Lecture 13

- Matched asymptotics
- Model ODE

$u(0) = 0 = B + C$

$u(Y \to \infty) = 1 = C$

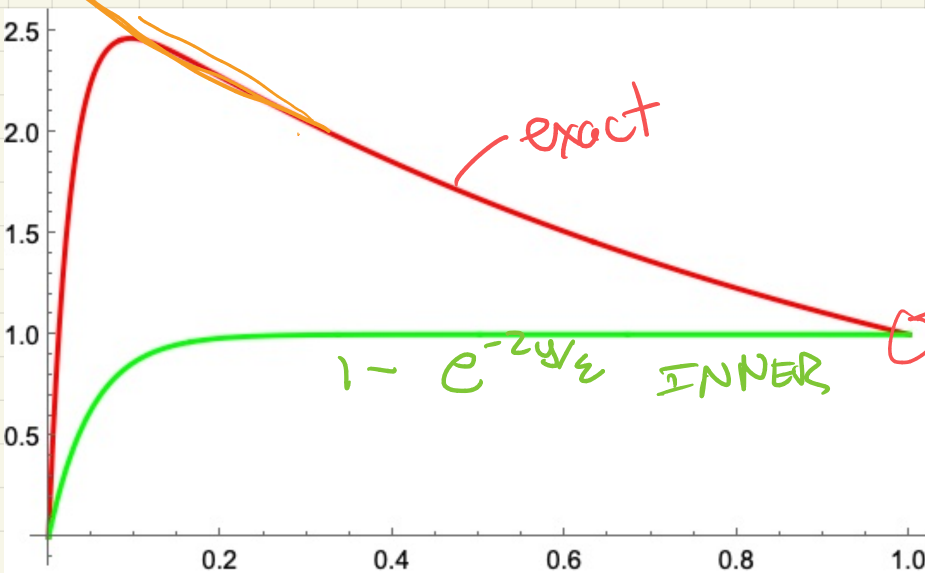
$B = -1$

$u(Y) = 1 - e^{-2Y}$

$u(y) = 1 - e^{-2y/\varepsilon}$

where things went wrong

e^{1-y} OUTER



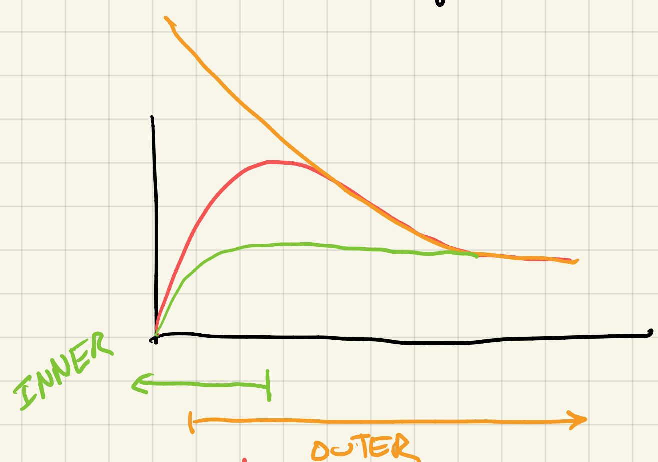
can't just add

need better way to match INNER and OUTER ...

Method of Matched Asymptotic Expansions

van Dyke

- can apply w/o exact solution
- facilitates higher-order $O(\varepsilon^n)$ approximations
- allows more general analysis



"match" where they both apply – not just values, but functional forms

Simple "just values" matching

INNER limit of (OUTER limit)

= OUTER limit of (INNER limit)

→ what we did so far for BL's

$u \to U$ as $y/\delta \to \infty$

→ ignored v

→ does not generalize to higher orders...

Better INNER representation of (OUTER representation)

= OUTER representation (INNER representation)

asymptotics → an expansion in some large/small

$$f(\varepsilon) = a_0 \varepsilon^0 + a_1 \varepsilon^1 + a_2 \varepsilon^2 + \dots$$

sum of terms of increasing order

"smallness hierarchy"

INNER n-term expansion of (OUTER n-term expansion)

= OUTER n-term expansion of (INNER n-term expansion)

Back to our example...

$$\underbrace{\overbrace{\varepsilon \frac{d^2u}{dy^2} + 2\frac{du}{dy}}^{} + 2u}_{} = 0 \qquad u(0)=0 \qquad u(1)=1$$

$Y = \varepsilon/y$

OUTER $\bar{w}$ OUTER b.c. $\Rightarrow$ $u(y) = e^{1-y}$

set one of 2 constants

INNER $\bar{w}$ INNER b.c. $\Rightarrow$ $u(Y) = B\left(e^{-2Y} - 1\right)$

remaining const

"ZOOM" for INNER

$$\boxed{Y = \frac{y}{\varepsilon}}$$

Y – INNER variable

y – OUTER variable

OUTER$^{(y)}$ expanded in INNER$^{(Y)}$ variables

$$u(y) = e^{1-y} \rightarrow u(Y) = e^{1-\varepsilon Y}$$

$e^x = 1 + x + \frac{1}{2}x^2 + \dots$

$$= e^1 e^{-\varepsilon Y}$$

$$= e^1\left(1 - \varepsilon Y + \tfrac{1}{2}\varepsilon^2 Y^2 + \dots\right)$$

$$= e^1 + O(\varepsilon)$$

INNER$^{(Y)}$ expanded in OUTER$^{(y)}$ variables

$$u(Y) = B(e^{-2Y} - 1) \rightarrow u(y) = B(e^{-2y/\varepsilon} - 1)$$

$\uparrow$ smaller than any power of ε as $\varepsilon \to 0$

$\varepsilon \to 0 \rightarrow e^{-2y/\varepsilon}$ smaller than any ε^n

$$u(y) = -B$$

matching $u(Y) = u(y)$ in "overlap"

$$B = -e^1$$

<u>combine</u> (all in y variable)

OUTER $u(y) = e^{1-y}$

INNER $u(y) = -e^1(e^{-2y/\varepsilon} - 1)$

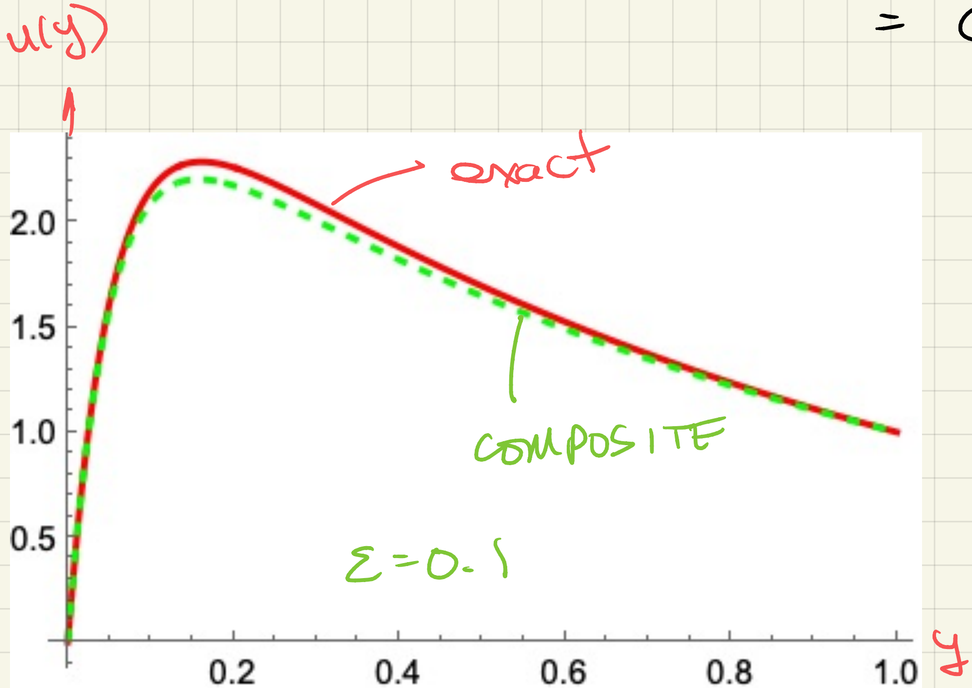
behavior in overlap $\rightarrow$ COMMON $u(y) = e^1 = -B$

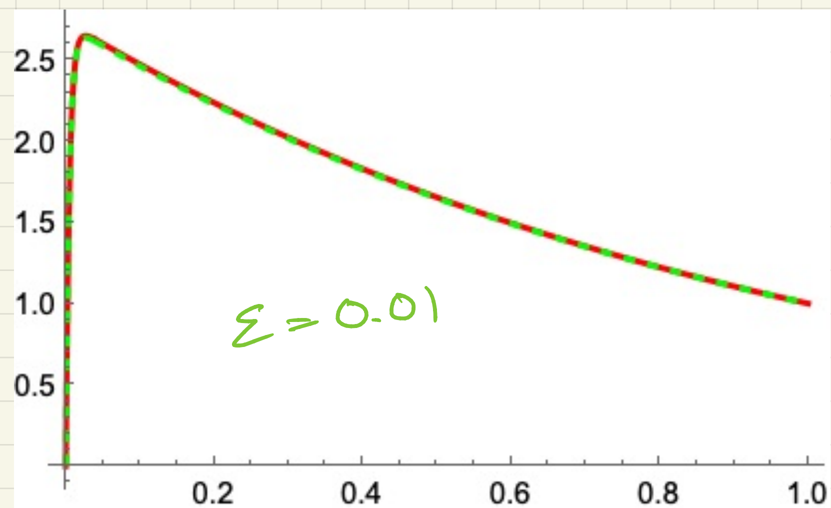
COMPOSITE

$$u(y) = \underbrace{u_{outer}}_{} + u_{inner} - \underbrace{u_{common}}_{}$$

$$= e^{1-y} - e^{1}e^{-2y/\varepsilon} + \cancel{e^{1}} - \cancel{e^{1}}$$

$$= e^{1-y} - e^{1-2y/\varepsilon}$$





Path to higher-order approximation ... why?

- more accurate
- gives a measure of what's missing
  - → might support different phenomenology

e.g. instability

$O(\varepsilon^2)e^t$ "beats" $O(\varepsilon)$ eventually

- "fix" qualitative "loose ends" — v-jump

same model $\varepsilon \frac{d^2u}{dy^2} + 2\frac{du}{dy} + 2u = 0$ $u(0)=0$, $u(1)=1$

assume $u(y) = u_0(y) + \varepsilon u_1(y) + \varepsilon^2 u_2(y) \cdots$

"works if it works"

sub in

$$\varepsilon \frac{d^2u_0}{dy^2} + \varepsilon^2 \frac{d^2u_1}{dy^2} + \cdots + 2\frac{du_0}{dy} + 2\varepsilon\frac{du_1}{dy} + \cdots + 2u_0 + 2\varepsilon u_1 + \cdots = 0$$

INNER $Y = y/\varepsilon$

$$\frac{\varepsilon}{\varepsilon^2}\frac{d^2u_0}{dY^2} + \frac{\varepsilon^2}{\varepsilon^2}\frac{d^2u_1}{dY^2} + \cdots + 2\frac{1}{\varepsilon}\frac{du_0}{dY} + 2\frac{\varepsilon}{\varepsilon}\frac{du_1}{dY} + \cdots + 2u_0 + 2\varepsilon u_1 + \cdots = 0$$

$\times\varepsilon$ rearrange

$$\underbrace{\frac{d^2u_0}{dY^2} + 2\frac{du_0}{dY}}_{O(1)} = \underbrace{-\varepsilon\frac{d^2u_1}{dY^2} - 2\varepsilon\frac{du_1}{dY} - 2\varepsilon u_0}_{O(\varepsilon)} + O(\varepsilon^2)$$

$O(1)$ — SAME as before for $u_0(0)=0$ $\quad u_0(y) = B_0\left(e^{-2y/\varepsilon} - 1\right)$

OUTER $\div$

$$\underbrace{\frac{\partial u_0}{dy} + u_0}_{O(1)} = \underbrace{-\frac{\varepsilon}{2}\frac{d^2u_0}{dy^2} - \varepsilon\frac{du_1}{dy} - \varepsilon u_1}_{O(\varepsilon)} + O(\varepsilon^2)$$

$O(1)$ — SAME used $u_0(1) = 1$ $\quad u_0(y) = e^{(1-y)}$

any change so far $u \to u_0$ $\Rightarrow B_0 = -e^1$

next order ...

$O(\varepsilon)$ INNER :
$$\varepsilon \frac{d^2 u_1}{dY^2} + 2\varepsilon \frac{du_1}{dY} = -2\varepsilon u_0$$

$$u(0)=0 = u_0(0) + \varepsilon u_1(0)$$
$$u_1(0)=0$$

sub in u_0

$\div \varepsilon$
$$\frac{d^2 u_1}{dY^2} + 2\frac{du_1}{dY} = -2e^1(1-e^{-2Y})$$

can solve
$$u_1(Y) = B_1(1-e^{-2Y}) - Ye^1(1+e^{-2Y})$$

only applied inner BC ... $\rightarrow$ B_1 const ....