Lecture 04

- Exact NS
  - Stagnation Flow
  - ω transport

... continued

$$u(y=0,t) = U \quad (\text{const})$$

$$u(y=0,t) = U_0 t^{\gamma} \qquad \gamma > 0 \qquad \cdots \quad \text{ODE} \quad f'' + \tfrac{1}{2}\eta' - \gamma f = 0$$

$\rightarrow$ numerically

more observations from Stokes $1^{\underline{st}}$

- δ_{99} : $\dfrac{u(y=\delta_{99})}{U} = .99 \rightarrow \delta_{99} \simeq 3.6\sqrt{\nu t}$

  $\nu_{wtr} = 0.15\, cm^2/s \qquad \delta_{99} = 10\,cm$ in 1 min

- ω - vorticity $\Rightarrow$ even more confined near wall

$$\omega_z = \cancel{\frac{\partial v}{\partial x}} - \frac{\partial u}{\partial y} = A t^{\alpha} f' \; B t^{\beta}$$

0

$$\vdots$$

$$\omega_z = \frac{-U}{\sqrt{\pi \nu t}}\, e^{-y^2/4\nu t}$$

note singular at $t \to 0 \Rightarrow$ vortex sheet on the wall note $\int \omega_z \, dy = -U$ constant vorticity

$$\int \left(\cancel{\frac{\partial v}{\partial x}} - \frac{\partial u}{\partial y} \right) dy = U$$ any $u(y)$ profile

- wall stress (~ drag component)

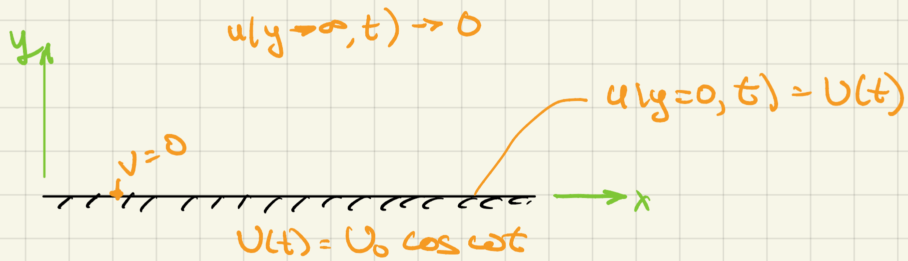
$$\tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}^{\,0} \right)$$

$$\tau_{xy}\Big|_{wall} = \tau_w = \mu \frac{\partial u}{\partial y}\Big|_{wall}^{y=0} = \frac{U\mu}{\sqrt{\pi \nu t}}$$

decreses on time

Stokes 2nd Problem or Stokes Oscillating Plate

- interested in time keeping $\rightarrow$ drag on a pendulum



- same assumptions: μ, ρ const, $\nabla \cdot \underline{u} = 0$
- same v B.C.

• same x-uniformity $\partial_x \to 0$

↳ same PDE $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$

$u(y=0,t) = U_0 \cos \omega t$
$u(\infty, t) = 0$

→ long t ⇒ assume any initial transient decays → sort of informed by Stokes 1st

↳ take $u(y,t) = \mathrm{Re}\{\hat{u}(y) e^{i\omega t}\}$

$$i\omega \hat{u} = \nu \frac{d^2\hat{u}}{dy^2} \to \frac{d^2\hat{u}}{dy^2} - \frac{i\omega}{\nu}\hat{u} = 0$$

$$\hat{u}(y) = A_+ e^{+\sqrt{\frac{i\omega}{\nu}}y} + A_- e^{-\sqrt{\frac{i\omega}{\nu}}y}$$

$$\hat{u}(0) = U_0$$

$$\hat{u}(y) = A_+ e^{+\sqrt{\frac{\omega}{2\nu}}y} e^{i\sqrt{\frac{\omega}{2\nu}}y} + A_- e^{-\sqrt{\frac{\omega}{2\nu}}y} e^{-i\sqrt{\frac{\omega}{2\nu}}y}$$

$A_+ = 0$
for $\hat{u}(y \to \infty) = 0$

↳ $A_- = U_0$

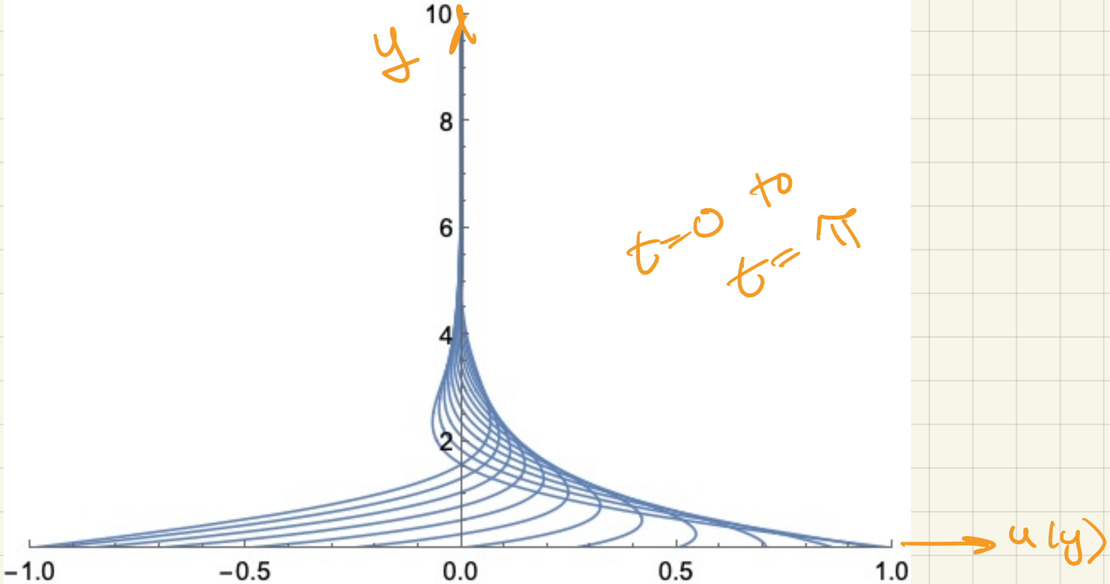
$\sqrt{i} = \frac{1+i}{\sqrt{2}}$

$i = e^{i\pi/2}$

$\sqrt{i} = e^{i\pi/4}$

$$u(y,t) = \mathrm{Re}\{U_0 e^{-\sqrt{\frac{\omega}{2\nu}}} e^{-i\sqrt{\frac{\omega}{2\nu}}y + i\omega t}\}$$

$$= U_0 e^{-\sqrt{\frac{\omega}{2\nu}}} \cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right)$$



$$\delta_{99} = 4.5\sqrt{\frac{2\nu}{\omega}}$$

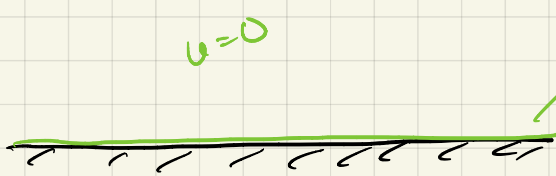
thinner for larger ω

$\sim \sqrt{\nu}$

Q: where is the τ_w peak (in phase)

→ easy to compute

Both Stokes 1 and 2 → for $\nu \to 0 / Re \to \infty$



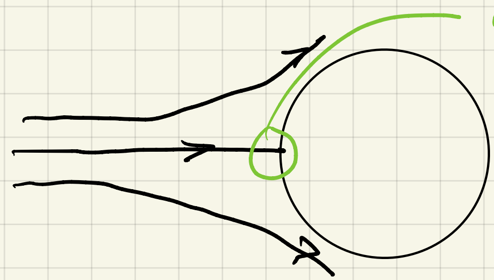
thin layer of ω

~1 times

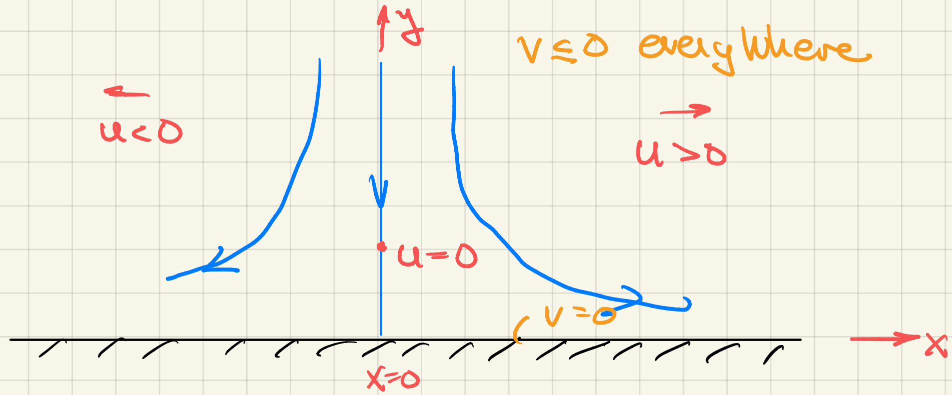
~1 frequencies

~1

Stagnation Pt Flow



analyze this region — recall P $\mu = 0$ solution was good here locally flat



potential flow solution $\quad \psi = kxy$ $\quad$ strength ($k>0$) $\quad$ stream function

$$\boxed{\nabla^2\psi = 0 \quad \omega = 0}$$

$$u = \frac{\partial \psi}{\partial y} = kx$$

$$v = -\frac{\partial \psi}{\partial x} = -ky$$

full N-S, can show $\boxed{\nabla^2\psi = -\omega}$ (2D) $\quad$ it "works"

do this?

potential flow $\nabla^2\psi = 0$

↳ ψ irrotational

"Add" no slip – make some assumptions

↳ inform strategy/setup – not simplify

main assumption

- Assume ω stays near the wall

- rationale I
  - ν small
  - diffusive process

  $\Big\}$ Stoke 1/2

  $\Rightarrow$ don't expect ω to "get far"

  a mild concern: steady flow sort of implies $t \to \infty$ $\to$ diffusion can go a long way

- rationale II – vorticity transport mechanics

$$\nabla \times \left(\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \underline{u} \right) \qquad \underline{\omega} = \nabla \times \underline{u}$$

$\vdots$ do this

The vorticity transport equation

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega} \cdot \nabla \underline{u} + \nu \nabla^2 \underline{\omega}$$

no P!

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega}$$

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} + \cdots$$

- advection – velocity advecting vorticity
- it moves like a scalar field in flow

dye concentration ξ

$$\frac{D\xi}{Dt} = 0 \quad \text{if no diffusion}$$

- note – actually nonlinear

redistribution / concentration by $\nabla\underline{u}$

→ between components of $\underline{\omega}$

→ intensify / diminish a component

"vortex stretching"

MAIN THING:

totally inactive if $\underline{\omega} = 0$

never active in 2D

$\underline{u} = (u, v, 0)$

$\underline{\omega} = (0, 0, \omega_z)$

$\underline{\omega}\cdot\nabla\underline{u} = 0$

$\nabla = (\partial_x, \partial_y, 0)$

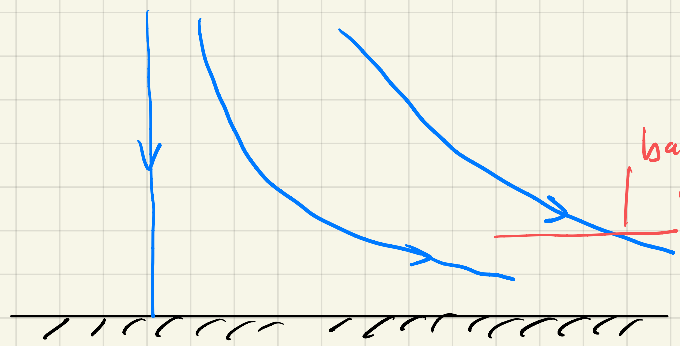
Fickian-like diffusion

like heat

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T$$

→ concentration of $\underline{\omega}$ → spreads but in a conserved way

no creation / destruction



$u = kx$
$v = -ky$

$\downarrow$ strong advection down

$\downarrow$ weaker advection down

balance in some region

no matter how long in time

$\uparrow\uparrow$ diffusion of ω (like Stokes I/2)

- rationale III – if "works" – we do find self consistent N-S solution

<u>Solution</u>

- take $\psi(x,y) = x f(y)$ in layer close to wall
- "blend" to $\psi(x,y) = kxy$ above $(f(y) \to ky)$

$$u = \overset{\frac{\partial\psi}{\partial y}}{x f'} \qquad v = \overset{-\frac{\partial\psi}{\partial x}}{-f}$$

$$\omega = \left(\underbrace{\frac{\partial v}{\partial x}}_{0} - \frac{\partial u}{\partial y}\right) = -x f''$$

ω transport in 2D

$$\cancel{\frac{\partial \omega}{\partial t}} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

steady

$$-xf'f'' + f\,xf''' = \nu\left(0 - x f''''\right)$$

$\div x$

$$-f'f'' + ff''' = -\nu f''''$$

$u(x, y=0) = 0 \rightarrow f'(0) = 0$

$v(x, y=0) = 0 \rightarrow f(0) = 0$

$f(y \to \infty) = ky$ matching

$f''(y \to 0) = 0$