

Math Refresher
Winter Institute in Data Science

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2025-01-04

Functions

Vectors

Matrix Algebra

Systems of Linear Equations

Warming Up

1. Let “ $A \circledast B$ ” be defined as $A^B + A \cdot B$. Calculate $4 \circledast 3$.
2. Solve this system of two linear equations:

$$2x - y = 4$$

$$x + y = 5$$

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- ▶ Input variable: *predictor, covariate, indep var*
- ▶ Output variable: *outcome, response, dep var*

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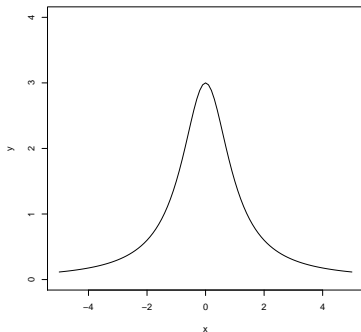
Range $f(X) =$

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Range $f(X) = (0, 3]$



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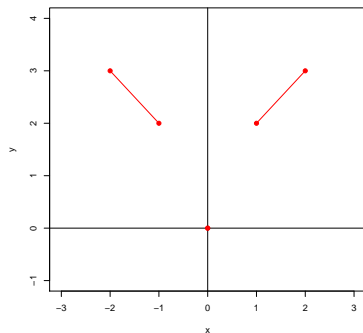
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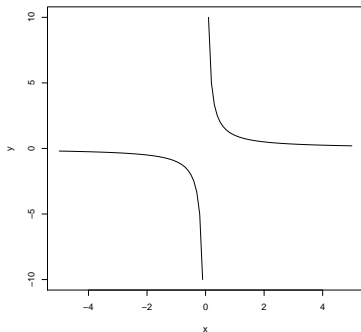
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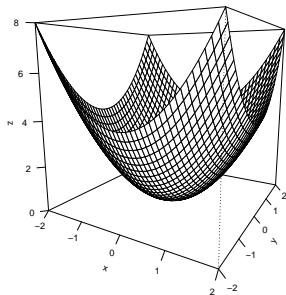
Image $f(X, Y) =$

Example 4

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Image $f(X, Y) = \mathbb{R}_+^1 \cup \{0\}$



General Types of Functions

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a is the coefficient. k is the degree.

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► **Trigonometric Functions:** Examples: $y = \cos(x)$,
 $y = 3 \sin(4x)$

Trigonometric Functions: Gill & Casella (2004)

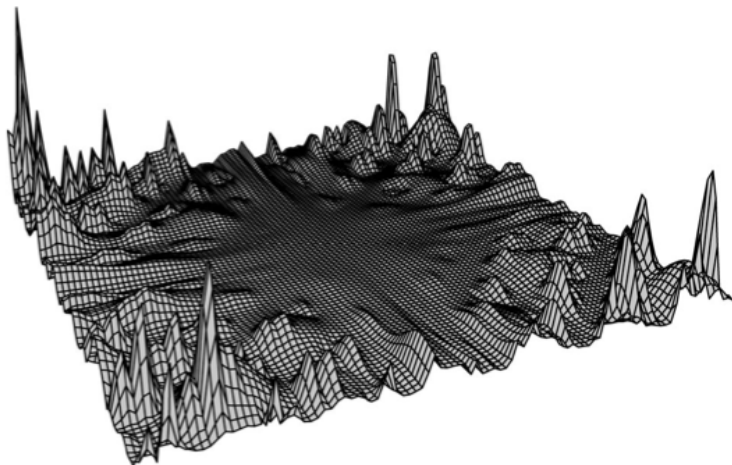


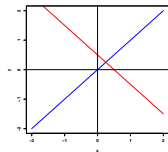
Fig. 1 A highly multimodal surface.

$$f(x, y) = |(x \sin(20y - 90) - y \cos(20x + 45))^3 a \cos(\sin(90y + 42)x) + (x \cos(10y + 10) - y \sin(10x + 15))^2 a \cos(\cos(10x + 24)y)|$$

Linear and Nonlinear Functions

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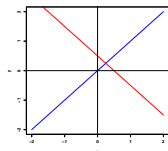
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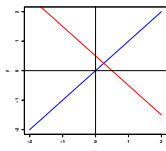
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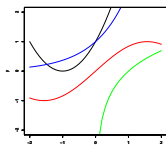
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Properties:

$$\blacktriangleright \sum_{i=1}^n cx_i = c \sum_{i=1}^n x_i$$

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$$\blacktriangleright \prod_{i=1}^n c = c^n$$

Sums, Products, and Logs

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Sums, Products, and Logs

Logarithms can help us change products into sums.

$$\begin{aligned}\log\left(\prod_{i=1}^n cx_i\right) &= \log(cx_1 \cdot cx_2 \cdot \dots \cdot cx_n) \\&= \log(cx_1) + \log(cx_2) + \dots + \log(cx_n) \\&= \sum_{i=1}^n \log(cx_i) \\&= [(\log c + \log x_1) + (\log c + \log x_2) + \\&\quad \dots + (\log c + \log x_n)] \\&= \sum_{i=1}^n \log c + \sum_{i=1}^n \log(x_i) \\&= n \log(c) + \sum_{i=1}^n \log(x_i)\end{aligned}$$

Vectors

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- ▶ **Vector:** A vector in n -space is an ordered list of n numbers. These numbers can be represented as either a row vector or a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

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- ▶ We can also think of a vector as defining a point in n -dimensional space, usually \mathbf{R}^n ; each element of the vector defines the coordinate of the point in a particular direction.

Vector Arithmetic

- ▶ **Vector Addition:** Vector addition is defined for two vectors \mathbf{u} and \mathbf{v} iff they have the same number of elements:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 & u_2 + v_2 & \cdots & u_n + v_n \end{pmatrix}$$

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$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 3 + 2 & -2 + 0 & 1 + 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 & 2 \end{pmatrix}$$

```
u <- c(3, -2, 1)
```

```
v <- c(2, 0, 1)
```

```
u + v
```

```
## [1] 5 -2 2
```

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$$c\mathbf{v} = \begin{pmatrix} 6 \cdot 3 & 6 \cdot -2 & 6 \cdot 1 \end{pmatrix}$$

Vector Arithmetic

- **Scalar Multiplication:** The product of a scalar c and vector \mathbf{v} is:

$$c\mathbf{v} = (cv_1 \quad cv_2 \quad \dots \quad cv_n)$$

Let $\mathbf{v} = (3 \quad -2 \quad 1)$, $c = 6$.

$$c\mathbf{v} = (6 \cdot 3 \quad 6 \cdot -2 \quad 6 \cdot 1) = (18 \quad -12 \quad 6)$$

```
c <- 6
```

```
v <- c(3, -2, 1)
```

```
c * v
```

```
## [1] 18 -12 6
```

Vector Properties

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors, c, d be constants, $\mathbf{0}$ be vector \mathbf{z} s.t.
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- ▶ Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- ▶ Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- ▶ Distributivity: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- ▶ Scalar Distributivity: $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

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- ▶ Scalar Multiplicative Identity: $1\mathbf{u} = \mathbf{u}$

Transpose

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Inner Product

- **Inner Product:** The Euclidean inner product (also, the “dot product”) of two vectors \mathbf{u} and \mathbf{v} is defined iff they have the same number of elements

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^n u_iv_i$$

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$$\mathbf{u} \cdot \mathbf{v} = 3 \cdot 2 + (-2) \cdot 0 + 1 \cdot 1 = 6 + 1 = 7.$$

```
u %*% v %>% drop() # Simplify array dims
```

```
## [1] 7
```


Inner Product and Orthogonality

If $\mathbf{u} \cdot \mathbf{v} = 0$, the vectors are *orthogonal* (or perpendicular).

Let $\mathbf{u} = \begin{pmatrix} 5 & 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 & -2 \end{pmatrix}$.

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$$\mathbf{u} \cdot \mathbf{v} = 5 \cdot 0 + 0 \cdot -2 = 0 + 0 = 0.$$

Inner Product

- Think about the $\mathbf{u} \cdot \mathbf{v}$ inner product as

$$\underbrace{\mathbf{u}}_{1 \times k} \cdot \underbrace{\mathbf{v}}_{k \times 1} = \underbrace{w}_{1 \times 1}$$

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$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v}' \mathbf{u} = \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

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$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v}' \mathbf{u} = \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 7$$

Inner Product Properties

- ▶ Commutativity: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- ▶ Associativity: $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
- ▶ Distributivity: $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- ▶ Zero Product: $\mathbf{u} \cdot \mathbf{0} = 0$

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$$\begin{aligned}\|\mathbf{v}\| &= \left\| \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \right\| \\ &= \sqrt{2^2 + 0^2 + 1^2} \\ &= \sqrt{5}\end{aligned}$$

Vector Norm Properties

- ▶ Scalar Multiplication: $\|c\mathbf{u}\| = |c| \cdot \|\mathbf{u}\|$
- ▶ Vector Distance: $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$
- ▶ Norm Squared: $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$
- ▶ Cosine Rule: $\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}$
- ▶ Difference Norm: $\begin{aligned} \|\mathbf{u} - \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 - 2\|\mathbf{u}\| \cdot \|\mathbf{v}\|(\cos \theta) + \|\mathbf{v}\|^2 \end{aligned}$
- ▶ Triangle Inequality: $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$
- ▶ Cauchy-Schwartz Inequality: $\|\mathbf{u} \cdot \mathbf{v}\| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$

Mahalanobis Distance

$$MD_{ij} = \sqrt{(\mathbf{x}_i - \mathbf{x}_j)' \Sigma^{-1} (\mathbf{x}_i - \mathbf{x}_j)}$$

Mahalanobis Distance

$$MD_{ij} = \sqrt{(\mathbf{x}_i - \mathbf{x}_j)' \Sigma^{-1} (\mathbf{x}_i - \mathbf{x}_j)}$$

Like Euclidean distance, but scaled by inverse covariances

$$\hat{\Sigma} = \frac{1}{n-1} \sum_{i=1}^n (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})'$$

where

$$\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i$$

is the vector of sample means.

A Bit of R

```
a <- c(3, 0, 0)
b <- c(0, 2, 0)
a %*% b ## inner product
```

```
##      [,1]
## [1,]    0
```


Dependence and Independence

- ▶ **Linear combinations:** The vector \mathbf{u} is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ if

$$\mathbf{u} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k$$

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$$\mathbf{u} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k$$

- ▶ **Linear independence:** A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is linearly independent if the only solution to the equation

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_k \mathbf{v}_k = \mathbf{0}$$

is $c_1 = c_2 = \dots = c_k = 0$. If another solution exists, the set of vectors is linearly dependent.

Linear Dependence

- ▶ A set S of vectors is linearly dependent iff at least one of the vectors in S can be written as a linear combination of the other vectors in S .
- ▶ Linear independence is only defined for sets of vectors with the same number of elements
- ▶ Any linearly independent set of vectors in n -space contains at most n vectors.

Example 1

Are the following sets of vectors linearly independent?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

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Yes. $(c_3 = 0) \Rightarrow (c_2 = 0) \Rightarrow (c_1 = 0)$

Example 2

Are the following sets of vectors linearly independent?

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Example 2

Are the following sets of vectors linearly independent?

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

No. $\mathbf{c} = \begin{pmatrix} 2 & 1 & -2 \end{pmatrix}$ (e.g.)

Matrix Algebra

Matrices

- **Matrix:** A matrix is an array of mn real numbers arranged in m rows by n columns.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Matrices

- ▶ Vectors are special cases of matrices
 - ▶ Col vector of length k is a $k \times 1$ matrix
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$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \cdots \\ \mathbf{b}_n \end{pmatrix}$$

Special Matrices

► Identity: $\mathbf{I}_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$

► $\mathbf{J}_n = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$

► Zero: $\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$

Special Matrices

► Diagonal:
$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

► Lower Triangular:
$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

► Upper Triangular:
$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

► Triangular: Either upper triangular or lower triangular

Matrix Equality

- Let \mathbf{A} and \mathbf{B} be two $m \times n$ matrices. Then

$$\mathbf{A} = \mathbf{B}$$

iff

$$a_{ij} = b_{ij}$$

$$\forall i \in \{1, \dots, m\}, \quad j \in \{1, \dots, n\}$$

Matrix Addition

- ▶ Let \mathbf{A} and \mathbf{B} be two $m \times n$ matrices. Then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

- ▶ \mathbf{A} and \mathbf{B} must be same size – *conformable* for addition

Matrix Addition Example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} =$$

Matrix Addition Example

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$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 4 & 4 \\ 6 & 6 & 8 \end{pmatrix}$$

A + B

```
##      [,1] [,2] [,3]
## [1,]    2    4    4
## [2,]    6    6    8
```

Scalar Multiplication

Scalar Multiplication: Given scalar c , the scalar multiplication $c\mathbf{A}$ is

$$c\mathbf{A} = c \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix}$$

Scalar Multiplication Example

$$c = 2 \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

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$$c\mathbf{A} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

```
c <- 2
```

```
c * A
```

```
##      [,1] [,2] [,3]  
## [1,]    2    4    6  
## [2,]    8   10   12
```

Matrix Multiplication

- **Matrix Multiplication:** If \mathbf{A} is $m \times k$ and \mathbf{B} is $k \times n$, then their product $\mathbf{C} = \mathbf{AB}$ is $m \times n$ matrix where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

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- ▶ Consider \mathbf{A} to be composed of stacked rows

$$\mathbf{a}_i = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix},$$

$$\mathbf{B} \text{ to be composed of stacked columns } \mathbf{b}_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix}.$$

Then, $\mathbf{AB} = \mathbf{C}$, where

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j$$

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- ▶ Given \mathbf{AB} , say \mathbf{B} is pre-multiplied by \mathbf{A} or \mathbf{B} is left-multiplied by \mathbf{A} or \mathbf{A} is post-multiplied by \mathbf{B} or \mathbf{A} is right-multiplied by \mathbf{B}

Warning!

- ▶ Commutative law for multiplication does **not** hold – order of multiplication matters:

$$\mathbf{AB} \neq \mathbf{BA}$$

- ▶ \mathbf{AB} may exist, while \mathbf{BA} does not.

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

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$$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ -2 & 2 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 1 & 7 \\ -1 & 3 \end{pmatrix}$$

Transpose

- ▶ **Transpose:** The transpose of the $m \times n$ matrix \mathbf{A} is $n \times m$ matrix \mathbf{A}^T (or \mathbf{A}') obtained by interchanging rows and columns of \mathbf{A} .

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$$\mathbf{B}^T \mathbf{A}^T =$$

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Identity Matrix

The $n \times n$ identity matrix \mathbf{I}_n has diagonal elements = 1 and off-diagonal elements = 0.

Examples:

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Systems of Linear Equations

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Systems of Linear Equations

Often interested in solving linear systems like

More Systems of Linear Equations

- More generally, we might have a system of m equations in n unknowns

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & \vdots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

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How many Solutions?

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 3. Infinite solutions: lines coincide.

Matrices to Represent Linear Systems

Matrices are an efficient way to represent linear systems such as

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as $\mathbf{Ax} = \mathbf{b}$

Coefficient Matrix

The $m \times n$ **coefficient matrix** \mathbf{A} is an array of mn real numbers arranged in m rows by n columns:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

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- The unknown quantities are represented by the vector

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- ▶ The RHS of the linear system is represented by the vector

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Inverse of a Matrix

- ▶ An $n \times n$ matrix \mathbf{A} is *nonsingular* or *invertible* if there exists an $n \times n$ matrix \mathbf{A}^{-1} such that

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- ▶ Then, \mathbf{A}^{-1} is the inverse of \mathbf{A} .
- ▶ If there is no such \mathbf{A}^{-1} , then \mathbf{A} is *singular* or *noninvertible*.

Example of Inverses

► Example: Let

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{pmatrix}$$

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Since

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

we conclude that \mathbf{B} is the inverse of \mathbf{A} , \mathbf{A}^{-1} , and that \mathbf{A} is nonsingular.

Calculating the Inverse in R

```
A <- matrix(c(1, 4, 5, 6), 2, 2, byrow = TRUE)
```

```
A
```

```
##      [,1] [,2]  
## [1,]    1    4  
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```
solve(A)
```

```
##      [,1]      [,2]  
## [1,] -0.4285714  0.28571429  
## [2,]  0.3571429 -0.07142857
```

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solve(A) %*% A
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```

Linear Systems and Inverses

- ▶ Matrix representation of a linear system

$$\mathbf{Ax} = \mathbf{b}$$

- ▶ If \mathbf{A} is an $n \times n$ matrix, then $\mathbf{Ax} = \mathbf{b}$ is a system of n equations in n unknowns.

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$$\mathbf{Xb} = \mathbf{y}$$

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 - ▶ \mathbf{y} is $n \times 1$ vector of outcome variable values
 - ▶ \mathbf{b} is $k \times 1$ vector of linear parameters (β 's)

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Befriend $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$. If you understand it, its cousins, and their properties (both strengths and weaknesses), your data-analytic future will be bright.

Solving a System in R ($n \times n$)

$$\mathbf{y} = \mathbf{X}\mathbf{b}$$

```
x1 <- c(1, 3, 5)
x2 <- c(3, 1, 2)
x3 <- c(1, 1, 1)
y <- 4 * x1 + 3 * x2 + x3
(X <- cbind(x1, x2, x3))
```

```
##      x1 x2 x3
## [1,]  1  3  1
## [2,]  3  1  1
## [3,]  5  2  1
```

Solving a System in R ($n \times n$)

```
Xinv <- solve(X)
b <- Xinv %*% y
b
```

```
##      [,1]
## x1      4
## x2      3
## x3      1
```