

## Use cases for integrals:

### Net change theorem

$\int_a^b f'(x) dx = f(b) - f(a)$  The integral of the rate of change ( $f'(x)$ ) is the net change ( $f(b) - f(a)$ ).

Ex: Suppose you have a bird's eye view of a car and know its  $x$  and  $y$  velocity in meters per sec.

$$V_x(t) = 1 \quad V_y(t) = 1 + \cos(2t)$$

What is its displacement and distance traveled after 5 sec?

$$\Delta x = \int_0^5 1 dt = t \Big|_0^5 = 5$$

$$\Delta y = \int_0^5 1 + \cos(2t) dt = t + \frac{1}{2} \sin(2t) \Big|_0^5 = 5 + \frac{1}{2} \sin(10)$$

$$\text{Displacement from origin: } \sqrt{(5)^2 + \left(5 + \frac{1}{2} \sin(10)\right)^2} = 6.88 \text{ meters}$$

Total distance traveled:

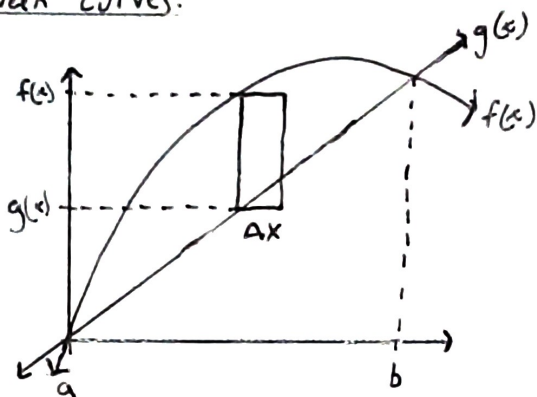
$$\text{Speed}(t) = \sqrt{V_x^2 + V_y^2}$$

$$\int_0^5 \text{Speed}(t) dt = \int_0^5 \sqrt{25 + (1 + \cos(2t))^2} dt \approx 7.42 \text{ meters}$$

$$\text{Total distance traveled on the } x = \int_0^5 |V_x| dt = 5 \text{ meters}$$

$$\text{Total distance traveled on the } y = \int_0^5 |V_y| dt = 4.73 \text{ meters}$$

## Area between curves:



$$\begin{aligned} \text{Area} &= \int_a^b f(x) - g(x) \, dx \\ &= \left| \int_a^b g(x) - f(x) \, dx \right| \end{aligned}$$

1. Find  $a$  and  $b$  usually by finding intersection points.
2. Find which function is on top or use absolute value.
3. Solve definite integral

Ex: Find area bounded by  $y = \sin x$  and  $y = \cos x$  on  $[0, \frac{\pi}{2}]$

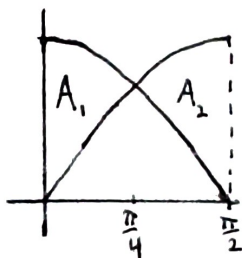
$$\sin x = \cos x \quad x = \frac{\pi}{4}$$

$$\begin{aligned} \text{Area} &= \left| \int_0^{\frac{\pi}{4}} \sin x - \cos x \, dx \right| + \left| \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x \, dx \right| \\ &= \left| -\cos x - \sin x \Big|_0^{\frac{\pi}{4}} \right| + \left| -\cos x - \sin x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right| \end{aligned}$$

$$= \left| -\cos \frac{\pi}{4} - \sin \frac{\pi}{4} - (-\cos 0 - \sin 0) \right| + \left| -\cos \frac{\pi}{2} - \sin \frac{\pi}{2} - (-\cos \frac{\pi}{4} - \sin \frac{\pi}{4}) \right|$$

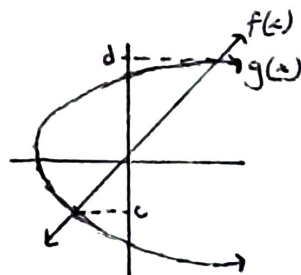
$$= \left| -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - (-1 - 0) \right| + \left| -0 - 1 - (-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) \right| = \left| -\sqrt{2} + 1 \right| + \left| -1 + \sqrt{2} \right|$$

$$= \sqrt{2} - 1 + \sqrt{2} - 1 = 2(\sqrt{2} - 1) = 2\sqrt{2} - 2$$



You can also integrate with respect to  $y$

$$\int_c^d f(x) - g(x) \, dy$$



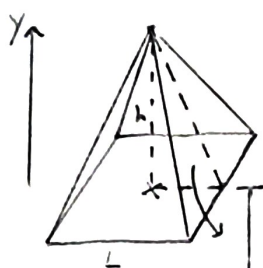
## Volumes with cylinders

Volume of a cylinder = Area of base  $\cdot$  Height  
(Even for irregular base)

The volume of objects other than cylinders can be found by cutting it into infinitely many cylinders where the area of its base is  $A(x)$  or  $A(y)$  and its height is  $dx$  or  $dy$ .

$$\int_a^b A(x) dx \quad \text{or} \quad \int_c^d A(y) dy$$

Ex 1: Find the volume of a pyramid with a square base of length  $L$  and height of  $h$ .



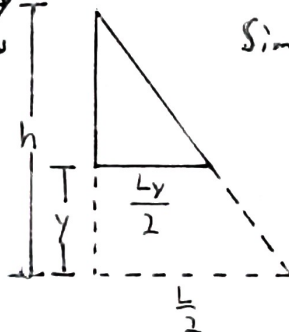
$$\int_0^h A(y) dy = \int_0^h L_y^2 dy$$

Cylinders stacked  
in the  $y$   
direction so  
it's  $dy$ .

Similar triangles:

$$\frac{h}{\frac{L}{2}} = \frac{h-y}{\frac{L_y}{2}} \Rightarrow \frac{2h}{L} = \frac{2(h-y)}{L_y}$$


$$L_y = \frac{L}{h} (h-y) = L - \frac{L}{h} y$$



$$\begin{aligned} \int_0^h \left( L - \frac{L}{h} y \right)^2 dy &= \int_0^h \left( L^2 - 2 \frac{L^2}{h} y + \frac{L^2}{h^2} y^2 \right) dy = L^2 \int_0^h \left( 1 - \frac{2}{h} y + \frac{1}{h^2} y^2 \right) dy \\ &= L^2 \left[ y - \frac{1}{h} y^2 + \frac{1}{3h^2} y^3 \right]_0^h = L^2 \left( h - \frac{h^2}{h} + \frac{h^3}{3h^2} \right) = \frac{1}{3} L^2 h \end{aligned}$$

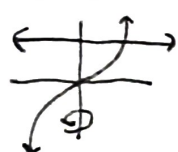
When an area is revolved around an axis, the area of the base is a circle ( $\pi r^2$ ) and you need to write  $r$  in terms of  $x$  or  $y$ .

Ex 2: Find the volume of the solid obtained by rotating  $y = \sqrt{x}$  around the  $x$ -axis from 0 to 1.



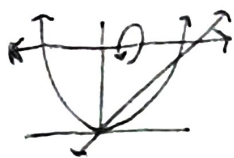
$$\int_0^1 \pi (\sqrt{x})^2 dx = \pi \int_0^1 x dx = \pi \left[ \frac{1}{2} x^2 \right]_0^1 = \frac{\pi}{2}$$

Ex 3: Find the volume of the solid obtained by rotating the area bounded by  $y = x^3$ ,  $y = 8$ , and  $x = 0$  around the  $y$ -axis.



$$\begin{aligned} \int_0^8 \pi (y^{\frac{1}{3}})^2 dy &= \pi \int_0^8 y^{\frac{2}{3}} dy = \pi \left[ \frac{3}{5} y^{\frac{5}{3}} \right]_0^8 = \pi \left( \frac{3}{5} 8^{\frac{5}{3}} \right) \\ &= \pi \frac{3}{5} (8^{\frac{1}{3}})^5 = \pi \frac{3}{5} 2^5 = \pi \frac{3 \cdot 32}{5} = \frac{96}{5} \pi \end{aligned}$$

Ex 4: Find the volume of the solid obtained by rotating the area enclosed by  $y = x$  and  $y = x^2$  around  $y = 2$ .



Find the bounds:  $x = x^2 \Rightarrow 0 = x^2 - x = x(x-1)$   
 $x = 0$  and  $1$

$$\int_0^1 \pi \underbrace{(2-x^2)^2}_{\text{outer radius}} - \pi \underbrace{(2-x)^2}_{\text{inner radius}} dx = \pi \int_0^1 4 - 4x^2 + x^4 - (4 - 4x + x^2) dx$$

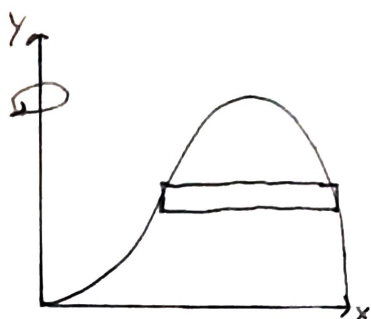
$$(2-x^2)(2-x^2) = 4 - 4x^2 + x^4 \quad (2-x)(2-x) = 4 - 4x + x^2$$

$$\begin{aligned} &= \pi \int_0^1 x^4 - 5x^2 + 4x dx = \pi \left[ \frac{1}{5} x^5 - \frac{5}{3} x^3 + \frac{4}{2} x^2 \right]_0^1 = \pi \left( \frac{1}{5} - \frac{5}{3} + 2 \right) \\ &= \pi \left( \frac{6}{30} - \frac{50}{30} + \frac{60}{30} \right) = \pi \left( -\frac{44}{30} + \frac{60}{30} \right) = \frac{16}{30} \pi = \frac{8}{15} \pi \end{aligned}$$

## Volume with shells

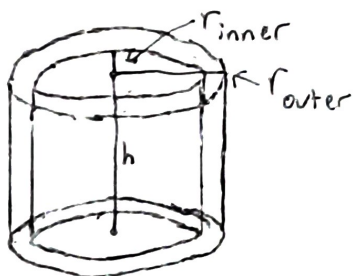
It can be hard to stack cylinders for some objects.

Ex: Find the volume obtained by rotating  $y = 2x^2 - x^3$  and  $y = 0$  around the  $y$ -axis



If we were to use cylinders, we would have to find the local max and then divide  $y = 2x^2 - x^3$  into two separate functions for the outer and inner radii of the circle.

Instead we could use an infinite number of shells.



$$\text{Volume} = \int_a^b 2\pi r \cdot f(r) \cdot dr$$

$$\text{Volume} = \text{Volume}_{\text{outer}} - \text{Volume}_{\text{inner}}$$

$$= \pi r_{\text{out}}^2 h - \pi r_{\text{in}}^2 h$$

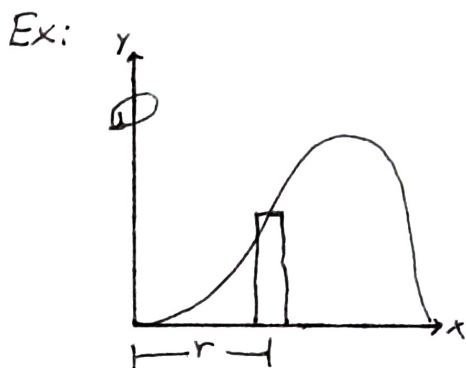
$$= \pi (r_{\text{out}}^2 - r_{\text{in}}^2) h$$

$$= \pi (r_{\text{out}} + r_{\text{in}}) (r_{\text{out}} - r_{\text{in}}) h$$

$$= \pi (r_{\text{out}} + r_{\text{in}}) \Delta r h$$

$$= 2\pi \frac{r_{\text{out}} + r_{\text{in}}}{2} h \Delta r$$

$$= \underbrace{2\pi r_{\text{ave}}}_{\text{Circumference}} \cdot \underbrace{h}_{\text{height}} \cdot \underbrace{\Delta r}_{\text{Thickness}}$$



$$\int_0^2 2\pi x \cdot (2x^2 - x^3) dx = \frac{16}{5}\pi$$

Use shells over cylinders when the axis of rotation (Ex:  $y$ -axis) is different from the independent variable (Ex:  $x$ ).  $y \neq x$

## Work

$$F = ma$$

$$W = F \cdot d \quad \text{when Force is constant}$$

$$\text{Springs: } F = kx$$

$$W = \int_{x_i}^{x_f} F(x) \cdot dx = \int_{x_i}^{x_f} h(x) \cdot dF(x) \quad \begin{array}{l} \text{when Force} \\ \text{is not constant.} \end{array}$$

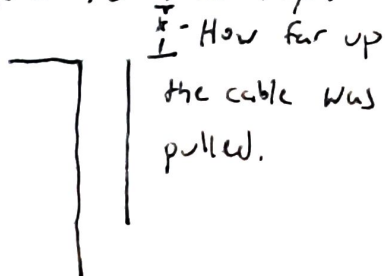
	Force	Distance	Work	g
Metric	N	m	J	9.8
Imperial	lbs	ft	ft-lbs	32.2

Ex 1: A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done by stretching it from 15 cm to 18 cm?

$$F = kx \quad 40 = k(15 - 10) \quad k = 8 \frac{\text{N}}{\text{cm}}$$

$$W = \int_5^8 8x \, dx = 4x^2 \Big|_5^8 = 4 \cdot 8^2 - 4 \cdot 5^2 = 156 \text{ Ncm} \cdot \frac{1 \text{ m}}{100 \text{ cm}} = 1.56 \text{ J}$$

Ex 2: A 200 lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work to lift the cable to the top?



$F(x)$  = Force of cable hanging

$$F(0) = 200 \quad F(100) = 0$$

Assuming uniform density of the cable, the Force is linear/a line.

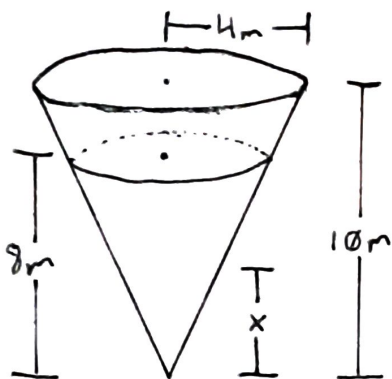
$$\text{slope} = \frac{F(0) - F(100)}{0 - 100} = \frac{200}{-100} = -2$$

$$F(x) - 0 = -2(x - 100)$$

$$W = \int_0^{100} -2(x - 100) \, dx = -2 \int_0^{100} x - 100 \, dx = -2 \left[ \frac{1}{2}x^2 - 100x \right]_0^{100} = 10,000 \text{ ft-lbs}$$



Ex 3: A tank has the shape of an inverted circular cone with height 10m and base radius 4m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank?  
 Density of water is  $1,000 \frac{\text{kg}}{\text{m}^3}$



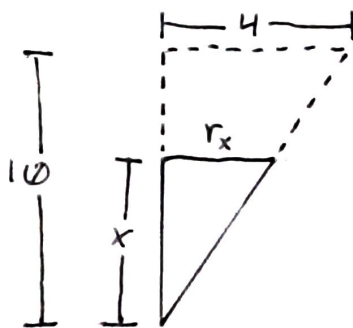
$x$  = Height of water left in the tank.

$$W = \int_0^8 h(x) dF(x) = \int_0^8 (10 - x) dF(x)$$

$dF(x)$  = Force from a small cylindrical slice.  
 = mass  $\cdot$  acc =  $dm(x) \cdot 9.8$

$dm(x)$  = Mass of small cylindrical slice.  
 = Density  $\cdot$  Volume =  $1,000 \cdot dV(x)$

$dV(x)$  = Volume of small cylindrical slice.  
 = Area of base  $\cdot$  height  
 =  $\pi r_x^2 \cdot dx$



$$\frac{4}{10} = \frac{r_x}{x}$$

$$r_x = \frac{2}{5}x$$

$$W = \int_0^8 (10 - x) \cdot 9.8 \cdot 1,000 \cdot \pi \left(\frac{2}{5}x\right)^2 dx = 3.36 \times 10^6 \text{ J}$$