CS 278 Lab 11

8.5.1 b) Prove that for any positive integer n, 6 evenly divides $7^n - 1$

1) Base case: $n = 1$ $\quad (7^1 - 1)\%6 = 0$

or $7^n - 1 = 6m \quad m \in \mathbb{Z}$

$7^{(1)} = 6m + 1$

$6 = 6m \quad m = 1$

2) Induction hypothesis: Assume 6 evenly divides $7^k - 1$, $k \geq 1$. So $\exists m$ such that $6m = 7^k - 1$; $7^k = 6m + 1$

3) Induction: We need to prove 6 evenly divides $7^{k+1} - 1$; $\exists n \in \mathbb{Z}$ such that $6n = 7^{k+1} - 1$

$$
\begin{aligned}
RHS &= 7^{k+1} - 1 \\
&= 7 \cdot 7^k - 1 \\
&= 7 \cdot (6m + 1) - 1 \\
&= 42m + 7 - 1 \\
&= 42m + 6 \\
&= 6(7m + 1)
\end{aligned}
$$

$m \in \mathbb{Z}$, so $7m + 1$ is an integer. So $7^{k+1} - 1 = 6$(some integer). QED

8.5.1 c) Prove that for any positive integer n, 4 evenly divides $11^n - 7^n$

1) Base case: Let $n=1$. We need to show

$4m = 11^n - 7^n$, $m \in \mathbb{Z}$.

$$4m = 11^1 - 7^1$$
$$4m = 4$$
$$m = 1$$

2) Inductive hypothesis: Let us assum for random value $k \in \mathbb{Z}^+$, 4 evenly divides $11^n - 7^n$, so $\exists\, m$ such that $4m = 11^k - 7^k$

3) Induction: We need to prove 4 evenly divides $11^{k+1} - 7^{k+1}$: $\exists\, n \in \mathbb{Z}$ such that $4n = 11^{k+1} - 7^{k+1}$.

$$\begin{aligned}
RHS &= 11^{k+1} - 7^{k+1} \\
&= 11 \cdot 11^k - 7 \cdot 7^k \\
&= 11(4m + 7^k) - 7 \cdot 7^k \\
&= 44m + 11(7^k) - 7(7^k) \\
&= 44m + 7^k \cdot 4 \\
&= 4(11m + 7^k)
\end{aligned}$$

$m \in \mathbb{Z}$, $k \geq 1$, so $11m + 7^k$ is some integer, so $11^{k+1} - 7^{k+1} = 4$(some integer), so 4 divides $11^{k+1} - 7^{k+1}$. QED

8.5.1 e) Prove that for any positive integer n, 2 evenly divides n^2-5n+2

1) Base case: Set $n=1$, We need to show 2 divides n^2-5n+2,

$2m = n^2-5n+2$, $m \in \mathbb{Z}$

$2m = 1^2-5(1)+2$

$2m = -2$

$m = -1$

2) Inductive hypothesis: Assume 2 evenly divides k^2-5k+2; $k \geq 1$. So $\exists m$ such that

$2m = k^2-5k+2$

3) Induction: We need to show 2 evenly divides $(k+1)^2-5(k+1)+2$. $\exists n \in \mathbb{Z}$ such that $2n = (k+1)^2-5(k+1)+2$

RHS $= (k+1)^2-5(k+1)-2$

$= k^2+2k+1-5k-5-2$

$= (k^2-5k+2)+2k-8$

$= 2m+2k-8$

$= 2(m+k-4)$

Since $m, k \in \mathbb{Z}$, $m+k-4$ is some integer, So $(k+1)^2-5(k+1)-2 = 2(\text{some integer})$

QED

8.5.3 b) $\{b_n\}$ is defined as: $b_0 = 1$, $b_n = 2b_{n-1} + 1$ for $n \geq 1$, Prove that for $n \geq 0$, $b_n = 2^{n+1} - 1$

Proof: 1) Base case: Set $n = 1$

$LHS = b_1 = 2(b_0) + 1 = 3.$

$RHS = 2^{1+1} - 1 = 4 - 1 = 3$

2) Inductive hypothesis: Let us assume for $k \geq 0$

$b_k = 2^{k+1} - 1$

3) Induction: We need to show for $k+1$; $k \geq 0$,

$b_{k+1} = 2^{(k+1)+1} - 1$

$b_{k+1} = 2(2^{(k+1)} - 1) + 1$

$= 2^{k+2} - 2 + 1$

$= 2^{k+2} - 1 = RHS$ QED

8.5.3 c) $\{a_n\}$ is defined as: $a_1 = 6$, $a_n = 2 \cdot a_{n-1} + 2n$ for $n \geq 2$

Prove that for any positive integer $n \geq 1$, $a_n = -2n - 4 + 6 \cdot 2^n$

Base case: Set $n = 2$

$LHS = a_2 = 2(6) + 2(2) = 12 + 4 = 16$

$RHS = -2(2) - 4 + 6 \cdot 2^2 = -4 - 4 + 24 = 16$

Inductive hypothesis: Let us assume for $k \geq 1$, $a_k = -2k - 4 + 6 \cdot 2^k$

Induction: We need to show for $k+1$; $k \geq 1$, $a_{k+1} = -2(k+1) - 4 + 6 \cdot 2^{(k+1)}$

$a_{k+1} = 2(-2k - 4 + 6 \cdot 2^k) + 2k$

$= -4k - 8 + 12 \cdot 2^k + 2k$

$= -2k - 8 + 6 \cdot 2^{k+1}$

$= -2(k+1) - 4 + 6 \cdot 2^{(k+1)}$ QED