MAT 230 EXAM TWO

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Directions: Type your solutions into this document and be sure to show all steps for arriving at your solution. Just giving a final number may not receive full credit.

Problem 1

This question has 2 parts.

Part 1: Suppose that F and X are events from a common sample space with $P(F) \neq 0$ and $P(X) \neq 0$.

(a) Prove that $P(X) = P(X|F)P(F) + P(X|\bar{F})P(\bar{F})$. Hint: Explain why $P(X|F)P(F) = P(X \cap F)$ is another way of writing the definition of conditional probability, and then use that with the logic from the proof of Theorem 4.1.1.

Considering that $P(X|F) = \frac{P(X \cap F)}{P(F)}$. Means, $P(X \cap F) = P(X|F)P(F)$.

Next, the sample space was then partitioned into the event: F and its complement $\bar{F}$. thus, the event X can occur either with F or with $\bar{F}$, and these two cases are mutually exclusive.

Next, by the Law of Total Probability, I noted that $P(X)$ can be obtained by adding $P(X)$ occurring with F and $P(X)$ occurring with $\bar{F}$: $P(X) = P(X \cap F) + P(X \cap \bar{F})$.

Now, I used the same technique that I used previously to express $P(X \cap \bar{F})$ in terms of conditional probability: $P(X \cap \bar{F}) = P(X|\bar{F})P(\bar{F})$.

Combining the previous expression into the Law of Total Probability, I got: $P(X) = P(X|F)P(F) + P(X|\bar{F})P(\bar{F})$.

(b) Explain why $P(F|X) = P(X|F)P(F)/P(X)$ is another way of stating Theorem 4.2.1 Bayes Theorem.

From the definition of Conditional Probability, I know that $P(F|X) = \frac{P(F \cap X)}{P(X)}$.
Next I noted that $P(F \cap X) = P(X \cap F)$, because the intersection is commutative. Then, I applied the definition of conditional probability again, to see that $P(X \cap F) = P(X|F)P(F)$. Then, I was able to substitute what I got earlier into the definition from the beginning, showing Bayes' Theorem: $P(F|X) = \frac{P(X|F)P(F)}{P(X)}$.

Part 2: A website reports that 70% of its users are from outside a certain country. Out of their users from outside the country, 60% of them log on every day. Out of their users from inside the country, 80% of them log on every day.

(a) What percent of all users log on every day? Hint: Use the equation from Part 1 (a).

The percent of all users who log on every day can be calculated using the formula from Part 1 (a), which is the Law of Total Probability:

$$P(\text{login}) = P(\text{login}|\text{outside})P(\text{outside}) + P(\text{login}|\text{inside})P(\text{inside})$$

Given that:

$$P(\text{outside}) = 0.70$$
$$P(\text{login}|\text{outside}) = 0.60$$
$$P(\text{login}|\text{inside}) = 0.80$$
$$P(\text{inside}) = 1 - P(\text{outside}) = 0.30$$

With this, the overall probability that a user logs on every day would be:

$$P(\text{login}) = 0.60 \times 0.70 + 0.80 \times 0.30 = 0.66$$

So, 66 percent of all users log on every day.

(b) Using Bayes Theorem, out of users who log on every day, what is the probability that they are from inside the country?

Using Bayes Theorem:

$$P(\text{inside}|\text{login}) = \frac{P(\text{login}|\text{inside})P(\text{inside})}{P(\text{login})}$$

Given my previous calculations, I got:

$$P(\text{inside}|\text{login}) = \frac{0.80 \times 0.30}{0.66} \approx 0.364$$

So, there is approximately a 36.4 percent probability that users who log on every day are from inside the country.

Problem 2

This question has 2 parts.

Part 1: The drawing below shows a Hasse diagram for a partial order on the set:
$\{A, B, C, D, E, F, G, H, I, J\}$

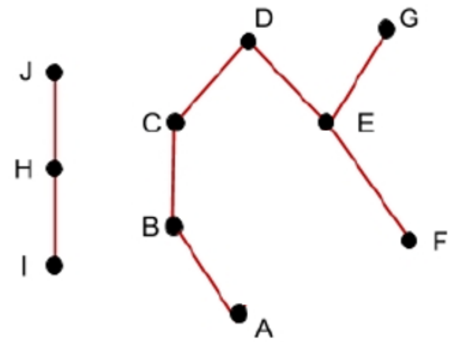


Figure 1: *A Hasse diagram shows 10 vertices and 8 edges. The vertices, represented by dots, are as follows: vertex J is upward of vertex H; vertex H is upward of vertex I; vertex B is inclined upward to the left of vertex A; vertex C is upward of vertex B; vertex D is inclined upward to the right of vertex C; vertex E is inclined upward to the left of vertex F; vertex G is inclined upward to the right of vertex E. The edges, represented by line segments between the vertices are as follows: 3 vertical edges connect the following vertices: B and C, H and I, and H and J; 5 inclined edges connect the following vertices: A and B, C and D, D and E, E and F, and E and G.*

Determine the properties of the Hasse diagram based on the following questions:

(a) What are the minimal elements of the partial order?

The **minimal elements** are those that have no edges descending to them. In this diagram, the minimal element is:

- **A**.

(b) What are the maximal elements of the partial order?

The **maximal elements** are those with no edges ascending from them. Here, the maximal elements are:

- **J**, **D**, and **G**.

(c) Which of the following pairs are comparable?

$$(A, D),\ (J, F),\ (B, E),\ (G, F),\ (D, B),\ (C, F),\ (H, I), (C, E)$$

Two elements are **comparable** if you can travel from one to the other following the edges without backtracking. Based on this diagram:

- **A** and **B** are comparable because you can move upwards from **A** to **B**.
- **B** and **C** are comparable because you can move upwards from **B** to **C**.
- **C** and **D** are comparable because you can move upwards from **C** to **D**.
- **D** and **E** are not comparable; there's no way to move from **D** to **E** or vice versa following the edges.
- **E** and **F** are comparable because you can move downwards from **E** to **F**.
- **E** and **G** are comparable because you can move upwards from **E** to **G**.

Part 2: Consider the partial order with domain {3, 5, 6, 7, 10, 14, 20, 30, 60, 70} and with $x \leq y$ if x evenly divides y. Select the correct Hasse diagram for the partial order.

(a)

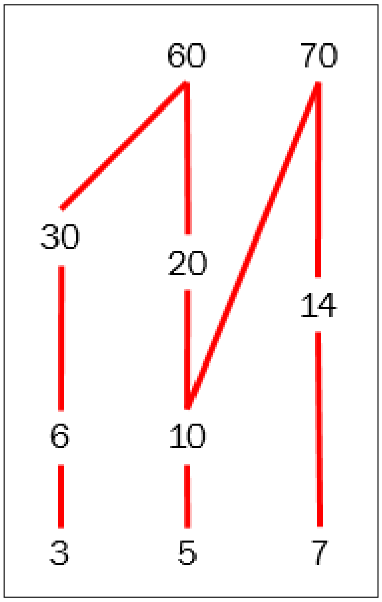


Figure 2: *A Hasse diagram shows a set of elements 3; 5; 6; 7; 10; 14; 20; 30; 60, 70. There are lines connecting 3 and 6, 6 and 30, 30 and 60, 5 and 10, 10 and 20, 20 and 60, 10 and 70, 7 and 14, 14 and 70.*

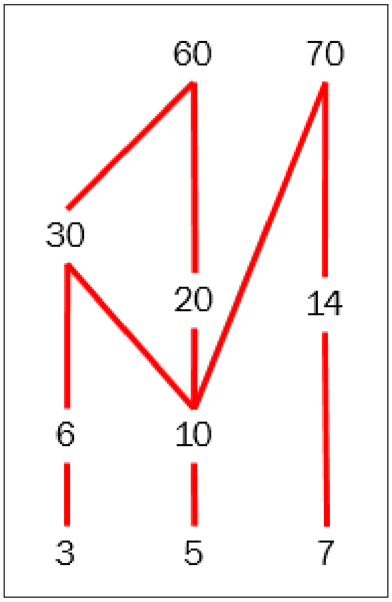
(b) 

Figure 3: *A Hasse diagram shows a set of elements 3; 5; 6; 7; 10; 14; 20; 30; 60, 70. There are lines connecting 3 and 6, 6 and 30, 30 and 60, 5 and 10, 10 and 30, 10 and 20, 20 and 60, 10 and 70, 7 and 14, 14 and 70.*

the correct Hasse diagram is described in **Figure 3**. This result resulted by comparing the explicit connections and relations depicted in the diagrams with the divisibility relations among the set elements.

The connections in **Figure 3** showcase the necessary relations that define our partial order, such as:

- 3 dividing 6, 30, and indirectly 60 through 30,
- 5 dividing 10, 20, 30, and indirectly 60 and 70 through 10 and 30,
- 6 dividing 30 and indirectly 60 through 30,
- 7 dividing 14 and indirectly 70 through 14,
- 10 dividing 20, 30, and indirectly 60 and 70 through 20 and 30,
- 14 dividing 70,
- 20 dividing 60,
- 30 dividing 60.

These relations are best represented in **Figure 3**, making it the best representation of the partial order based on divisibility among the given set elements.

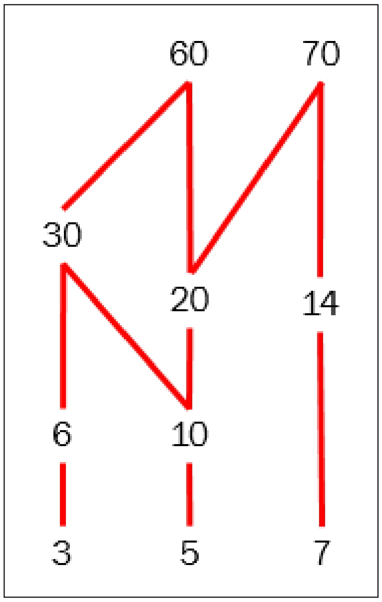
(c) 

Figure 4: *A Hasse diagram shows a set of elements 3; 5; 6; 7; 10; 14; 20; 30; 60, 70. There are lines connecting 3 and 6, 6 and 30, 30 and 60, 5 and 10, 10 and 30, 10 and 20, 20 and 60, 20 and 70, 7 and 14, 14 and 70.*

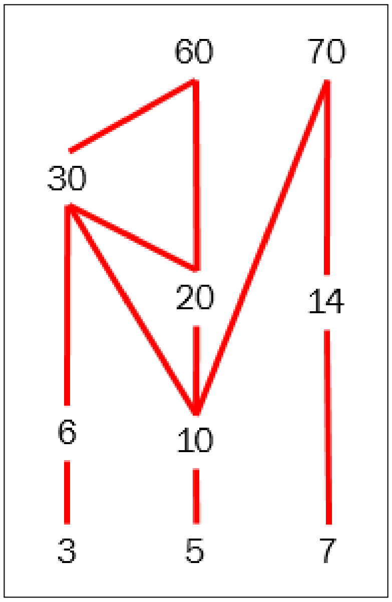
(d) 

Figure 5: *A Hasse diagram shows a set of elements 3; 5; 6; 7; 10; 14; 20; 30; 60, 70. There are lines connecting 3 and 6, 6 and 30, 30 and 60, 5 and 10, 10 and 30, 10 and 20, 20 and 30, 20 and 60, 10 and 70, 7 and 14, 14 and 70.*

Problem 3

A car dealership sells cars that were made in 2015 through 2020. Let the cars for sale be the domain of a relation R where two cars are related if they were made in the same year.

(a) Prove that this relation is an equivalence relation.

The relation R is an equivalence relation because it satisfies the following properties:

- **Reflexive**: Every car is related to itself since it is made in the same year as itself.

- **Symmetry**: If car a is related to car b, then car b is related to car a as they are made in the same year.
- **Transitivity**: If car a is related to car b, and car b is related to car c, then car a is also related to car c, as they are all made in the same year.

Therefore, R is an equivalence relation.

(b) Describe the partition defined by the equivalence classes.

The partition defined by the equivalence classes will consist of six sets, each containing cars from one of the years between 2015 and 2020. Each set, or equivalence class, can be represented as $[2015], [2016], [2017], [2018], [2019], [2020]$, where the cars in each class are all the cars made in that respective year.

PROBLEM 4

Analyze each graph below to determine whether it has an Euler circuit and/or an Euler trail.

- If it has an Euler circuit, specify the nodes for one.
- If it does not have an Euler circuit, justify why it does not.
- If it has an Euler trail, specify the nodes for one.
- If it does not have an Euler trail, justify why it does not.

(a)

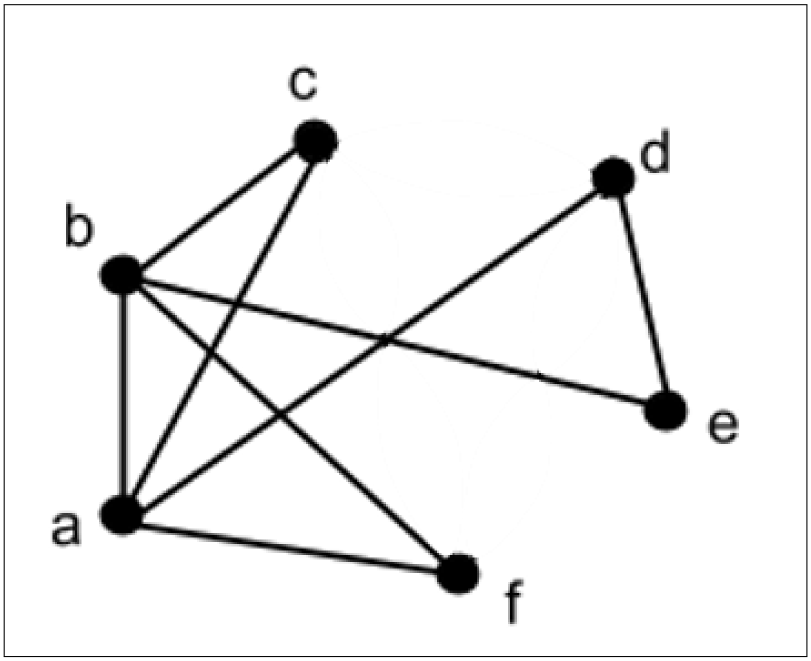


Figure 6: *An undirected graph has 6 vertices, a through f. There are 8-line segments that are between the following vertices: a and b, a and c, a and d, a and f, b and c, b and e, b and f, d and e.*

All vertices have an even degree. Therefore, it has an Euler circuit. An example of an Euler circuit in this graph starting at vertex 'a' could be: $a \to b \to c \to a \to d \to e \to b \to f \to a$.

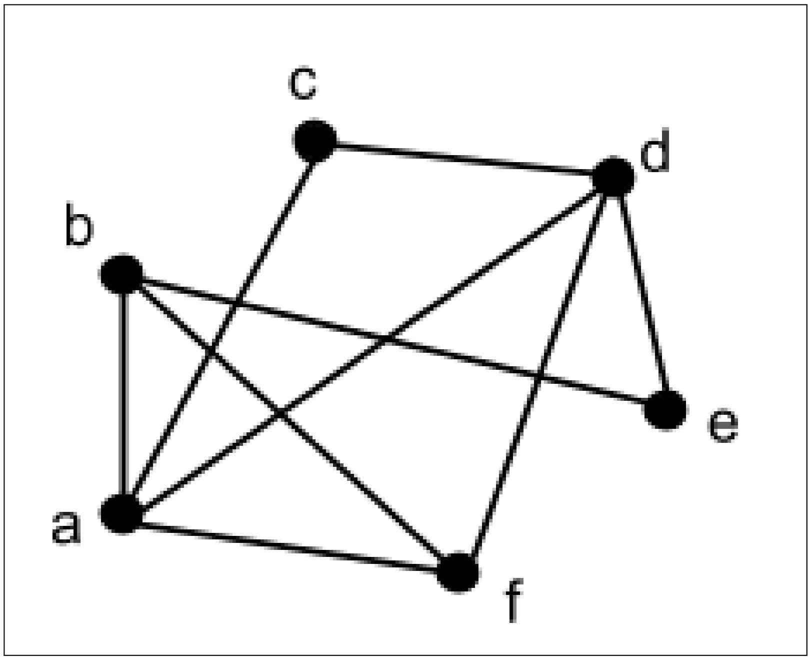
(b) 

Figure 7: *An undirected graph has 6 vertices, a through f. There are 9-line segments that are between the following vertices: a and b, a and c, a and d, a and f, b and e, b and f, c and d, d and e, d and f.*

All vertices have an even degree except vertices 'a' and 'd', which have an odd degree.

With this said, it does not have an Euler circuit but has an Euler trail. An example of an Euler trail in this graph could be: $a \to b \to f \to a \to c \to d \to e \to b \to e \to d$.

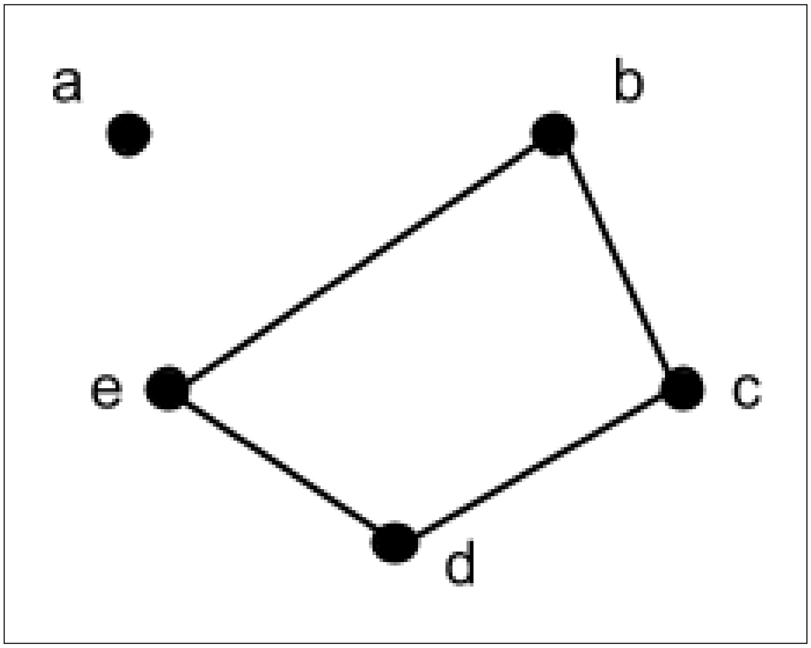
(c) 

Figure 8: *An undirected graph has 5 vertices, a through e. There are 4-line segments that are between the following vertices: b and c, b and e, c and d, d and e.*

All vertices have an even degree except vertices 'a' and 'e', which have an odd degree.

Therefore, it does not have an Euler circuit but has an Euler trail. An example of an Euler trail in this graph could be: $a \to b \to c \to d \to e$.

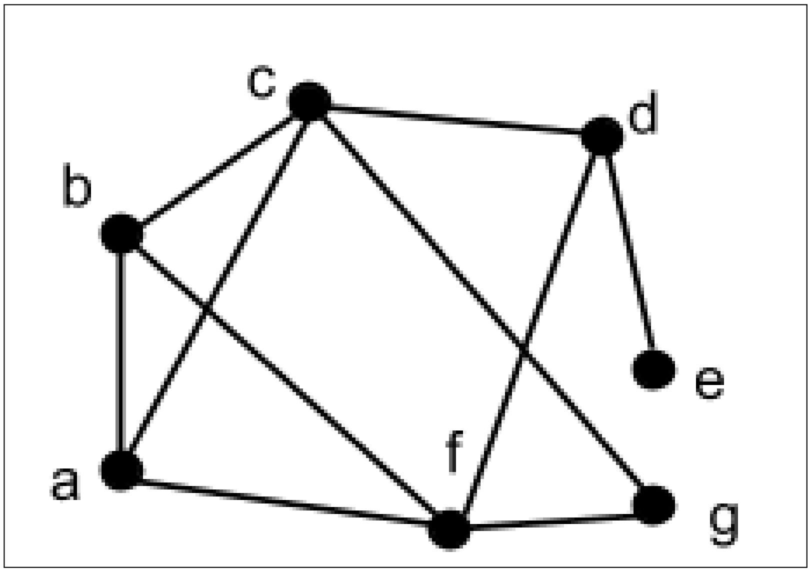
(d) 

Figure 9: *An undirected graph has 7 vertices, a through g. There are 10-line segments that are between the following vertices: a and b, a and c, a and f, b and c, b and f, c and d, c and g, d and e, d and f, f and g.*

All vertices have an even degree except for vertices 'c' and 'f', which have an odd degree.

Therefore, it does not have an Euler circuit but has an Euler trail. An example of an Euler trail in this graph could be: $c \to a \to b \to c \to d \to e \to d \to f \to b \to f \to g \to c$.

PROBLEM 5

Use Prim's algorithm to compute the minimum spanning tree for the weighted graph. Start the algorithm at vertex A. Explain and justify each step as you add an edge to the tree.

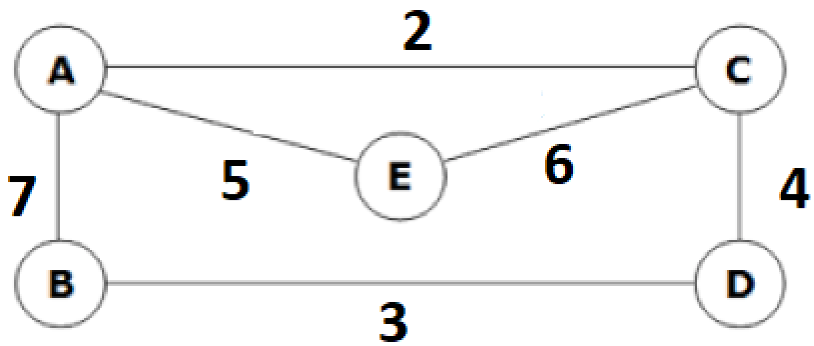


Figure 10: *A weighted graph shows 5 vertices, represented by circles, and 6 edges, represented by line segments. Vertices A, B, C, and D are placed at the corners of a rectangle, whereas vertex E is at the center of the rectangle. The edges, A B, B D, A C, C D, A E, and E C, have the weights, 7, 3, 2, 4, 5, and 6, respectively.*

- **Start at vertex A**: It's the starting point of the algorithm.
- **Add the smallest edge connected to A**: The smallest edge is A-C with a weight of 2. I added this edge to the MST.
- **Consider all edges connected to the MST**: Next I noted edges A-E (5), A-B (7), C-D (4), and C-E (6).
- **Add the smallest edge not yet in the MST**: The smallest edge is C-D with a weight of 4. Add C-D to the MST.
- **Consider new edges connected to the MST**: Now I noted edges A-E, A-B, and C-E since D has no other connections not already in the MST.
- **Add the smallest edge not yet in the MST**: The smallest edge is A-E with a weight of 5. Add A-E to the MST.
- **Consider new edges connected to the MST**: Now I noted the edges A-B and C-E.
- **Add the smallest edge not yet in the MST**: The smallest edge is C-E with a weight of 6. Nonetheless, an important note was how adding C-E would create a cycle, which is not allowed in an MST. Therefore, I essentially skipped this edge.
- **Add the smallest edge not yet in the MST**: The smallest remaining edge was A-B with a weight of 7. Add A-B to the MST.

At this point, all vertices are included in the MST, and I ensured not to create any cycles. The edges included in the MST are A-C, C-D, A-E, and A-B, with total weights of $2 + 4 + 5 + 7 = 18$.

The minimum spanning tree using Prim's algorithm from vertex A is thus made up of the edges with weights 2, 4, 5, and 7.

Problem 6

A lake initially contains 1000 fish. Suppose that in the absence of predators or other causes of removal, the fish population increases by 10% each month. However, factoring in all causes, 80 fish are lost each month.

Give a recurrence relation for the population of fish after n months. How many fish are there after 5 months? If your fish model predicts a non-integer number of fish, round down to the next lower integer.

The recurrence relation for the population of fish after n months is given by:

$$P(n) = P(n-1) \times 1.10 - 80$$

with the initial condition being $P(0) = 1000$.

This means that after 5 months, the fish population is predicted to be 1122 fish; rounding down to the nearest integer.