Handout #6
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AE 514: Supplemental Notes

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1 DRAFT: Second-order Flat Plate Boundary Layer

Though obviously successful, there were nagging issues about the standard Prandtl formulation for the boundary layer. For one, the wall normal v velocity was discontinuous at the edge of the boundary layer, however, that point is defined. Inside the boundary layer, v is finite and indeed plays and important role in the wall-normal advection of vorticity. Outside, the invisid-flow solution is a uniform with $u = U$ and $v = 0$. This wall-normal velocity is small, $v = O(1/Re^{\frac{1}{2}})$, but the discontinuity is fundamentally a violation of conservation of mass. We insisted that $\nabla \cdot \mathbf{u} = 0$ was enforced *exactly* in the boundary layer, but this condition is clearly violated to $O(1/Re^{\frac{1}{2}})$ right at its edge. This seems inconsistent.

Second, it is difficult to anticipate how accurate the boundary layer approximation is without knowing what it would take to improve it. Having the second-order solution tells us, importantly, how accurate the first-order solution is. Having the second-order approximation also extends the range of applicability to lower Reynolds numbers. Engineering Reynolds are in so many cases are so huge that the second-order improvement is rarely a factor in the accuracy of an approximate bounday layer solution solution. Other errors will often dominate any such inaccuracy. However, in some low-Reynolds number applications, improvements are potentially useful.

There is also a pedagogical objective. Understanding how the matching procedure—the Methods of Matched Asymptotic Expansions—can be applied to flow is useful well beyond momentum boundary layers on flat plates. There are copious problems in which a small parameter presents itself in ways similar to the Reynolds number for flow over a wall. Asymptotic analysis, as in the present example, is invaluable for understanding behaviors or improving numerical approximations. The cases where a part of a solution are challenging to resolve in space or time with a mesh-based discrete numerical solution often correspond to cases where asymptotic approximations are most powerful.

For simplicity, we limit our discussion to the semi-infinite flat plate boundary layer, but the principles apply more generally. We take the free stream velocity to be U, the kinematic viscosity to be ν, and L to be some length "of interest" on the plate. Non-dimensionalization by these parameters of the streamfunction form of the equations to yields

$$\left(\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y} - \frac{1}{Re} \nabla^2 \right) \nabla^2 \psi = 0. \tag{1}$$

The Reynolds number is $Re = UL/\nu$. Boundary conditions for a plate at $y = 0$, $x \geq 0$ are:

- a streamline on the plate,
$$\psi(x, 0) = 0, \tag{2}$$
which is consistent with a no-penetration condition $v = -\psi_x(x, 0) = 0$, and
- the no-slip condition
$$\psi_y(x, 0) = 0. \tag{3}$$

The free-stream condition, applicable for $y \to \pm\infty$ and $x \to -\infty$, is

$$\psi(x, y) = y, \tag{4}$$

which yields $u = \psi_y = 1$ in the far field, a non-dimensional uniform x-velocity. The following analysis follows the presentation of van Dyke.

1.1 First-order outer approximation

We assume a general form for the OUTER expansion:

$$\psi(x, y) = \delta_1(Re)\psi_1(x, y) + \delta_2(Re)\psi_2(x, y) + \cdots . \tag{5}$$

The ψ_n are all $O(1)$; the coefficients are included here as unknown functions of Re, which is assumed to be large. We will determine the δ_n as part of the analysis.

It is straightforward to anticipate that $\delta_1 = O(1)$. We know that in the free stream $\psi \sim y = O(1)$ and anticipate that this will provide the OUTER boundary condition to be satisfied by ψ_1 the OUTER expansion of (5). Thus, rearranging the left-most two terms of (5) in the limit of large Re yields

$$\psi_1 = \lim_{Re\to\infty} \frac{y}{\delta_1(Re)} = O(1), \tag{6}$$

which can be satisfied only if $\delta_1(Re) = O(1)$.

Substituting (5) into (1) and grouping terms that are small in the $Re \to \infty$ limit on the right-hand side yields

$$\left(\psi_{1y}\frac{\partial}{\partial x} - \psi_{1x}\frac{\partial}{\partial y}\right)\nabla^2\psi_1 = o(1), \tag{7}$$

where the $o(a)$ indicates that the remaining terms have order such that they tend to zero faster than a as, in this case, $Re \to \infty$. In this case, it includes all terms that become zero in the $Re \to \infty$ limit. This faster-than little 'o' notation is helpful when the δ_n dependencies are not yet determined. The key thing here is that the missing terms are fundamentally smaller; how much smaller is not yet important.

The left-hand side of (7) is simply a statement of advection of vorticity:

$$\frac{D\omega_1}{Dt} = 0, \tag{8}$$

where $\omega_1 = -\nabla^2\psi_1$ is the vorticity. This indicates that to first order, vorticity is advected unchanged by the flow. Since the vorticity is zero upstream, it therefore must also zero everywhere, which corresponds to

$$\nabla^2\psi_1 = 0. \tag{9}$$

Adding the OUTER boundary condition $\psi_1(x, y) \sim y$ in the free stream and the no penetration condition $\psi_1(x, 0) = 0$ yields the solution

$$\psi_1 = y. \tag{10}$$

We have dropped the INNER no slip boundary condition based upon our experience with ODE's: it is unreasonable to expect the OUTER problem to satisfy the INNER boundary condition since it no longer has a form that can satisfy all the original boundary conditions.

1.2 First-order inner approximation

From the form of (1) we anticipate that we need to expand the INNER region with a coordinate transform to zoom in on the region where diffusion is comparable to advection. This new coordinate is

$$Y = \frac{y}{\Delta_1(Re)}, \tag{11}$$

where $\Delta_1(Re)$ is, for now, assumed to be an unknown function of our large parameter Re. The goal is that $Y = O(1)$ in the INNER region.

This coordinate re-scaling imposes a consistency condition with the INNER expansion of ψ. We know that the streamwise velocity will be first order everywhere:

$$u = \psi_y = \frac{\partial \psi}{\partial y} = O(1). \tag{12}$$

In the INNER region, (11) indicates that $y = O(\Delta_1)$. From (12), it is thus clear that in the INNER region the stream function needs to scale as $\psi = O(\Delta_1)$ for u to remain $O(1)$. Thus, the coordinate stretching (11) needs to be consistent with the INNER expansion. We therefore take

$$\psi = \Delta_1(Re)\Psi_1(x,y) + \Delta_2(Re)\Psi_2(x,y) + \cdots . \tag{13}$$

Substituting this into (1) yields

$$\left(\Psi_{1Y}\frac{\partial}{\partial x} - \Psi_{1x}\frac{\partial}{\partial Y}\right)\left(\Delta_1\Psi_{xx} + \frac{1}{\Delta_1}\Psi_{YY}\right) - \frac{1}{Re}\frac{1}{\Delta_1^3}\Psi_{YYYY} = o(\Delta_1). \tag{14}$$

If we assume that momentum (the first terms) and viscosity (the final term on the left-hand side) are both important, then we require $\Delta_1 \sim Re^{-\frac{1}{2}}$. The resulting first approximation can be Y-integrated once to yield

$$\Psi_{1YYY} + \Psi_{1x}\Psi_{1YY} - \Psi_{1Y}\Psi_{1xY} = g(x), \tag{15}$$

where $g(x)$ resulted from this integration. We shall see that it corresponds to a pressure gradient imposed by the OUTER flow solution.

1.3 First-order inner-outer matching

We start with the OUTER solution. Matching is most straightforward if we match ψ_y rather than ψ itself. This is in line with the standard more *ad hoc* matching of tangential velocities. The matching proceeds as follows.

$$\begin{aligned}
\text{1-term OUTER expansion:} \quad & \psi_y \sim \psi_{1y}(x,y) \\
\text{represent with inner variable:} \quad & = \psi_{1y}(x, Y/Re^{\frac{1}{2}}) \\
Re \to \infty \text{ expansion:} \quad & = \psi_{1y}(x,0) + \tfrac{Y}{Re^{\frac{1}{2}}}\psi_{1yy}(x,0) + \cdots \\
\text{retain first term:} \quad & = \psi_{1y}(x,0)
\end{aligned}$$

The corresponding operations for the INNER solution are:

$$
\begin{aligned}
\text{1-term INNER expansion:} \quad & \psi_y \sim \Psi_{1Y}(x, Y) \\
\text{represent with OUTER variable:} \quad & = \Psi_{1Y}(x, Re^{\frac{1}{2}} y) \\
Re \to \infty \text{ expansion:} \quad & = \Psi_{1Y}(x, \infty) + \cdots \\
\text{retain first term:} \quad & = \Psi_{1Y}(x, \infty)
\end{aligned}
$$

The matching condition is thus

$$\psi_{1y}(x, 0) = \Psi_{1Y}(x, \infty), \tag{16}$$

which is a matching of the tangential velocity, as is usually done in less formal derivations of the boundary layer limit of the flow equations.

Integrating (16) allows us to make additional statements about the matching. Specifically, it will set $g(x)$ in (15) in terms of the outer solution. It is easy to see that (16) is consistent with differentiation with respect to Y in the $Y \to \infty$ limite:

$$\Psi_1(x, Y) = Y \psi_{1y}(x, 0) + o(Y) \qquad Y \to \infty. \tag{17}$$

This can be substituted into the left-hand side of (15), which yields

$$\Psi_{1YYY} + \Psi_{1x}\Psi_{1YY} - \Psi_{1Y}\Psi_{1xy} = -\psi_{1y}(x, 0)\psi_{1xy}(x, 0) = g(x). \tag{18}$$

Taking the OUTER flow velocity to be $u = \psi_y$, the rightmost equality is equivalent to

$$g(x) = -u\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}, \tag{19}$$

where the pressure relation is equivalent to the Bernoulli result, which is appropriate in the outer irrotational flow.

1.4 Solution of the inner problem

The INNER problem (15) has boundary conditions

$$\Psi_1(x, 0) = 0 \qquad \text{no penetration} \tag{20}$$
$$\Psi_{1Y}(x, 0) = 0 \qquad \text{no slip} \tag{21}$$

along with the matching result (16), which can be evaluated with the OUTER solution (10) to yield in effect third boundary condition that reflects the matching condition:

$$\Psi_{1Y}(x, \infty) = \psi_{1y}(x, 0) = 1. \tag{22}$$

While the outer solution was trivial, it is well known that the flat-plate boundary layer equations must be solved numerically. This is facilitated by the usual similarity transform,

$$\Psi_1 = \sqrt{x} f_1(\eta) \qquad \text{with} \qquad \eta = \frac{Y}{\sqrt{x}}, \tag{23}$$

which yields

$$f_1''' + \frac{1}{2} f_1 f_1'' = 0, \tag{24}$$

with corresponding boundary conditions

$$f(0) = f'(0) = 0 \qquad \text{and} \qquad f'(\infty) = 1. \tag{25}$$

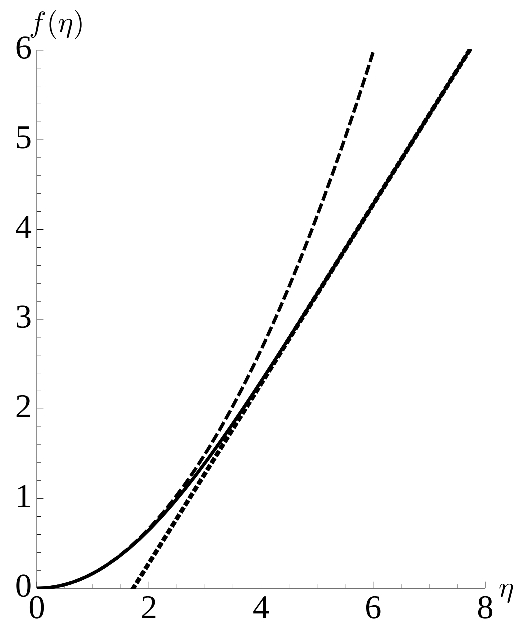

Figure 1: The numerical solution of (24) with large and small η asymptotic behaviors.

A numerical solution of $f(\eta)$ is plotted in figure 1.

From this numerical solution, we will need only one numerical parameter to complete the second-order approximation. It is from the $\eta \to \infty$ behavior of $f(\eta)$. For completeness, we quote both the large and small η behavior as

$$f(\eta) \sim \begin{cases} \frac{1}{2}\alpha_1 \eta^2 + \cdots & \text{for } \eta \to 0 \\ \eta + \beta_1 + \exp & \text{for } \eta \to \infty \end{cases} \tag{26}$$

These curves are both plotted in figure 1 and the numerical values of their parameters are $\alpha_1 = 0.33206$ and $\beta_1 = 1.72079$. The exp refers to terms that are exponentially small. The vorticity is governed essentially by a diffusive process and therefore decays exponentially, which is consistent with the exponential smallness of these terms.

1.5 Second-order inner-outer matching

Establishment of the second-order matching conditions involves first determining $\delta_2(Re)$ in (5). We follow the usual procedure, starting with manipulation of the INNER result and now seek to match for ψ itself

$$\begin{aligned}
\text{INNER solution:}\quad & \psi \sim \tfrac{1}{Re^{\frac{1}{2}}}\sqrt{x} f_1(\eta) \\
\text{represent with OUTER variable:}\quad & = \tfrac{1}{Re^{\frac{1}{2}}}\sqrt{x} f_1\left(\tfrac{Re^{\frac{1}{2}} y}{\sqrt{x}}\right) \\
Re \to \infty \text{ expansion:}\quad & = \tfrac{1}{Re^{\frac{1}{2}}}\sqrt{x}\left[\tfrac{Re^{\frac{1}{2}} y}{\sqrt{x}} - \beta_1\right] \\
\text{simplification:}\quad & = y - \tfrac{1}{Re^{\frac{1}{2}}}\sqrt{x}\beta_1 \\
\text{in terms of } Y \text{ for later convenience:}\quad & = \tfrac{Y}{Re^{\frac{1}{2}}} - \tfrac{1}{Re^{\frac{1}{2}}}\sqrt{x}\beta_1
\end{aligned}$$

The corresponding operations for the OUTER solution, recognizing that $\psi_1 = y$, are

$$\begin{aligned}
\text{OUTER solution expansion:}\quad & \psi \sim y + \delta_2(Re)\psi_2(x,y) \\
\text{represent with INNER variable:}\quad & = \tfrac{Y}{Re^{\frac{1}{2}}} + \delta_2(Re)\psi_2(x, Y/Re^{\frac{1}{2}}) \\
Re \to \infty \text{ expansion:}\quad & = \tfrac{Y}{Re^{\frac{1}{2}}} + \delta_2(Re)\left[\psi_2(x,0) + \psi_{2y}(x,0)Y/Re^{\frac{1}{2}} + \cdots\right] \\
\text{retain leading order:}\quad & = \tfrac{Y}{Re^{\frac{1}{2}}} + \delta_2(Re)[\psi_2(x,0) + o(1)]
\end{aligned}$$

Matching the final lines of these two series of manipulations indicates

$$\delta_2(Re) = \frac{1}{Re^{\frac{1}{2}}}. \tag{27}$$

This is necessary for there to be any matching consistency for the next order terms. It is also clear that

$$\psi_2(x,0) = -\beta_1\sqrt{x}, \tag{28}$$

which effectively a boundary condition that provides the correction to the OUTER due to the INNER solution.

The OUTER expansion $\psi \sim \psi_1 + \delta_2\psi_2 + \cdots$ is thus

$$\psi \sim y - \beta_1\frac{\sqrt{x}}{Re^{\frac{1}{2}}} + \cdots. \tag{29}$$

We note that $\psi = 0$ would occur on the curve $y = \beta_1\sqrt{x}/Re^{\frac{1}{2}}$. Thus, one interpretation of this second-order OUTER solution is that effective surface streamline is displaced into the shape of an $x \propto y^2$ parabola. The corrected OUTER flow then corresponds to the inviscid flow over a semi-infinite parabola.

An alternate interpretation is that the outer flow now has a finite wall-normal v velocity, matching that of the INNER boundary layer solution. This fixes the discontinuity that served in part to motivate the second-order analysis. In this case,

$$v = -\psi_x = \frac{\beta_1}{2\sqrt{Rex}}. \tag{30}$$

1.6 Second-order outer solution

We have a matching condition, but both the INNER and OUTER second-order problems still need to be solved. For the OUTER case, we substitute $\psi \sim \psi_1 + \delta_2\psi_2$ into the full flow equation (1). The

$O(1)$ terms are already satisfied by ψ_1. Keeping the $O(\delta_1) = O(1/Re^{\frac{1}{2}})$ terms yields

$$\frac{1}{Re^{\frac{1}{2}}}\left[\underbrace{\left(\psi_{2y}\frac{\partial}{\partial x} - \psi_{2x}\frac{\partial}{\partial y}\right)\nabla^2\psi_1}_{=\frac{D_2\omega_1}{Dt}} + \underbrace{\left(\psi_{1y}\frac{\partial}{\partial x} - \psi_{1x}\frac{\partial}{\partial y}\right)\nabla^2\psi_2}_{=\frac{D_1\omega_2}{Dt}}\right] = O(\frac{1}{Re}). \tag{31}$$

As the underbraces indicate, the first set of terms is advection of first-order vorticity ($\omega_1 = -\nabla^2\psi_1$) by the second-order flow ψ_2 ,and the second set of terms is advection of the second-order vorticity ($\omega_2 = -\nabla^2\psi_2$) by the first-order flow ψ_1. It was already established that the flow upstream was irrotational, so $\omega_1 = 0$ everywhere as discussed in section 1.1. The same advection by ψ_1 then leads to the same conclusion for ω_2 based upon the second set of terms in (31): $\omega_2 = -\nabla^2\psi_2 = 0$.

Based on this reasoning, the governing equations for the second-order OUTER solution is thus

$$\nabla^2\psi_2 = 0. \tag{32}$$

The boundary conditions come from the symmetry streamline upstream of the plate and matching condition we determined in (28):

$$\psi_2(x,0) = \begin{cases} 0 & \text{for } x < 0 \\ -\beta_1\sqrt{x} & \text{for } x > 0 \end{cases} \tag{33}$$

We do not, of course, enforce the no slip condition. The free-stream is such that the influence of the plate should decay:

$$\psi_2(x,y) = o(y). \tag{34}$$

It is straightforward to confirm that

$$\psi_2 = -\beta_1\Re\left(\sqrt{x+iy}\right) \tag{35}$$

satisfies the governing equation and boundary conditions, where $\Re(z)$ yields the real-valued component of z.

1.7 Inner expansion of second-order outer solution

For completing the overall solution, we seek the inner expansion of second-order outer solution. This proceeds as follows

$$\begin{aligned}
\text{two-term outer solution:} \quad & \psi \sim \psi_1 + \tfrac{1}{Re^{\frac{1}{2}}}\psi_2 = y - \tfrac{\beta_1}{Re^{\frac{1}{2}}}\Re\left(\sqrt{x+iy}\right) \\
y \text{ derivative to match } \psi_y\text{:} \quad & = 1 - \tfrac{\beta_1}{Re^{\frac{1}{2}}}\tfrac{i}{2}(x+iy)^{-\frac{1}{2}} \\
\text{switch to inner variable } Y\text{:} \quad & = 1 - \tfrac{\beta_1}{Re^{-\frac{1}{2}}}\tfrac{1}{2}\left(x + \tfrac{iY}{Re^{-\frac{1}{2}}}\right)^{-\frac{1}{2}} \\
\text{two-term expansion:} \quad & = 1 - \tfrac{\beta_1}{Re^{-\frac{1}{2}}}\tfrac{i}{2}x + O(Re^{-1}) \\
\text{real part:} \quad & = 1 + \tfrac{0}{Re^{\frac{1}{2}}}.
\end{aligned}$$

The semi-infinite parabolic cylinder is a special case in that it has no real-valued contribution for the $Re^{-\frac{1}{2}}$ term. In general, this term would be non-zero.

1.8 Outer expansion of second-order inner solution

Likewise, we need the OUTER expansion of the second-order INNER solution, with proceeds as follows. We undertake this before solving for Ψ_2 in order to identify the effective boundary condition that derives from the matching. We will use the relation $\Psi_1 = \sqrt{x} f_1(\eta)$ for $\eta = Y/\sqrt{x}$ in this.

$$\begin{aligned}
\text{two-term inner solution:} \quad & \psi \sim \Delta_1 \Psi_1(x, Y) + \Delta_2 \Psi_2(x, Y) + \cdots \\
\text{crafted in OUTER variable } y\text{:} \quad & = \Delta_1 \Psi_1(x, yRe^{\frac{1}{2}}) + \Delta_2 \Psi_2(x, yRe^{\frac{1}{2}}) + \cdots \\
y \text{ derivative to match } \psi_y\text{:} \quad & = \frac{1}{Re^{\frac{1}{2}}} Re^{\frac{1}{2}} \Psi_{1Y}(x, yRe^{\frac{1}{2}}) + Re^{\frac{1}{2}} \Delta_2(Re) \Psi_{2Y}(x, yRe^{\frac{1}{2}}) \\
\text{evaluate in terms of } f_1\text{:} \quad & = f_1'\left(\frac{yRe^{\frac{1}{2}}}{\sqrt{x}}\right) + Re^{\frac{1}{2}} \Delta_2(Re) \Psi_{2Y}(x, y) Re^{\frac{1}{2}}) \\
\text{two-term expansion for } Re \to \infty\text{:} \quad & = 1 + Re^{\frac{1}{2}} \Delta_2 \Psi_{2Y}(x, \infty)
\end{aligned}$$

Comparing this with the inner expansion of the outer solution, we determine that $\Delta_2 = O(1/Re)$. It also follows that

$$\Psi_{2Y}(x, \infty) = 0, \tag{36}$$

which serves in effect as an outer boundary condition on the inner problem.

1.9 Second-order inner solution

Substituting $\psi = \frac{1}{Re^{\frac{1}{2}}} \Psi_1(x, Y) + \frac{1}{Re} \Psi_2(x, Y) + \cdots$ into the full flow equations (1) allows use to develop the $O(\Delta_2) = O(1/Re)$ equation to solve:

$$\frac{\partial}{\partial Y} \left(\Psi_{2YYY} + \Psi_{1x} \Psi_{2YY} - \Psi_{1Y} \Psi_{2xY} + \Psi_{2x} \Psi_{1YY} - \Psi_{2Y} \Psi_{1xY} \right) = 0. \tag{37}$$

The boundary conditions are

$$\Psi_2(x, 0) = 0 \qquad \text{no penetration} \tag{38}$$

$$\Psi_{2Y}(x, 0) = 0 \qquad \text{no slip} \tag{39}$$

as before for the first-order solution, now with also a homogeneous outer boundary condition from (36). Since this is fully homogeneous—the equation and all the boundary conditions—it can only have non-trivial eigensolution. For it to be unique, these all must be zero. This uniqueness can be argued based upon the artificiality of the L parameter used in the non-dimensionalization, or just blindly accepted. Let's conclude by just accepting it. The overall solution to second order is then

$$\psi(x, y) \sim \begin{cases} y - \frac{1}{Re^{\frac{1}{2}}} \beta_1 \Re\left(\sqrt{x + iy}\right) & \text{OUTER} \\ \frac{\sqrt{x}}{Re^{\frac{1}{2}}} f_1\left(\frac{yRe^{\frac{1}{2}}}{\sqrt{x}} + \frac{0}{R}\right) & \text{INNER} \end{cases}. \tag{40}$$