CS 278 HW 11

8.4.2 ⓒ Prove that for any positive integer n,
$$\sum_{j=1}^{n} j(j-1) = \frac{n(n^2-1)}{3}$$

Sol: 1) Base case : when $n=1$;
$$LHS = \sum_{j=1}^{1} 1(1-1) = 0$$
$$RHS = \left(\frac{1(1^2-1)}{3}\right) = 0/3 = 0$$

2) Inductive hypothesis: Let us assume that for a random $n \in \mathbb{Z}^+$, $\sum_{j=1}^{n} j(j-1) = \left(\frac{n(n^2-1)}{3}\right)$

3) Induction: We need to prove when $n = k+1$, $\sum_{j=1}^{k+1} = \left(\frac{(k+1)((k+1)^2-1)}{3}\right)$

$$LHS = \sum_{j=1}^{k+1} j(j-1)$$
$$= \sum_{j=1}^{k} j(j-1) + (k+1)((k+1)-1)$$
$$= \frac{k(k^2-1)}{3} + \frac{3[(k+1)(k)]}{3}$$
$$= \frac{k^3-k+3k^2+3k}{3}$$
$$= \frac{k^3+3k^2+2k}{3}$$
$$= \frac{k(k+2)(k+1)}{3}$$

$$= \frac{(k+1)\left[(k^2+2k+1)-1\right]}{3}$$

$$= \frac{(k+1)\left((k+1)^2-1\right)}{3} = \text{RHS} \quad \text{QED}$$

(e) Prove for any non-negative integer n,

$$\sum_{j=0}^{n} j\cdot 3^j = 3/4\left[n\cdot 3^{n+1} - (n+1)3^n + 1\right]$$

SOL: 1) Base case: when $n=0$

$$\text{RHS} = 3/4\left[0\cdot 3^{0+1} - (0+1)3^0 + 1\right]$$
$$= 3/4\left[0 - 1 + 1\right] = 0$$
$$\text{LHS} = 0\cdot 3^0 = 0$$

2) Inductive hypothesis: Let us assume that for a random $n \in (\mathbb{Z} \geq 0)$, $\sum_{j=0}^{n} j\cdot 3^j = 3/4\left[n\cdot 3^{n+1} - (n+1)3^n + 1\right]$

3) We need to prove when $n = k+1$,

$$\sum_{j=0}^{k+1} j\cdot 3^j = 3/4\left[(k+1)\cdot 3^{(k+1)+1} - ((k+1)+1)3^{k+1} + 1\right]$$

$$\text{LHS} = \sum_{j=0}^{k+1} j\cdot 3^j$$
$$= \sum_{j=0}^{k} j\cdot 3^j + \left((k+1)\cdot 3^{(k+1)}\right)$$
$$= 3/4\left[k\cdot 3^{k+1} - (k+1)3^k + 1\right] + \left((k+1)\cdot 3^{(k+1)}\right)$$
$$= \frac{3}{4}\left[\left(k\cdot 3^{k+1} - (k+1)3^k + 1\right) + 4\left((k+1)3^k\right)\right]$$
$$= \frac{3}{4}\left[3k3^k + 3\left((k+1)3^k\right) + 1\right]$$
$$= \frac{3}{4}\left[3k3^k + \left((k+1)3^{(k+1)}\right) + 1\right]$$

$= (\ldots)$
$= RHS$ QED

8.4.2 f) Prove that for any positive integer $n \geq 2$,
$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$$

1) Base Case:
$LHS = 1 - \frac{1}{2^2}$ $\quad RHS = \frac{2+1}{2(2)}$
$= 3/4$ $\quad = 3/4$

2) Inductive hypothesis: Let us assume that for any $k \in (\mathbb{Z}^+ \geq 2)$
$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$$

3) Induction: We need to prove that for any $n = k+1$, $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right) = \frac{n+1+1}{2(n+1)}$

$$LHS = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)$$
$$= \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{(n+1)^2}\right)$$
$$= \frac{n+1}{2n} - \frac{n+1}{2n^3+4n^2+2n}$$
$$= \frac{(n+1)^3-(n+1)}{2n^3+4n^2+2n}$$
$$= \frac{(n+1)(n^2+2n+1)-(n+1)}{2n^3+4n^2+2n}$$
$$= \frac{(n+1)(n+1)(n+1)-(n+1)}{2n((n+1)(n+1)+1)}$$
$= \ldots$
$= RHS$ QED

8.4.3 b) For any $n \geq 1$, the factorial function for $n \geq 4$,
$n! \geq 2^n$

Base case: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$

Induction hypothesis: Let us assume that for random
$n \in (\mathbb{Z}^+ \geq 4)$, $n! \geq 2^n$

Induction:
$(n+1)! = n! \times (n+1)$
$(n+1)! \geq 2^n \times (n+1) \left(\frac{2}{2}\right)$
$\geq \frac{2^{n+1}(n+1)}{2} \geq 2^{n+1}$ QED

d) Prove that for any non-negative integer
$n \geq 3$, $3^n \geq n^3$

Base case $n = 3$ $\quad 3^3 = 27 \geq 3^3 = 27$

Inductive hypothesis: Let us assume that for random
$n \in (\mathbb{Z}^+ \geq 3)$, $3^n \geq n^3$

Induction:
$LHS = 3^{n+1}$
$\geq 3(3^n)$
$\geq 3n^3$
$\geq (n+1)^3$ QED

f) Prove that for any non-negative integer $n \geq 4$, $3^n \leq$
$(n+1)!$

Base case: $n = 4$, $3^4 = 81 \leq (4+1)! = 120$

inductive hypothesis: Let us assume for random $n \in (\mathbb{Z} \geq 4)$

$3^n \leq (n+1)!$

induction:

$LHS = 3^{n+1}$

$= 3(3^n)$

$\leq 3(n+1)!$

$\leq (n+1+1)!$ QED

a) Prove that for any non-negative integer $n \geq 0$ and

$a \geq 0$, $(1+na) \leq (1+a)^n$

Base case $n=0$ $1+0 \cdot 0 \leq (1+0)^0$

$1 \leq 1$

Inductive hypothesis: Let us assume random a

and random n, $\in (\mathbb{Z} \geq 0)$, $1+na \leq (1+a)^n$

Induction:

$LHS = 1 + (n+1)(a+1)$

$= (1+na) + n + a + 1$

$\leq (1+a)^n + n + a + 1$

$\leq (1+a+1)^{n+1}$ QED