On a Minimizing Crossing Problem

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Abstract

Here, we will be exploring the field of graph drawing in minimizing crossings. The problem of interest pertains to complete bipartite graphs $K_{m,n}$, with a special mention at the end for complete graphs K_m. Particularly, we will be dwelling on the Zarankiewicz Conjecture for minimizing crossings. In addition, I will dwell on the motivation for such problems and applications for studying minimizing crossings.

Keywords: Complete Bipartite Graphs; Minimizing Crossings; Turán's Brick Problem

1 Initial Motivation & History

Origin of Turán's Problem The problem originates from a Hungarian mathematician named Paul Turán during his time in a war-time labor camp. Paul Turán during a lecture posed the following question: "*There were some kilns where the bricks were made and some open storage yards where bricks were stored. All the kilns were connected by rail with all storage yards. ... the trouble was only the crossings. The Trucks generally jumped the rails there, and the bricks fell out of them; ... the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rail had been minimized. But what is the minimum number of crossings?*" He worked with a general problem with **m** kilns and **n** storage yards.

House-and-Utilities Problem (Special Case) A famous puzzle describing a undirected complete bipartite where $m = n = 3$ i.e. $K_{3,3}$. Here, we have two independent set called **N** (neighbors) and **U** (utilities). The three neighbors A, B, and C, wants their homes to be connected to water, gas, and electricity (**W**, **G**, and **E** respectively) in such a way that no connections cross.

Solution: Exist no solution unless they disregard the rules laid out i.e. no connections cross. First let's connect A to W, B to G, and C to E. Then to make all vertices 3-regular, first connect A to G and E. Then connect B to E and W. In this moment, neighbors A, B, and E are 3-regular. Then connect C to W. Then identify a cycle where each vertex is 3-regular and call it C_k for k $\geq$ 4. Here, k cannot be less than 4 as traversing independent set of vertices N and U, require at least length 4 to get back to starting vertex. In addition, notice in a complete bipartite, to create a cycle with minimal crossings, exist at least one edge that traverses the graph around the **center**. The center of $K_{3,3}$ is the set of straight constant/undefined gradient adjacent edges disregarding minimizing crossings. In this example, [1]$V(C_4)$ = {E, A, W, B} where G is inside the cycle. Since exist two of each vertex from vertex set N and U where for each vertex in N and U are 3-regular, all vertices in C_4 has at least degree 2 thus creating a cycle. Here, doing the following steps starting at any vertices will create a cycle. Thus to create an edge outside of C_4, we would have to cross at least one edge.

This is a solution utilizing cylic-ordering. Another way to approach this problem is by contradiction.

Proof: Suppose $K_{3,3}$ is planar i.e. exist no crossing for this context. Then utilizing Euler's formula $V - E + F = 2$, for a complete bipartite where $m = n = 3$, we have $m * n = 3 * 3 = 9$ edges and $m + n = 3 + 3 = 6$ vertices. Putting this into the Euler's formula, we get $9 - 6 + F = 2$ where $F = 5$.

We know that $K_{3,3}$ is simple thus no face has two sides, and bipartite where every face has an even number of sides. Because its simple and bipartite, every face must have at least 4 sides. Each sum of the sides over all the faces in $K_{3,3}$ is twice the number of edges as each edge is a side of two faces. Thus we can present the inequality $2E \geq 4 * F$ where $18 \geq 20$ exist a contradiction. Thus $K_{3,3}$ is non-planar and exist at least one crossing.

[1]$V(C_k)$ refers to set of vertices respectively

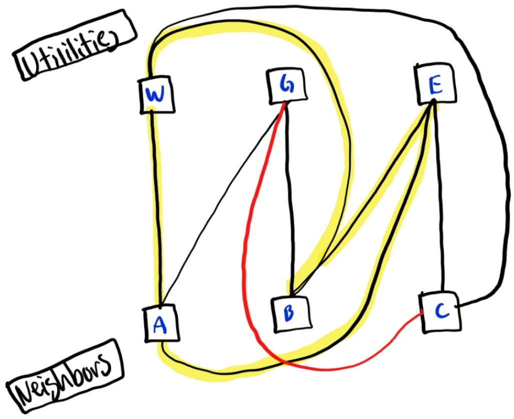

Figure 1: Example Solution for House-and-Utilities Problem

NOTE: Solution is on the basis of minimizing crossings i.e. if exist an edge that doesn't cross, we take that one rather the one that creates a crossing. Many alternate solutions exist, this is just one possibility utilizing the basic definitions and notions learned in class.

Above, we discussed the initial motivation for studying minimal crossings utilizing the setting of houses and utilities for $m = 3$ and $n = 3$ respectively. Looking further, we want to make a more general statement for the minimal number of crossing for all $K_{m,n}$ graphs utilizing the setting from Turán's Brick Problem. A good place to start is on the case for K_m i.e. a complete graph, which we will then transition into the Zaranckiewicz's Conjecture for the case for $K_{m,n}$.

2 On the case for K_m

For a quick special recognition, we take a look at minimal crossings for K_m. First introduced in 1960 by a Mathematician named Anthony Hill. Namely, we know that there always exist a drawing with the following upper bound for the minimal number of crossings.

$$cr(K_m) \leq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

The initial conjecture is that there can be no better drawing equivalent to the RHS thus the formula should give an optimal number of crossing for the complete graph. The formula gives the following values that have been proven to be minimal:

n	$cr(K_m)$
5	1
6	3
7	9
8	18
9	36
10	60
11	100
12	150

Table 1: Values for $n = 5, ..., 12$

Possible Algorithm for Finding Minimal crossing in K_m

For interior edges, draw edges with only positive/undefined gradient i.e. positive slope and vertical line edges. For all other edges, we can create n-1 regular vertices with the remaining edges traversing outside the interior of the graph. This is a proposed algorithm to find the minimal crossing. But observing the values in the table, there doesn't seem to be a clear solution for all complete graphs. Although solutions are easily verifiable, proposing general solution for n is similar in complexity but out of scope for this project.

Note: Unlike $K_{m,n}$ which mathematicians believe has an optimal outline by partitioning onto the x-axis and y-axis as shown above, conventional geometric shape drawing are often utilized for finding minimizing crossing for K_n. e.g. triangle, squares, pentagon etc.

3 Zarankiewicz Conjecture

To answer the question posed above by Turán, Zarankiewicz submitted his proposal of the problem after attending Turán's lecture in October 1952. Zarankiewicz construction is using rectilinear edges. In some form of construction, using curved edges might be able to minimize number of crossing. But if his conjecture is true, the number of crossing utilizing rectilinear edges vs curved edges would be equal. The way most minimal crossing proof works is by showing that their specific construction is the minimal number of crossings possible i.e. minimal number of crossings equals $cr(K_{m,n})$. Knowing this, we will state the necessary Theorem provided by Zaranckiewicz:

Theorem 1 *If*

(α) in the Euclidean plane two sets of points, A and B, are given, A consisting of p points a_1, a_2, a_3,...,a_q, and B consisting of q points b_1, b_2, b_3,...,b_q, (p and q are natural numbers);

(β) for each pair of points a_i, b_j, where i = 1,2,3,...,p, j = 1,2,3,...,q, there exists a simple arc lying in the plane and having the points a_i, b_j as its end points;

(γ) the arcs lie in such a way that no three arcs have an interior point (i.e. a point that is not an endpoint) in common;

(δ) K(p,q) denotes the smallest number of intersection points of arcs; then the following formulas hold:

$$K(2k, 2n) = (k^2 - k)(n^2 - n), \tag{1}$$

$$K(2k, 2n + 1) = (k^2 - k)n^2, \tag{2}$$

$$K(2k + 1, 2n + 1) = k^2 n^2 \tag{3}$$

Note: As stated above, Zarankiewicz and Urbanik independently came to a similar conclusion where formulas (1), (2), and (3) can be written into a single formula:

$$K(p, q) = \left(p - 1 - E\left(\frac{p}{2}\right)\right) E\left(\frac{p}{2}\right) \left(q - 1 - E\left(\frac{q}{2}\right)\right) E\left(\frac{q}{2}\right) \tag{4}$$

where E(x) denotes the greatest integer $\leq$ x i.e. floor function for integers. Notice for formulas (1), (2), and (3) have the inputs even/even, even/odd, and odd/odd. Utilizing the greatest integer for $\leq x$, we similarly account for all possibilities with formula (4). Take for example $p = q = 4$.

$$K(4, 4) = \left(4 - 1 - E\left(\frac{4}{2}\right)\right) E\left(\frac{4}{2}\right) \left(4 - 1 - E\left(\frac{4}{2}\right)\right) E\left(\frac{4}{2}\right)$$
$$K(4, 4) = (1)(1)(1)(1) = 1$$

Utilizing this, we can state Zarankiewicz's Conjecture.

Conjecture 3.1 (Zarankiewicz Conjecture) *For a complete bipartite graph $K_{m,n}$, we can denote $cr(K_{m,n})$ as the minimum crossing.*

$$cr(K_{m,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \tag{5}$$

Note: This is equivalent to the equation (4) above.

Moving forward, we will be utilizing the notation $cr(K_{m,n})$ to denote minimal crossings.

3.1 Proof Methods Utilized in the Zarankiewicz Conjecture

Note: In this section, I will redefine all necessary definitions. The following above was to restate the necessary statements to get started.

To establish the equation stated above, we first draw vertices at points with integer coordinates $(i, 0)$ for

$-\lfloor\frac{m+1}{2}\rfloor \leq i \leq \lfloor\frac{m}{2}\rfloor$, where for each vertex, we join by a straight edge to each vertices $(0, j)$ for $-\lfloor\frac{n+1}{2}\rfloor \leq j \leq \lfloor\frac{n}{2}\rfloor$. Imagine a Cartesian plane in $\mathbb{R}^2$ drawing m vertices equivalently on the negative and positive x-axis and similarly for the y-axis avoiding the origin. If we have even number of vertices, then it is clear we have same number of vertices on the negative and positive, but if we have odd number, we will have one side with one more vertex than the other.

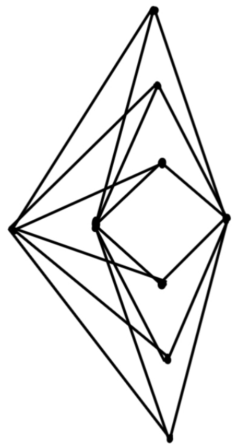

Figure 2: Example for $m = 3$ and $n = 6$

Now that we established our drawing, Zarankiewicz proceeds in his first lemma to establish the case for m = 3. To proceed, I will lay out some necessary definitions.

Here, to generalize all our arguments, I will reinstate some notations. For minimal number of crossings, we will denote it as $cr(K_{m,n})$. For equations (1),(2),(3) in theorem 1, the following is equivalent:

$$cr(K_{2p,2q}) = (p^2 - p)(q^2 - q)$$
$$cr(K_{2p,2q+1}) = (p^2 - p)q^2$$
$$cr(K_{2p+1,2q+1}) = p^2 q^2$$

Definition 1 *An edge from x and y will be denoted by xy. The graph $K_{m,n}$ will consist of two set of vertices, A and B, consisting of points $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_m$ respectively.*

Definition 2 *The set of interior points are called **indoor**. An indoor is denoted as $I_k(x)$ which are intersections that lie in the set of adjacent edges starting at x with endpoints $y_1, y_2, ..., y_k$.*
Note: *The set of interior points $I_k(x)$ does not include the endpoints x and $y_1, y_2, ..., y_k$.*

Definition 3 *A **star** denoted K_{1,k_i} for $i = 1, 2, ..., k$ with root vertex starting at x and ending at the endpoints $y_1, y_2, ..., y_k$. The star is the set of endpoints and interior i.e. $K_{1,k_i} = x + y_1 + y_2 + ... + y_k + I_k(x)$.*

Corollary 1 *For the most basic construction i.e. putting set of vertices A and B in a straight line and connecting them up, there exist $\binom{m}{2}\binom{n}{2}$ many crossings.*

Proof: This results from a counting argument of stars in $K_{m,n}$. Suppose m n-star i.e. $K_{1,n}$ which composes $K_{m,n}$ m times. Then for each pair of stars, we have a distinct crossing, then for another distinct pair of stars, we have another distinct crossing. We can do these for both sides and since we want to pair each distinct star to another to form distinct crossings, we can multiply the possible number of pairs which is similar to the principle for counting the number of edges in a complete bipartite graph i.e. $m * n$, thus there exist $\binom{m}{2}\binom{n}{2}$ many distinct pair of stars. This implies that we can have at most $\binom{m}{2}\binom{n}{2}$ many crossings via this construction. We can also proceed by induction as well for further clarification.

For $m = n = 1$, there exist no crossings so let's establish the base case for $m = n = 2$ i.e. $K_{2,2}$. Following our construction we establish two stars for each set A and B respectively and thus we get $\binom{2}{2}\binom{2}{2} = 1$ crossing as expected. Then suppose this holds from for m' and n'. We want to show this holds for $m' + 1$ and $n' + 1$

respectively. Consider a complete bipartite graph $K_{m'+1,n'+1}$. We construct from $K_{m',n'}$ by adding one vertex to each set in $K_{m',n'}$. When we add this new vertex, we create a new star in both sets s.t. the stars can be paired with every other star in $K_{m',n'}$. This results in $m'+1$ and $n'+1$ new pairings from each new vertex respectively. As each pair of star creates a distinct crossing, we now have $(m'+1)*(n'+1)$ ways to pair stars. Thus we can show that for $K_{m'+1,n'+1}$, this construction will form $\binom{m'+1}{2}\binom{n'+1}{2}$ as desired. Thus our proof is complete.

Comment: This is the most novice way to connect up $K_{m,n}$. Via the construction of the Zaranckiewicz's graph, we can reduce this by a factor of 4. This comes from the result of reducing the number of pairings of each star into 4 different quadrants. In other words, consider the construction of what Zaranckiewicz's deems produces the minimal number of crossings. One reasoning as to his construction is to reduce the number of pairs of stars that create distinct crossings. Observe the figure above which has separated the graph into 4 quadrants s.t. for each star, it isn't able to paired up with all other stars to create distinct crossings. With this in mind, I will state the following lemmas:

Lemma 1 *If three sets of indoors, starting at different vertices a_1, a_2, a_3, have the same three points b_1, b_2, b_3 as their end points, there exist at least at least two indoors that have a point in common.*

> **Remark:** Refer back to the House-and-Utilities Problem where we found that there exist at least one crossing. Essentially, this lemma is stating that there exist at least one interior point for $m = n = 3$.

Lemma 2 *In the graph $K_{3,n}$, which is the sum of three indoors with the vertices b_1, b_2, b_3, each of which has the same end points $a_1, a_2, a_3, ..., a_n$, the number of intersection amounts to at least*

$$q^2 - q \text{ when } n = 2q$$
$$q^2 \text{ when } n = 2q + 1$$

In the lemma above, he establishes that there exist at least one interior point for when $m = n = 3$. In addition, notice that for $m = 3$, equating it to when $m = 2p+1$ for where $3 = 2p+1$, we get $p = 1$. Thus when multiplying $q^2 - q$ or q^2 by $p^2 = (1)^2 = 1$, we can omit it for simplicity sake. Here, in this lemma, he attempts to show that for $m = 3$ and general n i.e. $K_{3,n}$. In this paper, this is the only thing that he shows successfully is true.

Proof: We can proceed via induction. For q = 1, the following equations hold from the result of lemma 1 i.e. when $m = n = 3$ case. Note that via the construction, $m = n = 2$ does not contain any crossings thus the base case is as mentioned above. Then assume the following equation holds for q. We want to show that the number of intersection for $K_{3,n}$ i.e. for $m = 3$ and general n, will amount to **at least** $(q+1)^2 - (q+1)$ when $n = 2(q+1)$ and $(q+1)^2$ when $n = 2(q+1)+1$ which simplifies to

$$q^2 + q \text{ when } n = 2q + 2 \tag{6}$$
$$q^2 + 2q + 1 \text{ when } n = 2q + 3 \tag{7}$$

Now consider the star $K_{1,3_i} = I_3(a_i) + a_i + b_1 + b_2 + b_3$ for $i = 1, 2, 3, ..., n$ which includes both endpoints and the interior points.

Note: Try to view $K_{m,n}$ as a collective of m n-star i.e. m many $K_{1,n}$ stars in $K_{m,n}$. The figure below with arbitrary selected $K_{1,3_1}$ and $K_{1,3_2}$, has one distinct common point excluding b_1, b_2, b_3 as demonstrated for $m = n = 3$.

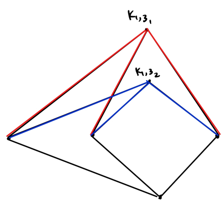

Figure 3: Example for $m = 3$ and $n = 3$

For each pair from among the stars having one point in common, distinct from b_1, b_2, b_3, then for different pairs of stars, those points would be distinct. Refer to assumption 1(γ) in Theorem 1 for further clarification. Therefore, the number of distinct intersection in $K_{3,n}$ would be as high as different pairs of stars otherwise put as:

$$n \text{ choose } 2 = \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2)!}{2!(n-2)!} = \frac{n(n-1)}{2!} = \frac{n(n-1)}{2}$$

Then let's consider for each case:

- For $n = 2q+2$, $\frac{(2q+2)(2q+1)}{2} = \frac{4q^2+2q+4q+2}{2} = 2q^2+3q+1 > q^2+q$
- For $n = 2q+3$, $\frac{(2q+3)(2q+2)}{2} = \frac{4q^2+4q+6q+6}{2} = 2q^2+5q+3 > q^2+2q+1$

Here we have shown via induction that for $q+1$, exist crossing greater than q^2+q and q^2+2q+1 for both even and odd cases. Thus lemma 2 is proved via induction. Now that we have our case for $m = 3$, we want to verify our inductive premise is true.

Inductive Premise: Let us assume that there is a pair of star $K_{1,3_1}$ and $K_{1,3_2}$ which have no points in common except b_1, b_2, b_3. Then for each star $i \neq 1, 2$ has a point distinct from b_1, b_2, b_3 in common with set $K_{1,3_1} + K_{1,3_2}$, where different stars having with $K_{1,3_1} + K_{1,3_2}$ common points. Here, assumption 1(γ) still holds. This implies that in the set of stars $K_{1,3_1} + K_{1,3_2}$, the number of crossings points that are distinct from each other, is at least as high as the number of stars where $i \neq 1, 2$. Therefore we can represent the number of intersection point as follow:

$$K_{3,n} = K_{1,3_1} + K_{1,3_2} + \sum_{i \neq 1,2} K_{1,3_i} = K_{1,3_1} + K_{1,3_2} + K_{3,(n-2)}$$

To interpret the equation above, it states that the number of intersection on $K_{3,n}$ will be at least equal to the number of crossing of the graph $K_{3,n-2}$ with $K_{1,3_1} + K_{1,3_2}$ plus the number of intersection in the graph $K_{3,(n-2)}$. Knowing this, let us assume that the number of intersection for $K_{3,(n-2)}$ can be expressed using the formulas (6) and (7) above. Then that implies that the number of intersection in $K_{3,n}$ when $n-2 = 2q$ is at least q^2+q and for $n-2 = 2q+1$ is at least $2q^2+2q+1$. Thus induction premise is complete and we show for $m = 3$, theorem 1 is true.

Alternative Explanation We can say that initially there exist two vertices in the set B denoted b_1 and b_2. Now consider adding the third vertex b_3. We can see that the b_3 and its incident edges with b_1 and b_2 gives us the drawing for $K_{3,3}$ as well as the crossing. Since there exist $n-2$ possible choices for b_3, we can find $n-2$ distinct crossings. Deleting vertices b_1 and b_2, the number of crossings is equivalent to $cr(K_{3,n-2)}$ where all possible crossings are different from $n-2$ that had b_1 and b_2 as endpoints. Thus, utilizing the induction hypothesis, $cr(K_{3,n-2}) + (n-2) = cr(K_{3,n})$. Here we are assuming that it is possible to find b_1 and b_2 such that none of their edges intersect. We will comment in more detail on this notion in the later section. But if it was false, then for all b_1 and b_2, there will exist two intersecting edges with b_1 and b_2 as the endpoints. Thus once again we have completed our induction premise and shown that $cr(K_{3,n})$ holds true.

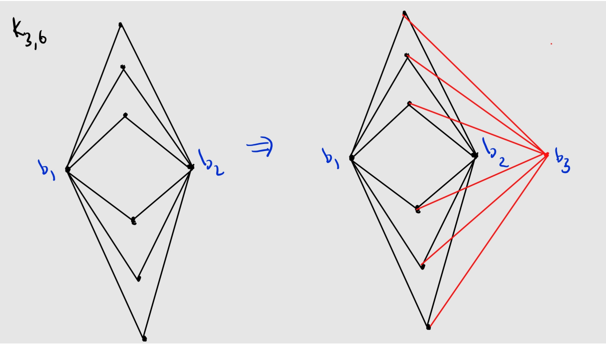

Figure 4: Visual for inductive premise for $K_{3,6}$

In the paper, Zaranckiewicz utilizes the inductive premise and attempts to show that if $K_{m,n}$ holds, then $K_{m,n+1}$, $K_{m+1,n}$ and $K_{m+1,n+1}$ must hold as well. The process is similar but is fallacious as we will discuss further.

3.2 Zarankiewicz Inductive Counterproof and Fallacy

When it comes to finding minimal numbers of crossing, the way it works is by demonstrating that a certain construction cannot have a smaller number of crossings. Here, Zarankiewicz attempts to establish an argument for the inductive premise used in Lemma 2 for the general equation 3.1. He finds that for the case $m = 3$, his inductive argument is true, but this idea does not generalize. There doesn't seem to be another construction that can give a minimum number of crossing nor is there a precise argument that says that the construction provided is as well. But what we can do is investigate the inductive premise and demonstrate that there exist an error in the inductive process.

A possible explanation why is given by K. Guy who points out that the inductive assumption demonstrates a fallacy for the general m n-stars i.e. $K_{1,n}$ of which the graph $K_{m,n}$ may be regarded as composed of, it is always possible to find two which do not contain a crossing. This is not true for all m, n. This is likely a consequence of the induction premise above where for all m, n, we assume that there exist two stars with no crossing. R. Guy remarks 11 years after the initial conjecture that this is not true.

In addition, as separately pointed out by Paul Kainen and Ringel, they were able to demonstrate that his inductive argument works easily from odd to even values but not for even to odd values. In essence, Zaranckiewicz attempts to utilize the inductive premise that goes to argue that his construction minimizes crossings but this is not true and is a fallacious argument. Although we believe that the construction provides the smallest number of crossing for complete bipartite graphs (utilizing rectilinear lines), it still remains a conjecture. If it is possible to show that the conjecture is true, all rectilinear construction and arc constructions would have equal number of minimal crossing.

K. Guy attempts to utilize what's left of the conjecture and conjectures that 3.1 is the upper bound for the minimum number of crossing in $K_{m,n}$. His approach is rather cumbersome and requires the use of a counting argument to yield a recursive lower bound.

4 Alternative Methods and Special Cases

Another proposed method to finding minimizing crossing is to throw away rectilinear construction and allow for arcs that are non-self-intersecting and identify cyclic orderings in the graph. This strategy is similar to the one utilized in the house-and-utilites 1. But this methodology of construction is also not yet proven to be the most optimal and doesn't allow us to generalize an elegant construction for all m and n. Therefore, mathematicians have been more focused on utilizing rectilinear construction thus the special cases will reflect such desires.

Induction similar to Zarankiewicz Case: $(m = n = 5)$ Mathematician Kleitman showed the following to be true for $m = n = 5$. We can proceed similarly to the inductive proof presented above. If for $m = n = 5$, then assume that every pair of 5 stars contains a crossing thus we can do the same thing inductively as shown above. We can also utilize the formula for $cr(K_{5,5}) = 16$ and say that there exist a pair of 5 stars containing only one crossing given the following inequality:

$$2\binom{5}{2} = 20 > 16 \geq cr(K_{5,5})$$

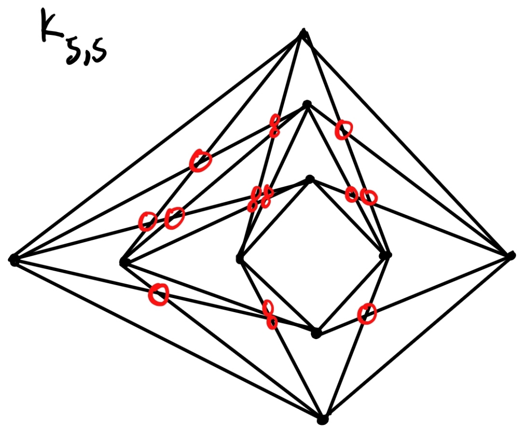

Figure 5: Figure for $K_{5,5}$

Then by considering the area in which $K_{2,5}$, containing just one crossing, divides the plane, and the possible ways to distribute the remaining three vertices in this area. Thus we establish for the case $m = n = 5$ is true. This method is not able to be generalized and the specifics are left to investigate in further papers. But similar methods of induction are used for $m = 5$ and $n = 7$ although the argument is much more tedious.

Special Mentions: Mathematician D.R. Woodall utilized computer aided searches to find for the case $m = n = 7$ and for $m = 7$ and $n = 9$.

Note: All cases that are proven have an isomorphism from $K_{m,n}$ to $K_{n,m}$ by definition.

5 Application

The obvious application for Turan's Brick Problem involving minimizing crossings for bipartite graphs is reducing crossings on factory floors (with less dire circumstances). But a true application for minimizing crossing in a bipartite setting is with VLSI (Very Large Scale Integration).

VLSI is the process of creating IC (Integrated Circuits) by combining billions of transistors into a single chip. On a two-dimensional chip, each crossing results in another layer of routing which results in extra cost to connect certain transistors to be routed to. Thus to create less bulky and cheaper chips, it is best to minimize the crossing for each transistors. If the conjecture was true, then for any routing of transistors has a given upper bound of crossing which can be useful for engineers and companies who want to minimize cost while improving performance via Moore's Law i.e. doubling transistors every 2 years.

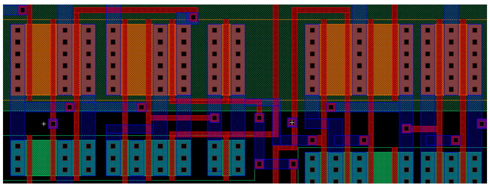

Figure 6: VLSI visual circuitry

Turning out attention back to the origin of the problem, although not as prevalent, it has been found effective to minimize crossings of work station and their pathway on factory floors. For example, in Amazon warehouse facilities, they have been able to improve their efficiency and time by preventing crossing with other workers and having the packing of the items in one workstation while having robots carry over products to their area. This minimizes the interaction between other workers which lead to increases in efficiency. Although more optimal and efficient, it has its downsides.

6 Special Mentions out of Scope for the Project

NP-Hard In 1983, mathematicians Garey and Johnson concluded that the crossing for $K_{m,n}$ NP-hard problem. In simple terms, an NP-hard problem is a type of computational problem that is very challenging to solve efficiently i.e. unable to solve in polynomial time. Although not a CS student by any means, a common NP-hard problem is with puzzles and games e.g. chess or Donkey Kong, which all have complex inputs and outputs that can't be determined as easily. Essentially, it is easy to check if the solution is correct but finding an algorithm or correct solution is extremely difficult and can have a non-deterministic time to complete. The description is true for solving for the minimal crossing for $K_{m,n}$ and K_m as pointed out by Garey and Johnson.

Limit of n In 1997, Richter and Thomassen showed that the following exist:

$$\lim_{n\to\infty} cr(K_{n,n}) \binom{n}{2}^{-2}$$

They determined that the limit when approaching ∞ is at most $1/4$. They also demonstrate that if the Zaranckiewicz's conjecture is true, this value is exactly $1/4$. This is a fascinating result that demonstrates to an extent the reduction of crossings by 4 (if the conjecture is true).

7 Conclusion

The goal to find the construction and a formula for a minimal crossing is still an ongoing open problem that has been enduring for half a century. Similar to other proven conjectures like the 4-color-theorem, more computation heavy solutions have been proposed. As mentioned quickly above, this problem of finding minimizing crossing for

$K_{m,n}$ being an NP-Hard problem signifies that there might not be a solution that can be proposed in a polynomial time. In this project, I was able to identify the inductive procedure for certain values of m and n that have been proven to be true for the Zarankiewicz Conjecture along with demonstrating the fallacy in it as well.

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