The Damped Driven Pendulum

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1 Introduction

(a) The damped harmonic oscillator. In classical mechanics, the non-linear damped harmonic oscillator equation describes any physical system that, when perturbed from it's stable equilibrium state, experiences a 'restoring force' which acts opposite to the direction of the perturbation. In the case of a damped pendulum, the corresponding equation of motion accounts for kinetic energy 'loss' through means of a damping term, which describes the rate kinetic energy dissipates into other forms of energy.

The equation of motion for a damped undriven pendulum is given by;

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2 \sin(\theta) = 0, \qquad \gamma \geq 0, \quad \omega > 0, \tag{1}$$

where $\theta = \theta(t)$ is the deflection angle of the pendulum at time t; $\dot{\theta} = d\theta/dt$ is the angular velocity of the pendulum at time t; and $\ddot{\theta} = d\dot{\theta}/dt$ is the angular acceleration at time t. Also, $\omega = \sqrt{g/l}$ is the angular frequency of the pendulum, with g representing the gravitational acceleration and l is the length of the pendulum rod; lastly $\gamma = \mu/2ml$ is the damping co-efficient[1], where μ is the friction co-efficient and m is the mass of the pendulum bob, and determines how quickly the pendulum's kinetic energy decays.

Equation (1) is derived from Newton's 2nd law of motion[2], $F = ma$, which is defined in the angular case as

$$T = I\ddot{\theta}.^{[3]} \tag{2}$$

Here, $T = T_g + T_f$, where $T_g = -mgl\sin(\theta)$ is the "restoring torque" due to gravity, and $T_f = -\mu l\dot{\theta}$ is the frictional torque responsible for the damping; and $I = ml^2$ is the Moment of Inertia, analogous to mass for angular motion. Dividing through (2) by I, we are left with (1).

(b) Linearizing the damped undriven oscillator equation. It's often useful to linearize an ODE to assess the stability around an equilibrium point. Consider the case where the maximum amplitude of the pendulum is constrained to low values, so θ always remains small. This allows us to linearize (1) by using the approximation $\sin(\theta) = \theta + \mathcal{O}(\theta^3)$ for $|\theta| \ll 1$. The success of this approximation can be shown from analysing the Taylor series of $\sin(\theta)$, i.e

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \tag{3}$$

Taking $|\theta| \ll 1$, the higher order terms become negligible in comparison to the first term of the series. The error associated with truncating (3) in this fashion is bounded by $\left|\theta^3/3!\right|$.

We deduce this by inspecting the behaviour of the partial sums. The terms of (3) clearly alternate and decrease for $|\theta| < 1$ (which is all we care about), so for all $n \geq 3$ the relationship $s_2 < s_n < s_1$ holds. This demonstrates that $|\theta - \theta^3/3!| < |\sin(\theta)| < |\theta|$, thus the maximum error $|\theta - \sin(\theta)| < \left|\theta^3/3!\right|$.

For $\theta = \pi/18$, (10 degrees), the maximum error is $\left|(\pi/18)^3/3!\right| = 0.000886$ (3.sf). Therefore even at 10 degrees the small angle approximation is still a very good approximation. So for $|\theta| \ll 1$ we may use the linearized version of (1), i.e,

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2\theta = 0. \tag{4}$$

The superposition principle of linear systems. The principle states that for any linear system, if $x_1(t)$ and $x_2(t)$ are solutions to the ODE that describes the linear system, then $x_1(t)+x_2(t)$, $Ax_1(t)$, $Bx_2(t)$, or any combination of these is also a solution to that ODE, where A, B are arbitrary constants. The principle allows us to find new solutions to linear systems with known solutions. So if $\theta_1(t)$ and $\theta_2(t)$ are solutions to (4), then by the superposition principle, $\theta_3(t)$ is also a solution to (4), where $\theta_3(t) \equiv A\theta_1(t) + B\theta_2(t)$ for some $A, B \in \mathbb{R}$.

To show this let θ_1 and θ_2 be solutions to (4), that is

$$\ddot{\theta_1} + 2\gamma\dot{\theta_1} + \omega^2\theta_1 = 0, \tag{5}$$

$$\ddot{\theta_2} + 2\gamma\dot{\theta_2} + \omega^2\theta_2 = 0. \tag{6}$$

Then for some $A, B \in \mathbb{R}$,

$$\begin{aligned} A(5) + B(6) &= A(\ddot{\theta_1} + 2\gamma\dot{\theta_1} + \omega^2\theta_1) + B(\ddot{\theta_2} + 2\gamma\dot{\theta_2} + \omega^2\theta_2), \\ &= (A\ddot{\theta_1} + B\ddot{\theta_2}) + 2\gamma(A\dot{\theta_1} + B\dot{\theta_2}) + \omega^2(A\theta_1 + B\theta_2), \\ &= (A\theta_1 \ddot{+} B\theta_2) + 2\gamma(A\theta_1 \dot{+} B\theta_2) + \omega^2(A\theta_1 + B\theta_2), \qquad (7.1) \\ &= 0,^{[4]} \end{aligned}$$

where (7.1) results from linearity of differentiation. Substituting θ_3 in (7.1) yields $(\ddot{\theta_3}) + 2\gamma(\dot{\theta_3}) + \omega^2(\theta_3) = 0$, so $\theta_3(t)$ is a new solution.

Similar oscillating systems. The pendulum is an example a rotational mechanical system. Rotational mechanical systems are not the only type of system which can be described by the equation of motion. Series and parallel RLC circuits have analogous equations to the pendulum, as do translational mechanical systems. The motion of an oscillating mass on a spring is an example of a translational mechanical system, described by Newton's Second Law, and given by the equation;

$$M\ddot{x} + \zeta\dot{x} + Kx = F. \tag{7}$$

Here, M is the mass of the object attached to the spring, ζ is the damping force, K is the spring constant, and F is the driving force. The displacement of the mass at time t is given by $x(t)$; the velocity and acceleration of the mass at time t is given by $\dot{x}$, and $\ddot{x}$ respectively. The stiffness of the spring is mainly responsible for the damping of the oscillations, the stiffer the spring, the larger the damping force and the quicker the mass comes to rest.

(c) Analytic solution of the linearized ODE. Linearizing (1) allows us to find the analytic solution to the equation, which enables us to easily see the qualitative behaviour of the system as time passes. The auxiliary equation for (4) is

$$\lambda^2 + 2\gamma\lambda + \omega^2 = 0, \tag{8}$$

so the roots are $\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$. There are 3 cases to consider here for the general solution; namely $\omega > \gamma$, $\gamma > \omega$, and $\gamma = \omega$.

Under-damping. Refers to the case where the pendulum overshoots the equilibrium point, $\theta = 0$, and experiences a restoring torque opposing the direction of motion. The pendulum continues to oscillate back and forth until it comes to rest. This occurs when $\omega > \gamma$, the roots of (8) are complex and the solution to (4) is

$$\theta(t) = Ae^{at}\cos(\sqrt{\gamma^2 - \omega^2}t) + \delta), \tag{9}$$

which is periodic and corresponds to the oscillating response. To demonstrate this is the general solution, take $\theta(t) = Ae^{-\gamma t}\cos(x + \delta)$, where $x^2 = \omega^2 - \gamma^2$. Then

$$\begin{aligned} \dot{\theta(t)} &= -Ae^{-\gamma t}(\gamma\cos(xt + \delta) + x\sin(xt + \delta)), \\ \ddot{\theta(t)} &= Ae^{-\gamma t}((\gamma^2 - x^2)(\cos(xt + \delta) + 2\gamma x\sin(xt + \delta)). \end{aligned}$$

Substituting θ, $\dot{\theta}$, and $\ddot{\theta}$ into (4) then equating the cos and sin components yields two equations,

$$\begin{aligned} &Ae^{-\gamma t}\cos(xt+\delta)(w^2-2\gamma^2+\gamma^2-w^2+\gamma^2)=0, \\ &Ae^{-\gamma t}\sin(xt+\delta)(-2\gamma x+2\gamma x)=0. \end{aligned} \tag{10}$$

Clearly the LHS's of (10) equal 0, showing $\theta(t)$ is indeed a solution.

Over-damping. Refers to the case where the frictional torque associated with a pendulum is so strong that the pendulum swings slowly towards the equilibrium point, but never crosses it. This occurs when $\gamma > \omega$, the roots of (8) are both real, implying that the general solution is $\theta(t) = Ae^{-(\gamma-\sqrt{\gamma^2-\omega^2})t} + Be^{-(\gamma+\sqrt{\gamma^2-\omega^2})t}$.

Critical damping. Refers to a special case of over-damping. The minimum time in which an over-damped pendulum can reach the equilibrium is the critically damped case. This happens precisely when $\gamma = \omega$. (8) has a repeated root which implies that the general solution is $\theta(t) = e^{-\gamma t}(A + Bt)$.

2 The undriven pendulum.

I used python code in order to investigate the motion of an under-damped, undriven pendulum released from rest at $\theta = \pi/2$. I used the `odeint()` function from the `scipy.integrate` module in order to calculate the numerical solution to (1), and then used `matplotlib.pyplot` to create the figures.

(a) First order reduction. One of the advantages of reducing an nth order ODE into n 1st order ODE's is that is becomes much easier to solve the system numerically with computers. I used the following decomposition of (1) into a system of 1st order ODE's to calculate the numerical solution. Define $\dot{\theta} = \eta$ and substitute into (1). This gives

$$\begin{aligned} &\dot{\eta} + 2\gamma\eta + \omega^2\sin(\theta) = 0, \\ &\dot{\eta} = -2\gamma\eta - \omega^2\sin(\theta). \end{aligned}$$

Therefore (1) is equivalent to the 1st order system

$$\dot{\theta} = \eta, \qquad \dot{\eta} = -2\gamma\eta - \omega^2\sin(\theta). \tag{11}$$

(b) Example solution. Using the initial conditions $\theta(0) = \pi/2$, $\eta(0) = 0$, and parameters $\gamma = 0.1$, $\omega = 1.0$; I compared the linear solution (blue) against the numerical solution (red) in order to examine the qualitative behaviour of the pendulum.

Fig.1(i) is a time portrait showing the amplitude of the pendulum at time t. The decrease in amplitude over time tells us that the pendulum is losing kinetic energy due to damping. In this particular case the damping of the pendulum is quite large despite being under-damped, and returns to it's equilibrium state after a small number of oscillations.

Fig.1(ii) is a time portrait showing the angular velocity of the pendulum at time t. The reduction in angular velocity is caused by friction, and is responsible for the decrease in amplitude seen in (i). The angular velocity decays to zero after sufficient time has passed.

Fig.1(iii) is a phase portrait combining (i) and (ii), showing the relationship between the amplitude and the angular velocity of the pendulum at time t. Here we see the inward spiral trajectory corresponds to the decrease angular velocity directly affecting the amplitude, resulting in the amplitude decreasing over time until the pendulum comes to rest.

(c) Fixed point at (0,0). The behaviour around this equilibrium point seen in Fig.1(iii) can be classified by inspecting the Jacobian matrix of (11). To show that (11) has an equilibrium point at (0,0), we require $\dot{\theta} = \dot{\eta} = 0$, signifying no change in the system. $\dot{\theta} = 0$ iff $\eta = 0$ for (11(i)). Substituting $\eta = 0$ into (11(ii)) implies $\dot{\eta} = 0$ iff $\sin(\theta) = 0$ where $\omega \neq 0$ by (1), which

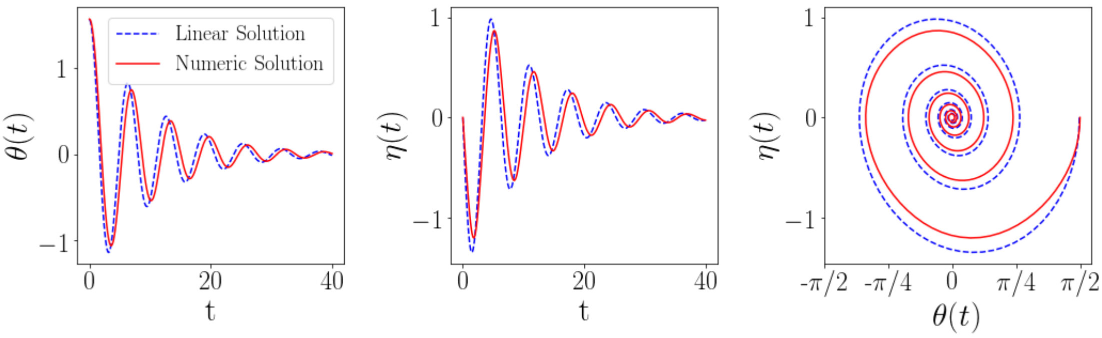

Figure 1: Motion dynamics for the damped undriven pendulum for initial conditions $\theta(0) = \pi/2$, $\eta(0) = 0$, parameters $\gamma = 0.1$, $\omega = 1$, with phase angle $\delta = -0.1$ (*See Appendix for derivation*) for the linear solution. (i) *Left*: A time domain plot showing the deflection angle over time. The damping causes the amplitude to decrease over time, decaying to 0 eventually. (ii) *centre*: A time domain plot showing the angular velocity over time. The damping causes the velocity to decay to zero over time, which is what causes the amplitude to decrease to 0. (iii) *Right*: A phase portrait showing the relationship between (i) and (ii) over time. The trajectories converge to the fixed point at (0,0) as time passes.

occurs iff $\theta = k\pi$ for $k \in \mathbb{Z}$. So the equilibrium points are given by $(k\pi, 0)$. This of course makes sense as the pendulum has an equilibrium at the bottom of it's swing, and another directly above when the pendulum is perfectly balanced. Therefore $(0, 0)$ is a fixed point.

The Jacobian matrix of (11) is;

$$J = \begin{bmatrix} 0 & 1 \\ -\omega^2\cos(\theta) & -2\gamma \end{bmatrix}.$$

The eigenvalues are $\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega^2\cos(\theta)}$, however since we are at $(0, 0)$, $\theta = 0$, so $\cos(\theta) = 1$. Therefore $\lambda = \gamma \pm \sqrt{\gamma^2 - \omega^2}$. The trace, $\mathrm{T}(J) = -2\gamma$, and the determinant $\mathrm{D}(J) = \omega^2$, since $\cos(\theta) = 1$.

For the under-damped case Fig.2(i) where $\omega > \gamma$, since $T < 0$, $D > 0$ by (1), and $T^2 < 4D$, we have a stable spiral. For the over-damped case Fig.2(iii) where $\gamma > \omega$, since $T < 0$, $D > 0$, and $T^2 > 4D$, we have a stable node. For the critically damped case Fig.2(ii) where $\gamma = \omega$, since $T = 0$, $D > 0$, and $T^2 = 4D$, we have a degenerate stable node.

3 The driven non-linear damped pendulum

The investigation now moves onto the case where a periodic driving force is introduced to the system. In particular, taking a modified version of (1) given by

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2\sin(\theta) = \alpha\cos(\Omega t), \tag{12}$$

where Ω is the frequency of the driving force and α is the amplitude. Define $\overrightarrow{A_d} = \alpha\cos(\Omega t)$ as the acceleration of the driving force, $\overrightarrow{A_g} = \omega^2\sin(\theta)$ as the acceleration gravitational torque, and ϕ as the period of $\overrightarrow{A_d}$. I focused on the long term behaviour of this system, after the transient response had decayed, in order to examine how changes in the driving force affect the eventual behaviour of the pendulum; more specifically, does the system respond to a periodic driving force by oscillating in a periodic way?

First order reduction. Again it is useful to reduce the equation into a 3-dimensional

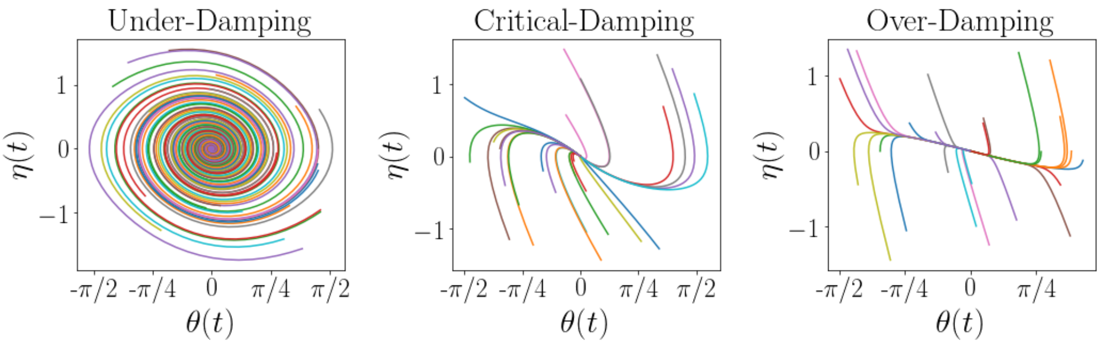

Figure 2: Motion dynamics of the damped undriven pendulum with randomized initial amplitudes and angular velocities with parameter $\omega = 1$. (i) *Left*: A phase portrait showing the under-damped case where $\gamma < \omega$, with parameter $\gamma = 0.1$. There is a fixed point at (0,0), and the spiral is inwards, indicating that the fixed point is a stable equilibrium. (ii) *Centre*: Shows the critically damped case where $\gamma = \omega$, with parameter $\gamma = 1$. The trajectories fall without spiralling into the fixed point at (0,0) and there is 1 direction where the trajectories falls straight towards the fixed point, making it degenerate stable node. (iii) *Right*: Shows the over-damped case where $\gamma > \omega$, with parameter $\gamma = 1.9$. Like the critically damped case the trajectories fall into the fixed point without spiralling, however there are two directions in this case where the trajectories fall straight into the fixed point (0,0), indicating this is a stable node.

first-order autonomous system to investigate the system numerically in python. Take $\dot{\theta} = \eta$, and $\dot{\phi} = \Omega$. Substituting these into (12) yields $\dot{\eta} + 2\gamma\eta + \omega^2 \sin(\theta) = \alpha\cos(\phi)$. So then (12) becomes the system;

$$\dot{\theta} = \eta, \qquad \dot{\eta} = -2\gamma\eta - \omega^2\sin(\theta) + \alpha\cos(\phi), \qquad \dot{\phi} = \Omega. \tag{13}$$

(a) Phase plots. I used python to simulate (13) numerically in order to model (12) and create phase portraits for varying driving force amplitudes, as seen in Fig.3. The phase portraits show the eventual state of the system once the transient response has decayed away, which I examined for evidence of periodic behaviour.

Fig.3(i) is a phase portrait for $\alpha = 0.9$. There is a single closed loop indicating that the post-transient response is resonant with the driving force, so we conclude the period of the oscillations is singular, and that the motion is predictable.

Fig.3(ii) is a phase portrait for $\alpha = 1.07$. The double loop indicates that the period of the oscillations has doubled, (i.e two oscillations are needed to return to the same point on the phase portrait). Note that α is now large enough to drive the pendulum over the top. This is what causes the doubling in period, as the driving force $\overrightarrow{T_d}$ now has to work against the gravitational torque $\overrightarrow{T_g}$ in order to reverse the motion of the pendulum (*See part 3(c)*). Consequently, this causes the accelerations $\overrightarrow{A_d}$ and $\overrightarrow{A_g}$ to "de-synchronize", which results in the pendulum failing to swing over the top on the next oscillation as it lacks the kinetic energy due to $\overrightarrow{A_d}$ working against $\overrightarrow{A_g}$ more often. This lack of momentum to reach the top allows $\overrightarrow{A_d}$ to re-synchronize with $\overrightarrow{A_g}$, which means $\overrightarrow{A_d}$ and $\overrightarrow{A_g}$ work together for longer on the second oscillation, giving the pendulum enough kinetic energy to swing over the top once again, completing the period.
(*See* [5] *for a visual animation.*)

Fig.3(iii) is a phase portrait for $\alpha = 1.15$. The post-transient response in this case is chaotic, although it's clear that there is a particular motion the pendulum likes to follow, as can be seen by the concentration of points along certain trajectories. The most interesting feature is the sharp points along the curves, which correspond to the motion caused by $\overrightarrow{T_d}$ working against $\overrightarrow{T_g}$, which halts the pendulum mid-swing as it swings towards the bottom. $\overrightarrow{T_d}$ then

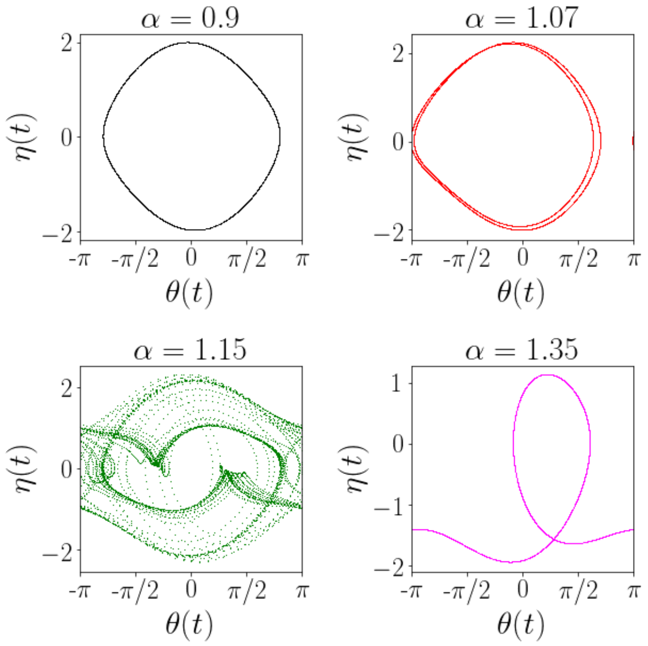

Figure 3: Motion dynamics of the post-transient response of the driven damped pendulum with parameters $\Omega = 0.6667$, $\gamma = 0.25$, $\omega = 1$ for $1000 \leq t \leq 2000$. (i) *Top left*: A phase portrait for $\alpha = 0.9$. The single loop indicates a singular period for the motion. (ii) *Top right*: $\alpha = 1.07$. The double loop indicates that the period of the motion has doubled, i.e it takes two oscillations to repeat the same trajectory. (iii) *Bottom left*: $\alpha = 1.15$. There is no loop which indicates the motion is no longer periodic, which implies the pendulum is moving unpredictably, i.e chaotic. (iv) *Bottom right*: $\alpha = 1.35$. The single loop indicates the motion has returned to periodic behaviour.

switches direction and the pendulum begins to move again, continuing in the same direction. This behaviour is qualitatively regular even though not exact, indicating a sort of crude, self-regulating behaviour by not behaving completely erratically.

The reason that the pendulum becomes non-periodic is because of where the pendulum is stopped in motion. $\overrightarrow{T_d}$ attempts to stop the pendulum when the velocity and $|\overrightarrow{A_g}|$ is large and working against $\overrightarrow{A_d}$. An analogy would be braking in a car travelling at high speed, any slight deviation in the point at which the brakes were applied would result in a much larger deviation on where the car comes to a halt. So it is easy to imagine that any slight deviation in where the pendulum is stopped on the subsequent oscillation leads to a rather big change in how the forces act on it next time, creating the chaos we observe.

Fig.3(iv) is a phase portrait for $\alpha = 1.35$. The oscillations become resonant with the driving force again, meaning $\overrightarrow{A_d}$ is in sync with $\overrightarrow{A_g}$ again, as indicated by the singular trajectory. Thus the pendulum has a single period, as in (i).

(b) Poincaré Section. The phase portraits seen in Fig.3 belong to a 3-dimensional autonomous system, in which a 3-dimensional plot could be made of the system, however I have chosen to omit these plots as they are not particularly informative. The Poincaré Section tells us much more; taking a slice through what would be the 3D-plot at $\sin(\phi) = 0$, i.e the (θ, η) plane, we get a cross section of each trajectory in Fig.3 which enables us to see easily if any periodic behaviour exists. If there are n distinct points on the graph, then the behaviour is considered non-periodic, i.e the trajectory does not follow the same path twice. Hence it appears Fig.3(iii) is an example of non-periodic behaviour, as expected. We can also see the double period for Fig.3(ii) and the single periods for Fig.3(i) & (iv).

(c) Further investigation. To investigate the system further, I created a pair of bifurcation diagrams (Fig.5) showing θ at times $t_0 = 2\pi/\Omega$ for varying α. The period at each α is given by the number of dots along the vertical. For $\alpha \leq 1$ there is 1 dot belonging to each α, so the period is singular. For $\alpha = 1.07$ there are 2 dots, indicating a double period. For $\alpha = 1.15$, there are n distinct dots for this α, indicating there is no periodic behaviour. We see from the diagrams that slight changes in α at certain values can lead to completely chaotic behaviour, which implies the driven pendulum is an example of a chaotic system. Fig.5(ii) shows us that the occurrence of chaotic behaviour irregular.

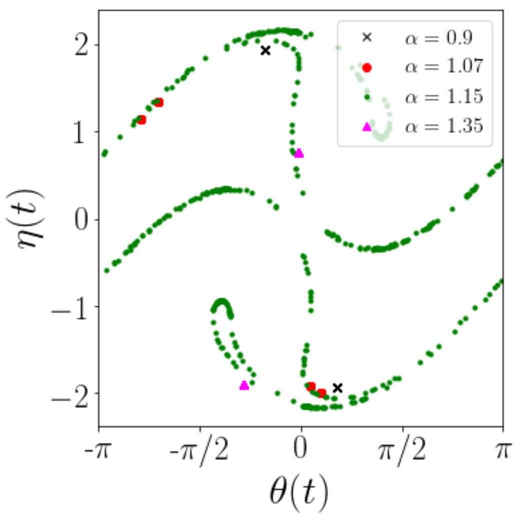

Figure 4: A Poincaré Section showing the periodicity of the motions in Fig.3, superimposed onto 1 plot, for parameter choice $\Omega = 0.6667$, $\gamma = 0.25$, and $\omega = 1$, at times $T_0 = \pi n/\Omega$ after the transient response has decayed. *Black*: Is a cross-section of the trajectory in Fig.3(i) at $\sin(\phi) = 0$. There are two dots, indicating a singular period. *Red*: Is a cross-section of Fig.3(ii). There are 4 dots, indicating the period has doubled. *Green*: Is a cross-section of Fig.3(iii). There are n dots here, indicating non-periodic motion. *Magenta*: Is is cross-section of Fig.3(iv). There are two dots again, indicating the period of the motion is singular.

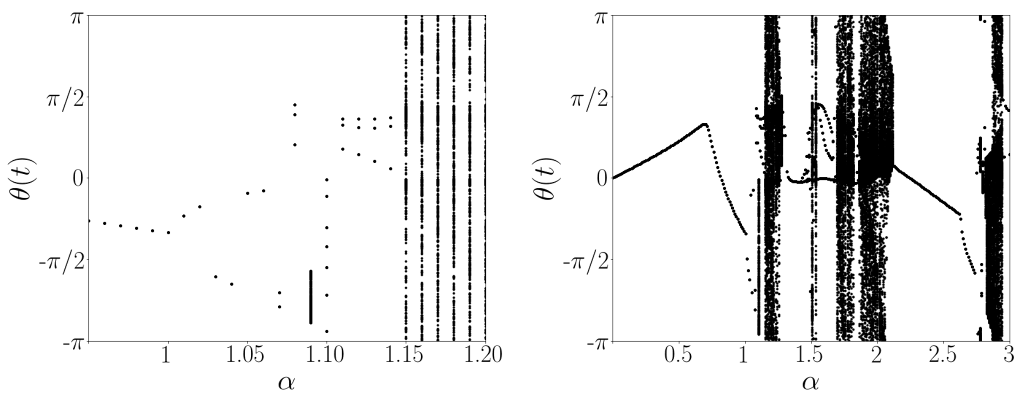

Figure 5: Bifurcation diagrams showing the period of the pendulum motion for parameter choice $\omega = 1$, $\Omega = 0.6667$, and $\gamma = 0.25$; at times $T_0 = 2\pi n/\Omega$, with α increasing in increments of 0.1. Each dot represents a single period, and multiple dots for a single α show a period equal to the number of dots along the vertical. (i)*Left*: A bifurcation diagram showing the periods for $0.95 \leq \alpha \leq 1.20$. The first bifurcation occurs at $\alpha \approx 1.07$, where the period doubles, as observed in Fig.4. At $\alpha \geq 1.15$ there are n distinct dots, indicating that the motion is chaotic. (ii)*Right*: $0 \leq \alpha \leq 3$. The diagram shows that there are several intervals of periodic motion when increasing the amplitude, and the transition between them gives rise to chaotic motion. Also, the occurrence of periodic and non-periodic motion is seemingly non-periodic itself.

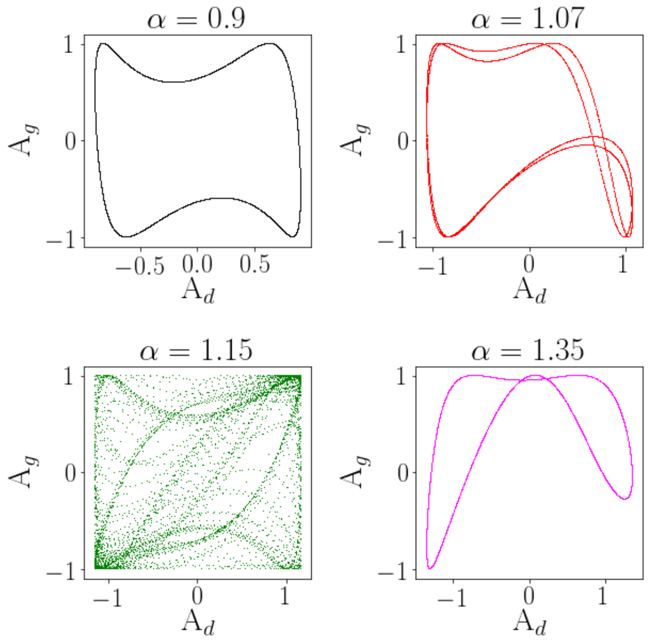

Figure 6: Dynamics of the driving force acceleration A_d vs the gravitational acceleration A_g for parameters $\omega = 1$, $\gamma = 0.25$, and $\Omega = 0.6667$. (i) *Top Left*: A phase portrait for $\alpha = 0.9$. The single loop indicates A_d and A_g are in sync with each other, implying the pendulum has regular motion, as seen in Fig.3(i). (ii) *Top Right*: $\alpha = 1.07$. The double loop indicates that A_d and A_g are partially synchronous with each other, implying that the pendulum motion is still periodic, but not singular, which leads to the double period observed in Fig.3(ii). (iii) *Bottom Left*: $\alpha = 1.15$. No loops exist indicating A_d and A_g are completely de-synchronized. This implies the pendulum is chaotic which is observed in Fig.3(iii). (iv) *Bottom Right*: $\alpha = 1.35$. The single loop indicates A_d and A_g have re-synchronized with each other, thus the pendulum motion is again regular, as seen in Fig.3(iv).

It was natural to then took a closer look at the relationship between $\overrightarrow{T_d}$ and $\overrightarrow{T_g}$ over time to investigate the nature of why the chaotic behaviour arises in the first place. I created phase portraits (Fig.6) comparing $\overrightarrow{A_d}$ against $\overrightarrow{A_g}$. Bifurcations in period indicate that the accelerations are becoming more de-synchronized with each other, which consequently affects the regularity of the motion of the pendulum. When the phase portrait shows a non-periodic trajectory, we know that the accelerations are completely de-synchronized, meaning $\overrightarrow{T_d}$ and $\overrightarrow{T_g}$ are sometimes working together and some times are not.

Conclusion. From the investigation it is clear that a non-linear driven damped pendulum is a chaotic system. There is no guarantee that introducing a periodic driving force will result in the post-transient response behaving in a periodic fashion. We explored why the chaotic behaviour arises, dependent upon the strength of resonance between $\overrightarrow{T_d}$ and $\overrightarrow{T_g}$ and can conclude that the chaotic behaviour is the greatest when of $\overrightarrow{A_d}$ is acting in the opposite direction to the motion, and the pendulum is moving at it's maximum speed. This ensures the subsequent oscillations have a lower probability of repeating exactly the same trajectory.

4 References

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5 Appendix

Phase angle derivation for the linear solution. We have $\theta(t) = Ae^{-\gamma t}\cos(xt+\delta)$, where $x = \sqrt{\omega^2 - \gamma^2}$ as a solution to (4). Then differentiate to get

$$\dot{\theta(t)} = -Ae^{-\gamma t}(\gamma\cos(xt+\delta) + x\sin(xt+\delta)). \tag{14}$$

Given the initial conditions $A = \pi/2$ and $\eta(0) = 0$, substituting into (14) yields

$$0 = \frac{-\pi}{20}(\cos(\delta) + 3\sqrt{11}\sin(\delta)). \tag{15}$$

Re-arranging and cancelling (15) gives $\tan(\delta) = -3\sqrt{11}$, and so $\delta = 0.100167 \approx 0.1$.