PHY 329 Homework 2

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Problem 9.16

Given the equations

$$\begin{aligned} 2x_1 - 6x_2 - x_3 &= -38 \\ -3x_1 - x_2 + 7x_3 &= -34 \\ -8x_1 + x_2 - 2x_3 &= -20 \end{aligned}$$

Part a

Solve by Gaussian elimination with partial pivoting, using the diagonal elements to calculate the determinant.

First, I will create an augmented matrix such that

$$\mathbf{A}\vec{x} = \vec{b},\ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ \vec{b} = \begin{bmatrix} -38 \\ -34 \\ -20 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -6 & -1 & -38 \\ -3 & -1 & 7 & -34 \\ -8 & 1 & -2 & -20 \end{bmatrix}$$

Now, to create a pivot in the first entry of the first row, we divide the entire first row by 2.

$$\begin{bmatrix} 1 & -3 & -0.5 & -19 \\ -3 & -1 & 7 & -34 \\ -8 & 1 & -2 & -20 \end{bmatrix}$$

Now add 3 times the first row to the second to eliminate the first entry of the second row. Similarly, add 8 times the first row to the third to eliminate the first entry of the third row.

$$\begin{bmatrix} 1 & -3 & -0.5 & -19 \\ 0 & -10 & 5.5 & -91 \\ 0 & -23 & -6 & -172 \end{bmatrix}$$

Now replace the second row with -1/10 times itself to normalize the second entry to 1. Similarly, replace the third row with 1/23 times itself. This will create a -1 in the second column of the third row.

$$\begin{bmatrix} 1 & -3 & -0.5 & -19 \\ 0 & 1 & -0.55 & 9.1 \\ 0 & -1 & -6/23 & -172/23 \end{bmatrix}$$

Now, we can easily replace the third row with the sum of itself and the second row.

$$\begin{bmatrix} 1 & -3 & -0.5 & -19 \\ 0 & 1 & -0.55 & 9.1 \\ 0 & 0 & -0.811 & 1.622 \end{bmatrix}$$

We could continue to normalize the third row, but this is in upper triangular form, allowing us to calculate the determinant as the product of the diagonal which is -0.811, and to solve the system via back substitution. Using the matrix equation, we can see the following:

$$\begin{aligned} -0.811x_3 &= 1.622 \therefore x_3 = -2 \\ x_2 - 0.55x_3 &= 9.1 \therefore x_2 = 8 \\ x_1 - 3x_2 - x_3/2 &= -19 \therefore x_1 = 4 \end{aligned}$$

This yields a solution $x_1 = 4, x_2 = 8, x_3 = -2.$

Part b

Substituting our results back into the original three equations, we yield:

$$\begin{aligned} 2(4) - 6(8) - (-2) &= -38 \\ -3(4) - 8 + 7(-2) &= -34 \\ -8(4) + 8 - 2(-2) &= -20 \end{aligned}$$

We find that our results are exact and solve the system.

Problem 9.16

We are tasked with solving a pentadiagonal system without pivoting and to test it for $\mathbf{A}\vec{x} = \vec{b}$.

$$\mathbf{A} = \begin{bmatrix} 8 & -2 & -1 & 0 & 0 \\ -2 & -9 & -4 & -1 & 0 \\ -1 & -3 & 7 & -1 & -2 \\ 0 & -4 & -2 & 12 & -5 \\ 0 & 0 & -7 & -3 & -15 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 1 \\ 5 \end{bmatrix}$$

The solution to this matrix is $x_1 = 0.8178, x_2 = 0.5837, x_3 = 0.3748, x_4 = 0.1187, x_5 = -0.5320.$

Problem 10.10

Solve the following system with LU factorization with pivoting. (I will represent it as an augmented matrix equation).

$$\begin{bmatrix} 3 & -2 & 1 & -10 \\ 2 & 6 & -4 & 44 \\ -1 & -2 & 5 & -26 \end{bmatrix}$$

Using MATLAB's LU factorization function, we find the solution to this matrix is $x_1 = 1, x_2 = 5, x_3 = -3.$

Problem 10.11

Part a) LU Factorization

We are tasked with determining the LU factorization without pivoting of the following matrix by hand and checking the results such that $\mathbf{LU} = \mathbf{A}$.

$$\begin{bmatrix} 8 & 2 & 1 \\ 3 & 7 & 2 \\ 2 & 3 & 9 \end{bmatrix}$$

We define LU decomposition as creating two matrices, one lower diagonal, $\mathbf{L}$, and one upper diagonal, $\mathbf{U}$. Since these two matrices are not a unique factorization for $\mathbf{A}$, I will choose the diagonal elements of $\mathbf{L} = 1$. To prevent over-constraining my problem, I will let $\mathbf{U}$ be composed of any values forming an upper triangular matrix. This creates matrices as follows.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Since $\mathbf{A} = \mathbf{LU}$, we simply multiply these two matrices to produce the following matrix, which by definition is equal to $\mathbf{A}$.

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}$$

One can simply see that the first row of this product matrix, which is equal to the first row of $\mathbf{U}$ is simply the first row of $\mathbf{A}$. Then, by substitution, we can find the other terms of the matrix simply.

Ultimately, we arrive at the following factorization:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.375 & 1 & 0 \\ 0.25 & 0.4 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 8 & 2 & 1 \\ 0 & 6.25 & 1.625 \\ 0 & 0 & 8.1 \end{bmatrix}$$

If we multiply these matrices, we can check to see that $\mathbf{A} = \mathbf{LU}$.

Part b) Determinant

We can compute the determinant of $\mathbf{A}$ quite simply as the determinant of $\mathbf{L}$ multiplied by the determinant of $\mathbf{U}$. Since these are triangular matrices, the determinants are the products of their diagonals. Therefore $\det(\mathbf{A}) = 8 \times 6.25 \times 8.1 = 405$.

Part c)

We are then asked to repeat the steps using MATLAB. I implemented an LU factorization M-file that did not use pivoting and called it. I was able to check that my factorization and determinant are correct.

Problem 10.12

Given the following LU factorization, compute the determinant and solve for $\mathbf{A}x^T = [-10, 44, -26]$.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.6667 & 1 & 0 \\ -0.3333 & -0.3636 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 7.3333 & -4.6667 \\ 0 & 0 & 3.6364 \end{bmatrix}$$

Part a

As we did with our LU factorization, we can compute the determinant as the product of the diagonal entries of both matrices. This yields $det(\mathbf{LU}) = 3 \times 7.3333 \times 3.6364 = 80.0004$.

Part b

We can solve the system by defining an intermediary solution vector, $\vec{d}$ such that $\mathbf{L}\vec{b} = \vec{d}$. Then, we can find our solution as $\mathbf{U}\vec{x} = \vec{b}$. Using MATLAB, we find a solution of $\vec{b} = [0.9999, 4.9997, -3.0004]$.

Problem 11.2

We are asked to find the inverse of the matrix corresponding to the system:

$$\begin{aligned} -8x_1 + x_2 - 2x_3 &= -20 \\ 2x_1 - 6x_2 - x_3 &= -38 \\ -3x_1 - x_2 + 7x_3 &= -34 \end{aligned}$$

Evidently, the corresponding matrix equation $\mathbf{A}\vec{x} = \vec{b}$ would be

$$A = \begin{bmatrix} -8 & 1 & -2 \\ 2 & -6 & -1 \\ -3 & -1 & 7 \end{bmatrix}$$

and $\vec{b} = [-20, -38, -34]$. Using MATLAB's built-in inv function, we find that the inverse of matrix $\mathbf{A}$ which we will call $\mathbf{A^{-1}}$ is

$$\begin{bmatrix} -0.1153 & -0.0134 & -0.0349 \\ -0.0295 & -0.1662 & -0.0322 \\ -0.0536 & -0.0295 & 0.1233 \end{bmatrix}$$

This yields a solution $\vec{x} = \mathbf{A^{-1}}\vec{b}$.

Problem 11.6

We are asked with finding the Frobenius, Column-Sum, and Row-Sum norms of the following matrix after scaling them such that the maximum value of each row is 1.

$$A = \begin{bmatrix} 8 & 2 & -10 \\ -9 & 1 & 3 \\ 15 & -1 & 6 \end{bmatrix}$$

To scale this matrix, we simply divide each row by its greatest element.

$$A = \begin{bmatrix} -0.8 & -0.2 & 1 \\ 1 & -1/9 & -1/3 \\ 1 & -1/15 & 2/5 \end{bmatrix}$$

Then, we can use MATLAB's norm function to calculate the following norms. Frobenius Norm = 1.9920. Column-Sum Norm = 2.8000. Row-Sum Norm = 2.0000.

Problem 11.8

We are asked to use MATLAB to determine the spectral condition number of the following matrix without normalization.

$$\begin{bmatrix} 1 & 4 & 9 & 16 & 25 \\ 4 & 9 & 16 & 25 & 36 \\ 9 & 16 & 25 & 36 & 49 \\ 16 & 25 & 36 & 49 & 64 \\ 25 & 36 & 49 & 64 & 81 \end{bmatrix}$$

We can use the cond function to find the spectral condition number to be 8.037e16 and the row-sum condition number to be 1.723e18.

Problem 12.2

Part a) Gauss-Seidel

We are asked to solve the system $\mathbf{A}\vec{x} = [41, 25, 105]$ where

$$A = \begin{bmatrix} 0.8 & -0.4 & 0 \\ -0.4 & 0.8 & -0.4 \\ 0 & -0.4 & 0.8 \end{bmatrix}$$

Using an M-file and a relative error of 5%, we can calculate the solution to be $\vec{x} = [167.8711, 239.1211, 250.8105]$.

Part b) Overrelaxatioon

Now, we are asked to solve the system with Gauss-Seidel and overrelaxation of lambda = 1.2.

After implementing a relaxation term into my Gauss-Seidel M-file, I found the solution $\vec{x} = [79.6875, 150.9375, 206.7188]$.

Problem 12.8

Using Newton-Raphson with initial guesses $x = y = 1.2$, we are asked to solve the system:

$$y = -x^2 + x + 0.75$$
$$y + 5xy = x^2$$

By rearranging both equations into multivariate root finding problems, we can use a modified version of Newton-Raphson which uses the Jacobian of the two equations instead of a normal derivative of one function. Ultimately, our results are $x = 1.3721, y = 0.2395$.

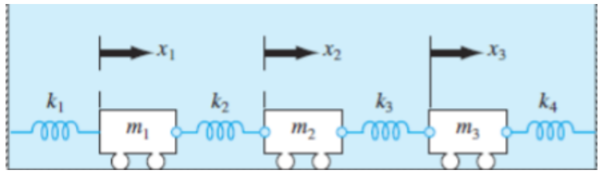

Figure 1: A simple schematic corresponding to Problem 13.4

Problem 13.4

We are asked to derive the system of equations for a three mass and four spring system in which all masses are equal, $k_1 = k_4$ N/m, and $k_2 = k_3 = 35$ N/m. A schematic of this problem can be found in Figure 1.

I will drop the subscript on mass since they are all equal ($m = 1.5$ kg) while denoting their position according to the mass subscripts in the figure. By using Newton's second law on each mass, we can express the problem as a system of ordinary differential equations:

$$\begin{aligned} m\ddot{x_1} &= (-k_1 - k_2)x_1 + k_2 x_2 \\ m\ddot{x_2} &= k_2 x_1 + (-k_2 - k_3)x_2 + k_3 x_3 \\ m\ddot{x_3} &= k_3 x_2 + (-k_4 - k_3)x_3 \end{aligned}$$

These equations can be rewritten such that $0 = [\ddot{x_1}, \ddot{x_2}, \ddot{x_3}]$ - $\mathbf{M}[x_1, x_2, x_3]$ where

$$M = \begin{bmatrix} \frac{-k_1-k_2}{m} & \frac{k_2}{m} & 0 \\ \frac{k_2}{m} & \frac{-k_2-k_3}{m} & \frac{k_3}{m} \\ 0 & \frac{k_3}{m} & \frac{-k_3-k_4}{m} \end{bmatrix} = \begin{bmatrix} -3.3333 & 2.3333 & 0 \\ 2.3333 & -4.6667 & 2.3333 \\ 0 & 2.3333 & -3.3333 \end{bmatrix}$$

Solving for this matrix's eigenvalues will give us the negative square of the resonant frequencies of the system as shown by the following ansatz:

$$\begin{aligned} x(t) &= A \sin \omega t \\ \ddot{x(t)} &= -A\omega^2 \sin \omega t = -\omega^2 x(t) \\ -\omega^2 x(t) &= \begin{bmatrix} -3.3333 & 2.3333 & 0 \\ 2.3333 & -4.6667 & 2.3333 \\ 0 & 2.3333 & -3.3333 \end{bmatrix} \\ x(t) &= \begin{bmatrix} -3.3333 + \omega^2 & 2.3333 & 0 \\ 2.3333 & -4.6667 + \omega^2 & 2.3333 \\ 0 & 2.3333 & -3.3333 + \omega^2 \end{bmatrix} \end{aligned}$$

Since eigenvalues are typically subtracted across the diagonal, we will multiply the eigenvalues MATLAB gives us by -1 before taking the square root. Using MATLAB, we find corrected eigenvalues 7.3665, 3.3333, 0.6335. This yields resonant frequencies of 2.7141, 1.8257, 0.7959 rad/s.

Problem 13.5

Considering a mass spring system described the the following matrix equation, we are asked to use MATLAB to find the eigenvalues and the frequencies of the system.

$$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This simply can be entered into MATLAB which produces eigenvalues 0 and 6 with multiplicity 2. Therefore, the non-trivial resonant frequency is 2.4495 rad/s.

Problem 13.6

We are asked to solve the following system of equations given $L = 1$ H, $C = 0.25$ F:

$$
\begin{aligned}
L_1\ddot{i_1} + \frac{1}{C_1}(i_1 - i_2) &= 0 \\
L_2\ddot{i_2} + \frac{1}{C_2}(i_2 - i_3) - \frac{1}{C_1}(i_1 - i_2) &= 0 \\
L_3\ddot{i_3} + \frac{1}{C_3}i_3 - \frac{1}{C_2}(i_2 - i_3) &= 0
\end{aligned}
$$

This can be rewritten as an eigenvalue problem of the form:

$$
\begin{bmatrix}
\frac{1}{L_1C_1} & \frac{-1}{L_1C_1} & 0 \\
\frac{-1}{L_1C_1} & \frac{1}{L_2C_1} + \frac{1}{L_2C_2} & \frac{-1}{L_2C_2} \\
0 & \frac{-1}{L_3C_2} & \frac{1}{L_3C_3} + \frac{1}{L_3C_2}
\end{bmatrix}
$$

Using our values for L and C and MATLAB, we find eigenvalues 0.7922, 6.2198, 12.9879. Taking the square root, we find resonant frequencies of 0.8901, 2.4940, 3.6039 rad/s.