Lecture 21

- Viscous dissipation
- One way coupling $\to T$

Schlichting

Add energy transport ... compressible

- heat transfer → v. important at high Mach #
- compressibility + heating → affect BC flow

Strategy

① first: one-way coupled
$\underline{u}$ affects T but
T does not affect $\underline{u}$

② fully coupled – equation of state
– $\mu(T)$, $k(T)$

expectation for "action" on T

– advection $\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \cdots = \frac{DT}{Dt}$

– diffusion – heat conduction

$$\underset{\sim}{q} = -k\nabla T$$

thermal conductivity

– BC on walls T_w or q_w

– viscous dissipation → friction heating
KE → thermal energy

momentum

$$\rho \frac{D\underline{u}}{Dt} = \nabla \cdot \underline{\underline{\sigma}}$$

$$\underline{\underline{\sigma}} = -p\underline{\underline{\delta}} + \underbrace{\mu\left(\nabla\underline{u} + \nabla\underline{u}^T\right) + \lambda\underline{\underline{\delta}}\nabla\cdot\underline{u}}_{\text{general Netonian fluid}}$$

pressure on diagonal

$$\underline{\underline{\delta}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{\tau}} = \underline{\underline{\underline{\underline{A}}}} : \nabla\underline{u}$$

linear stress - strain-rate relation

reduce for isotropic material

λ 2nd coeficient of viscosity

$\underline{u}\cdot$

$$\underline{u}\cdot\rho\frac{D\underline{u}}{Dt} = \underline{u}\cdot\nabla\cdot\underline{\underline{\sigma}}$$

$$\rho\left(u_i\frac{\partial u_i}{\partial t} + u_i u_j\frac{\partial u_i}{\partial x_j}\right) = u_i\frac{\partial\sigma_{ij}}{\partial x_j}$$

$$\rho\left(\frac{\partial\frac{1}{2}u_iu_i}{\partial t} + u_j\frac{\partial\frac{1}{2}u_iu_i}{\partial x_j}\right) =$$

$$\rho\frac{De_k}{Dt} = \underline{u}\cdot\nabla\cdot\underline{\underline{\sigma}}$$

e_k - $\frac{KE}{mass}$

KE advected

$\bar{p}$ mechanical pressure - isotropic part of $\underline{\underline{\sigma}}$ [same in all direction]

traces of tensors are invariant [does not matter coordinates]

$\llcorner$ σ_{ii} - sum of diagonal elements

$$\bar{p} = -\frac{1}{3}\sigma_{ii} \qquad \longrightarrow \qquad tr(\bar{p}\underline{\underline{\delta}}) = -tr(\underline{\underline{\sigma}})$$

$tr(\underline{\underline{\delta}}) = 3$

$$\underline{\underline{\sigma}} = -p\underline{\underline{\delta}} + \mu\underbrace{(\nabla\underline{u} + \nabla\underline{u}^T)}_{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} + \lambda\underline{\underline{\delta}}\,\nabla\cdot\underline{u}$$

$\llcorner tr(\underline{\underline{\delta}}) = 3$

$$\bar{p} = -\tfrac{1}{3}\sigma_{ii} = p - \frac{1}{3}\left(2\mu\frac{\partial u_i}{\partial x_i}\right) - \tfrac{1}{3}\lambda\, 3\,\frac{\partial u_i}{\partial x_i}$$

$$\bar{p} = p - \frac{1}{3}(2\mu + 3\lambda)\,\nabla\cdot\underline{u}$$

mechanical pressure

thermodynamic pressure

possibility that they deviate

"bulk viscosity"

$\mu_B = \lambda + \tfrac{2}{3}\mu$

Often not important

Often $\nabla\cdot\underline{u}$ small

If μ_B matters, Newtonian model is probably poor

Physics → some molecular relaxation

Vincenti + Kruger

$$\rho \frac{De_k}{Dt} = \underline{u} \cdot \nabla \cdot \underline{\underline{\sigma}} \rightarrow u_i \frac{\partial \sigma_{ij}}{\partial x_j} = \overset{*}{\frac{\partial u_i \sigma_{ij}}{\partial x_j}} - \overset{*}{\sigma_{ij} \frac{\partial u_i}{\partial x_j}}$$

the loss of KE must be in this term

$$\rho \frac{De_k}{Dt} = \overset{*}{\nabla \cdot (\underline{\underline{\sigma}} \cdot \underline{u})} - \overset{*}{\underline{\underline{\sigma}} : \nabla \underline{u}}$$

product rule differenciation

$$\frac{dab}{dx} = a\frac{db}{dx} + b\frac{da}{dx}$$

flux → moves around, integrates to zero ...

$$\underline{\underline{\sigma}} = -p\underline{\underline{\delta}} + \mu(\nabla \underline{u} + \nabla \underline{u}^T) + \lambda \underline{\underline{\delta}}\, \nabla \cdot \underline{u}$$

symmetric

symmetric part of $\nabla \underline{u}$

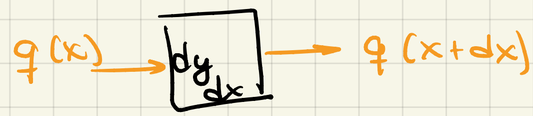
asy $\frac{1}{2}(\nabla \underline{u} - \nabla \underline{u}^T)$

$$= \underline{\underline{\sigma}} : \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$$

$\underline{\underline{\delta}} : \nabla \underline{u}$

$$\underline{\underline{\sigma}} : \nabla \underline{u} = -p\, \nabla \cdot \underline{u} + \underbrace{\overbrace{\mu \frac{1}{2}\underbrace{(\nabla \underline{u} + \nabla \underline{u}^T):(\nabla \underline{u} + \nabla \underline{u}^T)}_{\geq 0 \text{ "squared term"}} + \underbrace{\lambda (\nabla \cdot \underline{u})^2}_{\geq 0 \text{ "squared term"}}}^{\Phi}}_{\text{always "sink" terms}}$$

→ KE → thermal energy



total energy eq:

$$\rho \frac{De_t}{Dt} = -\nabla\cdot\underline{q} + \nabla\cdot\underline{\underline{\sigma}}\cdot\underline{u}$$

$\frac{\text{total energy}}{\text{mass}}$

"attached" to advecting fluid

heat flux

$\underline{q} = -k\nabla T$

surfaces "forces" x $\underline{u}$ ⇒ work

subtract

$$\rho \frac{De_k}{Dt} = \nabla\cdot\underline{\underline{\sigma}}\cdot\underline{u} - \underline{\underline{\sigma}}:\nabla\underline{u}$$

internal (thermal) energy

$$e_i = e_t - e_k$$

$$\rho \frac{De_i}{Dt} = -\nabla\cdot\underline{q} + \underline{\underline{\sigma}}:\nabla\underline{u}$$

e_i $\boxed{\rho \frac{De_i}{Dt} = -\nabla\cdot \underline{q} - p\nabla\cdot\underline{u} + \Phi}$

pressure work

viscous dissipation ≥ 0 source

enthalpy $h \equiv e_i + p/\rho$

mass $\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\underline{u} = 0$

$$\underbrace{\frac{\partial\rho}{\partial t} + \underline{u}\cdot\nabla\rho}_{\frac{D\rho}{Dt}} + \rho\nabla\cdot\underline{u} = 0$$

add $\rho \frac{D\, p/\rho}{Dt}$ to both sides

$$\rho \frac{D e_i + p/\rho}{Dt} = -\nabla\cdot\underline{q} + \frac{\rho}{\rho}\frac{Dp}{Dt} \cancel{- \frac{p\rho}{\rho^2}\frac{D\rho}{Dt} + \frac{p}{\rho}\frac{D\rho}{Dt}} + \Phi$$

$\nabla\cdot\underline{u} = -\frac{1}{\rho}\frac{D\rho}{Dt}$

h $\boxed{\rho\frac{Dh}{Dt} = -\nabla\cdot\underline{q} + \frac{Dp}{Dt} + \Phi}$

$1/\rho$ specific volume

Gibbs eq: $T ds = du + p dv$

$h = u + pv$

$dh = du + p dv + v dp$

$du = dh - p dv - v dp$

$T ds = dh - v dp$

$$T\frac{Ds}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}$$

$\rho = 1/v$

S

$$\boxed{\rho\frac{Ds}{Dt} = -\frac{1}{T}\nabla\cdot\underline{q} + \frac{1}{T}\underline{\Phi}}$$

source of entropy

if no thermal conductivity or μ $\rightarrow$ $\frac{Ds}{Dt} = 0$ just advects

specific heat

$h = c_p T$ take c_p const

$$\rho c_p \frac{DT}{Dt} = -\nabla\cdot q + \cancel{\frac{Dp}{Dt}} + \Phi$$

assume isobaric

$p \approx$ const

$\frac{\Delta p}{p_0} \ll 1$

small Δp sufficient preserve flow w/o thermodynamic effects

ΔP's don't change T

BL limit ?