

Use cases for integrals:

Net change theorem

$$\int_a^b f'(x) dx = f(b) - f(a)$$

The integral of the rate of change ( $f'(x)$ ) is the net change ( $f(b) - f(a)$ ).

Ex: Suppose you have a bird's eye view of a car and know its  $x$  and  $y$  velocity in meters per sec.

$$v_x(t) = 1 \quad v_y(t) = 1 + \cos(2t)$$

What is its displacement and distance traveled after 5 sec?

$$\Delta x = \int_0^5 1 dt = t \Big|_0^5 = 5$$

$$\Delta y = \int_0^5 1 + \cos(2t) dt = t + \frac{1}{2} \sin(2t) \Big|_0^5 = 5 + \frac{1}{2} \sin(10)$$

Displacement from origin:  $\sqrt{(5)^2 + (5 + \frac{1}{2} \sin(10))^2} = 6.88$  meters

Total distance traveled:

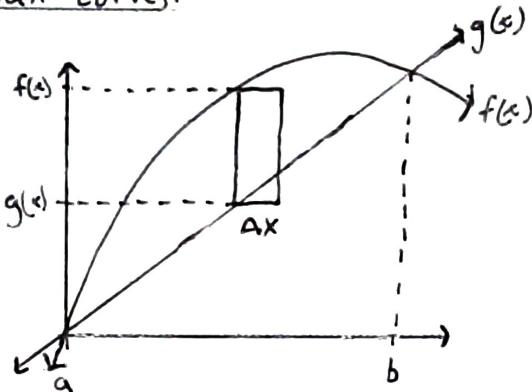
$$\text{Speed}(t) = \sqrt{v_x^2 + v_y^2}$$

$$\int_0^5 \text{Speed}(t) dt = \int_0^5 \sqrt{25 + (1 + \cos(2t))^2} dt \approx 7.42 \text{ meters}$$

Total distance traveled on the  $x$  =  $\int_0^5 |v_x| dt = 5$  meters

Total distance traveled on the  $y$  =  $\int_0^5 |v_y| dt = 4.73$  meters

## Area between curves:



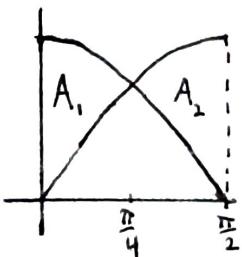
$$\begin{aligned} \text{Area} &= \int_a^b [f(x) - g(x)] dx \\ &= \left| \int_a^b [g(x) - f(x)] dx \right| \end{aligned}$$

1. Find  $a$  and  $b$  usually by finding intersection points.
2. Find which function is on top or use absolute value.
3. Solve definite integral

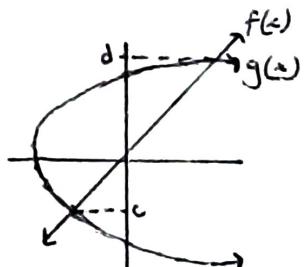
Ex: Find area bounded by  $y = \sin x$  and  $y = \cos x$  on  $[0, \frac{\pi}{2}]$

$$\sin x = \cos x \quad x = \frac{\pi}{4}$$

$$\begin{aligned} \text{Area} &= \left| \int_0^{\frac{\pi}{4}} [\sin x - \cos x] dx \right| + \left| \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} [\sin x - \cos x] dx \right| \\ &= \left| -\cos x - \sin x \Big|_0^{\frac{\pi}{4}} \right| + \left| -\cos x - \sin x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right| \\ &= \left| -\cos \frac{\pi}{4} - \sin \frac{\pi}{4} - (-\cos 0 - \sin 0) \right| + \left| -\cos \frac{\pi}{2} - \sin \frac{\pi}{2} - (-\cos \frac{\pi}{4} - \sin \frac{\pi}{4}) \right| \\ &= \left| -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - (-1 - 0) \right| + \left| -0 - 1 - \left( -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) \right| = \left| -\sqrt{2} + 1 \right| + \left| -1 + \sqrt{2} \right| \\ &= \sqrt{2} - 1 + \sqrt{2} - 1 = 2(\sqrt{2} - 1) = 2\sqrt{2} - 2 \end{aligned}$$



You can also integrate with respect to  $y$

$$\int_c^d [f(y) - g(y)] dy$$


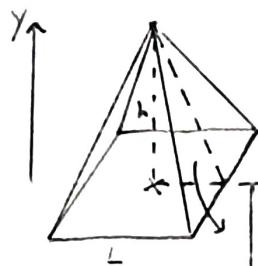
## Volumes with cylinders

Volume of a cylinder = Area of base · Height  
 (Even for irregular base)

The volume of objects other than cylinders can be found by cutting it into infinity many cylinders where the area of its base is  $A(x)$  or  $A(y)$  and its height is  $dx$  or  $dy$ .

$$\int_a^b A(x) dx \text{ or } \int_c^d A(y) dy$$

Ex 1: Find the volume of a pyramid with a square base of length  $L$  and height of  $h$ .



$$\int_0^h A(y) dy = \int_0^h Ly^2 dy$$

Cylinders stacked in the  $y$  direction so it's  $dy$ .

Similar triangles:

$$\frac{h}{\frac{L}{2}} = \frac{h-y}{\frac{Ly}{2}} \Rightarrow \frac{2h}{L} = \frac{2(h-y)}{Ly}$$

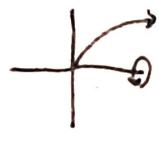
$$Ly = \frac{L}{h}(h-y) = L - \frac{L}{h}y$$

$$\int_0^h \left( L - \frac{L}{h}y \right)^2 dy = \int_0^h L^2 - 2 \frac{L^2}{h}y + \frac{L^2}{h^2}y^2 dy = L^2 \int_0^h 1 - \frac{2}{h}y + \frac{1}{h^2}y^2 dy$$

$$= L^2 \left[ y - \frac{1}{h}y^2 + \frac{1}{3h^2}y^3 \right]_0^h = L^2 \left( h - \frac{h^2}{h} + \frac{h^3}{3h^2} \right) = \frac{1}{3}L^2h$$

When an area is revolved around an axis, the area of the base is a circle ( $\pi r^2$ ) and you need to write  $r$  in terms of  $x$  or  $y$ .

Ex 2: Find the volume of the solid obtained by rotating  $y=\sqrt{x}$  around the  $x$ -axis from  $0$  to  $1$ .



$$\int_0^1 \pi(\sqrt{x})^2 dx = \pi \int_0^1 x dx = \pi \left[ \frac{1}{2}x^2 \right]_0^1 = \frac{\pi}{2}$$

Ex 3: Find the volume of the solid obtained by rotating the area bounded by  $y=x^3$ ,  $y=8$ , and  $x=0$  around the  $y$ -axis.



$$\begin{aligned} \int_0^8 \pi(y^{\frac{1}{3}})^2 dy &= \pi \int_0^8 y^{\frac{2}{3}} dy = \pi \left[ \frac{3}{5}y^{\frac{5}{3}} \right]_0^8 = \pi \left( \frac{3}{5}8^{\frac{5}{3}} \right) \\ &= \pi \frac{3}{5}(8^{\frac{1}{3}})^5 = \pi \frac{3}{5}2^5 = \pi \frac{3 \cdot 32}{5} = \frac{96}{5}\pi \end{aligned}$$

Ex 4: Find the volume of the solid obtained by rotating the area enclosed by  $y=x$  and  $y=x^2$  around  $y=2$



Find the bounds:  $y=x \Rightarrow 0=x^2-x=x(x-1)$   
 $x=0$  and  $1$

$$\int_0^1 \pi(2-x^2)^2 - \pi(2-x)^2 dx = \pi \int_0^1 4-4x^2+x^4 - (4-4x+x^2) dx$$

$\underbrace{\hspace{2cm}}$  Outer radius       $\underbrace{\hspace{2cm}}$  Inner radius

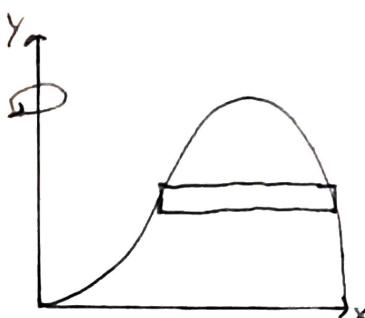
$$(2-x^2)(2-x^2) = 4-4x^2+x^4 \quad (2-x)(2-x) = 4-4x+x^2$$

$$\begin{aligned} &= \pi \int_0^1 x^4 - 5x^2 + 4x dx = \pi \left[ \frac{1}{5}x^5 - \frac{5}{3}x^3 + \frac{4}{2}x^2 \right]_0^1 = \pi \left( \frac{1}{5} - \frac{5}{3} + 2 \right) \\ &= \pi \left( \frac{6}{30} - \frac{50}{30} + \frac{60}{30} \right) = \pi \left( -\frac{44}{30} + \frac{60}{30} \right) = \frac{16}{30}\pi = \frac{8}{15}\pi \end{aligned}$$

## Volume with shells

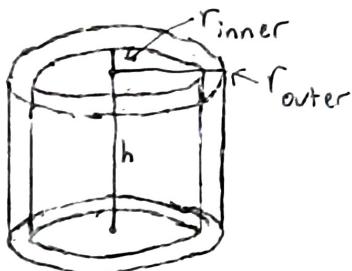
It can be hard to stack cylinders for some objects.

Ex: Find the volume obtained by rotating  $y=2x^2-x^3$  and  $y=0$  around the  $y$ -axis



If we were to use cylinders, we would have to find the local max and then divide  $y=2x^2-x^3$  into two separate functions for the outer and inner radii of the circle.

Instead we could use an infinite number of shells.



$$\text{Volume} = \int_a^b 2\pi r \cdot f(r) \cdot dr$$

$$\text{Volume} = \text{Volume}_{\text{outer}} - \text{Volume}_{\text{inner}}$$

$$= \pi r_{\text{out}}^2 h - \pi r_{\text{in}}^2 h$$

$$= \pi (r_{\text{out}}^2 - r_{\text{in}}^2) h$$

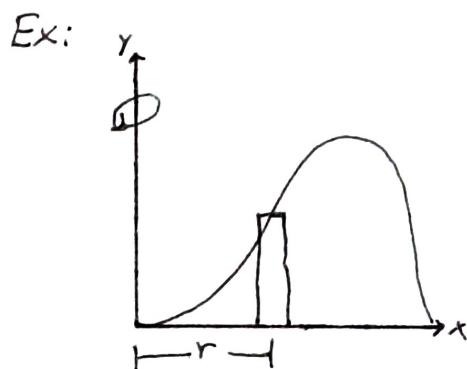
$$= \pi (r_{\text{out}} + r_{\text{in}})(r_{\text{out}} - r_{\text{in}}) h$$

$$= \pi (r_{\text{out}} + r_{\text{in}}) \Delta r h$$

$$= 2\pi \frac{r_{\text{out}} + r_{\text{in}}}{2} h \Delta r$$

$$= \underbrace{2\pi r_{\text{ave}}}_{\text{Circumference}} \cdot \underbrace{h}_{\text{height}} \cdot \underbrace{\Delta r}_{\text{Thickness}}$$

$$\int_0^2 2\pi x \cdot (2x^2 - x^3) dx = \frac{16}{5}\pi$$



Use shells over cylinders when the axis of rotation (Ex:  $y$ -axis) is different from the independent variable (Ex:  $x$ ).  $y \neq x$

## Work

$$F=ma \quad W=F \cdot d \quad \text{when Force is constant}$$

$$\text{Springs: } F=kx$$

$$W = \int_{x_i}^{x_f} F(x) \cdot dx = \int_{x_i}^{x_f} h(x) \cdot dF(x) \quad \text{when Force isn't constant.}$$

	Force	Distance	Work	g
Metric	N	m	J	9.8
Imperial	lbs	ft	ft-lbs	32.2

Ex 1: A force of 40N is required to hold a spring that has been stretched from its natural length of 10cm to a length of 15cm. How much work is done by stretching it from 15cm to 18cm?

$$F=kx \quad 40 = k(15-10) \quad k = 8 \frac{N}{cm}$$

$$W = \int_5^8 8x \, dx = 4x^2 \Big|_5^8 = 4 \cdot 8^2 - 4 \cdot 5^2 = 156 \text{ Ncm} \cdot \frac{1 \text{ m}}{100 \text{ cm}} = 1.56 \text{ J}$$

Ex 2: A 200 lb cable is 100ft long and hangs vertically from the top of a tall building. How much work to lift the cable to the top?

 How far up  $F(x)$  = Force of cable hanging

the cable was pulled.

$$F(0) = 200 \quad F(100) = 0$$

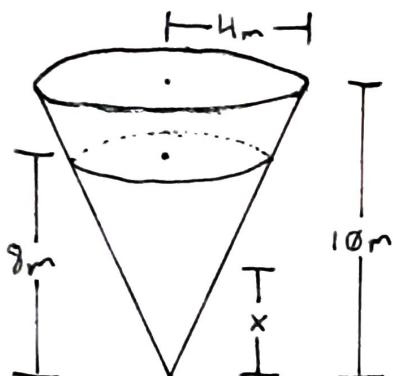
Assuming uniform density of the cable, the Force is linear/a line.

$$\text{Slope} = \frac{F(0) - F(100)}{0 - 100} = \frac{200}{-100} = -2 \quad F(x) - 0 = -2(x - 100)$$

$$W = \int_0^{100} -2(x-100) \, dx = -2 \int_0^{100} x - 100 \, dx = -2 \left[ \frac{1}{2}x^2 - 100x \right]_0^{100} = 10,000 \text{ ft-lbs}$$

Ex 3: A tank has the shape of an inverted circular cone with height 10m and base radius 4m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank?

Density of water is  $1000 \frac{\text{kg}}{\text{m}^3}$



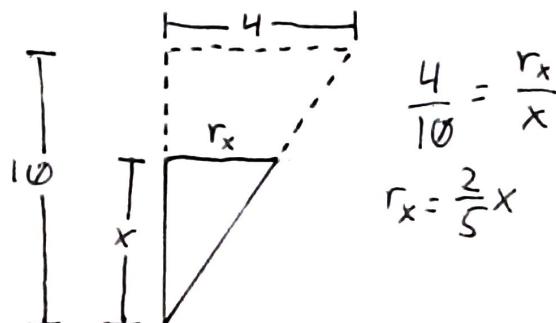
$x$  = Height of water left in the tank.

$$W = \int_0^8 h(x) dF(x) = \int_0^8 (10-x) dF(x)$$

$dF(x)$  = Force from a small cylindrical slice.  
 $= \text{mass} \cdot \text{acc} = dm(x) \cdot 9.8$

$dm(x)$  = Mass of small cylindrical slice.  
 $= \text{Density} \cdot \text{Volume} = 1000 \cdot dV(x)$

$dV(x)$  = Volume of small cylindrical slice.  
 $= \text{Area of base} \cdot \text{height}$   
 $= \pi r_x^2 \cdot dx$



$$W = \int_0^8 (10-x) \cdot 9.8 \cdot 1000 \cdot \pi \left(\frac{2}{5}x\right)^2 dx = 3.36 \times 10^6 \text{ J}$$