Math Refresher
Winter Institute in Data Science

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Warming Up

1. Let "$A \circledast B$" be defined as $A^B + A \cdot B$. Calculate $4 \circledast 3$.
2. Solve this system of two linear equations:

$$\begin{aligned} 2x - y &= 4 \\ x + y &= 5 \end{aligned}$$

Functions

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  - Function of two variables: $f : \mathbb{R}^2 \to \mathbb{R}^1$, $f : \mathbb{R}^2 \to \mathbb{R}^2$

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- Input variable: *predictor*, *covariate*, *indep var*
- Output variable: *outcome*, *response*, *dep var*

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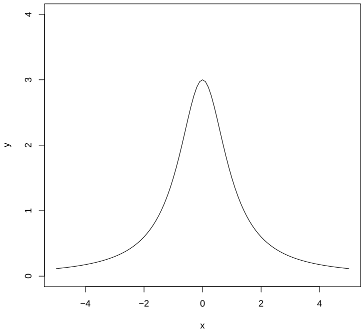

Example 2

$$f(x) = \begin{cases} x+1, & 1 \le x \le 2 \\ 0, & x = 0 \\ 1-x & -2 \le x \le -1 \end{cases}$$

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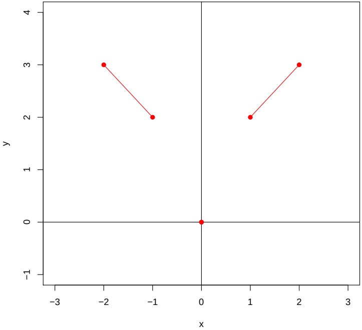

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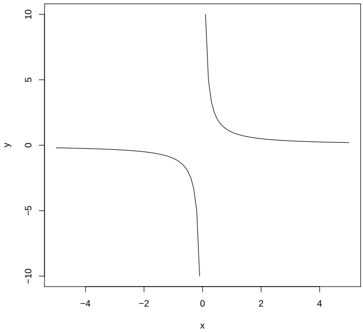

Example 4

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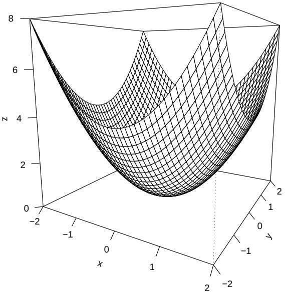

General Types of Functions

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a is the coefficient. k is the degree.

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- **Trigonometric Functions**: Examples: $y = \cos(x)$, $y = 3\sin(4x)$

Trigonometric Functions: Gill & Casella (2004)

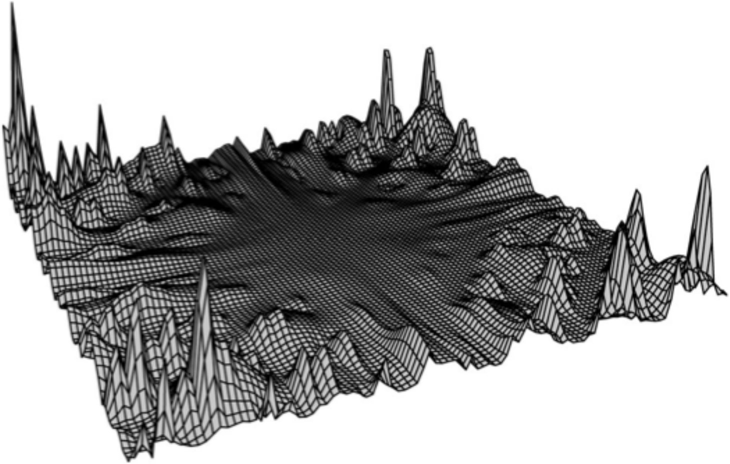


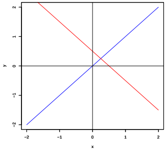
Fig. 1 A highly multimodal surface.

$$\begin{aligned} f(x,y) &= |(x\sin(20y-90) - y\cos(20x+45))^3 a\cos(\sin(90y+42)x) \\ &\quad +(x\cos(10y+10) - y\sin(10x+15))^2 a\cos(\cos(10x+24)y)| \end{aligned}$$

Linear and Nonlinear Functions

- **Linear**: polynomial of degree 1.

Example: $y = mx + b$, where $m =$ slope and $b = y$-intercept.



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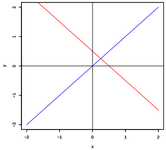

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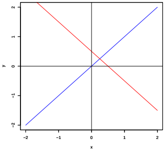

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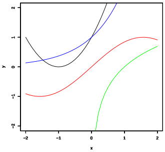

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$$(\log_b P = Q) \iff (b^Q = P)$$

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3. $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$
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- $\sum_{i=2}^{3} (x_i + y_{i-1}) = (-2 + -1) + (3 + 0) = 0$

Summation Notation

Properties:

- $\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$
- $\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$
- $\sum_{i=1}^{n} c = nc$

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- $\prod_{i=2}^{3} (x_i + y_{i-1}) = (-2 + -1) \cdot (3 + 0) = -9$

Product Notation

Properties:

- $\prod_{i=1}^{n} cx_i = c^n \prod_{i=1}^{n} x_i$
- $\prod_{i=1}^{n} (x_i + y_i) = (x_1 + y_1)(x_2 + y_2) \dots$
- $\prod_{i=1}^{n} c = c^n$

Sums, Products, and Logs

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\end{aligned}
$$

Sums, Products, and Logs

Logarithms can help us change products into sums.

$$
\begin{aligned}
\log\left(\prod_{i=1}^{n} cx_i\right) &= \log(cx_1 \cdot cx_2 \cdot \ldots \cdot cx_n) \\
&= \log(cx_1) + \log(cx_2) + \ldots + \log(cx_n) \\
&= \sum_{i=1}^{n} \log(cx_i) \\
&= [(\log c + \log x_1) + (\log c + \log x_2) + \\
&\quad \ldots + (\log c + \log x_n)] \\
&= \sum_{i=1}^{n} \log c + \sum_{i=1}^{n} \log(x_i) \\
&= n \log(c) + \sum_{i=1}^{n} \log(x_i)
\end{aligned}
$$

Vectors

Vectors

- **Vector**: A vector in n-space is an ordered list of n numbers. These numbers can be represented as either a row vector or a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

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- We can also think of a vector as defining a point in n-dimensional space, usually $\mathbf{R}^n$; each element of the vector defines the coordinate of the point in a particular direction.

Vector Arithmetic

- **Vector Addition**: Vector addition is defined for two vectors $\mathbf{u}$ and $\mathbf{v}$ iff they have the same number of elements:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 & u_2 + v_2 & \cdots & u_n + v_n \end{pmatrix}$$

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$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 3+2 & -2+0 & 1+1 \end{pmatrix} = \begin{pmatrix} 5 & -2 & 2 \end{pmatrix}$$

```r
u <- c(3, -2, 1)
v <- c(2, 0, 1)
u + v
```

```
## [1]  5 -2  2
```

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$c\mathbf{v} = \begin{pmatrix} 6 \cdot 3 & 6 \cdot -2 & 6 \cdot 1 \end{pmatrix}$

Vector Arithmetic

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$$c\mathbf{v} = \begin{pmatrix} cv_1 & cv_2 & \dots & cv_n \end{pmatrix}$$

Let $\mathbf{v} = \begin{pmatrix} 3 & -2 & 1 \end{pmatrix}$, $c = 6$.

$$c\mathbf{v} = \begin{pmatrix} 6 \cdot 3 & 6 \cdot -2 & 6 \cdot 1 \end{pmatrix} = \begin{pmatrix} 18 & -12 & 6 \end{pmatrix}$$

```r
c <- 6
v <- c(3, -2, 1)
c * v
```

```
## [1]  18 -12   6
```

Vector Properties

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors, c, d be constants, $\mathbf{0}$ be vector $\mathbf{z}$ s.t. $z_i = 0$.

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- Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- Distributivity: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- Scalar Distributivity: $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

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- Scalar Multiplicative Identity: $1\mathbf{u} = \mathbf{u}$

Transpose

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Let $\mathbf{u} = \begin{pmatrix} 3 & -2 & 1 \end{pmatrix}$ be 1×3. Then, $\mathbf{u}' = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ is 3×1.

Inner Product

- **Inner Product**: The Euclidean inner product (also, the "dot product") of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined iff they have the same number of elements

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

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$\mathbf{u} \cdot \mathbf{v} = 3 \cdot 2 + -2 \cdot 0 + 1 \cdot 1 = 6 + 1 = 7.$

```
u %*% v |> drop() # Simplify array dims
```

```
## [1] 7
```

If $\mathbf{u} \cdot \mathbf{v} = 0$, the vectors are *orthogonal* (or perpendicular).

Let $\mathbf{u} = \begin{pmatrix} 5 & 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 & -2 \end{pmatrix}$.

Inner Product and Orthogonality

If $\mathbf{u} \cdot \mathbf{v} = 0$, the vectors are *orthogonal* (or perpendicular).

Let $\mathbf{u} = \begin{pmatrix} 5 & 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 & -2 \end{pmatrix}$.

$\mathbf{u} \cdot \mathbf{v} = 5 \cdot 0 + 0 \cdot -2 = 0 + 0 = 0.$

Inner Product

- Think about the $\mathbf{u} \cdot \mathbf{v}$ inner product as

$$\underbrace{\mathbf{u}}_{1\times k} \cdot \underbrace{\mathbf{v}}_{k\times 1} = \underbrace{w}_{1\times 1}$$

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$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v}'\mathbf{u} = \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 7$$

Inner Product Properties

- Commutativity: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- Associativity: $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
- Distributivity: $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- Zero Product: $\mathbf{u} \cdot \mathbf{0} = 0$

Vector Norm

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$$\begin{aligned} ||\mathbf{v}|| &= || \begin{pmatrix} 2 & 0 & 1 \end{pmatrix} || \\ &= \sqrt{2^2 + 0^2 + 1^2} \\ &= \sqrt{5} \end{aligned}$$

Vector Norm Properties

- Scalar Multiplication: $||c\mathbf{u}|| = |c| \cdot ||\mathbf{u}||$
- Vector Distance: $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$
- Norm Squared: $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$
- Cosine Rule: $\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||}$
- Difference Norm: $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 - 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2$
  $= ||\mathbf{u}||^2 - 2||\mathbf{u}|| \cdot ||\mathbf{v}||(\cos\theta) + ||\mathbf{v}||^2$
- Triangle Inequality: $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$
- Cauchy-Schwartz Inequality: $||\mathbf{u} \cdot \mathbf{v}|| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$

Mahalanobis Distance

$$MD_{ij} = \sqrt{(\mathbf{x}_i - \mathbf{x}_j)'\Sigma^{-1}(\mathbf{x}_i - \mathbf{x}_j)}$$

Mahalanobis Distance

$$MD_{ij} = \sqrt{(\mathbf{x}_i - \mathbf{x}_j)'\Sigma^{-1}(\mathbf{x}_i - \mathbf{x}_j)}$$

Like Euclidean distance, but scaled by inverse covariances

$$\hat{\Sigma} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{x})'$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i$$

is the vector of sample means.

A Bit of R

```r
a <- c(3, 0, 0)
b <- c(0, 2, 0)
a %*% b  ## inner product
```

```
##      [,1]
## [1,]    0
```

Dependence and Independence

- **Linear combinations**: The vector $\mathbf{u}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ if

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

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- **Linear independence**: A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ is linearly independent if the only solution to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

  is $c_1 = c_2 = \cdots = c_k = 0$. If another solution exists, the set of vectors is linearly dependent.

Linear Dependence

- A set S of vectors is linearly dependent iff at least one of the vectors in S can be written as a linear combination of the other vectors in S.
- Linear independence is only defined for sets of vectors with the same number of elements
- Any linearly independent set of vectors in n-space contains at most n vectors.

Example 1

Are the following sets of vectors linearly independent?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

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Yes. $(c_3 = 0) \Rightarrow (c_2 = 0) \Rightarrow (c_1 = 0)$

Example 2

Are the following sets of vectors linearly independent?

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Example 2

Are the following sets of vectors linearly independent?

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

No. $\mathbf{c} = \begin{pmatrix} 2 & 1 & -2 \end{pmatrix} \qquad \text{(e.g.)}$

Matrix Algebra

Matrices

- **Matrix**: A matrix is an array of mn real numbers arranged in m rows by n columns.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Matrices

- Vectors are special cases of matrices
  - Col vector of length k is a $k \times 1$ matrix
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$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \cdots \\ \mathbf{b}_n \end{pmatrix}$$

Special Matrices

- Identity: $\mathbf{I_n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$
- $\mathbf{J_n} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$
- Zero: $\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$

Special Matrices

- Diagonal: $\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$

- Lower Triangular: $\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

- Upper Triangular: $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$

- Triangular: Either upper triangular or lower triangular

Matrix Equality

- Let $\mathbf{A}$ and $\mathbf{B}$ be two $m \times n$ matrices. Then

$$\mathbf{A} = \mathbf{B}$$

iff

$$a_{ij} = b_{ij}$$

$\forall i \in \{1, \dots, m\}, \quad j \in \{1, \dots, n\}$

Matrix Addition

- Let $\mathbf{A}$ and $\mathbf{B}$ be two $m \times n$ matrices. Then

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

- $\mathbf{A}$ and $\mathbf{B}$ must be same size – *conformable* for addition

Matrix Addition Example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} =$$

Matrix Addition Example

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$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 4 & 4 \\ 6 & 6 & 8 \end{pmatrix}$$

```
A + B
```

```
##      [,1] [,2] [,3]
## [1,]    2    4    4
## [2,]    6    6    8
```

Scalar Multiplication

Scalar Multiplication: Given scalar c, the scalar multiplication $c\mathbf{A}$ is

$$c\mathbf{A} = c\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix}$$

Scalar Multiplication Example

$$c = 2 \qquad \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

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$$c\mathbf{A} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

```r
c <- 2
c * A
```

```
##      [,1] [,2] [,3]
## [1,]    2    4    6
## [2,]    8   10   12
```

Matrix Multiplication

- **Matrix Multiplication**: If $\mathbf{A}$ is $m \times k$ and $\mathbf{B}$ is $k \times n$, then their product $\mathbf{C} = \mathbf{AB}$ is $m \times n$ matrix where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

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- Consider $\mathbf{A}$ to be composed of stacked rows $\mathbf{a}_i = \begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{pmatrix}$,

  $\mathbf{B}$ to be composed of stacked columns $\mathbf{b}_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \dots \\ b_{mj} \end{pmatrix}$.

Then, $\mathbf{AB} = \mathbf{C}$, where

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j$$

Notes on Matrix Multiplication

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- Given $\mathbf{AB}$, say $\mathbf{B}$ is pre-multiplied by $\mathbf{A}$ or $\mathbf{B}$ is left-multiplied by $\mathbf{A}$ or $\mathbf{A}$ is post-multiplied by $\mathbf{B}$ or $\mathbf{A}$ is right-multiplied by $\mathbf{B}$

Warning!

- Commutative law for multiplication does not hold – order of multiplication matters:

$$\mathbf{AB} \neq \mathbf{BA}$$

- **AB** may exist, while **BA** does not.

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

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$$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ -2 & 2 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 1 & 7 \\ -1 & 3 \end{pmatrix}$$

Transpose

- **Transpose**: The transpose of the $m \times n$ matrix $\mathbf{A}$ is $n \times m$ matrix $\mathbf{A}^T$ (or $\mathbf{A}'$) obtained by interchanging rows and columns of $\mathbf{A}$.

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$$(\mathbf{AB})^T =$$

$$\left[\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{pmatrix}\right]^T = \begin{pmatrix} 12 & 7 \\ 5 & -3 \end{pmatrix}$$

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Example Continued

$$\mathbf{B}^T\mathbf{A}^T =$$

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Identity Matrix

The $n \times n$ identity matrix $\mathbf{I}_n$ has diagonal elements $= 1$ and off-diagonal elements $= 0$.

Examples:

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Systems of Linear Equations

Linear Equations

- **Linear Equation**: $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$
  a_i are parameters or coefficients. x_i are variables or unknowns.

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  3. $\mathbb{R}^n$: hyperplane

Systems of Linear Equations

Often interested in solving linear systems like

More Systems of Linear Equations

- More generally, we might have a system of m equations in n unknowns

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
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How many Solutions?

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  3. Infinite solutions: lines coincide.

Matrices to Represent Linear Systems

Matrices are an efficient way to represent linear systems such as

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\end{array}$$

as $\mathbf{Ax} = \mathbf{b}$

Coefficient Matrix

The $m \times n$ **coefficient matrix A** is an array of mn real numbers arranged in m rows by n columns:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Variable & Output Vectors

- The unknown quantities are represented by the vector

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$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$
- The RHS of the linear system is represented by the vector
$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Inverse of a Matrix

- An $n \times n$ matrix $\mathbf{A}$ is *nonsingular* or *invertible* if there exists an $n \times n$ matrix $\mathbf{A}^{-1}$ such that

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- Then, $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$.
- If there is no such $\mathbf{A}^{-1}$, then $\mathbf{A}$ is *singular* or *noninvertible*.

Example of Inverses

- Example: Let

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{pmatrix}$$

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Since

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

we conclude that $\mathbf{B}$ is the inverse of $\mathbf{A}$, $\mathbf{A}^{-1}$, and that $\mathbf{A}$ is nonsingular.

Calculating the Inverse in R

```r
A <- matrix(c(1, 4, 5, 6), 2, 2, byrow = TRUE)
A
```

```
##      [,1] [,2]
## [1,]    1    4
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```r
solve(A)
```

```
##            [,1]        [,2]
## [1,] -0.4285714  0.28571429
## [2,]  0.3571429 -0.07142857
```

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```
##                [,1]         [,2]
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```
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## [2,] -3.330669e-16    1
```

Linear Systems and Inverses

- Matrix representation of a linear system

$$\mathbf{Ax} = \mathbf{b}$$

- If $\mathbf{A}$ is an $n \times n$ matrix,then $\mathbf{Ax} = \mathbf{b}$ is a system of n equations in n unknowns.

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\mathbf{I}_n\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b}
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$$\mathbf{Xb} = \mathbf{y}$$

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  - $\mathbf{X}$ is $n \times k$ matrix of predictor variable data
  - $\mathbf{y}$ is $n \times 1$ vector of outcome variable values
  - $\mathbf{b}$ is $k \times 1$ vector of linear parameters (β's)
- Then

$$\mathbf{Xb} = \mathbf{y}$$

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$$
\begin{aligned}
\mathbf{Xb} &= \mathbf{y} \\
\mathbf{X}'\mathbf{Xb} &= \mathbf{X}'\mathbf{y}
\end{aligned}
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$$\begin{aligned} \mathbf{X}\mathbf{b} &= \mathbf{y} \\ \mathbf{X}'\mathbf{X}\mathbf{b} &= \mathbf{X}'\mathbf{y} \end{aligned}$$

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How to isolate $\mathbf{b}$? Multiply by an inverse.

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Befriend $(\mathbf{X'X})^{\mathbf{-1}}\mathbf{X'y}$.

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Befriend $(\mathbf{X}'\mathbf{X})^{\mathbf{-1}}\mathbf{X}'\mathbf{y}$. If you understand it, its cousins, and their properties (both strengths and weaknesses), your data-analytic future will be bright.

Solving a System in R ($n \times n$)

$$\mathbf{y} = \mathbf{X}\mathbf{b}$$

```r
x1 <- c(1, 3, 5)
x2 <- c(3, 1, 2)
x3 <- c(1, 1, 1)
y <-  4 * x1 + 3 * x2 + x3
(X <- cbind(x1, x2, x3))
```

```
##      x1 x2 x3
## [1,]  1  3  1
## [2,]  3  1  1
## [3,]  5  2  1
```

Solving a System in R ($n \times n$)

```
Xinv <- solve(X)
b <- Xinv %*% y
b
```

```
##    [,1]
## x1    4
## x2    3
## x3    1
```