PHY 329 Homework 6

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Problem 24.1

We are asked to solve the boundary value problem $\frac{d^2T}{dx^2} - 0.15T = 0, T(0) = 240, T(10) = 150$ using various methods.

Part a) Analytically

We will use the method of characteristic equation by making an Ansatz, $T(x) = e^{rx}$. Therefore, $T' = re^{rx}$, $T'' = r^2e^{rx}$.

$$
\begin{aligned}
\frac{d^2T}{dx^2} - 0.15T = 0 \\
r^2e^{rx} - 0.15e^{rx} = 0 \\
e^{rx}(r^2 - 0.15) = 0 \\
e^{rx} \neq 0 \; \forall x, r \in \mathbb{R} \\
r^2 - 0.15 = 0 \\
r = \pm\sqrt{0.15} \\
T(x) = Ae^{\sqrt{0.15}x} + Be^{-\sqrt{0.15}x} \\
T(0) = A + B = 240 \therefore A = 240 - B \\
T(10) = (240 - B)e^{\sqrt{0.15}(10)} + Be^{-\sqrt{0.15}(10) = 150} \\
B = (150 - 240e^{\sqrt{0.15}(10)})/(e^{-\sqrt{0.15}(10)}) - e^{\sqrt{0.15}(10)} \approx 236.983 \\
A = 240 - B \approx 3.017 \\
T(x) \approx 3.017e^{\sqrt{0.15}x} + 236.983e^{-\sqrt{0.15}x}
\end{aligned}
$$

This exact solution is denoted in Figure 1.

Part b) Shooting Method

Using a function I developed to perform the shooting method using MATLAB's ode45 solver, then plot the results, I found the solution illustrated in Figure 2.

Part c) Finite Differences ($\mathbf{\Delta x = 1}$)

Since we know the values of $T(x)$ at the boundary, we can set up a problem for the 9 interior nodes defined by our given spacing Δx. This creates a 9x9 tridiagonal matrix.

First, we discretize our differential equation by letting $y'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$. Since $h = \Delta x = 1$, the denominator can be ignored.

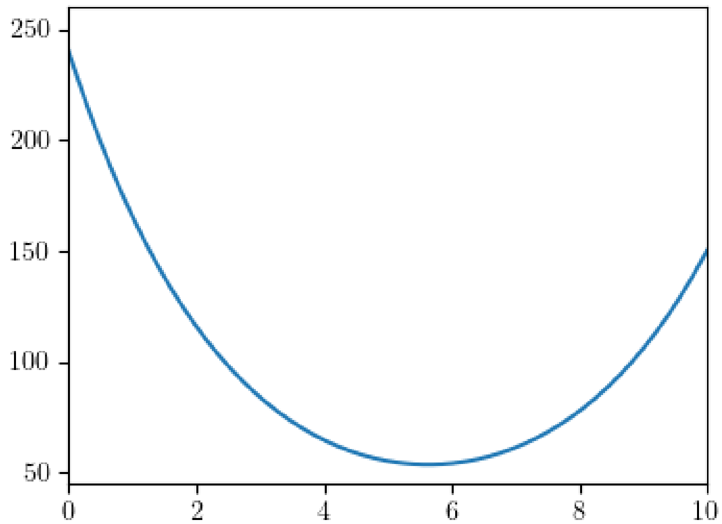

Figure 1: Plot of solution to BVP denoted by Problem 24.1 solved analytically, horizontal x axis and vertical T axis

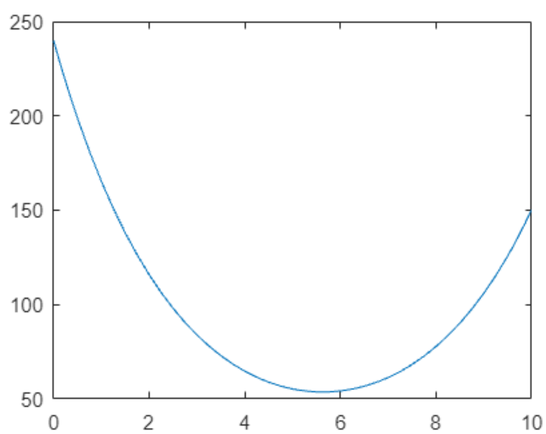

Figure 2: Plot of solution to BVP denoted by Problem 24.1 solved using shooting method, horizontal x axis and vertical T axis

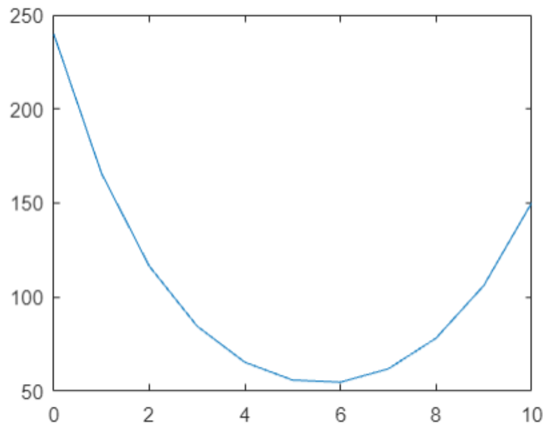

Figure 3: Plot of solution to BVP denoted by Problem 24.1 solved using finite differences, horizontal x axis and vertical T axis

Rewriting the differential equation, we find: $T_{i+1} - 2T_i + T_{i-1} - 0.15T_i = 0$ which can be rewritten as $T_{i-1} - 2.15T_i + T_{i+1} = 0$. This allows us to set up our three diagonals. For the first node, we know $T_{i-1} = T(0) = 240$. Therefore, the equation simplifies to $-2.15T_i + T_{i+1} = -240$. For the last node, we know $T_{i+1} = T(10) = 150$. Therefore, the equation simplifies to $T_{i-1} - 2.15T_i = -150$. This creates a system $\mathbf{A}\vec{x} = \vec{b}$.

$$\begin{pmatrix} -2.15 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & -2.15 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & & & & & & & & 0 \\ 0 & & & & & & & & 0 \\ 0 & & & & & & & & 0 \\ 0 & & & & & & & & 0 \\ 0 & & & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2.15 & 1 \end{pmatrix} \begin{pmatrix} T(1) \\ T(2) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ T(9) \end{pmatrix} \begin{pmatrix} -240 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ -150 \end{pmatrix}$$

Using MATLAB to perform the matrix calculations on my tridiagonal matrix, I found solution illustrated in Figure 3.

This method could perform better given a larger matrix with more interior nodes.

Problem 24.3

We are asked to use the shooting method to solve the boundary value problem $7\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y + x = 0, y(0) = 5, y(20) = 8$.

Using the same script I developed, I produced a result as shown in Figure 4.

Problem 24.21

We are asked to use the shooting method to solve the boundary value problem $\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} - g = 0, x(0) = 0, x(12) = 500$. We are given parameters $c = 12.5$ kg/s, $m = 70$ kg, $g = 9.81$ m/s^2.

Using the same script I developed, I produced a result as shown in Figure 5.

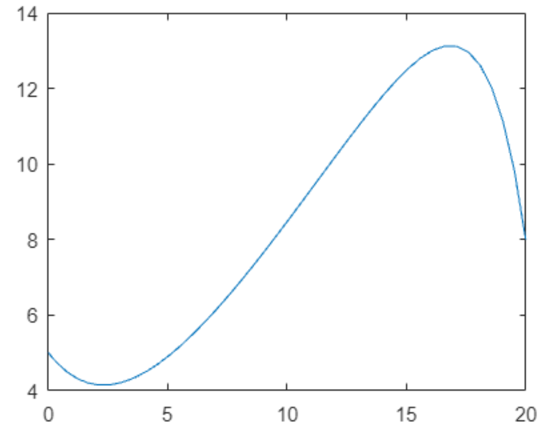

Figure 4: Plot of solution to BVP denoted by Problem 24.3 solved using shooting method, horizontal x axis and vertical y axis

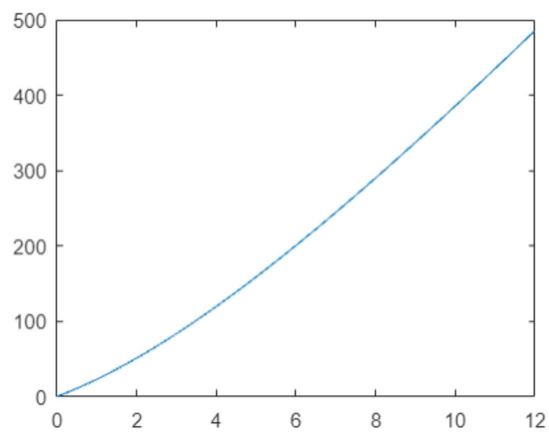

Figure 5: Plot of solution to BVP denoted by Problem 24.21 solved using shooting method, horizontal t axis and vertical x axis

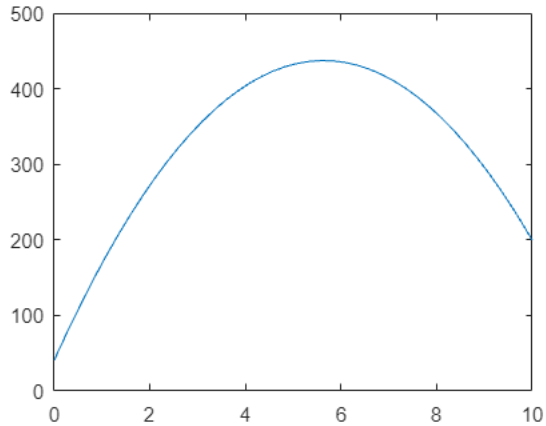

Figure 6: Plot of solution to BVP denoted by Problem 24.28 solved using shooting method, horizontal x axis and vertical T axis

Problem 24.28

We are asked to solve the boundary value problem $\frac{d^2T}{dx^2} = -f(x), T(0) = 40, T(10) = 200$ where $f(x) = 25$ using various methods.

Part a) Shooting Method

Using the same script I developed, I produced a result as shown in Figure 6.

Part b) Finite Difference Method

By discretizing the differential equation using 9 interior nodes as before, I arrive at the matrix equation:

$$\begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & & & & & & & & 0 \\ 0 & & & & & & & & 0 \\ 0 & & & & & & & & 0 \\ 0 & & & & & & & & 0 \\ 0 & & & & & & & & 0 \\ 0 & & & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} T(1) \\ T(2) \\ \vdots \\ \\ \\ \\ \\ \\ \\ T(9) \end{pmatrix} \begin{pmatrix} -65 \\ -25 \\ \vdots \\ \\ \\ \\ \\ \\ -25 \\ -225 \end{pmatrix}$$

Using MATLAB to perform the matrix calculations on my tridiagonal matrix, I found solution illustrated in Figure 7.

Part c) bvp4c

Using MATLAB's built in boundary value problem solver, I found the solution shown in Figure 8.

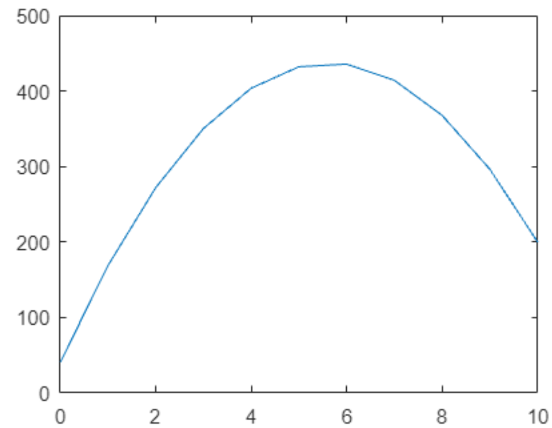

Figure 7: Plot of solution to BVP denoted by Problem 24.28 solved using finite differences, horizontal x axis and vertical T axis

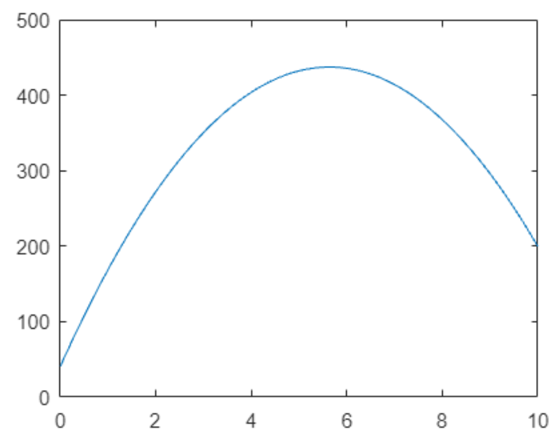

Figure 8: Plot of solution to BVP denoted by Problem 24.1 solved using MATLAB bvp4c function, horizontal x axis and vertical T axis