

## Finding Integrals

### 1. Limit definition

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \overbrace{\sum_{i=1}^n f(x_i) \Delta x}^{\text{Riemann sum}}$$

$f(x_i) = \text{Height}$

$\Delta x = \text{Width} = \frac{b-a}{n}$

$x_i = a + i\Delta x$

- We use an infinite number of rectangles to get the area under the curve.

- If we don't use an infinite number, we can use the left, right, midpoint, smallest  $y$  value (lower sum), or largest  $y$  value (upper sum) to estimate the area under the curve.

- We can also get a rough estimate by finding the  $\max(M)$  and  $\min(m)$  of  $f(x)$ .

$$\text{If } m \leq f(x) \leq M, \text{ then } m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

$$\sum_{i=1}^n c = c \sum_{i=1}^n 1 = cn$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n a_i + b_i = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^n a_i - b_i = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$$

### 2. Properties of integrals

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\int_a^a f(x) dx = 0$$

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx$$

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

$$\int_a^b (f(x) - g(x)) dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

### 3. Fundamental theorem of calculus

Part 1: Deriving the integral of  $f$  gives  $F$

$$\frac{d}{dx} \int_a^x f(t) dt = f(x) \quad a \text{ is any constant}$$

Ex:  $\frac{d}{dx} \int_1^{x^4} \sec t \, dt$

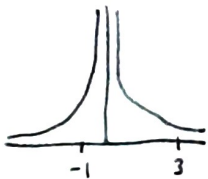
$$u = x^4 \quad \frac{d}{dx} = \frac{du}{dx} \cdot \frac{d}{du}$$
$$\frac{du}{dx} = 4x^3$$

$$4x^3 \cdot \frac{d}{du} \int_1^u \sec t \, dt = 4x^3 \cdot \sec u = 4x^3 \sec x^4$$

Part 2:

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

Ex:  $\int_{-1}^3 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_{-1}^3 = -\frac{4}{3}$



$f(x)$  has to be continuous or have a finite number of jump discontinuities over  $[a, b]$  to be integrable.

Integration and differentiation are inverse processes.

### 4 U-Substitution

1. Define  $u$

2. Find  $\frac{du}{dx}$

3. Solve for  $dx$

4. Plug  $u$  and  $dx$  into the integral and simplify

If definite integral: Re-write bounds for  $u$

5. Solve integral

If indefinite integral: Substitute  $u$  so it's in terms of  $x$ .

$$\text{Ex 1: } \int x^3 \cos(x^4+2) dx \quad u = x^4 + 2$$

$$\frac{du}{dx} = 4x^3 \Rightarrow dx = \frac{du}{4x^3}$$

$$\int x^3 \cos(u) \frac{du}{4x^3} = \int \frac{1}{4} \cos u \, du = \frac{1}{4} \sin u + C = \frac{1}{4} \sin(x^4+2) + C$$

$$\text{Ex 2: } \int \sqrt{1+x^2} x^5 dx \quad u = 1+x^2 \quad u-1 = x^2$$

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$\int \sqrt{u} x^5 \frac{du}{2x} = \int \sqrt{u} x^4 \frac{1}{2} du = \int \frac{1}{2} \sqrt{u} (u-1)^2 du$$

$$= \int \frac{1}{2} \sqrt{u} (u^2 - 2u + 1) du = \int \frac{1}{2} u^{\frac{5}{2}} - u^{\frac{3}{2}} + \frac{1}{2} u^{\frac{1}{2}} du$$

$$= \frac{1}{2} \cdot \frac{2}{\frac{7}{2}} u^{\frac{7}{2}} - \frac{2}{\frac{5}{2}} u^{\frac{5}{2}} + \frac{1}{2} \cdot \frac{2}{\frac{3}{2}} u^{\frac{3}{2}} + C = \frac{1}{7} (1+x^2)^{\frac{7}{2}} - \frac{2}{5} (1+x^2)^{\frac{5}{2}} + \frac{1}{3} (1+x^2)^{\frac{3}{2}} + C$$

$$\text{Ex 3: } \int \tan x \, dx = \int \frac{\sin x}{\cos x} dx \quad u = \cos x$$

$$\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$$

$$\int \frac{\sin x}{u} \left(-\frac{du}{\sin x}\right) = \int -\frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C$$

$$\text{Ex 4: } \int_1^e \frac{\ln x}{x} dx \quad u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x \, du$$

$$\text{Lower: } u = \ln(1) = 0 \quad \text{Upper: } u = \ln(e) = 1$$

$$\int_0^1 \frac{u}{x} x \, du = \int_0^1 u \, du = \frac{1}{2} u^2 \Big|_0^1 = \frac{1}{2} (1^2) - \frac{1}{2} (0^2) = \frac{1}{2}$$

5. Symmetry: If  $f(x)$  is continuous on  $[-a, a]$

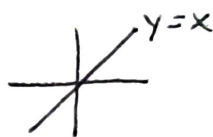
If  $f(-x) = f(x)$  then



$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

If  $f(-x) = -f(x)$  then

$$\int_{-a}^a f(x) dx = 0$$



6. Integration by parts

Just like U-substitution is the inverse of the chain rule, integration by parts is the inverse of the product rule.

$$\frac{d}{dx} (f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g(x) = \int f(x)g'(x) dx + \int f'(x)g(x) dx$$

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

$$\begin{aligned} u &= f(x) & dv &= g'(x) dx \\ du &= f'(x) dx & v &= g(x) \end{aligned}$$

$$\boxed{\int u dv = uv - \int v du}$$

Use LIPET for picking  $u$ :

L -  $\ln$

I - Inverse trig

P - Powers of  $x$

E - Exponential

T - Trig

The goal of integration by parts is to convert the integral to a simpler integral.

This may not always be the case.

$$\text{Definite integrals: } \int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

$$\text{Ex 1: } \int x \sin x \, dx \quad \begin{array}{ll} u = x & dv = \sin x \, dx \\ du = dx & v = -\cos x \end{array}$$

$$= x(-\cos x) - \int -\cos x \, dx = -x \cos x + \sin x + C$$

$$\text{Ex 2: } \int \ln x \, dx \quad \begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x} dx & v = x \end{array}$$

$$= \ln x(x) - \int x \frac{1}{x} dx = x \ln x - x + C$$

$$\text{Ex 3: } \int t^2 e^t \, dt \quad \begin{array}{ll} u = t^2 & dv = e^t \, dt \\ du = 2t \, dt & v = e^t \end{array}$$

$$= t^2 e^t - \int e^t 2t \, dt = t^2 e^t - 2 \int t e^t \, dt \quad \begin{array}{ll} u = t & dv = e^t \, dt \\ du = dt & v = e^t \end{array}$$

$$= t^2 e^t - 2 \left[ t e^t - \int e^t \, dt \right] = t^2 e^t - 2 t e^t + 2 e^t + C$$

$$\text{Ex 4: } \int e^x \sin x \, dx \quad \begin{array}{ll} u = e^x & dv = \sin x \, dx \\ du = e^x \, dx & v = -\cos x \end{array}$$

$$= e^x(-\cos x) - \int -\cos x e^x \, dx = -e^x \cos x + \int \cos x e^x \, dx$$

$$\begin{array}{ll} u = e^x & dv = \cos x \, dx \\ du = e^x \, dx & v = \sin x \end{array}$$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int \sin x e^x \, dx \quad \leftarrow \text{Add to the other side.}$$

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

## 7. Powers of sin and cos

$$\int \sin^m x \cos^n x dx$$

$\sin^m x$	$\cos^n x$	Solution
Odd	Odd	Either
Odd	Even	$u = \cos x$
Even	Odd	$u = \sin x$
Even	Even	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$\left. \begin{array}{l} \text{Odd} \\ \text{Odd} \\ \text{Even} \end{array} \right\} \sin^2 x + \cos^2 x = 1$$

Ex 1:  $\int \cos^5 x dx$        $\cos^5 x$  is odd,  $\sin^0 x$  is even

$$u = \sin x$$

$$du = \cos x dx \Rightarrow dx = \frac{du}{\cos x}$$

$$\int \cos^5 x \frac{du}{\cos x} = \int \cos^4 x du = \int (\cos^2 x)^2 du$$

$$\cos^2 x = 1 - \sin^2 x \quad = \int (1 - \sin^2 x)^2 du = \int (1 - u^2)^2 du$$

$$(1 - u^2)(1 - u^2) = 1 - 2u^2 + u^4 \quad = \int 1 - 2u^2 + u^4 du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Ex 2:  $\int \sin^4 x dx$        $\sin^4 x$  is even and  $\cos^0 x$  is even

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1}{2}(1 - \cos 2x)\right)^2 = \frac{1}{4}(1 - \cos 2x)^2$$

$$= \int \frac{1}{4}(1 - \cos 2x)^2 dx = \frac{1}{4} \int 1 - 2\cos 2x + \cos^2 2x dx$$

$$= \frac{1}{4} \left[ \int dx - 2 \int \cos 2x dx + \int \cos^2 2x dx \right]$$

$$= \frac{1}{4} \left[ x - \sin 2x + \int \cos^2 2x dx \right]$$

$$\cos^2(2x) = \frac{1}{2} (1 + \cos 4x)$$

$$= \frac{1}{4} \left[ x - \sin 2x + \frac{1}{2} \int 1 + \cos 4x dx \right]$$

$$= \frac{1}{4} x - \frac{1}{4} \sin 2x + \frac{1}{8} \left[ \int dx + \int \cos 4x dx \right]$$

$$= \frac{1}{4} x - \frac{1}{4} \sin 2x + \frac{1}{8} x + \frac{1}{32} \sin 4x + C = \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$$

## 8. Powers of tan and sec

$$\int \tan^m x \sec^n x dx$$

This also applies for cot and csc

$\tan^m x$	$\sec^n x$	Solution
Odd	Odd	$u = \sec x$
Odd	Even	Either
Even	Odd	other
Even	Even	$u = \tan x$

$$\tan^2 x + 1 = \sec^2 x$$

- Integration by parts
- $\int \sec x = \ln |\sec x + \tan x| + C$
- $\int \tan x = \ln |\sec x| + C$
- Move to other side and divide

$$\text{Ex 1: } \int \tan^2 x \sec^4 x dx$$

$$u = \tan x$$

$$du = \sec^2 x dx \Rightarrow dx = \frac{du}{\sec^2 x}$$

$$\int \tan^2 x \sec^4 x \frac{du}{\sec^2 x} = \int u^2 \sec^2 x du$$

$$\sec^2 x = 1 + \tan^2 x = \int u^2 (1 + u^2) du = \int u^2 + u^4 du$$

$$= \frac{1}{3} u^3 + \frac{1}{5} u^5 + C = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C$$

$$\text{Ex 2: } \int \sec^3 x dx = \int \sec^2 x \sec x dx = \int (\tan^2 x + 1) \sec x dx$$

$$= \int \tan^2 x \sec x + \sec x \, dx = \int \tan^2 x \sec x \, dx + \int \sec x \, dx$$

$$\int \sec x \, dx = \int \frac{\sec x}{1} \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$u = \sec x + \tan x \quad du = \sec x \tan x + \sec^2 x \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{u} \cdot \frac{du}{\sec^2 x + \sec x \tan x} = \int \frac{1}{u} du = \ln |\sec x + \tan x| + C$$

$$\int \tan^2 x \sec x \, dx \quad \begin{array}{ll} u = \tan x & du = \sec x \tan x \, dx \\ du = \sec^2 x \, dx & v = \sec x \end{array}$$

$$= \sec x \tan x - \int \sec^3 x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \ln |\sec x + \tan x| + C$$

$$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$$

9.  $\sin A$  and  $\cos B$

$$\int \sin A \cos B \, dx \quad \text{or} \quad \int \sin A \sin B \, dx \quad \text{or} \quad \int \cos A \cos B \, dx$$

$$\sin A \cos B = \frac{1}{2} [\sin(A-B) + \sin(A+B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$

$$\int \sin 4x \cos 5x \, dx = \int \frac{1}{2} [\sin(-x) + \sin(9x)] \, dx$$

$$= \frac{1}{2} \left[ \int -\sin x \, dx + \int \sin(9x) \, dx \right] = \frac{1}{2} \left[ \cos x - \frac{1}{9} \cos 9x \right] + C$$