Use cases for integrals:

Net change theorem

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

The integral of the rate of change $(f'(x))$ is the net change $(f(b) - f(a))$.

Ex: Suppose you have a bird's eye view of a car and know it's x and y velocity in meters per sec.

$$V_x(t) = 1 \qquad V_y(t) = 1 + \cos(2t)$$

What is its displacement and distance traveled after 5 sec?

$$\Delta x = \int_0^5 1\,dt = t\Big|_0^5 = 5$$

$$\Delta y = \int_0^5 1 + \cos(2t)\,dt = t + \frac{1}{2}\sin(2t)\Big|_0^5 = 5 + \frac{1}{2}\sin(10)$$

Displacement from origin: $\sqrt{(5)^2 + \left(5 + \frac{1}{2}\sin(10)\right)^2} = 6.88$ meters
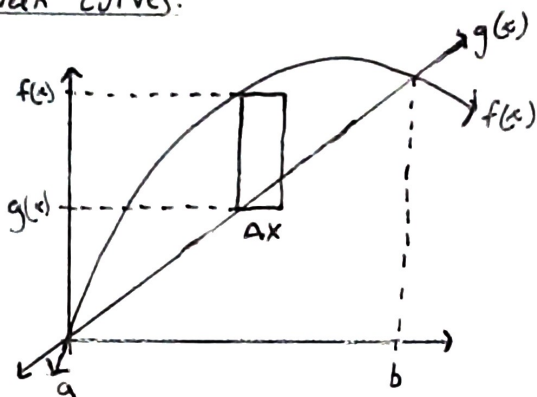
Total distance traveled:

$$\text{Speed}(t) = \sqrt{V_x^2 + V_y^2}$$

$$\int_0^5 \text{Speed}(t)\,dt = \int_0^5 \sqrt{25 + (1 + \cos(2t))^2}\,dt \approx 7.42 \text{ meters}$$

Total distance traveled on the $x = \int_0^5 |V_x|\,dt = 5$ meters

Total distance traveled on the $y = \int_0^5 |V_y|\,dt = 4.73$ meters

Area between curves:

$$\text{Area} = \int_a^b f(x) - g(x)\, dx$$

$$= \left| \int_a^b g(x) - f(x)\, dx \right|$$

1. Find a and b usually by finding intersection points.
2. Find which function is ontop or use absolute value.
3. Solve definite integral

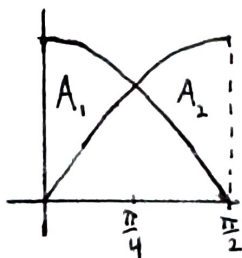Ex: Find area bounded by $y = \sin x$ and $y = \cos x$ on $\left[0, \frac{\pi}{2} \right]$

$$\sin x = \cos x \qquad x = \frac{\pi}{4}$$

$$\text{Area} = \left| \int_0^{\frac{\pi}{4}} \sin x - \cos x\, dx \right| + \left| \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x\, dx \right|$$

$$= \left| -\cos x - \sin x \Big|_0^{\frac{\pi}{4}} \right| + \left| -\cos x - \sin x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right|$$

$$= \left| -\cos \frac{\pi}{4} - \sin \frac{\pi}{4} - \left(-\cos 0 - \sin 0 \right) \right| + \left| -\cos \frac{\pi}{2} - \sin \frac{\pi}{2} - \left(-\cos \frac{\pi}{4} - \sin \frac{\pi}{4} \right) \right|$$
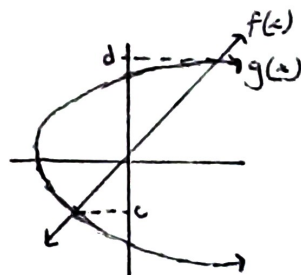
$$= \left| -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \left(-1 - 0 \right) \right| + \left| -0 - 1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \right| = \left| -\sqrt{2} + 1 \right| + \left| -1 + \sqrt{2} \right|$$

$$= \sqrt{2} - 1 + \sqrt{2} - 1 = 2 \left(\sqrt{2} - 1 \right) = 2\sqrt{2} - 2$$

You can also integrate with respect to y

$$\int_c^d f(x) - g(x)\, dx$$

Volumes with cylinders

Volume of a cylinder = Area of base · Height
(Even for irregular base)

The volume of objects other than cylinders can be found by cutting it into infinitly many cylinders where the area of its base is $A(x)$ or $A(y)$ and its height is dx or dy.

$$\int_a^b A(x)\, dx \quad \text{or} \quad \int_c^d A(y)\, dy$$

Ex 1: Find the volume of a pyramid with a square base of length L and height of h.
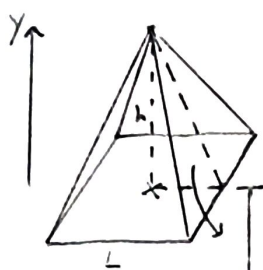
$$\int_0^h A(y)\, dy = \int_0^h L_y^2\, dy$$

Cylinders stuck in the y direction so it's dy.

Similar triangles:

$$\frac{h}{\frac{L}{2}} = \frac{h-y}{\frac{L_y}{2}} \Rightarrow \frac{2h}{L} = \frac{2(h-y)}{L_y}$$

$$L_y = \frac{L}{h}(h-y) = L - \frac{L}{h}y$$

$$\int_0^h \left(L - \frac{L}{h}y\right)^2 dy = \int_0^h L^2 - 2\frac{L^2}{h}y + \frac{L^2}{h^2}y^2\, dy = L^2\int_0^h 1 - \frac{2}{h}y + \frac{1}{h^2}y^2\, dy$$

$$= L^2\left[y - \frac{1}{h}y^2 + \frac{1}{3h^2}y^3\right]_0^h = L^2\left(h - \frac{h^2}{h} + \frac{h^3}{3h^2}\right) = \frac{1}{3}L^2 h$$
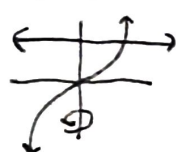
When an area is revolved around an axis, the area of the base is a circle (πr^2) and you need to write r in terms of x or y.

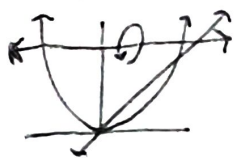Ex 2: Find the volume of the solid obtained by rotating $y=\sqrt{x}$ around the x-axis from $\emptyset$ to 1.

$$\int_{\emptyset}^{1} \pi\left(\sqrt{x}\right)^2 dx = \pi\int_{\emptyset}^{1} x\, dx = \pi\left[\frac{1}{2}x^2\right]_{\emptyset}^{1} = \frac{\pi}{2}$$

Ex 3: Find the volume of the solid obtained by rotating the area bounded by $y=x^3$, $y=8$, and $x=\emptyset$ around the y-axis.

$$\int_{\emptyset}^{8} \pi\left(y^{\frac{1}{3}}\right)^2 dy = \pi\int_{\emptyset}^{8} y^{\frac{2}{3}} dy = \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_{\emptyset}^{8} = \pi\left(\frac{3}{5}8^{\frac{5}{3}}\right)$$

$$= \pi\frac{3}{5}\left(8^{\frac{1}{3}}\right)^5 = \pi\frac{3}{5}2^5 = \pi\frac{3\cdot 32}{5} = \frac{96}{5}\pi$$

Ex 4: Find the volume of the solid obtained by rotating the area enclosed by $y=x$ and $y=x^2$ around $y=2$.

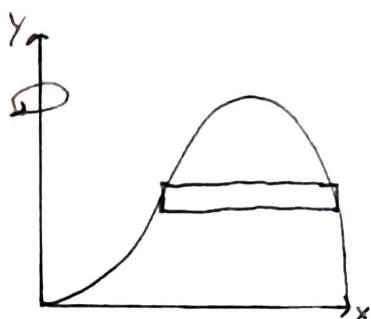Find the bounds: $x=x^2 \Rightarrow \emptyset = x^2 - x = x(x-1)$

$x=\emptyset$ and 1

$$\underbrace{\int_{\emptyset}^{1} \pi\left(2-x^2\right)^2}_{\text{Outer radius}} - \underbrace{\pi\left(2-x\right)^2 dx}_{\text{Inner radius}} = \pi\int_{\emptyset}^{1} 4 - 4x^2 + x^4 - \left(4 - 4x + x^2\right) dx$$

$$(2-x^2)(2-x^2) = 4 - 4x^2 + x^4 \qquad (2-x)(2-x) = 4 - 4x + x^2$$

$$= \pi\int_{\emptyset}^{1} x^4 - 5x^2 + 4x\, dx = \pi\left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + \frac{4}{2}x^2\right]_{\emptyset}^{1} = \pi\left(\frac{1}{5} - \frac{5}{3} + 2\right)$$

$$= \pi\left(\frac{6}{30} - \frac{50}{30} + \frac{60}{30}\right) = \pi\left(-\frac{44}{30} + \frac{60}{30}\right) = \frac{16}{30}\pi = \frac{8}{15}\pi$$
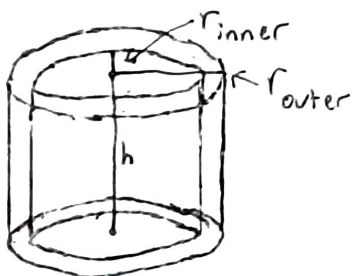
Volume with shells

It can be hard to stack cylinders for some objects.

Ex: Find the volume obtained by rotating $y = 2x^2 - x^3$ and $y = 0$ around the y-axis

If we were to use cylinders, we would have to find the local max and then divide $y = 2x^2 - x^3$ into two separate functions for the outer and inner radius of the circle.

Instead we could use an infinite number of shells.

$$\text{Volume} = \text{Volume}_{outer} - \text{Volume}_{inner}$$

$$= \pi r_{out}^2 h - \pi r_{in}^2 h$$
$$= \pi (r_{out}^2 - r_{in}^2) h$$
$$= \pi (r_{out} + r_{in})(r_{out} - r_{in}) h$$
$$= \pi (r_{out} + r_{in}) \Delta r \, h$$
$$= 2\pi \frac{r_{out} + r_{in}}{2} h \, \Delta r$$

$$\text{Volume} = \int_a^b 2\pi r \cdot f(r) \cdot dr$$

$$= \underbrace{2\pi r_{ave}}_{\text{Circumference}} \cdot h \cdot \underbrace{\Delta r}_{\text{Thickness}}$$

height

Ex:
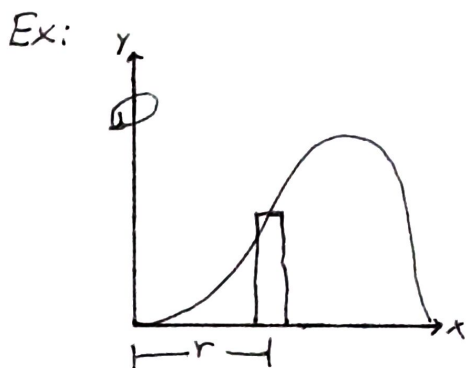
$$\int_0^2 2\pi x \cdot (2x^2 - x^3) \, dx = \frac{16}{5}\pi$$

Use shells over cylinders when the axis of rotation (Ex: y-axis) is different from the independent variable (Ex: x). $y \neq x$