Use cases for integrals:

Net change theorem

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

The integral of the rate of change ($f'(x)$) is the net change ($f(b) - f(a)$).

Ex: Suppose you have a bird's eye view of a car and know it's x and y velocity in meters per sec.

$V_x(t) = 1$ $\quad$ $V_y(t) = 1 + \cos(2t)$

What is its displacement and distance traveled after 5 sec?

$$\Delta x = \int_0^5 1\,dt = t\Big|_0^5 = 5$$

$$\Delta y = \int_0^5 1 + \cos(2t)\,dt = t + \tfrac{1}{2}\sin(2t)\Big|_0^5 = 5 + \tfrac{1}{2}\sin(10)$$

Displacement from origin: $\sqrt{(5)^2 + \left(5 + \tfrac{1}{2}\sin(10)\right)^2} = 6.88$ meters

Total distance traveled:

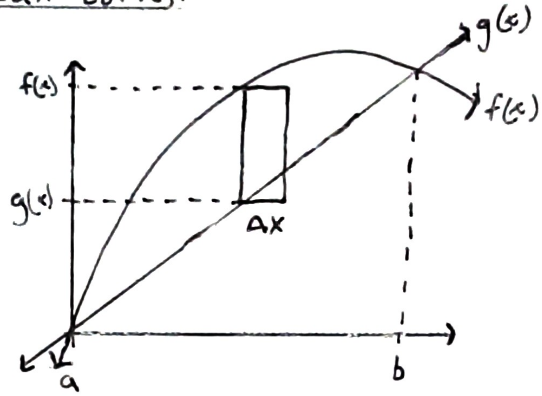
$$\text{Speed}(t) = \sqrt{V_x^2 + V_y^2}$$

$$\int_0^5 \text{Speed}(t)\,dt = \int_0^5 \sqrt{25 + (1 + \cos(2t))^2}\,dt \approx 7.42 \text{ meters}$$

Total distance traveled on the $x = \int_0^5 |V_x|\,dt = 5$ meters

Total distance traveled on the $y = \int_0^5 |V_y|\,dt = 4.73$ meters

Area between curves:



$$\text{Area} = \int_a^b f(x) - g(x)\,dx$$

$$= \left| \int_a^b g(x) - f(x)\,dx \right|$$

1. Find a and b usually by finding intersection points.
2. Find which function is ontop or use absolute value.
3. Solve definite integral

Ex: Find area bounded by $y = \sin x$ and $y = \cos x$ on $[0, \frac{\pi}{2}]$

$\sin x = \cos x \qquad x = \frac{\pi}{4}$

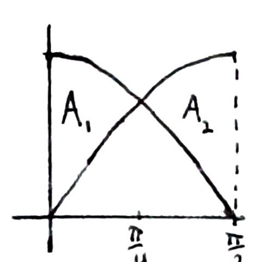
$$\text{Area} = \left| \int_0^{\frac{\pi}{4}} \sin x - \cos x\,dx \right| + \left| \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x\,dx \right|$$

$$= \left| -\cos x - \sin x \Big|_0^{\frac{\pi}{4}} \right| + \left| -\cos x - \sin x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right|$$

$$= \left| -\cos\frac{\pi}{4} - \sin\frac{\pi}{4} - \left(-\cos 0 - \sin 0\right) \right| + \left| -\cos\frac{\pi}{2} - \sin\frac{\pi}{2} - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) \right|$$

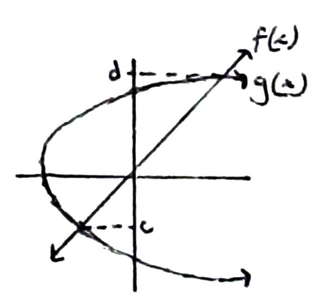
$$= \left| -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - (-1 - 0) \right| + \left| -0 - 1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \right| = \left| -\sqrt{2} + 1 \right| + \left| -1 + \sqrt{2} \right|$$

$$= \sqrt{2} - 1 + \sqrt{2} - 1 = 2(\sqrt{2} - 1) = 2\sqrt{2} - 2$$



You can also integrate with respect to y

$$\int_c^d f(x) - g(x)\,dx$$



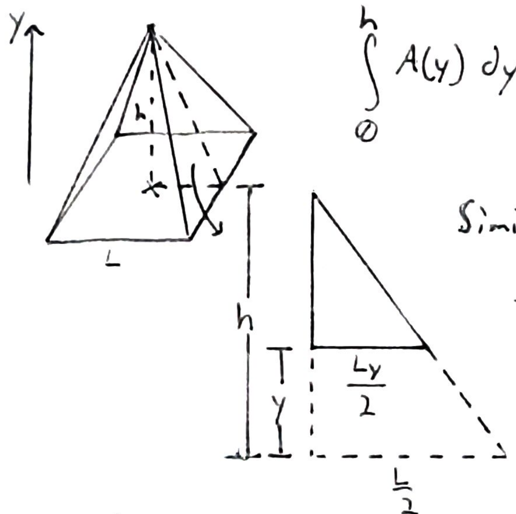
Volumes with cylinders

Volume of a cylinder = Area of base · Height
(Even for irregular base)

The volume of objects other than cylinders can be found by cutting it into infinitly many cylinders where the area of its base is $A(x)$ or $A(y)$ and its height is dx or dy.

$$\int_a^b A(x)\,dx \quad \text{or} \quad \int_c^d A(y)\,dy$$

Ex 1: Find the volume of a pyramid with a square base of length L and height of h.



$$\int_0^h A(y)\,dy = \int_0^h L_y^2\,dy$$

Cylinders stack in the y direction so it's dy.

Similar triangles:

$$\frac{h}{\frac{L}{2}} = \frac{h-y}{\frac{L_y}{2}} \Rightarrow \frac{2h}{L} = \frac{2(h-y)}{L_y}$$

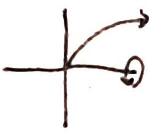
$$L_y = \frac{L}{h}(h-y) = L - \frac{L}{h}y$$

$$\int_0^h \left(L - \frac{L}{h}y\right)^2 dy = \int_0^h L^2 - 2\frac{L^2}{h}y + \frac{L^2}{h^2}y^2\,dy = L^2\int_0^h 1 - \frac{2}{h}y + \frac{1}{h^2}y^2\,dy$$

$$= L^2\left[y - \frac{1}{h}y^2 + \frac{1}{3h^2}y^3\right]_0^h = L^2\left(h - \frac{h^2}{h} + \frac{h^3}{3h^2}\right) = \frac{1}{3}L^2h$$

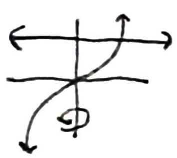
When an area is revolved around an axis, the area of the base is a circle (πr^2) and you need to write r in terms of x or y.

Ex 2: Find the volume of the solid obtained by rotating $y=\sqrt{x}$ around the x-axis from 0 to 1.



$$\int_0^1 \pi(\sqrt{x})^2\,dx = \pi\int_0^1 x\,dx = \pi\left[\frac{1}{2}x^2\right]_0^1 = \frac{\pi}{2}$$

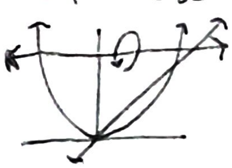
Ex 3: Find the volume of the solid obtained by rotating the area bounded by $y=x^3$, $y=8$, and $x=0$ around the y-axis.



$$\int_0^8 \pi\left(y^{\frac{1}{3}}\right)^2 dy = \pi\int_0^8 y^{\frac{2}{3}}\,dy = \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^8 = \pi\left(\frac{3}{5}8^{\frac{5}{3}}\right)$$

$$= \pi\frac{3}{5}\left(8^{\frac{1}{3}}\right)^5 = \pi\frac{3}{5}2^5 = \pi\frac{3\cdot 32}{5} = \frac{96}{5}\pi$$

Ex 4: Find the volume of the solid obtained by rotating the area enclosed by $y=x$ and $y=x^2$ around $y=2$.



Find the bounds: $x=x^2 \Rightarrow 0=x^2-x=x(x-1)$

$x=0$ and 1

$$\int_0^1 \pi\underbrace{(2-x^2)}_{\text{Outer radius}}{}^2 - \pi\underbrace{(2-x)}_{\text{Inner radius}}{}^2\,dx = \pi\int_0^1 4-4x^2+x^4-(4-4x+x^2)\,dx$$

$$(2-x^2)(2-x^2)=4-4x^2+x^4 \qquad (2-x)(2-x)=4-4x+x^2$$

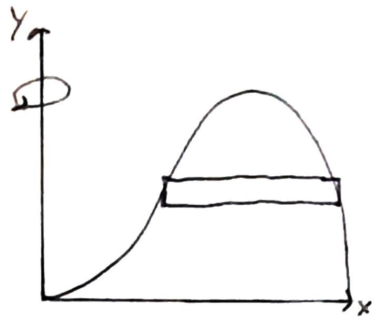
$$=\pi\int_0^1 x^4-5x^2+4x\,dx = \pi\left[\frac{1}{5}x^5-\frac{5}{3}x^3+\frac{4}{2}x^2\right]_0^1 = \pi\left(\frac{1}{5}-\frac{5}{3}+2\right)$$

$$=\pi\left(\frac{6}{30}-\frac{50}{30}+\frac{60}{30}\right)=\pi\left(-\frac{44}{30}+\frac{60}{30}\right)=\frac{16}{30}\pi=\frac{8}{15}\pi$$

Volume with shells

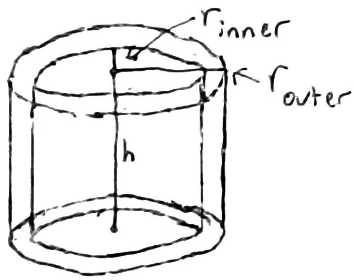
It can be hard to stack cylinders for some objects.

Ex: Find the volume obtained by rotating $y = 2x^2 - x^3$ and $y = 0$ around the y-axis



If we were to use cylinders, we would have to find the local max and then divide $y = 2x^2 - x^3$ into two separate functions for the outer and inner radius of the circle.

Instead we could use an infinite number of shells.



$$\text{Volume} = \text{Volume}_{outer} - \text{Volume}_{inner}$$

$$= \pi r_{out}^2 h - \pi r_{in}^2 h$$

$$= \pi (r_{out}^2 - r_{in}^2) h$$

$$= \pi (r_{out} + r_{in})(r_{out} - r_{in}) h$$

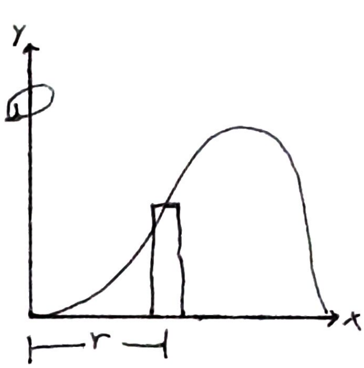
$$= \pi (r_{out} + r_{in}) \Delta r \, h$$

$$= 2\pi \frac{r_{out} + r_{in}}{2} h \, \Delta r$$

$$= \underbrace{2\pi r_{ave}}_{\text{Circumference}} \cdot \underset{\downarrow \text{height}}{h} \cdot \underbrace{\Delta r}_{\text{Thickness}}$$

$$\text{Volume} = \int_a^b 2\pi r \cdot f(r) \cdot dr$$

Ex:



$$\int_{\emptyset}^{2} 2\pi x \cdot (2x^2 - x^3)\, dx = \frac{16}{5}\pi$$

Use shells over cylinders when the axis of rotation (Ex: y-axis) is different from the independent variable (Ex: x). $y \neq x$

Work

$F = ma$ $\qquad W = F \cdot d$ when Force

$$W = \int_{x_i}^{x_f} F(x)\,dx$$ when Force

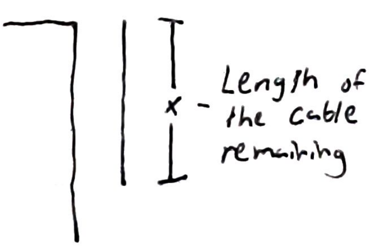
	Force	Distance	Work	g
Metric	N	m	J	9.8
Imperial	lbs	ft	ft-lbs	32.2

Ex 1: A force of 40N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15cm. How much work is done by stretching it from 15cm to 18cm?

$F = kx \qquad 40 = k(15-10) \qquad k = 8\ \frac{N}{cm}$

$$W = \int_5^8 8x\,dx = 4x^2\Big|_5^8 = 4 \cdot 8^2 - 4 \cdot 5^2 = 156\ Ncm \cdot \frac{1m}{100cm} = 1.56\ J$$

Ex 2: A 200 lb cable is 100ft long and hangs vertically from the top of a tall building. How much work to lift the cable to the top?



$F(x)$ = Force of cable hanging

$F(100) = 200 \qquad F(0) = 0$

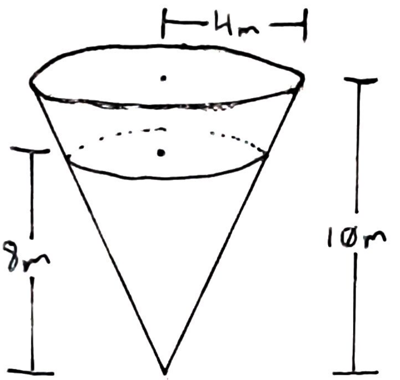
Assuming uniform density of the cable, the Force is linear/a line.

$$\text{slope} = \frac{F(100) - F(0)}{100 - 0} = \frac{200}{100} = 2$$

$F(x) - 0 = 2(x - 0) \Rightarrow F(x) = 2x$

$$\int_{100}^{0} 2x\,dx = x^2\Big|_{100}^{0} = 10{,}000\ \text{ft-lbs}$$

Ex 3: A tank has the shape of an inverted circular cone with height 10m and base radius 4m. It is filled with water to a height of 8m Find the work required to empty the tank by pumping all of the water to the top of the tank? Density of water is $1{,}000 \frac{kg}{m^3}$



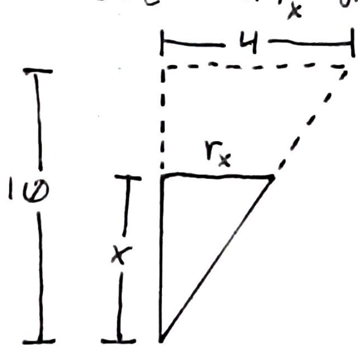
x = Height of water left in the tank.

W = Add up the force of a slice of at height x times its height to the top.
$= \int (10-x)\, dF(x)$

$dF(x)$ = Mass of a small cylindrical slice times acceleration due to gravity.
$= 9.8\, dm(x)$

$dm(x)$ = Density of water times the volume of tiny cylindrical slice.
$= 1{,}000\, dV(x)$

$dV(x) = \pi r_x^2\, dx$



$\frac{4}{10} = \frac{r_x}{x}$

$r_x = \frac{2}{5}x$

$$W = \int_0^8 (10-x)\cdot 9.8 \cdot 1{,}000 \cdot \pi \left(\frac{2}{5}x\right)^2 dx = 3.36 \times 10^6\ J$$