



厦门大学《微积分 I-2》课程期中试卷

试卷类型：(理工类 A 卷) 考试日期 2021.4.17

一、求下列各题 (每小题 7 分, 共 21 分) :

1. 设向量 \vec{a} 与 $\vec{b} = 2\vec{i} - \vec{j} + 3\vec{k}$ 平行, 并满足 $\vec{a} \cdot \vec{b} = 28$, 求 \vec{a} ;

解: 由向量 \vec{a} 与 $\vec{b} = 2\vec{i} - \vec{j} + 3\vec{k}$ 平行, 有 $\vec{a} = \lambda \vec{b}$ 。再由 $\vec{a} \cdot \vec{b} = 28$, 知 $\lambda |\vec{b}|^2 = 28$,

又 $|\vec{b}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$, 得到 $\lambda = 2$ 。于是 $\vec{a} = 2\vec{b} = (4, -2, 6)$ 。

2. 已知三角形顶点为 $A(1,1,1)$ 、 $B(2,3,4)$ 、 $C(4,3,2)$, 求此三角形 $\triangle ABC$ 的面积;

$$\text{解: } S_{\triangle ABC} = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = \frac{1}{2} |(-4, 8, -4)|$$

$$= \frac{1}{2} |(-4, 8, -4)| = 2 |(-1, 2, -1)| = 2\sqrt{6}$$

3. 求通过直线 $L: \begin{cases} x+y=0 \\ x-y+z=0 \end{cases}$ 且平行于直线 $x=y=z$ 的平面。

解一: 通过直线 $L: \begin{cases} x+y=0 \\ x-y+z=0 \end{cases}$ 的平面束方程为 $x+y+\lambda(x-y+z)=0$,

即 $(1+\lambda)x + (1-\lambda)y + \lambda z = 0$ 选择 λ 使得此平面平行于已知直线 $x=y=z$, 则

$$(1+\lambda) + (1-\lambda) + \lambda = 0 \text{ 解得 } \lambda = -2, \text{ 所求平面为 } x-3y+2z=0$$

解二: 求直线 $L: \begin{cases} x+y=0 \\ x-y+z=0 \end{cases}$ 上一点, 令 $x=0$, 得到 $y=0, z=0$ 。此直线的方向向量为

$$\vec{s}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = (1, -1, -2), \text{ 直线 } x=y=z \text{ 的方向向量为 } \vec{s}_2 = (1, 1, 1)$$

$$\text{所求平面的法向量为 } \vec{n} = \vec{s}_1 \times \vec{s}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = (1, -3, 2)$$

$$\text{所求平面为 } x-3y+2z=0$$

二、求下列各题（每小题 8 分，共 24 分）：

1. $\iint_D \sqrt{x^2 + y^2} dx dy$, 其中 $D = \{(x, y) | 0 \leq y \leq x, x^2 + y^2 \leq 2x\}$;

解: $\iint_D \sqrt{x^2 + y^2} dx dy = \int_0^{\frac{\pi}{4}} d\theta \int_0^{2\cos\theta} \rho \cdot \rho d\rho = \frac{8}{3} \int_0^{\frac{\pi}{4}} \cos^3 \theta d\theta = \frac{10}{9} \sqrt{2}$

2. $\int_1^2 dx \int_{\sqrt{x}}^x \sin \frac{\pi x}{2y} dy + \int_2^4 dx \int_{\sqrt{x}}^2 \sin \frac{\pi x}{2y} dy$;

解: $\int_1^2 dx \int_{\sqrt{x}}^x \sin \frac{\pi x}{2y} dy + \int_2^4 dx \int_{\sqrt{x}}^2 \sin \frac{\pi x}{2y} dy = \int_1^2 dy \int_y^{y^2} \sin \frac{\pi x}{2y} dx$

$= \int_1^2 \left(-\frac{2y}{\pi} \cos \frac{\pi x}{2y} \right) \Big|_y^{y^2} dy = \int_1^2 \frac{2y}{\pi} (\cos \frac{\pi}{2} - \cos \frac{\pi}{2} y) dy$

$= -\frac{2}{\pi} \int_1^2 y \cos(\frac{\pi}{2} y) dy = -\frac{8}{\pi^3} \int_{\frac{\pi}{2}}^{\pi} t \cos t dt = -\frac{8}{\pi^3} (t \sin t + \cos t) \Big|_{\frac{\pi}{2}}^{\pi} = \frac{8}{\pi^3} + \frac{4}{\pi^2}$

3. 求函数 $u = \frac{\sqrt{6x^2 + 8y^2}}{z}$ 在点 $P(1,1,1)$ 处沿 $\vec{n} = (2,3,1)$ 的方向导数。

解: $\vec{n} = (2,3,1)$ 方向余弦为 $(\cos \alpha, \cos \beta, \cos \gamma) = \frac{1}{\sqrt{14}} (2,3,1)$

函数 $u = \frac{\sqrt{6x^2 + 8y^2}}{z}$ 在点 $P(1,1,1)$ 处的梯度为

$(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) = (\frac{6x}{z\sqrt{6x^2 + 8y^2}}, \frac{8y}{z\sqrt{6x^2 + 8y^2}}, -\frac{\sqrt{6x^2 + 8y^2}}{z^2}) \Big|_{(1,1,1)} = \sqrt{14}(\frac{3}{7}, \frac{4}{7}, -1)$

所求的方向导数为 $\frac{\partial u}{\partial \vec{n}} = \frac{6}{\sqrt{14}} \cdot \frac{2}{\sqrt{14}} + \frac{8}{\sqrt{14}} \cdot \frac{3}{\sqrt{14}} - \frac{1}{\sqrt{14}} \sqrt{14} = \frac{6}{7} + \frac{12}{7} - 1 = \frac{11}{7}$ 。

三、(8 分) 求由曲线 $\begin{cases} 3x^2 + 2y^2 = 12 \\ z = 0 \end{cases}$ 绕 y 轴旋转一周得到的曲面在点 $(0, \sqrt{3}, \sqrt{2})$ 处的切平面方程和法线方程。

解: 曲线 $\begin{cases} 3x^2 + 2y^2 = 12 \\ z = 0 \end{cases}$ 绕 y 轴旋转一周得到的曲面方程为 $3(x^2 + z^2) + 2y^2 = 12$

此曲面在 $(0, \sqrt{3}, \sqrt{2})$ 处的法向量为 $\vec{n} = (6x, 4y, 6z) \Big|_{(0, \sqrt{3}, \sqrt{2})} = 2(0, 2\sqrt{3}, 3\sqrt{2})$

曲面在点 $(0, \sqrt{3}, \sqrt{2})$ 处的切平面方程为 $2\sqrt{3}(y - \sqrt{3}) + 3\sqrt{2}(z - \sqrt{2}) = 0$

即 $2\sqrt{3}y + 3\sqrt{2}z - 12 = 0$

所求的法线方程为 $\frac{x}{0} = \frac{y - \sqrt{3}}{2\sqrt{3}} = \frac{z - \sqrt{2}}{3\sqrt{2}}$, 即 $\begin{cases} 3\sqrt{2}y - 2\sqrt{3}z - \sqrt{6} = 0 \\ x = 0 \end{cases}$

四、(10 分) 设 $u = y, v = \frac{y}{x}$, 试将方程 $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0$ 变换成以 u, v 为自变量的方程, 其中二元函数 z 具有连续的一阶偏导数。

解法一: z 函数可看为复合函数 $z = z(u, v) = z(u(y), v(x, y))$, 则由链式法则有

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = -\frac{y}{x^2} \frac{\partial z}{\partial v},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{1}{x} \frac{\partial z}{\partial v},$$

由 $u = y, v = \frac{y}{x}$, 有 $y = u, x = \frac{u}{v}$,

于是 $\frac{\partial z}{\partial x} = -\frac{v^2}{u} \frac{\partial z}{\partial v}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{v}{u} \frac{\partial z}{\partial v}$

则 $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{u}{v} \left(-\frac{v^2}{u} \frac{\partial z}{\partial v} \right) + u \left(\frac{\partial z}{\partial u} + \frac{v}{u} \frac{\partial z}{\partial v} \right) = u \frac{\partial z}{\partial u},$ 即 $\frac{\partial z}{\partial u} = 0$

解法二: z 函数可看为复合函数 $z = z(u, v) = z(u(y), v(x, y))$, 则由链式法则有

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = -\frac{y}{x^2} \frac{\partial z}{\partial v},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{1}{x} \frac{\partial z}{\partial v},$$

则 $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = x \left(-\frac{y}{x^2} \frac{\partial z}{\partial v} \right) + y \left(\frac{\partial z}{\partial u} + \frac{1}{x} \frac{\partial z}{\partial v} \right) = y \frac{\partial z}{\partial u} = u \frac{\partial z}{\partial u},$ 因此有 $\frac{\partial z}{\partial u} = 0$

五、(10 分) 设方程组 $\begin{cases} e^{\frac{u}{x}} \cos \frac{v}{y} = \frac{x}{\sqrt{2}} \\ e^{\frac{u}{x}} \sin \frac{v}{y} = \frac{y}{\sqrt{2}} \end{cases}$ 确定了函数 $u = u(x, y), \quad v = v(x, y)$ 。求在点 $x = 1,$

$y = 1, \quad u = 0, \quad v = \frac{\pi}{4}$ 处的 du 和 dv 。

解法一: 微分法。将 u, v, x, y 看成独立变量, 对原方程组取全微分得到相应的方程:

$$\begin{cases} (\cos \frac{v}{y} e^{\frac{u}{x}} - \frac{u}{x^2} - \frac{\sqrt{2}}{2}) dx + e^{\frac{u}{x}} \sin \frac{v}{y} \frac{v}{y^2} dy + \cos \frac{v}{y} e^{\frac{u}{x}} \frac{1}{x} du + e^{\frac{u}{x}} \sin \frac{v}{y} \frac{1}{y} dv = 0 \\ \sin \frac{v}{y} e^{\frac{u}{x}} \frac{u}{x^2} dx + (e^{\frac{u}{x}} \cos \frac{v}{y} \cdot \frac{-v}{y^2} - \frac{\sqrt{2}}{2}) dy + \sin \frac{v}{y} e^{\frac{u}{x}} \frac{1}{x} du + e^{\frac{u}{x}} \cos \frac{v}{y} \frac{1}{y} dv = 0 \end{cases}$$

代入 $x=1, y=1, u=0, v=\frac{\pi}{4}$, 得
$$\begin{cases} -\frac{\sqrt{2}}{2} dx + \frac{\sqrt{2}}{2} \frac{\pi}{4} dy + \frac{\sqrt{2}}{2} du - \frac{\sqrt{2}}{2} dv = 0 \\ (-\frac{\sqrt{2}}{2} \cdot \frac{-\pi}{4} - \frac{\sqrt{2}}{2}) dy + \frac{\sqrt{2}}{2} du + \frac{\sqrt{2}}{2} dv = 0 \end{cases}$$

解得 $du = \frac{1}{2}(dx + dy), dv = -\frac{1}{2}dx + (\frac{\pi}{4} + \frac{1}{2})dy$ 。

解法二：通过求偏导数公式来得到微分。

设
$$\begin{cases} F(x, y, u, v) = e^{\frac{u}{x}} \cos \frac{v}{y} - \frac{x}{\sqrt{2}} \\ G(x, y, u, v) = e^{\frac{u}{x}} \sin \frac{v}{y} - \frac{y}{\sqrt{2}} \end{cases}$$
, 在点 $x=1, y=1, u=0, v=\frac{\pi}{4}$ 处:

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F, G)}{\partial(x, v)}}{\frac{\partial(F, G)}{\partial(u, v)}} = -\frac{\begin{vmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{vmatrix}}{\begin{vmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix}} = \frac{1}{2}, \quad \frac{\partial u}{\partial y} = -\frac{\frac{\partial(F, G)}{\partial(y, v)}}{\frac{\partial(F, G)}{\partial(u, v)}} = -\frac{\begin{vmatrix} \frac{\sqrt{2}}{8}\pi & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{8}\pi - \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix}}{\begin{vmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix}} = \frac{1}{2}$$

$$\frac{\partial v}{\partial x} = -\frac{\frac{\partial(F, G)}{\partial(u, x)}}{\frac{\partial(F, G)}{\partial(u, v)}} = -\frac{\begin{vmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{vmatrix}}{\begin{vmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix}} = -\frac{1}{2}, \quad \frac{\partial v}{\partial y} = -\frac{\frac{\partial(F, G)}{\partial(u, y)}}{\frac{\partial(F, G)}{\partial(u, v)}} = -\frac{\begin{vmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{8}\pi \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{8}\pi - \frac{\sqrt{2}}{2} \end{vmatrix}}{\begin{vmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix}} = \frac{\pi}{4} + \frac{1}{2}$$

从而 $du = \frac{1}{2}(dx + dy), dv = -\frac{1}{2}dx + (\frac{\pi}{4} + \frac{1}{2})dy$ 。

解法三：通过方程两边求导数得到导数后再求微分

方程组
$$\begin{cases} e^{\frac{u}{x}} \cos \frac{v}{y} = \frac{x}{\sqrt{2}} \\ e^{\frac{u}{x}} \sin \frac{v}{y} = \frac{y}{\sqrt{2}} \end{cases}$$
 两边对 x 求导数, 可得

$$\begin{cases} e^{\frac{u}{x}} \cdot \frac{x \frac{\partial u}{\partial x} - u}{x^2} \cdot \cos \frac{v}{y} - e^{\frac{u}{x}} \sin \frac{v}{y} \cdot \frac{1}{y} \frac{\partial v}{\partial x} = \frac{1}{\sqrt{2}} \\ e^{\frac{u}{x}} \cdot \frac{x \frac{\partial u}{\partial x} - u}{x^2} \cdot \sin \frac{v}{y} + e^{\frac{u}{x}} \cos \frac{v}{y} \cdot \frac{1}{y} \frac{\partial v}{\partial x} = 0 \end{cases}$$

代入 $x=1$, $y=1$, $u=0$, $v=\frac{\pi}{4}$, 得

$$\begin{cases} \frac{1}{\sqrt{2}} \frac{\partial u}{\partial x} - \frac{1}{\sqrt{2}} \frac{\partial v}{\partial x} = \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \frac{\partial u}{\partial x} + \frac{1}{\sqrt{2}} \frac{\partial v}{\partial x} = 0 \end{cases} \text{ 或 } \begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 1 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0 \end{cases}$$

解得 $\frac{\partial u}{\partial x} = \frac{1}{2}$, $\frac{\partial v}{\partial x} = -\frac{1}{2}$.

方程组 $\begin{cases} e^{\frac{u}{x}} \cos \frac{v}{y} = \frac{x}{\sqrt{2}} \\ e^{\frac{u}{x}} \sin \frac{v}{y} = \frac{y}{\sqrt{2}} \end{cases}$ 两边对 y 求导数, 可得

$$\begin{cases} e^{\frac{u}{x}} \cdot \frac{1}{x} \cdot \frac{\partial u}{\partial y} \cos \frac{v}{y} - e^{\frac{u}{x}} \sin \frac{v}{y} \cdot \frac{y \frac{\partial v}{\partial y} - v}{y^2} = 0 \\ e^{\frac{u}{x}} \cdot \frac{1}{x} \cdot \frac{\partial u}{\partial y} \cdot \sin \frac{v}{y} + e^{\frac{u}{x}} \cos \frac{v}{y} \cdot \frac{y \frac{\partial v}{\partial y} - v}{y^2} = \frac{1}{\sqrt{2}} \end{cases}$$

代入 $x=1$, $y=1$, $u=0$, $v=\frac{\pi}{4}$, 得

$$\begin{cases} \frac{1}{\sqrt{2}} \frac{\partial u}{\partial y} - \frac{1}{\sqrt{2}} \left(\frac{\partial v}{\partial y} - \frac{\pi}{4} \right) = 0 \\ \frac{1}{\sqrt{2}} \frac{\partial u}{\partial y} + \frac{1}{\sqrt{2}} \left(\frac{\partial v}{\partial y} - \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}} \end{cases} \text{ 或 } \begin{cases} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -\frac{\pi}{4} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 1 + \frac{\pi}{4} \end{cases}$$

解得 $\frac{\partial u}{\partial y} = \frac{1}{2}$, $\frac{\partial v}{\partial y} = \frac{1}{2} \left(1 + \frac{\pi}{2} \right)$.

故 $du = \frac{1}{2}(dx + dy)$, $dv = -\frac{1}{2}dx + \left(\frac{\pi}{4} + \frac{1}{2} \right)dy$.

解法四: 由 $\begin{cases} e^{\frac{u}{x}} \cos \frac{v}{y} = \frac{x}{\sqrt{2}} \\ e^{\frac{u}{x}} \sin \frac{v}{y} = \frac{y}{\sqrt{2}} \end{cases}$ 可得 $\begin{cases} e^{\frac{2u}{x}} = \frac{1}{2}(x^2 + y^2) \\ \tan \frac{v}{y} = \frac{y}{x} \end{cases}$,

由 $F(x, y, u) = e^{\frac{2u}{x}} - \frac{1}{2}(x^2 + y^2)$ 可得

$$\frac{\partial u}{\partial x} = -\frac{F_x}{F_u} = -\frac{(-\frac{2u}{x^2})e^{\frac{2u}{x}} - x}{\frac{2}{x}e^{\frac{2u}{x}}} = \frac{2ue^{\frac{2u}{x}} + x^3}{2xe^{\frac{2u}{x}}},$$

$$\frac{\partial u}{\partial y} = -\frac{F_y}{F_u} = -\frac{-y}{\frac{2}{x}e^{\frac{2u}{x}}} = \frac{xy}{2e^{\frac{2u}{x}}}$$

代入 $x=1, y=1, u=0, v=\frac{\pi}{4}$, 得 $\frac{\partial u}{\partial x} = \frac{1}{2}, \frac{\partial u}{\partial y} = \frac{1}{2}$.

由 $G(x, y, v) = \tan \frac{v}{y} - \frac{y}{x}$ 可得

$$\frac{\partial v}{\partial x} = -\frac{G_x}{G_v} = -\frac{-\frac{y}{x^2}}{\frac{1}{y}\sec^2 \frac{v}{y}} = -\frac{y^2}{x^2} \cos^2 \frac{v}{y},$$

$$\frac{\partial v}{\partial y} = -\frac{G_y}{G_v} = -\frac{\sec^2 \frac{v}{y} \cdot (-\frac{v}{y^2}) - \frac{1}{x}}{\frac{1}{y}\sec^2 \frac{v}{y}} = \frac{v}{y} + \frac{y}{x} \cos^2 \frac{v}{y}$$

代入 $x=1, y=1, u=0, v=\frac{\pi}{4}$, 得 $\frac{\partial v}{\partial x} = -\frac{1}{2}, \frac{\partial v}{\partial y} = \frac{\pi}{4} + \frac{1}{2} = \frac{\pi+2}{4}$.

故 $du = \frac{1}{2}(dx + dy), dv = -\frac{1}{2}dx + (\frac{\pi}{4} + \frac{1}{2})dy$.

六、(12 分) 求二元函数 $z = f(x, y) = x^2 y(4 - x - y)$ 在由直线 $x + y = 6$, x 轴和 y 轴所围成的有界闭区域 D 上的极值与最值。

解: 先求区域 D 内部的极值

令 $\begin{cases} f_x = 2xy(4 - x - y) - x^2 y = 0 \\ f_y = x^2(4 - x - y) - x^2 y = 0 \end{cases}$, 解得唯一内部驻点 $(2, 1)$ 。

$$A = f_{xx}(2, 1) = (8y - 6xy - 2y^2)|_{(2,1)} = -6, B = f_{xy}(2, 1) = (8x - 3x^2 - 4xy)|_{(2,1)} = -4,$$

$$C = f_{yy}(2, 1) = -2x^2|_{(2,1)} = -8, \text{ 知 } A < 0, AC - B^2 > 0, \text{ 因此 } f(x, y) \text{ 在 } (2, 1) \text{ 取得极大值 } 4.$$

当 $x=0(0 \leq y \leq 6)$ 和 $y=0(0 \leq x \leq 6)$ 上 $f(x, y) = 0$ 。由边界方程 $x + y = 6$ 解出 $y = 6 - x$, 代入

$f(x, y)$ 中得 $z = 2x^3 - 12x^2$ ($0 \leq x \leq 6$)，令 $\frac{dz}{dx} = 6x^2 - 24x = 0$ ，解得 $x = 4$ ，即 D 边界上点 $(4, 2)$ 。

比较下列函数值 ($0 \leq x \leq 6, 0 \leq y \leq 6$)：

$$f(2, 1) = 4, f(x, 0) = 0, f(0, y) = 0, f(4, 2) = -64,$$

由此知 $f(x, y)$ 在 D 上最大值为 $f(2, 1) = 4$ ，最小值为 $f(4, 2) = -64$ 。

七、(10 分) 试问函数 $f(x, y) = \begin{cases} xy \sin \frac{1}{xy}, & x \cdot y \neq 0 \\ 0, & x \cdot y = 0 \end{cases}$ ，在点 $(0, 0)$ 处是否可微？请说明理由。

解： $f_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = 0$ ，同理 $f_y(0, 0) = 0$ 。故在点 $O(0, 0)$ 处一阶偏导数存在。

$$\text{又由于 } \frac{\Delta f - f_x(0, 0)\Delta x - f_y(0, 0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \begin{cases} \frac{\Delta x \Delta y \sin \frac{1}{\Delta x \Delta y}}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}, & \Delta x \Delta y \neq 0, \\ 0, & \Delta x \Delta y = 0 \end{cases}$$

于是，

$$\left| \frac{\Delta f - f_x(0, 0)\Delta x - f_y(0, 0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right| \leq \frac{|\Delta x \Delta y|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \leq \frac{1}{2} \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

因此当 $(\Delta x, \Delta y) \rightarrow (0, 0)$ 时，由 $\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \sqrt{(\Delta x)^2 + (\Delta y)^2} = 0$ 可得

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta f - f_x(0, 0)\Delta x - f_y(0, 0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

因此在点 $(0, 0)$ 处 $f(x, y)$ 可微。

八、(5 分) 设 $f(x)$ 、 $g(x)$ 为 $[a, b]$ 上的连续函数。利用二重积分证明以下的 Cauchy-Schwartz

不等式： $(\int_a^b f(x)g(x)dx)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx$ 。

证明：设 $D = [a, b] \times [a, b]$ ，则

$$(\int_a^b f(x)g(x)dx)^2 = \int_a^b f(x)g(x)dx \int_a^b f(y)g(y)dy = \iint_D f(x)g(x)f(y)g(y)dxdy,$$

$$\int_a^b f^2(x)dx \int_a^b g^2(x)dx = \int_a^b f^2(x)dx \int_a^b g^2(y)dy = \iint_D f^2(x)g^2(y)dxdy,$$

$$\int_a^b f^2(x)dx \int_a^b g^2(x)dx = \int_a^b f^2(y)dy \int_a^b g^2(x)dx = \iint_D f^2(y)g^2(x)dxdy.$$

又因为

$$2 \int_a^b \iint_D f(x)g(x)f(y)g(y)dxdy \leq \iint_D [f^2(x)g^2(y) + f^2(y)g^2(x)]dxdy$$

所以有 $(\int_a^b f(x)g(x)dx)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx$ ，得证。