Use cases for derivatives

Find a tangent line at a point

1. Find the slope at the point
   $m = f'(a)$
2. Point slope form: $y - f(a) = m(x-a)$
3. Slope intercept form: $y = mx + b$ ← y-intercept

Normal line: Perpendicular to the tangent line.
- Slope is the opposite reciprical.

Ex: m of tangent: $\frac{3}{2}$ m of normal: $-\frac{2}{3}$

Related rates: Converting the rate of one variable into another through a common equation.

Ex: Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50cm?

$\frac{dV}{dt} = 100 \frac{cm^3}{s}$ $V = \frac{4}{3}\pi r^3$ $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

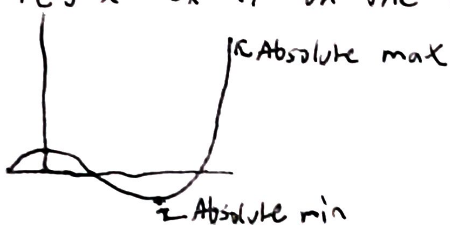
$\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{1}{4\pi r^2}$ $d = 2r$ $50 = 2r$ $r = 25$

$\frac{dr(r=25)}{dt} = (100) \cdot \frac{1}{4\pi (25)^2} = \frac{1}{25\pi}$ cm/s

Find the absolute max and min values of $f(x)$ in an interval:

0. Graph if possible
1. Check if $f(x)$ is contirious on the interval.
2. Find $f'(x)$
3. Find x values where $f'(x) = 0$ or DNE.
4. Check if those x values are in the interval.
5. Find $f(x)$ for those x values.
6. Find $f(x)$ for the endpoints.
7. Absolute max is the lagest $f(x)$.
   Absolute min is the lowest $f(x)$.

Ex: $f(x) = x^3 - 3x^2 + 1$ on the interval $[-\frac{1}{2}, 4]$.



$f(x)$'s domain is $(-\infty, \infty)$ so it's contirious on the interval.

$f'(x) = 3x^2 - 6x$

$0 = 3x^2 - 6x = (3x)(x-2)$

$x = 0, 2$ which are both in the interval.

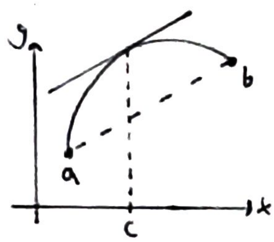
$f(0) = 1 \quad f(2) = -3$

Endpoints: $f(-\frac{1}{2}) = \frac{1}{8} \quad f(4) = 17$

Absolute max: $(4, 17)$ Absolute min: $(2, -3)$

Mean value theorem

If $f(x)$ is continious on $[a,b]$ and differentiable on (a,b), then there exists a C in the interval such that $f'(c)$ = the slope of a and b.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Ex: Find all Cs that satisfy MVT for $f(x) = x^3 - x$ on $(0,2)$.

$f(x)$ is continious on the interval

$f'(x) = 3x^2 - 1$

$f(0) = 0 \quad f(2) = 6 \qquad f'(c) = \frac{f(0) - f(2)}{0 - 2} = \frac{-6}{-2} = 3$

$-\sqrt{\frac{4}{3}}$ isn't in the interval.

$3 = 3c^2 - 1 \qquad 4 = 3c^2 \qquad c^2 = \frac{4}{3} \qquad c = \pm\sqrt{\frac{4}{3}}$ so $c = \sqrt{\frac{4}{3}}$

Graphing a function

Finding local minimums and maximums

1. Find $f'(x)$
2. Find x values that cause $f'(x) = 0$ or DNE
3. Draw number line with x values
4. Pick nums between to see if $f'(x)$ is pos or neg
5. Local min: $f'(x) = 0$ and $f'(x)$ changes from neg to pos.
   Local max: $f'(x) = 0$ and $f'(x)$ changes from pos to neg.

Finding inflection points

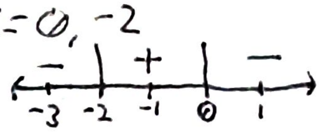
Same steps as local min/max, but with $f''(x)$.

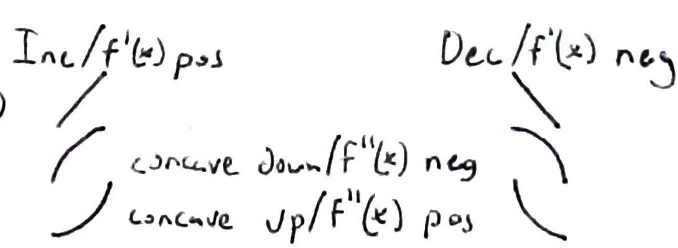
Concave up: $f''(x)$ pos $\qquad$ Concave down: $f''(x)$ neg

Ex: $f(x) = -4x^3 - 12x^2 + 5$

$f'(x) = -12x^2 - 24x = (-12x)(x+2)$

$x = 0, -2$



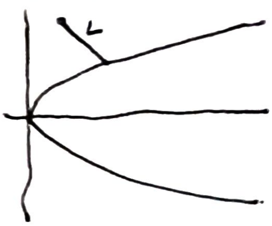


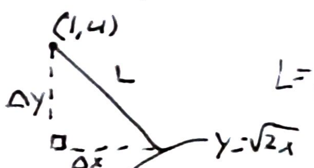
Other useful things for graphing a function:
- Finding its domain
- Finding y-intercept (when $x=0$) and x-intercept (when $y=0$)
- Testing symetry
  - If $f(-x)=f(x)$, then it's symetric across the y-axis
  - If $f(-x)=-f(x)$, then it's symetric when rotated $180°$
- Finding asymptotes
  - $\lim_{x\to\pm\infty} f(x)=L$ Horizontal asymptote
  - $\lim_{x\to a} f(x)=\pm\infty$ Vertical asymptotes (Test x values that're DNE
- Find local mins/maxes and intervals of inc & dec
- Find points of inflection and concavity

Optimization problems

1. Find equation for var you're optimizing for in terms of one other var.
2. Find local mins/maxes.

Ex: Find the point on $y=\sqrt{2x}$ that's closest to $(1,4)$.





$L=\sqrt{\Delta y^2+\Delta x^2}$ Minimize L

$\Delta y=|4-y|$ $\Delta x=|1-x|$

$\Delta y=|4-\sqrt{2x}|$

$$L=\sqrt{(4-\sqrt{2x})^2+(1-x)^2}$$

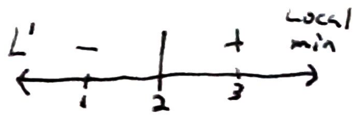
$$L=\sqrt{2x-8\sqrt{2x}+16+x^2-2x+1}$$

$$(4-\sqrt{2x})^2=16-4\sqrt{2x}-4\sqrt{2x}+2x=2x-8\sqrt{2x}+16$$

$$(1-x)^2=1-x-x+x^2=x^2-2x+1$$

$$L=\sqrt{x^2-8\sqrt{2x}+17}$$

$$L'=\frac{1}{2\sqrt{x^2-8\sqrt{2x}+17}}\left(2x-\frac{8}{2\sqrt{2x}}(2)\right)=\frac{x-\frac{4}{\sqrt{2x}}}{\sqrt{x^2-8\sqrt{2x}+17}}=0 \text{ at } x=2$$



$y=\sqrt{2(2)}=2$ So $(2,2)$ is the point on y closest to $(1,4)$