THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Term One 2022

MATH5901/3901/3801

(Higher) Probability and Stochastic Processes

(1) TIME ALLOWED – THREE (3) HOURS

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER FOUR (4) QUESTIONS

(4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE

YOU ARE TO SUBMIT ONLINE HANDWRITTEN/TYPED SOLUTIONS AND MATLAB/R/PYTHON CODE, WHERE APPROPRIATE.

EACH AND EVERY FILE/DOCUMENT/PHOTO YOU SUBMIT MUST BE NAMED AS EITHER "Q1...", "Q2..., "Q3...", OR "Q4...". For example, a file with name "Q1_ii)" is acceptable, and must only contain an answer to Q1, part ii).

FAILURE TO NAME ANY FILE ACCORDING TO THE INSTRUCTIONS ABOVE WILL RESULT IN THAT FILE NOT BEING MARKED (LOSS OF MARKS).

YOU CAN DELETE AND/OR RELOAD FILES UNTIL THE DEADLINE.

Upload Separate/New File(s) clearly marked Q1

1. [**18 marks**] Let $W_1, W_2, \ldots$ be exponentially distributed iid random variables with mean $1/\lambda$, and define $T_n = W_1 + \cdots + W_n$ ($T_0 = 0$). Let $(N_t, t \geq 0)$ be a stochastic process defined as $N_0 = 0$ and for $t \in \mathbb{R}^+$

$$N_t = \max\{n \in \mathbb{N} : W_1 + \cdots + W_n \leq t\}.$$

i) [**3 marks**] Use the moment generating function method to derive the distribution of T_n. Write down the formula for its pdf.

ii) [**3 marks**] Let x be an integer. Derive a formula for $\mathbb{P}(N_t \leq x)$ in terms of the cdf of T_n.

iii) [**6 marks**] Assume (N_t) is a Poisson arrival-counting process with intensity $\lambda = 2$.

a) [**3 marks**] Find the numerical value (up to four significant figures) of the following probabilities:
(1) $\mathbb{P}(N_2 = 1, N_3 = 2, N_5 = 4)$,
(2) $\mathbb{P}(N[2, 6] = 4, N[3, 8] = 6)$.

b) [**3 marks**] Prove that the process defined via

$$M_t = \exp(\ln(1 - u)N_t + u\lambda t)$$

is a martingale for $u \in (0, 1)$.

iv) [**6 marks**] Define the process (R_t) via $R_t = T_{N_t+1} - t$. Find the complementary cdf $\mathbb{P}(R_t \geq u)$ of R_t.

MATH5901/3901 only: Having computed the cdf of R_t, find the cdf of the random variable $M_t := T_{N_t+1} - T_{N_t}$. What is the limiting distribution of M_t as $t \uparrow \infty$?

Upload Separate/New File(s) clearly marked Q2

2. [**12 marks**]

i) [**5 marks**] Suppose that

$$M_n = X_1 + \cdots + X_n,$$

where (X_n) are iid and zero mean with

$$\sum_{n=1}^{\infty} \mathbb{V}\mathrm{ar}\, X_n < \infty.$$

a) [**2 marks**] Explain what it means for (M_n) to be a Cauchy sequence in L^2-norm. Write down a relevant equation in your explanation.

b) [**3 marks**] Show that (M_n) converges in L^2-norm to some M, that is,

$$M_n \xrightarrow{L^2} M.$$

ii) [**7 marks**] Suppose that (Z_n) are iid with

$$\mathbb{P}(Z_n = 1) = \mathbb{P}(Z_n = -1) = 1/2,$$

and define the random variable

$$S_n := \sum_{k=1}^{n} Z_k / k.$$

a) Simulate S_{100} one thousand times on a computer (using Matlab/R/Python) and then plot a histogram of the distribution of S_{100}.

Do not forget to include the figure in your submission (either embedded within your typeset document, or submitted as a separate file named "Q2").

b) Show that S_n converges in L^2-norm to some S.

MATH5901/3901 only: Does (S_n) converge almost surely? You may refer to results given in the textbook.

Upload Separate/New File(s) clearly marked Q3

3. [**12 marks**]

i) [**6 marks**] Suppose that $U_1, U_2, \ldots$ are iid uniform on $(0, 1)$.

a) [**3 marks**]
Find the limiting distribution of

$$M_n = n(1 - \max\{U_1, \ldots, U_n\}).$$

b) [**3 marks**] Verify your answer above by estimating the distribution (cdf and/or pdf) of M_{100} using computer simulation.

Do not forget to include any figures in your submission (either embedded within your typeset document, or submitted as a separate file named "Q3").

ii) [**6 marks**] Suppose that (X_n) is a sequence of pairwise independent random variables such that

$$\mathbb{P}(X_n = n) = 1/n = 1 - \mathbb{P}(X_n = 0).$$

a) Find the expectation $\lim_n \mathbb{E}X_n$ and the limit of (X_n) in probability.

b) What is the reason $\lim_n \mathbb{E}X_n$ and the limit of (X_n) in probability are not the same? Explain using mathematical concepts covered during lectures.

MATH5901/3901 only: Does (X_n) converge almost surely? Why or why not? Explain using a mathematical argument.

Upload Separate/New File(s) clearly marked Q4

4. [**18 marks**] Let $(W_t, t \geq 0)$ be a Wiener process with natural filtration $(\mathcal{F}_t, t \geq 0)$, that is, $\mathcal{F}_t$ is the history of the process up until time t.

i) [**4 marks**] Compute the numerical values of $\mathbb{V}\mathrm{ar}(2W_1 - 3W_2 + W_3)$ and $\mathbb{P}(2W_1 - 3W_2 + W_3 > 1)$. You may use Matlab/R/Python to help you with any numerical computations.

MATH5901/3901 only: Compute the value of

$$\mathbb{P}(2W_1 - 3W_2 + W_3 > 1 \,|\, W_1 = 0).$$

ii) [**4 marks**] Suppose that we know the values of the Wiener process at time instants $t = 1, 2, 3, 4$ and $(W_1, W_2, W_3, W_4) = (1, 2, 3, 4)$. Write a computer script in Matlab/R/Python to simulate the values of this Wiener process at times $t = 0.5, 1.5, 2.5, 3.5$. Upload a plot of the result and your code.

iii) [**3 marks**] Define the process $(M_t, t \geq 0)$, where $M_t = \exp(t/2)\sin(W_t)$. What are the three conditions that (M_t) has to satisfy to be a **sub-martingale** with respect to $(\mathcal{F}_t)$? Write them down.

MATH5901/3901 only: Prove that (M_t) is, in fact, a **martingale** with respect to $(\mathcal{F}_t)$.

iv) [**7 marks**] Define the partition of the interval $[0, 1]$ using the set of points

$$\Pi_n = \{t_0, \dots, t_n : 0 = t_0 < t_1 < \cdots < t_n = 1\}.$$

The "norm" of the partition

$$\|\Pi_n\| = \max_{0 \leq k \leq n-1} \{t_{k+1} - t_k\}$$

is the interval with largest length. Assume that $\|\Pi_n\| \downarrow 0$ as $n \uparrow \infty$. Define the stochastic integrals

$$I_n := \sum_{k=0}^{n-1} W_{t_k}(W_{t_{k+1}} - W_{t_k}), \quad J_n := \sum_{k=0}^{n-1} W_{t_{k+1}}(W_{t_{k+1}} - W_{t_k}).$$

a) Find $J_n + I_n =?$ and $\mathbb{E}(J_n - I_n) =?$.

b) Show that $J_n - I_n$ converges in L^2-norm to a constant (as $n \uparrow \infty$) and find this constant.

c) As $n \uparrow \infty$, the integral J_n converges in L^2-norm to a random variable J. Use the previous results about $J_n + I_n$ and $J_n - I_n$ to find a formula for J.

MATH5901/3901 only: Assume the sequence of partitions (Π_n) is such that $\sum_n \|\Pi_n\| < \infty$. Show that $J_n \xrightarrow{\text{a.s.}} J$.

END OF EXAMINATION