# 4.5 — Bayesian Players

## ECON 316 • Game Theory • Fall 2021

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# Outline

# Bayesian Statistics

# Bayesian Statistics

- Most people's understanding & intuitions of probability are about the **objective** **frequency** of events occurring
    - "If I flip a fair coin many times, the probability of Heads is 0.50"
    - "If this election were repeated many times, the probability of Biden winning is 0.60"
- This is known as the **"frequentist"** interpretation of probability
    - And is almost entirely the only thing taught to students (because it's easier to explain)

# Bayesian Statistics

- Another valid (competing) interpretation is probability represents our **subjective belief** about an event
  - "I am 50% certain the next coin flip will be Heads"
  - "I am 60% certain that Biden will win the election"
  - This is particularly useful for **unique** events (that occur once...and really, isn't that every event in the real world?)

- This is known as the "Bayesian" interpretation of probability

# Bayesian Statistics

- In **Bayesian statistics**, probability measures the degree of certainty about an event
    - Beliefs range from impossible ($p = 0$) to certain ($p = 1$)
- This conditions probability on your **beliefs** about an event

Rev. Thomas Bayes

1702—1761

# Bayesian Statistics

- The bread and butter of thinking like a Bayesian is **updating your beliefs in response to new evidence**
    - You have some **prior** belief about something
    - New evidence should **update** your belief (level of certainty) about it
    - Updated belief known as your **posterior** belief
- Your beliefs are *completely* determined by the latest evidence, new evidence just *slightly* changes your beliefs, proportionate to how compelling the evidence is
- **This is fundamental to modern science and having rational beliefs**
    - And some mathematicians will tell you, the *proper* use of statistics

# Bayesian Statistics Examples

1. You are a bartender. If the next person that walks in is wearing a kilt, what is the probability s/he wants to order Scotch?

2. What is the probability that someone has watched the superbowl? What if you know that persoon is a man?

3. You are a policymaker deciding foreign policy, and get a new intelligence report.

4. You are trying to buy a home and make an offer, which the seller declines.

# Conditional Probability

- All of this revolves around **conditional probability**: the probability of some event $B$ occurring, given that event $A$ has already occurred

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

- $P(B|A)$: “Probability of $B$ given $A$”

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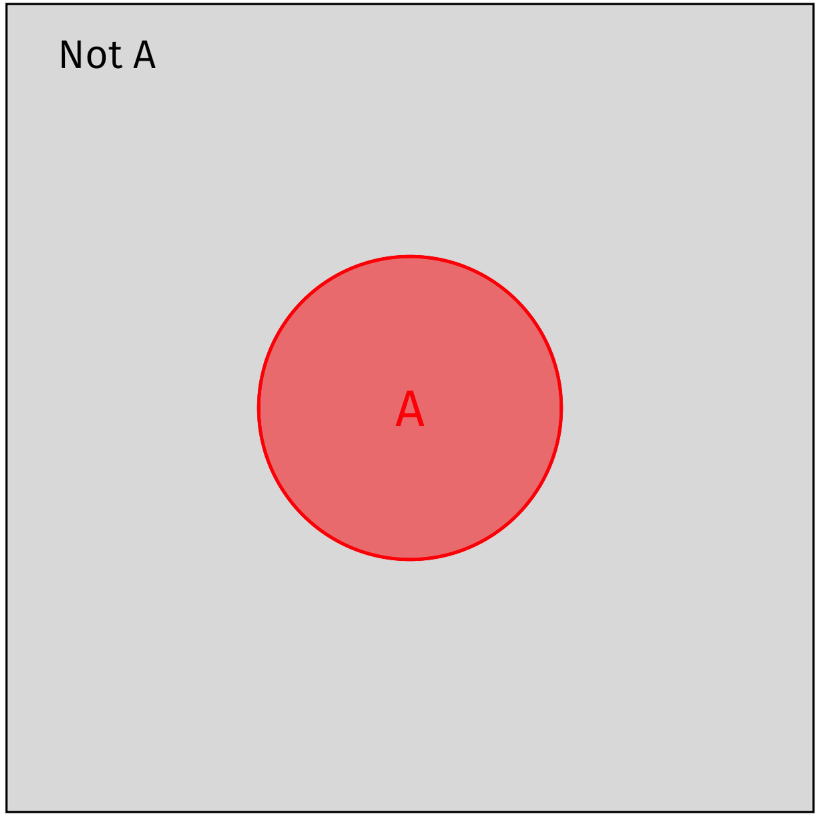
- If we know $A$ has occurred, $P(A) > 0$, and then every outcome that is $\neg A$ ("not A") cannot occur ($P(\neg A) = 0$)



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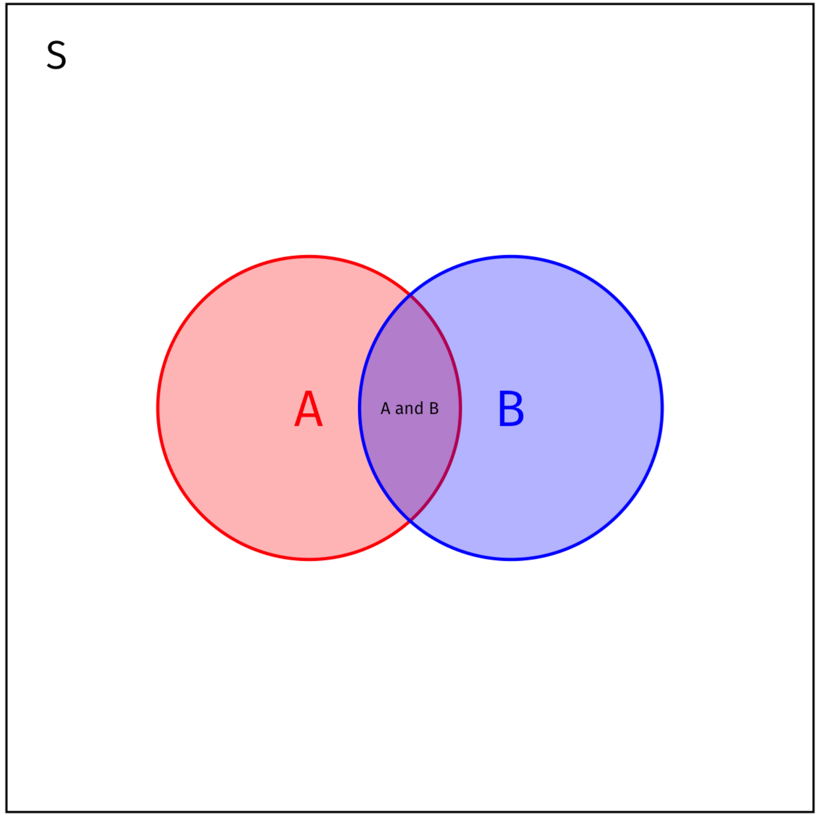

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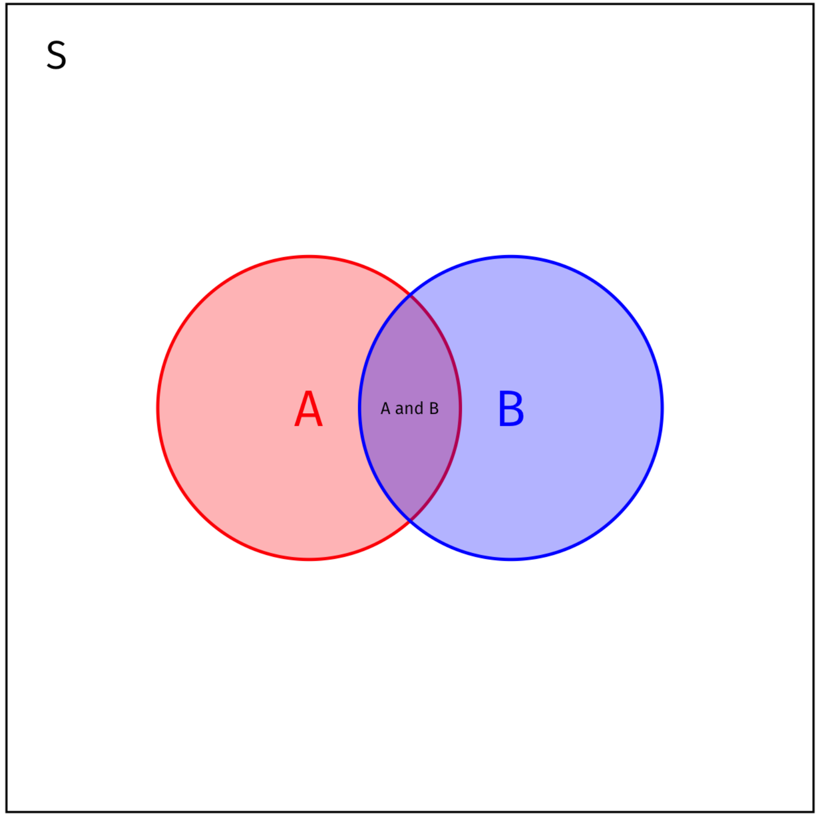
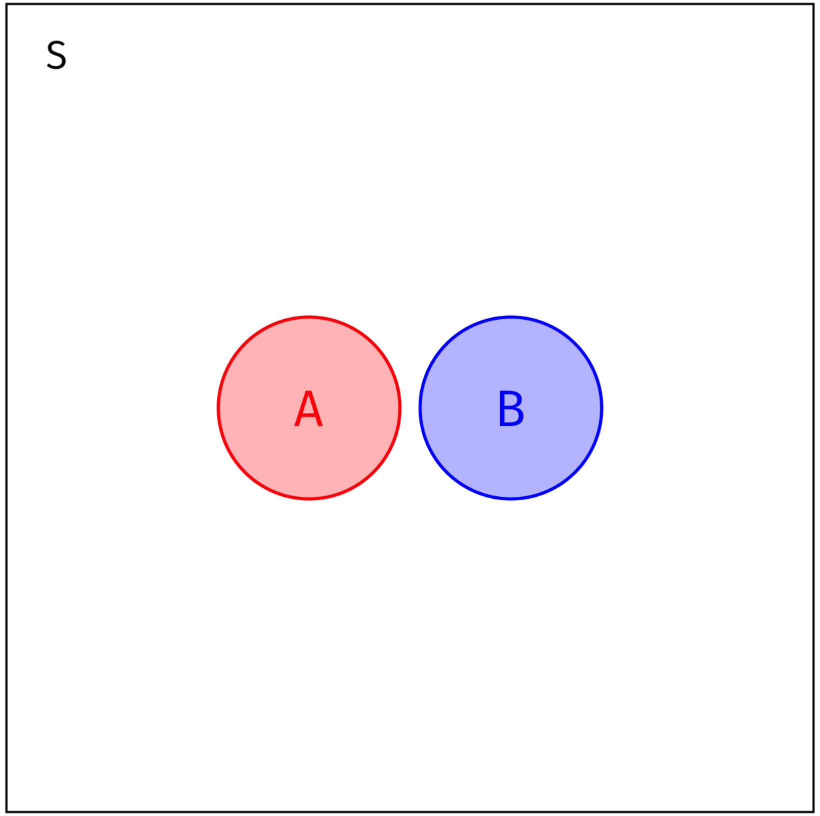
- If we know $A$ has occurred, $P(A) > 0$, and then every outcome that is $\neg A$ ("not A") cannot occur $(P(\neg A) = 0)$
  - The only part of $B$ which can occur if $A$ has occurred is $A$ and $B$
  - Since the sample space $S$ must equal 1, we've reduced the sample space to $A$, so we must rescale by $\frac{1}{P(A)}$



# Conditional Probability

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

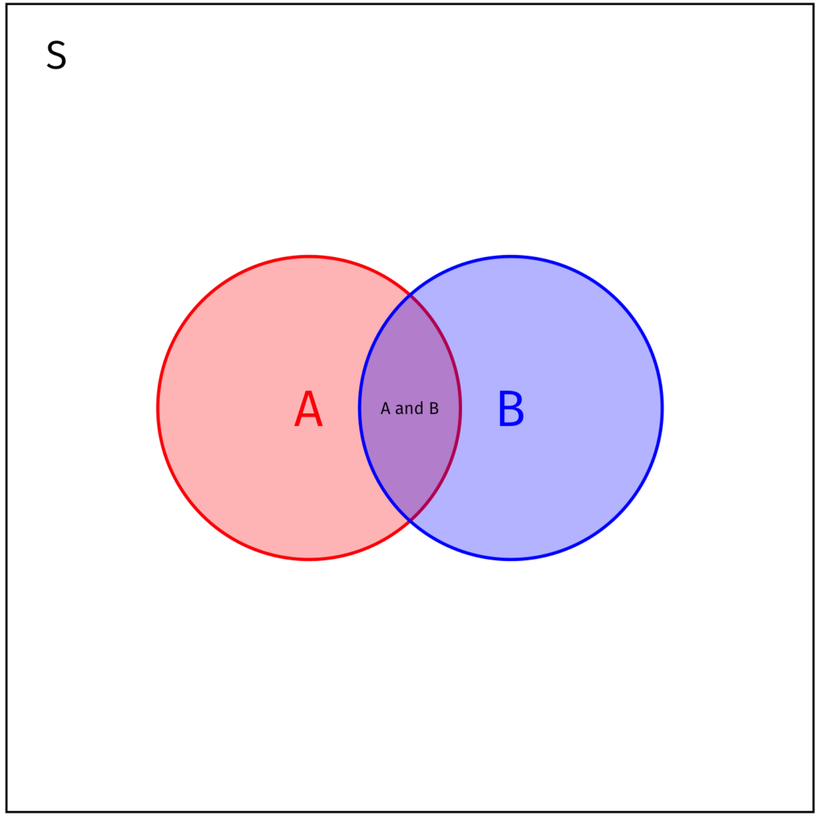
- If events $A$ and $B$ were **independent**, then the probability $P(A \text{ and } B)$ happening would be just $P(A) \times P(B)$
  - $P(A|B) = P(A)$
  - $P(B|A) = P(B)$



# Conditional Probability

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

- But if they are *not* independent, it's $P(A \text{ and } B) = P(A) \times P(B|A)$
  - (Just multiplying both sides above by the denominator, $P(A)$)



# Conditional Probability and Bayes' Rule

- Bayes realized that the conditional probabilities of two non-independent events are proportionately related

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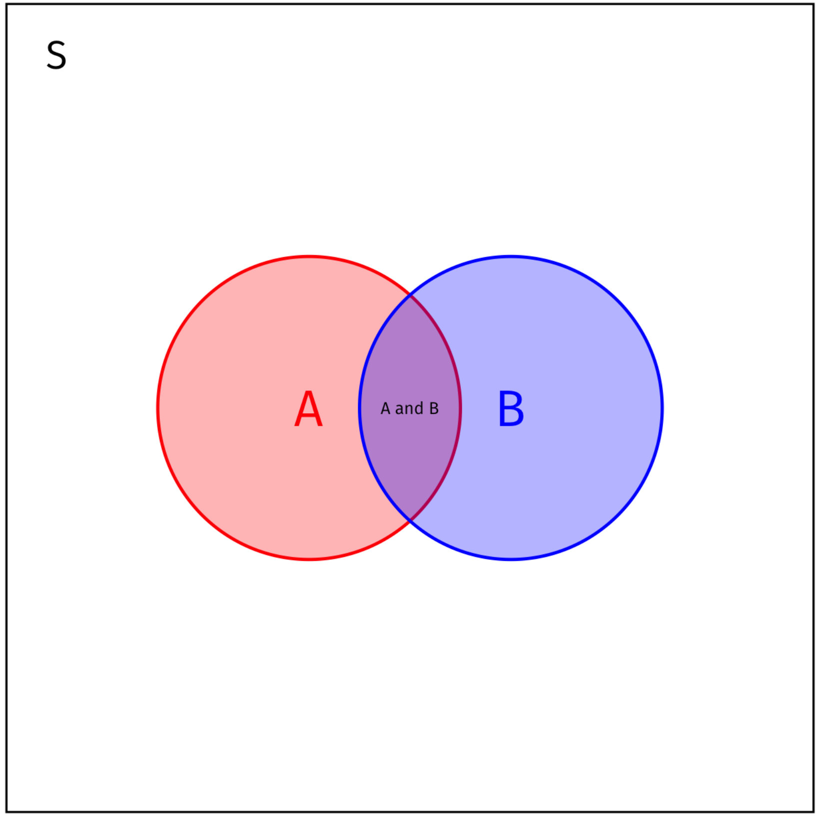

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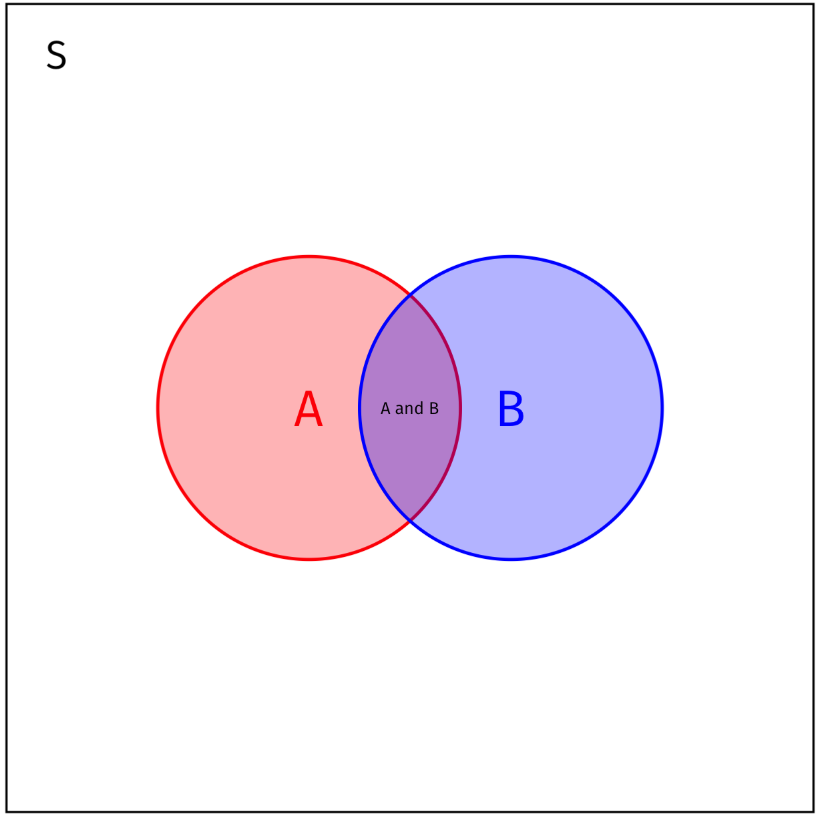

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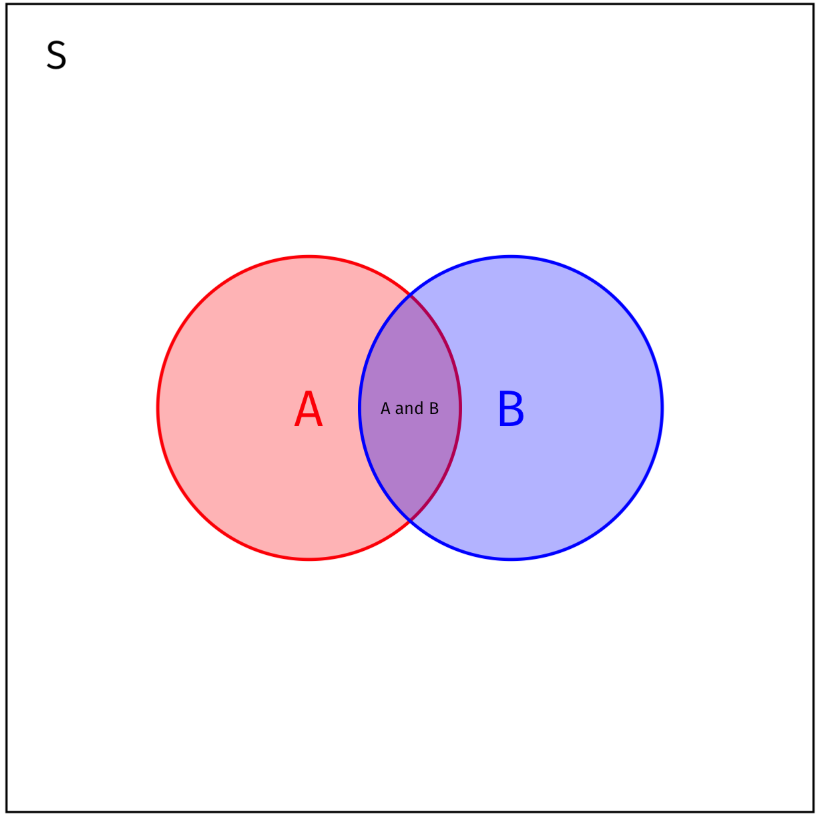
- If you divide everything by P(B), you get...**Bayes' rule**:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$



# Bayes’ Rule Example

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**Example**: Suppose 1% of the population has a rare disease. A test that can diagnose the disease is 95% accurate. What is the probability that a person who takes the test and comes back positive has the disease?

- What would you guess the probability is?

# Bayes' Rule Example

**Example**: Suppose 1% of the population has a rare disease. A test that can diagnose the disease is 95% accurate. What is the probability that a person who takes the test and comes back positive has the disease?


- $P(\text{Disease}) = 0.01$
- $P(+|\text{Disease}) = 0.95 = P(-|\neg\text{Disease})$
- We know $P(+|\text{Disease})$ but want to know $P(\text{Disease}|+)$
  - **These are not the same thing!**
  - Related by Bayes' Rule:

$$P(\text{Disease}|+) = \frac{P(+|\text{Disease})P(\text{Disease})}{P(+)}$$

# Bayes' Rule Example

- $P(\text{Disease}) = 0.01$
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$$P(\text{Disease}|+) = \frac{P(+|\text{Disease})P(\text{Disease})}{P(+)}$$

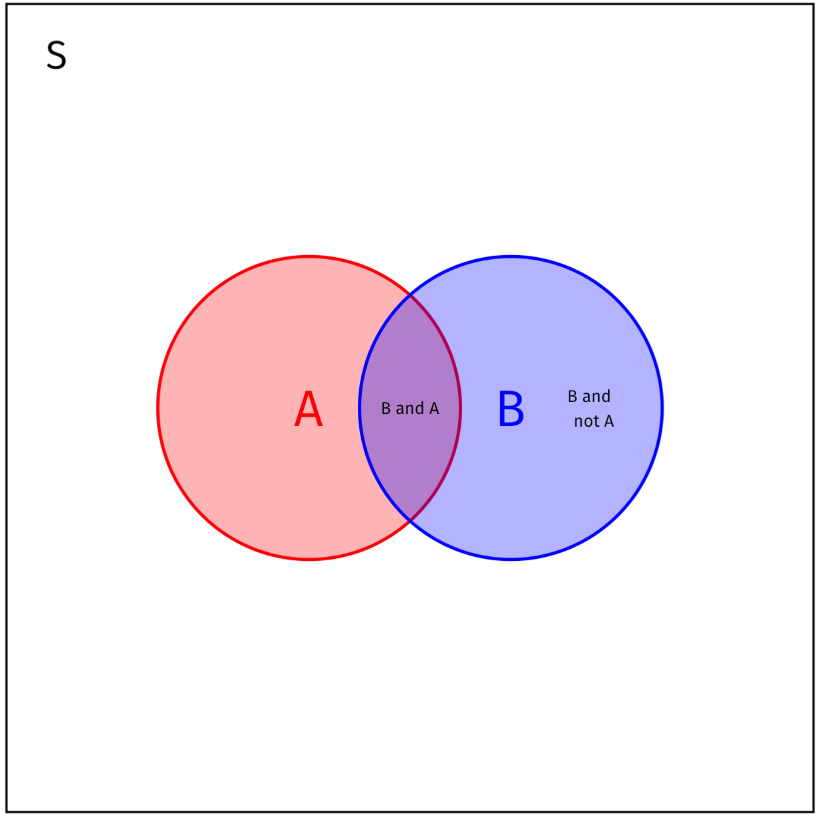
- What is $P(+)$??

# Bayes' Rule Example

- What is the total probability of $B$ in the diagram?

$$\begin{aligned} P(B) &= P(B \text{ and } A) + P(B \text{ and } \neg A) \\ &= P(B|A)P(A) + P(B|\neg A)P(\neg A) \end{aligned}$$

- This is known as the law of total probability

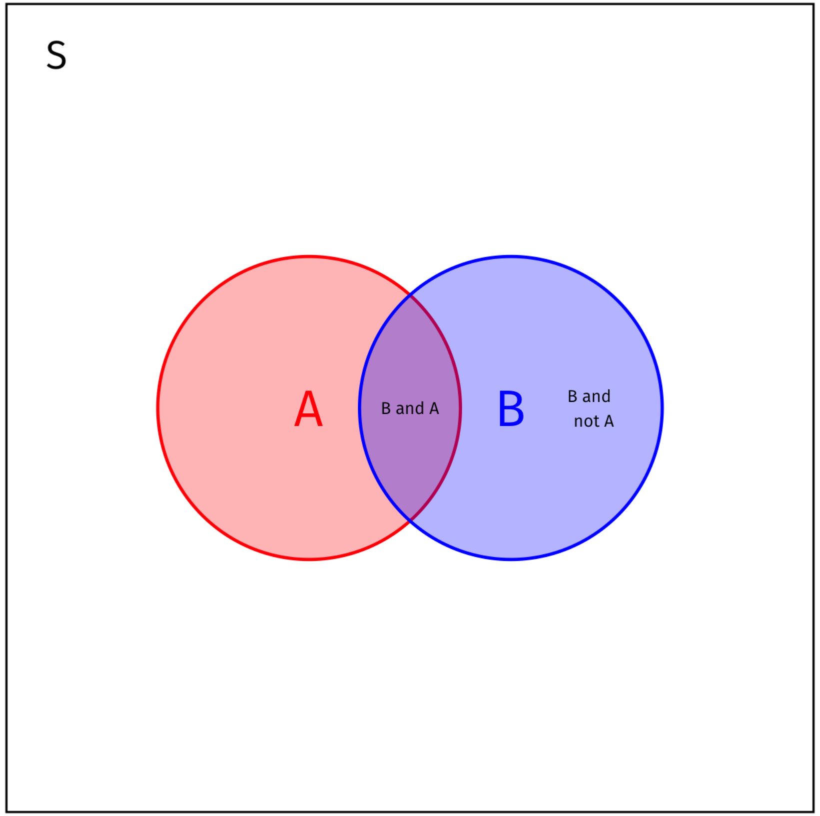


# Bayes' Rule Example: Aside

- Because we usually have to figure out $P(B)$ (the denominator), Bayes' rule is often expanded to

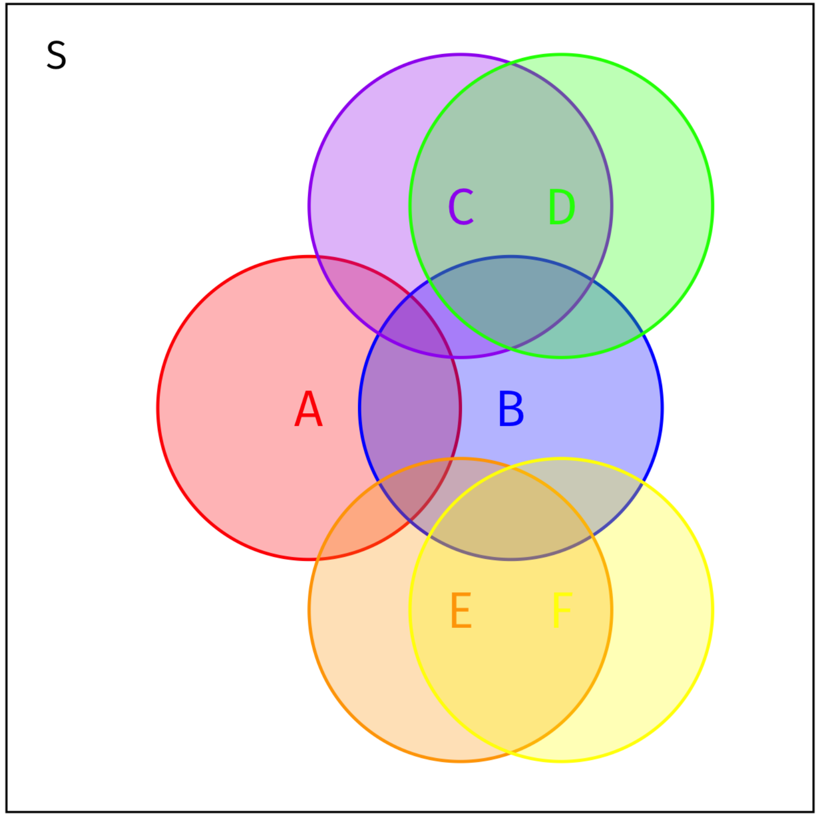
$$P(B|A) = \frac{P(A|B)P(A)}{P(B|A)P(A) + P(B|\neg A)P(\neg A)}$$

- Assuming there are two possibilities ($A$ and $\neg A$), e.g. True or False



# Bayes' Rule Example: Aside

- If there are more than two possibilities, you can further expand it to $\sum_{i=1}^{n} P(B|A_i)P(A_i)$ for $n$ number of possible alternatives to $A$

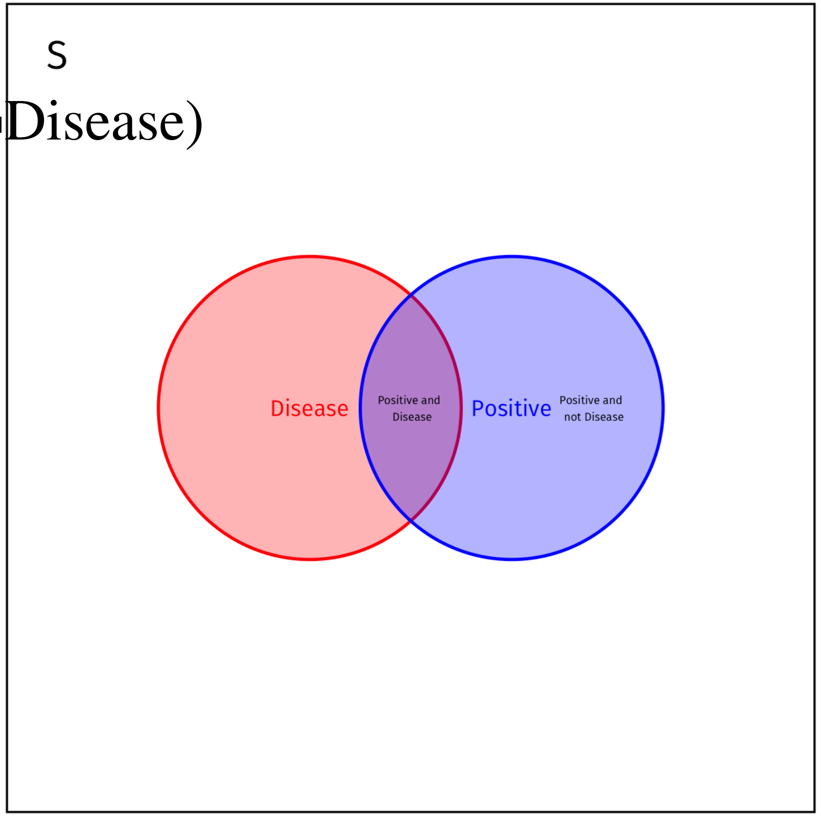


# Bayes' Rule Example

- What is the total probability of +?

$$
\begin{aligned}
P(+) &= P(+ \text{ and Disease}) + P(+ \text{ and } \neg \text{ Disease}) \\
&= P(+|\text{Disease})P(\text{Disease}) + P(+|\neg\text{Disease})P(\neg\text{Disease})
\end{aligned}
$$

- $P(\text{Disease}) = 0.01$
- $P(+|\text{Disease}) = 0.95$



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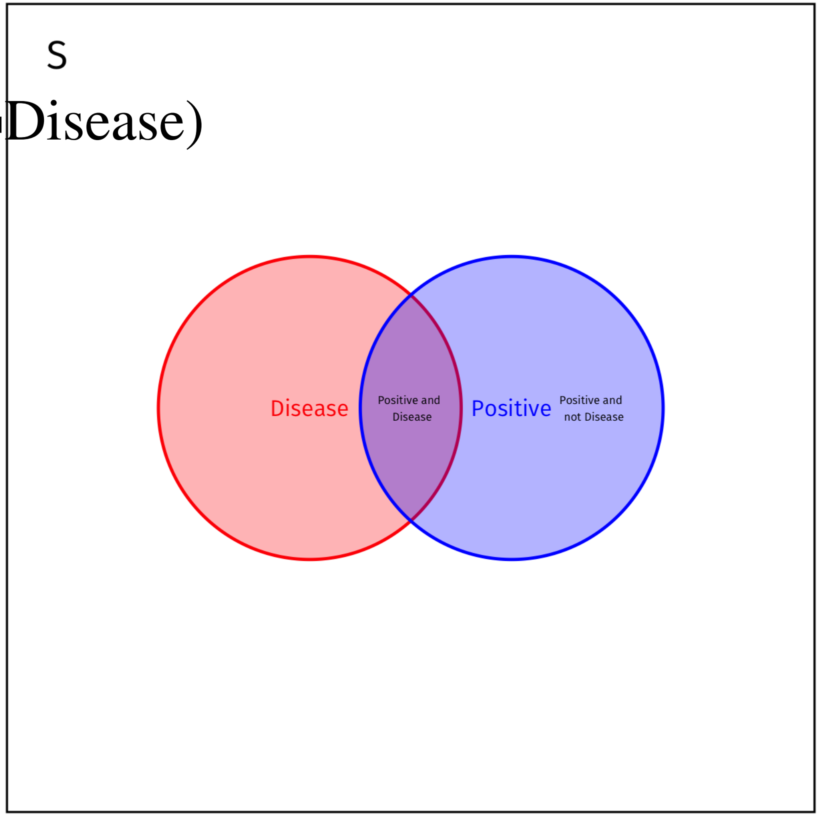
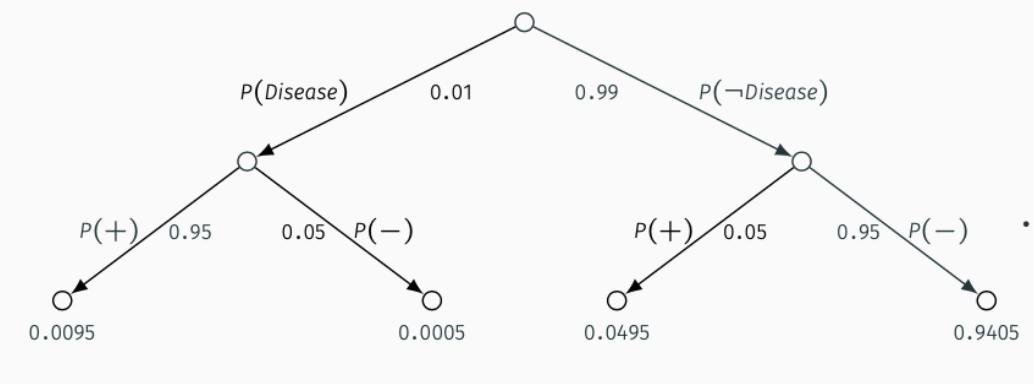
- $P(\text{Disease}) = 0.01$
- $P(+|\text{Disease}) = 0.95$

$$P(+) = 0.95(0.01) + 0.05(0.99) = 0.0590$$



# Bayes' Rule Example

| | Disease | ¬ Disease | Total |
|---|---|---|---|
| + | 0.0095 | 0.0495 | **0.0590** |
| - | 0.0005 | 0.9405 | 0.9410 |
| Total | 0.0100 | 0.9900 | 1.0000 |



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$$P(\text{Disease}|+) = \frac{P(+|\text{Disease})P(\text{Disease})}{P(+)}$$

$$P(\text{Disease}|+) = \frac{0.95 \times 0.01}{0.0590}$$

$$= 0.16$$

- The probability you have the disease is only 16%!
  - Most people vastly overestimate because they forget the base rate of the disease, $P(\text{Disease})$ is so low (1%)!

# Bayes’ Rule and Bayesian Updating

- Bayes Rule tells us how we should update our beliefs given new evidence