

Use cases for derivatives

Find a tangent line at a point

1. Find the slope at the point

$$m = f'(a)$$

2. Point slope form: $y - f(a) = m(x - a)$

3. Slope intercept form: $y = mx + b$ ← y-intercept

Normal line: Perpendicular to the tangent line.

- Slope is the opposite reciprocal.

Ex: m of tangent: $\frac{3}{2}$ m of normal: $-\frac{2}{3}$

Related rates: Converting the rate of one variable into another through a common equation.

Ex: Air is being pumped into a spherical balloon so that its volume increases at a rate of $100 \text{ cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the diameter is 50 cm ?

$$\frac{dV}{dt} = 100 \frac{\text{cm}^3}{\text{s}} \quad V = \frac{4}{3}\pi r^3 \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{1}{4\pi r^2} \quad d = 2r \quad 50 = 2r \quad r = 25$$

$$\frac{dr(r=25)}{dt} = (100) \cdot \frac{1}{4\pi(25)^2} = \frac{1}{2\pi} \text{ cm/s}$$

Find the absolute max and min values of $f(x)$ in an interval:

0. Graph if possible

1. Check if $f(x)$ is continuous on the interval.

2. Find $f'(x)$

3. Find x values where $f'(x) = 0$ or DNE.

4. Check if those x values are in the interval.

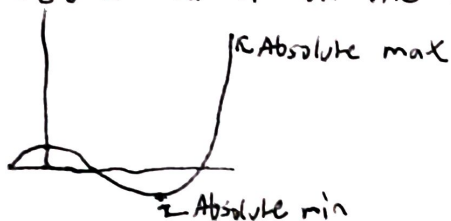
5. Find $f(x)$ for those x values.

6. Find $f(x)$ for the endpoints.

7. Absolute max is the largest $f(x)$.

Absolute min is the lowest $f(x)$.

Ex: $f(x) = x^3 - 3x^2 + 1$ on the interval $[-\frac{1}{2}, 4]$.



$f(x)$'s domain is $(-\infty, \infty)$ so it's continuous on the interval.

$$f'(x) = 3x^2 - 6x$$

$$0 = 3x^2 - 6x = (3x)(x - 2)$$

$x = 0, 2$ which are both in the interval.

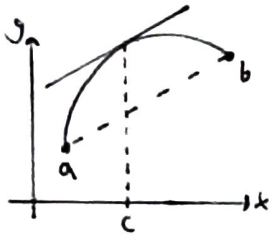
$$f(0) = 1 \quad f(2) = -3$$

$$\text{Endpoints: } f(-\frac{1}{2}) = \frac{1}{8} \quad f(4) = 17$$

$$\text{Absolute max: } (4, 17) \quad \text{Absolute min: } (2, -3)$$

Mean value theorem

If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) , then there exists a c in the interval such that $f'(c)$ = the slope of a and b .



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Ex: Find all c s that satisfy MVT for $f(x) = x^3 - x$ on $(0, 2)$.

$f(x)$ is continuous on the interval

$$f'(x) = 3x^2 - 1$$

$$f(0) = 0 \quad f(2) = 6$$

$$f'(c) = \frac{f(0) - f(2)}{0 - 2} = \frac{-6}{-2} = 3$$

$-\sqrt{\frac{4}{3}}$ isn't in the interval.

$$3 = 3c^2 - 1 \quad 4 = 3c^2 \quad c^2 = \frac{4}{3} \quad c = \pm \sqrt{\frac{4}{3}} \quad \text{so } c = \sqrt{\frac{4}{3}}$$

Graphing a function

Finding local minimums and maximums

1. Find $f'(x)$

2. Find x values that cause $f'(x) = 0$ or DNE

3. Draw number line with x values

4. Pick nums between to see if $f'(x)$ is pos or neg

5. Local min: $f'(x) = 0$ and $f'(x)$ changes from neg to pos.

Local max: $f'(x) = 0$ and $f'(x)$ changes from pos to neg.

Finding inflection points

Same steps as local min/max, but with $f''(x)$.

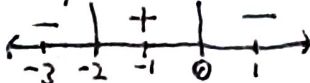
Concave up: $f''(x)$ pos

Concave down: $f''(x)$ neg

$$\text{Ex: } f(x) = -4x^3 - 12x^2 + 5$$

$$f'(x) = -12x^2 - 24x = (-12x)(x+2)$$

$$x = 0, -2$$



Inc/ $f'(x)$ pos

Dec/ $f'(x)$ neg

concave down/ $f''(x)$ neg
concave up/ $f''(x)$ pos

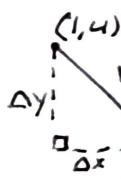
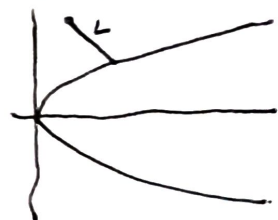
Other useful things for graphing a function:

- Finding its domain
- Finding y-intercept (when $x=0$) and x-intercept (when $y=0$)
- Testing symmetry
 - If $f(-x) = f(x)$, then it's symmetric across the y-axis
 - If $f(-x) = -f(x)$, then it's symmetric when rotated 180°
- Finding asymptotes
 - $\lim_{x \rightarrow \pm\infty} f(x) = L$ Horizontal asymptote
 - $\lim_{x \rightarrow a} f(x) = \pm\infty$ Vertical asymptotes (Test x values that're DNE)
- Find local mins/maxes and intervals of inc & dec
- Find points of inflection and concavity

Optimization problems

1. Find equation for var you're optimizing for in terms of one other var.
2. Find local mins/maxes.

Ex: Find the point on $y = \sqrt{2x}$ that's closest to $(1, 4)$.



$$L = \sqrt{\Delta y^2 + \Delta x^2} \quad \text{Minimize } L$$

$$\Delta y = |4 - y| \quad \Delta x = |1 - x|$$

$$\Delta y = |4 - \sqrt{2x}|$$

$$L = \sqrt{(4 - \sqrt{2x})^2 + (1 - x)^2}$$

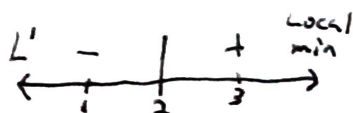
$$L = \sqrt{2x - 8\sqrt{2x} + 16 + x^2 - 2x + 1}$$

$$L = \sqrt{x^2 - 8\sqrt{2x} + 17}$$

$$(4 - \sqrt{2x})^2 = 16 - 4\sqrt{2x} - 4\sqrt{2x} + 2x = 2x - 8\sqrt{2x} + 16$$

$$(1 - x)^2 = 1 - x - x + x^2 = x^2 - 2x + 1$$

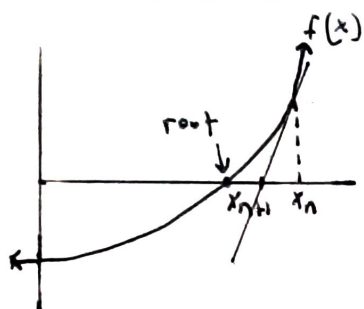
$$L' = \frac{1}{2\sqrt{x^2 - 8\sqrt{2x} + 17}} \left(2x - \frac{8}{2\sqrt{2x}} (2) \right) = \frac{x - \frac{4}{\sqrt{2x}}}{\sqrt{x^2 - 8\sqrt{2x} + 17}} = 0 \text{ at } x = 2$$



$$y = \sqrt{2(2)} = 2$$

So $(2, 2)$ is the point on y closest to $(1, 4)$

Newton's Method:



Used to find roots/zeros of a function.

1. Take a guess of x . $(x_n, f(x_n))$

2. Find tangent line from your guess.

$$y - y_1 = m(x - x_1)$$

$$y - f(x_n) = f'(x_n)(x - x_n)$$

3. Solve for x_{n+1} at the point $(x_{n+1}, 0)$

$$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$$

$$\frac{-f(x_n)}{f'(x_n)} = x_{n+1} - x_n$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Ex: What is $\sqrt[6]{2}$ to 8 decimal places?

$$\sqrt[6]{2} = x \quad 2 = x^6 \quad 0 = x^6 - 2 = f(x) \quad f'(x) = 6x^5$$

$$x_0 = 1.5$$

$$x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.293895748$$

↙ Don't round until the end

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.170160594$$

$$x_3 = 1.127066579$$

$$x_4 = 1.122508821$$

$$x_5 = 1.122462053$$

$$x_6 = 1.122462048$$

$$x_7 = 1.122462048$$