^{1-y}}_{O(\varepsilon)}$$

expand in INNER Y $\longrightarrow$ into overlap

$$u(Y) = e^{1-\varepsilon Y} + \tfrac{\varepsilon}{2}(1-\varepsilon Y)e^{1-\varepsilon Y}$$

use: $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$

$$e^{1-\varepsilon Y} = e^1 e^{-\varepsilon Y} = e^1(\overset{A}{1} - \overset{B}{\varepsilon Y} + \tfrac{1}{2}\varepsilon^2 Y^2 + \cdots)$$

$$\tfrac{\varepsilon}{2}(1-\varepsilon Y)e^1(1-\varepsilon Y + O(\varepsilon^2)) = \overset{C}{\tfrac{\varepsilon}{2}e^1} + O(\varepsilon^2)$$

$$(*)\quad u(Y) = \overset{A}{e^1} - \overset{B}{\varepsilon Y e^1} + \overset{C}{\tfrac{\varepsilon}{2}e^1} + O(\varepsilon^2)$$

INNER ... $$u(Y) = \underbrace{e^1(1-e^{-2Y})}_{O(1)} + \underbrace{\varepsilon B_1(1-e^{-2Y}) - \varepsilon Y e^1(1+e^{-2Y})}_{O(\varepsilon)}$$

switch to OUTER and expand

$$u(y) = e^1(1-\cancel{e^{-2y/\varepsilon}}) + \varepsilon B_1(1-\cancel{e^{-2y/\varepsilon}}) - y e^1(1+\cancel{e^{-2y/\varepsilon}}) + O(\varepsilon^2)$$

$\varepsilon \to 0$ $\quad e^{-2y/\varepsilon}$ is smaller than any order of ε

$$u(y) = e^1 + \varepsilon B_1 - y e^1 + O(\varepsilon^2)$$

(*) $$u(Y) = e^1 + \varepsilon B_1 - \varepsilon Y e^1 + O(\varepsilon^2)$$

(*) compare $\rightarrow$ $B_1 = \frac{1}{\varepsilon} e^1$

(y)

INNER:

$$u(y) = e^1\left(1 - \overset{C}{e^{-2y/\varepsilon}}\right) + \frac{\varepsilon}{2} e^1\left(1 - \overset{D}{e^{-2y/\varepsilon}}\right) - y e^1\left(1 + \overset{E}{e^{-2y/\varepsilon}}\right)$$

OUTER:

$$u(y) = \overset{A}{e^{1-y}} + \overset{B}{\frac{\varepsilon}{2}(1-y)e^{1-y}}$$

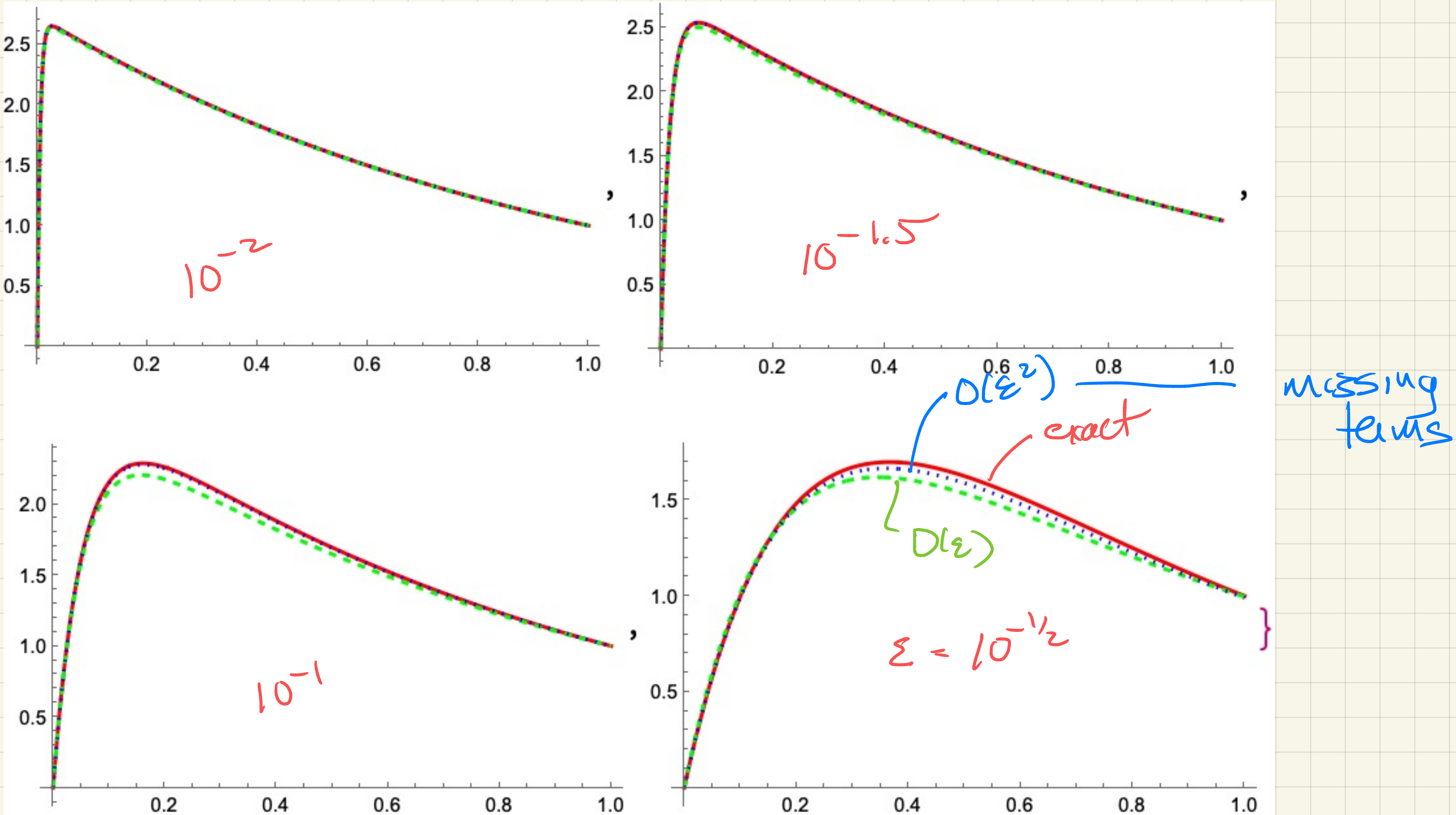
COMMON $\circledast$:

$$y(y) = e^1 + \frac{\varepsilon}{2} e^1 - y e^1$$

COMPOSITE: INNER + OUTER − COMMON

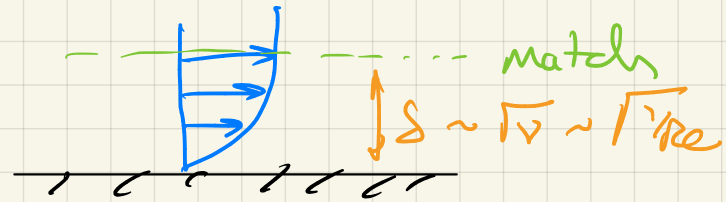
$$u(y) = \overset{A}{e^{1-y}} + \overset{B}{\frac{\varepsilon}{2}(1-y)e^{1-y}} - \overset{C}{e^{1-2y/\varepsilon}} - \overset{D}{\frac{\varepsilon}{2} e^{1-2y/\varepsilon}} - \overset{E}{y e^{1-2y/\varepsilon}}$$

$$u(y) = \underbrace{e^{1-y} - (1+y)e^{1-2y/\varepsilon}}_{O(1)} + \underbrace{\frac{\varepsilon}{2}\left[(1-y)e^{1-y} - e^{1-2y/\varepsilon}\right]}_{O(\varepsilon)}$$

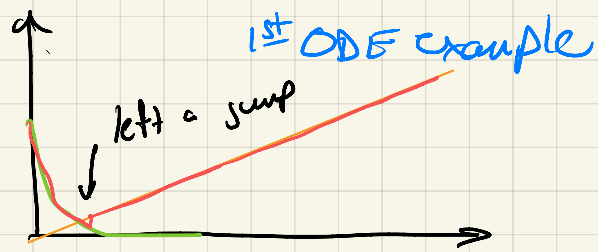


10^-2
10^-1.5
10^-1
O(ε²)
exact
missing terms
O(ε)
ε = 10^-½

Flow



- $u(y) \rightarrow$ only matched U value
  - $\rightarrow$ same as least order matching (essentially)
- $v(y) \rightarrow$ did nothing $\rightarrow$ $\sim \sqrt{\nu} \sim \sqrt{1/Re}$ "jump"



Misses

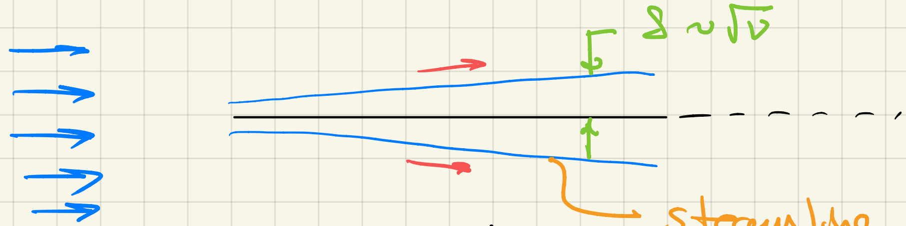
$\rightarrow$ effect of BL on OUTER flow

- displacement cause $V \sim \sqrt{\nu}$ $\rightarrow$ should extend beyond δ $(y > \delta)$
- should alter U

$Re \to \infty$ — no effect on outer flow

$\delta \to 0$, no displacement

Re large



$\delta \sim \sqrt{\nu}$

streamline at "edge" of BL → it's dispaced by δ

effects missing in first-order BL theory

2nd Order BL theory ⇒ how to include these rigorously

- BL approximation due to some U, V — INNER
- U, V response to the BL — OUTER

MATCH, each responding to the other