Problem Set 2

Answer Key

ECON 480 — Fall 2021

Answers generally go above and beyond what I expect from you. They are meant to show you the correct answer, explain *why* it is correct, and potentially show *several methods* by which you can reach the answer.

Theory and Concepts

Question 1

In your own words, explain the difference between endogeneity and exogeneity. An **exogenous** model is one where the independent variable (X) is not associated with any other factors that affect the dependent variable (Y). If a model is truly exogenous, we can estimate the **causal effect** of X on Y.

An **endogenous** model is one where the independent variable (X) *is* associated with any other factors that affect the dependent variable (Y). If a model is endogenous, we have no accurately estimated the causal effect of X on Y, since other factors are getting entangled with X and Y.

Question 2

Part A

In your own words, explain what (sample) standard deviation *means*. Sample standard deviation measures the average deviation (distance) of any given value of a variable from the variable's mean.

Part B

In your own words, explain how (sample) standard deviation *is calculated*. You may also write the formula, but it is not necessary. The formula is

$$sd(X) = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{n-1}}$$

It helps to consider the calculation as a series of steps:

1. Find the mean of X, ($\bar{X}$)
2. Subtract the mean from each value of X in the data, to get deviations, ($x_i - \bar{X}$)
3. Square the deviations to ensure they are all positive, $(x_i - \bar{X})^2$
4. Take the average of the squared deviations: add them all up and divide by $n-1$, $\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{X})^2$
5. What you have in step 4 is variance (measured in units of X^2), square root to get standard deviation (measured in original units of X): $\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{X})^2}$

Note on Step 4: because this is a *sample*, we have do deal with **degrees of freedom (df)** loss. We use up one df to calculate the mean, $\bar{x}$, which is needed before calculating variance or standard deviation. Hence, instead of averaging like normal: $\frac{1}{n}\sum x_i$, we need to divide by $n-1$.

Problems

Question 3

Suppose you have a very small class of four students that all take a quiz. Their scores are reported as follows:

$$\{83, 92, 72, 81\}$$

For the remaining questions, you may use `R` to *verify*, but please calculate all sample statistics by hand and show all work.

Part A

Calculate the median. Arrange the values in numerical order from smallest to largest. Find the value in the middle (i.e. an equal number of values are on either side); possibly by crossing-out one number on either side at a time (like in Elementary School).

$$\underline{72}, 81, 83, \underline{92}$$

Since we have an even number of observations, we have two numbers in the middle, 81 and 83, so we must take the average of them:

$$\frac{81+83}{2} = 82$$

Using R:

```
median(c(83,92,72,81))
```

```
## [1] 82
```

Part B

Calculate the sample mean, $\bar{x}$.

$$\begin{aligned}
\bar{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\
\bar{x} &= \frac{72+81+83+92}{4} \\
\bar{x} &= \frac{328}{4} \\
\bar{x} &= 82
\end{aligned}$$

Using R:

```
mean(c(83,92,72,81))
```

```
## [1] 82
```

Part C

Calculate the sample standard deviation, s. My suggestion is to use the "table" method, and follow the 5 steps described in problem 2b.

x_i	$x_i - \bar{X}$	$(x_i - \bar{X})^2$
72	-10	100
81	-1	1
83	1	1
92	10	100
$\sum$		202
$\frac{1}{3} \times \sum$		≈ 67.33
$\sqrt{\frac{1}{3} \times \sum}$		$\approx \mathbf{8.21}$

In R:

```
sd(c(83,92,72,81))
```

```
## [1] 8.205689
```

Part D

```
# load tidyverse (for tibble and ggplot2)
library(tidyverse)
```

Make or sketch a rough histogram of this data, with the size of each bin being 10 (i.e. 70's, 80's, 90's, 100's). You can draw this by hand or use R.[1] Is this distribution roughly symmetric or skewed? What would we expect about the mean and the median?

```
## -- Attaching packages --------------------------------------- tidyverse 1.3.1 --
```

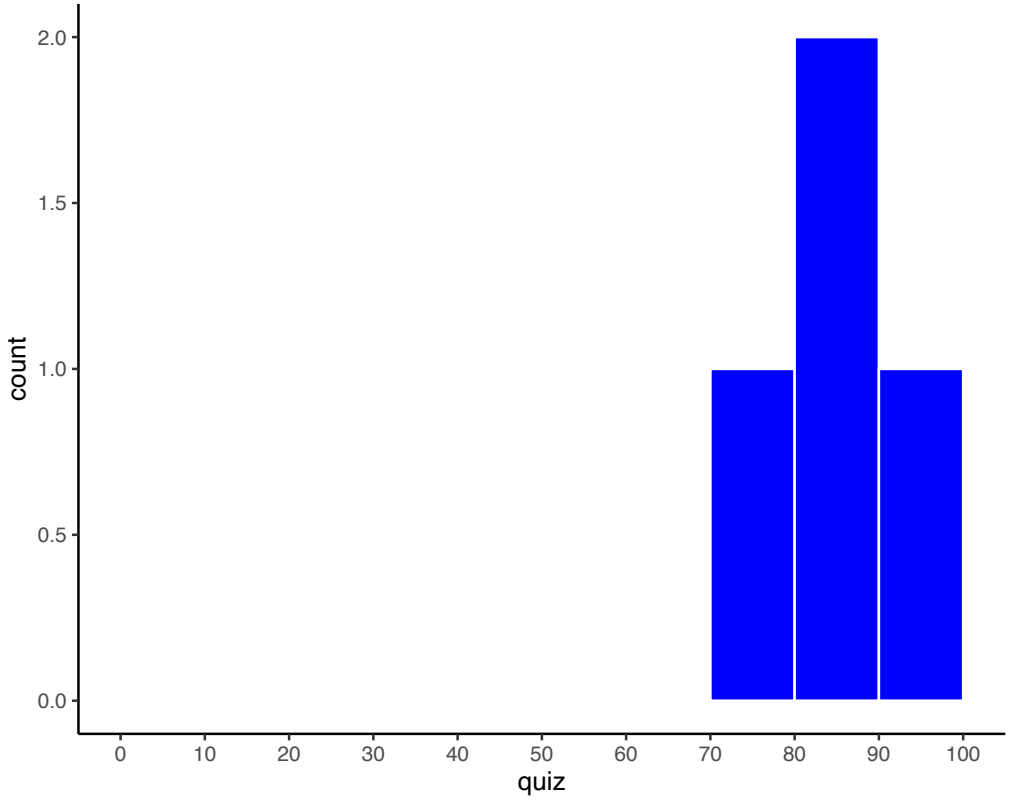
```
## v ggplot2 3.3.5     v purrr   0.3.4
## v tibble  3.1.5     v dplyr   1.0.7
## v tidyr   1.1.3     v stringr 1.4.0
## v readr   2.0.0     v forcats 0.5.1
```

```
## -- Conflicts ------------------------------------------ tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
```

```
# make a dataframe of our data,
# called df
# one variable in it, called quiz

df <- tibble(quiz = c(83,92,72,81))

# use this as our data for plot
ggplot(data = df)+
  aes(x = quiz)+
  geom_histogram(breaks=seq(0,100,10), # make bins of size 10 between 0 and 100
                 color = "white", # color is for borders
```

[1] If you are using `ggplot`, you want to use `+geom_histogram(breaks=seq(start,end,by))` and add `+scale_x_continuous(breaks=seq(start,end,by))`. For each, it creates bins in the histogram, and ticks on the x axis by creating a `sequence` starting at `start` (a number), ending at `end` (number), `by` a certain interval (i.e. by `10`s.).

```
                fill = "blue")+ # fill is for area
scale_x_continuous(breaks=seq(0,100,10))+ # have x axix ticks same as breaks
theme_classic()
```



We can see it is roughly symmetric. We would therefore expect the mean and the median to be approximately the same (which parts A and B showed was true).

Part E

Suppose instead the person who got the 72 did not show up that day to class, and got a 0 instead. Recalculate the mean and median. What happened and why?

$$\underline{0}, 81, 83, \underline{92}$$

$$\frac{81+83}{2} = 82$$

Replacing the 72 with a 0, and keeping the same number of observations does not change the median!

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$\bar{x} = \frac{0+81+83+92}{4}$$
$$\bar{x} = \frac{256}{4}$$
$$\bar{x} = 64$$

The mean is pulled down significantly by the outlier.

In R:

```
mean(c(83,92,0,81))
```

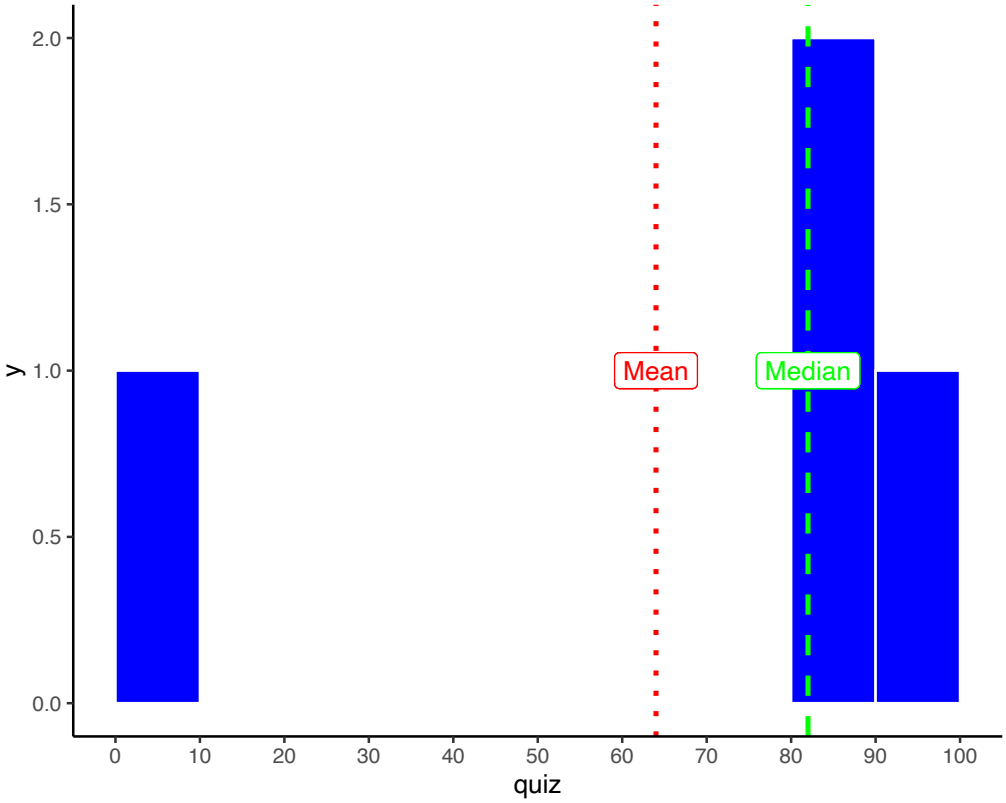
```
## [1] 64
```

```
median(c(83,92,0,81))
```

```
## [1] 82
```

If we were to look at the histogram, it would be skewed, and the mean would be lower than the mean:

```
# make new tibble called df_2
df_2 <- tibble(quiz = c(83,92,0,81)) # replace 72 with 0

# use this as our data for plot
ggplot(data = df_2)+
  aes(x = quiz)+
  geom_histogram(breaks=seq(0,100,10), # make bins of size 10 between 0 and 100
                 color = "white", # color is for borders
                 fill = "blue")+ # fill is for area
  geom_vline(aes(xintercept = median(quiz)), size = 1, color = "green", linetype = "dashed")+ # green d
  geom_label(aes(x = median(quiz), y = 1), label = "Median", color = "green")+ # label median line
  geom_vline(aes(xintercept = mean(quiz)), size = 1, color = "red", linetype = "dotted")+ # green dashe
  geom_label(aes(x = mean(quiz), y = 1), label = "Mean", color = "red")+ # label mean line
  scale_x_continuous(breaks=seq(0,100,10))+ # have x axix ticks same as breaks
  theme_classic()
```



Question 4

Suppose the probabilities of a visitor to Amazon's website buying 0, 1, or 2 books are 0.2, 0.4, and 0.4 respectively.

Part A

Calculate the *expected number* of books a visitor will purchase. Define X to be the number of books a visitor to Amazon's website purchases. The pdf of X is as follows:

x_i	$P(X = x_i)$
0	0.20
1	0.40
2	0.40

The expected value of X is the probability weighted average of X:

$$\begin{aligned} E(X) &= \sum_{i=1}^{n} p_i x_i \\ &= 0.2(0) + (0.4)1 + (0.4)2 \\ &= 0 + 0.4 + 0.8 \\ &= 1.2 \end{aligned}$$

Part B

Calculate the *standard deviation* of book purchases. The formula(s) for standard deviation of a random variable is:

$$\sigma_X = sd(X) = \sqrt{E[(X - E[X])^2]} = \sqrt{\sum_{i=1}^{n} p_i (x_i - E[X])^2}$$

I suggest using the table method, again. Working from the inside out of the formula, the steps are:

1. Find the expected value of X, $E[X]$.
2. Subtract the expected value from each value of X in the data, to get deviations, $(x_i - E[X])$
3. Square the deviations to ensure they are all positive, $(x_i - E[X])^2$
4. Take the probability-weighted average of the squared deviations: multiply each squared deviation by the probability of its associated x value and add them all up $\sum_{i=1}^{n} p_i (x_i - E[X])^2$
5. What you have in step 4 is variance (measured in units of X^2), square root to get standard deviation (measured in original units of X): $\sqrt{\sum_{i=1}^{n} p_i (x_i - E[X])^2}$

x_i	$P(X = x_i)$	$x_i - E[X]$	$(x_i - E[X])^2$	$p_i(x_i - E[X])^2$
0	0.20	−1.20	1.44	0.288
1	0.40	−0.20	0.04	0.016
2	0.40	0.80	0.64	0.256
$\sum$				0.560
$\sqrt{\sum}$				**0.748**

Part C

```
# make a dataframe called "amazon" of # of books and associated probabilities
amazon<-tibble(books = c(0,1,2),
               prob = c(0.2,0.4,0.4))

# look at it
amazon
```

Bonus: try doing this in R by making an initial dataframe of the data, and then making new columns to the "table" like we did in class.

```
## # A tibble: 3 x 2
##   books  prob
##   <dbl> <dbl>
## 1     0   0.2
## 2     1   0.4
## 3     2   0.4
```

```
# find expected value
amazon %>%
  summarize(exp_value = sum(books*prob))
```

```
## # A tibble: 1 x 1
##   exp_value
##       <dbl>
## 1       1.2
```

```
# it's 1.2, let's save exp_value

exp_value <- 1.2

# make new columns: devs, devs_sq, p_weight_devs_sq
# save to new tibble
amazon_table<-amazon %>%
  mutate(devs = books - exp_value,
         devs_sq = devs^2,
         p_weight_devs_sq = prob*devs^2)

# look at the tibble
amazon_table
```

```
## # A tibble: 3 x 5
##   books  prob  devs devs_sq p_weight_devs_sq
##   <dbl> <dbl> <dbl>   <dbl>            <dbl>
## 1     0   0.2  -1.2  1.44             0.288
## 2     1   0.4  -0.2  0.0400           0.016
## 3     2   0.4   0.8  0.64             0.256
```

```
# now let's take these and summarize
amazon_table %>%
  summarize(var = sum(p_weight_devs_sq), # variance
            sd = sqrt(var)) # sqrt to get sd, confirm its same!
```

```
## # A tibble: 1 x 2
##     var    sd
##   <dbl> <dbl>
```

```
## 1  0.56 0.748
```

Question 5

Scores on the SAT (out of 1600) are approximately normally distributed with a mean of 500 and standard deviation of 100.

Part A

What is the probability of getting a score between a 400 and a 600? Let random variable S be the score earned on the SAT.

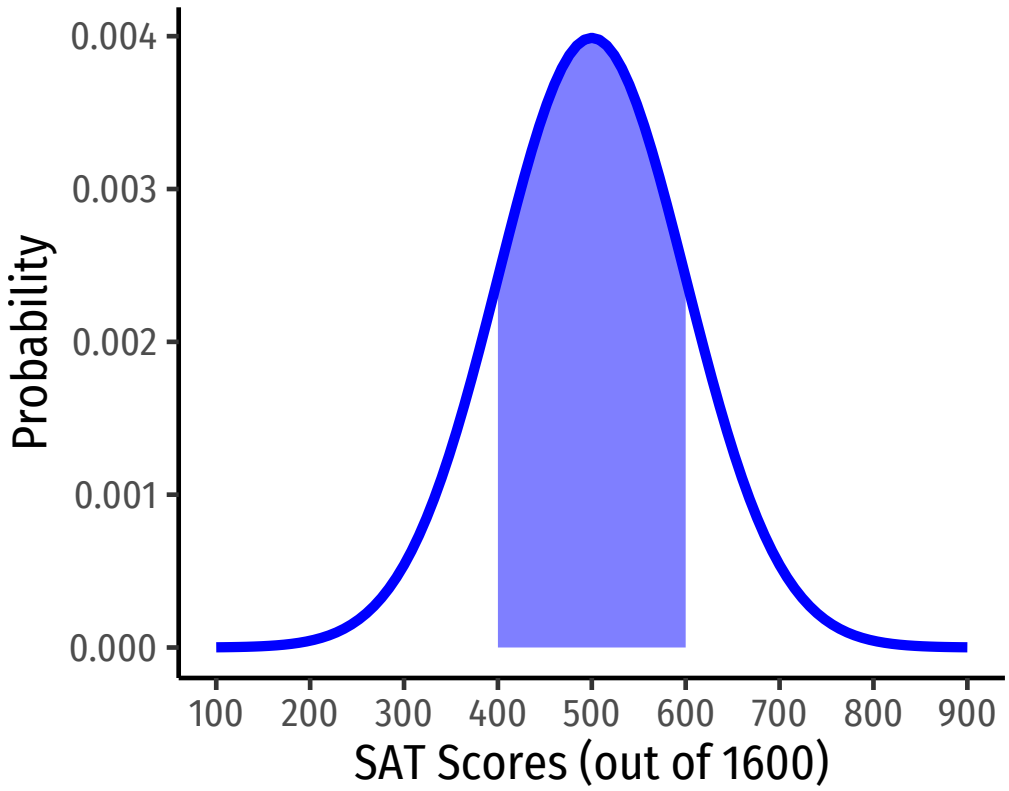
Convert thse numbers to Z-scores.

$$\begin{aligned} P(400 \leq S \leq 600) &= P(\frac{400-500}{100} \leq \frac{S-500}{100} \leq \frac{600-500}{100}) \\ &= P(-1 \leq Z \leq 1) \\ &\approx 0.68 \end{aligned}$$

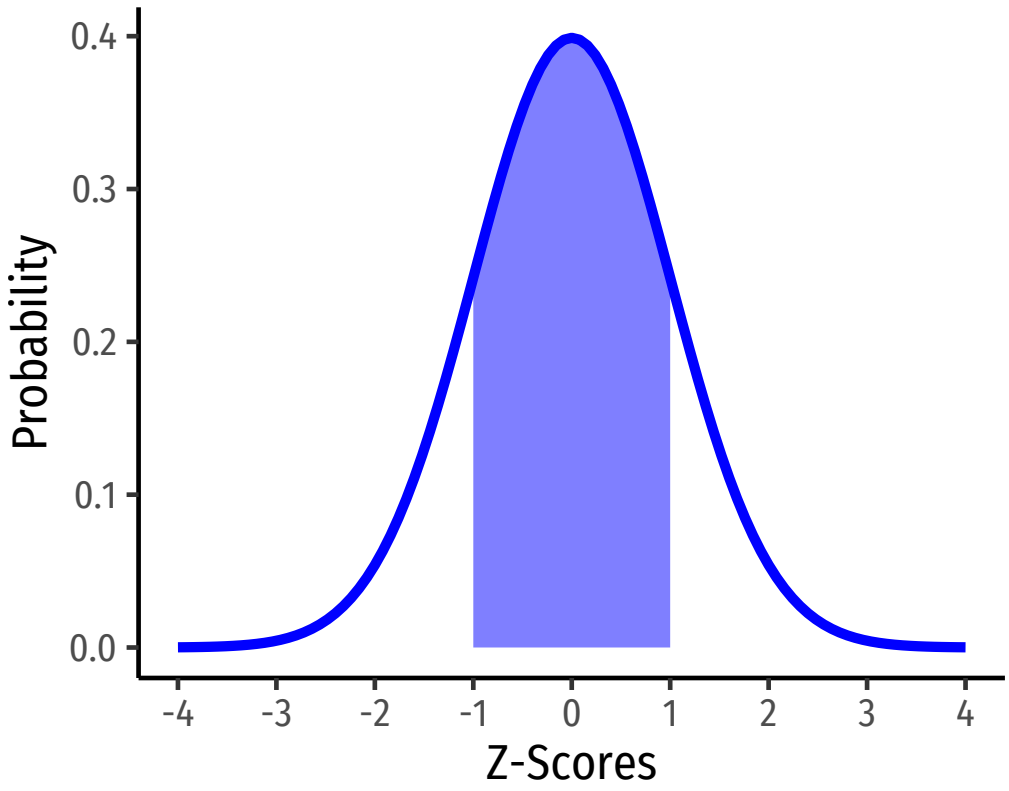
Using the 68-95-99.7 rule: about 68% of the values fall within one standard deviation (± 1 Z-score) of the mean.

You don't need to draw the pdf, but it helps to visualize what we're looking for, and how converting to Z-scores helps:

```
# see class 2.3 notes on how to graph and shade stats graphs

# it helps to first figure out where the x-axis ticks should be
# show about 4 standard deviations above and below the mean (mu +/- 4*sd)
# then have ticks in intervals of one sd

# in this case, with mean 500 and sd 100, it should be seq(100,900,100)

s_plot<-ggplot(data = tibble(scores=seq(from = 100,
                            to = 900,
                            by = 100)))+
  aes(x = scores)+
  stat_function(fun = dnorm,
                args = list(mean = 500, sd = 100),
                size = 2, color = "blue")+
  labs(x = "SAT Scores (out of 1600)",
       y = "Probability")+
  scale_x_continuous(breaks=seq(from  = 100,
                            to = 900,
                            by = 100))+
  theme_classic(base_family = "Fira Sans Condensed",
            base_size=20)

s_plot+stat_function(fun = dnorm,
                args = list(mean = 500, sd = 100),
                geom = "area",
                xlim = c(400,600),
                size = 2, fill = "blue", alpha = 0.5)
```




```
Z<-ggplot(data = tibble(Z=seq(from = -4,
                              to = 4,
                              by = 1)))+
  aes(x = Z)+
  stat_function(fun = dnorm,
                size = 2, color = "blue")+
  labs(x = "Z-Scores",
       y = "Probability")+
  scale_x_continuous(breaks=seq(from  = -4,
                              to = 4,
                              by = 1))+
  theme_classic(base_family = "Fira Sans Condensed",
            base_size=20)

Z+stat_function(fun = dnorm,
                geom = "area",
                xlim = c(-1,1),
                size = 2, fill = "blue", alpha = 0.5)
```



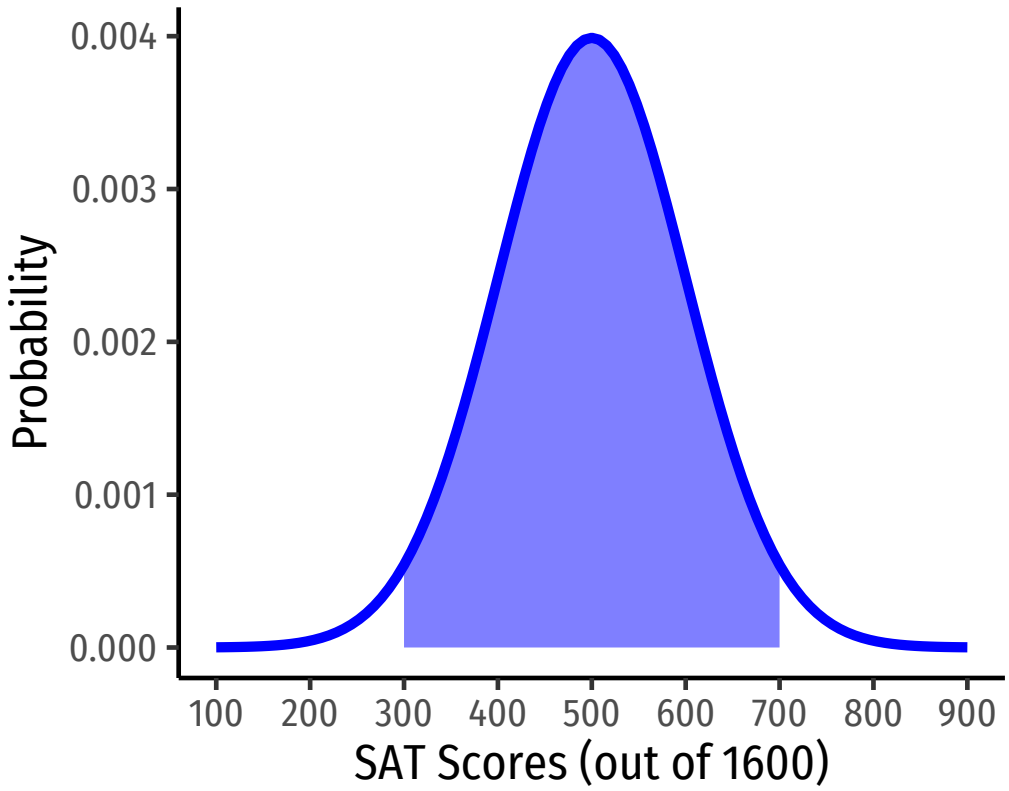
Part B

What is the probability of getting a score between a 300 and a 700? Convert thse numbers to Z-scores.

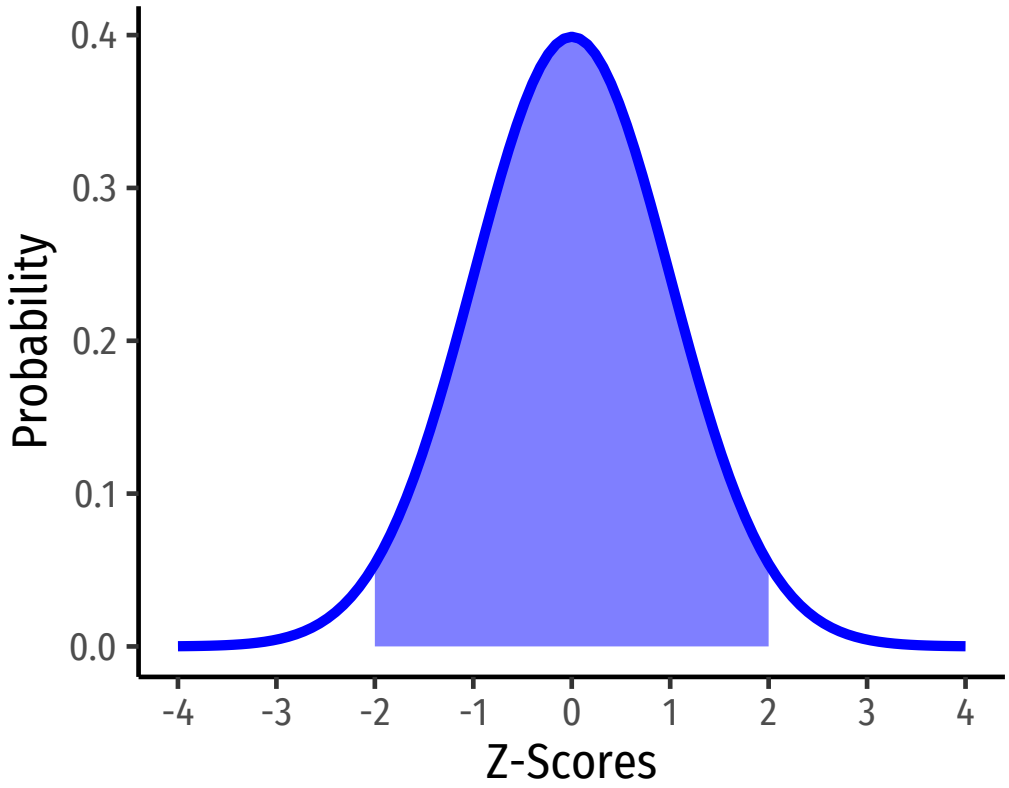
$$\begin{aligned} P(300 \leq S \leq 700) &= P(\frac{300-500}{100} \leq \frac{S-500}{100} \leq \frac{700-500}{100}) \\ &= P(-2 \leq Z \leq 2) \\ &\approx 0.95 \end{aligned}$$

Using the 68-95-99.7 rule: about 95% of the values fall within two standard deviations (± 2 Z-score) of the mean.

```
s_plot+stat_function(fun = dnorm, args = list(mean = 500, sd = 100), geom = "area", xlim = c(300,700), s
```




```
Z+stat_function(fun = dnorm, geom = "area", xlim = c(-2,2), size = 2, fill = "blue", alpha = 0.5)
```

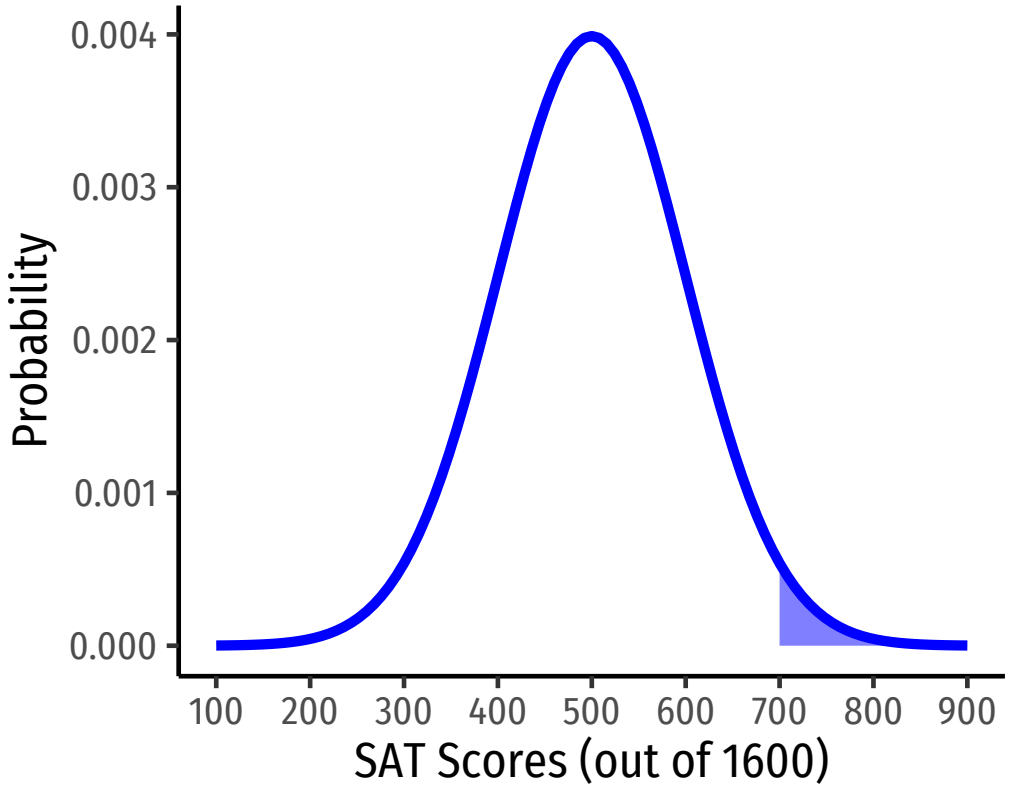


Part C

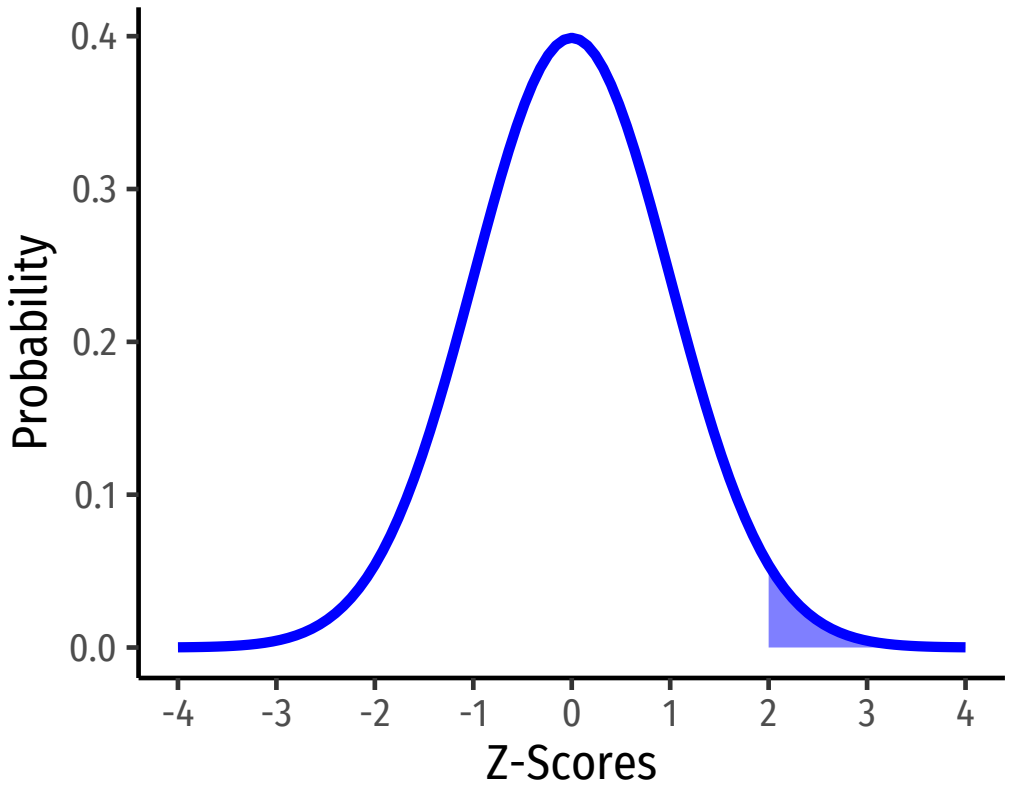
What is the probability of getting *at least* a 700? We saw in part B that Z for 700 is 2.

Using the 68-95-99.7 rule, we know about 95% of the values fall within two standard deviations (± 2 Z-score) of the mean. That means that 5% of values fall *beyond* a Z score of ± 2, or 2.5% in each direction. We only want the right-tail (probability above $Z = 2$, so 2.5%).

```
s_plot+stat_function(fun = dnorm, args = list(mean = 500, sd = 100), geom = "area", xlim = c(700,900), s
```



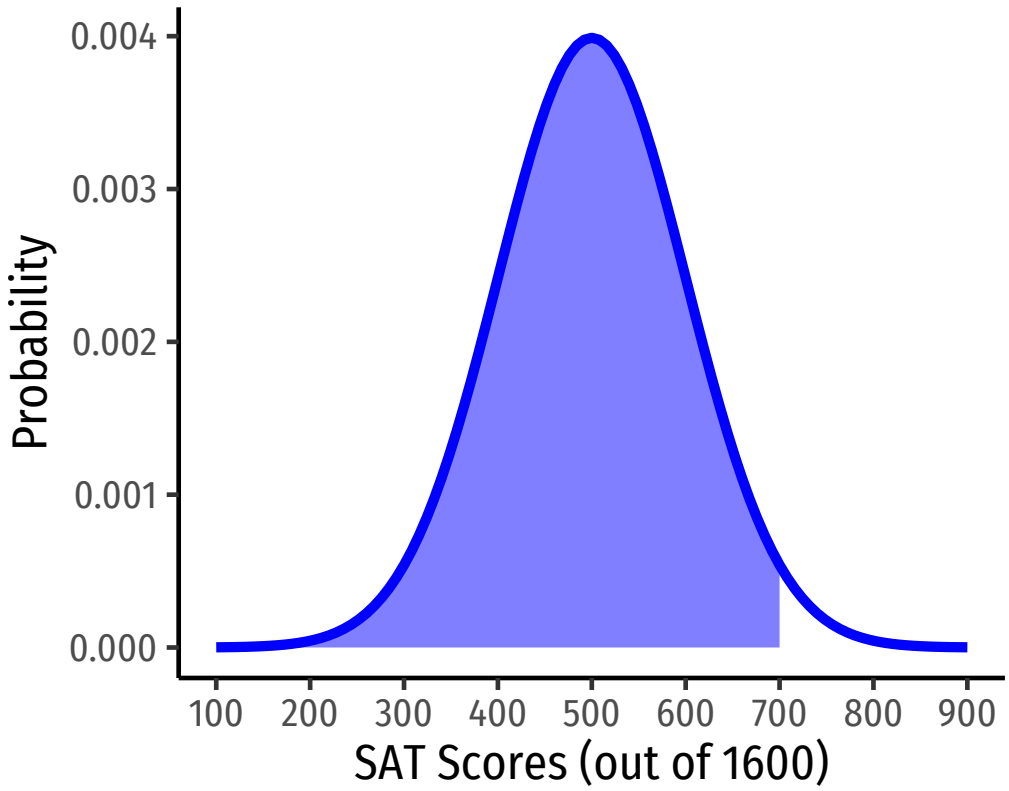

```
Z+stat_function(fun = dnorm, geom = "area", xlim = c(2,4), size = 2, fill = "blue", alpha = 0.5)
```



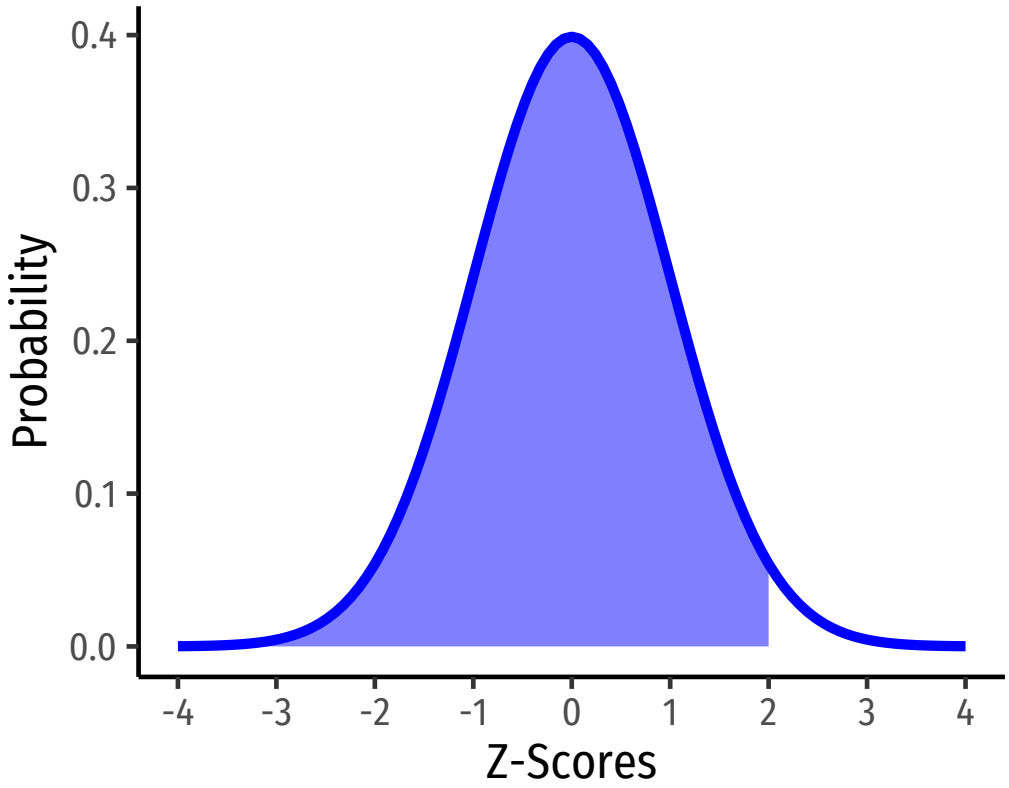
Part D

What is the probability of getting *at most* a 700? We saw in Part C that the area to the right of $Z = 2$ is 2.5%, so the remaining 97.5% falls to the left of $Z = 2$.

```
s_plot+stat_function(fun = dnorm, args = list(mean = 500, sd = 100), geom = "area", xlim = c(100,700), s
```




```
Z+stat_function(fun = dnorm, geom = "area", xlim = c(-4,2), size = 2, fill = "blue", alpha = 0.5)
```



Part E

What is the probability of getting exactly a 500? This is a trick question! For a continuous random variable, the probability of any one specific value is 0.

Question 6

Redo problem 5 by using the `pnorm()` command in `R`.[2]

Part A

Look back to the graphs of the pdfs in Question 5 to visualize what we are looking for.

`pnorm` converts X to Z and takes the **standard normal cdf**, Φ of a variable, i.e.

$$\Phi(k) = P(Z \leq k)$$

That means, it calculates the probability of everything up to (to the *left* of) that value on the **pdf.** So, if you want to calculate the area between values j and k, you need to take the cdf of the larger number and subtract the cdf of the smaller number:

$$P(j \leq Z \leq k) = \Phi(k) - \Phi(j)$$

In this case (in Z-scores):

$$P(-1 \leq Z \leq 1) = \Phi(1) - \Phi(-1)$$

```
pnorm(600, mean = 500, sd = 100, lower.tail = TRUE) - pnorm(400, mean = 500, sd = 100, lower.tail = TRU
```

```
## [1] 0.6826895
```

Part B

In this case (in Z-scores):

$$P(-2 \leq Z \leq 2) = \Phi(2) - \Phi(-2)$$

```
pnorm(700, mean = 500, sd = 100, lower.tail = TRUE) - pnorm(300, mean = 500, sd = 100, lower.tail = TRU
```

```
## [1] 0.9544997
```

Part C

In this case (in Z-scores):

$$P(Z \geq 2) = 1 - P(Z \leq 2) = 1 - \Phi(2)$$

```
1- pnorm(700, mean = 500, sd = 100, lower.tail = TRUE)
```

```
## [1] 0.02275013
```

[2]Hint: This function has four arguments: 1. the value of the random variable, 2. the mean of the distribution, 3. the sd of the distribution, and 4. `lower.tail` `TRUE` or `FALSE`.

Part D

In this case (in Z-scores):

$$P(Z \leq 2) = \Phi(2)$$

```
pnorm(700, mean = 500, sd = 100, lower.tail = TRUE)
```

```
## [1] 0.9772499
```

We can see that the 68-95-99.7 rule is *close*, but not *exactly* equal to the true probabilities of Z. As such, it's just a good rule of thumb!

Part E

Again, the probability is 0, no need to do anything here.