MA442 FINAL EXAM

RYAN WANS

Problem 1 (Mappings Between $\mathbb{R}^n \to \mathbb{R}^m, n \neq m$)**.** My solutions to parts (a)-(d) follow.

(a) Suppose for the sake of contradiction that f is injective. Then by continuity of f, connectedness is preserved, so by connectedness of $\mathbb{R}^2$, we know $f(\mathbb{R}^2)$ is connected, and thus an interval in $\mathbb{R}$. By injectivity, $f(\mathbb{R}^2)$ is dense in $\mathbb{R}$, and thus has nonempty interior by connectedness. Choose any $y \in f(\mathbb{R}^2)^{\circ}$ with respective $x = f^{-1}(y) \in \mathbb{R}^2$. Uniqueness gaurunteed by injectivity.

Claim 1.1. $\mathbb{R}^2 \setminus \{x\}$ is connected.

Proof. Take any distinct $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{R}^2 \setminus \{x\}$. They are path connected by the path $(a_1, a_2) \to (a_1, b_2) \to (b_1, b_2)$. If x lies anywhere in this path, take instead the path given by $(a_1, a_2) \to (b_1, a_2) \to (b_1, b_2)$. The two path's only common points are a and b themselves, which are by definition not equal to x. Thus one can always be taken, so $\mathbb{R}^2 \setminus \{x\}$ is path connected, and thus also connected in general. $\square$

By the injectivity assumption, $f(\mathbb{R}^2 \setminus \{x\}) = f(\mathbb{R}^2) \setminus \{y\}$, and since y was assumed to be an interior point of the image interval, f is mapping a connected set (by Claim 1.1) to a disconnected set (two intervals), which is a contradicition, so f cannot be injective.

(b) Consider now any general $f \colon \mathbb{R}^n \to \mathbb{R}^m$ of class C^1 where $m < n$. Then $\operatorname{rank}[Df(x)] = r \leq m < n$ for any $x \in \mathbb{R}^n$. We invoke the Rank Theorem. Suppose for sake of contradiction that f is injective and take any $x \in \mathbb{R}^n$ with respective local neighborhood $x \in V \subset \mathbb{R}^n$. By Rank Theorem (iii), we know that there exists C^1 diffeomorphisms $\sigma \colon V \to V'$ and $\tau \colon Z \to Z'$ in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, such that $f(x) = \tau^{-1} \circ g(x)$ where $g(x) = \iota_r \circ \sigma(x)$, and

$$\iota_r(y_1, \ldots, y_r, y_{r+1}, \ldots, y_n) = (y_1, \ldots, y_r, 0, \ldots, 0) \in \mathbb{R}^m.$$

Since f is injective over $\mathbb{R}^n$ and thus $V \subset \mathbb{R}^n$, we have that $f(x_1) = f(x_2) \implies x_1 = x_2$ for $x_1, x_2 \in V$. Since τ and thus τ^{-1} are diffeomorphic, they are injective, and thus

$$f(x_1) = f(x_2) \iff \tau^{-1} \circ g(x_1) = \tau^{-1} \circ g(x_2) \iff g(x_1) = g(x_2)$$

such that g must also be injective over V. Then since σ^{-1} is also a diffeomorphism, $g = \iota_r \circ \sigma \iff \iota_r = g \circ \sigma^{-1}$ composition of injective functions, so ι_r itself is injective over $\sigma(V)$. This of course is a contradtiction since, for example,

$$\iota_r(y_1, \ldots, y_r, y_{r+1}, \ldots, y_n) = \iota_r(y_1, \ldots, y_r, y_{r+1} + \epsilon, \ldots, y_n + \epsilon) = (y_1, \ldots, y_r, 0, \ldots, 0).$$

where $\epsilon > 0$ is chosen such that both points are distinct and lie in $\sigma(V)$, which can be done since $\sigma(V)$ is open by property of diffeomorphisms and V open in $\mathbb{R}^n$. Thus it must be that f cannot be injective over V. Since we can apply this to any $x \in \mathbb{R}^n$ with the existence of open $V \subset \mathbb{R}^n$ guarunteed by the Rank Theorem, f cannot be injective over all of $\mathbb{R}^n$.

(c) Take any $f \colon \mathbb{R} \to \mathbb{R}^2$ to be of class C^1. We construct a new function $g \colon \Omega \to \mathbb{R}^2$ such that $\Omega = \mathbb{R}^2$ is open and thus $g(x, y) = f(x)$ for any $(x, y) \in \mathbb{R}^2$. Since we have that

$$[Dg] = \begin{bmatrix} D_1 g_1 & D_2 g_1 \\ D_1 g_2 & D_2 g_2 \end{bmatrix} = \begin{bmatrix} D_1 f_1 & 0 \\ D_1 f_2 & 0 \end{bmatrix}$$

and each of $D_1 f_1, D_1 f_2$ is continuous since f is $C^1(\mathbb{R})$, then g is also $C^1(\Omega)$. Then the critical set $\Sigma = \{x \in \Omega \mid \det[Dg(x)] = 0\} = \Omega$ since $\det[Dg]$ is trivially zero. In other words, the entire domain of g is the critical set of g. By Sard's Theorem, for *any* $K \subset \mathbb{R}^2$ compact, we have that $g(K \cap \Sigma) = g(K)$ has Jordan content zero and $g(\mathbb{R}^2) = f(\mathbb{R})$ has measure zero. Accordingly, for any $K \subset \mathbb{R}^2$ compact and K_x it's projection onto the x-axis, $g(K) = f(K_x)$ has Jordan content zero. Now to the main results of the problem.

Claim 1.2. $f(\mathbb{R})$ contains no open sets in $\mathbb{R}^2$.

Proof. Suppose $A \subset f(\mathbb{R}) \subset \mathbb{R}^2$ is open. Then for any $x \in A$, there exists some $\delta > 0$ such that $B_\delta(x) \subset A$ and we can even take an inscribed closed cube $Q_\epsilon(x) \subset B_\delta(x)$ of side length $\epsilon = \frac{\delta}{4} > 0$ centered at x. Of course $Q_\epsilon(x)$ is Jordan measurable (2-cell) with nonzero Jordan content of $\epsilon^2 > 0$. But $Q_\epsilon(x) \subset A$ means that $f^{-1}(Q_\epsilon(x)) \subset f^{-1}(A)$ where f^{-1} means the set of preimages/fiber, not the inverse. Since f is C^1, by preservation of compactness, then $f^{-1}(Q_\epsilon(x))$ was originally compact. But we know that if $f^{-1}(Q_\epsilon(x))$ is compact, then $f \circ f^{-1}(Q_\epsilon(x))$ has Jordan content zero by Sard, which is a contradiction. $\square$

From Claim 1.2, we know that $f(\mathbb{R})$ has empty interior. Alas, $\mathbb{R}^2$ has a very much nonempty interior, so $f(\mathbb{R})$ cannot be $\mathbb{R}^2$, and the f cannot be surjective. Another approach is to consider any open set in $\mathbb{R}^2$, where we've shown there cannot exist a suitable preimage in $\mathbb{R}$ such that f maps to this open set. Thus f cannot be surjective.

(d) Consider now any general C^1 function $f\colon \mathbb{R}^m \to \mathbb{R}^n$ with $m < n$. We construct again our function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g(x_1, \ldots, x_m, \ldots, x_n) = f(x_1, \ldots, x_m)$ with $\Omega = \mathbb{R}^n$. From our previous analysis, we easily observe that $\Sigma = \{x \in \Omega \mid \det[Dg(x)] = 0\} = \mathbb{R}^n$ since $\operatorname{rank}[Dg(x)] = \operatorname{rank}[Df(x)] \leq m < n$ for $x \in \mathbb{R}^n$. Moreover, g is also C^1 by continuity of it's derivative as a function of Df. By Sard's Theorem, for any $K \subset \mathbb{R}^n$ compact, we know that $g(K) = f(K_x)$ has Jordan content zero for K_x the projection of K into $\mathbb{R}^m$. Claim 1.2 is easily modified such that if $A \subset \mathbb{R}^n$ is an open image under $f(\mathbb{R}^m)$, one can construct a closed n-cell $Q_\epsilon(x) \subset A$ of content $\epsilon^n > 0$ for $x \in A$, contradicting our result from Sard since the preimage of $Q_\epsilon(x)$ is compact in $\mathbb{R}^n$ and thus $\mathbb{R}^m$. Finally, we use this to say that $f(\mathbb{R}^m)^{\circ}$ is empty, so for any $A \subset \mathbb{R}^n$ open, there does not exist a preimage $A' \subset \mathbb{R}^m$ such that $A' \mapsto A$ under f, so it is not surjective.

Problem 2 (Centroids)**.** My solutions to parts (a)-(d) follow.

(a) Let $S \subset \mathbb{R}^n$ have nonzero content $v(S) > 0$, which we've previously shown to mean that S^{o} is nonempty. Suppose that $h(S) = S$ where h is as given in the problem, that is, S is symmetric over $x_k = 0$ for some k. Since S has content, $h : \mathbb{R}^n \to \mathbb{R}^n$ is bijective and of class C^∞ with $\det Dh = -1 \neq 0$, and $x_k : \mathbb{R}^n \to \mathbb{R}$ is continuous, we apply Weak Change of Variables

$$c_k(S) = c_k(h(S)) = \frac{1}{v(h(S))}\int_{h(S)} x_k = \frac{1}{v(h(S))}\int_S (x_k \circ h)|-1| = \frac{1}{v(h(S))}\int_S -x_k = -c_k(S)$$

This condition is only satisfied if $c_K(S) = 0$. Since S^{o} is nonempty, we know that the set $S_k = \{x_k \mid (x_1, \ldots, x_k, \ldots, x_n) \in S\}$ is also nonempty and has nonzero $(n-1)$-dimensional content. Since the action of h is just a sign flip, we conclude that $v(h(S)) = v(S)$.

(b) Let $L = Ax + B$ be an invertible affine transformation over $\mathbb{R}^n$. By linearity, L is of class C^∞ and bijective. Moreover, invertibility implies A must have full rank, and thus $DL = A$ is also full rank and thus bijective with $\det DL \neq 0$. We again invoke Weak Change of Variables such that

$$c(LS) = \frac{1}{v(LS)}\int_{LS} x = \frac{1}{v(LS)}\int_S (x \circ L)|J_L| = \frac{1}{v(LS)}\int_S L|J_L|$$

where $v(LS) = v(AS + B) = v(AS) = |J_A|v(S)$ by translation invariance of Jordan content and transformation of S under linear mapping A. Thus we get

$$c(LS) = \frac{1}{|J_A|v(S)}\int_S L|J_A| = \frac{1}{v(S)}\int_S L = B + \frac{A}{v(S)}\int_S x = Lc(S)$$

by linearity of the integral.

(c) Define the parameterization $\phi(x, y) = (1-y)(x, 0) + yp$ such that $\phi : \overline{A} \times [0, 1] \to S$ and is injective over the domain $\overline{A} \times [0, 1)$, a restriction of ϕ to $y \neq 1$. We thus claim that ϕ is injective over the interior of the domain of ϕ. We recognize that our Jacobian J_ϕ has the form

$$J_\phi = \begin{bmatrix} (1-y)I_{n-1} & \mathbf{p} \\ \mathbf{0} & p_n \end{bmatrix}$$

where $\mathbf{p}$ is just the first $(n-1)$ coordinates of p. By assumption, $p_n > 0$ and over the interior of the domain of y, so is $(1-y)$. Thus by it's upper triangular form, we conclude that $\det J_\phi$ is nonzero, and thus diffeomorphic over the interior of the domain of ϕ. So ϕ is injective and smooth over this domain. We now express $v(S)$ in terms of $v(A)$ to show that it has content which is nonzero. By differentiability of ϕ and it's injectivity, we can invoke Strong Change of Variables since the set of x with $J_\phi(x) = 0$ has content zero in $\mathbb{R}^n$. Thus we get that

$$v(S) = \int_{\mathbb{R}^n} \mathbb{1}_S = \int_{\phi(A,[0,1])} 1 = \int_A (1 \circ \phi)|J_\phi| = |J_\phi|\int_A 1 = |J_\phi|v(A)$$

where of course $v(A) > 0$ by assumption and $|J_\phi| > 0$ by nondegeneracy. Hence the centroid must be defined. In fact, we can go further to say that since $|J_\phi| = p_n(1-y)^{n-1}$, and $\int_{[0,1]}(1-y)^{n-1}dy = \frac{1}{n}$, then we get $v(S) = \frac{p_n}{n}v(A)$.

(d) We start with

$$c(S) = \frac{1}{v(S)}\int_S x dx$$

and use our parameterization from part (c) to split this integral into

$$\begin{aligned} c(S) &= \int_{[0,1]}\int_A (1-y)(a, 0)y^{n-1} da dy + \int_{[0,1]}\int_A ypy^{n-1} da dy \\ &= \left(\int_{[0,1]}(1-y)y^{n-1}dy\right)\left(\int_A (a, 0)da\right) + \left(\int_{[0,1]} y^n dy\right)\left(\int_A p da\right) \end{aligned}$$

where we can do some computations to get that $\int_{[0,1]}(1-y)y^{n-1}dy = \frac{1}{n(n+1)}$ and $\int_{[0,1]} y^n dy = \frac{1}{n+1}$. Now we do some algebra (running out of time) and get $c(S) = \frac{1}{n+1}(c(A), 0) + \frac{n}{n+1}p$.

Ran out of time due to 5 other finals :(

Problem 3 (Orientable Manifolds)**.** My solutions to parts (a)-(c) follow.

(a) Let M be any n-dimensional manifold in $\mathbb{R}^n$ of class C^r and consider any arbitrary atlas $\mathcal{A}$ of M with charts $\alpha_i : U_i \to V_i$. By definition (Munkres), each $\operatorname{rank} D\alpha_i(x) = n$ for $x \in U_i$ and, by homeomorphic property of α_i, each α_i^{-1} has similar rank for $y \in \alpha_i(U_i)$. By a trivial linear algebra result, we know then that each $\det D\alpha_i$ and $\det D\alpha_i^{-1}$ do not vanish over U_i and $\alpha(U_i)$, respectively. We now construct our new atlas $\mathcal{A}'$ as follows. Consider any arbitrary chart $\alpha_0 \in \mathcal{A}$; this will serve as our "reference chart", or "reference orientation" for M. Take now any α_i for which $V_0 \cap V_i$ is nonempty and consider the corresponding transition map $\alpha_0^{-1} \circ \alpha_i$. By chain rule, we know that

$$D(\alpha_0^{-1} \circ \alpha_i) = D(\alpha_0^{-1}(\alpha_i)) \cdot D\alpha_i \implies \det D(\alpha_0^{-1} \circ \alpha_i) = \det D(\alpha_0^{-1}(\alpha_i)) \cdot \det D\alpha_i =: \Pi_i$$

each of which we've shown above is nonsingular, so Π_i has sign. If $\Pi_i > 0$, great, we append α_i into our new $\mathcal{A}'$ and can move on to another α that intersects one of α_0 or α_i to repeat the algorithm. If $\Pi_i < 0$, we can alter α_i as follows. Let $\widetilde{\alpha_i} = \alpha_i \circ R$ be our new chart which is precomposed with some reflection R. By property of orthogonal matrices and, specifically, reflections, we know $\det R = -1$ and R is smooth, so $\det D\widetilde{\alpha_i} = \det(D\alpha_i(R)) \cdot \det DR = -\det D\alpha_i$ by property of determinants, chain rule, and the derivative of a linear function. So thus $\det D(\alpha_0^{-1} \circ \widetilde{\alpha_i}) = \widetilde{\Pi}_i > 0$, so our new chart preserves the chosen orientation of α_0, and we append $\widetilde{\alpha}_i$ into our new $\mathcal{A}'$. We now wrap up by showing that such an R will always exist when needed.

Claim 3.1. For any $W = V_i \cap V_j \subset M$, an appropriate reflection R can be found.

Proof. Consider first when $W \cap \partial M$ is empty. Then W lies fully within the interior of M, in which case *any reflection R will suffice.* Consider $\alpha : U \to V$. Then $\widetilde{\alpha}$, will no longer send $U \mapsto V$, but instead $R^{-1}U \mapsto V$. In other words, we are free to alter the domain of α so long as we preserve its image V in M. Thus, $\widetilde{\alpha}(R^{-1}U) = \alpha \circ R \circ R^{-1}U = \alpha \circ U = V$ with the additional structure that we have flipped the sign of the Jacobian's determinant.

Now consider if $W \cap \partial M$ is nonempty. Then for our $\alpha : U \to V$, we have that U is relatively open in $\mathbb{H}^n$. We can then trivially take R to be the reflection across the axis along $\mathbf{e}_n$, where $R : \mathbb{H}^n \to \mathbb{H}^n$ and $\partial\mathbb{H}^n \mapsto \partial\mathbb{H}^n$ with preservation of how U interacts (intersects) with $\mathbb{R}^{n-1} \times \{0\}$. By the same properties as before, such $\widetilde{\alpha}$ will preserve the image V while alternating the sign of the Jacobian's determinant. $\square$

With this, we simply iterate over every pairwise intersecting charts such that $\mathcal{A}'$ has the same number of charts as $\mathcal{A}$, but such that for every $\alpha_i, \alpha_i \in \mathcal{A}'$, we get $\det D(\alpha_i^{-1} \circ \alpha_j) > 0$. It's somewhat important that we start with charts that intersect with α_0 and then only move "one intersection away" such that the next iteration has intersection with a chart from the previous iteration. Thus we have constructed an atlas $\mathcal{A}'$ that makes M orientable.

(b) Let M be now a k-dimensional manifolds in $\mathbb{R}^n$ of class C^r which is orientable and whose boundary is nonempty. By orientability, we can take an atlas $\mathcal{A}$ of M that satisfies the orientable condition. Now consider any $p \in \partial M$ such that WLOG there are two $\alpha_1 : U_1 \to V_1$ and $\alpha_2 : U_2 \to V_2$ such that $p \in V_1 \cap V_2 = W$. By proposition in class, we know that, in both U_1 and U_2, we have $\alpha_1^{-1}(p) \in U_1 \cap \partial\mathbb{H}^k$ and $\alpha_2^{-1}(p) \in U_2 \cap \partial\mathbb{H}^k$. Consider the transition map $\tau : \alpha_2^{-1} \circ \alpha_1$ which is a homeomorphism of class C^r. By the orientability of M, we know $\det D\tau > 0$. Then if $W_1 = \alpha_1^{-1}(W)$ and $W_2 = \alpha_2^{-1}(W)$, then we have that

$$W_1 \cap \partial\mathbb{H}^k \xrightarrow{\alpha_1} W \cap \partial M \xrightarrow{\alpha_2^{-1}} W_2 \cap \partial\mathbb{H}^k$$

is the action of τ. Thus if we consider the restrictions of α_1 to $W_1 \cap \partial\mathbb{H}^k$ and α_2 to $W_2 \cap \partial\mathbb{H}^k$, say β_1 and β_2, then $\beta_2^{-1} \circ \beta_1 = \tau|_{W_1 \cap \partial\mathbb{H}^k}$ is a transition map of ∂M around p. Since τ is C^r and $\det D\tau > 0$, then it's restriction τ' to the $(k-1)$-dimensional subspace $\partial\mathbb{H}^k \cong \mathbb{R}^{k-1}$ shares the same determinant sign. This comes from the fact this our restriction τ' is nothing but the $(k-1) \times (k-1)$ principle minor of $D\tau$, which since $\det D\tau > 0$, all principle minor determinant will also be positive. Hence, τ' is a transition map over ∂M which inherents the same positive Jacobian determinant property as it's parent τ. Thus, using our atlas $\mathcal{A}$ of orientable M, we've induced the orientability of ∂M through the transition maps over M.

(c) Select any point $p = (a, b) \in M \subset \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ on the level set M. By assumption, we know that $Dg(p)$ has full rank of $n - k$ at any such p. This allows us to define the bijection

$$L_2(v)\colon \{0\}^k \times \mathbb{R}^{n-k} \cong \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$$

such that $L_2(v) = Dg(p)(0, v)$ where Dg is of class C^{r-1} by $g \in C^r$. From the Implicit Function Theorem (Bartle 41.9), we know then that there exists an open $W \subset \mathbb{R}^k$, $a \in W$ and unique function $\phi\colon W \to \mathbb{R}^{n-k}$ of class $C^r(W)$ such that $b = \phi(a)$ and $g(x, \phi(x)) = 0$ for $x \in W$. Thus locally, the level set $g^{-1}(0)$ is the graph of ϕ in W. Also from the theorem, we get that there exists an open $V \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ containing p that satisfies $g(x, \phi(x)) = 0$ if $x \in W$ and $(x, \phi(x)) \in V$. Then we can construct the homeomorphic chart

$$\alpha(x) = (x, \phi(x)), \qquad \alpha\colon W \to V \ni p$$

where, since V is more generally open in $\mathbb{R}^n$, we have that V is relatively open in M. Also, α is a Cartesian product of a C^∞ function and C^r function, so is of class C^r. Let $\mathcal{A}$ be the atlas of all such charts α which allow us to define M as an $(n-k)$-dimensional manifold of class C^r. Let $\psi = (\alpha^{-1}, g)\colon V \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ where $\alpha^{-1} : V \to W$ by simply taking $\alpha^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$ and $g : V \to \mathbb{R}^{n-k}$ as defined before. Then ϕ is trivially of class C^r by differentiability of its constituents. Since $\psi(x, y) = (x, g(x, y))$, we have

$$D\psi(x, y) = \begin{bmatrix} I_k & \mathbf{0} \\ * & Dg_y(x, y) \end{bmatrix}$$

where, since $Dg_y(x, y)$ was initially assumed to have full rank, ψ has full rank. The first principle minor block trivially has determinant 1, and thus since $D\psi$ is lower triangular, $\det D\psi$ is fully determined by $\det Dg_y(x, y)$, which is constant from the definition of g. Thus if the sign of $D\psi$ is not to our liking (negative) we can trivially flip it using some reflection R as in the techniques from part (b).

Ran out of time due to 5 other finals :(