Handout #6
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AE 514: Supplemental Notes

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1 DRAFT: Streaming

Consider figure 1. A small amplitude but rapid oscillation of a cylinder appears to generate a steady flow at a distance. This is curious, and it is potentially important in some applications especially when there is otherwise no transport in this oscillation dominated flow. The principal reference for this is Batchelor.

1.1 The approximation

Flows of this class are predominantly oscillating: the oscillation can be high frequency but the peak velocities achieved are small. Thus,

$$\left|\frac{\partial \mathbf{u}}{\partial t}\right| \gg |(\mathbf{u}\cdot\nabla)\mathbf{u}| \tag{1}$$

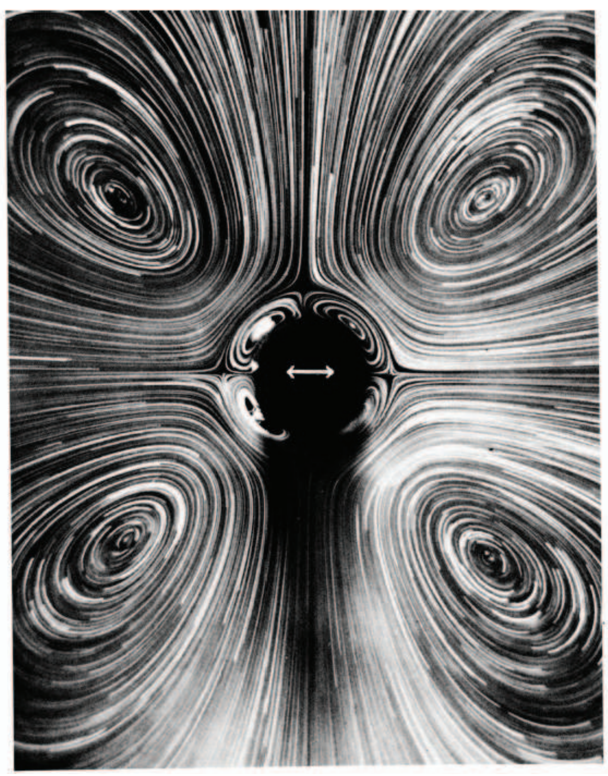

Figure 1: From M. Van Dyke collection: streaming due to an oscillating cylinder

is the key approximation. We take n to be the frequency of the oscillation, U_o to be its amplitude, and L to be the distance along a streamline over which $\mathbf{u}$ changes. For the oscillating cylinder, this L would correspond roughly with its diameter.

Anywhere on a solid object in such an oscillating flow we expect the formation of a thin boundary. Stokes oscillating plate is the iconic problem of this class, but here we allow the possibility that the exterior driving flow, at least as seen from the perspective of the object, can vary slowly (on scale L) along the boundary (see figure 2). As in the Stokes oscillating plate problem, we anticipate that $\delta \sim \sqrt{\nu t}$, or, for oscillations of frequency n,

$$\delta \sim \left(\frac{\nu}{n}\right)^{\frac{1}{2}}. \tag{2}$$

A high-Reynolds-number condition is consistent with $\delta \ll L$, or alternatively, since the boundary layer thickness Reynolds number is near unity,

$$\frac{\delta U_o}{\nu} \approx 1, \tag{3}$$

this is equivalent to

$$\frac{L U_o}{\nu} \gg 1. \tag{4}$$

A Reynolds number formed with a velocity scale made up of L and frequency n must be considered to be even larger since $nL \gg U_o$:

$$\frac{L^2 n}{\nu} \ggg 1. \tag{5}$$

1.2 The simplified boundary layer equations

According to (1), we neglect the inertia terms in the incompressible-flow boundary layer momentum equation,

$$\frac{\partial u}{\partial t} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\approx 0} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}. \tag{6}$$

The pressure is determined from the external stream at some location x in the boundary layer,

$$U(x,t) = \Re\left(\hat{U}(x)e^{int}\right), \tag{7}$$

which yields

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial}{\partial t}\left(\hat{U}e^{int}\right). \tag{8}$$

The boundary conditions on u are the usual no slip condition,

$$u(y=0) = 0, \tag{9}$$

and matching to the oscillating flow outside the boundary layer,

$$u(y \to \infty) = \hat{U}e^{int}. \tag{10}$$

This system is solved by

$$u(x,y,t) = \hat{U}(x)\left(1 - e^{(i+1)y/\delta}\right)e^{int}, \tag{11}$$

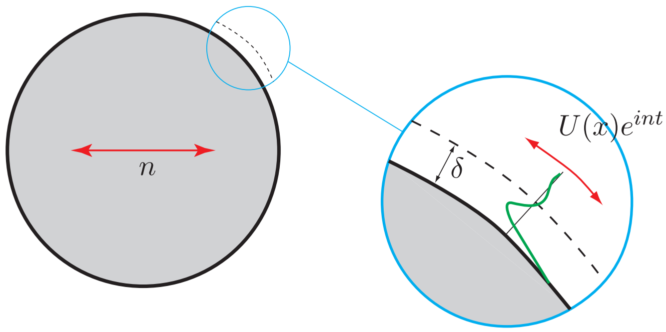

Figure 2: Oscillating flow.

where we have taken

$$\delta = \sqrt{\frac{2\nu}{n}}. \tag{12}$$

It is clear that (11) will time average to zero and hence can not explain the streaming flow observed.

Because U varies, albeit slowly, in x, there is also a finite v velocity. This is a key difference from Stokes oscillating plate, which was homogeneous in x. Integrating $\nabla \cdot \mathbf{u} = 0$ in y yields

$$v = -\int_0^y \frac{\partial u}{\partial y}\, dy. \tag{13}$$

Substituting (11) into this and integrating gives the wall-normal velocity

$$v(x, y, t) = -e^{int}\frac{d\hat{U}}{dx}\left[y - \frac{\delta}{i+1} + \frac{\delta}{i+1}e^{-\frac{i+1}{\delta}y}\right]. \tag{14}$$

1.3 Higher-order interactions and momentum transport

Though as for any wall-normal velocity in the boundary layer this wall normal v velocity in (14) is small, it plays an important role in the y-direction transport of momentum. It is for this reason that the y-direction advection term is retained in the boundary layer approximation to start with. Its capacity to transport momentum from within the boundary layer that can then be consistent with a streaming flow beyond depends upon the relative phasing of u and v. The time averaged y-flux of x-momentum is

$$J_y = \rho\int_0^{\frac{2\pi}{n}} uv\, dt. \tag{15}$$

If u and v are 90° out of phase, this integrates to zero. However, it is non-zero for any other relative phasing. Calculation of the actual momentum flux requires calculation of contributions throughout the boundary layer. Since this is fundamentally nonlinear, inclusion of weak (at least) nonlinearity in the analysis is therefore essential.

We take u_1 and U_1 to be the velocities determined by the linear analysis just completed: U_1 is imposed outside the boundary layer and in the boundary layer they satisfy

$$\frac{\partial u_1}{\partial t} - \frac{\partial U_1}{\partial t} = \nu\frac{\partial^2 u_1}{\partial y^2}. \tag{16}$$

We take u_1+u_2 with exterior flow U_1+U_2 to satisfy the full nonlinear equation:

$$\begin{aligned}\frac{\partial(u_1+u_2)}{\partial t}+(u_1+u_2)\frac{\partial(u_1+u_2)}{\partial x}+(v_1+v_2)\frac{\partial(u_1+u_2)}{\partial y}\\ =-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2(u_1+u_2)}{\partial y^2}\end{aligned} \tag{17}$$

The pressure term must satisfy the usual inviscid/irrotational relation outside the boundary layer,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\partial(U_1+U_2)}{\partial t}+(U_1+U_2)\frac{\partial(U_1+U_2)}{\partial x}. \tag{18}$$

Weakly nonlinear terms are retained by considering u_1 and U_1 to be $O(\varepsilon)$ and u_2 and U_2 to be $O(\varepsilon^2)$. Subtracting (16) from (17) and retaining only $O(\varepsilon^2)$ terms yields

$$\frac{\partial u_2}{\partial t}-\nu\frac{\partial^2 u_2}{\partial y^2}-\frac{\partial U_2}{\partial t}=U_1\frac{\partial U_1}{\partial x}-u_1\frac{\partial u_1}{\partial x}-v_1\frac{\partial u_1}{\partial y}. \tag{19}$$

The terms on the right-hand side of (19) can be considered to be known, whereas the terms on the left-hand side contain the unknown u_2.

Because of the products involved, the notational short cuts facilitated by considering the real part of a complex-valued solution as in (11) are no longer possible. Hence, we specify that U_1 and u_1 be real valued, which can be accomplished simply as

$$U_1 = \frac{1}{2}\left\{\hat{U}e^{int}+\hat{U}^*e^{-int}\right\} \tag{20}$$

$$u_1 = \frac{1}{2}\left\{\hat{U}(1-e^{-\alpha y})e^{int}+\hat{U}^*(1-e{-}\alpha^* y)e^{-int}\right\}, \tag{21}$$

where $\alpha=(1+i)/\delta$. All products invoking these quantities will, of course, have a form

$$\text{product}=c_\pm e^{\pm 2int}+c_2. \tag{22}$$

The first term time averages to zero whereas the second is constant in time and is thus unaffected by time averaging. Terms of this type, which persist though a long time average, are expected to constitute the long-time streaming motion.

Our strategy is thus to time average (19) and seek $\bar{u}$ directly:

$$\overline{\frac{\partial u_2}{\partial t}}-\nu\overline{\frac{\partial^2 u_2}{\partial y^2}}-\overline{\frac{\partial U_2}{\partial t}}=\overline{U_1\frac{\partial U_1}{\partial x}}-\overline{u_1\frac{\partial u_1}{\partial x}}-\overline{v_1\frac{\partial u_1}{\partial y}}. \tag{23}$$

The time derivative terms are both zero since the time average zeros the terms of the form $c_\pm e^{\pm 2int}$ and the time derivative zeros the constant terms of the form c_2.

Direct substitution is used to evaluated the right-hand side of (23). The first term is thus

$$\begin{aligned}\overline{U_1\frac{\partial U_1}{\partial x}} &= \frac{1}{2}\overline{\frac{\partial U_1^2}{\partial x}}\\ &= \frac{1}{8}\overline{\frac{\partial}{\partial x}\left[\left(\hat{U}e^{int}+\hat{U}^*e^{-int}\right)\left(\hat{U}e^{int}+\hat{U}^*e^{-int}\right)\right]}\\ &= \frac{1}{4}\frac{d\hat{U}\hat{U}^*}{dx}.\end{aligned} \tag{24}$$

From (21), a similar procedure yields

$$\overline{u_1 \frac{\partial u_1}{\partial x}} = \frac{1}{4} \frac{d\hat{U}\hat{U}^*}{dx} \left(1 - e^{-\alpha y}\right) \left(1 - e^{-\alpha^* y}\right) \tag{25}$$

for the second term on the right-hand side of (23). The the non-zero component of the third term on the right-hand side of (23) can be written compactly as

$$\overline{v_1 \frac{\partial u_1}{\partial y}} = \Re\left\{\overline{v \frac{\partial u_1}{\partial y}}\right\} = -\frac{1}{2}\Re\left\{\hat{U}^* \frac{d\hat{U}}{dx} \frac{\alpha^*}{\alpha} e^{-\alpha^* y} \left(y\alpha - 1 + e^{-\alpha y}\right)\right\}. \tag{26}$$

Substituting (24), (25) and (26) into (23) and recognizing the null contribution of the time-averaged time derivatives on the left-hand side yields

$$\begin{aligned} -\nu \frac{\partial^2 \overline{u_2}}{\partial y^2} &= \frac{1}{4} \frac{d\hat{U}\hat{U}^*}{dx} \left[1 - (1 - e^{-\alpha y})(1 - e^{-\alpha^* y})\right] + \frac{1}{2}\Re\left\{\hat{U}^* \frac{d\hat{U}}{dx} \frac{\alpha^*}{\alpha} e^{-\alpha^* y} \left(\alpha y - 1 + e^{-\alpha y}\right)\right\} \\ &\equiv G(x, y). \end{aligned} \tag{27}$$

Note that the $G(x, y)$ depends only on the $U(x, t)$ for any particular problem. It is a body force like term which balances viscous stresses (the left-hand side of 27) and decays as $e^{-y/\delta}$ outside the boundary layer. Since $G \to 0$ outside the boundary layer, we expect $\bar{u}_2 \to \overline{U}_2 = const$, or equivalently

$$\frac{\partial \bar{u}_2}{\partial y} = 0 \qquad \text{for} \qquad y/\delta \to \infty, \tag{28}$$

which thus serves as a boundary condition. The other boundary condition is $\bar{u}_2(y = 0) = 0$, which is no slip applied through $O(\varepsilon^2)$.

It is important to recognize that though these boundary conditions are similar to conventional boundary layer behavior, this situation for $\bar{u}_2$ is quite different. An effective body force in $0 < y < \delta$ drives the flow. The δ-scale boundary layer is already established by the $O(\varepsilon)$ solution for u_1. Thus, the resulting flow does not need to be high Reynolds number. There is no requirement that

$$Re_2 = \frac{\bar{u}_2 L}{\nu} \tag{29}$$

be large.

A general solution of (27) is

$$\overline{U}_2 = \frac{1}{\nu} \int_0^\infty yG(y)\, dy. \tag{30}$$

Recognizing $\nu = \delta^2 n/2$ and integrating for the G in (27) yields

$$\bar{u}_2(y \to \infty) = \overline{U}_2 = \frac{3}{8n}\left[-\frac{d\hat{U}\hat{U}^*}{dx} + i\left(\hat{U}^* \frac{d\hat{U}}{dx} - \hat{U}\frac{d\hat{U}^*}{dx}\right)\right]. \tag{31}$$

It is clear that this is a small streaming velocity. Applying the length and velocity scales of the problem to (31) shows us that

$$\frac{\overline{U}_2}{U_o} \frac{U_o}{nL}, \tag{32}$$

which is small since we took $nL/U_o \gg 1$ in defining the problem. So in that sense it is a slow flow. However, for the linear system, there is no streaming velocity. Therefore, this small-but-finite streaming flow is significant because it is persistent. It is the most significant component of the time averaged flow.

1.4 Amplitude and phase effects

It is clear in (31) the $\overline{U}_2$ depends only upon the flow field outside the boundary layer contained in $\hat{U}$, the inviscid flow corresponding to the oscillations. More detailed understanding of the nature of this dependence can be achieved by separately considering the the amplitude and phase properties of $\hat{U}$. We take

$$\hat{U}(x) = A(x)e^{i\gamma(x)}. \tag{33}$$

An oscillating cylinder will have $A(x)$ varying between the stagnation point to the points of peak velocity, but constant phase $\gamma(x)$. In contrast, on the bottom of a wave channel with constant wavelength and amplitude surface waves, the amplitude $A(x)$ would be fixed while the phase $\gamma(x)$ would vary substantially. Substituting (33 into (31) yields

$$\overline{U}_2 = -\frac{8}{3n}\left(\frac{dA^2}{dx} + 2A^2\frac{d\gamma}{dx}\right). \tag{34}$$

Thus we see a distinct dependence of $\overline{U}_2$ upon variations in oscillation amplitude and phase.

1.5 Drift

One of the principal applications of this streaming motion is the transport of mass, such as in the visualization in figure 1. However, there is a subtle aspect of this: the $\overline{U}_2$ is not the speed of material at the edge of the boundary layer. Just as it is the material derivative of velocity that yields the acceleration of fluid particles,

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt}, \tag{35}$$

the drift velocity $\mathbf{W}_2$ must be calculated in acknowledgment of the motion of the particles.

This motion is called *drift* and is discussed at length by Lighthill in volume 1 of the *Journal Fluid Mechanics* (1956), which I understand to be to this day the *JFM* paper with the shortest title. Take $\mathbf{w}(\mathbf{x}_o, t)$ to be the velocity of a material particle at $\mathbf{x}_o$ at some reference time t_o. At some later time, this same material fluid particle (or some spec of material whose velocity matches that of the local fluid material) moves at speed

$$\mathbf{w}(\mathbf{x}_o, t) = \mathbf{u}\left(\mathbf{x}_o + \int_{t_o}^{t} \mathbf{w}(\mathbf{x}_o, t)\, dt, t\right). \tag{36}$$

This is the fluid velocity, but evaluated at the current location of the particle. For small $t - t_o$ (and hence $|\mathbf{x} - \mathbf{x}_o|$), we can Taylor expand (36)

$$\mathbf{w}(\mathbf{x}_o, t) = \mathbf{u}(\mathbf{x}_o, t) + \left(\int_{t_o}^{t} \mathbf{w}(\mathbf{x}_o, t)\, dt\right) \cdot \nabla\mathbf{u}\Big|_{\mathbf{x}=\mathbf{x}_o} + O\left((t - t_o)^2\right). \tag{37}$$

The $\mathbf{w}$ in the integrand of (37) differs from $\mathbf{u}$ by $O(t - t_o)$. The bounds of the integration impose another factor of $O(t - t_o)$. Thus, to the same level of approximation

$$\mathbf{w}(\mathbf{x}_o, t) = \mathbf{u}(\mathbf{x}_o, t) + \left(\int_{t_o}^{t} \mathbf{u}(\mathbf{x}_o, t)\, dt\right) \cdot \nabla\mathbf{u}\Big|_{\mathbf{x}=\mathbf{x}_o} + O\left((t - t_o)^2\right). \tag{38}$$

We can apply this to the $\mathbf{U}_2$ velocity we determined just outside the boundary layer to determine the particle drift velocity there. Applying (38) for the flow at the edge of the boundary layer, we have

$$W = U + \frac{\partial U}{\partial x}\int_{t_o}^{t} U\, dt. \tag{39}$$

Substituting $U = U_1 + U_2$ as before, now also with $W = W_1 + W_2$, retaining terms $O(\varepsilon^2)$, and time averaging yields

$$\overline{W}_2 = \overline{U}_2 + \overline{\frac{\partial U_1}{\partial x} \int_{t_o}^{t} U_1 \, dt}. \tag{40}$$

The time integration for U_1 in (20) is straightforward as is the x-direction differentiation. The result for the drift velocity is

$$\overline{W}_2 = \overline{U}_2 + \frac{1}{4in}\left(\hat{U}\frac{d\hat{U}^*}{dx} - \hat{U}^*\frac{d\hat{U}}{dx}\right). \tag{41}$$

Taking $\hat{U} = Ae^{i\gamma}$ as before to differentiate phase and amplitude dependence yields

$$\overline{W}_2 = -\frac{3}{8n}\frac{dA^2}{dx} - \frac{5}{4n}A^2\frac{d\gamma}{dn}. \tag{42}$$

When there is no spatial dependence of the phase γ, the streaming velocity and the drift speed of material are the same, but they differ when the phase has a spatial dependence.

1.6 Examples

1.6.1 Flow due to an oscillating cylinder

Consider a cylinder of radius a as shown in figure 3 oscillating with frequency n and velocity amplitude U_o. If the oscillations are high enough frequency and sufficiently low amplitude, we expect that the cylinder will reverse direction before any separation occurs. The vorticity of one sign, generated at its surface, will diffusively mix with the opposite sign vorticity generated subsequently. Thus we take the inviscid flow solution,

$$U = 2U_o e^{int} \sin\theta, \tag{43}$$

to provide the velocity at the edge of a thin viscous boundary layer. The surface parallel coordinate x can be taken to be $x = a\theta$. For this flow, every point in x has the same phase. Thus $A = 2U_o \sin\theta$ and $\gamma = const$. The drift velocity at the edge of the boundary layer is thus

$$\overline{W}_2 = -\frac{3}{8n}\frac{dA^2}{dx}, \tag{44}$$

which for $x = a\theta$ is

$$\overline{W}_2 = -\frac{3}{8n}\frac{1}{a}\frac{d}{d\theta}\left(4U_o^2 \sin^2\theta\right) = -\frac{3}{2}\frac{U_o^2}{an}\sin 2\theta, \tag{45}$$

which corresponds to the colored arrows in figure 3 and the flow visualized in figure 1.

1.6.2 Flow due to a wave train

The amplitude of water waves decays exponentially with depth from the free surface. Still, for relatively shallow channels there is a finite oscillatory flow $U_o e^{int}$ just outside a thin viscous boundary layer on the bottom, as shown in figure 4. This too can drive a streaming flow. The phase of this velocity is

$$\gamma(x) = -\frac{2\pi x}{\lambda}, \tag{46}$$

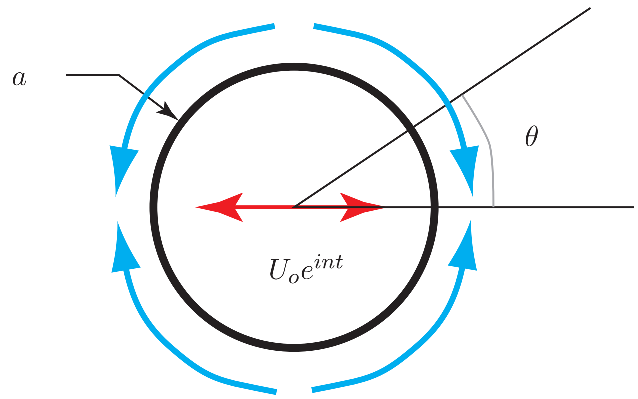

Figure 3: Oscillating cylinder.

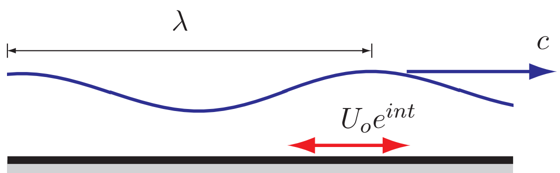

Figure 4: Wave train.

where λ is the wavelength and the negative giving waves that travel in $+x$ since the time dependence is e^{+int}. The amplitude $A(x)$ is constant for this flow, so the drift speed just outside the viscous boundary layer is

$$\overline{W}_2 = -\frac{5}{4n} A^2 \frac{d\gamma}{dx}, \tag{47}$$

which is equivalent to

$$\overline{W}_2 = \frac{5}{4} A^2 \frac{1}{c}, \tag{48}$$

where c is the surface wave speed. The amplitude A will be small at any significant depth due to the rapid decay of the surface wave's influence, but it is persistent and can drive significant sediment transport over long times.