Lectur 07

finish exact NS

- jet

$Re \to 0 \neq \nu = 0$

Office Hours

Options for Consideration

Who	When	Where
Freund	T 11:00-12:30	306F Talbot / Zoom
Freund	W 11:00-12:00	306F Talbot / Zoom
Freund	Th 2:00-3:00	306F Talbot / Zoom
Freund	M evening ???	Zoom
Freund	By appointment	—

tie for favorites

worked for most

1/2 hour from each

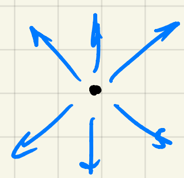
?

A final (for us) exact N-S — A Laminar Jet

- hard / impossible to realize — remarkably unstable
- illustrates properties of governing equations
- no "boundary", but BL approximation can be tested, "thin layer"

<u>Outline</u> — self-consistent "solution"

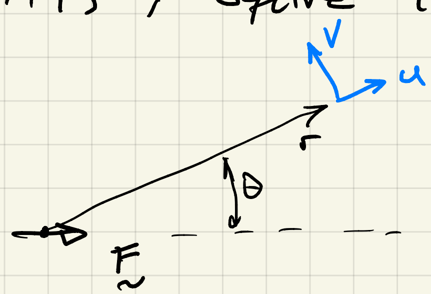
- similar to a point source / sink (3D)



$$v_r = \frac{Q}{4\pi r^2}$$ ← strength

solution

- Landau (1944), Squire (1951) — point momentum source jet



axisymmetric spherical coordinates

- mass

$$\nabla \cdot \underline{u} = \frac{1}{r^2}\frac{\partial r^2 u}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \sin\theta\, v}{\partial \theta} = 0$$

Stokes Stream Fct: $u = \frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta}$

$v = \frac{-1}{r\sin\theta}\frac{\partial \psi}{\partial r}$

$$\psi(r,\theta) = r\nu f(\theta)$$

$\vdots$ sub in / time passes

ODE: $f^2 - 2(1-\xi) f' - 4\xi f = 0$ $\qquad \xi = \cos\theta$

$$f(\xi) = \frac{2(1-\xi^2)}{1+c-\xi} \quad \text{solution}$$

C large $\qquad f(\xi) = \dfrac{2\sin^2\theta}{1+\underbrace{c}_{\text{large}} - \underbrace{\cancel{2\cos\theta}}_{\pm 1 \text{ at most}}}$ neglect

$\longrightarrow$ symmetric

(like low Re#)

C small $f(\xi) = \dfrac{2\sin^2\theta}{\underbrace{1-\cos\theta}_{0 \to 2} + \underbrace{c}_{\text{small}}}$

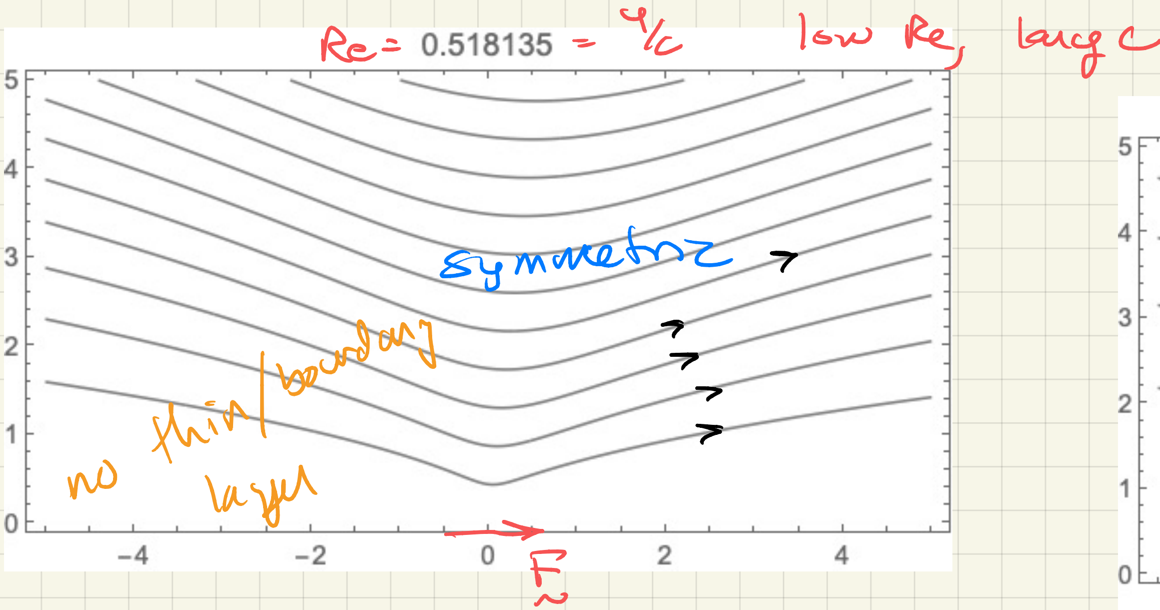
→ asymmetrict

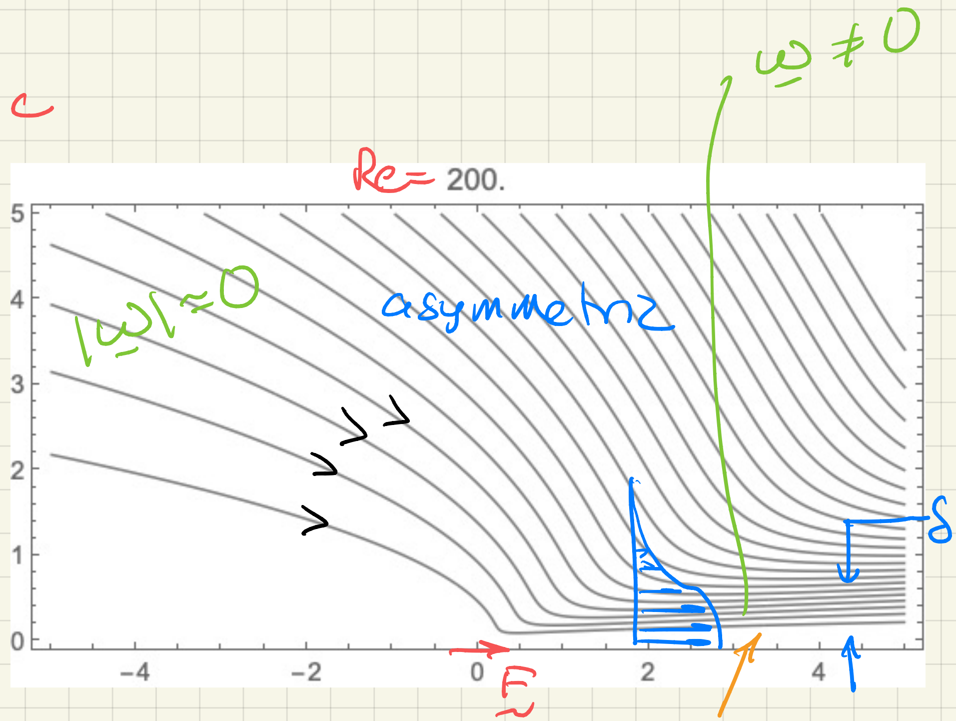
$\dfrac{2}{2} \to \dfrac{2}{\text{tiny}(c)}$

Bachelor

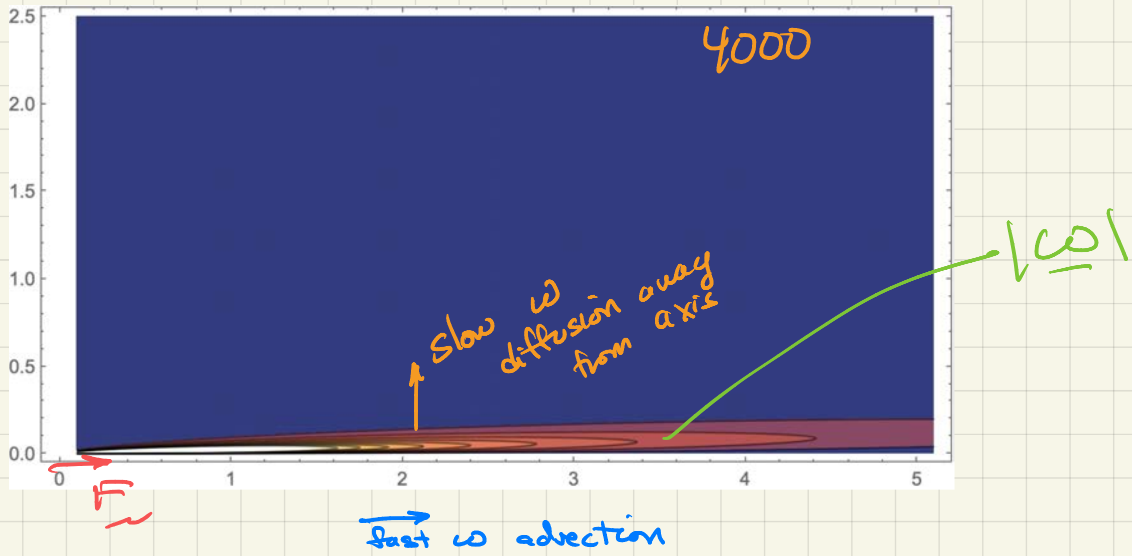
Re # $= 4/c$ based on ref momentum flux a nominal orifice ('ct exit)

√





nearly ∥, slowly changing, like "BL"



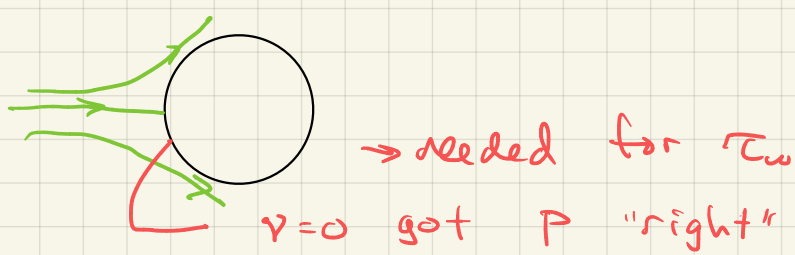
BL

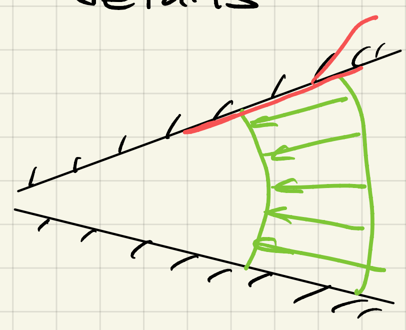
"Re → ∞" ≠ "ν = 0"

"high Re#" ≠ "inviscid"

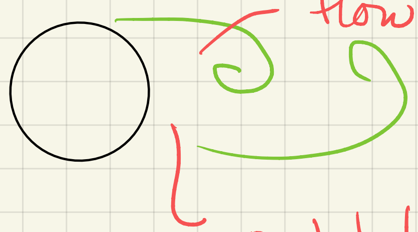
2 way ν = 0 fails ....

- some times just needed for details



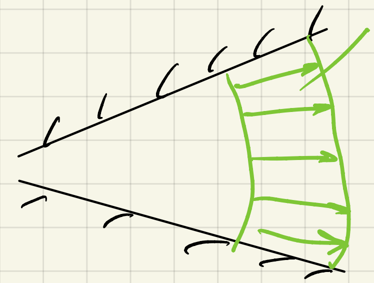


- other times totally wrong



flow qualitatively different

p totally wrong



$\nu = 0$ fine

no solution found for $\nu \simeq 0$

"something" either (1) needs correction or (2) "breaks" when $Re \to \infty$ is approximated by $\nu = 0$

$$\frac{D\underline{u}}{Dt} = -\nabla p + \cancelto{0}{\frac{1}{Re}\nabla^2 \underline{u}}$$

same problem

$$\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} + \cancelto{0}{\frac{1}{Re}\nabla^2 \underline{\omega}}$$

A model:

$$\varepsilon \frac{du}{dx} + u = x \qquad u(0) = 1$$

ε like $\frac{1}{Re}$ for large Re

Ockendon & Ockendon

$$u(x) = C e^{-x/\varepsilon} + x - \varepsilon$$

$$\frac{du}{dx} = -\frac{C}{\varepsilon} e^{-x/\varepsilon} + 1$$

sub in: $\cancel{\varepsilon}\left(-\frac{C}{\cancel{\varepsilon}} \cancel{e^{-x/\varepsilon}} + \cancel{\varepsilon}\right) + C\cancel{e^{-x/\varepsilon}} + x - \cancel{\varepsilon}$

$= x$?

$x = x$ ✓

BC $\rightarrow$ $u(0) = 1$ $\quad 1 = C - \varepsilon$

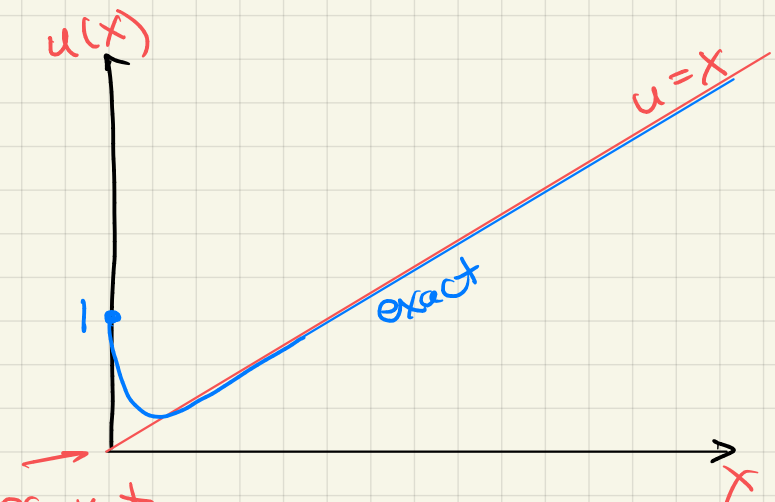
$$C = 1 + \varepsilon$$

$$\boxed{u(x) = (1+\varepsilon) e^{-x/\varepsilon} + x - \varepsilon}$$

$$\cancel{\varepsilon \frac{du}{dx}} + u = x$$

$\varepsilon \to 0$

$u(x) = x$ solution



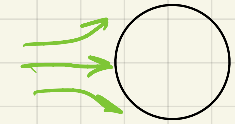
does not satisfy BC $u(0)=1$

approximate $u(x) = x$ solution

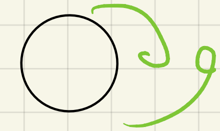
good, except when

$x = O(\varepsilon)$ $\longrightarrow$ $\lim\limits_{\varepsilon \to 0} \frac{x}{\varepsilon} = c$ $\qquad e^{-x/\varepsilon}$ not small $O(1)$

→ like case where OUTER flow looks good, even though doesn't satisfy BC



→ bigger problem is when physics of the tiny "wrong" region has global impact



Systematic Procedure for Gov. Eq.

- "zoom in" on problem region — with scaled variable(s)

  → define an INNER problem

$$\xi = x/\varepsilon$$

→ have $\xi = O(1)$ when $x = O(\varepsilon)$ (problem region)

$$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{1}{\varepsilon}\frac{du}{d\xi}$$

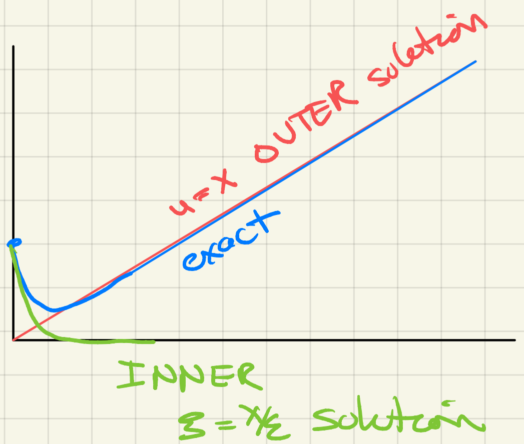
$$\varepsilon\frac{du}{dx} + u = x$$

$$\varepsilon\left(\frac{1}{\varepsilon}\frac{du}{d\xi}\right) + u = \underbrace{\varepsilon\xi}_{x}$$

$$\underbrace{\frac{du}{d\xi} + u}_{O(1)} = \underbrace{0}_{\text{small } O(\varepsilon)}$$

$u(\xi) = e^{-\xi}$ $\qquad u(0) = 1 \qquad$ [same BC]

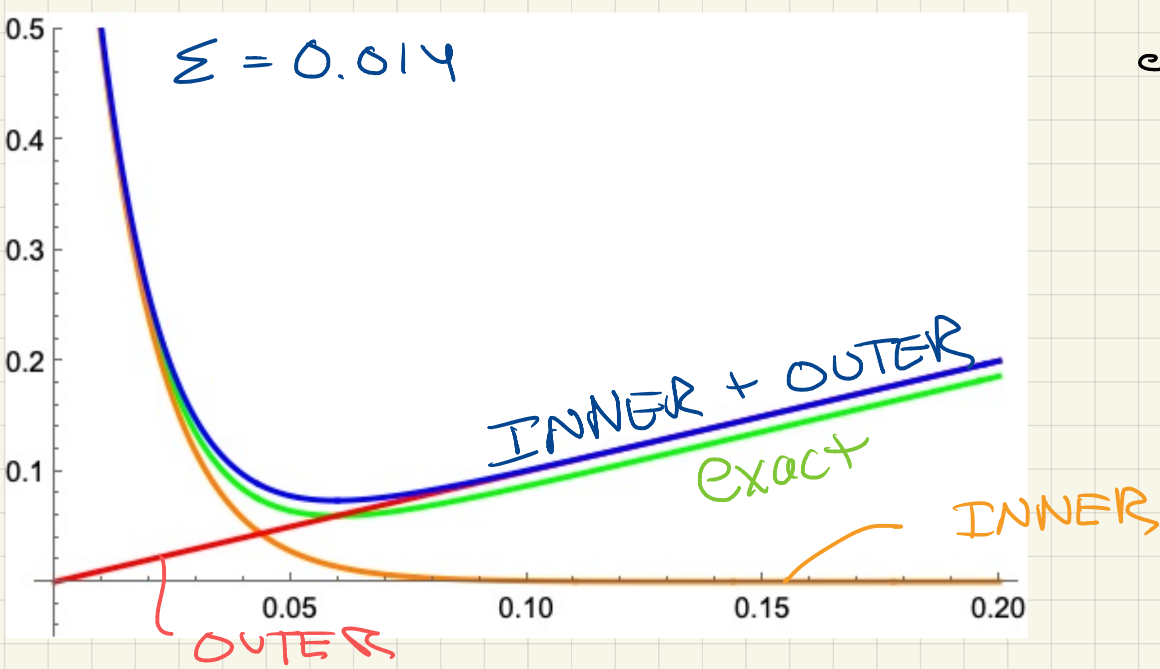
$u(x) = e^{-x/\varepsilon}$ $\qquad$ INNER SOLUTION



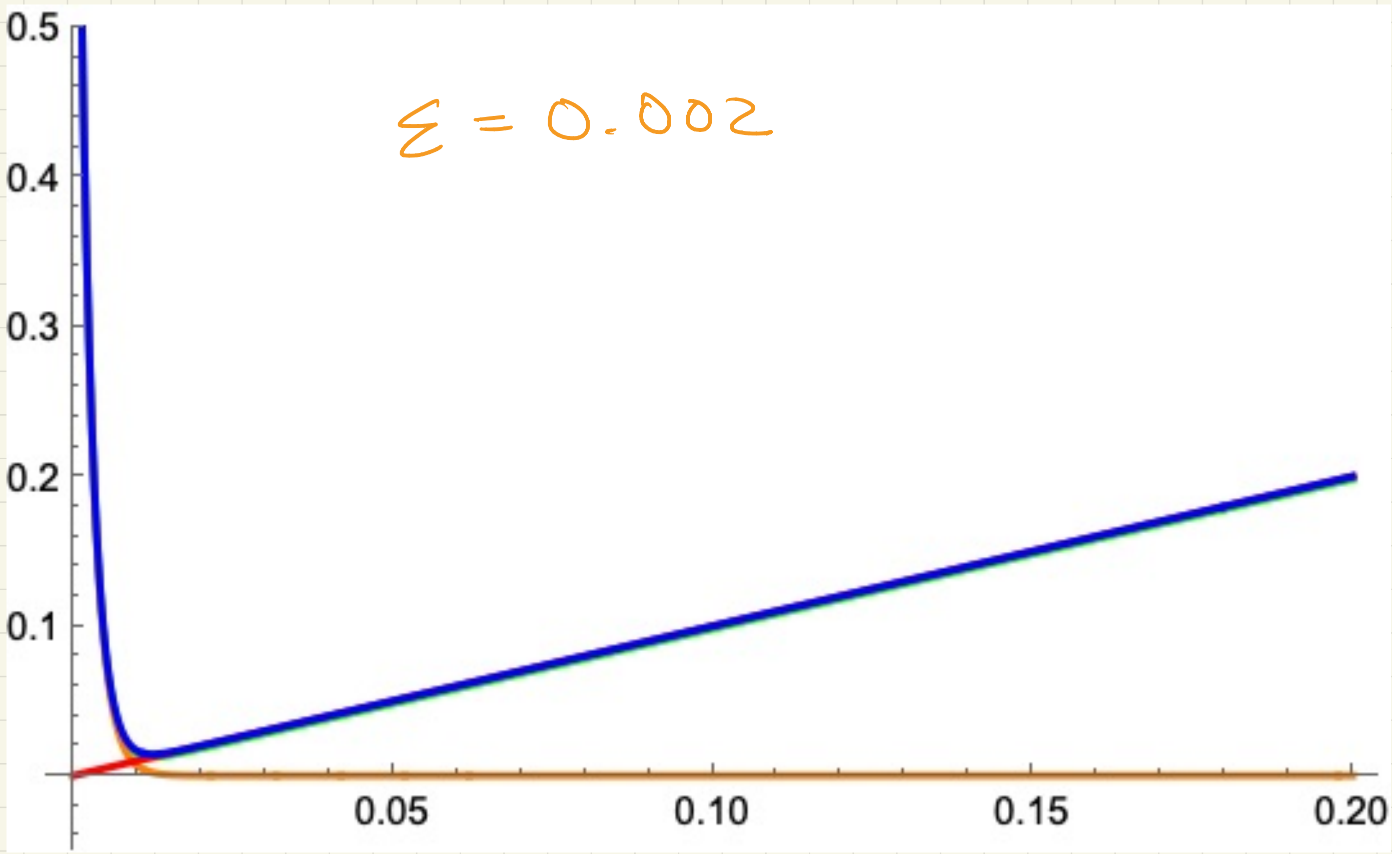
add INNER + OUTER ... $\qquad u(x) = e^{-x/\varepsilon} + x$

exact $\qquad u(x) = (1+\varepsilon)e^{-x/\varepsilon} + x - \varepsilon$

$O(\varepsilon)$ different



CAUTION / APPOLOGY:
can't in general
just add INNER
+OUTER



One way to view for flow: a new length scale forms near boundary $\delta \sim \varepsilon$

- N-S: $Re = \frac{\delta U}{\nu} = O(1)$ δ adusts so we can't cross out $\frac{1}{Re\#}$ term

$\rightarrow$ δ changes to balance ν and δU advection