Finding Integrals

1. Limit definition

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\,\Delta x}_{\text{Riemann sum}}$$

$f(x_i) = \text{Height}$

$\Delta x = \text{Width} = \frac{b-a}{n}$

$x_i = a + i\Delta x$

- We use an infinite number of rectangles to get the area under the curve.
- If we don't use an infinite number, we can use the left, right, midpoint, smallest y value (lower sum), or largest y value (upper sum) to estimate the area under the curve.
- We can also get a rough estimate by finding the max (M) and min (m) of f(x).

If $m \le f(x) \le M$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$$\sum_{i=1}^{n} c = c\sum_{i=1}^{n} = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \qquad \sum_{i=1}^{n} a_i - b_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

2. Properties of integrals

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad \int_a^a f(x)\,dx = 0$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx \qquad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad \int_a^b \big(f(x)-g(x)\big)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

3. Fundamental theorem of calculus

Part 1: Deriving the integral of f gives f

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$ a is any constant

Ex: $$\frac{d}{dx}\int_1^{x^4} \sec t\,dt$$

$u = x^4$

$\frac{du}{dx} = 4x^3$

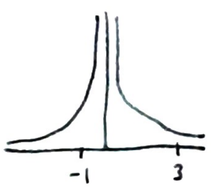
$\frac{d}{dx} = \frac{du}{dx}\cdot\frac{d}{du}$

$$4x^3 \cdot \frac{d}{du}\int_1^{u} \sec t\,dt = 4x^3 \cdot \sec u = 4x^3 \sec x^4$$

Part 2:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Ex: $$\int_{-1}^{3} \frac{1}{x^2}\,dx \neq -\frac{1}{x}\Big|_{-1}^{3} = -\frac{4}{3}$$



f(x) has to be contirious or have a finite number of jump discontinuities over [a, b] to be integrable.

Integration and differentiation are inverse processes.

4 U-Substitution

1. Define u
2. Find $\frac{du}{dx}$
3. Solve for dx
4 Plug u and dx into the integral and simplify

If definite integral: Re-write bounds for u

5. Solve integral

If indefinte integral: Substitute u so it's in terms of x.

Ex 1: $\int x^3 \cos(x^4+2)\,dx$

$u = x^4+2$

$\frac{du}{dx} = 4x^3 \Rightarrow dx = \frac{du}{4x^3}$

$$\int x^3 \cos(u)\,\frac{du}{4x^3} = \int \frac{1}{4}\cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2) + C$$

Ex 2: $\int \sqrt{1+x^2}\,x^5\,dx$

$u = 1+x^2$ $\qquad u-1 = x^2$

$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$$\int \sqrt{u}\,x^5\,\frac{du}{2x} = \int \sqrt{u}\,x^4\,\frac{1}{2}\,du = \int \frac{1}{2}\sqrt{u}\,(u-1)^2\,du$$

$$= \int \frac{1}{2}\sqrt{u}\,(u^2-2u+1)\,du = \int \frac{1}{2}u^{\frac{5}{2}} - u^{\frac{3}{2}} + \frac{1}{2}u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{7}u^{\frac{7}{2}} - \frac{2}{5}u^{\frac{5}{2}} + \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{7}(1+x^2)^{\frac{7}{2}} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$$

Ex 3: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$

$$\int \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right) = \int -\frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C$$

Ex 4: $\int_1^e \frac{\ln x}{x}\,dx$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$

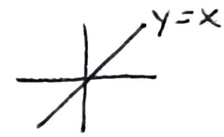
Lower: $u = \ln(1) = 0$ $\qquad$ Upper: $u = \ln(e) = 1$

$$\int_0^1 \frac{u}{x}\,x\,du = \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}$$

5. Symmetry: If $f(x)$ is continious on $[-a, a]$

If $f(-x) = f(x)$ then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

If $f(-x) = -f(x)$ then $\int_{-a}^{a} f(x)\,dx = 0$



6. Integration by parts

Just like U-substitution is the inverse of the chain rule, integration by parts is the inverse of the product rule.

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$u = f(x)$ $\quad dv = g'(x)\,dx$

$du = f'(x)\,dx$ $\quad v = g(x)$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Use LIPET for picking u:

L - ln

I - Inverse trig

P - Powers of x

E - Exponential

T - Trig

The goal of integration by parts is to convert the integral to a simpler integral. This may not always be the case.

Definite integrals: $\int_{a}^{b} u\,dv = uv\Big|_{a}^{b} - \int_{a}^{b} v\,du$

Ex 1: $\int x \sin x \, dx$ $\quad u = x \quad dv = \sin x \, dx$

$du = dx \quad v = -\cos x$

$= x(-\cos x) - \int -\cos x \, dx = -x\cos x + \sin x + C$

Ex 2: $\int \ln x \, dx$ $\quad u = \ln x \quad dv = dx$

$du = \frac{1}{x} dx \quad v = x$

$= \ln x (x) - \int x \frac{1}{x} dx = x \ln x - x + C$

Ex 3: $\int t^2 e^t \, dt$ $\quad u = t^2 \quad dv = e^t \, dt$

$du = 2t \, dt \quad v = e^t$

$= t^2 e^t - \int e^t 2t \, dt = t^2 e^t - 2\int t e^t \, dt$ $\quad u = t \quad dv = e^t \, dt$

$du = dt \quad v = e^t$

$= t^2 e^t - 2\left[t e^t - \int e^t dt\right] = t^2 e^t - 2te^t + 2e^t + C$

Ex 4: $\int e^x \sin x \, dx$ $\quad u = e^x \quad dv = \sin x \, dx$

$du = e^x dx \quad v = -\cos x$

$= e^x(-\cos x) - \int -\cos x \, e^x \, dx = -e^x \cos x + \int \cos x \, e^x \, dx$

$u = e^x \quad dv = \cos x \, dx$

$du = e^x dx \quad v = \sin x$

Add to the other side.

$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int \sin x \, e^x \, dx$

$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$

$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$