Use cases for integrals:

Net change theorem

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

The integral of the rate of change $(f'(x))$ is the net change $(f(b) - f(a))$.

Ex: Suppose you have a bird's eye view of a car and know it's x and y velocity in meters per sec.

$V_x(t) = 1$ $\quad V_y(t) = 1 + \cos(2t)$

What is its displacement and distance traveled after 5 sec?

$$\Delta x = \int_0^5 1\,dt = t\Big|_0^5 = 5$$

$$\Delta y = \int_0^5 1 + \cos(2t)\,dt = t + \frac{1}{2}\sin(2t)\Big|_0^5 = 5 + \frac{1}{2}\sin(10)$$

Displacement from origin: $\sqrt{(5)^2 + \left(5 + \frac{1}{2}\sin(10)\right)^2} = 6.88$ meters

Total distance traveled:

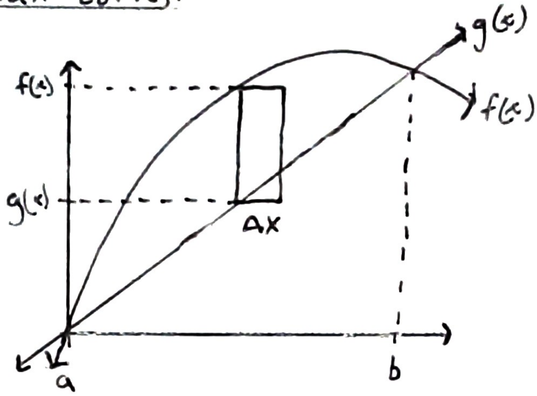
$$\text{Speed}(t) = \sqrt{V_x^2 + V_y^2}$$

$$\int_0^5 \text{Speed}(t)\,dt = \int_0^5 \sqrt{25 + (1+\cos(2t))^2}\,dt \approx 7.42 \text{ meters}$$

Total distance traveled on the $x = \int_0^5 |V_x|\,dt = 5$ meters

Total distance traveled on the $y = \int_0^5 |V_y|\,dt = 4.73$ meters

Area between curves:



$$\text{Area} = \int_a^b f(x) - g(x)\,dx$$

$$= \left| \int_a^b g(x) - f(x)\,dx \right|$$

1. Find a and b usually by finding intersection points.
2. Find which function is ontop or use absolute value.
3. Solve definite integral

Ex: Find area bounded by $y = \sin x$ and $y = \cos x$ on $[0, \frac{\pi}{2}]$

$\sin x = \cos x \quad x = \frac{\pi}{4}$

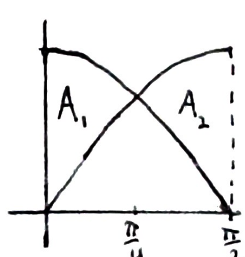
$$\text{Area} = \left| \int_0^{\pi/4} \sin x - \cos x\,dx \right| + \left| \int_{\pi/4}^{\pi/2} \sin x - \cos x\,dx \right|$$

$$= \left| -\cos x - \sin x \Big|_0^{\pi/4} \right| + \left| -\cos x - \sin x \Big|_{\pi/4}^{\pi/2} \right|$$

$$= \left| -\cos\frac{\pi}{4} - \sin\frac{\pi}{4} - (-\cos 0 - \sin 0) \right| + \left| -\cos\frac{\pi}{2} - \sin\frac{\pi}{2} - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) \right|$$

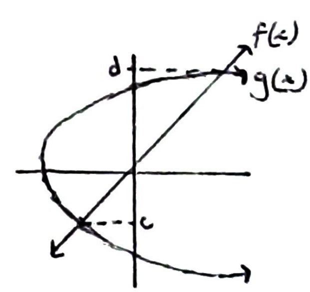
$$= \left| -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - (-1 - 0) \right| + \left| -0 - 1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \right| = \left| -\sqrt{2} + 1 \right| + \left| -1 + \sqrt{2} \right|$$

$$= \sqrt{2} - 1 + \sqrt{2} - 1 = 2(\sqrt{2} - 1) = 2\sqrt{2} - 2$$



You can also integrate with respect to y

$$\int_c^d f(x) - g(x)\,dx$$



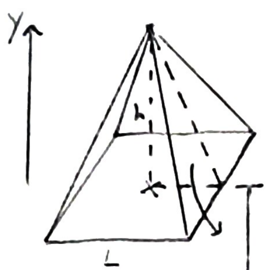
Volumes with cylinders

Volume of a cylinder = Area of base · Height
(Even for irregular base)

The volume of objects other than cylinders can be found by cutting it into infinitly many cylinders where the area of its base is $A(x)$ or $A(y)$ and its height is dx or dy.

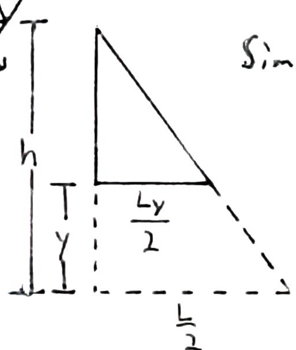
$$\int_a^b A(x)\,dx \quad \text{or} \quad \int_c^d A(y)\,dy$$

Ex 1: Find the volume of a pyramid with a square base of length L and height of h.



$$\int_0^h A(y)\,dy = \int_0^h L_y^{\,2}\,dy$$

Cylinders stack in the y direction so it's dy.



Similar triangles:

$$\frac{h}{\frac{L}{2}} = \frac{h-y}{\frac{L_y}{2}} \Rightarrow \frac{2h}{L} = \frac{2(h-y)}{L_y}$$

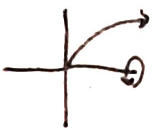
$$L_y = \frac{L}{h}(h-y) = L - \frac{L}{h}y$$

$$\int_0^h \left(L - \frac{L}{h}y\right)^2 dy = \int_0^h L^2 - 2\frac{L^2}{h}y + \frac{L^2}{h^2}y^2\,dy = L^2\int_0^h 1 - \frac{2}{h}y + \frac{1}{h^2}y^2\,dy$$

$$= L^2\left[y - \frac{1}{h}y^2 + \frac{1}{3h^2}y^3\right]_0^h = L^2\left(h - \frac{h^2}{h} + \frac{h^3}{3h^2}\right) = \frac{1}{3}L^2 h$$

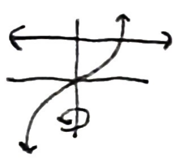
When an area is revolved around an axis, the area of the base is a circle (πr^2) and you need to write r in terms of x or y.

Ex 2: Find the volume of the solid obtained by rotating $y=\sqrt{x}$ around the x-axis from $\emptyset$ to 1.



$$\int_\emptyset^1 \pi(\sqrt{x})^2\,dx = \pi\int_\emptyset^1 x\,dx = \pi\left[\frac{1}{2}x^2\right]_\emptyset^1 = \frac{\pi}{2}$$

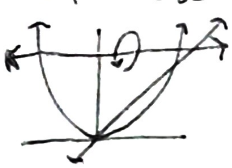
Ex 3: Find the volume of the solid obtained by rotating the area bounded by $y=x^3$, $y=8$, and $x=\emptyset$ around the y-axis.



$$\int_\emptyset^8 \pi\left(y^{\frac{1}{3}}\right)^2 dy = \pi\int_\emptyset^8 y^{\frac{2}{3}}\,dy = \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_\emptyset^8 = \pi\left(\frac{3}{5}8^{\frac{5}{3}}\right)$$

$$= \pi\frac{3}{5}\left(8^{\frac{1}{3}}\right)^5 = \pi\frac{3}{5}2^5 = \pi\frac{3\cdot 32}{5} = \frac{96}{5}\pi$$

Ex 4: Find the volume of the solid obtained by rotating the area enclosed by $y=x$ and $y=x^2$ around $y=2$



Find the bounds: $x=x^2 \Rightarrow \emptyset = x^2 - x = x(x-1)$

$x=\emptyset$ and 1

$$\int_\emptyset^1 \pi\underbrace{(2-x^2)}_{\text{Outer radius}}{}^2 - \pi\underbrace{(2-x)}_{\text{Inner radius}}{}^2\,dx = \pi\int_\emptyset^1 4-4x^2+x^4-(4-4x+x^2)\,dx$$

$(2-x^2)(2-x^2) = 4-4x^2+x^4 \qquad (2-x)(2-x) = 4-4x+x^2$

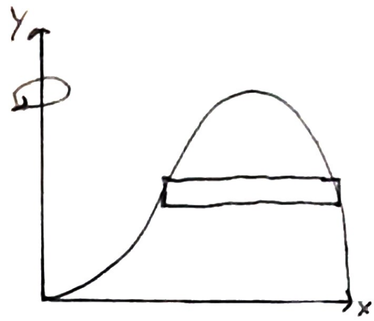
$$=\pi\int_\emptyset^1 x^4-5x^2+4x\,dx = \pi\left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + \frac{4}{2}x^2\right]_\emptyset^1 = \pi\left(\frac{1}{5}-\frac{5}{3}+2\right)$$

$$=\pi\left(\frac{6}{30}-\frac{50}{30}+\frac{60}{30}\right) = \pi\left(-\frac{44}{30}+\frac{60}{30}\right) = \frac{16}{30}\pi = \frac{8}{15}\pi$$

Volume with shells

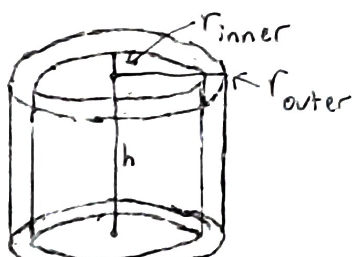
It can be hard to stack cylinders for some objects.

Ex: Find the volume obtained by rotating $y=2x^2-x^3$ and $y=0$ around the y-axis



If we were to use cylinders, we would have to find the local max and then divide $y=2x^2-x^3$ into two separate functions for the outer and inner radius of the circle.

Instead we could use an infinite number of shells.



$$\text{Volume} = \text{Volume}_{outer} - \text{Volume}_{inner}$$

$$= \pi r_{out}^2 h - \pi r_{in}^2 h$$

$$= \pi (r_{out}^2 - r_{in}^2) h$$

$$= \pi (r_{out} + r_{in})(r_{out} - r_{in}) h$$

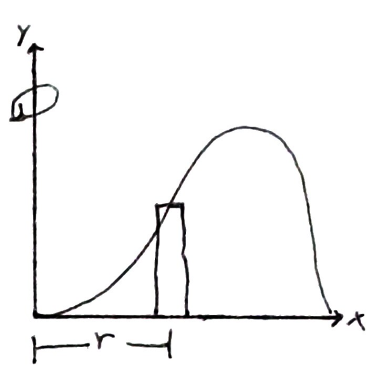
$$= \pi (r_{out} + r_{in}) \Delta r\, h$$

$$= 2\pi \frac{r_{out} + r_{in}}{2} h \,\Delta r$$

$$= \underbrace{2\pi r_{ave}}_{\text{Circumference}} \cdot \underset{\text{height}}{h} \cdot \underbrace{\Delta r}_{\text{Thickness}}$$

$$\text{Volume} = \int_a^b 2\pi r \cdot f(r) \cdot dr$$

Ex:



$$\int_0^2 2\pi x \cdot (2x^2 - x^3)\, dx = \frac{16}{5}\pi$$

Use shells over cylinders when the axis of rotation (Ex: y-axis) is different from the independent variable (Ex: x). $y \neq x$

<u>Work</u>

$F = ma$ $\qquad W = F \cdot d$ when Force is constant

Springs: $F = kx$ $\qquad W = \int_{x_i}^{x_f} F(x) \cdot dx = \int_{x_i}^{x_f} h(x) \cdot dF(x)$ when Force isn't constant.

	Force	Distance	Work	g
Metric	N	m	J	9.8
Imperial	lbs	ft	ft-lbs	32.2

Ex 1: A force of 40N is required to hold a spring that has been stretched from its natural length of 10cm to a length of 15cm. How much work is done by stretching it from 15cm to 18cm?

$F = kx \qquad 40 = k(15-10) \qquad k = 8 \frac{N}{cm}$

$$W = \int_5^8 8x\,dx = 4x^2 \Big|_5^8 = 4 \cdot 8^2 - 4 \cdot 5^2 = 156 \text{ Ncm} \cdot \frac{1\text{m}}{100\text{cm}} = 1.56 \text{ J}$$

Ex 2: A 200 lb cable is 100ft long and hangs vertically from the top of a tall building. How much work to lift the cable to the top?

$\bar{x}$ - How far up the cable was pulled.

$F(x)$ = Force of cable hanging

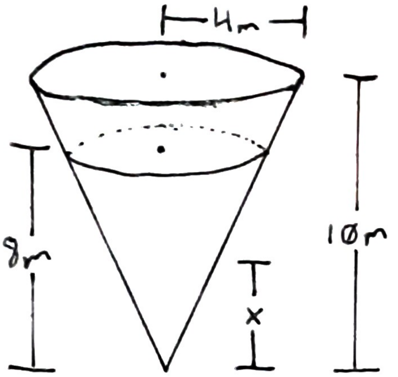
$F(0) = 200 \qquad F(100) = 0$

Assuming uniform density of the cable, the Force is linear/a line.

$$\text{slope} = \frac{F(0) - F(100)}{0 - 100} = \frac{200}{-100} = -2 \qquad F(x) - 0 = -2(x - 100)$$

$$W = \int_0^{100} -2(x-100)\,dx = -2\int_0^{100} x - 100\,dx = -2\left[\tfrac{1}{2}x^2 - 100x\right]_0^{100} = 10{,}000 \text{ ft-lbs}$$

Ex 3: A tank has the shape of an inverted circular cone with height 10m and base radius 4m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank?
Density of water is $1{,}000 \frac{kg}{m^3}$



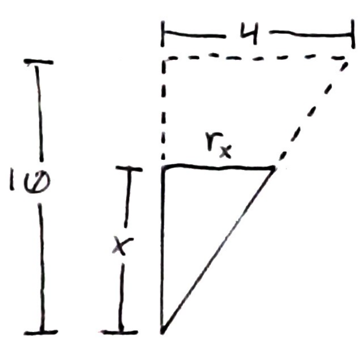
x = Height of water left in the tank.

$$W = \int_0^8 h(x)\, dF(x) = \int_0^8 (10 - x)\, dF(x)$$

$dF(x)$ = Force from a small cylindrical slice.
$= \text{mass} \cdot \text{acc} = dm(x) \cdot 9.8$

$dm(x)$ = Mass of small cylindrical slice.
$= \text{Density} \cdot \text{Volume} = 1000 \cdot dV(x)$

$dV(x)$ = Volume of small cylindrical slice.
$= \text{Area of base} \cdot \text{height}$
$= \pi r_x^2 \cdot dx$



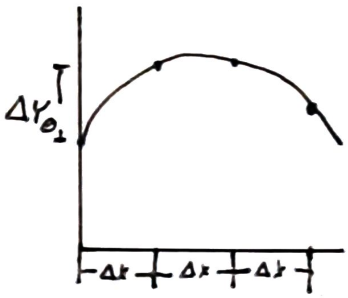
$$\frac{4}{10} = \frac{r_x}{x}$$

$$r_x = \frac{2}{5}x$$

$$W = \int_0^8 (10 - x) \cdot 9.8 \cdot 1{,}000 \cdot \pi \left(\frac{2}{5}x\right)^2 dx = 3.36 \times 10^6 \text{ J}$$

Arc Length

- Divide arc into infinity many line segments with equal width (Δx)
- Add all the line segment's distances up using the distance formula



$$\text{Distance of one segment} = \sqrt{(x_i - x_{i-1})^2 + (Y_i - Y_{i-1})}$$

$$= \sqrt{(\Delta x)^2 + (\Delta Y_i)^2}$$

$$\text{Slope} = \frac{\Delta Y_i}{\Delta x} = f'(x_i) \Rightarrow \Delta Y_i = f'(x_i)\,\Delta x$$

$$= \sqrt{(\Delta x)^2 + (f'(x_i)\,\Delta x)^2} = \sqrt{\Delta x^2\left(1 + (f'(x_i))^2\right)} = \sqrt{1 + (f'(x_i))^2}\,\Delta x$$

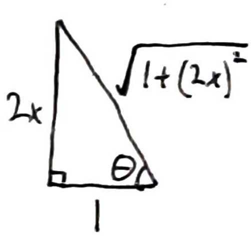
Sum them up $\Rightarrow$ $\underline{L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx}$

Ex: Find the arc length of $y = x^2$ from 0 to 5

$$L = \int_0^5 \sqrt{1 + (2x)^2}\, dx \qquad 2x = \tan\theta \qquad \int \sqrt{1 + \tan^2\theta}\, \frac{\sec^2\theta}{2}\, d\theta$$

$$2dx = \sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int \sec^3\theta\, d\theta = \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$$



$$\sec\theta = \frac{hyp}{adj} = \sqrt{1 + (2x)^2}$$

$$= \frac{1}{4}\left(\sqrt{1+(2x)^2}\, 2x + \ln\left|\sqrt{1+(2x)^2} + 2x\right|\right)\Big|_0^5$$

$= 25.8742$

You can also have a constant Δy and have Δx_i

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$