Spectral Theory and Geometry

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The Laplacian

Definition (Laplacian)

The Laplacian $\Delta\colon C^\infty(\Omega) \to C^\infty(\Omega)$ is the differential operator

$$\Delta u = \nabla \cdot \nabla u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

- Invariant under translation and rotation

Motivation

View Δ as a matrix acting on a vector space and diagonlize this operator. Specifically, we're interested in L^2 (Hilbert space with inner product).

The Helmholtz Equation on S^1

The eigenvalue problem of Δ is the Helmholtz equation: $\boxed{-\Delta u = \lambda u}$

- Domain $\Omega = S^1 = \mathbb{R}/\mathbb{Z}$
- Functions $u \in L^2(S^1) \cong L^2_{\text{per}}(\mathbb{R})$ are 1-periodic
- Assume that $u'' \in L^2(S^1)$

Example ($\Omega = S^1$)

$$-\Delta u(x) = -u''(x) = \lambda u(x)$$

has characteristic equation $1r^2 + 0r + \lambda = 0$. For the case of $\lambda < 0$

$$u(x) = c_1 e^{\sqrt{|\lambda|}x} + c_2 e^{-\sqrt{|\lambda|}x}$$

which are not periodic!

On The Circle S^1

Example (contd.)

Hence we assume that $\lambda > 0$, and our solutions

$$u(x) = c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x}$$

for some $c_1, c_2 \in \mathbb{C}$. For u to be 1-periodic, we must have $\lambda = 4\pi^2 k^2$. Hence our solutions $-\Delta u_k(x) = 4\pi^2 k^2 u_k(x)$ form eigenspaces

$$\begin{aligned} \lambda_0 = 0 &: \ \text{span}\left\{1\right\}, \\ \lambda_k = 4\pi^2 k^2 &: \ \text{span}\left\{e^{2\pi ikx}, e^{-2\pi ikx}\right\} = \text{span}\left\{\cos(2\pi kx), \sin(2\pi kx)\right\}. \end{aligned}$$

Our spectrum is given by $\text{spec}_{S^1}(-\Delta) = \{4\pi^2 k^2 \mid k \in \mathbb{Z}\}$

- We've decomposed a periodic function on the basis of complex exponentials (Fourier series!)

On The ℓ-Length Circle S^1_ℓ

- Domain $\Omega = S^1_\ell = \mathbb{R}/\ell\mathbb{Z}$
- Functions $u \in L^2(S^1_\ell)$ are ℓ-periodic
- Apply reparameterization $y = \ell x$

Theorem (Laplacian on S^1_ℓ)

The eigenvalues have the form $\mathrm{spec}_{S^1_\ell}(-\Delta) = \{4\pi^2 k^2/\ell^2 \mid k \in \mathbb{Z}\}$ with

$$\lambda_0 = 0 : \ \mathrm{span}\,\{1\}\,,$$
$$\lambda_k = 4\pi^2 k^2/\ell^2 : \ \mathrm{span}\left\{e^{2\pi i k x/\ell}, e^{-2\pi i k x/\ell}\right\}.$$

Weyl's Law in Dimension 1

- How do these eigenvalues distribute themselves?
- Eigenvalue counting function: $N(\mu) = \#\{\lambda \in \mathrm{spec}(-\Delta) \mid \lambda \leq \mu\}$ including multiplicity

Theorem (Weyl's Law on S^1_ℓ)

For an ℓ-length circle S^1_ℓ,

$$N(\mu) \sim \mu^{1/2}\frac{\ell}{\pi} = c_d \mu^{d/2}\,\mathrm{vol}(S^1_\ell) \quad \text{as } \mu \to \infty$$

Inverse Problem: We can "hear the perimeter of a circle"!

Weyl's Law in Dimension 1

Proof. (Kinda easy!)

For $\Omega = S^1_\ell$ such that $\mathrm{vol}(\Omega) = \ell$. Then for fixed μ,

$$\left(\frac{2\pi}{\ell}\right)^2 k^2 \leq \mu \iff k^2 \leq \mu\left(\frac{\ell}{2\pi}\right)^2 \iff k \leq \left\lfloor \mu^{1/2}\frac{\ell}{2\pi}\right\rfloor.$$

Every λ has multiplicity 2 except 0, of which there is 1, so

$$N(\mu) = 1 + 2\left\lfloor \mu^{1/2}\frac{\ell}{2\pi}\right\rfloor \sim \mu^{1/2}\frac{\ell}{\pi} \quad \text{as } \mu \to \infty$$

$\square$

Extending to The Torus $\mathbb{T}^2$

- Apply separation of variables and use the case of S^1
- Domain $\Omega = \mathbb{T}^2 = S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$
- Functions $u \in L^2(\mathbb{T}^2)$
- Freeze x_2 such that $[(S^1 \times S^1) \to \mathbb{T}^2] \to (S^1 \to [S^1 \to \mathbb{T}^2])$ and decompose on the basis of complex exponentials again

Take single-variable Fourier transform

$$\mathcal{F}_{x_1}[u(x_1, x_2)] = \sum_{k \in \mathbb{Z}} c_k(x_2) e^{2\pi i k x_1}$$

And plug into eigenvalue problem

$$\begin{aligned} -\Delta \mathcal{F}_{x_1}[u] &= \sum_{k \in \mathbb{Z}} (c_k(x_2) 4\pi^2 k^2 - c_k''(x_2)) e^{2\pi i k x_1} \\ &= \lambda \mathcal{F}_{x_1}[u] = \lambda \sum_{k \in \mathbb{Z}} c_k(x_2) e^{2\pi i k x_1} \end{aligned}$$

Extending to The Torus $\mathbb{T}^2$

$$c_k''(x_2) = c_k(x_2)(4\pi^2 k^2 - \lambda) \quad \implies \quad -\Delta c_k(x_2) = \mu c_k(x_2)$$

- Another 1D Laplacian eigenvalue problem with $\mu = \lambda - 4\pi^2 k^2$
- One can show that each $c_k(x_2) \in L^2(S^1)$
- Since $c_k(x_2)$ are eigenfunctions of $-\Delta$ on S^1, then $\mu \in \text{spec}_{S^1}(-\Delta)$
- Each μ takes form $\mu = 4\pi^2 n^2$ and solutions take form $e^{2\pi i(kx_1 + nx_2)}$

Theorem (Laplacian on $\mathbb{T}^2$)

The eigenvalues have the form $\text{spec}_{\mathbb{T}^2}(-\Delta) = \{4\pi^2 \|m\|^2 \mid m \in \mathbb{Z}^2\}$ for $\|m\|^2 = m_1^2 + m_2^2$ and hence our (nonzero) eigenspace is

$$\lambda = 4\pi^2 \|m\|^2 : \text{span}\left\{e^{2\pi i\langle m', x\rangle} \mid m' \in \mathbb{Z}^2, \|m'\| = \|m\|\right\}.$$

Extending to The Torus $\mathbb{T}^2$

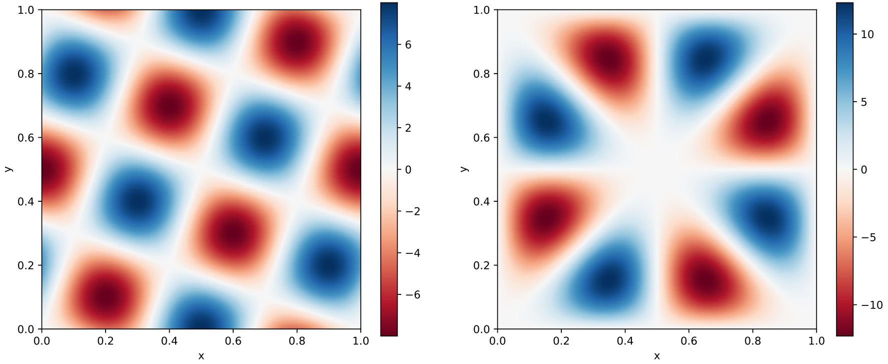


Figure: Some eigenfunctions on $\mathbb{T}^2$ shown in $\mathbb{R}^2$

Weyl's Law on $\mathbb{T}^2$

- Gauss circle problem: no formula for the multiplicty of arbitrary λ

$$N(\mu) = \#\left\{ z \in \mathbb{Z}^2 \mid \|z\| \leq \frac{\sqrt{\mu}}{2\pi} \right\}$$

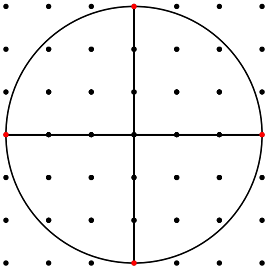


Figure: Multiplicity of $\lambda = 4\pi^2 9$ is 4

Weyl's Law on $\mathbb{T}^2$

Theorem (Weyl's Law on $\mathbb{T}^2$)

$$N(\mu) \approx \text{Area}\left(\left\{z \mid \|z\| \leq \frac{\sqrt{\mu}}{2\pi}\right\}\right) = \frac{\mu}{4\pi} = c\mu \operatorname{vol}(\mathbb{T}^2) \quad \text{as } \mu \to \infty$$

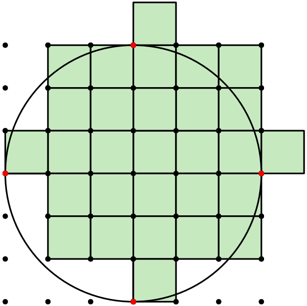


Figure: Cells in $\lambda = 4\pi^2 9$

Visualizing Weyl's Law

Gauss Circle Problem Conjecture: $N(\mu) \approx c\mu\,\mathrm{vol}(\mathbb{T}^2) + \mathcal{O}(\mu^{\frac{1}{4}+\epsilon})$

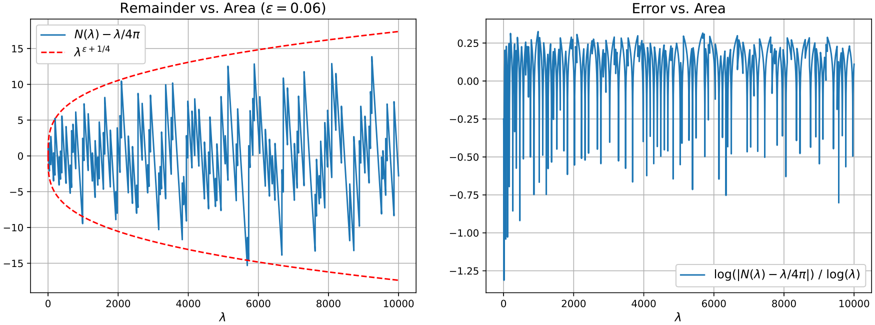


Figure: Estimates of the remainder

The Distorted Torus $\mathbb{T}^2_\Lambda$

- Take some diffeomorphism $A \in \mathrm{GL}_2(\mathbb{R})$ as our distortion
- Define the lattice $\Lambda = A\mathbb{Z}^2 = \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$ and it's respective dual lattice $\Lambda^* = A^{-T}\mathbb{Z}^2 = \mathrm{Hom}_{\mathrm{Ab}}(\Lambda, \mathbb{Z})$
- Domain $\Omega = \mathbb{T}^2_\Lambda \cong \mathbb{R}^2/\Lambda$ and functions $u \in L^2(\mathbb{T}^2_\Lambda)$
- Apply reparameterization again, this time $x \mapsto Ax =: y$

Theorem (Laplacian on $\mathbb{T}^2_\Lambda$)

The eigenvalues have the form

$$\mathrm{spec}_{\mathbb{T}^2_\Lambda}(-\Delta) = \{4\pi^2 \|m\|^2 \mid m \in \Lambda^*\}$$

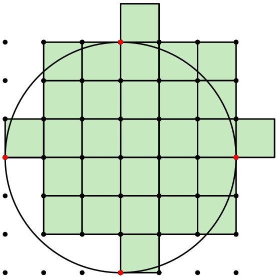
and the eigenspaces are given by

$$\lambda = 4\pi^2 \|m\|^2 : \mathrm{span}\left\{e^{2\pi i \langle m', y\rangle} \mid m' \in \Lambda^*, \|m'\| = \|m\|\right\}.$$

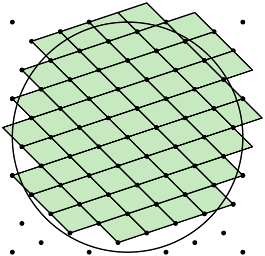
Weyl's Law on $\mathbb{T}^2_\Lambda$

Theorem (Weyl's Law on $\mathbb{T}^2_\Lambda$)

$\mathrm{vol}(\mathbb{T}^2_\Lambda) = \det A$, hence $N(\mu) \sim \frac{\mu}{4\pi} \det A = c\mu \det A$ as $\mu \to \infty$.



$\mathbb{Z}^2$



$\Lambda = A\mathbb{Z}^2$

Thank you!