Proposition 1. *If N is a natural number then it is a perfect square or $\sqrt{N}$ is irrational.*

Proof. Suppose $N \in \mathbb{N}$ and N is not a perfect square. By the prime factorisation theorem, $N = 2^{m_1}3^{m_2}...p^{m_K}$ for some $m \in \mathbb{Z}_+$.
Let $E^2 = 2^{2n_1}3^{2n_2}...p^{2n_K}$ for some $n \in \mathbb{Z}_+$ be all the factorised even powers of N, and let $O = 2^{M_1}3^{M_2}...p^{M_K}$ for $M = \{0, 1\}$ be the leftover single powers of N. (E.g. if $N = 3^2 2^3$, then $E^2 = 3^2 2^2$ and $O = 2^1$).

We get $N = E^2 O$, and $\sqrt{N} = E\sqrt{O}$.

Now $E = 2^{n_1}3^{n_2}...p^{n_K}$, so $E \in \mathbb{Q}$. We know that the product of a rational and irrational number is irrational, so if $\sqrt{N} \in \mathbb{Q}$, then $E\sqrt{O} \in \mathbb{Q}$, so $\sqrt{O} \in \mathbb{Q}$.

Assume $\sqrt{O} \in \mathbb{Q}$, that is $\sqrt{O} = \dfrac{\alpha}{\beta}$ for some $\alpha, \beta \in \mathbb{N}$. Re-arranging we find $\alpha^2 = \beta^2 O$. Rewriting α, β, and O in terms of their prime factor decompositions we have $2_\alpha^{2A_1}3_\alpha^{2A_2}...r_\alpha^{2A_K} = 2_\beta^{2B_1}3_\beta^{2B_2}...q_\beta^{2B_K}(2^{M_1}3^{M_2}...p^{M_K})$ where q and r are prime and $A, B \in \mathbb{Z}_+$.

By the prime factorisation theorem, the LHS and the RHS must have the same prime factor decompositions, therefore $2_\alpha^{2A_1} = 2_\beta^{2B_1}(2^{M_1})$, and $3_\alpha^{2A_2} = 3_\beta^{2B_2}(3^{M_2})$ etc... So the equalities only hold if $M_k = 0$, otherwise we have a contradiction. This implies $O = 1$, which implies $N = E^2$ is a square number.

Therefore if N is a natural number and N is not a perfect square, then $\sqrt{N}$ is irrational, as required. □