MA353 HW10 CHALLENGE PROBLEM

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Problem 1. Suppose that $A \in M_3(\mathbb{R})$ is orthogonal with $\det A = 1$. Then A is a rotation matrix.

Proof. A orthogonal implies that A^{-1} exists and $A^{-1} = A^T$ such that $A^TA = AA^T = \mathrm{Id}$. We begin with the following claim:

Claim 1.1. $1 \in \operatorname{spec} A$.

Proof. By Theorem 4.5 in the lecture notes, we have that A orthogonal implies that, for any $u \in \mathbb{R}^3$, u satisfies $\|Au\| = \|u\|$. Thus, $\lambda \in \operatorname{spec} A$ must satisfy $\|\lambda\| = 1$. By dimension of A, we can have maximally 3 distinct complex eigenvalues, and since complex eigenvalues come in conjugate pairs, $\lambda \in \operatorname{spec} A$ with $\mathrm{Im}(\lambda) \neq 0 \implies \overline{\lambda} \in \operatorname{spec} A$. Thus we must have either 1 or 3 real eigenvalues. Suppose λ_1 is one such real eigenvalue and that $\lambda_1 = -1$ and break into 2 disjoint, exhaustive cases:

(1) $\lambda_2, \lambda_3 \in \operatorname{spec} A$ with $\mathrm{Im}(\lambda_2) \neq 0$ and $\lambda_3 = \overline{\lambda_2}$. Then
$$\det A = +1 = \lambda_1\lambda_2\lambda_3 = -\|\lambda_2\|^2,$$
where $\|\lambda_2\|$ is strictly positive (and thus it's square is too) since A invertible implies that none of our eigenvalues are zero. Thus $-\|\lambda_2\|^2 > 0$ is a contradiction, and it must be the case that $\lambda_1 = +1$.

(2) $\lambda_2, \lambda_3 \in \operatorname{spec} A$ with $\lambda_2, \lambda_3 \in \mathbb{R}$. As stated above, we know that $|\lambda_2| = |\lambda_3| = 1$. Then $\det A = +1 = \lambda_1\lambda_2\lambda_3 \implies \lambda_2\lambda_3 = -1$. Thus WLOG if $\lambda_2 = -1$, we must have $\lambda_3 = 1$.

We have shown that, no matter the composition of the spectrum of A, at least one eigenvalue is equal to 1. □

Ok, cool. From here on, we assume WLOG that only one such $\lambda \in \operatorname{spec} A$ is equal to $+1$ since otherwise $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and we have that $A = \mathrm{Id}$ which is trivially a rotation by zero degrees. Now to the meat of the proof.

Let E be the eigenspace associated with $\lambda = 1 \in \operatorname{spec} A$ such that $E = \{u \in \mathbb{R}^3 \mid Au = u\}$ which we've assumed above has $\dim E = 1$. (As an aside, we know that $\dim E > 0$ since, by the very existence of an eigenvalue, there is atleast one nonzero $u \in \mathbb{R}^3$ satisfying $Au = u$). Then since $\mathbb{R}^3$ is a normed, inner product space and $E < \mathbb{R}^3$ is a subspace, we have that the orthogonal complement $E^\perp$ exists and $\operatorname{codim} E^\perp = 1 \implies E^\perp$ is a subspace (a plane embedded) in $\mathbb{R}^3$. For any $u \in \mathbb{R}^3$, consider it's orthogonal decomposition $u = \operatorname{proj}_{E^\perp}(u) + \operatorname{proj}_E(u)$ and define $w = \operatorname{proj}_{E^\perp}(u)$. By linearity of A and the construction of E, we have
$$Au = Aw + A\operatorname{proj}_E(u) = Aw + \operatorname{proj}_E(u).$$
Let's focus just on w and Aw then since the projection of u in E is invariant under A.

Claim 1.2. $E^\perp$ is also invariant under A.

Proof. Consider some $u \in E$ and $v \in E^\perp$. By definition of orthogonally complimentary spaces, $\langle u, v\rangle = 0$. But from Theorem 4.5, we know that A orthogonal implies that $\langle u, v\rangle = \langle Au, Av\rangle = 0$ and thus since we know that E is invariant under A, $Au \in E \implies Av \in E^\perp$. □

Claim 1.3. We can re-express A as a block matrix under change of basis as
$$\mathrm{Rep}_C(A) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$
such that $\mathbf{R} \in M_2(R)$ and $\mathbf{R}$ is orthogonal and automorphic over $\mathrm{Rep}_C(E^\perp) \cong \mathbb{R}^2$.

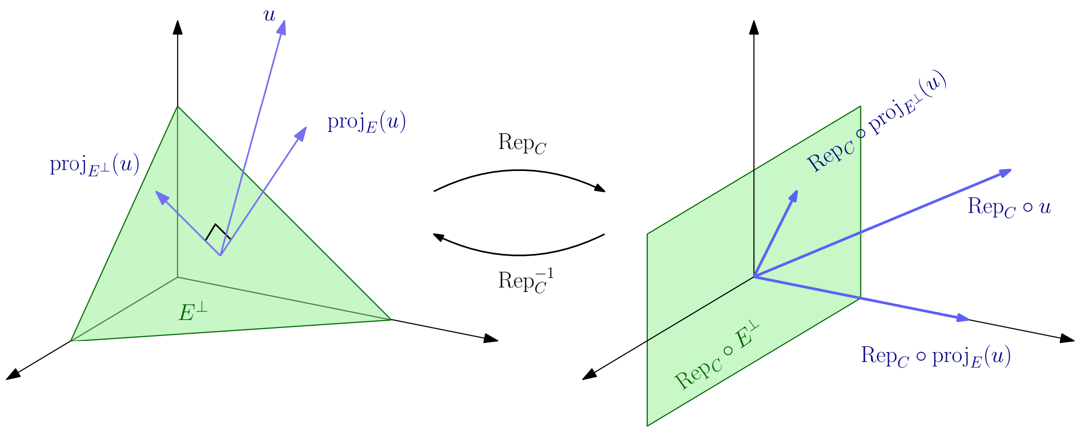

FIGURE 1. Depiction of orthogonal decomposition of u over E eigenspace of $\lambda = 1$

Proof. Consider a unit vector $u \in E$. Further consider that since $E^\perp$ is a subspace with $\dim E^\perp = 2$, we know there exists a (not necessarily unique) set of vectors $\{v, w\} \subset E^\perp$ such that v and w form an orthonormal basis for $E^\perp$ by Gram-Schmidt. By definition of orthogonally complimentary spaces, we have then that u, v, w are all pair-wise orthogonal, of unit length, and span $E \cup E^\perp = \mathbb{R}^3$ such that they form an orthonormal basis for $\mathbb{R}^3$. Let $B = \text{Id} = [\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3]$ be the standard basis for $\mathbb{R}^3$ and $C = [u\ v\ w]$ be our newly constructed basis. Since the columns of both B, C are orthonormal, both matrices are orthogonal and thus invertible by Theorem 4.5 again. Then since we have that

$$AC = \begin{bmatrix} Au & Av & Aw \end{bmatrix} = \begin{bmatrix} u & Av & Aw \end{bmatrix},$$

we know that in representation C (that is, $C^{-1}AC = C^TAC$), A takes on the form

$$\operatorname{Rep}_C(A) = C^TAC = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

Some quick intuition behind the presence of the zeroes. First, since A, C, and C^T are all orthogonal, so is their composition, meaning rows/columns of $\operatorname{Rep}_C(A)$ are orthonormal. Second, we know that both E and $E^\perp$ are invariant under A and thus $\operatorname{Rep}_C(E)$ and $\operatorname{Rep}_C(E^\perp)$ are invariant under $\operatorname{Rep}_C(A)$ as well. Thus for any $x \in E^\perp$, we know that $\langle \operatorname{Rep}_C(x), \operatorname{Rep}_C(u) \rangle = 0$. Trivially, by structure of the block matrix in representation C, we know that $\mathbf{R}$ is endomorphic over $\operatorname{Rep}_C(E^\perp)$. The isomorphic property of $\mathbf{R}$ follows from the orthonormality of it's rows/columns, and thus it's invertibility in $M_2(\mathbb{R})$. Then by inspection, all of $\operatorname{Rep}_C(A)$ is orthogonal and thus invertible. $\square$

Let $T = \operatorname{Rep}_C(A)$ as in Claim 1.3. It trivial to see that $u \in E \implies Tu = u$. We've also shown that the matrix $\mathbf{R}$ within T is orthogonal. Then we get access to the nice property that $\det A = \det C \det T \det C^T = \det T = 1 \cdot \det \mathbf{R} = 1$, so $\mathbf{R}$ has determinant 1. We have visualized our work thus far in Figure 1. A quick auxillary claim...

Claim 1.4. *A orthogonal implies that A is diagonalizable.*

Proof. By invertibility of $A \in M_n(\mathbb{R})$, each of our eigenvalues satisfy equivalence of geometric multiplicity with algebraic multiplicity. Thus we have n lineary independent eigenvalues, satisfying the sufficient condition for diagonalizability. As suggested, this is an equivalent problem to that of Claim 1.3, where we find a change of basis to the standard basis. $\square$

We arrive now at the main result of the proof.

Claim 1.5. *$\mathbf{R}$ is the rotation matrix in $\mathbb{R}^2$.*

Proof. We have that $\mathbf{R}$ is orthogonal and has determinant 1. As shown earlier on, $\lambda \in \operatorname{spec} \mathbf{R} \implies |\lambda| = 1$ and thus each λ lies somwhere on the complex unit circle such that there exists a $\theta \in [0, 2\pi)$ where $\lambda = e^{i\theta}$. Moreover, $\lambda \in \operatorname{spec} \mathbf{R} \implies \overline{\lambda} \in \operatorname{spec} \mathbf{R}$. Suppose $\theta = 0$ such that $\lambda_1 = \lambda_2 = 1$. By diagonalizability of orthogonal matrices, we know that there must exists two distinct eigenvectors, and in fact $\mathbf{R} = I$ rotation by zero degrees. Similarly, if $\theta = \pi$ where $\lambda_1 = \lambda_2 = -1$, we ge that $\mathbf{R} = -I$, the rotation by 180 degrees. Now we can WLOG assume that $\lambda_1 = \overline{\lambda_2}$ are distinct complex eigenvalues with parameter θ and distinct eigenvectors v and $\overline{v}$, respectively. Thus $\mathbf{R}v = e^{i\theta}v$ and $\mathbf{R}\overline{v} = e^{-i\theta}\overline{v}$. Suppose that $v = v_1 + iv_2$ with $v_1, v_2 \in \mathbb{R}^2$. Then we have that

$$\mathbf{R}v = (\cos\theta + i\sin\theta)v$$
$$\mathbf{R}\overline{v} = (\cos\theta - i\sin\theta)\overline{v}$$

by Euler's identity and the fact that sin is odd. Re-expressing this as

$$\mathbf{R}v_1 + i\mathbf{R}v_2 - i\mathbf{R}v_2 + \mathbf{R}v_1 = 2v_1\cos\theta - 2v_2\sin\theta$$
$$\mathbf{R}v_1 + i\mathbf{R}v_2 - \mathbf{R}v_1 + i\mathbf{R}v_2 = 2iv_1\sin\theta + 2iv_2\cos\theta$$

and thus $\mathbf{R}v_1 = v_1\cos\theta - v_2\sin\theta$ and $\mathbf{R}v_2 = v_1\sin\theta + v_2\cos\theta$. Further, we know that $\langle v_1, v_2\rangle = 0$ since otherwise, $v = (1+ic)v_1$ and thus $\mathbf{R}v_1 = e^{i\theta}v_1$ which is a contradiction since multiplication by $e^{i\theta}$ would leave the image in $\mathbb{C}^2$, not $\mathbb{R}^2$. Thus $\{v_1, v_2\}$ forms an orthogonal basis for $\mathbb{R}^2$. Now consider any $u \in \mathbb{R}^2$. We can express $u = av_1 + bv_2$ and thus,

$$\mathbf{R}u = a\mathbf{R}v_1 + b\mathbf{R}v_2 = (a\cos\theta + b\sin\theta)v_1 + (b\cos\theta - a\sin\theta)v_2.$$

If T is the change of basis such that $v_1 \mapsto \mathbf{e}_1$ and $v_2 \mapsto \mathbf{e}_2$, then we get that

$$T^{-1}\mathbf{R}\begin{bmatrix} a \\ b \end{bmatrix} T = T^{-1}\begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix} T$$

and thus $\mathbf{R}$ is the rotation matrix in $\mathbb{R}^2$. □

Wrapping things up now, we've shown that, for the same $C = [u\ v\ w]$ as before,

$$\operatorname{Rep}_C(A) = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}.$$

Consider now any $x \in \mathbb{R}^3$. If we apply orthogonal decomposition such that

$$x = \operatorname{proj}_E(x) + \operatorname{proj}_{E^\perp}(x) = c_1u + c_2v + c_3w,$$

we have that

$$\operatorname{Rep}_C(Ax) = \operatorname{Rep}_C(A) \circ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ \mathbf{R}\begin{bmatrix} c_2 \\ c_3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} c_2 \\ c_2\cos\theta - c_3\sin\theta \\ c_2\sin\theta + c_3\cos\theta \end{bmatrix}$$

and thus

$$Ax = \operatorname{Rep}_C(A)^{-1} \circ \begin{bmatrix} c_2 \\ c_2\cos\theta - c_3\sin\theta \\ c_2\sin\theta + c_3\cos\theta \end{bmatrix}$$

which corresponds to a rotation in the yz-plane in representation C of $\mathbb{R}^3$ or, in our standard basis, a rotation of vector x around the vector u by θ radians in the counterclockwise direction. □