

Finding the limit

1. If the function is continuous at the point, then you can use direct substitution.

- You can do direct substitution for the left and right limits and see if they're equal.

Ex: $f(x) = \begin{cases} x+1 & \text{if } x \neq 0 \\ 5 & \text{if } x = 0 \end{cases}$ $\lim_{x \rightarrow 0^-} f(x) = 1$ $\lim_{x \rightarrow 0^+} f(x) = 1$

- You can use algebra to convert the function to one that's continuous at the point.

- Factoring: $\lim_{x \rightarrow 0} \frac{(3+x)^2 - 9}{x} = \lim_{x \rightarrow 0} x+6 = 6$

$$\frac{(3+x)^2 - 9}{x} = \frac{9 + 6x + x^2 - 9}{x} = \frac{x(x+6)}{x} = x+6$$

- Rationalizing: $\lim_{x \rightarrow 0} \frac{\sqrt{x^2+9} - 3}{x^2} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2+9} + 3} = \frac{1}{6}$

$$\frac{\sqrt{x^2+9} - 3}{x^2} \cdot \frac{\sqrt{x^2+9} + 3}{\sqrt{x^2+9} + 3} = \frac{x^2+9-9}{x^2(\sqrt{x^2+9} + 3)} = \frac{1}{\sqrt{x^2+9} + 3}$$

- Least common denominator: $f(x) = \frac{1-x}{2+x}$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1-x-h}{2+x+h} - \frac{1-x}{2+x}}{h} \cdot \frac{(2+x+h)(2+x)}{(2+x+h)(2+x)}$$

$$= \lim_{h \rightarrow 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)}$$

$$(1-x-h)(2+x) = 1(2+x) - x(2+x) - h(2+x) = 2+x-2x-x^2-2h-xh$$

$$(2+x+h)(1-x) = 2(1-x) + x(1-x) + h(1-x) = 2-2x+x-x^2+h-xh$$

$$\lim_{h \rightarrow 0} \frac{2+x-2x-x^2-2h-xh - (2-2x+x-x^2+h-xh)}{h(2+x+h)(2+x)}$$

$$= \lim_{h \rightarrow 0} \frac{-2h-h}{h(2+x+h)(2+x)} = -\frac{3}{(2+x)(2+x)} = -\frac{3}{(2+x)^2}$$

2. Squeeze theorem: Finding functions that are greater and less than that converge to the same limit

Ex: $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$

$$\begin{aligned} -1 &\leq \sin x \leq 1 && \text{Since } \lim_{x \rightarrow 0} -x^2 = 0 \text{ and } \lim_{x \rightarrow 0} x^2 = 0 \\ -1 &\leq \sin \frac{1}{x} \leq 1 && \text{and} \\ -x^2 &\leq x^2 \sin \frac{1}{x} \leq x^2 && \text{then } \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0 \end{aligned}$$

3. Composite limit: $\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right)$

Ex: $\lim_{x \rightarrow 2} (3x+1)^2 = \left(\lim_{x \rightarrow 2} (3x+1)\right)^2 = 7^2 = 49$

4. Guess and check from the left and right

- Too far causes calculator errors
- Too short may lead to a wrong guess.

Left

0.9

0.99

0.999

0.9999

Right

1.1

1.01

1.001

1.0001

5. Finding limits at infinity

- Multiplying numerator and denominator by greatest power of x in the denominator.

$$-\lim_{x \rightarrow \infty} |x| = \sqrt{x^2}$$

$$\lim_{x \rightarrow -\infty} |x| = -\sqrt{x^2}$$

You want it in the form of $\frac{1}{\infty}$ which $= 0$.

- ONE cases

- Oscillating Ex: $\lim_{x \rightarrow \infty} \sin x$

- ∞/∞ Ex: $\lim_{x \rightarrow \infty} \sqrt{x^2+1}$

Ex: $\lim_{x \rightarrow \infty} \sqrt{x^2+1} - x$

$$\frac{\sqrt{x^2+1} - x}{1} \cdot \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x} = \frac{x^2+1 - x^2}{\sqrt{x^2+1} + x}$$

$$\frac{1}{\sqrt{x^2+1} + x} \cdot \frac{1}{x} = \frac{\frac{1}{x}}{\frac{\sqrt{x^2+1}}{x} + 1} = \frac{\frac{1}{x}}{\frac{\sqrt{x^2+1}}{x} + 1} = \frac{\frac{1}{x}}{\sqrt{\frac{x^2+1}{x^2}} + 1}$$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}} + 1} = \frac{0}{\sqrt{1+0} + 1} = 0$$