MA442 MIDTERM EXAM 2

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Problem 1 (Cutoff Functions)**.** My solutions to parts (a)-(e) follow.

(a) Let $c(x) = e^{-\frac{1}{x}}$ for $x > 0$ and 0 for $x \le 0$ which we know is of class $C^\infty(\mathbb{R})$ from homework problem Munkres 16.1. Define $B(x) = c(x)c(1-x)$ as the usual bump function which is a product of C^∞ functions, thus B is also of class $C^\infty(\mathbb{R})$. Since then B is Riemann integrable, let $M = \int_{\mathbb{R}} B$ for which $M \ge 0$ and finite by boundedness of B. It should be noted that this and all further integrals over $\mathbb{R}$ are proper since B can be made to have compact support without alteration to the integral value. Finally, let $\widehat{B}(x) = \frac{1}{M}B(x)$ such that $\int_{\mathbb{R}} \widehat{B} = \frac{1}{M}\int_{\mathbb{R}} B = 1$, which is trivially C^∞ as well. Then we can define

$$h_\epsilon(x) = \begin{cases} 0, & x \le 0 \\ \frac{1}{\epsilon}\int_0^x \widehat{B}\left(\frac{y}{\epsilon}\right) dy, & x \in (0,\epsilon) \\ 1, & x \ge \epsilon \end{cases}$$

which satisfies the desired properties. h_ϵ being of class $C^\infty(\mathbb{R})$ follows trivially from the fact that the mapping $x \mapsto \int_0^x f$ is continuous given f is continuous over $[0,x]$.

(b) Here, K need not be connected, but can instead consist of countably-many connected, compact sets K_i such that $\bigcup_i K_i = K$. For each i, we conduct the following. Let $P_i = \{K_{i1},\ldots,K_{ip}\}$ be the projections of K_i onto the p coordinate axes where each K_{ij} is compact and connected. Further, let $0 < \delta = \frac{1}{4}\min\{d(K_i,K_j) \mid i,j \in \mathbb{N}, i \ne j, K_i, K_j \text{ exist}\}$ be a quarter of the smallest distance between every pair-wise combination of K_i subsets. To ensure we stay within Ω, let $\epsilon = \min\{\delta, a\}$ where we have $a = \frac{1}{4}\min\{d(K_i,\partial\Omega) \mid i \in \mathbb{N}, K_i \text{ exists}\}$. We know $\epsilon > 0$ since each K_i and K_j are disconnected for $i \ne j$ and Ω is open in $\mathbb{R}^p$. Then for a given P_i, we have $4\epsilon \le d(K_{ij},K_{ik})$ for each $j,k \le p$ where $j \ne k$. For each $K_{ij} = [x_1,x_2]$, define $\ell_{ij} = c((x_1-\epsilon, x_2+\epsilon)) = (x_2-x_1)+2\epsilon$ and

$$D_{ij}(x) = \frac{1}{\ell_{ij}} B\left(\frac{x-(x_1-\epsilon)}{\ell_{ij}}\right)$$

where B is the bump function from part (a). Thus D_{ij} is a bump function with $\operatorname{supp} D_{ij} = (x_1-\epsilon, x_2+\epsilon)$ and is therefore of class $C^\infty(\mathbb{R})$. It's also important to note that the closure $\overline{\operatorname{supp} D_{ij}}$ is compact here. We now define

$$D_i(x) = \bigtimes_{j=1}^{p} D_{ij}(x \cdot \mathbf{e}_j) \quad \text{and} \quad \Psi_i = D_i\big|_{K_i}$$

where D_i is a cartesian product of C^∞ functions, and thus is C^∞, and nonnegative over the cartesian product of each $\operatorname{supp} D_{ij}$, which contains K_i. Thus D_i and it's restriction Ψ_i are each nonnegative over K_i, where Ψ_i is also C^∞ by connectedness of K_i. Define $\Psi(x) = \sum_i \Psi_i(x)$ where $\Psi\colon \Omega \to \mathbb{R}$ has $\operatorname{supp}\Psi = \bigcup_i \operatorname{supp} D_i$ where each $\operatorname{supp} D_i \subset \Omega$ is an open cell. Then we have constructed Ψ such that $\Psi > 0$ in each K_i and thus K in general, and $\Psi(x) = 0$ for $x \notin \overline{\operatorname{supp}\Psi}$ which is compact and strictly contained in Ω.

(c) Consider Ψ from part (b). Since each $K_{ij} \subset \operatorname{supp} D_{ij}$ and D_{ij} is strictly positive over its open support, we know there exists an $\epsilon_{ij} > 0$ satisfying $D_{ij}(x) \ge \epsilon_{ij}$ for $x \in K_{ij}$. Let $\epsilon = \min_{ij}\epsilon_{ij}$ such that $\Psi(x) \ge \epsilon$ for $x \in K$. Then from part (a), we can reconstruct some $\widehat{\Psi}(x) = h_\epsilon \circ \Psi(x)$ such that $\widehat{\Psi}(x) = 1$ for $x \in K$. Moreover, since $h_\epsilon\colon \mathbb{R} \to [0,1]$, we have that $\widehat{\Psi}(x) \in [0,1]$ for each $x \in \Omega$.

(d) Consider any $\Omega \subset \mathbb{R}^p$ open. First, Ω need not be connected, so consider its constituent open, connected subsets Ω_i such that $\bigcup_i \Omega_i = \Omega$. For each Ω_i, consider the exhaustion by compacts

$\{K_n^i\}_{n=1}^\infty$ where $K_n^i = \{x \in \Omega_i \mid d(x, \partial\Omega_i) \geq \frac{1}{n}\}$ is compact and connected, and we have that $K_1^i \Subset \cdots \Subset K_n^i \Subset \cdots$ with $\Omega_i = \bigcup_{n=1}^\infty K_n^i$. Let $\widehat{\Psi}_n^i$ be defined as in part (c) such that $\widehat{\Psi}_n^i(x) = 1$ for $x \in K_n^i$ and $\operatorname{supp} \widehat{\Psi}_n^i \subset \Omega_i$. We claim that $\{\widehat{\Psi}_n^i\}_{n=1}^\infty$ is point-wise nondecreasing since each $K_n^i \Subset K_{n+1}^i \implies \operatorname{supp} \widehat{\Psi}_n^i \subset \operatorname{supp} \widehat{\Psi}_{n+1}^i$. Then for $\widehat{\Psi}_n = \sum_i \widehat{\Psi}_n^i$, we have that $\{\widehat{\Psi}_n\}_{n=1}^\infty$ is point-wise nondecreasing over Ω with $\operatorname{supp} \widehat{\Psi}_n \subset \Omega$ by disjointness of each $\operatorname{supp} \widehat{\Psi}_n^i$. Cool. Now consider any $K \Subset \Omega$. Again, such Ω and thus K may be disconnected, so consider the constituent $K_i \Subset \Omega_i$ such that $K = \bigcup_i K_i$. Each Ω_i is open and $K_i \Subset \Omega_i \implies \exists \epsilon_i > 0$ such that $\epsilon_i = \frac{1}{2} \min\{d(x, \partial\Omega_i) \mid x \in K_i\}$. Then for $\epsilon = \min_i \epsilon_i$, we know $\epsilon > 0$ and can actually take $\epsilon' = \min\{\epsilon, 1\}$ such that $0 < \epsilon' \leq 1$. Then we know there exists some $N_K \in \mathbb{N}$ such that $0 < \frac{1}{N_K} \leq \epsilon'$ and thus $K_i \subset K_{N_K}^i \Subset \Omega_i$ for each i. Therefore, for $n \geq N_K$, we have that each $\widehat{\Psi}_n^i(x) = 1$ and thus $\widehat{\Psi}_n(x) = 1$ for $x \in K$.

(e) Consider some $f \colon \Omega \to \mathbb{R}$ for which (improper) $\int_\Omega f$ exists. Consider the sequence $\{f\widehat{\Psi}_n\}_{n=1}^\infty$ which is point-wise monotonically nondecreasing and bounded above by f. We know that $\operatorname{supp} \widehat{\Psi}_n \subset \Omega$ for each n and thus by homework problem Munkres 16.1, if we extend $\widehat{\Psi}_n(x) = 0$ for $x \in \mathbb{R}^p \setminus \operatorname{supp} \widehat{\Psi}_n$, we know that $\widehat{\Psi}_n$ maintains its continuity over $\mathbb{R}^p$. Then the product of two integrable functions $f\widehat{\Psi}_n$ is also integrable over Ω. Moreover, since $\bigcup_{n=1}^\infty K_n = \Omega$, we have that $\widehat{\Psi}(x) = \lim_{n\to\infty} \widehat{\Psi}_n(x) = 1$ for $x \in \Omega$. Thus $\lim_{n\to\infty} f\widehat{\Psi}_n = f \lim_{n\to\infty} \widehat{\Psi}_n = f\widehat{\Psi}$ which is identically f over Ω. Since f is also assumed to be (improperly) integrable over Ω, we can apply the Monotone Convergence Theorem. Then by Munkres Thm. 15.2, a Prop. given in lecture, and MCT, we know that

$$\text{(improper)} \int_\Omega f = \text{(extended)} \int_\Omega f = \lim_{n\to\infty} \text{(extended)} \int_\Omega f\widehat{\Psi}_n = \lim_{n\to\infty} \text{(improper)} \int_\Omega f\widehat{\Psi}_n$$

exist where $\{K_n\}_{n=1}^\infty$ is the exhaustion by compacts from part (d).

Problem 2 (Sard's Theorem)**.** My solutions to parts (a) and (b) follow.

(a) First, $g \in C^1(\Omega)$ and $\overline{\Omega_1} \subset \Omega$ compact implies that $M = \sup_{\overline{\Omega_1}} \|Dg\|$ is finite. By Bartle Lemma 45.1, we have then since $Q \subset \Omega_1 \Subset \Omega$ is a cube with side length $h > 0$ and $\|Dg\| \le M$ over Ω_1, then if $g(Q)$ is covered by N cubes $\widetilde{K}_i$ such that $g(Q) \subset \bigcup_{i=1}^N \widetilde{K}_i$, we have that

$$c(g(Q)) \le \sum_{i=1}^{N} c(\widetilde{K}_i) \le (M\sqrt{p})^p c(Q).$$

by finite subadditivity of Jordan content. We proceed with an auxillary claim.

Claim 2.1. $x_0 \in Q \cap \Sigma \implies Dg(x_0)(\mathbb{R}^p) \subset \Pi_0$ where Π_0 is at most a $(p-1)$-dimensional linear subspace of $\mathbb{R}^p$.

Proof. $\det[Dg(x_0)] = 0 \implies \operatorname{rank}[Dg(x_0)] = k \le p-1$ such that $Dg(x_0)\colon \mathbb{R}^p \to V$ where $V \le \mathbb{R}^p$ (meaning V is a subspace of $\mathbb{R}^p$) with $\dim V = k$. Since $g \in C^1(\Omega)$, then $[Dg(x_0)]$ is continuous, well-defined, and linear since it is given by a matrix representation. $\square$

From here one we assume WLOG that $\dim \Pi_0 = p-1$ since if not, it would still be embedded in the same hyperplane.

Claim 2.2. Any point $x_1 \in g(Q)$ is at most distance $\epsilon h\sqrt{p}$ from the hyperplane $g(x_0) + \Pi_0$.

Proof. From the Approximation Lemma, we have that since Ω is open and $g \in C^1(\Omega)$, then for $x_0 \in Q \cap \Sigma$ and any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, \Omega_1) > 0$ such that $x_1 \in \overline{B_\delta(x_0)}$ implies that $\|g(x_1) - g(x_0) - Dg(x_0)(x_1 - x_0)\| \le \epsilon \|x_1 - x_0\|$ and $x_1 \in \Omega$. In fact, we since g is C^1 over $\Omega \supset Q \supset Q^{\mathrm{o}}$ is open in $\mathbb{R}^p$, we can alter this result such that

$$x_1 \in \overline{B_\delta(x_0)} \cap Q^{\mathrm{o}} \implies \|g(x_1) - (g(x_0) + Dg(x_0)(x_1 - x_0))\| \le \epsilon \|x_1 - x_0\| \le \epsilon h\sqrt{p} \qquad (\star)$$

and $x_1 \in Q^{\mathrm{o}}$. The final inequality comes from the fact that since both $x_0, x_1 \in Q$, their distance from each other is maximally the p-cube's diagonal, or $h\sqrt{p}$. Of course, $Dg(x_0)(x) \in \Pi_0$ for $x \in \mathbb{R}^p$, so the first part of the inequality just represents *at least* the distance between $g(x_1)$ and the affine hyperplane $g(x_0) + \Pi_0$ for any given $x_1 \in Q$, and thus for $g(Q)$ in general. $\square$

We now know that all of $g(Q)$ has bounded distance from the affine hyperplane $\Pi_0 + g(x_0)$ where $[Dg(x_0)]$ degenerates. We now provide a geometric bound for this distance which will allow us to rigorize the content of such $g(Q)$ later.

Claim 2.3. $g(Q)$ is contained fully within a cylinder of height $2\epsilon h\sqrt{p}$ and base radius $Mh\sqrt{p}$.

Proof. Equation $(\star)$ from above is equivalent to saying that the vector $v(x_1) = g(x_1) - g(x_0)$ is at most distance $\epsilon h\sqrt{p}$ from affine hyperplane Π_0. Consider the orthogonal decomposition of $v(x_1)$ such that

$$v(x_1) = \operatorname{proj}_{\Pi_0} v(x_1) + \operatorname{proj}_{\Pi_0^\perp} v(x_1) = v_\parallel(x_1) + v_\perp(x_1)$$

where $\Pi_0^\perp$ is the orthogonal complement of Π_0. First, we know that for $x_1 \in Q$, $\|v(x_1)\| \le \epsilon h\sqrt{p}$ implies that $\|v_\perp(x_1)\| \le \epsilon h\sqrt{p}$ as well since $\|v_\parallel(x_1)\|$ is at least 0. Then $-\epsilon h\sqrt{p} \le v_\perp(x_1) \le \epsilon h\sqrt{p}$, which dictates that the height of our local cylinder is at most $2\epsilon h\sqrt{p}$. We also know that since both $x_1, x_0 \in Q$, then $\|x_1 - x_0\| \le h\sqrt{p}$ implies that $\|g(x_1) - g(x_0)\| = \|v(x_1)\| \le M\|x_1 - x_0\| \le Mh\sqrt{p}$ from the Mean Value Theorem. This is similar to the arguments given in the proof of Bartle Lemme 45.1. Thus we can bound $\|v_\parallel(x_1)\| \le \|v(x_1)\| \le Mh\sqrt{p}$ by property of projections. This gives us a bound for the radius of our cylinder (since we can allow $v_\parallel$ to point in any direction within the plane Π_0) as $Mh\sqrt{p}$. Thus $g(Q)$ is countained fully in a cylinder of height at most $2\epsilon h\sqrt{p}$ and base radius at most $Mh\sqrt{p}$ around $g(x_0)$. $\square$

Let U be the set of the cylinder as defined in Claim 2.3. We now approximate the content of the cylinder stated above as $c(U) = k_p l r^p$ where l is the height, r is the base radius, and k_p is some constant dependent on p. Then by property of Jordan content, we have that $g(Q) \subset U \implies c(g(Q)) \le c(U)$. Using the bounds of our cylinder, we get that

$$c(g(Q)) \le k_p(Mh\sqrt{p})^p 2\epsilon h\sqrt{p} = (2k_p\epsilon h\sqrt{p})(M\sqrt{p})^p c(Q) = \epsilon'(M\sqrt{p})^p c(Q)$$

since $h^p = c(Q)$ and ϵ is arbitrary, so we can assign the group of constants $2k_p\epsilon h\sqrt{p}$ as some new arbitrarily small ϵ'. The upper bound for $c(g(Q))$ that we've attained here differs slightly (though negligably) from the one given in the problem statement. We claim that either inequality will be suitable to arrive at the result of part (b). In any case, we will use the one we've proved here.

(b) Consider any $K \Subset \Omega$ and the set $\Sigma = \{x \in \Omega \mid \det[Dg(x)] = 0\}$. Since K is compactly contained, we know that it is bounded, and thus $\overline{K} \subset \Omega$ is compact. WLOG assume that $K \cap \Sigma$ is nonempty since otherwise $g(K \cap \Sigma)$ trivially has content zero by Bartle Thm. 45.2.

Claim 2.4. The set $K \cap \Sigma$ is compact.

Proof. $K \cap \Sigma \subset K \subset \overline{K} \implies K \cap \Sigma$ is bounded since $\overline{K}$ is compact. Moreover, $g \in C^1(\Omega)$ implies that Dg is continuous over Ω as discussed in Claim 2.1. Moreover, since det is a polynomial in the entires of the matrix, it is smooth. So the composition $\det \circ Dg$ is also continuous. By assumption that $K \cap \Sigma$ is nonempty, we know there exists $x \in K$ such that $\det[Dg(x)] = 0$. Since $\det[Dg(x)]$ is continuous, by preservation of compactness, we must have that the set $\{x \in K \cap \Sigma\} \mapsto \{0\}$ is also compact. □

Then we can always find a finite open subcover $\{Q_i\}_{i=1}^N$ of $K \cap \Sigma$ such that $K \cap \Sigma \subset \bigcup_{i=1}^N Q_i$. WLOG assume each Q_i is a cube of side length $h_i < \delta$ from part (a). As each h_i shrinks, we can just continue to partition each Q_i into 2^p sub-cubes and remove the ones that do not intersect with $K \cap \Sigma$. This ensures that, no matter how small h_i becomes, we will always have a finite open subcover of $K \cap \Sigma$. We do not care if such Q_i intersect with other Q_j. From part (a),

$$c(g(Q_i)) \leq \epsilon(M\sqrt{p})^p c(Q_i) \implies \sum_{i=1}^N c(g(Q_i)) \leq \sum_{i=1}^N \epsilon(M\sqrt{p})^p c(Q_i)$$

for each $i \in [1, N]$ since each p-cube has side lengths bounded above by δ.

Claim 2.5. g is injective over $K \setminus \Sigma$.

Proof. $x \in K \setminus \Sigma \implies \operatorname{rank}[Dg(x)] = p$ so Dg is injective at x. Then by the Injective Mapping Theorem, there exists an $\eta > 0$ such that f restricted to $K \cap B_\eta(x)$ is an injection. We can apply this to every $x \in K \setminus \Sigma$ such that these $B_\eta(x)$ will cover $K \setminus \Sigma$. Thus f restricted to $K \setminus \Sigma$ is injective. □

Claim 2.6. $c(g(Q)) \leq \sum_{i=1}^N c(g(Q_i))$.

Proof. This follows from subadditivity of Jordan content, since

$$c(g(Q)) = c\left(g\left(\bigcup_{i=1}^N Q_i\right)\right) = c\left(\bigcup_{i=1}^N g(Q_i)\right) \leq \sum_{i=1}^N c(g(Q_i)).$$

□

Claim 2.7. $c(g(K \cap \Sigma)) \leq c(g(Q))$.

Proof. If $Q = K \cap \Sigma$, then $g(Q) = g(K \cap \Sigma) \implies c(g(Q)) = c(g(K \cap \Sigma))$. Otherwise, each Q_i is allowed to cover portions of $K \setminus \Sigma$ as well, so we may also have $K \cap \Sigma \subset Q$. Thus $Q \cap (K \setminus \Sigma)$ is nonempty, and g is injective over this part of K, so we have

$$c(g(Q)) = c(g(K \cap \Sigma) \cup g(Q \setminus (K \cap \Sigma))) = c(g(K \cap \Sigma)) + c(g(Q \setminus (K \cap \Sigma)) \geq c(g(K \cap \Sigma))$$

which follows from $g(K \cap \Sigma)$ being disjoint from $g(Q \setminus (K \cap \Sigma))$ as a result of g injective. □

Finally, we have the results to show that

$$c(g(K \cap \Sigma)) \leq c(g(Q)) \leq \sum_{i=1}^N c(g(Q_i)) \leq \epsilon(M\sqrt{p})^p \sum_{i=1}^N c(Q_i)$$

and thus for any arbitrarily small choice of $\epsilon > 0$, we can bound $c(g(K \cap \Sigma))$, which will converge from above to content zero by nonnegativity of Jordan content. Naturally, the sum $\sum_i c(Q_i)$ will converge to $c(K \cap \Sigma)$ which is finite given Q is compact.

Problem 3 (Additive Functions)**.** My solutions to parts (a)-(f) follow.

(a) f continuous implies that f is Riemann integrable. Since $A \in \mathcal{D}(\Omega)$ has content, $F(A) = \int_A f$ exists and is well-defined. For $A, B \in \mathcal{D}$ with $A \cap B = \emptyset$, we have that $\int_{A\cup B} f = \int_A f + \int_B f$ by Bartle Thm. 44.9, and thus $F(A \cup B) = F(A) + F(B)$ is additive. Choose any $A \subset K \Subset \Omega$ with $A \in \mathcal{D}(\Omega)$ and $\epsilon > 0$ such that $|F(A)| < \epsilon$. Since f is continuous and compactly supported over K, we know f is bounded by some $M > 0$. By Bartle Thm. 44.10, we know that $|F(A)| = |\int_A f| \le Mc(A)$. Then choose $\delta = \epsilon/M$ such that if $c(A) < \delta$, we have that $|F(A)| \le Mc(A) < M\delta = \epsilon$. Thus F is absolutely continuous. Finally, consider any $K \Subset \Omega$, $Q \subset \Omega$ closed cube of side length h, and any $\epsilon > 0$. Since f is continuous and compactly supported over K and Q, it's uniformly contiuous over K and Q, and thus for $\epsilon > 0$ there's some $\delta = \min\{\delta_K, \delta_Q\}$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Choose the cube side length $h < \delta/\sqrt{p}$ which bounds the diameter of Q by δ. Then all such $x, y \in Q$ satisfy $|f(x) - f(y)| < \epsilon$ and thus for $x \in K \cap Q$, we have

$$\left|\frac{F(Q)}{c(Q)} - f(x)\right| = \left|\frac{1}{c(Q)}\int_Q (f(y) - f(x))dy\right| \le \frac{1}{c(Q)}\int_Q |f(y) - f(x)|dy < \epsilon.$$

(b) Let G have a strong density g. Then for any $\epsilon > 0$ and $K \Subset \Omega$ there exists a $\delta_x = \delta(\epsilon, K, x) > 0$ such that if $Q \subset \Omega$ is a closed cube of side length $0 < 2h < \delta_x$, then for $x \in K \cap Q$, we have $|g(x) - G(Q)/c(Q)| < \epsilon$. We know $\overline{Q} \subset \Omega$, so if x_0 is the center of Q, we have that $B_h(x_0) \subset \overline{Q} \subset \Omega$. Pick any $y \in B_h(x_0)$ such that we have $|x - y| < h < \delta_x$. WLOG, choose K such that K covers Q completely. Then since both x, y satisfy the strong density condition of $x, y \in K \cap Q$, we have

$$|g(x) - (y)| \le \left|g(x) - \frac{G(Q)}{c(Q)}\right| + \left|\frac{G(Q)}{c(Q)} - g(y)\right| < 2\epsilon = \epsilon'$$

from the triangle inequality. Thus we have achieved continuity of g over Ω since, by openness of Ω, for any $x \in \Omega$ we can always construct an appropriate K and Q with δ_x that satisfy this condition. For example, if $B_\eta(x) \subset \Omega$, we simply just inscribe Q in this ball and can choose $K = Q$. Since δ depends on x and K, this is *not* uniform continuity over Ω.

(c) Choose any $Q \subset \Omega$ cube of side length h and set $K = Q$. We know Q has content since it's a cube. We start by partitioning Q into $P_\delta = \{Q_i\}_{i=1}^N$ where each Q_i is *closed* and has equal side lengths δ. Since each $Q_i \subset Q$ and is accordingly bounded, they have Jordan content. Then we start by claiming that

$$\begin{aligned} G(Q) &= G\left(\bigcup_{i=1}^N G_i\right) = G\left(\bigcup_{i=1}^N {G_i}^\circ \cup \bigcup_{i=1}^N \partial G_i\right) = G\left(\bigcup_{i=1}^N {G_i}^\circ\right) + G\left(\bigcup_{i=1}^N \partial G_i\right) \\ &= G\left(\bigcup_{i=1}^N {Q_i}^\circ\right) = \sum_{i=1}^N G({Q_i}^\circ). \end{aligned}$$

The first step comes from our construction of the partition P_δ. The next follows from the disjointness of each Q_i's boundary and interior. This disjointness allows us to use the additivity property of G since each subset of G_i has Jordan content. We then have that G of the union of the boundaries vanishes as a result of absolute continuity of G. In other words, since $Q = K$ is compactly contained, then no matter what $\epsilon > 0$ we choose, $c(\bigcup_{i=1}^N \partial Q_i)$ will always have content zero (since each G_i has content, the boundary has content zero) and thus from absolute continuity, we know that $|G(\bigcup_{i=1}^N \partial Q_i)| < \epsilon$ will also vanish. The last equality of our claim comes from the fact that every ${G_i}^\circ$ are disjoint and have content, and thus we apply additivity. Now we apply the stong density g of G to bound this sum. For any $\epsilon > 0$ and $K = Q \subset \Omega$, there exists a $\delta > 0$ such that Q having side length less than δ implies that $|G(Q)|/c(Q) < \epsilon$ since g is identically zero over Ω and content is nonnegative. Choose now any $\epsilon > 0$. Then for $\delta(\epsilon) > 0$ from the strong density, we create the partition $P_{\delta(\epsilon)}$ of Q such that each $Q_i \subset Q$ satisfies the side length requirement by $\delta(\epsilon)$. Then we have that for each $Q_i \in P_{\delta(\epsilon)}$,

$$\frac{|G(Q_i)|}{c(Q_i)} < \epsilon \iff |G(Q_i)| < \epsilon c(Q_i) \implies |G({Q_i}^{\circ})| < \epsilon c(Q_i),$$

where the last implication follows from the same interior/boundary decomposition and application of absolute continuity as before. Also note that, by property of the Jordan content, we have $c({Q_i}^{\circ}) = c(Q_i)$. We now apply this bound such that

$$|G(Q)| = \left|\sum_{i=1}^{N} G({Q_i}^{\circ})\right| \leq \sum_{i=1}^{N} |G({Q_i}^{\circ})| < \sum_{i=1}^{N} \epsilon c(Q_i) = \epsilon c(Q),$$

where we've used triangle inequality, the bound recently attained, and the finite additivity of Jordan content. Thus for any $\epsilon > 0$, we can bound $|G(Q)|$ by a constant multiple of ϵ, which will vanish. Thus $|G(Q)| = 0 \implies G(Q) = 0$.

(d) Let G be as in part (c) and take any $A \in \mathcal{D}(\Omega)$. Then necessarily $A \subset \Omega$ and is bounded since it has content (Bartle Defn. 44.2). We assume first that A° is nonempty since otherwise $A = \partial A$ which trivially has content zero, and thus by our absolute continuity argument from before $G(A)$ is also zero. Using this same logic, we further ignore ∂A. Consider now an approximation of A° from within using a set of cubes P_δ as defined in part (c) such that $Q_i \in P_\delta \implies Q_i \subset A^{\circ}$ and $c(A) - c(P_\delta)$ approaches zero as ϵ approaches zero. We start by choosing such $\epsilon > 0$ as in part (c) and decrement from there. Then for each $Q_i \in P_\delta$, we have $|G({Q_i}^{\circ})| < \epsilon c(Q_i)$ and

$$|G(A)| = |G(A \setminus P_\delta) + G(P_\delta)| < |G(A \setminus P_\delta)| + \epsilon c(A)$$

where the $|G(A \setminus P_\delta)|$ term vanished, leaving us which a $|G(A)|$ that vanishes as $\epsilon \to 0$.

(e) Suppose $G\colon \mathcal{D}(\Omega) \to \mathbb{R}$ has strong density $f\colon \Omega \to \mathbb{R}$ which is continuous over Ω, and that G is additive and absolutely continuous, where $G \neq F$ from part (a). Then if we construct $H\colon \mathcal{D}(\Omega) \to \mathbb{R}$ where $H(A) = G(A) - F(A)$, it's obvious that H is all of additive, absolutely continuous, and has strong density given by $f - f = 0$ over Ω. By part (d), for any $A \in \mathcal{D}(\Omega)$, we then have that $H(A) = 0 \implies G(A) = F(A)$ over Ω, contradiction.

(f) f is Riemann integrable over Ω by continuity. Take $A, B \in \mathcal{D}(\Omega)$ disjoint. Then by injectivity of ϕ over Ω, we get $\phi(A) \cap \phi(B) = \emptyset$ as well. Thus we have

$$F_\phi(A \cup B) = \int_{\phi(A \cup B)} f = \int_{\phi(A)} f + \int_{\phi(B)} f = F_\phi(A) + F_\phi(B)$$

by Bartle Thm. 44.9(a), so F_ϕ is additive. Consider now any $A \subset K \Subset \Omega$ and $\epsilon > 0$ such that $c(A) < \delta = \delta_{\epsilon,K}$. We know that since K compact $\implies$ $\phi(K)$ is compact, and f cont., so is bounded over $\phi(K)$, say by M. By Bartle Thm. 44.10, we have that $|\int_{\phi(A)} f| \leq Mc(\phi(A))$. Since $A \subset K$ compact, we have $\overline{A} \subset K$ where $\overline{A}$ is compact. Thus we can cover $\overline{A}$ with finitely-many open cubes $\{K_i\}_{i=1}^{N}$ where WLOG each cube K_i is centered at some $x_i \in A$. Then by The Jacobian Theorem, we have that $c(\phi(K_i)) \leq W(x_i, \phi, \epsilon, p)c(K_i)$ for $W(x_i, \phi, \epsilon, p) \in \mathbb{R}$. Since these cover $\overline{A}$ and thus A, we recover some W' such that $c(\phi(A)) \leq W'c(A)$. Hence for any $\epsilon > 0$, we can choose $\delta = \epsilon/MW'$ such that $|F_\phi(A)| \leq Mc(\phi(A)) \leq MW'c(A) < \epsilon$ for $c(A) < \delta$. Thus F_ϕ is absolutely continuous over Ω. Let $\widetilde{f} = f \circ \phi |J_\phi|$ be our supposed strong density function. Since Q is a closed cube, we can apply the Jacobian Theorem directly now, providing now some $E(x_i, \phi, \epsilon, p) \in \mathbb{R}$ such that $c(\phi(Q)) \leq E(x_i, \phi, \epsilon, p)c(Q)$. Then for Q with side lengths bounded by δ, we have that ??

By part (e), the only F that can satisfy these requirements is given by $\int_A \widetilde{f} = \int_A f \circ \phi |J_\phi|$ for any $A \in \mathcal{D}(\Omega)$.