

Finding Integrals

1. Limit definition

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

Riemann sum

$f(x_i) = \text{Height}$
 $\Delta x = \text{Width} = \frac{b-a}{n}$
 $x_i = a + i\Delta x$

- We use an infinite number of rectangles to get the area under the curve.

- If we don't use an infinite number, we can use the left, right, midpoint, smallest y value (lower sum), or largest y value (upper sum) to estimate the area under the curve.

- We can also get a rough estimate by finding the max(M) and min(m) of $f(x)$.

$$\text{If } m \leq f(x) \leq M, \text{ then } m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

$$\sum_{i=1}^n c = c \sum_{i=1}^n = cn$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n a_i + b_i = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2} \right)^2$$

$$\sum_{i=1}^n a_i - b_i = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$$

2. Properties of integrals

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\int_a^a f(x) dx = 0$$

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx$$

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

$$\int_a^b (f(x) - g(x)) dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

3. Fundamental theorem of calculus

Part 1: Deriving the integral of f gives F

$$\frac{d}{dx} \int_a^x f(t) dt = F(x) \quad a \text{ is any constant}$$

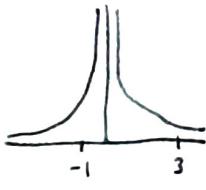
Ex: $\frac{d}{dx} \int_1^{x^4} \sec t dt$ $u = x^4$ $\frac{d}{dx} = \frac{du}{dx} \cdot \frac{d}{du}$
 $\frac{du}{dx} = 4x^3$

$$4x^3 \cdot \frac{d}{du} \int_1^u \sec t dt = 4x^3 \cdot \sec u = 4x^3 \sec x^4$$

Part 2:

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

Ex: $\int_{-1}^3 \frac{1}{x^2} dx \neq -\frac{1}{x} \Big|_{-1}^3 = -\frac{4}{3}$



$f(x)$ has to be continuous or have a finite number of jump discontinuities over $[a, b]$ to be integrable.

Integration and differentiation are inverse processes.

4 U-Substitution

1. Define u

2. Find $\frac{du}{dx}$

3. Solve for dx

4. Plug u and dx into the integral and simplify

If definite integral: Re-write bounds for u

5. Solve integral

If indefinite integral: Substitute u so its in terms of x .

$$Ex 1: \int x^3 \cos(x^4+2) dx \quad u = x^4 + 2 \\ \frac{du}{dx} = 4x^3 \Rightarrow dx = \frac{du}{4x^3}$$

$$\int x^3 \cos(u) \frac{du}{4x^3} = \int \frac{1}{4} \cos u du = \frac{1}{4} \sin u + C = \frac{1}{4} \sin(x^4+2) + C$$

$$Ex 2: \int \sqrt{1+x^2} x^5 dx \quad u = 1+x^2 \quad u-1 = x^2 \\ \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$\begin{aligned} \int \sqrt{u} x^5 \frac{du}{2x} &= \int \sqrt{u} x^4 \frac{1}{2} du = \int \frac{1}{2} \sqrt{u} (u-1)^2 du \\ &= \int \frac{1}{2} \sqrt{u} (u^2 - 2u + 1) du = \int \frac{1}{2} u^{\frac{5}{2}} - u^{\frac{3}{2}} + \frac{1}{2} u^{\frac{1}{2}} du \\ &= \frac{1}{2} \cdot \frac{2}{7} u^{\frac{7}{2}} - \frac{2}{5} u^{\frac{5}{2}} + \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + C = \frac{1}{7} (1+x^2)^{\frac{7}{2}} - \frac{2}{5} (1+x^2)^{\frac{5}{2}} + \frac{1}{3} (1+x^2)^{\frac{3}{2}} + C \end{aligned}$$

$$Ex 3: \int \tan x dx = \int \frac{\sin x}{\cos x} dx \quad u = \cos x \\ \frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$$

$$\int \frac{\sin x}{u} \left(-\frac{du}{\sin x} \right) = \int -\frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C$$

$$Ex 4: \int_1^e \frac{\ln x}{x} dx \quad u = \ln x \\ \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x du$$

$$\text{Lower: } u = \ln(1) = 0 \quad \text{Upper: } u = \ln(e) = 1$$

$$\int_0^1 \frac{u}{x} x du = \int_0^1 u du = \frac{1}{2} u^2 \Big|_0^1 = \frac{1}{2}(1) - \frac{1}{2}(0)^2 = \frac{1}{2}$$

5. Symmetry: If $f(x)$ is continuous on $[-a, a]$

$$\text{If } f(-x) = f(x) \text{ then } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

$y = |x|$

$$\text{If } f(-x) = -f(x) \text{ then } \int_{-a}^a f(x) dx = 0$$

$y = -x$

$y = x$

6. Integration by parts

Just like U-substitution is the inverse of the chain rule, integration by parts is the inverse of the product rule.

$$\frac{d}{dx} (f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g(x) = \int f(x)g'(x) dx + \int f'(x)g(x) dx$$

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

$$\begin{array}{ll} u = F(x) & dv = g'(x) dx \\ du = F'(x) dx & v = g(x) \end{array}$$

$$\boxed{\int u dv = uv - \int v du}$$

Use LIPET for picking u :

L - Ln

I - Inverse trig

P - Powers of x

E - Exponential

T - Trig

The goal of integration by parts is to convert the

integral to a simpler integral.

This may not always be the case.

Definite integrals: $\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$

$$Ex 1: \int x \sin x \, dx \quad u = x \quad dv = \sin x \, dx \\ du = dx \quad v = -\cos x$$

$$= x(-\cos x) - \int -\cos x \, dx = -x \cos x + \sin x + C$$

$$Ex 2: \int \ln x \, dx \quad u = \ln x \quad dv = dx \\ du = \frac{1}{x} dx \quad v = x$$

$$= \ln x(x) - \int x \frac{1}{x} dx = x \ln x - x + C$$

$$Ex 3: \int t^2 e^t dt \quad u = t^2 \quad dv = e^t dt \\ du = 2t dt \quad v = e^t$$

$$= t^2 e^t - \int e^t 2t dt = t^2 e^t - 2 \int t e^t dt \quad u = t \quad dv = e^t dt \\ = t^2 e^t - 2 \left[t e^t - \int e^t dt \right] = t^2 e^t - 2t e^t + 2e^t + C \quad v = e^t$$

$$Ex 4: \int e^x \sin x \, dx \quad u = e^x \quad dv = \sin x \, dx \\ du = e^x dx \quad v = -\cos x$$

$$= e^x(-\cos x) - \int -\cos x e^x dx = -e^x \cos x + \int \cos x e^x dx \\ u = e^x \quad dv = \cos x dx \\ du = e^x dx \quad v = \sin x$$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int \sin x e^x dx \quad \text{Add to the other side.}$$

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

7. Powers of sin and cos

$$\int \sin^m x \cos^n x dx$$

$\sin^m x$	$\cos^n x$	Solution
Odd	Odd	Either
Odd	Even	$u = \cos x$
Even	Odd	$u = \sin x$
Even	Even	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$\sin^2 x + \cos^2 x = 1$$

Ex 1: $\int \cos^5 x dx$ $\cos^5 x$ is odd, $\sin^0 x$ is even

$$u = \sin x \quad du = \cos x dx \Rightarrow dx = \frac{du}{\cos x}$$

$$\int \cos^5 x \frac{du}{\cos x} = \int \cos^4 x du = \int (\cos^2 x)^2 du$$

$$\cos^2 x = 1 - \sin^2 x \quad = \int (1 - \sin^2 x)^2 du = \int (1 - u^2)^2 du$$

$$(1 - u^2)(1 - u^2) = 1 - 2u^2 + u^4 \quad = \int 1 - 2u^2 + u^4 du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Ex 2: $\int \sin^4 x dx$ $\sin^4 x$ is even and $\cos^0 x$ is even

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1}{2}(1 - \cos 2x)\right)^2 = \frac{1}{4}(1 - \cos 2x)^2$$

$$= \int \frac{1}{4}(1 - \cos 2x)^2 dx = \frac{1}{4} \int 1 - 2\cos 2x + \cos^2 2x dx$$

$$= \frac{1}{4} \left[\int dx - 2 \int \cos 2x dx + \int \cos^2 2x dx \right]$$

$$= \frac{1}{4} \left[x - \sin 2x + \int \cos^2 2x dx \right]$$

$$\cos^2(2x) = \frac{1}{2} (1 + \cos 4x)$$

$$= \frac{1}{4} \left[x - \sin 2x + \frac{1}{2} \int 1 + \cos 4x dx \right]$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8} \left[\int dx + \int \cos 4x dx \right]$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

8. Powers of tan and sec

$$\int \tan^m x \sec^n x dx \quad \text{This also applies for cot and csc}$$

$\tan^m x$	$\sec^n x$	Solution
Odd	Odd	$u = \sec x$
Odd	Even	Either
Even	Odd	Other
Even	Even	$v = \tan x$

$$\tan^2 x + 1 = \sec^2 x$$

Integration by parts

$$\int \sec x = \ln |\sec x + \tan x| + C$$

$$\int \tan x = \ln |\sec x| + C$$

Move to other side and divide

$$\text{Ex 1: } \int \tan^2 x \sec^4 x dx \quad v = \tan x \\ dv = \sec^2 x dx \Rightarrow dx = \frac{dv}{\sec^2 x}$$

$$\int \tan^2 x \sec^4 x \frac{dv}{\sec^2 x} = \int v^2 \sec^2 x dv$$

$$\sec^2 x = 1 + \tan^2 x \quad = \int v^2 (1 + v^2) dv = \int v^2 + v^4 dv$$

$$= \frac{1}{3}v^3 + \frac{1}{5}v^5 + C = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

$$\text{Ex 2: } \int \sec^3 x dx = \int \sec^2 x \sec x dx = \int (\tan^2 x + 1) \sec x dx$$

$$= \int \tan^2 x \sec x \, dx + \sec x \, dx = \int \tan^2 x \sec x \, dx + \int \sec x \, dx$$

$$\int \sec x \, dx = \int \frac{\sec x}{1} \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$u = \sec x + \tan x \quad du = \sec x \tan x + \sec^2 x \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{u} \cdot \frac{du}{\sec^2 x + \sec x \tan x} = \int \frac{1}{u} \, du = \ln |\sec x + \tan x| + C$$

$$\int \tan^2 x \sec x \, dx \quad u = \tan x \quad du = \sec x \tan x \, dx \\ \quad \quad \quad \quad \quad \quad v = \sec x$$

$$= \sec x \tan x - \int \sec^3 x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \ln |\sec x + \tan x| + C$$

$$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$$

9. $\sin A$ and $\cos B$

$$\int \sin A \cos B \, dx \text{ or } \int \sin A \sin B \, dx \text{ or } \int \cos A \cos B \, dx$$

$$\sin A \cos B = \frac{1}{2} [\sin(A-B) + \sin(A+B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$

$$\int \sin 4x \cos 5x \, dx = \int \frac{1}{2} [\sin(-x) + \sin(9x)] \, dx$$

$$= \frac{1}{2} \left[\int -\sin x \, dx + \int \sin(9x) \, dx \right] = \frac{1}{2} \left[\cos x - \frac{1}{9} \cos 9x \right] + C$$

10. Trig Inverse Substitution

Expression in integrand

$$\sqrt{a^2 - x^2}$$

$$\sqrt{a^2 + x^2}$$

$$\sqrt{x^2 - a^2}$$

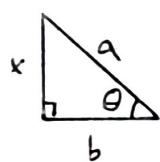
Trig Substitution

$$x = a \sin \theta$$

$$x = a \tan \theta$$

$$x = a \sec \theta$$

$$x = a \sin \theta \Rightarrow \sin \theta = \frac{x}{a} = \frac{\text{opp}}{\text{hyp}}$$



$$\text{Pythagorean: } a^2 = x^2 + b^2 \Rightarrow b^2 = a^2 - x^2 \Rightarrow b = \sqrt{a^2 - x^2}$$

$$\text{Ex 1: } \int \frac{\sqrt{9-x^2}}{x^2} dx$$

$$x = 3 \sin \theta \\ dx = 3 \cos \theta d\theta$$

$$\sqrt{9-x^2} = \sqrt{3^2 - x^2} = \sqrt{3^2 - (3 \sin \theta)^2} = \sqrt{3^2 - 3^2 \sin^2 \theta} = \sqrt{3^2 (1 - \sin^2 \theta)}$$

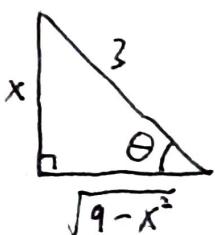
$$= \sqrt{3^2 \cos^2 \theta} = 3 |\cos \theta|$$

$$\int \frac{\sqrt{9-x^2}}{x^2} dx = \int \frac{3|\cos \theta|}{3^2 \sin^2 \theta} 3 \cos \theta d\theta = \int \frac{\cos^2 \theta}{\sin^2 \theta} d\theta$$

$$= \int \cot^2 \theta d\theta = \int \csc^2 \theta - 1 d\theta = -\cot \theta - \theta + C$$

$$x = 3 \sin \theta \Rightarrow \frac{x}{3} = \sin \theta \Rightarrow \theta = \sin^{-1} \left(\frac{x}{3} \right)$$

soh cah toa



$$\cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{9-x^2}}{x}$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1} \left(\frac{x}{3} \right) + C$$

$$Ex 2: \int \frac{x}{\sqrt{3-2x-x^2}} dx$$

$$3-2x-x^2 = 3-(2x+x^2) = 3+1-(2x+x^2+1) = 4-(x+1)^2$$

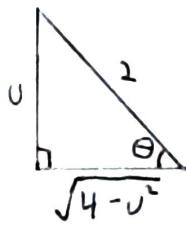
$$= \int \frac{x}{\sqrt{2^2-(x+1)^2}} dx \quad u=x+1 \Rightarrow x=u-1 \\ du = dx$$

$$= \int \frac{u-1}{\sqrt{2^2-u^2}} du \quad u=2\sin\theta \\ du = 2\cos\theta d\theta$$

$$\sqrt{2^2 - 2^2 \sin^2 \theta} = \sqrt{2^2(1-\sin^2 \theta)} = \sqrt{2^2 \cos^2 \theta} = 2|\cos \theta|$$

$$= \int \frac{2\sin\theta - 1}{2|\cos\theta|} 2\cos\theta d\theta = \int 2\sin\theta - 1 d\theta = -2\cos\theta - \theta + C$$

$$u=2\sin\theta \Rightarrow \sin\theta = \frac{u}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{u}{2}\right)$$

$$\cos\theta = \frac{\sqrt{4-u^2}}{2}$$


$$= -2 \frac{\sqrt{4-u^2}}{2} - \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

11. Method of partial fractions

This method works for rational functions which are polynomial/polynomial

① Sometimes you need to do a U-sub on $\sqrt{g(x)}$

$$\text{Ex: } \int \frac{\sqrt{x+4}}{x} dx \quad u = \sqrt{x+4} \quad x = u^2 - 4 \quad = 2 \int \frac{u^2}{u^2 - 4} du$$

$$du = \frac{1}{2\sqrt{x+4}} dx \quad dx = 2u du$$

1. If the degree of the numerator is \geq degree of denominator

, then use long division.

$$\text{Ex: } \int \frac{5x^3}{x^2 + x + 1} dx \quad \frac{\text{denominator}}{\text{numerator}} = \frac{\text{answer}}{\text{denominator}} + \frac{\text{remainder}}{\text{denominator}}$$

$$\begin{array}{r} 5 \\ x^2 + x + 1 \overline{) 5x^3 + 0x^2 + 0} \\ - (5x^3 + 5x^2 + 5) \\ \hline -5x - 5 \end{array} = \int 5 + \frac{-5x - 5}{x^2 + x + 1} dx$$

2. Factor the denominator as far as possible.

$$\text{Ex: } \int \frac{1}{(x^2 - 1)^2} dx = \int \frac{1}{((x-1)(x+1))^2} dx = \int \frac{1}{(x-1)^2(x+1)^2} dx$$

- Sometimes you can try numbers and see if they equal 0

$$\text{Ex: } \int \frac{1}{x^3 + x^2 + x + 1} dx \quad \begin{aligned} &\text{Try } x = -1: -1 + 1 - 1 + 1 = 0 \\ &\text{Therefore } (x+1) \text{ is a factor.} \end{aligned}$$

$$\begin{array}{r} x^2 + 1 \\ x+1 \overline{) x^3 + x^2 + x + 1} \\ - (x^3 + x^2) \\ \hline x + 1 \end{array} = \int \frac{1}{(x+1)(x^2+1)} dx$$

- You also may need to find the square.

$$\text{Ex: } \int \frac{1}{4x^2 - 4x + 3} dx = \int \frac{1}{(2x-1)^2 + 2} dx$$

3. Find the partial fraction decomposition.

$$\frac{\text{numerator}}{(ax+b)^n} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \dots + \frac{C}{(ax+b)^n}$$

$$\frac{\text{numerator}}{(ax^2+bx+c)^n} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \cdots + \frac{Ex+F}{(ax^2+bx+c)^n}$$

4. Add all coefficients with a least common denominator.

5. Set numerators equal to each other.

6. Find coefficients (A, B, C, \dots)

A) Plug in a value for x , preferably one that sets other coefficients to 0.

$$\text{Ex: } 1 = A(x+1) + B(x-1)$$

$$\text{Try } x=1: 1 = A(2) + B(0) \Rightarrow A = \frac{1}{2}$$

$$\text{Try } x=-1: 1 = A(0) + B(-2) \Rightarrow B = -\frac{1}{2}$$

B) Expand, add like terms together, set polynomial terms equal.

$$1 = Ax + A + Bx - B$$

$$0x + 1 = (A+B)x + (A-B)$$

$$0 = A+B \quad \text{and} \quad 1 = A-B$$

$$A = 1+B \Rightarrow 0 = 1+B+B = 1+2B$$

$$-1 = 2B \quad B = -\frac{1}{2} \quad \text{and} \quad A = \frac{1}{2}$$

? Write coefficients in the integral and solve.

$$\text{Ex: } \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$$

$$\begin{array}{r} x+1 \\ \hline x^3 - x^2 - x + 1 \end{array} \left/ \begin{array}{r} x^4 + 0x^3 - 2x^2 + 4x + 1 \\ -(x^4 - x^3 - x^2 + x) \\ \hline x^3 - x^2 + 3x + 1 \\ - (x^3 - x^2 - x + 1) \\ \hline 4x \end{array} \right.$$

$$= \int x+1 + \frac{4x}{x^3 - x^2 - x + 1} dx$$

$$= \frac{1}{2}x^2 + x + \int \frac{4x}{x^3 - x^2 - x + 1} dx$$

$$\textcircled{1} \quad \int \frac{4x}{x^3 - x^2 - x + 1} dx = \int \frac{4x}{(x+1)(x-1)^2} dx$$

Try $x = -1: -1 - 1 + 1 + 1 = 0$

so $(x+1)$ is a factor

$$x+1 \overline{)x^3 - x^2 - x + 1} \\ - (x^3 + x^2) \downarrow \\ - 2x^2 - x \\ - (-2x^2 - 2x) \downarrow \\ x+1$$

Try $x = 1: 1 - 2 + 1 = 0$

so $(x-1)$ is a factor

$$x^2 - 2x + 1 = (x-1)(x-1)$$

$$= \int \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} dx$$

$$\frac{4x}{(x+1)(x-1)^2} = \frac{A(x-1)^2 + B(x-1)(x+1) + C(x+1)}{(x+1)(x-1)^2}$$

$$4x = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$$

$$x=1: 4 = A(0) + B(0)(2) + C(2) = 2C \Rightarrow C=2$$

$$x=-1: -4 = A(-2)^2 + B(-2)(0) + C(0) = 4A \Rightarrow A=-1$$

$$x=0: 0 = A(-1)^2 + B(-1)(1) + C(1) = A - B + C = -1 - B + 2$$

$$0 = 1 - B \Rightarrow B=1$$

$$= \int \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2} dx$$

$$= - \int \frac{1}{x+1} dx + \int \frac{1}{x-1} dx + 2 \int \frac{1}{(x-1)^2} dx$$

$$= -\ln|x+1| + \ln|x-1| - 2 \cdot \frac{1}{x-1} + C$$

$$\textcircled{1} \quad = \frac{1}{2}x^2 + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + C$$