Black Scholes Paper

Ryan McPherson

July 2021

Abstract

This paper is intended to shed light on the importance of the Black-Scholes model and formula. We will provide an introduction to Itô calculus, making use of it to derive Black-Scholes model. The Black-Scholes equation will be used to calculate option prices. A overview of common option trading terms and indicators will be defined. We will discuss and display the relation of Black-Scholes model to the heat equation. Finally modern alternatives to the model and shortcomings of the model will be considered.

1 History and Importance

Economists Fischer Black and Myron Scholes developed a groundbreaking model of European Option pricing. Referred to as the Black-Scholes model, Robert Merton was responsible for publishing it in the 1973 issue of Bell Journal of Economics. [1]. In 1992 Merton and Scholes published further research on the equation, giving a more in depth explanation of the mechanics. Upon this publishing the model The major importance was expanding on how the Black-Scholes formula can be derived and used [2]. The model itself makes use of Itô calculus to interpret a stock price's erratic movements. The developments sparked a hiring frenzy of mathematicians and physicists in the financial sector. Itô calculus was a familiar topic in math and physics, rapidly expanding the field of quantitative analysis in finance.

The Black-Scholes model is a representation of perfectly optimized risk hedging. Assuming that a market can be freely entered or exited without any costs, or that the market is friction-less, Black-Scholes measures the risk between a risk-free and volatile asset. In the case of our model, the volatile asset is a stock price and the risk-free asset is the risk-free interest rate. An important note to make is that Black-Scholes models European style options, where the contract can only be executed on the strike date. In a perfect environment, this would allow for risk-free trading, as anytime the price becomes unfavorable, traders will choose to make use of the interest rate instead. By analyzing the model, a pricing equation for option pricing can be created, known as the Black-Scholes equation. The Black-Scholes equation provides for option pricing of a given stock. The model is used by hedge funds to mitigate risk of certain positions. Options allow for large firms to fine tune their risk levels by counter balancing their long term holdings.

2 Itô Calculus

The inception of the Black-Scholes model would not have been possible without the development of Itô Calculus [1]. **Itô Calculus**, also known as **Stochastic Calculus**, applies the principles of calculus to stochastic processes, or over functions in the Wiener space. A function in the Wiener space is a **Wiener process**, a continuous-time stochastic process [3]. In the case of Black-Scholes we will be focusing on differentiation of functions of Brownian Motion as their stochastic process. **Brownian Motion** is defined as a discrete wiener process that starts all paths at 0 and each increment is mutually independent of the rest. Standard Brownian motion also has a normal distribution, and can be seen clearly through a graph. Given the start of 5000 paths of standard Brownian motion start at the same place, the distribution of their points after t time will form a normal distribution.

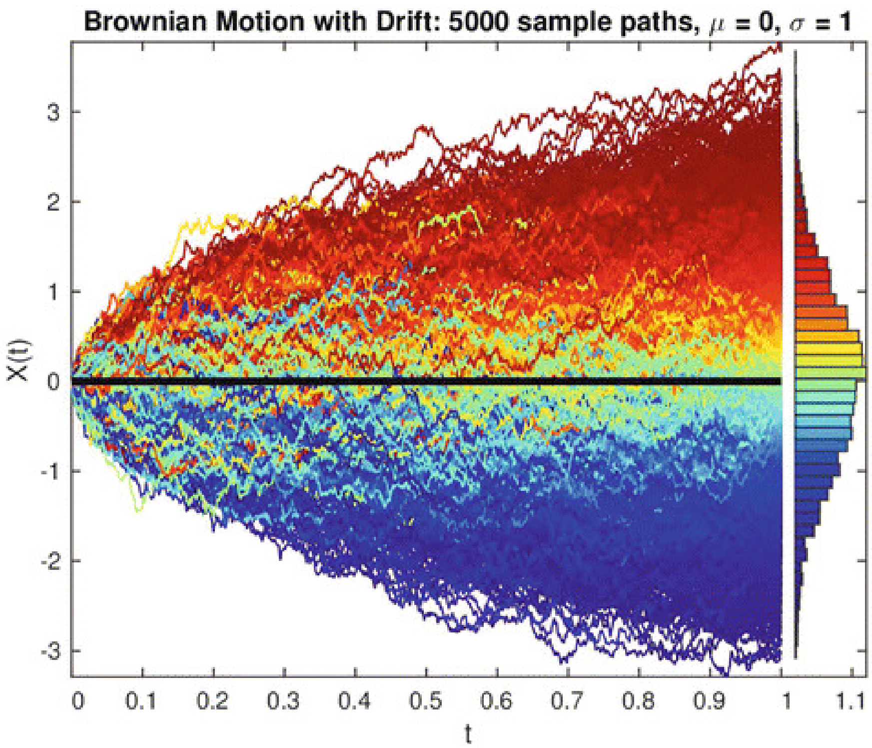

Figure 1: Normal Distribution by Standard Brownian Motion [3]

With a distribution in mind, a standard deviation σ and an expected value μ can be found. For the purposes of stochastic calculus, μ is often referred to as the drift coefficient and σ is commonly referred to as the diffusion coefficient. In figure 1, the drift coefficient μ is set to 0 and the diffusion coefficient σ is set to 1. We can observe the effect of μ as our mean is approximately the same as our starting value of 0. The σ of 1 results in the bell curve shape displayed on the right of Fig 1. By **Itô's Lemma**, the derivative of a function X(t) of a Wiener process W_t is given by

$$dX = \mu X_t dt + \sigma X_t dW_t. \tag{1}$$

Additionally, Itô's Lemma states that $dt^2 = 0$, $dtdB = 0$ and $dB^2 = dt$. This is expanded upon further in Nicholas Privault's notes [4].

3 Black-Scholes Model

The goal of Black-Scholes model is to optimize the profit of an option when compared to the risk-free interest rate. We begin by building a profit equation. The difference in the value of the option and the change in stock price will give us our profit, represented as π. Key variables to the value of an option are: time until maturity in years as t and stock price as S. We define a function $V(t,\, S)$that represents the value of an option. The importance of a European style option presents itself here, as the option only is executed on the expiration date. The change in value for a stock can be represented as ΔS where $\Delta S = S_t - S_0$. Profit is given by $\pi = V(t, S) - \Delta S$. We now want to find the marginal profit, which we can then use to optimize. Marginal profit is given by $d\pi = dV(t, S) - d\Delta S$. Differentiating dV first gives the following:

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}dt^2 + \frac{1}{2}\frac{\partial^2 V}{\partial S \partial t}dSdt.$$

If we focus on the term dS we can use (1) Itô's lemma:

$$dS = \mu S dt + \sigma S dW.$$

Here W is the Wiener process involved, which in Black-Scholes is standard Brownian motion. While Brownian Motion has a normal distribution curve, asset pricing cannot be negative so we will observe a log-normal distribution instead. The terms can be described as $\mu S dt$ being the average growth rate for the stock price and $\sigma S dW$ being the effects of volatility. $\sigma_{dt} = \sigma * \sqrt{dt}$. The value for dW can be represented by N(1,0)$\sqrt{dt}$, giving us the expected value of dW^2 by $1 * dt$. Additionally, if we take dS^2 we have $\mu^2 dt^2 + 2\mu\sigma dt dW + \sigma^2 S^2 dt$. By Itô's Lemma we know terms with dt^2 and $dtdW$ will equal 0, meaning dS^2 simplifies to $\sigma^2 S^2 dt$. Inserting the value of dS^2 to dV results in the following:

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}dt + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}dt^2 + \frac{1}{2}\frac{\partial^2 V}{\partial S \partial t}dSdt.$$

We can simplify dV by eliminating terms with dt^2 and $dSdt$ by Itô's Lemma:

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}dt.$$

Returning to $d\pi$, differentiating $d\Delta S$ gives us:

$$d\Delta S = \Delta dS.$$

We can view the risk as the difference in stock price and the difference in the value of the option. In order to minimize risk Δ is set equal to $\frac{\partial V}{\partial S}$. Intuitively, the change stock price is equated to the change in value of option due to the stock price. By equating them, we are able to achieve the minimum amount of risk.

$$d\pi = \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial S}dS - \frac{\partial V}{\partial S}dS.$$

We will reduce further:

$$d\pi = \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt.$$

The core of Black-Scholes is comparing the profitability of an option to the risk-free interest rate. Operating under the assumption that our option profit equation is now risk-free, we should see that our profit is equal to that of the risk-free interest rate over the same period of time. We can express this assumption as $d\pi = r\pi dt$, where r is the risk-free interest rate. We can expand the equations as follows

$$\frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt = r(V - \Delta S)dt$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r(V - \frac{\partial V}{\partial S}S).$$

Finally we arrive at the Black-Scholes model:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}.$$

Which can be rewritten in more traditional Partial Differential Equation ("PDE") form:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2}$$

The solution equation to this model and PDE is the Black-Scholes formula. After working through the steps, the model can be described as equating the profits of borrowing at the risk-free interest rate to the price increase of a given stock over the same period of time.

4 The Black-Scholes Formula

The **Black-Scholes formula**, or **Black-Scholes equation**, is the solution equation for the function V in the Black-Scholes model. The Black-Scholes equation for a call option is given by equation 2. First to establish the variables at hand we have the following:

S	The price of the stock or asset.
t	Time given in years.
$C(t, S)$	The Call option price, a function of S and t
r	Risk-free interest rate given as a decimal.
K	The strike price, the asset price the contract enables the buyer to purchase at.
$N(d)$	The distribution from a lognormal z-table, d is our z value.
σ_s	The standard deviation of S.

Which can be assembled into the Black-Scholes equation:

$$C(t, S) = SN(d) - Ke^{-rt}N(d - \sigma_s\sqrt{t}). \tag{3}$$

At first glance a lot is going on, however the equation can be broken into parts and digested a little more easily. To start with the variables at play, C is the which we would like to increase. The current stock price, S, will cause an increase in the call value by increasing. Strike price, represented by K, will increase option value by decreasing. We also have r as risk free interest rate, t as time. The term Ke^{-rt} represents the strike price discounted by the risk-free interest rate. By using r as the risk-free interest rate and t as the time until maturity in years, e^{-rt} is a present value equation. An point of inefficiency here comes from assuming the risk-free interest rate will remain constant. Assuming the risk-free interest rate used is the interest rate provided by US Treasury yields, the rate could change during the lifetime of an option. Additionally with an increase to both t and r we can observe an increase in call value [10].

The terms $N(d)$ and $N(d - \sigma\sqrt{t})$ represent the probability that the option will be exercised, with N being the function of the log-normal distribution created by Brownian Motion. We define d as $\frac{ln(\frac{S}{K})+t(r+\frac{\sigma_s^2}{2})}{\sigma_s\sqrt{t}}$. The derivation of d falls outside the scope of this paper, however we can understand the two terms as risk-adjusted probabilities [6]. The value of the call is tied directly to the idea that the probability of the strike price, in today's dollar value, must outweigh the current stock price's probability of increasing. The value of d is strongly tied to the volatility of the stock price, a high volatility will create a higher call price.

5 Examples

Starting with a stock of our choosing, we can calculate the call price for a set of condition. For this example we will be using Amazon.com, inc. (AMZN) as our stock, which as of July 15th, 2021 was trading at \$ 3,581.00 USD. That provides us with our value S; now we must select a strike price and period of time. We can select a strike price, K, of \$ 3,700.00 at 90 days. The value of t is found by $\frac{90}{365}$ which comes to $t = 0.2466$. This leaves us to find the volatility and the current risk-free interest rate. Using the close-to-close 180 day historic volatility [7], we find Amazon has a volatility of 0.2389. Using the US treasury 10 year rate as our risk-free interest rate we have r is equal to 1.31%. We start by calculating the value of d in the Black-Scholes equation:

$$\begin{aligned} d &= \frac{ln(\frac{S}{K}) + t(r + \frac{\sigma_s^2}{2})}{\sigma_s\sqrt{t}} \\ &= \frac{ln(\frac{3581}{3700}) + 0.2466(0.00131 + \frac{(0.2389)^2}{2})}{(0.2389)\sqrt{0.2466}} \\ &= \frac{-0.03269 + 0.00736}{0.11863} \\ &= -0.21352. \end{aligned}$$

Now we find $N(d)$ and $N(d - \sigma\sqrt{t})$ by using a log normal distribution Z table:

$$N(-0.21) = 0.4168$$

$$N(-0.21352 - (0.2389)\sqrt{0.2466}) = N(-0.33) = 0.3707.$$

Finally we use these values in equation 3 to find the call value:

$$\begin{aligned} C &= 3581 * 0.4168 - 3700e^{-0.2466*0.00131} * 0.3707 \\ &= 1492.5608 - 3698.80492 * 0.3707 \\ &= \$121.41. \end{aligned}$$

Contrarily, we can search for a stock with high volatility and assign similar parameters to see how Black-Scholes treats a different case. We can use Penn National Gaming, Inc. (PENN). This stock trades for \$66.21 USD as of July 15th, 2021, however has a 180 day close-to-close volatility of 0.6296. We will once again use a 90 day contract and to match the increase in strike price over the stock price we will use a strike price of \$68.41 USD. We will solve for d to begin,

$$d = \frac{ln(\frac{66.21}{68.41}) + 0.2466(0.00131 + \frac{(0.6296)^2}{2})}{(0.6296)\sqrt{0.2466}} = 0.05281.$$

Now we can find $N(d)$ and $N(d - \sigma\sqrt{t})$ by using a log normal distribution Z table,

$$N(0.05) = 0.4801$$

$$N(0.05 - (0.6296)\sqrt{0.2466}) = N(-0.26) = 0.3974.$$

Finally we can substitute these values to equation 3 to find the call value:

$$\begin{aligned} C &= 66.21 * 0.4801 - 68.41e^{-0.2466*0.00131} * 0.3974 \\ &= 31.7871 - 68.39 * 0.3974 \\ &= \$4.61. \end{aligned}$$

Now comparing the two pricings, we can find the call price in terms of the stock price. For AMZN we have $\frac{121.41}{3581} = 0.0339$ and for PENN we have $\frac{4.61}{66.21} = 0.06962$. Looking at the percentage of the stock price, we see that PENN has a volatility of 2.6 times more than AMZN and demands a call price that is comparatively 2.1 times more than amazon. This is a display of how the volatility of a stock has a direct effect on the Black-Scholes equation, and is accounted for in the Black-Scholes model.

6 Indicators and Terminology

There are many buzzwords and terms that float around in the world of option trading; for this paper we will only scratch the surface terminology. Important terms being used ahead are both **in-the-money**, ITM, and **out-of-the-money**, OOTM. In-the-money calls are when the strike price of the call is lower than the stock price, meaning that the option will be profitable to the buyer of the call. Out-of-the-money is the opposite, in that if the strike price is above the stock price then a call then the buyer will "out-of-the-money" if the call were to expire in these conditions. Additionally we have "at-the-money" where a the strike price and stock price are equal, however this is more often used as a dividing line between ITM and OOTM.

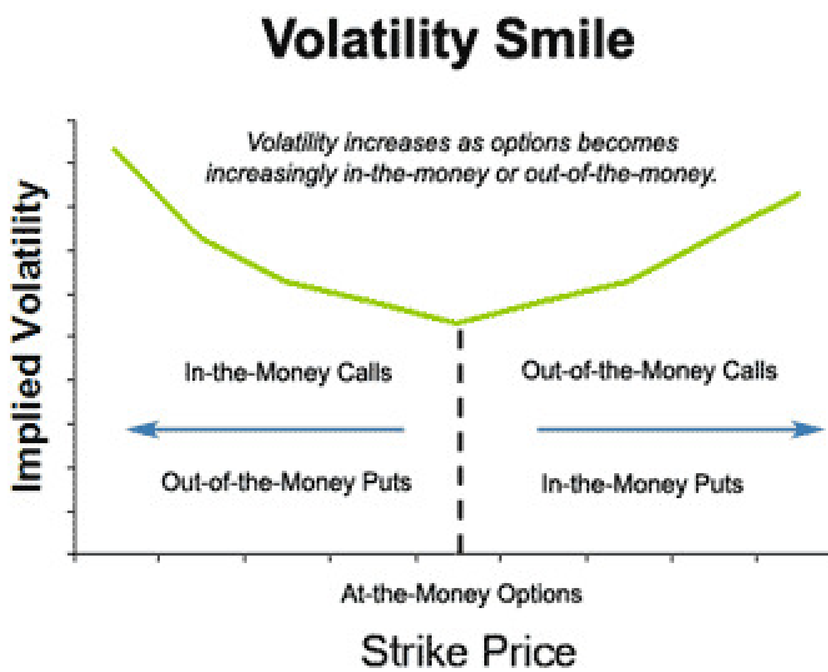

Figure 2: Volatility Smile [9]

When it comes to ITM and OOTM, it is relevant to discuss the volatility smile **(See Figure 2 below)**. Named after the shape of the graph, its easy to remember how implied volatility is correlated to the strike price of a call. At the lowest point we have the at-the-money calls, where the predicted amount of change would be low.

A large emphasis is put on the so called "Greeks" as indicators of an option's profitability. **The Greeks** are first, second, and third order derivatives of the Black-Scholes' model, which give useful insight to the pricing of a given option [11]. The most popular and what are commonly meant by "the Greeks" are composed of Delta, Gamma, Rho, Theta and Vega. The Greeks help to give insight into an option's mechanics at a glance. We will continue to focus on the call pricing equation.

Delta, $\Delta = \frac{\partial C}{\partial S}$

Given by the differentiation of C, the option call price in relation to S, the stock price. Delta is the rate at which the Call option price changes in regard to the stock price. If the option is set to expire on the same day, Delta can be viewed as the probability of a call finishing in ITM. Delta will sometimes be shown as a value in the range of 0 to 100. This value is normalized to the dollar change of a call contract, as the contracts are for 100 underlying shares. For example if a call has a delta of 75, for every dollar increase to the share price the call increases by $75. A Delta value near 100 indicates that the call value is nearly identical to the share price value, and

Gamma, $\Gamma = \frac{\partial C}{\partial K}$

Provided by the differentiation of in regards to the strike price. Gamma is used to measure the rate of change for the Delta of an option. Gamma is highest when the call is at-the-money, almost like an inverse of the volatility smile. Gamma will remain positive but decrease if the option falls into being ITM or OOTM.

Theta, $\Theta = \frac{\partial C}{\partial t}$

Theta measures the rate of change in the call price in relation to time. Theta represents how options with less time to expire will command a premium. As the expiration date increased, t will decrease and ITM calls will see Theta increase while OOTM calls see Theta decrease. Intuitively this makes sense, as the closer to expiration a given contract is, the less time random motion has to cause the price to drift.

Vega, V$= \frac{\partial C}{\partial \sigma}$

Vega, the change in call price due to volatility, is another highly valued Greek. Often confused with the volatility of an option, Vega instead is the way the option responds to change in volatility. The price of a option will correspond alongside Vega, with an increase causing an increase in price and a decrease causing a decrease in price. However, Vega is unique to the option at hand and usually has the greatest effect on options that are at-the-money.

Rho, $\rho = \frac{\partial C}{\partial r}$

The rate at which a change in the risk-free interest rate affects the pricing of an option. Returning to the Black-Scholes model, an increase in the risk-free interest rate would command that the option price to change in order to equalize. Rho is given as the dollar amount that will follow a 1% change in interest rate. Rho plays a much larger role in options that operate over a longer time period as the interest rate is more likely to see change.

Greeks

Call (AMZN 211015C03700000)	
Delta	0.42214
Gamma	0.00087
Rho	4.15845
Theta	-1.18675
Vega	7.72437

Figure 3: Greeks of an Amazon Call with the same parameters as used in the example [13]

The Greeks discussed above are known as the major Greeks, all of which being first order derivatives. More Greeks exist, they are referred to as minor Greeks and are composed of second and third order derivatives. In the process of analyzing an option, the minor Greeks play a much smaller role. Many represent a rate of change of one of the major Greeks discussed above. In that sense, they are used in a more niche environment and go beyond the scope being discussed here.

7 The Heat Equation

Within the topic of partial differential equations, the Heat Equation is bound to arrise. Many partial differential equations are derived from or can be transformed into the Heat Equation. The Heat Equation is as follows:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}. \tag{4}$$

The Heat Equation can be used to model the flow of heat through a homogeneous material with a heat transfer of k, over a distance of x, and given t time. Solution equations for this PDE take on the form u(x, t), providing the amount of heat transferred at the given distance. The Black-Scholes model can be elegantly transformed into the Heat Equation [5].

We start with (3), and a given set of boundary conditions:

$$C(t, S) = max(S - K, 0),$$

$$C(t, 0) = 0,$$

$$C(t, S) \approx S \text{ When } S \to \infty.$$

Now we must make a change of variables S and t as follows,

$$S = e^y,$$

$$t = T - \tau.$$

We can then insert into (1),

$$\frac{\partial V}{\partial (T-\tau)} + \frac{1}{2}\sigma^2 (e^y)^2 \frac{\partial^2 V}{\partial (e^y)^2} + r(e^y)\frac{\partial V}{\partial (e^y)} - rV = 0.$$

Simplifying, we have $\frac{\partial V}{\partial(T-\tau)} = -\frac{\partial V}{\partial \tau}$ and $(e^y)\frac{\partial V}{\partial(e^y)} = \frac{\partial V}{\partial y}$. Now we make an insertion of these equivalencies:

$$-\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial V}{\partial y^2} + r\frac{\partial V}{\partial y} - rV = 0.$$

We can rewrite the equation as follows:

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial y^2} - (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial y} + rV = 0.$$

Now we can replace our value equation with an equation for u or heat that is given by $u = e^{r\tau}V$. This allows for the following:

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2} - (r - \frac{1}{2}\sigma^2)\frac{\partial u}{\partial y} + rV = 0.$$

The final step is another substitution. We have $x = y + \tau(r - \frac{\sigma^2}{2})$ which will allow us to eliminate the first order term from our current equation,

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}.$$

Leaving us with a heat equation where the diffusion rate, K, is $\frac{1}{2}\sigma^2$. We have also changed our time variable to be in terms of the difference in total time. This interesting observation comes as a byproduct of Brownian motion being used to derive both Black-Scholes and the heat equation. Abstracting a little further, we understand the heat equation models atoms transferring energy at random to their neighboring atoms. In a sense the Black-Scholes is modeling a stock price's market orders, with buy and sell orders being randomly placed transferring increase and decreases in price. While the two are applied to very different cases, their mathematical basis is very similar.

8 Conclusion

The Black-Scholes model only opened the door to the mathematical side of the stock market. Modern Financial Theory has built upon the concepts introduced by Black-Scholes to explore further possibilities of modeling the stock market. In the case of American options, one such model that stands out is the **Binomial model**. Because American options are able to be executed at any point up until their expiration date, binomial option modeling sets out with the goal to provide optimal pricing given these conditions. The binomial pricing model is based in the idea that for every time increment the price has two options, increase or decrease [12]. This allows for the binomial option model to provide option pricing for a range of times, an important aspect of modeling American options.

The downsides of Black-Scholes are important to explore as well. Previously discussed, the idea that the risk-free interest rate is assumed to be constant is one point of arbitrage.

Additionally, Black-Scholes assumes that the market is frictionless and has no barriers to entry, in reality this is not true. Many brokerages place additional charges on the buying and selling of option contracts. The most debated shortcoming is the inability for the Black-Scholes to model the true volatility of the stock market, known as the implied volatility. The solutions to these problems are current day issues and have been subject to many different approaches in finding a solution. Making use of today's computational power for data analysis has provided promising results in attempts to refine and improve Black-Scholes. However, given the fact that modern industries are built on using the original model created in the 1970's speaks volumes to the signifigance of Black, Scholes and Merton's work.

References

[1] Merton, Robert C. "Theory of Rational Option Pricing." Bell Journal of Economics and Management Science 4, no. 1 (Spring 1973): 141-183. (Chapter 8 in Continuous-Time Finance.)

[2] Press release. NobelPrize.org. Nobel Prize Outreach AB 2021. Thu. 15 Jul 2021. https://www.nobelprize.org/prizes/economic-sciences/1997/press-release/

[3] Karatzas, Ioannis, Shreve, Steven. "Brownian Motion." *Brownian Motion and Stochastic Calculus*, Springer, 2012.

[4] Privault, Nicolas. "Stochastic Calculus." *Chapter 4 Brownian Motion and Stochastic Calculus*, 4 July 2021.

[5] Stecher, Mike. *Converting the Black-Scholes PDE to The Heat Equation.* https://www.math.tamu.edu/ stecher/425/Sp12/blackScholesHeatEquation.pdf

[6] Tyge Nielsen, Lars. *Understanding N(d1) and N(d2): Risk-Adjusted Probabilities in the Black-Scholes Model*, 1992.

[7] Alphaquery. *Amazon.com, Inc. (AMZN).* Retrieved from https://www.alphaquery.com/stock/AMZN/volatility-option-statistics/180-day/historical-volatility.

[8] Alphaquery. *Penn National Gaming, Inc. (PENN).* Retrieved from www.alphaquery.com/stock/PENN/volatility-option-statistics/180-day/historical-volatility.

[9] "Volatility Smiles & Smirks." *Volatility Smiles & Smirks Explained — The Options & Futures Guide* , www.theoptionsguide.com/volatility-smile.aspx.

[10] Hayes, Adam. "Black-Scholes Model Definition." *Investopedia*, Investopedia, 6 July 2021, www.investopedia.com/terms/b/blackscholes.asp.

[11] Hall, Mary. "Using the 'Greeks' to Understand Options." *Investopedia*, Investopedia, 3 July 2021, www.investopedia.com/trading/using-the-greeks-to-understand-options/.

[12] Chen, James. "How the Binomial Option Pricing Model Works." *Investopedia*, Investopedia, 7 July 2021, www.investopedia.com/terms/b/binomialoptionpricing.asp.

[13] *(AMZN) Call Put Options.* Nasdaq, www.nasdaq.com/market-activity/stocks/amzn/option-chain/call-put-options/amzn–211015c03700000.