Finding Integrals

1. Limit definition

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\Delta x}_{\text{Riemann sum}}$$

$f(x_i) = \text{Height}$

$\Delta x = \text{Width} = \dfrac{b-a}{n}$

$x_i = a + i\Delta x$

- We use an infinite number of rectangles to get the area under the curve.
- If we don't use an infinite number, we can use the left, right, midpoint, smallest y value (lower sum), or largest y value (upper sum) to estimate the area under the curve.
- We can also get a rough estimate by finding the max (M) and min (m) of $f(x)$.

If $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

$$\sum_{i=1}^{n} c = c\sum_{i=1}^{n} = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \qquad \sum_{i=1}^{n} a_i - b_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

2. Properties of integrals

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad \int_a^a f(x)\,dx = 0$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx \qquad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad \int_a^b \big(f(x)-g(x)\big)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

3. Fundamental theorem of calculus

Part 1: Deriving the integral of f gives f

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$ a is any constant

Ex: $$\frac{d}{dx}\int_1^{x^4} \sec t\,dt$$

$u = x^4$

$\frac{du}{dx} = 4x^3$

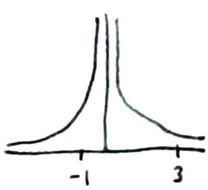
$\frac{d}{dx} = \frac{du}{dx} \cdot \frac{d}{du}$

$$4x^3 \cdot \frac{d}{du}\int_1^{u} \sec t\,dt = 4x^3 \cdot \sec u = 4x^3 \sec x^4$$

Part 2:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Ex: $$\int_{-1}^{3} \frac{1}{x^2}\,dx \neq -\frac{1}{x}\Big|_{-1}^{3} = -\frac{4}{3}$$



f(x) has to be continious or have a finite number of jump discontinuities over [a, b] to be integrable.

Integration and differentiation are inverse processes.

4 U-Substitution

1. Define u
2. Find $\frac{du}{dx}$
3. Solve for dx
4 Plug u and dx into the integral and simplify

If definite integral: Re-write bounds for u

5. Solve integral

If indefinite integral: Substitute u so it's in terms of x.

Ex 1: $\int x^3 \cos(x^4+2)\,dx$

$u = x^4 + 2$

$\frac{du}{dx} = 4x^3 \Rightarrow dx = \frac{du}{4x^3}$

$$\int x^3 \cos(u) \frac{du}{4x^3} = \int \frac{1}{4}\cos u \, du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2) + C$$

Ex 2: $\int \sqrt{1+x^2}\, x^5\, dx$

$u = 1 + x^2$ $\qquad u - 1 = x^2$

$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$$\int \sqrt{u}\, x^5 \frac{du}{2x} = \int \sqrt{u}\, x^4 \frac{1}{2} du = \int \frac{1}{2}\sqrt{u}(u-1)^2\, du$$

$$= \int \frac{1}{2}\sqrt{u}(u^2 - 2u + 1)\, du = \int \frac{1}{2}u^{\frac{5}{2}} - u^{\frac{3}{2}} + \frac{1}{2}u^{\frac{1}{2}}\, du$$

$$= \frac{1}{2}\cdot\frac{2}{7}u^{\frac{7}{2}} - \frac{2}{5}u^{\frac{5}{2}} + \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{7}(1+x^2)^{\frac{7}{2}} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$$

Ex 3: $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$

$$\int \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right) = \int -\frac{1}{u}\, du = -\ln|u| + C = -\ln|\cos x| + C$$

Ex 4: $\int_1^e \frac{\ln x}{x}\, dx$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\, du$

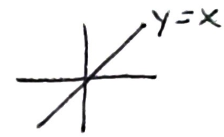
Lower: $u = \ln(1) = 0$ $\qquad$ Upper: $u = \ln(e) = 1$

$$\int_0^1 \frac{u}{x}\, x\, du = \int_0^1 u\, du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}$$

5. Symmetry: If $f(x)$ is continious on $[-a,a]$

If $f(-x) = f(x)$ then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

If $f(-x) = -f(x)$ then $\int_{-a}^{a} f(x)\,dx = 0$



6. Integration by parts

Just like U-substitution is the inverse of the chain rule, integration by parts is the inverse of the product rule.

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$u = f(x)$ $\quad dv = g'(x)\,dx$

$du = f'(x)\,dx$ $\quad v = g(x)$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Use LIPET for picking u:

L - ln
I - Inverse trig
P - Powers of x
E - Exponential
T - Trig

The goal of integration by parts is to convert the integral to a simpler integral. This may not always be the case.

Definite integrals: $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

Ex 1: $\int x \sin x \, dx$ $\quad u = x \quad dv = \sin x \, dx$

$du = dx \quad v = -\cos x$

$= x(-\cos x) - \int -\cos x \, dx = -x\cos x + \sin x + C$

Ex 2: $\int \ln x \, dx$ $\quad u = \ln x \quad dv = dx$

$du = \frac{1}{x} dx \quad v = x$

$= \ln x (x) - \int x \frac{1}{x} dx = x \ln x - x + C$

Ex 3: $\int t^2 e^t \, dt$ $\quad u = t^2 \quad dv = e^t \, dt$

$du = 2t \, dt \quad v = e^t$

$= t^2 e^t - \int e^t 2t \, dt = t^2 e^t - 2\int t e^t \, dt$ $\quad u = t \quad dv = e^t \, dt$

$du = dt \quad v = e^t$

$= t^2 e^t - 2\left[t e^t - \int e^t dt\right] = t^2 e^t - 2te^t + 2e^t + C$

Ex 4: $\int e^x \sin x \, dx$ $\quad u = e^x \quad dv = \sin x \, dx$

$du = e^x dx \quad v = -\cos x$

$= e^x(-\cos x) - \int -\cos x \, e^x \, dx = -e^x \cos x + \int \cos x \, e^x \, dx$

$u = e^x \quad dv = \cos x \, dx$

$du = e^x dx \quad v = \sin x$

Add to the other side.

$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int \sin x \, e^x \, dx$

$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$

$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$

7. Powers of sin and cos

$$\int \sin^m x \cos^n x \, dx$$

$\sin^m x$	$\cos^n x$	Solution
Odd	Odd	Either
Odd	Even	$u = \cos x$
Even	Odd	$u = \sin x$
Even	Even	$\sin^2 x = \frac{1}{2}(1-\cos 2x)$ $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

(first three rows) — $\sin^2 x + \cos^2 x = 1$

Ex 1: $\int \cos^5 x \, dx$ $\quad$ $\cos^5 x$ is odd, $\sin^0 x$ is even

$u = \sin x$

$du = \cos x \, dx \Rightarrow dx = \frac{du}{\cos x}$

$$\int \cos^5 x \frac{du}{\cos x} = \int \cos^4 x \, du = \int (\cos^2 x)^2 \, du$$

$\cos^2 x = 1 - \sin^2 x$ $\qquad = \int (1-\sin^2 x)^2 \, du = \int (1-u^2)^2 \, du$

$(1-u^2)(1-u^2) = 1 - 2u^2 + u^4$ $\qquad = \int 1 - 2u^2 + u^4 \, du$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Ex 2: $\int \sin^4 x \, dx$ $\quad$ $\sin^4 x$ is even and $\cos^0 x$ is even

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1}{2}(1-\cos 2x)\right)^2 = \frac{1}{4}(1-\cos 2x)^2$$

$$= \int \frac{1}{4}(1-\cos 2x)^2 = \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x \, dx$$

$$= \frac{1}{4}\left[\int dx - 2\int \cos 2x \, dx + \int \cos^2 2x \, dx\right]$$

$$= \frac{1}{4}\left[x - \sin 2x + \int \cos^2 2x \, dx\right]$$

$$\cos^2(2x) = \frac{1}{2}(1 + \cos 4x)$$

$$= \frac{1}{4}\left[x - \sin 2x + \frac{1}{2}\int 1 + \cos 4x \, dx\right]$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}\left[\int dx + \int \cos 4x \, dx\right]$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

8. Powers of tan and sec

$$\int \tan^m x \sec^n x \, dx$$

This also applies for cot and csc

$\tan^m x$	$\sec^n x$	Solution
Odd	Odd	$u = \sec x$
Odd	Even	Either
Even	Odd	other
Even	Even	$u = \tan x$

$$\tan^2 x + 1 = \sec^2 x$$

- Integration by parts
- $\int \sec x = \ln|\sec x + \tan x| + C$
- $\int \tan x = \ln|\sec x| + C$
- Move to other side and divide

Ex 1: $\int \tan^2 x \sec^4 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx \Rightarrow dx = \frac{du}{\sec^2 x}$

$$\int \tan^2 x \sec^4 x \frac{du}{\sec^2 x} = \int u^2 \sec^2 x \, du$$

$$\sec^2 x = 1 + \tan^2 x \qquad = \int u^2(1 + u^2)\, du = \int u^2 + u^4 \, du$$

$$= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

Ex 2: $\int \sec^3 x \, dx = \int \sec^2 x \sec x \, dx = \int (\tan^2 x + 1)\sec x \, dx$

$= \int \tan^2 x \sec x + \sec x \, dx = \int \tan^2 x \sec x \, dx + \int \sec x \, dx$

$\int \sec x \, dx = \int \frac{\sec x}{1} \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

$u = \sec x + \tan x \qquad du = \sec x \tan x + \sec^2 x \, dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{u} \cdot \frac{du}{\sec^2 x + \sec x \tan x} = \int \frac{1}{u} du = \ln|\sec x + \tan x| + C$

$\int \tan^2 x \sec x \, dx \qquad u = \tan x \qquad dv = \sec x \tan x \, dx$

$du = \sec^2 x \, dx \qquad v = \sec x$

$= \sec x \tan x - \int \sec^3 x \, dx$

$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \ln|\sec x + \tan x| + C$

$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$

9. Sin A and Cos B

$\int \sin A \cos B \, dx$ or $\int \sin A \sin B \, dx$ or $\int \cos A \cos B \, dx$

$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

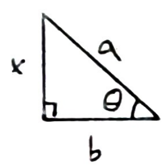
$\int \sin 4x \cos 5x \, dx = \int \frac{1}{2}[\sin(-x) + \sin(9x)] dx$

$= \frac{1}{2}\left[\int -\sin x \, dx + \int \sin(9x) dx\right] = \frac{1}{2}\left[\cos x - \frac{1}{9}\cos 9x\right] + C$

10. Trig Inverse Substitution

Expression in integrand	Trig Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

$$x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a} = \frac{opp}{hyp}$$



Pythagorean: $a^2 = x^2 + b^2 \Rightarrow b^2 = a^2 - x^2 \Rightarrow b = \sqrt{a^2 - x^2}$

Ex 1: $\int \frac{\sqrt{9 - x^2}}{x^2}\,dx$

$$x = 3\sin\theta$$
$$dx = 3\cos\theta\,d\theta$$

$$\sqrt{9 - x^2} = \sqrt{3^2 - x^2} = \sqrt{3^2 - (3\sin\theta)^2} = \sqrt{3^2 - 3^2\sin^2\theta} = \sqrt{3^2(1 - \sin^2\theta)}$$

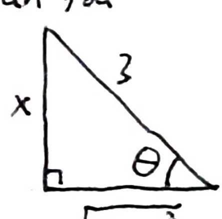
$$= \sqrt{3^2\cos^2\theta} = 3|\cos\theta|$$

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx = \int \frac{3|\cos\theta|}{3^2\sin^2\theta}\,3\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\,d\theta = \int \csc^2\theta - 1\,d\theta = -\cot\theta - \theta + C$$

$$x = 3\sin\theta \Rightarrow \frac{x}{3} = \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$$

soh cah toa



$$\cot\theta = \frac{adj}{opp} = \frac{\sqrt{9 - x^2}}{x}$$

$$= -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

Ex 2: $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

$$3-2x-x^2 = 3-(2x+x^2) = 3+1-(2x+x^2+1) = 4-(x+1)^2$$

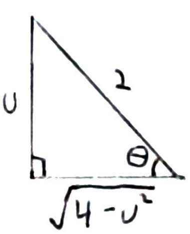
$$= \int \frac{x}{\sqrt{2^2-(x+1)^2}}\,dx \qquad \begin{array}{l} u = x+1 \Rightarrow x = u-1 \\ du = dx \end{array}$$

$$= \int \frac{u-1}{\sqrt{2^2-u^2}}\,du \qquad \begin{array}{l} u = 2\sin\theta \\ du = 2\cos\theta\, d\theta \end{array}$$

$$\sqrt{2^2-2^2\sin^2\theta} = \sqrt{2^2(1-\sin^2\theta)} = \sqrt{2^2\cos^2\theta} = 2|\cos\theta|$$

$$= \int \frac{2\sin\theta-1}{2|\cos\theta|}\,2\cos\theta\, d\theta = \int 2\sin\theta - 1\, d\theta = -2\cos\theta - \theta + C$$

$$u = 2\sin\theta \Rightarrow \sin\theta = \frac{u}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{u}{2}\right)$$



$$\cos\theta = \frac{\sqrt{4-u^2}}{2}$$

$$= -2\,\frac{\sqrt{4-u^2}}{2} - \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

$$= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

11. Method of partial fractions

This method works for rational functions which are $\frac{\text{polynomial}}{\text{polynomial}}$

0. Sometimes you need to do a U-sub on $\sqrt[n]{g(x)}$

Ex: $\int \frac{\sqrt{x+4}}{x}\,dx$ $u=\sqrt{x+4}$ $x=u^2-4$ $=2\int \frac{u^2}{u^2-4}\,du$

$du=\frac{1}{2\sqrt{x+4}}\,dx$ $dx=2u\,du$

1. If the degree of the numerator is $\geq$ degree of denominator, then use long division.

$\text{denominator}\overline{)\text{numerator}}$ with answer on top $=\int \text{answer}+\frac{\text{remainer}}{\text{denominator}}$

Ex: $\int \frac{5x^2}{x^2+x+1}\,dx$

$$\begin{array}{r} 5 \\ x^2+x+1 \overline{)5x^2+0x+0} \\ -(5x^2+5x+5) \\ \hline -5x-5 \end{array}$$

$=\int 5+\frac{-5x-5}{x^2+x+1}\,dx$

2. Factor the denominator as far as possible.

Ex: $\int \frac{1}{(x^2-1)^2}\,dx=\int \frac{1}{((x-1)(x+1))^2}\,dx=\int \frac{1}{(x-1)^2(x+1)^2}\,dx$

- Sometimes you can try numbers and see if they equal 0.

Ex. $\int \frac{1}{x^3+x^2+x+1}\,dx$ Try $x=-1$: $-1+1-1+1=0$ therefore $(x+1)$ is a factor.

$$\begin{array}{r} x^2+1 \\ x+1 \overline{)x^3+x^2+x+1} \end{array}$$

$=\int \frac{1}{(x+1)(x^2+1)}\,dx$

- You also may need to find the square.

Ex: $\int \frac{1}{4x^2-4x+3}\,dx=\int \frac{1}{(2x-1)^2+2}\,dx$

3. Find the partial fraction decomposition.

$$\frac{\text{numerator}}{(ax+b)^n}=\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\dots+\frac{C}{(ax+b)^n}$$

$$\frac{\text{numerator}}{(ax^2+bx+c)^n} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \cdots + \frac{Ex+F}{(ax^2+bx+c)^n}$$

4. Add all coefficients with a least common denominator.

5. Set numerators equal to eachother.

6. Find coefficients $(A, B, C, \ldots)$

A) Plug in a value for x, preferably one that sets other coefficients to $\emptyset$.

Ex: $1 = A(x+1) + B(x-1)$

Try $x=1$: $1 = A(2) + B(\emptyset) \Rightarrow A = \frac{1}{2}$

Try $x=-1$: $1 = A(\emptyset) + B(-2) \Rightarrow B = -\frac{1}{2}$

B) Expand, add like terms together, set polynomial terms equal.

$1 = Ax + A + Bx - B$

$\emptyset x + 1 = (A+B)x + (A-B)$

$\emptyset = A+B$ and $1 = A-B$

$A = 1+B \Rightarrow \emptyset = 1+B+B = 1+2B$

$-1 = 2B \quad B = -\frac{1}{2}$ and $A = \frac{1}{2}$

7. Write coefficients in the integral and solve.

Ex: $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

$$\begin{array}{r} x+1 \\ x^3-x^2-x+1 \overline{\smash{)}\, x^4+\emptyset x^3-2x^2+4x+1} \\ -(x^4-x^3-x^2+x) \downarrow \\ x^3-x^2+3x+1 \\ -(x^3-x^2-x+1) \\ \hline 4x \end{array}$$

$$= \int x+1 + \frac{4x}{x^3-x^2-x+1}\,dx$$

①

$$= \frac{1}{2}x^2 + x + \int \frac{4x}{x^3-x^2-x+1}\,dx$$

① $\displaystyle\int \frac{4x}{x^3-x^2-x+1}\,dx \qquad = \int \frac{4x}{(x+1)(x-1)^2}\,dx$

Try $x=-1$: $-1-1+1+1=0$

so $(x+1)$ is a factor

$$\begin{array}{r}
x^2-2x+1 \\
x+1 \overline{)\,x^3-x^2-x+1} \\
-(x^3+x^2) \downarrow \quad \\
-2x^2-x \quad \\
-(-2x^2-2x) \downarrow \\
x+1
\end{array}$$

Try $x=1$: $1-2+1=0$

so $(x-1)$ is a factor

$x^2-2x+1=(x-1)(x-1)$

$$= \int \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\,dx$$

$$\frac{4x}{(x+1)(x-1)^2} = \frac{A(x-1)^2 + B(x-1)(x+1) + C(x+1)}{(x+1)(x-1)^2}$$

$$4x = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$$

$x=1$: $4 = A(0) + B(0)(2) + C(2) = 2C \Rightarrow C=2$

$x=-1$: $-4 = A(-2)^2 + B(-2)(0) + C(0) = 4A \Rightarrow A=-1$

$x=0$ $\quad 0 = A(-1)^2 + B(-1)(1) + C(1) = A - B + C = -1 - B + 2$

$\quad 0 = 1 - B \Rightarrow B=1$

$$= \int \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}\,dx$$

$$= -\int \frac{1}{x+1}\,dx + \int \frac{1}{x-1}\,dx + 2\int \frac{1}{(x-1)^2}\,dx$$

$$= -\ln|x+1| + \ln|x-1| - 2\cdot\frac{1}{x-1} + C$$

①

$$= \frac{1}{2}x^2 + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + C$$

12. Trapezoidal and Midpoint Rules

Used to approximate definite integrals and gives a better aprox with fewer points than left or right endpoints.

Trapezoidal: $\frac{\text{left} + \text{right}}{2} = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + f(x_n)\right)$

Midpoint: $\Delta x\left(f(a + \frac{\Delta x}{2}) + f(a + \frac{\Delta x}{2} + \Delta x) + \ldots + f(a + \frac{\Delta x}{2} + n\Delta x)\right)$

Ex: $\int_0^1 e^{x^2}\,dx \quad n=4 \quad \Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = \frac{2}{8} \quad \frac{\Delta x}{2} = \frac{1}{8}$

Trap: $\frac{\Delta x}{2}\left(f(0) + 2f(\frac{1}{4}) + 2f(\frac{2}{4}) + 2f(\frac{3}{4}) + f(1)\right) = 1.49068$

Mid: $\Delta x\left(f(\frac{1}{8}) + f(\frac{3}{8}) + f(\frac{5}{8}) + f(\frac{7}{8})\right) = 1.44875$

Actual answere: 1.46265 * The Midpoint tends to be more accurate.

13. Improper integrals

Either bound is $\pm\infty$ and $f(x)$ is continious on bounds.

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad \int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx$$

c is usually chosen to be 0 or 1.

Ex: $$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx$$

$$= \lim_{a\to-\infty}\left[\tan^{-1}x\right]_a^0 + \lim_{b\to\infty}\left[\tan^{-1}x\right]_0^b = 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \pi$$

Verticle asymptotes

On left endpoint: $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$

On right endpoint: $\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$

In the center: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Ex: $\int_1^4 \frac{1}{x-2}\,dx$ $\quad x \neq 2$ and it's in $[1,4]$

$$= \int_1^2 \frac{1}{x-2}\,dx + \int_2^4 \frac{1}{x-2}\,dx = \lim_{c \to 2^-} \int_1^c \frac{1}{x-2}\,dx + \lim_{c \to 2^+} \int_c^4 \frac{1}{x-2}\,dx$$

$$= \lim_{c \to 2^-} \Big[\ln|x-2|\Big]_1^c + \lim_{c \to 2^+} \Big[\ln|x-2|\Big]_c^4$$

$$= \lim_{c \to 2^-} \left(\ln|c-2| - \ln 1\right) + \lim_{c \to 2^+} \left(\ln 2 - \ln|c-2|\right)$$

$= -\infty + \ln 2 + \infty =$ DNE. If one side goes to infinity, then it DNE.

Comparison test.

If $f(x)$ is ontop of $g(x)$ and they're both continuous on $[a,\infty)$

$\int_a^\infty f(x)\,dx$ converges $= \int_a^\infty g(x)\,dx$ converges

$\int_a^\infty g(x)\,dx$ diverges $= \int_a^\infty f(x)\,dx$ diverges

Isolate the dominate terms:

compare

Ex: $\int_0^\infty \frac{1}{x+1}\,dx \to \frac{1}{x}$

Ex: $\int_0^1 \frac{\sec^2 x}{\sqrt{x}\,(x+1)}\,dx \to \frac{1}{x^{3/2}}$