# Everything You Need To Know About the Linear Model

JEFF GILL
Distinguished Professor
Departments of Government and Mathematics & Statistics
*American University*

## Why We Need to Start with the Linear Model For This Course

- We are interested in Bayesian Hierarchical Models.

- Start with a standard linear model specification indexed by subjects and a first level of grouping, the *context* level.

- Now use a single explanatory variable that has the form:

$$y_i = \beta_{j0[i]} + \beta_{j1[i]} X_i + \epsilon_i.$$

- Add a second level to the model that explicitly nests effects within groups and index these groups $j = 1$ to $J$:

$$\beta_{j0} = \gamma_{00} + \gamma_{10} Z_{j0} + u_{j0}$$

$$\beta_{j1} = \gamma_{01} + \gamma_{11} Z_{j1} + u_{j1},$$

  where all individual level variation is assigned to groups producing department level residuals: $u_{j0}$ and $u_{j1}$.

- These $Z_{j\cdot}$ are group-level variables in that their effect is assumed to be measured at the aggregated rather than at the individual level.

## Why We Need to Start with the Linear Model For This Course

- The two-level model is produced by inserting the context level specifications into the original linear expression for the outcome variable of interest:

$$y_i = \gamma_{00} + \gamma_{01}X_i + \gamma_{10}Z_{j0} + \gamma_{11}X_iZ_{j1} + u_{j1}X_i + u_{j0} + \epsilon_i.$$

- This equation shows that the composite error structure, $u_{j1}X_i + u_{j0} + \epsilon_i$, is now clearly heteroscedastic since it is conditioned on levels of the explanatory variable, causing additional estimation complexity.
- Notice that there is an "automatic" interaction component: $\gamma_{11}X_iZ_{j1}$.
- Now we are going model *distributions* for $y$, $\beta_{j0}$, and $\beta_{j1}$.
- Thus it is important to review the linear regression model in some detail.
- Even though this lecture proves or derives every quantity of interest, it is assumed that you've seen linear regression models before.

## Precursor To Linear Models: Following Trends

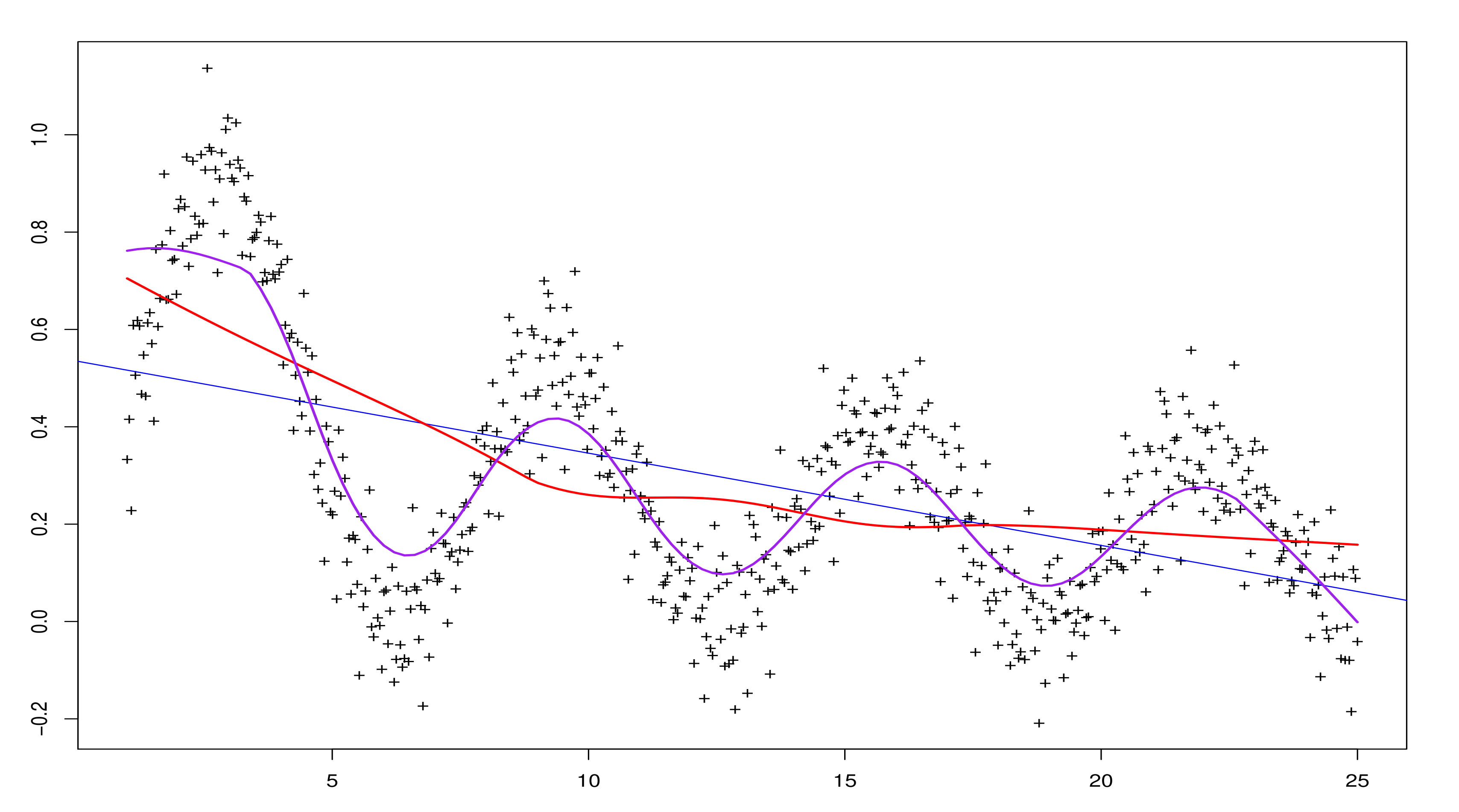
- Sometimes trends are obvious and easy to follow in data, but often they are not.
- Two standard tools: smoothing and linear regression.
- Usually one *or* the other is appropriate.
- Smoothers simply follow the trends in the data, with a given smoothing parameter.
- Main smoother: lowess, "locally weighted running line smoother."

- *Is it possible for a linear model result to look like it fits when it is the wrong specification?*

# Running Lowess

```
x <- seq(1,25,length=600)
y <- (2/(pi*x))^(0.5)*(1-cos(x)) + rnorm(100,0,1/10)
summary(lm(y~x))$coef
             Estimate Std. Error t value    Pr(>|t|)
(Intercept)  0.538054   0.019850  27.105 3.8061e-106
x           -0.018702   0.001347 -13.884  3.4425e-38

postscript("Class.Multilevel/trends1.ps")
par(mar=c(3,3,2,2), bg="white")
plot(x,y,pch="+")
ols.object <- lm(y~x)
abline(ols.object,col="blue")
lo.object <- lowess(y~x,f=2/3)
lines(lo.object$x,lo.object$y,lwd=2,col="red")
lo.object <- lowess(y~x,f=1/5)
lines(lo.object$x,lo.object$y,lwd=2,col="purple")
dev.off()
```

# Running Lowess

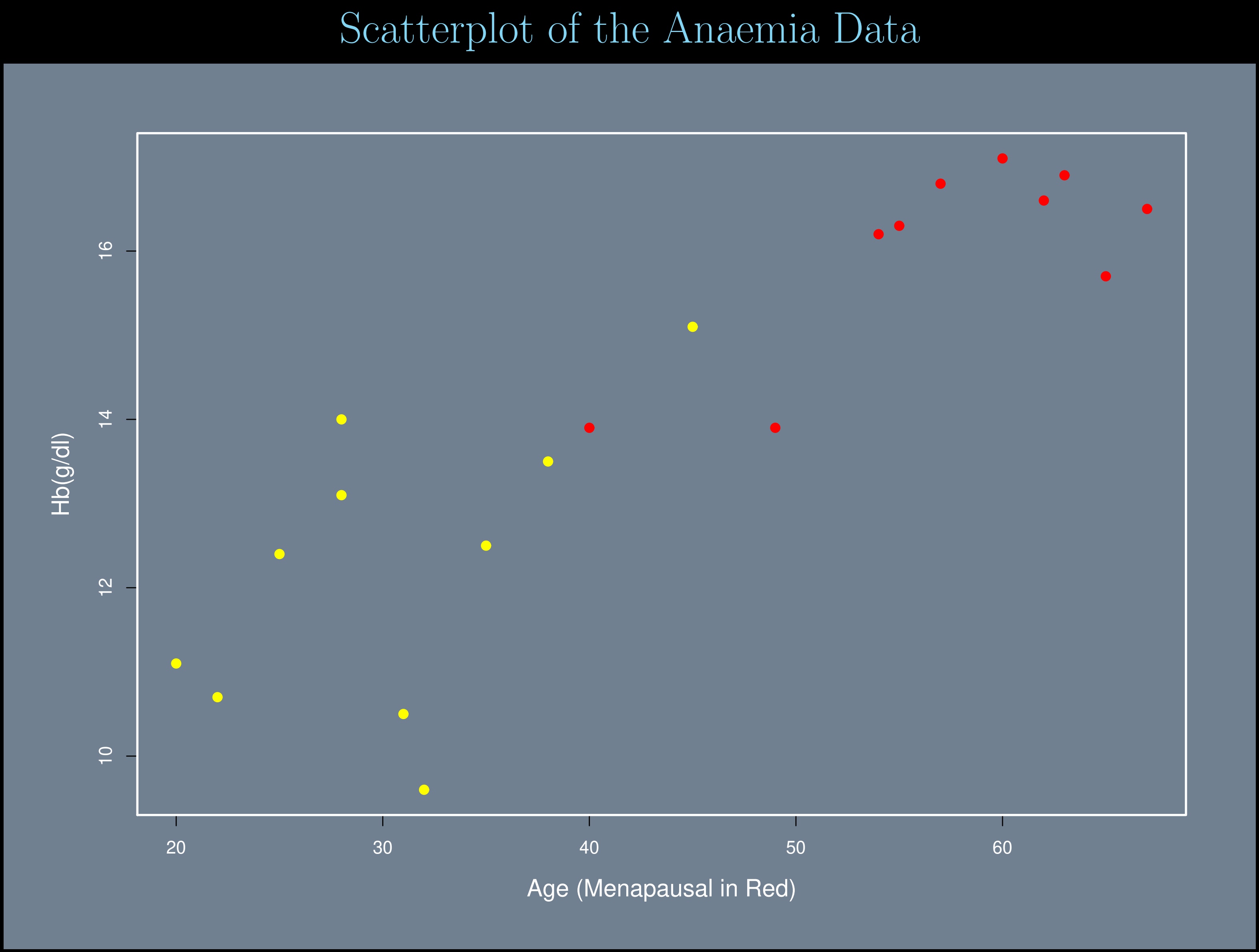


## What Does the Linear Model Get You? An Example

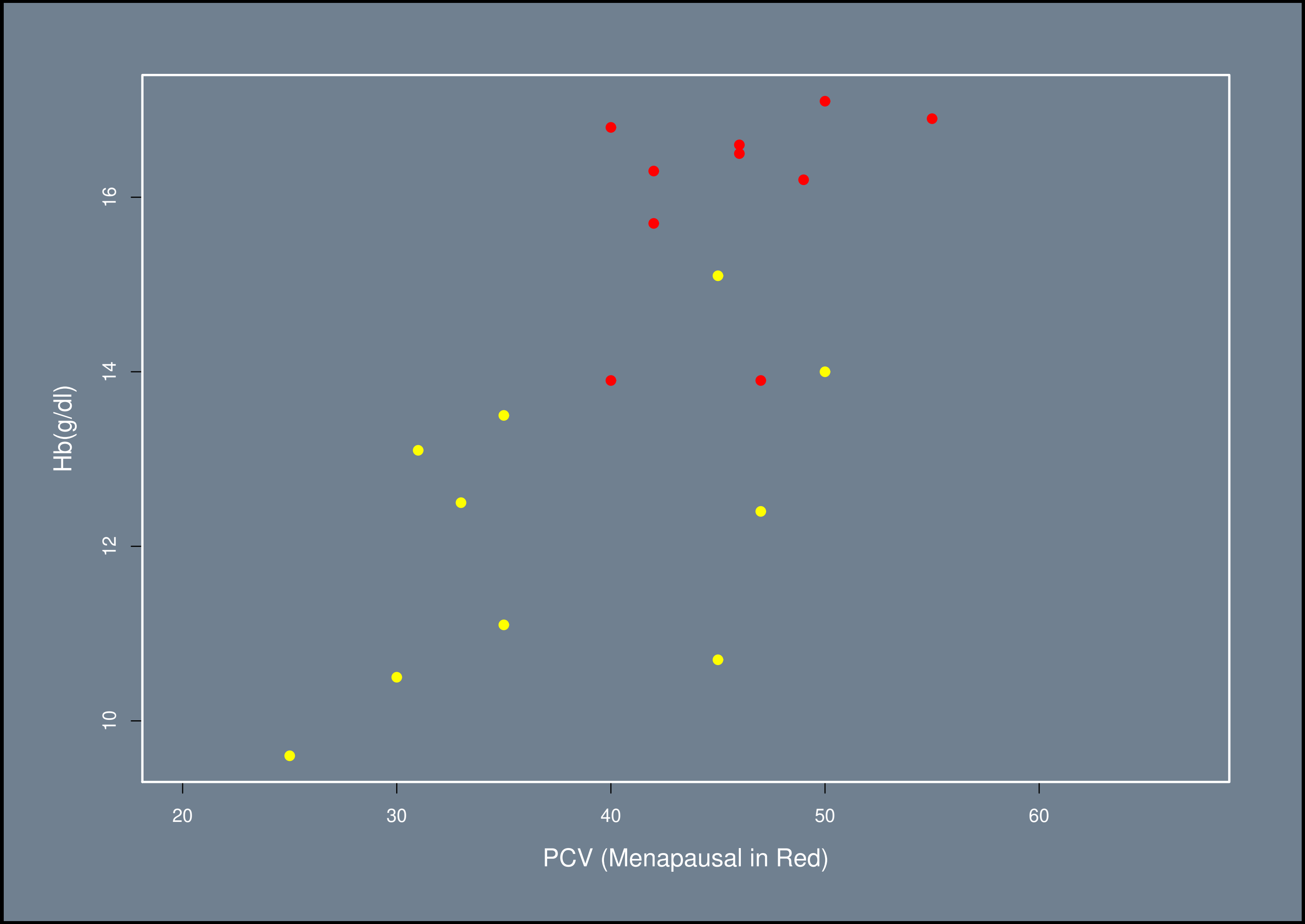
- Consider a study of anaemia in women in a given clinic, perhaps in St. Louis, where 20 cases are chosen at random from the full study to get the data here.
- From a blood sample we get:
  - haemoglobin level (Hb) in grams per deciliter (12–15 g/dl is normal in adult females)
  - packed cell volume (hematocrit) in percent (38% to 46% is normal in adult females)
- We also have:
  - age in years
  - menopausal status $(0 = no, 1 = yes)$
- There is an obvious endogeneity problem in modeling Hb(g/dl) versus PCV(%).

## What Does the Linear Model Get You? An Example

| Subject | Hb(g/dl) | PCV(%) | Age | Menopausal |
|---|---|---|---|---|
| 1 | 11.1 | 35 | 20 | 0 |
| 2 | 10.7 | 45 | 22 | 0 |
| 3 | 12.4 | 47 | 25 | 0 |
| 4 | 14.0 | 50 | 28 | 0 |
| 5 | 13.1 | 31 | 28 | 0 |
| 6 | 10.5 | 30 | 31 | 0 |
| 7 | 9.6 | 25 | 32 | 0 |
| 8 | 12.5 | 33 | 35 | 0 |
| 9 | 13.5 | 35 | 38 | 0 |
| 10 | 13.9 | 40 | 40 | 1 |
| 11 | 15.1 | 45 | 45 | 0 |
| 12 | 13.9 | 47 | 49 | 1 |
| 13 | 16.2 | 49 | 54 | 1 |
| 14 | 16.3 | 42 | 55 | 1 |
| 15 | 16.8 | 40 | 57 | 1 |
| 16 | 17.1 | 50 | 60 | 1 |
| 17 | 16.6 | 46 | 62 | 1 |
| 18 | 16.9 | 55 | 63 | 1 |
| 19 | 15.7 | 42 | 65 | 1 |
| 20 | 16.5 | 46 | 67 | 1 |


Scatterplot of the Anaemia Data
16
14
12
10
Hb(g/dl)
20
30
40
50
60
Age (Menapausal in Red)

# Scatterplot of the Anaemia Data



## Scatterplot of the Anaemia Data

```
postscript("Class.PreMed.Stats/Images/anaemia2.fig.ps")
par(mfrow=c(1,1),mar=c(5,5,2,2),lwd=2,col.axis="white",col.lab="white",
    col.sub="white",col="white",bg="slategray", cex.lab=1.3)
plot(anaemia$PCV[anaemia$Menapause==0],anaemia$Hb[anaemia$Menapause==0],
        pch=19,col="yellow", xlim=range(anaemia$Age),ylim=range(anaemia$Hb),
        xlab="PCV (Menapausal in Red)",ylab="Hb(g/dl)")
points(anaemia$PCV[anaemia$Menapause==1],anaemia$Hb[anaemia$Menapause==1],
        pch=19,col="red")
dev.off()
```

## What Does the Linear Model Get You? An Example

```
anaemia <-
    read.table("https://jeffgill.org/files/jeffgill/files/anaemia.txt",
    header=TRUE,row.names=1)
a.lm.out <- lm(Hb ~ Age + PCV, data=anaemia)
summary(a.lm.out)

Residuals:
   Min     1Q Median     3Q    Max
-1.600 -0.676  0.216  0.558  1.759

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   5.2388     1.2064    4.34  0.00044
Age           0.1104     0.0164    6.74  3.5e-06
PCV           0.0971     0.0326    2.98  0.00847

Residual standard error: 0.979 on 17 degrees of freedom
Multiple R-squared: 0.851,   Adjusted R-squared: 0.834
F-statistic: 48.6 on 2 and 17 DF,  p-value: 9.26e-08
```

## What Happens When it Doesn't Work?

- The New York Times Magazine, August 7, 2011, page 13.
- 24 countries: average survey review of restaurant service quality and a tipping index from three travel etiquette web sites.
- The data

| Country | Quality | Tip | Country | Quality | Tip |
|---|---|---|---|---|---|
| Japan | 4.4 | 0.00 | Thailand | 3.9 | 0.03 |
| Canada | 3.7 | 0.16 | New_Zealand | 3.7 | 0.07 |
| UAE | 3.6 | 0.10 | Germany | 3.6 | 0.08 |
| USA | 3.6 | 0.18 | South_Africa | 3.5 | 0.11 |
| Australia | 3.4 | 0.08 | Argentina | 3.4 | 0.10 |
| Morocco | 3.4 | 0.07 | Turkey | 3.4 | 0.08 |
| India | 3.3 | 0.10 | Brazil | 3.3 | 0.07 |
| Vietnam | 3.2 | 0.05 | England | 3.2 | 0.10 |
| Greece | 3.2 | 0.08 | Spain | 3.1 | 0.08 |
| France | 3.1 | 0.08 | Italy | 3.0 | 0.07 |
| Egypt | 3.0 | 0.08 | Mexico | 3.0 | 0.13 |
| China | 2.9 | 0.03 | Russia | 1.7 | 0.10 |

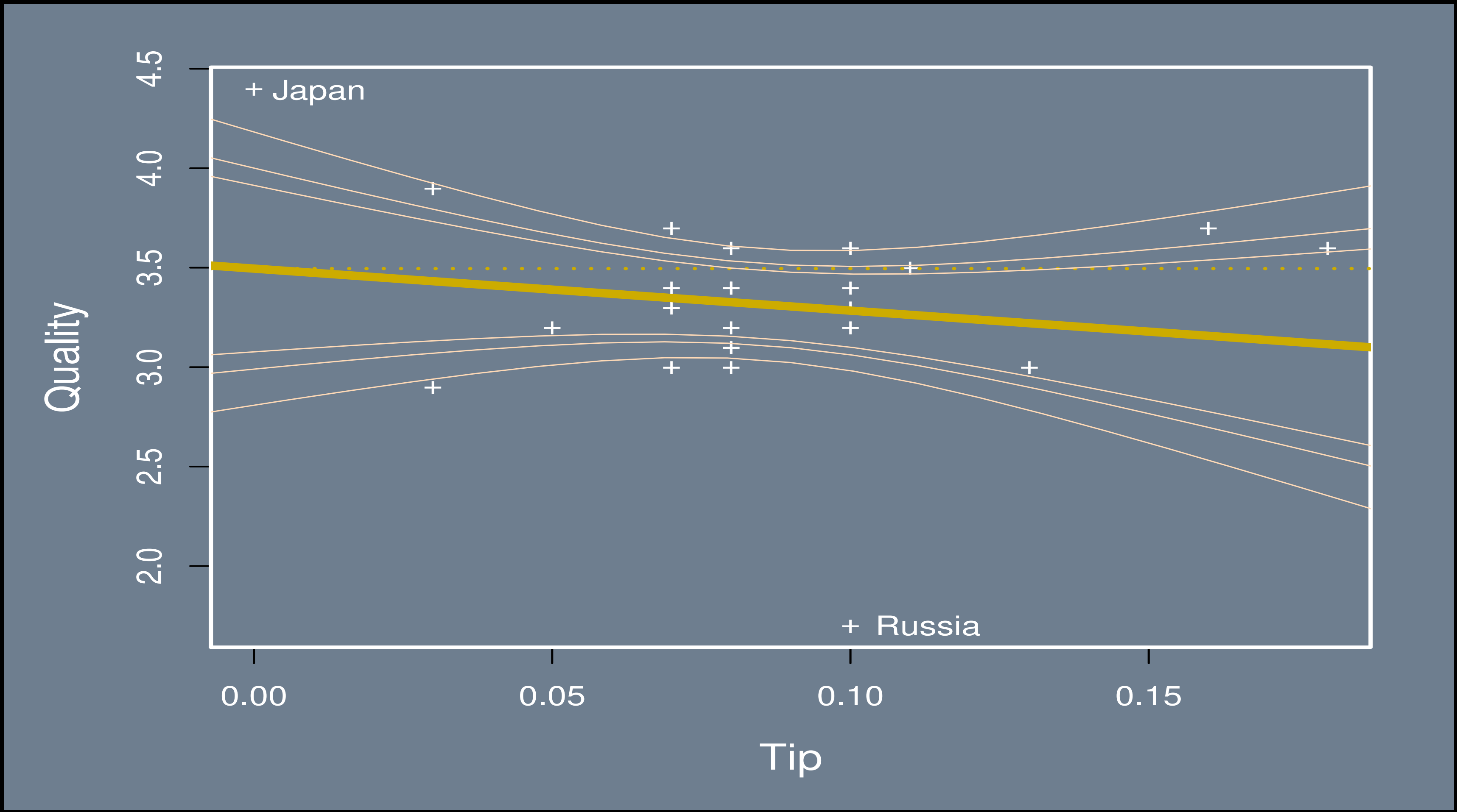
## What Happens When it Doesn't Work?

```
service <-
    read.table("https://jeffgill.org/files/jeffgill/files/service.dat_.txt",
    header=TRUE,row.names=1)
service.lm <- lm(Quality ~ Tip, data=service)
source("./Class.MLE/graph.summary.R")
graph.summary(service.lm)

Family: gaussian
Link function: identity

              Coef Std.Err. 0.95 Lower 0.95 Upper   CIs:  ZE+RO
(Intercept)  3.495    0.244      3.018      3.973             |o|
Tip         -2.113    2.632     -7.272      3.046  |-----o-----|

N: 24    Estimate of Sigma: 0.485
```

## What Happens When it Doesn't Work?

```
postscript("Class.Multilevel/Images/tipping.ps",height=5,width=7)
par(mfrow=c(1,1),mar=c(5,5,2,2),lwd=2,col.axis="white",col.lab="white",
    col.sub="white",col="white",bg="slategray", cex.lab=1.3)
# PLOT POINTS AND REGRESSION LINES
plot(service$Tip, service$Quality, pch="+",xlab="Tip",ylab="Quality")
abline(service.lm,col="gold3",lwd=5)
abline(h=service.lm$coef[1],col="gold3",lty=3,lwd=2)
# ADD CONFIDENCE BOUNDS AT THREE LEVELS
ruler.df <- data.frame(Tip = seq(-0.1, 2,length=200))
for (k in c(0.99,0.95,0.90))  {
    confidence.interval <- predict(service.lm, ruler.df, interval="confidence",
        level=k)
    lines(ruler.df[,1],confidence.interval[,2],col="peachpuff",lwd=0.75)
    lines(ruler.df[,1],confidence.interval[,3],col="peachpuff",lwd=0.75)
}
# IDENTIFY POTENTIAL OUTLIERS
text(0.113,1.7,"Russia")
text(0.011,4.38,"Japan")
dev.off()
```

# What Happens When it Doesn't Work?



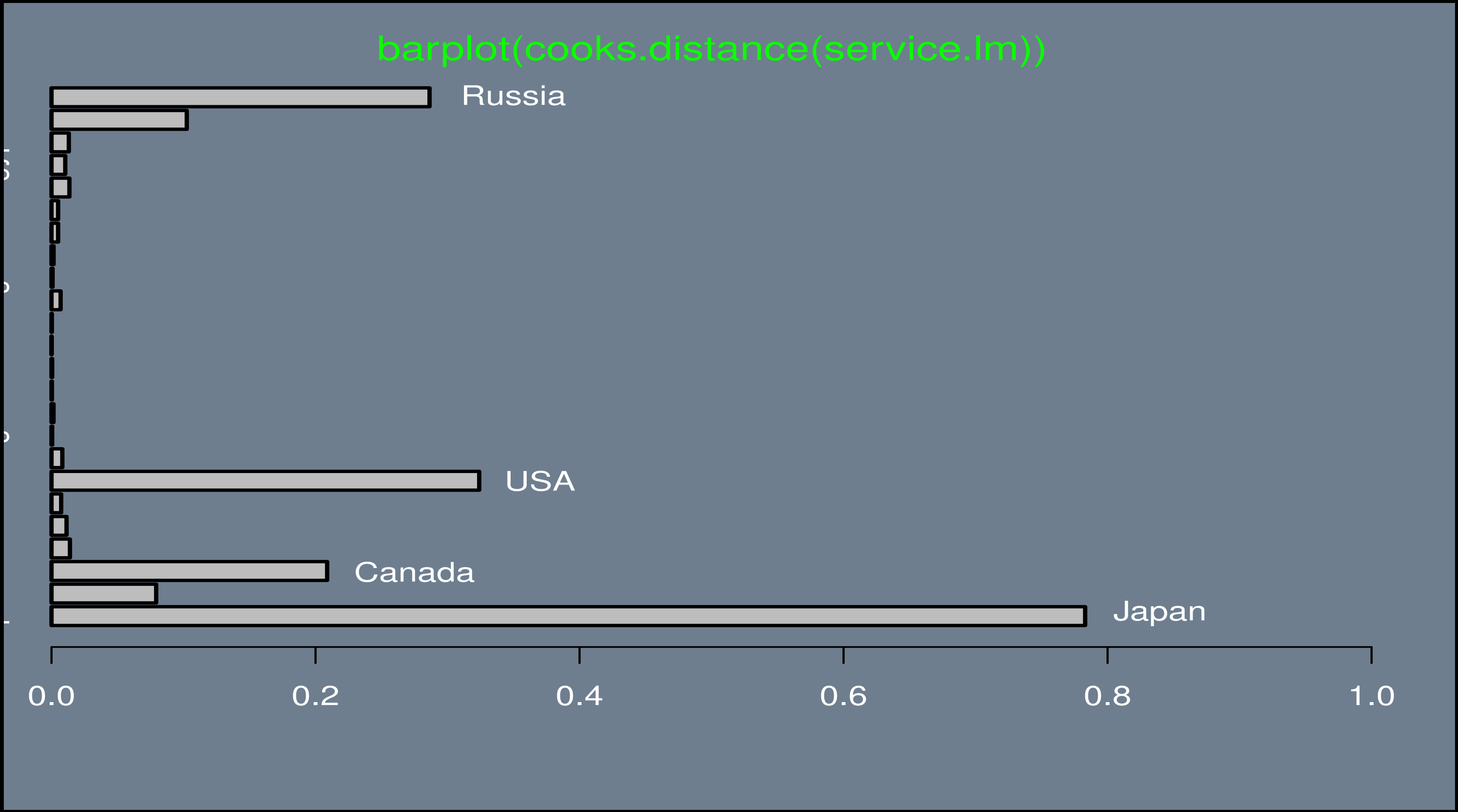
## Are Some Cases More Influential in LM Results

- Leverage is defined by the ability to be influential and is related to the distance that a point is from $\bar{X}$ on the x-axis.
  - Define the Hat Matrix as $H = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ (more on this later), which is square and symmetric, and $\mathbf{X}$ is the matrix containing the explanatory variables.
  - The the diagonal value $h_{ii}$ is the potential of the $i$th point to to influential, which is its leverage.
- Influence is defined as realized leverage, meaning how much does a case with large leverage exert influence over the estimated coefficients relative to other cases.
  - The most common way to measure influence is *Cook's Distance*:

$$D_i = \frac{e_i^2}{ks^2}\left(\frac{h_{ii}}{(1-h_{ii})^2}\right)$$

    where: $e_i$ is the $i$th residual, $k$ is the number of columns in the $\mathbf{X}$ matrix, and $s^2 = \mathbf{e}'\mathbf{e}/(n-k)$ is the MSE of the regression.
  - The function in `R` is `cooks.distance()`.
- This is getting just a little ahead of ourselves here, so consider it a preview.

# How Influential is Japan?



## Bi-Directional Symmetric Bivariate Relationships

- It is useful to start with bivariate regression where the quantities calculated are in scalar rather than matrix notation.

- Suppose the vectors $\mathbf{X}$ and $\mathbf{Y}$ have equal standard deviation, denoted $s$ (obviously a special case).

- Then the regression of $\mathbf{Y}$ on $\mathbf{X}$ (`y ~ x`) has the same slope as the regression of $\mathbf{X}$ on $\mathbf{Y}$ (`x ~ y`) since:
$$\hat{\beta}_{y|x} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(y_i - \bar{y})^2} = \frac{\text{Cov}(X,Y)}{s^2} = \hat{\beta}_{x|y}$$
which is interesting in its own right.

- But now recall that the correlation coefficient is:
$$cor(X,Y) = r_{x,y} = \frac{\text{Cov}(X,Y)}{s_X s_Y}$$
which is the equation above rescaled for different standard deviations and letting $(n-1)$ in the numerator and denominator cancel each other since:
$$\text{Cov}(X,Y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{n-1} \qquad s_X = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}} \qquad s_Y = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}}$$

## Correlation and Regression

- So the above point means that regression *is* correlation, since:

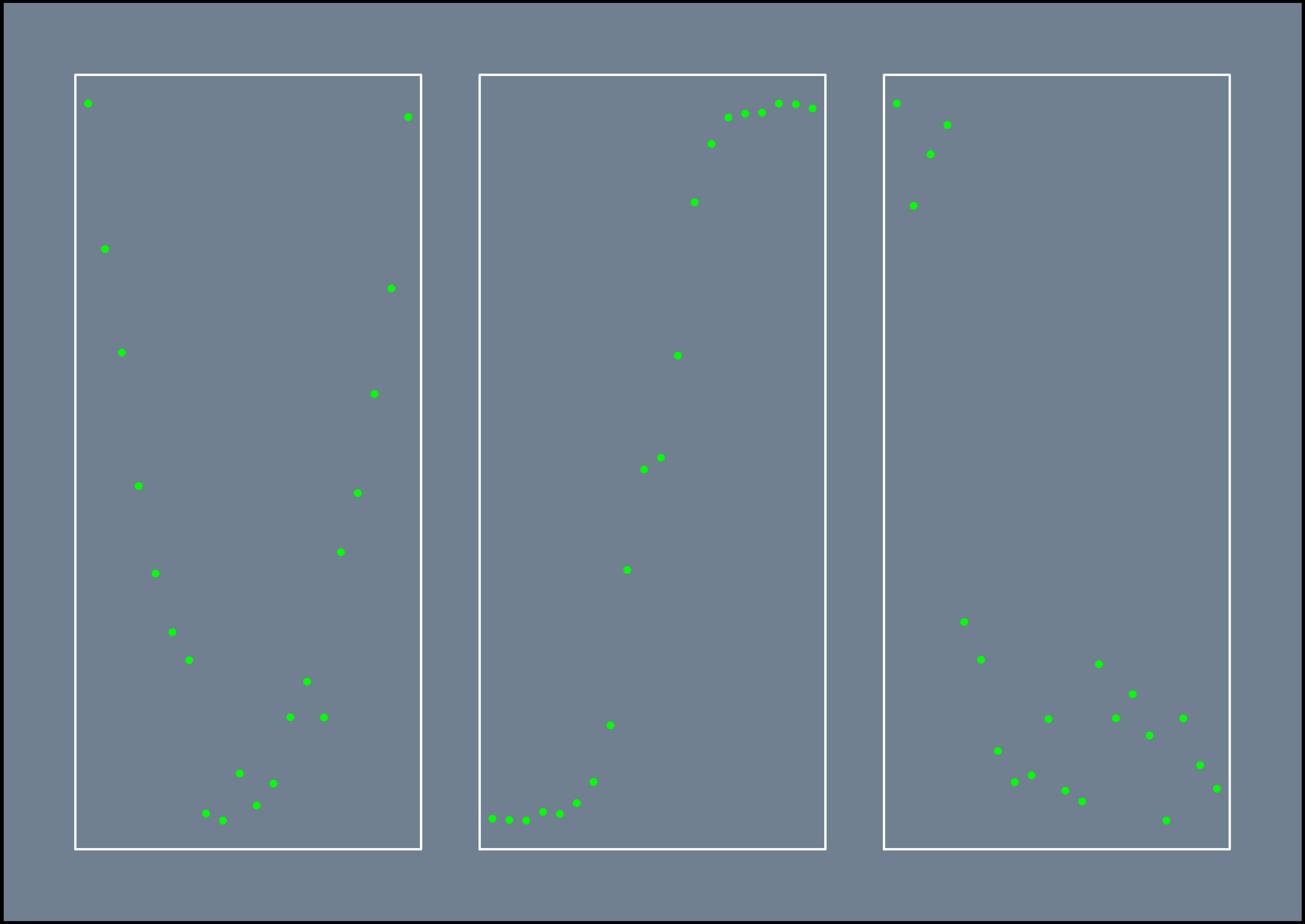
$$cor(X,Y) = \frac{s_X}{s_Y}\hat{\beta}_{y|x}$$

  in the more general case where $s_X \neq s_Y$(and they are identical in the special case).

- From our anaemia example picking `Age`:

```
coef(a.lm.out)[2]
0.1342515
apply(anaemia,2,sd)
        Hb        PCV        Age  Menapause
 2.4018852  7.8958683 15.7366485  0.5129892

cor(anaemia[,1],anaemia[,3])
[1] 0.8795875

(15.7366485/2.4018852) * 0.1342515
[1] 0.8795877
```

## When Not To Use Correlation



## Correlation: Tests of Significance

- Hypotheses: $H_0\!:\!\rho = 0$ versus $H_1\!:\!\rho \neq 0$.
- Test statistic:

$$t = r/SE(r), \qquad\qquad SE(r) = \sqrt{\frac{1-r^2}{n-2}}.$$

- HB and PCV from the anaemia data:

$$r = 0.6733745 \qquad SE(r) = \sqrt{\frac{1-0.6733745^2}{20-2}} = 0.1742551 \qquad t = 3.864304 \qquad p \approx 0.001.$$

- HB and Age from the anaemia data:

$$r = 0.8795875 \qquad SE(r) = \sqrt{\frac{1-0.6733745^2}{20-2}} = 0.1121323 \qquad t = 7.844191 \qquad p \approx 0.0001.$$

## Data Structures in Matrix/Vector Form

- The vector $\mathbf{Y}$ contains values of the outcome variable in a column vector:

$$\mathbf{Y} = [y_1, y_2, \ldots, y_n]'$$

- The matrix $\mathbf{X}$ contains the explanatory variables down the columns:

$$\mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \ldots & x_{1k} \\ 1 & x_{21} & x_{22} & \ldots & x_{2k} \\ 1 & x_{31} & x_{32} & \ldots & x_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \ldots & x_{nk} \end{bmatrix},$$

  with a leading column of $1$s (e.g. there are $k-1$ explanatory variables).

- The vector $\mathbf{e}$ contains values of the observed residuals (disturbances, errors) in a column vector:

$$\mathbf{e} = [e_1, e_2, \ldots, e_n]'$$

## Gauss-Markov Assumptions for Classical Linear Regression

- Functional Form: $\underset{(n\times 1)}{\mathbf{Y}} = \underset{(n\times k)(k\times 1)}{\mathbf{X}\boldsymbol{\beta}} + \underset{n\times 1}{\boldsymbol{\epsilon}}$ (recall $\mathbf{X}$ has a leading column of $1$'s)
- Mean Zero Errors: $\mathbb{E}[\boldsymbol{\epsilon}] = \mathbf{0}$
- Homoscedasticity: $\text{Var}[\boldsymbol{\epsilon}] = \sigma^2\mathbf{I}$
- Non-Correlated Errors: $\text{Cov}[\epsilon_i, \epsilon_j] = 0, \quad \forall i \neq j$
- Exogeneity of Explanatory Variables: $\text{Cov}[\epsilon_i, \mathbf{X}] = 0, \qquad \forall i$

▷ Note that every one of these lines has $\boldsymbol{\epsilon}$ in it, meaning that these are assumptions about the underlying population values.

## Other Considerations

- Requirements:
  - conformability of matrix/vector objects
  - $\mathbf{X}$ has full rank $k$, so $\mathbf{X}'\mathbf{X}$ is invertible (non-zero determinant, nonsingular)
  - identification condition: not all points lie on a vertical line.
- Freebee: eventual normality... $\boldsymbol{\epsilon}|\mathbf{X} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$.
- Toughness: the linear model is both *robust* to minor violations of the Gauss-Markov assumptions and *resistant* to outlying values.

## Estimation With OLS:

- Define the following function:

$$\begin{aligned} S(\boldsymbol{\beta}) &= \boldsymbol{\epsilon}'\boldsymbol{\epsilon} \\ &= (\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) \\ &= \underset{(1\times n)(n\times 1)}{\mathbf{Y}'\mathbf{Y}} - \underset{(1\times n)(n\times k)(k\times 1)}{2\mathbf{Y}'\mathbf{X}\boldsymbol{\beta}} + \underset{(1\times k)(k\times n)(n\times k)(k\times 1)}{\boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta}} \end{aligned}$$

- Take the derivative of $S(\boldsymbol{\beta})$ with respect to $\boldsymbol{\beta}$:

$$\frac{\partial}{\partial \boldsymbol{\beta}} S(\boldsymbol{\beta}) = 0 - 2 \underset{(k\times n)(n\times 1)}{\mathbf{X}'\mathbf{Y}} + \underset{(k\times n)(n\times k)(k\times 1)}{2\mathbf{X}'\mathbf{X}\boldsymbol{\beta}} \equiv 0,$$

  (think about what sign you would get by taking another derivative).

- So there exists a (minimizing) solution at some value $\hat{\boldsymbol{\beta}}$ (or notationally $\hat{\boldsymbol{\beta}}$) of $\boldsymbol{\beta}$: $\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{Y}$ which is the Normal Equation.

- Premultiplying the Normal Equation by $(\mathbf{X}'\mathbf{X})^{-1}$, gives: $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$, where we can call $\hat{\boldsymbol{\beta}}$ as $\hat{\boldsymbol{\beta}}$ for notational convenience (this is where the requirement for $\mathbf{X}'\mathbf{X}$ to be nonsingular comes in).

## OLS Estimator Notes

- Another way to express the OLS estimator is to say that we want the $\boldsymbol{\beta}$ that minimizes the *squared prediction error*:
$$S(\boldsymbol{\beta}) = \mathbb{E}[(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})^2],$$
which is a restatement of $S(\boldsymbol{\beta}) = \boldsymbol{\epsilon}'\boldsymbol{\epsilon}$.

- Sometimes the solution is called the *linear projection coefficient*:
$$\hat{\boldsymbol{\beta}} = \underset{\mathbf{b}\in\Re}{\text{argmin}} S(\mathbf{b}).$$

- And yet another expression for this quantity is:
$$\hat{\boldsymbol{\beta}} = \mathbb{E}(\mathbf{XY}) \left(\mathbb{E}(\mathbf{XX}')\right)^{-1}$$
meaning that:
$$\hat{\mathbf{Y}} = \mathbf{X}\mathbb{E}(\mathbf{XY}) \left(\mathbb{E}(\mathbf{XX}')\right)^{-1}.$$

## Estimation With MLE:

- Assume: $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N(0, \boldsymbol{\Sigma}), \quad \boldsymbol{\Sigma} = \sigma^2 I$
  (notice that normality is an *added* assumption here, since MLE calculations require a distribution to work with).

- The likelihood function for iid $\boldsymbol{\epsilon}$:

$$L(\boldsymbol{\epsilon}) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}\boldsymbol{\epsilon}'\boldsymbol{\epsilon}\right].$$

- Plug-in: $\epsilon_i = y_i - \mathbf{X}_i\boldsymbol{\beta}$ (where $\boldsymbol{\beta}$ to be estimated):

$$L(\boldsymbol{\beta}) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})\right].$$

- Which in log-likelihood form is:

$$\ell(\boldsymbol{\beta}) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}).$$

## Estimation With MLE:

- Now take the first derivative with respect to $\boldsymbol{\beta}$ and set it equal to zero:

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{\beta}}\ell(\boldsymbol{\beta}) = -\frac{1}{2\sigma^2}(-\mathbf{X})'(2)(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) \equiv 0 \\
&\mathbf{X}'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) = 0 \\
&\mathbf{X}'\mathbf{Y} - \mathbf{X}'\mathbf{X}\boldsymbol{\beta} = 0 \\
&\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \mathbf{X}\mathbf{Y} \qquad\qquad \text{the ``normal'' equation} \\
&\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}
\end{aligned}$$

- We can also take the first derivative with respect to $(\sigma^2)$ and set it equal to zero:

$$\begin{aligned}
\frac{\partial}{\partial \sigma^2}\ell(\sigma^2) &= -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2}\frac{1}{\sigma^4}(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}) \\
&= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\boldsymbol{\epsilon}'\boldsymbol{\epsilon} \equiv 0 \\
0 &= -n\sigma^2 + \boldsymbol{\epsilon}'\boldsymbol{\epsilon} \\
\hat{\sigma^2} &= \frac{\boldsymbol{\epsilon}'\boldsymbol{\epsilon}}{n}
\end{aligned}$$

which is slightly biased in finite samples for $\boldsymbol{\epsilon}'\boldsymbol{\epsilon}/(n-k)$, more on this later.

## Implications

- Normal Equation: $\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} - \mathbf{X}'\mathbf{Y} = -\mathbf{X}'(\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}) = -\mathbf{X}'\mathbf{e} \equiv \mathbf{0}$ (by assumption)
- Summation of errors: $\sum e_i \approxeq 0$
- The regression hyperplane passes through the mean vectors: $\bar{\mathbf{Y}} = \bar{\mathbf{X}}\hat{\boldsymbol{\beta}}$
- Equivalence of means: $\text{mean}(\hat{\mathbf{Y}}) = \text{mean}(\mathbf{Y})$
- The hat matrix with rank and trace $k$ ($\mathbf{H}$, $\mathbf{P}$, or $(\mathbf{I} - \mathbf{M})$) starts with:

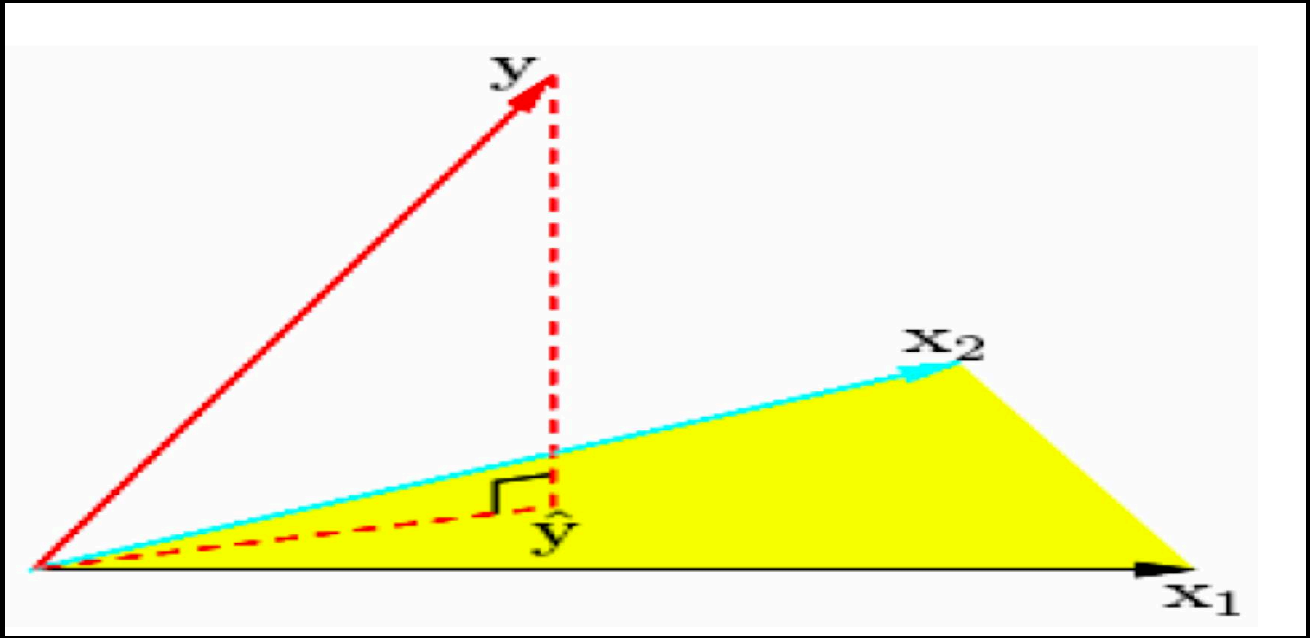
$$
\begin{aligned}
\mathbf{e} &= \mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}} \\
&= \mathbf{Y} - \mathbf{X}((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}) \\
&= \mathbf{Y} - (\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{Y} \\
&= \mathbf{Y} - \mathbf{H}\mathbf{Y} \\
&= (\mathbf{I} - \mathbf{H})\mathbf{Y} \\
&= \mathbf{M}\mathbf{Y}
\end{aligned}
$$

  where $\mathbf{M}$ and $\mathbf{H}$ are symmetric and idempotent.

- For example: $\mathbf{H}\cdot\mathbf{H} = (\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') = \mathbf{X}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}](\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$.

# The HAT Matrix

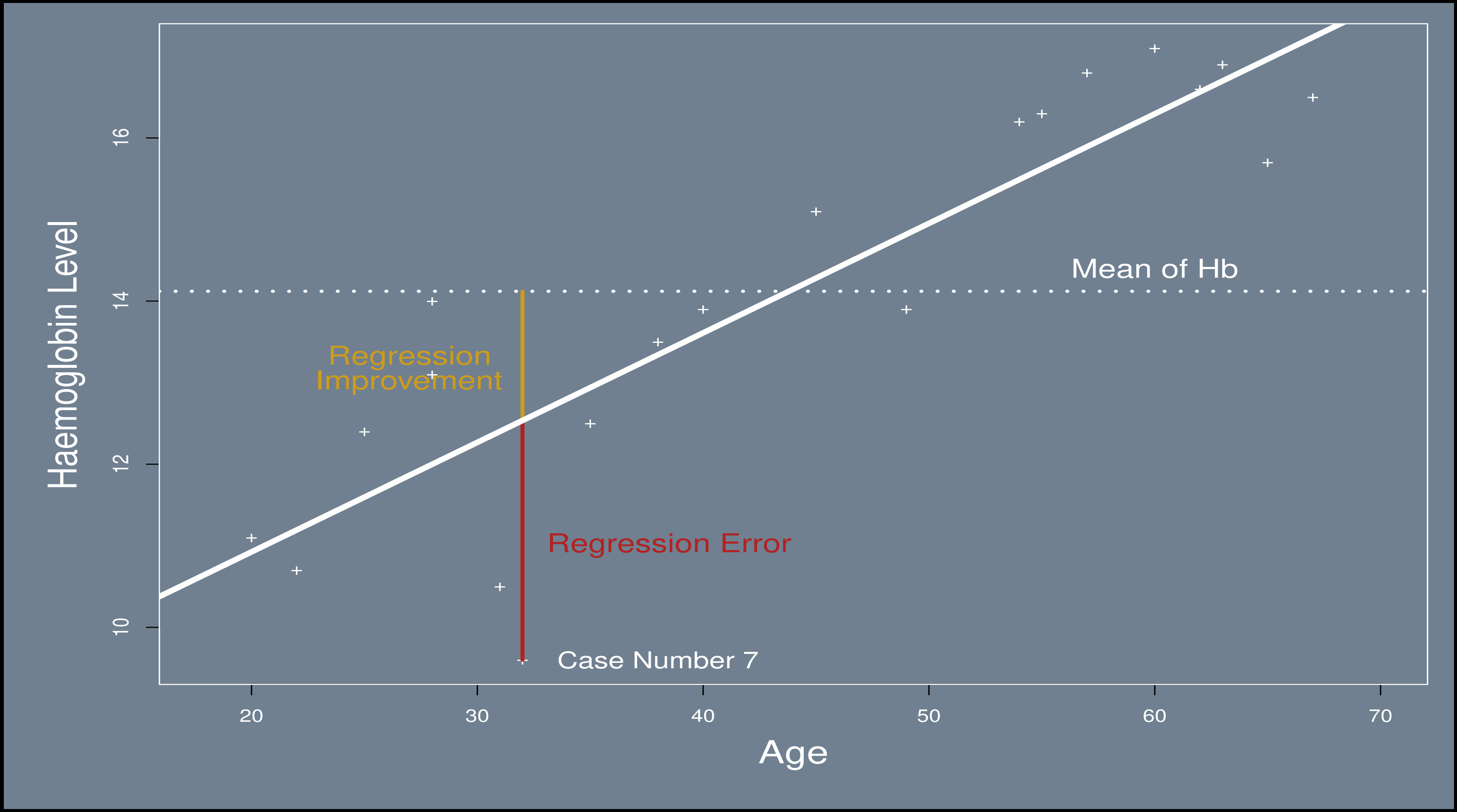
- The name is because $\hat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}) = (\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{Y} = \mathbf{HY}$, but "projection matrix" ($\mathbf{P}$) is better for geometric reasons.



## The HAT Matrix

- Related properties of interest:
  - $\mathbf{I} - \mathbf{M} = \mathbf{P}$, $\mathbf{I} - \mathbf{P} = \mathbf{M}$
  - $\mathbf{PX} = \mathbf{X}$ (an orthogonal projection onto $\mathbf{X}$)
  - $\mathbf{PM} = \mathbf{MP} = \mathbf{0}$ and $\mathbf{P}(\mathbf{I} - \mathbf{P}) = 0$ (orthogonality)
  - $\mathbf{e}'\mathbf{e} = \mathbf{Y}'\mathbf{M}'\mathbf{MY} = \mathbf{Y}'\mathbf{MY} = \mathbf{Ye}$ (sum of squares)
  - $\mathbf{Y}'\mathbf{Y} = \hat{\mathbf{Y}}'\hat{Y} + \boldsymbol{\epsilon}'\boldsymbol{\epsilon}$ (the Pythagorean Theorem!)
- Interestingly, the hat matrix also comes up in some nonlinear model theory.

# Fit & Decomposition, Illustration



## Fit & Decomposition, Variability Definitions

- *Sum of Squares Total*, all the variability to obtain over the mean estimate,

$$\text{SST} = \sum_{i=1}^{n}(\mathbf{Y}_i - \bar{\mathbf{Y}})^2$$

- *Sum of Squares Regression*, the variability accounted for by the regression,

$$\text{SSR} = \sum_{i=1}^{n}(\hat{\mathbf{Y}}_i - \bar{\mathbf{Y}})^2$$

- *Sum of Squares Error*, the remaining variability not accounted for by the regression,

$$\text{SSE} = \sum_{i=1}^{n}(\hat{\mathbf{Y}}_i - \mathbf{Y}_i)^2$$

## Fit & Decomposition, Total Variability

- Interesting manipulations of the sum of squares total:

$$
\begin{aligned}
\text{SST} &= \sum_{i=1}^{n}(\mathbf{Y}_i^2 - 2\mathbf{Y}_i\bar{\mathbf{Y}} + \bar{\mathbf{Y}}^2) \\
&= \sum_{i=1}^{n}\mathbf{Y}_i^2 - 2\sum_{i=1}^{n}\mathbf{Y}_i\bar{\mathbf{Y}} + n\bar{\mathbf{Y}}^2 \\
&= \sum_{i=1}^{n}\mathbf{Y}_i^2 - 2n\bar{\mathbf{Y}}^2 + n\bar{\mathbf{Y}}^2 \\
&= \sum_{i=1}^{n}\mathbf{Y}_i^2 - n\bar{\mathbf{Y}}^2 \qquad \text{(scalar description)} \\
&= \boxed{\mathbf{Y}'\mathbf{Y} - \frac{1}{n}\mathbf{Y}'\mathbf{J}\mathbf{Y}} \qquad \text{(matrix algebra description)}
\end{aligned}
$$

  where $\mathbf{J}$ is a $n \times n$ matrix of all $1$'s.

- Note that pre-multiplying by $\mathbf{J}$ produces a same-sized matrix where the values are the sum by column, and post-multiplying by $\mathbf{J}$ produces a same-sized matrix where the values are the sum by row.

## A Small Demonstration of the **J** Matrix

```
Y <- c(1,3,5)
J <- matrix(rep(1,9),ncol=3)
J
          [,1] [,2] [,3]
     [1,]    1    1    1
     [2,]    1    1    1
     [3,]    1    1    1

3*(mean(Y))^2
[1] 27

t(Y) %*% J %*% Y/3
[1,]   27
```

- Demonstrating the last line from scalar to matrix form

$$\sum_{i=1}^{n} \mathbf{Y}_i^2 - n\bar{\mathbf{Y}}^2 = \mathbf{Y}'\mathbf{Y} - \frac{1}{n}\mathbf{Y}'\mathbf{J}\mathbf{Y}.$$

## Fit & Decomposition, Regression Variability

- Sum of Squares Regression:

$$\begin{aligned}
\text{SSR} &= \sum_{i=1}^{n}(\hat{\mathbf{Y}}_i^2 - 2\hat{\mathbf{Y}}_i\bar{\mathbf{Y}} + \bar{\mathbf{Y}}^2) \\
&= \hat{\mathbf{Y}}'\hat{Y} - 2\bar{\mathbf{Y}}\sum_{i=1}^{n}\hat{Y}_i + n\bar{\mathbf{Y}}^2 \\
&= (\hat{\boldsymbol{\beta}}'\mathbf{X}')(\hat{Y}) - 2n\bar{\mathbf{Y}}^2 + n\bar{\mathbf{Y}}^2 \\
&= \hat{\boldsymbol{\beta}}'\mathbf{X}'\hat{Y} - n\bar{\mathbf{Y}}^2 \\
&= \boxed{\hat{\boldsymbol{\beta}}'\mathbf{X}'\hat{\mathbf{Y}} - \frac{1}{n}\mathbf{Y}'\mathbf{J}\mathbf{Y}}
\end{aligned}$$

## Fit & Decomposition, Remaining Variability

- Sum of Squares Error (using the Normal Equation, $\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{Y}$):

$$\begin{aligned}
\text{SSE} &= \sum_{i=1}^{n}(\mathbf{Y}_i - \hat{\mathbf{Y}}_i)^2 = \mathbf{e}'\mathbf{e} \\
&= (\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}) \\
&= \mathbf{Y}'\mathbf{Y} - \mathbf{Y}'\mathbf{X}\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{Y} + \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} \\
&\quad \text{now do the Normal Equation substitution...} \\
&= \mathbf{Y}'\mathbf{Y} - (\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}})'\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}'(\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}}) + \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} \\
&= \mathbf{Y}'\mathbf{Y} - (\hat{\boldsymbol{\beta}}'\mathbf{X}')(\mathbf{X}\hat{\boldsymbol{\beta}}) \\
&= \boxed{\mathbf{Y}'\mathbf{Y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\hat{\mathbf{Y}}}
\end{aligned}$$

## Total %*&^#%ing Magic!

- Adding Total Sum of Squares Regression to Total Sum of Squares Error:

$$\begin{aligned} SSR + SSE &= (\hat{\boldsymbol{\beta}}'\mathbf{X}'\hat{\mathbf{Y}} - n\mathbf{Y}'\mathbf{J}\mathbf{Y}) + (\mathbf{Y}'\mathbf{Y} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\hat{\mathbf{Y}}) \\ &= \mathbf{Y}'\mathbf{Y} - n\mathbf{Y}'\mathbf{J}\mathbf{Y} \\ &= \text{SST} \end{aligned}$$

- Because in general sums of squares do not equal squares of sums, for example:

```
7^2 + 3^2
[1] 58
(7+3)^2
[1] 100
```

(except in unusual or pathological circumstances).

## A Measure of Fit

- The "R-Square" or "R-Squared" measure:

$$R^2 = \frac{SSR}{SST} = \frac{SST - SSE}{SST} = 1 - \frac{\mathbf{e}'\mathbf{e}}{\mathbf{Y}'\mathbf{M}^o\mathbf{Y}} = \frac{\hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{M}^o\mathbf{X}\hat{\boldsymbol{\beta}}}{\mathbf{Y}'\mathbf{M}^o\mathbf{Y}}$$

where $M^o = \mathbf{I} - \frac{1}{n}\mathbf{i}\mathbf{i}'$, $\quad \mathbf{i} = c(1, 1, \ldots, 1)$.

- Note: $\mathbf{M}^o$ is idempotent and transforms means to deviances for the explanatory variables:

```
M.0 <- diag(3) - (1/3)*c(1,1,1)%*%t(c(1,1,1))
M.0
         [,1]     [,2]     [,3]
[1,]  0.66667 -0.33333 -0.33333
[2,] -0.33333  0.66667 -0.33333
[3,] -0.33333 -0.33333  0.66667
```

## A Measure of Fit

- Also, there is another version that accounts for sample size and the number of explanatory variables ($k$):

$$R^2_{adj} = 1 - \frac{\mathbf{e}'\mathbf{e}/(n-k)}{\mathbf{Y}'M^o\mathbf{Y}/(n-1)}$$

  which is useful with small datasets.

- Bivariate relationships for $R^2$ from `lm(Y ~ X)`:

$$\frac{s_Y}{s_X}\mathrm{cor}X,Y = \beta$$

$$R^2 = \mathrm{cor}(X,Y)^2$$

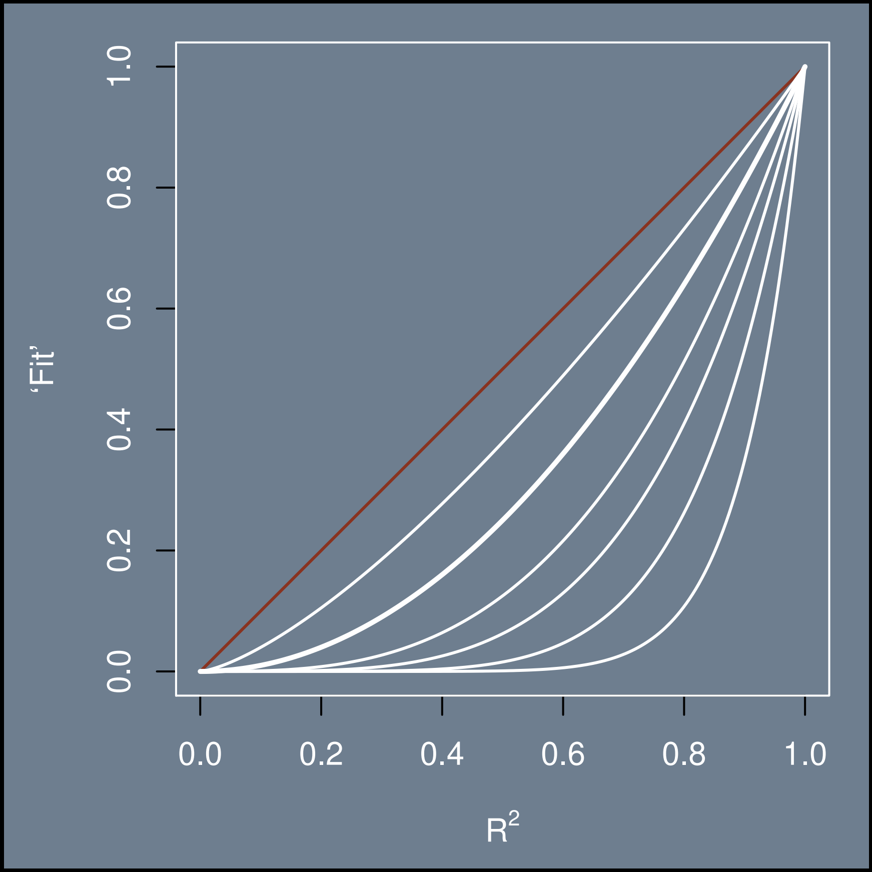
$$\frac{s_Y}{s_X}\sqrt{R^2} = \beta$$

$$\sqrt{R^2} = \frac{s_X}{s_Y}\beta$$

$$R^2 = \left(\frac{s_X}{s_Y}\beta\right)^2$$

# Warnings about $R^2$

- There is not a *population* analog.
- It can never be reduced by adding more explanatory variables.
- It is a *quadratic* form in $[0 : 1]$ space.
- Therefore it does not have quite the meaning that people expect due to the nonlinearity.



## Does $R^2$ Have a Distribution?

- Surprisingly, yes.
- Note that the F-statistic can be expressed as:

$$F = \frac{(n-k)}{(k-1)} \frac{R^2}{1-R^2}.$$

- Algebraically rearranging:

$$R^2 = \frac{(k-1)F}{(n-k)+(k-1)F}.$$

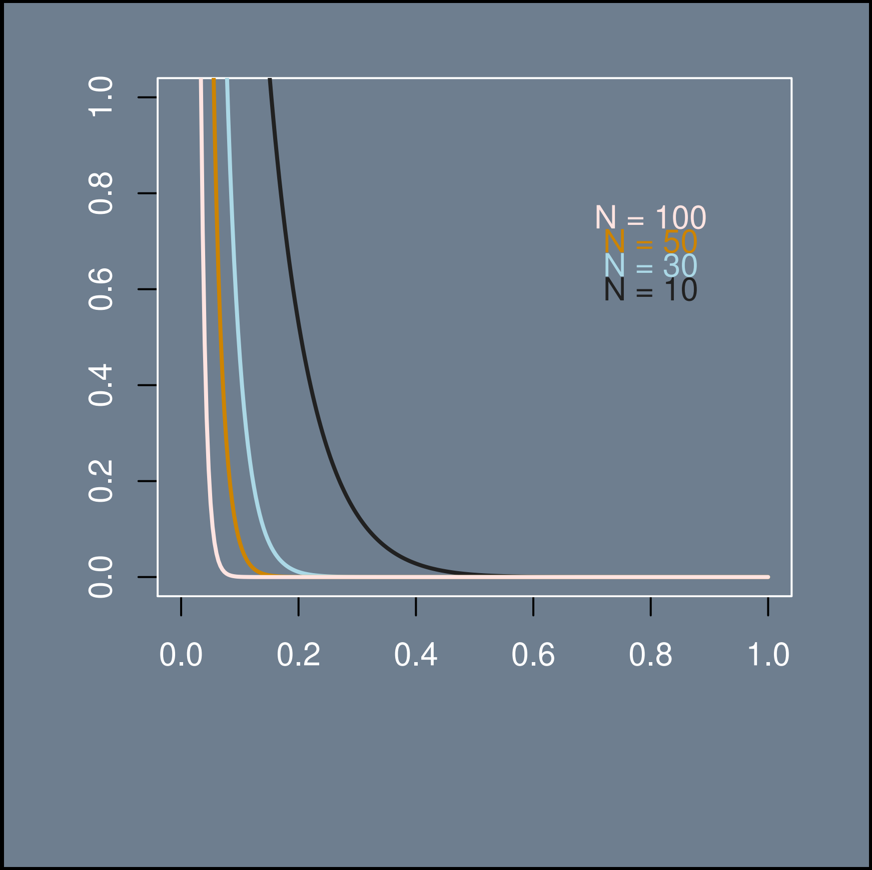
- This means that:

$$R^2 \sim \text{Beta}\left(\frac{k-1}{2}, \frac{n-k}{2}\right)$$

under the assumption that the effect of all the regressors is zero ($\boldsymbol{\beta} = 0$), inherited from the condition of the F-test.

## Does $R^2$ Have a Distribution?

- For simplicity restrict ourselves to $k = 1$, and again $H_0$ : all $\boldsymbol{\beta} = 0$.
- What does $R^2$ look like for different sample sizes?



## Properties of the Estimator, Unbiasedness

- $\hat{\boldsymbol{\beta}}$ is an estimator for $\boldsymbol{\beta}$, which can be rewritten according to:

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}) \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\epsilon} \\ &= \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\epsilon}\end{aligned}$$

  which also implies $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\epsilon}$.

- Taking expectations:

$$\begin{aligned}\mathbb{E}[\hat{\boldsymbol{\beta}}] &= \mathbb{E}[\boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\epsilon}] \\ &= \mathbb{E}[\boldsymbol{\beta}] + \mathbb{E}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\epsilon}] \\ &= \boldsymbol{\beta} + \mathbb{E}[K\boldsymbol{\epsilon}] \\ &= \boldsymbol{\beta} + K\mathbb{E}[\boldsymbol{\epsilon}] = \boldsymbol{\beta}\end{aligned}$$

  shows that it is unbiased.

## Properties of the Estimator, Variance

- By definition (using an outer product):

$$\begin{aligned}\text{Var}[\hat{\boldsymbol{\beta}}|\mathbf{X}] &= \mathbb{E}[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'|\mathbf{X}] - \mathbb{E}[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}|\mathbf{X})]^2 \\ &= \mathbb{E}[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'|\mathbf{X}] - \mathbb{E}[0]^2.\end{aligned}$$

  (using the elementary property $\text{Var}[A] = \mathbb{E}[A^2] - (\mathbb{E}[A])^2$).

- Now using $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\epsilon}$ from the previous slide,

$$\begin{aligned}\text{Var}[\hat{\boldsymbol{\beta}}|\mathbf{X}] &= \mathbb{E}\left[((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\epsilon})((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\epsilon})'|\mathbf{X}\right] \\ &= \mathbb{E}\left[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\epsilon}\boldsymbol{\epsilon}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}|\mathbf{X}\right] \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbb{E}[\boldsymbol{\epsilon}\boldsymbol{\epsilon}'|\mathbf{X}]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\text{Var}[\boldsymbol{\epsilon}|\mathbf{X}] + \mathbb{E}[\boldsymbol{\epsilon}|\mathbf{X}]^2)\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\sigma^2\mathbf{I})\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\end{aligned}$$

## Properties of the Estimator, General

- The OLS estimator is consistent (converges in probability):

$$\underset{n\to\infty}{\text{plim}}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta},$$

  with Gauss-Markov assumption #5: $\text{Cov}[\epsilon_i, \mathbf{X}] = 0, \forall i$, and there is no perfect multicollinearity.

- The OLS estimate is optimal:

$$\text{Var}(\hat{\boldsymbol{\beta}}_{\text{OLS}}) \leq \text{Var}(\hat{\boldsymbol{\beta}}_{\text{All Other}})$$

  with Gauss-Markov assumptions #3 $\text{Var}[\boldsymbol{\epsilon}] = \sigma^2\mathbf{I}$, and #4 $\text{Cov}[\epsilon_i, \epsilon_j] = 0, \quad \forall i \neq j$.

- Given all of the Gauss-Markov assumptions and $\sigma^2 < \infty$, we say that $\hat{\boldsymbol{\beta}}$ is BLUE (Best Linear Unbiased Estimator) for $\boldsymbol{\beta}$ if calculated from OLS or MLE.

- Given sufficient sample size $\hat{\boldsymbol{\beta}}|\mathbf{X} \sim N(\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1})$.

## Dealing With the Variance

- We want to use some $\text{Est.Var}[\hat{\boldsymbol{\beta}}]$, since $\sigma^2$ is generally not known.
- What about calculating the standard error of the regression?
- We will use the sample quantities for estimation:

$$\mathbb{E}[\mathbf{e}'\mathbf{e}|\mathbf{X}] = \text{Var}[\mathbf{e}|\mathbf{X}] + \mathbb{E}[\mathbf{e}|\mathbf{X}]^2 = \text{Var}[\mathbf{e}|\mathbf{X}] + 0 = \sigma^2\mathbf{I}$$

(using the elementary property $\text{Var}[A] = \mathbb{E}[A^2] - (\mathbb{E}[A])^2$) meaning that we that we only need to manipulate $\mathbb{E}[\mathbf{e}'\mathbf{e}|\mathbf{X}]$ to get an estimate of $\sigma^2$.

- Perspective here: $\mathbf{X}$ is fixed once observed and $\boldsymbol{\epsilon}$ is the random variable (to be estimated with $\mathbf{e}$):
  - since $\text{Var}[\boldsymbol{\epsilon}] = \sigma^2\mathbf{I}$, then no single term dominates and we get the Lindeberg-Feller CLT result,
  - so $\mathbf{e}$ (the sample quantity) is IID normal and we write the joint PDF as:

$$f(\mathbf{e}) = \prod_{i=1}^{n} f(\mathbf{e}_i) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp[-\mathbf{e}'\mathbf{e}/2\sigma^2]$$

  based on sample quantities.

## Estimating From Sample Quantities

- Population derived variance/covariance matrix: $\text{Var}[\hat{\boldsymbol{\beta}}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$.
- We also know: $\mathbb{E}[\mathbf{e}_i] = \boldsymbol{\epsilon}_i$, although in practice we always have finite samples.
- And by assumption: $\mathbb{E}[\mathbf{e}_i^2] = \text{Var}[\boldsymbol{\epsilon}_i] + (\mathbb{E}[\boldsymbol{\epsilon}_i])^2 = \sigma^2$
  (again using the elementary property $\text{Var}[A] = \mathbb{E}[A^2] - (\mathbb{E}[A])^2$).
- Therefore the sum is $\sum \mathbb{E}[\mathbf{e}_i^2] = \text{tr}(\sigma^2\mathbf{I}) = n\sigma^2$.
- So why not use: $\hat{\sigma}^2 \approx \frac{1}{n}\sum \mathbf{e}_i^2$.
- But:

$$\begin{aligned} \mathbf{e}_i &= \mathbf{Y}_i - \mathbf{X}_i\hat{\boldsymbol{\beta}} \qquad \text{(now insert population values for } \mathbf{Y}_i\text{)} \\ &= (\mathbf{X}_i'\boldsymbol{\beta} + \boldsymbol{\epsilon}_i) - \mathbf{X}_i\hat{\boldsymbol{\beta}} \\ &= \boldsymbol{\epsilon}_i - \mathbf{X}_i(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \end{aligned}$$

  meaning that $\text{plim}[\mathbf{e}_i] = \boldsymbol{\epsilon}_i$ since $\hat{\boldsymbol{\beta}} \underset{n\to\infty}{\longrightarrow} \boldsymbol{\beta}$.
- So asymptotically this substitution is fine, but is it okay in finite samples?

## Some Needed Relations

- Recall that:

$$\mathbf{M} = \mathbf{I} - \mathbf{H} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'.$$

- So that we can derive this in the opposite direction from the way we did before:

$$\begin{aligned}
\mathbf{MY} &= (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{Y} \\
&= \mathbf{Y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\
&= \mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}} \\
&= \mathbf{e}
\end{aligned}$$

- From similar calculations we get:

$$\begin{aligned}
\mathbf{Me} &= (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{e} \\
&= \mathbf{e} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{e} \\
&= \mathbf{e} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}) \\
&= \mathbf{e} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} + \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} \\
&= \mathbf{e} - \mathbf{X}\hat{\boldsymbol{\beta}} + \mathbf{X}\hat{\boldsymbol{\beta}} \\
&= \mathbf{e}
\end{aligned}$$

## Some Needed Relations

- Equating $\mathbf{MY} = \mathbf{e}$ and $\mathbf{Me} = \mathbf{e}$ from the last slide, we get $\mathbf{MY} = \mathbf{Me}$
- We could also get this from the corresponding population values:

$$\begin{aligned}
\mathbf{MY} &= \mathbf{M}[\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}] \\
&= \mathbf{MX}\boldsymbol{\beta} + \mathbf{M}\boldsymbol{\epsilon} \\
&= (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{X}\boldsymbol{\beta} + \mathbf{M}\boldsymbol{\epsilon} \\
&= \mathbf{X}\boldsymbol{\beta} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} + \mathbf{M}\boldsymbol{\epsilon} \\
&= \mathbf{M}\boldsymbol{\epsilon}
\end{aligned}$$

- So $\mathbf{e}'\mathbf{e} = (\mathbf{M}\boldsymbol{\epsilon})'\mathbf{M}\boldsymbol{\epsilon} = \boldsymbol{\epsilon}'\mathbf{M}'\mathbf{M}\boldsymbol{\epsilon} = \boldsymbol{\epsilon}'\mathbf{M}\boldsymbol{\epsilon}$.

## Estimating From Sample Quantities

- This means that we can use $\mathbf{e}'\mathbf{e} = \mathbf{e}'\mathbf{M}\mathbf{M}\mathbf{e} = \mathbf{e}'\mathbf{M}\mathbf{e}$ accordingly:

$$
\begin{aligned}
\mathbb{E}[\mathbf{e}'\mathbf{e}|\mathbf{X}] &= \mathbb{E}[\mathbf{e}'\mathbf{M}\mathbf{e}|\mathbf{X}] \\
&= \mathbb{E}[\mathrm{tr}(\mathbf{e}'\mathbf{M}\mathbf{e})|\mathbf{X}] && \text{(Gauss-Markov assumption \#4: } \mathrm{Cov}[\epsilon_i, \epsilon_j] = 0, \forall i \neq j) \\
&= \mathbb{E}[\mathrm{tr}(\mathbf{M}\mathbf{e}'\mathbf{e})|\mathbf{X}] && \text{(property of traces: } \mathrm{tr}(ABC) = \mathrm{tr}(BAC)) \\
&= \mathrm{tr}(\mathbf{M}\mathbb{E}[\mathbf{e}'\mathbf{e}|\mathbf{X}]) && (\mathbf{M} \text{ is fixed for observed } \mathbf{X}) \\
&= \mathrm{tr}(\mathbf{M})\mathbf{I}\sigma^2 && \text{(Gauss-Markov assumption \#3: } \mathrm{Var}[\boldsymbol{\epsilon}] = \sigma^2\mathbf{I}) \\
&= \mathrm{tr}(\mathbf{I} - \mathbf{H})\mathbf{I}\sigma^2 \\
&= [\mathrm{tr}(\mathbf{I}_{n\times n}) - \mathrm{tr}(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')]\mathbf{I}\sigma^2 && \text{(property of traces: } \mathrm{tr}(A - B) = \mathrm{tr}(A) - \mathrm{tr}(B)) \\
&= [\mathrm{tr}(\mathbf{I}_{n\times n}) - k)]\sigma^2 && \text{(for linear models the trace of the hat matrix} \\
& && \text{is the of rank } X) \\
&= [n - k]\sigma^2
\end{aligned}
$$

## Estimating From Sample Quantities

- From $\mathbb{E}[\mathbf{e}'\mathbf{e}|\mathbf{X}] = (n-k)\sigma^2$, we algebraically get an unbiased estimator:

$$\hat{\sigma}^2 = \frac{\mathbf{e}'\mathbf{e}}{n-k} = s^2,$$

so that a *finite sample* estimator of $\text{Var}[\hat{\boldsymbol{\beta}}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$ is:

$$\widehat{\text{Var}}[\hat{\boldsymbol{\beta}}] = s^2(\mathbf{X}'\mathbf{X})^{-1}.$$

- The Wald-style traditional linear inference, for the $k$th coefficient is:

$$z_k = \frac{\hat{\boldsymbol{\beta}}_k - \beta_k^{\text{null}}}{\sqrt{\sigma^2(\mathbf{X}'\mathbf{X})_k^{-1}}} \sim N(0,1),$$

with the assumption that we know $\sigma^2$ (which we usually do not).

- But we can use a well-known distributional relation to modify the above form:

$$\text{we know that } X^2 = \frac{(n-k)s^2}{\sigma^2} \sim \chi^2_{n-k} \qquad \text{then } \frac{z_k}{\sqrt{X^2/df}} \sim t_{(n-k)}(0)$$

provided the random variables $z_k$ and $X^2$ are independent.

## Estimating From Sample Quantities

- Making the obvious substitution gives:

$$t_{(n-k)} = \frac{\hat{\boldsymbol{\beta}}_k - \beta_k^{\text{null}}}{\sqrt{\sigma^2(\mathbf{X}'\mathbf{X})^{-1}}} \times \frac{1}{\sqrt{\frac{(n-k)s^2}{\sigma^2}/(n-k)}} = \frac{\hat{\boldsymbol{\beta}}_k - \beta_k^{\text{null}}}{\sqrt{s^2(\mathbf{X}'\mathbf{X})^{-1}}}$$

- Typical (Wald) regression test:

$$H_0\colon\ \beta_k = 0 \qquad\qquad H_1\colon\ \beta_k \neq 0$$

making:

$$t_{(n-k)} = \frac{\hat{\boldsymbol{\beta}}_k - \beta_k^{\text{null}}}{\sqrt{s^2(\mathbf{X}'\mathbf{X})^{-1}}} = \frac{\hat{\boldsymbol{\beta}}_k}{SE(\beta_k)}$$

- Alternatives usually look like:

$$H_0\colon\ \beta_k < 7 \qquad\qquad H_1\colon\ \beta_k \geq 7$$

making:

$$t_{(n-k)} = \frac{\hat{\boldsymbol{\beta}}_k - 7}{SE(\beta_k)}$$

## Summary Statistics

- $(1-\alpha)$ Confidence Interval for $\hat{\boldsymbol{\beta}}_k$:

$$\left[\hat{\boldsymbol{\beta}}_k - SE(\hat{\boldsymbol{\beta}})t_{\alpha/2,df} : \hat{\boldsymbol{\beta}}_k + SE(\hat{\boldsymbol{\beta}})t_{\alpha/2,df}\right]$$

- $(1-\alpha)$ Confidence Interval for $\sigma^2$:

$$\left[\frac{(n-k)s^2}{\chi^2_{1-\alpha/2}} : \frac{(n-k)s^2}{\chi^2_{\alpha/2}}\right]$$

- F-statistic test for all but $\hat{\beta}_0$ equal to zero:

$$F = \frac{SSR/(k-1)}{SSE/(n-k)} \sim F_{k-1,n-k} \text{ under the null.}$$

## Linear Model Confidence Bands

- We want the predicted value of the outcome variable for $\mathbf{x}_i$ *in the sample*:

$$y_i = \mathbf{x}_i\boldsymbol{\beta} + \boldsymbol{\epsilon}_i \qquad\qquad \hat{y}_i = \mathbf{x}_i\hat{\boldsymbol{\beta}}$$

- The variance at this point on the regression line, bivariate, is:

$$\text{Var}(\hat{b}|x_i) = s^2\left(\frac{1}{n} + \frac{(x_i - \bar{x})}{\sum(x_j - \bar{x})}\right).$$

- The variance at this point on the regression line, multivariate, is:

$$\begin{aligned}
\text{Var}[\mathbf{e}_i|\mathbf{X}, \mathbf{x}_i] &= \text{Var}[\mathbf{x}_i(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})] \\
&= \text{Var}[\mathbf{x}_i\boldsymbol{\beta}] + \text{Var}[\mathbf{x}_i\hat{\beta}] \\
&= s^2 + \mathbf{x}_i\text{Var}[\hat{\boldsymbol{\beta}}]\mathbf{x}_i' \\
&= s^2 + \mathbf{x}_i(s^2(\mathbf{X}\mathbf{X})^{-1})\mathbf{x}_i' \\
&= s^2[1 + \mathbf{x}_i(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i']
\end{aligned}$$

## Linear Model Predictions/Forecasts

- We want the predicted value for $\mathbf{x}^0$ *not in the sample*:

$$y^0 = \mathbf{x}^0\boldsymbol{\beta} + \boldsymbol{\epsilon}^0 \qquad\qquad \hat{y}^0 = \mathbf{x}^0\hat{\boldsymbol{\beta}}$$

since $\hat{y}^0$ is the LMVUE of $\mathbb{E}[\hat{y}^0|\mathbf{x}^0]$.

- The *prediction error* is:

$$e^0 = y^0 - \hat{y}^0 = \mathbf{x}^0(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) = \boldsymbol{\epsilon}^0.$$

(notationally suppressing the conditionality on $\mathbf{X}$ here).

- The Prediction variance is:

$$\text{Var}[e^0|\mathbf{X},\mathbf{x}^0] = s^2 + \text{Var}[\mathbf{x}^0(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})|\mathbf{X},\mathbf{x}^0] = s^2 + s^2(\mathbf{x}^0)(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{x}^0)'$$

and if we have a constant term in the regression, this is:

$$\text{Var}[e^0|\mathbf{X},\mathbf{x}^0] = s^2\left[1 + \frac{1}{n} + \sum_{j=1}^{K-1}\sum_{k=1}^{K-1}(\mathbf{x}_j^0 - \bar{\mathbf{x}}_j)(\mathbf{x}_k^0 - \bar{\mathbf{x}}_k)(\mathbf{X}_{-1}\mathbf{M}^0\mathbf{X}_{-1})^{jk}\right],$$

where $\mathbf{X}_{-1}$ is $\mathbf{X}$ omitting the first column, $K$ is the number of explanatory variables (including the constant), and $\mathbf{M}^0 = \mathbf{I} - \frac{1}{n}\mathbf{i}\mathbf{i}'$.

## Linear Model Predictions/Forecasts

- The prediction interval (in the vertical direction) is created from

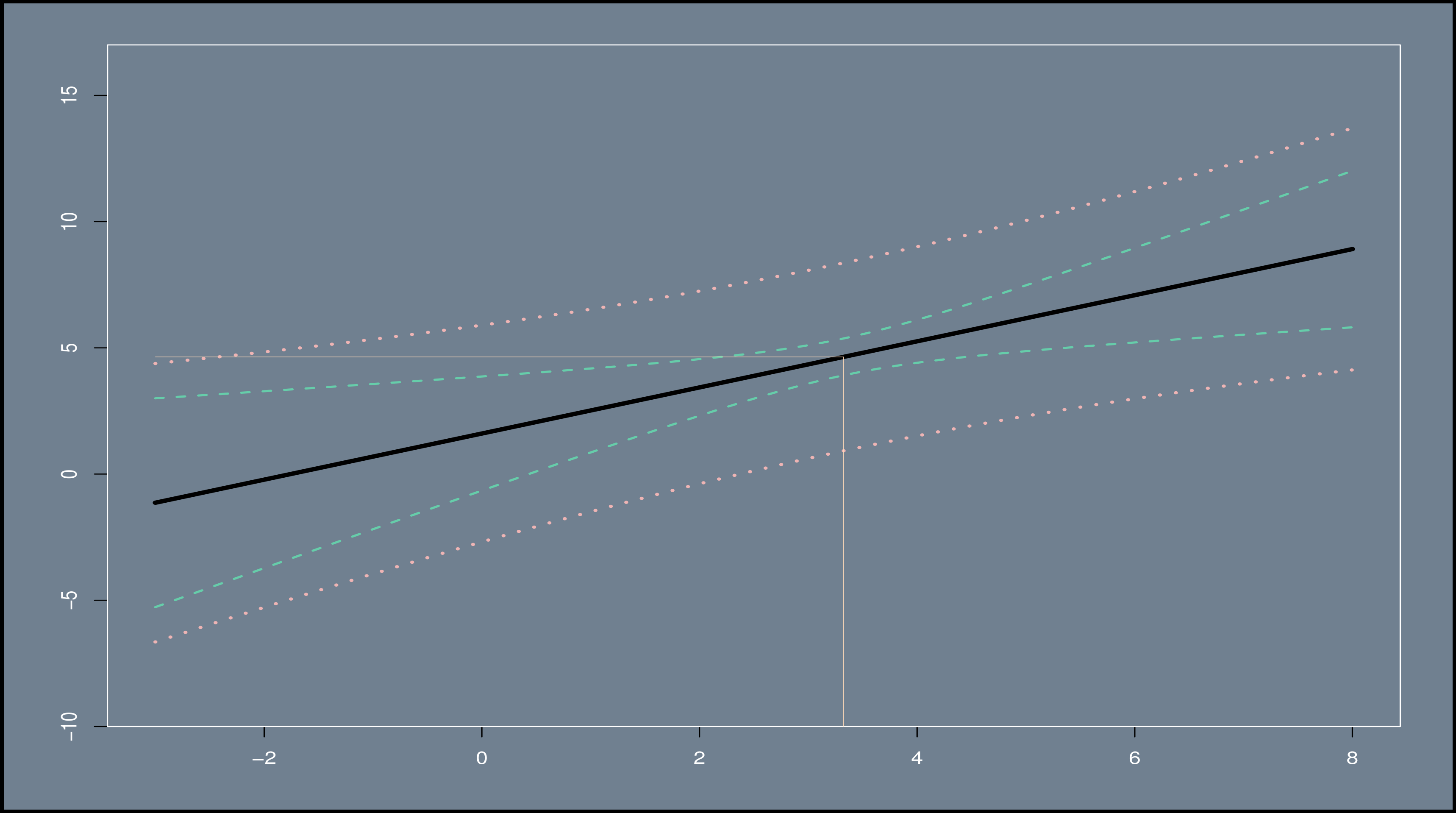
$$CI[\hat{y}^0] = \hat{y}^0 \pm t_{\alpha/2}\sqrt{\text{Var}[e^0|\mathbf{X}, \mathbf{x}^0]}.$$

- Note that the value of $\mathbf{x}^0$ is buried in there, and like the CI for $\boldsymbol{\beta}$, it is smallest around $\bar{x}$.

- It is important to also distinguish between two interval estimates around the regression line: the CI for $\hat{y} = \mathbf{X}\boldsymbol{\beta}$ and the CI for $\hat{y}^0$.

- Where the prediction interval is always wider than the regression confidence interval.

# Linear Model Predictions/Forecasts



## Linear Model Predictions/Forecasts

- The `R` code for these intervals can be produced by:

```
postscript("Class.Multilevel/linear.prediction.ps")
X <- rnorm(25,3,1); Y <- X + rnorm(25,2,2)
ruler <- data.frame(X = seq(-3, 8,length=200))
confidence.interval <- predict(lm(Y ~ X), ruler, interval="confidence")
predict.interval <- predict(lm(Y ~ X), ruler, interval="prediction")
par(mar=c(1,1,1,1),oma=c(3,3,1,1),mfrow=c(1,1),col.axis="white",col.lab="white",
   col.sub="white",col="white",bg="slategray")
# REGRESSION LINE
plot(ruler[,1], confidence.interval[,1], type="l",lwd=4,ylim=c(-9,16),col="black")
# UPPER AND LOWER CONFIDENCE INTERVALS
lines(ruler[,1],confidence.interval[,2], lwd=2, lty=2, col="aquamarine3")
lines(ruler[,1],confidence.interval[,3], lwd=2, lty=2, col="aquamarine3")
# UPPER AND LOWER PREDICTION INTER ALS
lines(ruler[,1],predict.interval[,2], lwd=3, lty=3, col="rosybrown2")
lines(ruler[,1],predict.interval[,3], lwd=3, lty=3, col="rosybrown2")
segments(mean(X),-10,mean(X),mean(Y), lwd=0.5, col="peachpuff")
segments(-3,mean(Y),mean(X),mean(Y),  lwd=0.5, col="peachpuff")
dev.off()
```

## Multicollinearity Issues

- If one explanatory variable is a linear combination of another then $\text{rank}(\mathbf{X}) = k - 1$.
- Therefore $\text{rank}(\mathbf{X}'\mathbf{X}) = k - 1$ (matrix size $k \times k$), and it is singular and non-invertible.
- Now no parameter estimates are possible, and the model is now unidentified.


- More typically: 2 explanatory variables are highly but not perfectly correlated.
- Symptoms:
  - small changes in data give large changes in parameter estimates
  - coefficients have large standard errors and poor $t$-statistics even if F-statistics and $R^2$ are okay
  - coefficients seem illogical (obviously wrong sign, huge magnitude).

## Multicollinearity Remedies

- Respecify model (if reasonable): add/drop variables, add data cases that break the pattern, restrict the range of some variables, combine variables possibly with PCA.

- Center explanatory variables, or standardize (slope coefficient is interpreted in units of standard deviations of the covariate, the intercept is the mean of the outcome $y$ when all covariate values are zero).

- Create a new variable that is a weighted combination of highly correlated variables and use it to replace both (two variables to one variable in the model).

- Ridge regression (add a little bias):

$$\hat{\boldsymbol{\beta}} = [\mathbf{X}'\mathbf{X} + \mathbf{RI}]^{-1}\mathbf{X}'\mathbf{Y}$$

  such that the [ ] part barely inverts, and can involve a penalty function.

- `R` packages that do this: `ridge`, `bigRR`, `genridge`, `parcor`, and more.

- See also: Jeff Gill and Gary King (SMR 2004), "What to do When Your Hessian is Not Invertible: Alternatives to Model Respecification in Nonlinear Estimation."

## More on Ridge Regression

- Suppose that we are concerned with a single problematic explanatory variable, so that $\mathbf{R}$ is just a vector of ones with $\lambda$ in the  place.
- Then:

$$\lambda \longrightarrow 0, b_j^{\text{ridge}} \longrightarrow b_j^{\text{OLS}} \qquad \lambda \longrightarrow \infty, b_j^{\text{ridge}} \longrightarrow 0.$$

- Under the assumption of an orthonormal $\mathbf{X}$ matrix (each column vector has length 1 and is orthogonal to all the other column vectors and the inverse is the transpose):

$$b_j^{\text{ridge}} = \frac{b_j^{\text{OLS}}}{1+\lambda},$$

  which shows that increasing $\lambda$ *shrinks* the estimator towards zero, increasing bias but reducing the variance.

## Even More on Ridge Regression

- Define $\mathbf{W} = (\mathbf{X}'\mathbf{X} + \lambda\mathbf{I})^{-1}$, such that:

$$\text{Var}(b) = \sigma^2\mathbf{W}\mathbf{X}'\mathbf{X}\mathbf{W}.$$

- This leads to an important result:

$$\text{bias}(b) = -\lambda\mathbf{W}\boldsymbol{\beta}.$$

- Also there always exists a $\lambda$ such that the MSE ridge estimator is always less than the the MSE of the regular OLS estimator, where:

$$\text{MSE}(b_j) = \mathbb{E}_{b_j}[(b_j - \beta_j)^2] = \text{Var}(b_j) + \text{bias}(b_j)^2.$$

## Simple Ridge Regression Example

```
anaemia <- read.table("https://jeffgill.org/files/jeffgill/files/anaemia.txt",
                      header=TRUE,row.names=1)
library(MASS)
a.lm3.out <- lm.ridge(Hb ~ Age + Menapause + I(Age+rnorm(nrow(anaemia))),
                      data=anaemia)
a.lm3.out$GCV
0.07936452

cbind(a.lm3.out$coef, sqrt(a.lm3.out$scales))
                                  [,1]    [,2]
Age                            1.06565 3.91640
Menapause                      0.29350 0.70711
I(Age + rnorm(nrow(anaemia))) 0.73817 3.89488

summary(lm(Hb ~ Age + Menapause,data=anaemia))$coef[2:3,1:2]
          Estimate Std. Error
Age        0.11716   0.035881
Menapause  0.60002   1.100703
```

## Summary of Asymptotic Results, Meeting the "Grenander Conditions:"

G1: For each column of $\mathbf{X}$: $\mathbf{X}_k'\mathbf{X}_k \longrightarrow +\infty$: sums of squares grow as $n$ grows, no columns of all zeros.

G2: No single observation dominates each explanatory variable $k$ in the limit:

$$\lim_{n\longrightarrow\infty} \frac{\mathbf{X}_{ik}^2}{\mathbf{X}_k'\mathbf{X}_k} = 0, \quad i = 1, \ldots, n, \; i \neq k$$

G3: $\mathbf{R}$ is the sample correlation matrix of the observed columns of $\mathbf{X}$, excluding the leading column of $1$s. Then $\lim_{n\longrightarrow\infty} \mathbf{R} = \mathbf{C}$, where $\mathbf{C}$ is a positive definite matrix ($\mathbf{q}'\mathbf{X}\mathbf{q} > 0$ for any conformable, non-null $\mathbf{q}$).

- Now G1 + G2 + G3 give:

$$\hat{\boldsymbol{\beta}} \overset{\text{asym.}}{\sim} N\left[\boldsymbol{\beta}, \frac{\sigma^2}{n}\mathbf{Q}^{-1}\right]$$

where:

$$\mathbf{Q} = \lim_{n\rightarrow\infty}\left[\frac{1}{n}\mathbf{X}'\mathbf{X}\right].$$

- See: Grenander and Rosenblatt (1957).

## In the Limit What About $s^2$

- Recall:
$$s^2 = \frac{1}{n-k}\boldsymbol{\epsilon}'\mathbf{M}\boldsymbol{\epsilon} = \frac{1}{n-k}\boldsymbol{\epsilon}'(\mathbf{I}-\mathbf{H})\boldsymbol{\epsilon} = \frac{1}{n-k}\left[\boldsymbol{\epsilon}'\boldsymbol{\epsilon} - \boldsymbol{\epsilon}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\epsilon}\right]$$
$$= \frac{n}{n-k}\left[\frac{\boldsymbol{\epsilon}'\boldsymbol{\epsilon}}{n} - \left(\frac{\boldsymbol{\epsilon}'\mathbf{X}}{n}\right)\left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1}\left(\frac{\mathbf{X}'\boldsymbol{\epsilon}}{n}\right)\right]$$
where $n/(n-k)$ goes to $1$ as $n$ goes to $\infty$.

- Taking the limit now as $n \rightarrow \infty$:
$$\lim s^2 = \lim \frac{\boldsymbol{\epsilon}'\boldsymbol{\epsilon}}{n} - (0)\mathbf{Q}^{-1}(0) = \lim \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{\epsilon}_i^2 = \frac{1}{n}\mathrm{tr}(\sigma^2\mathbf{I}) = \frac{1}{n}(n\sigma^2) = \sigma^2$$

- Summarizing:
$$\lim\ s^2\left[\frac{\mathbf{X}'\mathbf{X}}{n}\right]^{-1} = \sigma^2\mathbf{Q}^{-1}$$
$$\lim\ s^2(\mathbf{X}'\mathbf{X})^{-1} = \frac{\sigma^2}{n}\mathbf{Q}^{-1} = \frac{\sigma^2}{n}\left[\frac{1}{n}\mathbf{X}'\mathbf{X}\right]^{-1}$$
$$\therefore\ Est.Asy.Var[\hat{\boldsymbol{\beta}}] = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$$

## Testing Linear Restrictions

- A theory has *testable implications* if it implies some testable restrictions on the model definition:

$$H_0\colon\ \beta_k = 0 \qquad \text{versus} \qquad H_1\colon\ \beta_k \neq 0,$$

  for example.

- Most restrictions involve *nested* parameter spaces:

$$\text{unrestricted:}[\beta_0, \beta_1, \beta_2, \beta_3]$$

$$\text{restricted:}[\beta_0, 0, \beta_2, \beta_3]$$

  although the restriction does not have to be $\beta_1 = 0$.

- Note that *non*-nested comparisons cause problems for *non*-Bayesians.

- Likelihood-based non-nested comparisons require use of a "super model."

## Testing Linear Restrictions

- Continuing with the simple example:

$$\text{unrestricted: } [\beta_0, \beta_1, \beta_2, \beta_3] \qquad \text{restricted: } [\beta_0, 0, \beta_2, \beta_3]$$

- This example can be notated with $\mathbf{R} = [0, 1, 0, 0]$ to indicate the location of the restriction, and $\mathbf{q} = 0$ to indicate the value of the restriction, so that $\mathbf{R}\boldsymbol{\beta} = \mathbf{q}$ gives the full specification of the restriction in linear algebra terms:

$$[0, 1, 0, 0]\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = 0 \times \beta_0 + 1 \times \beta_1 + 0 \times \beta_2 + 0 \times \beta_3 = \beta_1 = \mathbf{q}$$

  which forces the restriction.

- The test statistic that we will build here, after estimating the regression model, has tail-values that indicate that the restriction is *not* supported: "it modifies the model more than the data wants".

## Testing Linear Restrictions

- More generally we express these restrictions as:

$$\begin{aligned} &r_{11}\beta_1 + r_{12}\beta_2 + \ldots + r_{1k}\beta_k = q_1 \\ &r_{21}\beta_1 + r_{22}\beta_2 + \ldots + r_{2k}\beta_k = q_2 \\ &\vdots \\ &r_{j1}\beta_1 + r_{j2}\beta_2 + \ldots + r_{jk}\beta_k = q_j \end{aligned}$$

  or in more succinct matrix algebra form: $\underset{(J\times k)}{\mathbf{R}}\,\underset{(k\times 1)}{\boldsymbol{\beta}} = \underset{(J\times 1)}{\mathbf{q}}$.

- Notes:
  - Each row of $\mathbf{R}$ is one restriction.
  - $J < k$ for $\mathbf{R}$ to be full rank.
  - This setup imposes $J$ restrictions on $k$ parameters, so there are $k - J$ free parameters left.
  - We are still assuming that $\boldsymbol{\epsilon}_i \sim N(0, \sigma^2)$.
- General test where tail values reject the restriction set:

$$H_0\colon \mathbf{R}\boldsymbol{\beta} - \mathbf{q} = \mathbf{0} \qquad\qquad H_1\colon \mathbf{R}\boldsymbol{\beta} - \mathbf{q} \neq \mathbf{0}$$

## Testing Linear Restrictions, Examples

- One of the coefficients is zero, $\beta_j = 0$, $J = 1$:

$$\mathbf{R} = [0, 0, 0, \underset{j}{1}, \dots, 0, 0, 0], \qquad \mathbf{q} = 0$$

- Two coefficients are equal, $\beta_j = \beta_k$, $J = 2$:

$$\mathbf{R} = [0, 0, 0, \underset{j}{1}, \dots, \underset{k}{-1}, \dots, 0, 0, 0], \qquad \mathbf{q} = 0$$

- First three coefficients are zero, $J = 3$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots 0 \\ 0 & 1 & 0 & 0 & \dots 0 \\ 0 & 0 & 1 & 0 & \dots 0 \end{bmatrix} = [\mathbf{I}_3 : 0], \qquad \mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

## Testing Linear Restrictions, Examples

- A set of coefficients sum to one, $\beta_2 + \beta_3 + \beta_4 = 1$:

$$\mathbf{R} = [0, 1, 1, 1, 0, \ldots, 0], \qquad \mathbf{q} = 1$$

- Several restrictions, $\beta_2 + \beta_3 = 4$, $\beta_4 + \beta_6 = 0$, $\beta_5 = 9$:

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} 4 \\ 0 \\ 9 \end{bmatrix}$$

- All coefficients except the constant are zero:

$$\mathbf{R} = [0{:}\mathbf{I}], \qquad \mathbf{q} = [\mathbf{0}]$$

## Testing Linear Restrictions, Examples

- Create the *discrepancy vector* dictated by the null hypothesis:

$$\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q} = \mathbf{m} \approx 0,$$

which asks whether $\mathbf{m}$ is sufficiently different from zero. Note that $\mathbf{m}$ is a linear function of $\hat{\boldsymbol{\beta}}$ and therefore also normally distributed ($\mathbf{R}$ and $\mathbf{q}$ are full of constants we set).

- This makes it straightforward to think about:

$$\mathbb{E}[\mathbf{m}|\mathbf{X}] = R\mathbb{E}[\hat{\boldsymbol{\beta}}|\mathbf{X}] - \mathbf{q} = \mathbf{R}\hat{B} - \mathbf{q} = 0$$

$$\mathrm{Var}[\mathbf{m}|\mathbf{X}] = \mathrm{Var}[\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q}|\mathbf{X}] = \mathbf{R}\mathrm{Var}[\hat{\boldsymbol{\beta}}|\mathbf{X}]\mathbf{R}' = \sigma^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'$$

- Wald Test:

$$W = \mathbf{m}'[\mathrm{Var}[\mathbf{m}|\mathbf{X}]]^{-1}\mathbf{m} = (\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q})'[\sigma^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']^{-1}(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q}) \sim \chi^2_J$$

where $J$ is the number of rows of $\mathbf{R}$, i.e. the number of restrictions.

## Testing Linear Restrictions, Examples

- Unfortunately we do not have $\sigma^2$, so we use a test with $s^2$, by modifying $W$:

$$F = W \;\times\; \frac{1}{J}\frac{s^2}{\sigma^2}\left(\frac{n-k}{n-k}\right)$$

$$F = (\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q})'(\sigma^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}')(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q}) \;\times\; \frac{1}{J}\frac{s^2}{\sigma^2}\left(\frac{n-k}{n-k}\right)$$

$$= \frac{(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q})'(\sigma^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}')^{-1}(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q})/J}{[(n-k)\sigma^2/s^2]/(n-k)} = \frac{X_n/J}{X_d/(n-k)}$$

$$= \frac{1}{J}(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q})'(s^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}')^{-1}(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q})$$

- Where:

$$X_{\text{numerator}} = X_n/J \sim \chi^2_J$$

$$X_{\text{denominator}} = X_d/(n-k) \sim \chi^2_{n-k}.$$

- Which is useful because:

$$F = \frac{X_n/J}{X_d/(n-k)} \sim F_{J,n-k}$$

## Testing Linear Restrictions, Examples

- With only 1 linear restriction, this simplifies down to:

$$H_0\colon r_1\beta_1 + r_2\beta_2 + \ldots + r_k\beta_k = \mathbf{r}\boldsymbol{\beta} = q$$

$$F_{1,n-k} = \frac{\sum_j (r_j\hat{\boldsymbol{\beta}}_j - \mathbf{q})^2}{\sum_j \sum_k r_j r_k Est.Cov.[b_j, b_k]}$$

and suppose the restriction is on the $\ell$th coefficient:

$$\beta_\ell = 0, \quad \text{so} \quad \mathbf{R} = [0, 0, \ldots, 0, 1, 0, \ldots, 0, 0], \quad \mathbf{q} = [\mathbf{0}]$$

so that $\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}$ is just the $\ell$ th diagonal value of $(\mathbf{X}'\mathbf{X})^{-1}$.

- We use $\hat{\boldsymbol{\beta}}$ to test for $\boldsymbol{\beta}$.

- Giving:

$$\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q} = \hat{\boldsymbol{\beta}}_\ell - \mathbf{q}, \qquad\qquad F_{1,n-k} = \frac{(\hat{\boldsymbol{\beta}}_\ell - \mathbf{q})^2}{Est.Var.[\hat{\boldsymbol{\beta}}_\ell]^{-\frac{1}{2}}}.$$

## Testing a Restriction with a Slightly Different Anaemia Model

```
anaemia <-
    read.table("https://jeffgill.org/files/jeffgill/files/anaemia.txt",
    header=TRUE,row.names=1)
a.lm2.out <- lm(Hb ~ Age + Menapause, data=anaemia)
summary(a.lm2.out)

Residuals:
    Min      1Q  Median      3Q     Max
-2.8375 -0.5839  0.1495  0.7821  2.0312

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  8.68823    1.15466   7.525 8.32e-07
Age          0.11716    0.03588   3.265  0.00456
Menapause    0.60002    1.10070   0.545  0.59275

Residual standard error: 1.198 on 17 degrees of freedom
Multiple R-squared: 0.7776,	Adjusted R-squared: 0.7514
F-statistic: 29.71 on 2 and 17 DF,  p-value: 2.827e-06
```

## Testing a Restriction with the Anaemia Data

- Test the hypothesis that `Menopause` is equal to zero:

$$\beta_2 = 0, \qquad J = 1, \qquad \mathbf{R} = [0, 0, 1], \qquad \mathbf{q} = 0$$

- Pieces:

$$(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q}) = [0, 0, 1] \begin{bmatrix} 8.688 \\ 0.117 \\ 0.600 \end{bmatrix} - 0 = 0.6, \qquad s^2 = 1.198, \qquad \mathbf{X}'\mathbf{X} = \begin{bmatrix} 20 & 876 & 10 \\ 876 & 43074 & 572 \\ 10 & 572 & 10 \end{bmatrix}$$

- Test Statistic:

$$F[J, n-K] = \frac{1}{J}(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q})'(s^2\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}')^{-1}(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{q})$$

$$F[1, 20-3] = \frac{1}{1}(0.6)' \left( (1.198)[0, 0, 1] \begin{bmatrix} 0.930 & -0.027 & 0.631 \\ -0.027 & 0.001 & -0.024 \\ 0.631 & -0.024 & 0.845 \end{bmatrix} [0, 0, 1]' \right)^{-1} (0.6)$$

$$F[1, 17] = 0.356$$

which is *not* in the tail, meaning that the restriction is *supported*.

## Testing a Restriction with the Anaemia Data

```
R <- c(0,0,1); q <- 0
b <- coef(a.lm2.out)
s.2 <- summary(a.lm2.out)$sigma
X <- cbind(rep(1,length=nrow(anaemia)),as.matrix(anaemia[,3:4]))
```

```
t(X) %*% X
                    Age Menapause
           20   876        10
Age       876 43074       572
Menapause  10   572        10
```

```
round(solve(t(X) %*% X),3)
                      Age Menapause
           0.930 -0.027     0.631
Age       -0.027  0.001    -0.024
Menapause  0.631 -0.024     0.845
```

```
( F <- t(R%*%b-q) %*% solve(s.2*R %*% solve(t(X) %*% X) %*% R) %*% (R%*%b-q) )
0.3558787

pf(F,1,17,lower.tail=FALSE)
0.5586653
```

## Testing *Nonlinear* Restrictions

- $H_0$: $c(\boldsymbol{\beta}) = q$ where $c()$ is some nonlinear function.

- Simple 1-restriction case:
$$z = \frac{c(\hat{\boldsymbol{\beta}})}{est.SE(c(\hat{\boldsymbol{\beta}}))} \sim t_{n-k}$$
(or equivalently $z^2 \sim F_{1,n-k}$).

- But getting $est.SE(c(\hat{\boldsymbol{\beta}}))$ is hard, so start with a Taylor series expansion:
$$f(b) = f(a) + f'(a)\frac{(b-a)^1}{1!} + f''(a)\frac{(b-a)^2}{2!} + f'''(a)\frac{(b-a)^3}{3!} + \dots$$
and then drop all but the first two terms to get an approximation:
$$f(b) \approx f(a) + f'(a)\frac{(b-a)^1}{1!} = f(a) + f'(a)(b-a).$$

- Substituting $\boldsymbol{\beta} = a$, $\hat{\boldsymbol{\beta}} = b$, and $c() = f()$, gives:
$$c(\hat{\boldsymbol{\beta}}) \approx c(\boldsymbol{\beta}) + \left(\frac{\partial c(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}}\right)^T (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}),$$
and also

## Testing *Nonlinear* Restrictions

- Now we can calculate the needed variance term:

$$\begin{aligned}
\mathrm{Var}(c(\hat{\boldsymbol\beta})) &= \mathbb{E}\left[c(\hat{\boldsymbol\beta})^2\right] - (\mathbb{E}[c(\hat{\boldsymbol\beta})])^2 \\
&\approx \mathbb{E}\left[\left(c(\boldsymbol\beta) + \left(\frac{\partial c(\boldsymbol\beta)}{\partial\boldsymbol\beta}\right)^T (\hat{\boldsymbol\beta} - \boldsymbol\beta)\right)^2\right] - (\mathbb{E}[c(\hat{\boldsymbol\beta})])^2 \\
&= \mathbb{E}\left[c(\boldsymbol\beta)^2 - 2c(\boldsymbol\beta)\left(\frac{\partial c(\boldsymbol\beta)}{\partial\boldsymbol\beta}\right)^T (\hat{\boldsymbol\beta} - \boldsymbol\beta) + \left(\frac{\partial c(\boldsymbol\beta)}{\partial\boldsymbol\beta}\right)^T (\hat{\boldsymbol\beta} - \boldsymbol\beta)^2 \left(\frac{\partial c(\boldsymbol\beta)}{\partial\boldsymbol\beta}\right)\right] - (\mathbb{E}[c(\hat{\boldsymbol\beta})])^2 \\
&= c(\boldsymbol\beta)^2 - 2c(\boldsymbol\beta)^2\left(\frac{\partial c(\boldsymbol\beta)}{\partial\boldsymbol\beta}\right)^T (0) + \mathbb{E}\left[\left(\frac{\partial c(\boldsymbol\beta)}{\partial\boldsymbol\beta}\right)^T (\hat{\boldsymbol\beta} - \boldsymbol\beta)^2 \left(\frac{\partial c(\boldsymbol\beta)}{\partial\boldsymbol\beta}\right)\right] - c(\boldsymbol\beta)^2 \\
&= \left(\frac{\partial c(\boldsymbol\beta)}{\partial\boldsymbol\beta}\right)^T \mathrm{Var}(\hat{\boldsymbol\beta}) \left(\frac{\partial c(\boldsymbol\beta)}{\partial\boldsymbol\beta}\right)
\end{aligned}$$

since $\mathbb{E}[(\hat{\boldsymbol\beta} - \boldsymbol\beta)^2] = \mathbb{E}[\hat{\boldsymbol\beta}^2 - 2\hat{\boldsymbol\beta}\boldsymbol\beta + \boldsymbol\beta^2] = \mathbb{E}\hat{\boldsymbol\beta}^2 - \boldsymbol\beta^2 = \left[\mathrm{Var}[\hat{\boldsymbol\beta}] + (\mathbb{E}[\hat{\boldsymbol\beta}])^2\right] - \boldsymbol\beta^2 = \mathrm{Var}(\hat{\boldsymbol\beta})$.

- This means that we can use sample estimates for $\partial c(\boldsymbol\beta)/\partial\boldsymbol\beta$ and plug in $s^2(\mathbf{X}'\mathbf{X})^{-1}$ for $\mathrm{Var}(\hat{\boldsymbol\beta})$ and then test with a normal distribution, provided reasonable sample size.

## Testing a Nonlinear Restriction with the Anaemia Data

- Test: $H_0$: $c(\boldsymbol{\beta}) = q$ where $c(\beta) = \beta^{\frac{1}{2}}$ and $q = 0$, versus $H_1$: $c(\boldsymbol{\beta}) \neq q$.
- Calculate: $\frac{\partial c(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = \frac{1}{2}(\boldsymbol{\beta})^{-\frac{1}{2}}$.
- From the model fit we have:

```
        Estimate Std. Error t value Pr(>|t|)
Age      0.11716    0.03588   3.265  0.00456
```

- Since this is a scalar test, use (with $c(\boldsymbol{\beta}) = 0$ as the restriction):

$$z = \frac{c(\hat{\boldsymbol{\beta}}) - 0}{est.SE(c(\hat{\boldsymbol{\beta}}))} \approx \frac{c(\hat{\boldsymbol{\beta}})}{\sqrt{\left(\frac{\partial c(\hat{\boldsymbol{\beta}})}{\partial \hat{\boldsymbol{\beta}}}\right)' \mathrm{Var}(\hat{\boldsymbol{\beta}}) \left(\frac{\partial c(\hat{\boldsymbol{\beta}})}{\partial \hat{\boldsymbol{\beta}}}\right)}}$$

$$= \frac{(0.11716)^{\frac{1}{2}}}{\sqrt{\frac{1}{2}(0.11716)^{-\frac{1}{2}}(0.03588)^2\frac{1}{2}(0.11716)^{-\frac{1}{2}}}} = 0.38257 \sim t_{n-k}$$

- We *fail* to reject the restriction since:

```
pt(0.38257,17,lower.tail=FALSE)
[1] 0.35339
```

## Heteroscedasticity

- If the variance of the residuals in the linear model is not constant over the range of one more **X** variables, then the regression model is heteroscedastic.

- Consider this old example from FiveThirtyEight.com:

## Weighted Least Squares for Known Heteroscedasticity

- A standard technique for compensating for non-constant error variance in LMs is to insert a diagonal matrix of weights, $\mathbf{\Omega}$, into the calculation of $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ such that the heteroscedasticity is mitigated.

- The $\mathbf{\Omega}$ matrix is created by taking the error variance of the $i^{\text{th}}$ case (estimated or known), $v_i$, and assigning the inverse to the $i^{\text{th}}$ diagonal: $\mathbf{\Omega}_{ii} = \frac{1}{v_i}$. The idea is that large error variances are reduced by multiplication of the reciprocal.

- Starting with $\mathbf{Y}_i = \mathbf{X}_i\boldsymbol{\beta} + \epsilon_i$, observe that there is heteroscedasticity in the error term so: $\epsilon_i = \epsilon v_i$, where the shared (minimum) variance is $\epsilon$ (i.e. non-indexed), and differences are reflected in the $v_i$ term.

- Really simple example: a heteroscedastic error vector: $\mathbf{E} = [1, 2, 3, 4]$. Then $\epsilon = 1$, and the $\mathbf{v}$ vector is $[1, 2, 3, 4]$. So by the logic above, the $\mathbf{\Omega}$ matrix for this example is:

$$\mathbf{\Omega} = \begin{bmatrix} \frac{1}{v_1} & 0 & 0 & 0 \\ 0 & \frac{1}{v_2} & 0 & 0 \\ 0 & 0 & \frac{1}{v_3} & 0 \\ 0 & 0 & 0 & \frac{1}{v_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

## Review of Weighted Least Squares

- Premultiply each term by the square root of the $\mathbf{\Omega}$ matrix (a Cholesky factorization given that $\mathbf{A}$ is a positive definite, but greatly simplified here since $\mathbf{\Omega}$ is diagonal).

$$\mathbf{\Omega}^{\frac{1}{2}}\mathbf{Y} = \mathbf{\Omega}^{\frac{1}{2}}\mathbf{X}\boldsymbol{\beta} + \mathbf{\Omega}^{\frac{1}{2}}\boldsymbol{\epsilon}.$$

- So if the heteroscedasticity in the error term is expressed as the diagonals of a matrix: $\boldsymbol{\epsilon} \sim (0, \sigma^2\mathbf{V})$, then this gives: $\boldsymbol{\epsilon} \sim (0, \mathbf{\Omega}\sigma^2\mathbf{V}) = (0, \sigma^2)$, and the heteroscedasticity is "removed."

- Now instead of minimizing

$$(\mathbf{Y} - \boldsymbol{X}\boldsymbol{\beta})'(\mathbf{Y} - \boldsymbol{X}\boldsymbol{\beta}),$$

  we minimize

$$(\mathbf{Y} - \boldsymbol{X}\boldsymbol{\beta})'\mathbf{\Omega}(\mathbf{Y} - \boldsymbol{X}\boldsymbol{\beta}),$$

  and the weighted least squares estimator is found by

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{\Omega}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Omega}\mathbf{Y}.$$

## GLS in R

```
diastolic.pressure.df <-
    read.table("https://jeffgill.org/files/jeffgill/files/bloodpressure.txt",
    header=FALSE)
dimnames(diastolic.pressure.df)[[2]] <- c("age","pressure")
summary(diastolic.pressure.df)

        age              pressure
Min.   :20.00   Min.   : 63.00
1st Qu.:30.25   1st Qu.: 71.00
Median :40.00   Median : 77.00
Mean   :39.57   Mean   : 79.11
3rd Qu.:49.00   3rd Qu.: 85.75
Max.   :59.00   Max.   :109.00
```

## GLS in R

```
attach(diastolic.pressure.df)
unweighted.lm <- lm(pressure~age)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 56.15693    3.99367  14.061  < 2e-16
age          0.58003    0.09695   5.983 2.05e-07
---
Residual standard error: 8.146 on 52 degrees of freedom
Multiple R-Squared: 0.4077,     Adjusted R-squared: 0.3963
F-statistic: 35.79 on 1 and 52 DF,  p-value: 2.05e-07

plot(age,pressure,pch=3)
abline(unweighted.lm)
```

## GLS in R

```
# REGRESS ABSOLUTE VALUE RESIDUALS ON PREDICTOR -> SD FUNCTION
resid.fit <- lm(abs(unweighted.lm$residuals)~age)

# OBTAIN FITTED VALUES FOR THE WEIGHTS
weights.fit <- 1/(resid.fit$fitted.values)^2

# USE THESE WEIGHTS FOR A GLS REGRESSION
weighted.lm <- lm(pressure~age,weights=weights.fit)
```

## GLS in R

```
summary(weighted.lm)

Residuals:
    Min      1Q  Median      3Q     Max
-2.0230 -0.9939 -0.0327  0.9250  2.2008

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 55.56577    2.52092  22.042  < 2e-16
age          0.59634    0.07924   7.526 7.19e-10
---
Residual standard error: 1.213 on 52 degrees of freedom
Multiple R-Squared: 0.5214,     Adjusted R-squared: 0.5122
F-statistic: 56.64 on 1 and 52 DF,  p-value: 7.187e-10
```

## Dealing with Heteroscedasticity: The *TWEED* Dataset

- TWEED = Terrorism in Western Europe: Events Data (Jan Oskar Engene)
- Contains information on events related to internal (domestic) terrorism in 18 countries in Western Europe.
- The time period covered is 1950 to 2004.
- By focusing on internal terrorism, the TWEED data set only includes events initiated by agents originating in the West European countries.
- Terrorism data is characterized by observable and latent groupings/clusters.
- So there is likely to be extra heteroscedasticity in the linear model from the presence of these groups.

## Dealing With Heteroscedascity From Group Effects

- What if there is heterogeneity in the standard errors from a group definition.
- This does not bias the coefficient estimates but will affect the estimated standard errors.
- Suppose there are $M$ groups, with modified degrees of freedom for the model now equal to

$$df_{\text{robust}} = \frac{M}{(M-1)}\frac{(N-1)}{(N-K)}$$

- Cluster-Robust standard errors (White, Huber, etc.) adjust the variance-covariance matrix with a "sandwich estimation" approach:

$$VC^{*} = f_{\text{robust}} \underbrace{(\mathbf{X}'\mathbf{X})^{-1}}_{\text{bread}} \underbrace{(\mathbf{U}'\mathbf{U})}_{\text{meat}} \underbrace{(\mathbf{X}'\mathbf{X})^{-1}}_{\text{bread}}$$

where:

- $\mathbf{U}$ is an $M \times k$ matrix,
- such that each row is produced by $\mathbf{X}_m * \mathbf{e}_m$ for group/cluster $m$, the element-wise product of the $N_m \times k$ matrix of observations in group $m$,
- and the $N_m$-length $\mathbf{e}_m$ corresponding residuals vector.

## Dealing With Heteroscedascity From Group Effects

- A non-clustered, non-robusted model:

```
tweed <- read.table("https://jeffgill.org/files/jeffgill/files/tweed2.txt",
    header=TRUE)
tweed.lm <- lm(I(killed+injured)~year+arrests+factor(attitude), data=tweed)
summary(tweed.lm)
                                                Estimate Std. Error t value Pr(>|t|)
(Intercept)                                      252.070    213.799    1.18    0.239
year                                              -0.123      0.108   -1.15    0.253
arrests                                            5.359      2.188    2.45    0.015
factor(attitude)Ethnic/regionalist Separatist     -6.373      7.137   -0.89    0.373
factor(attitude)Left wing extremist Other         -6.788      3.631   -1.87    0.063
factor(attitude)Right wing extremist Other        -4.710      3.632   -1.30    0.196

Residual standard error: 12.4 on 283 degrees of freedom
Multiple R-squared: 0.0374, Adjusted R-squared: 0.0204
F-statistic:  2.2 on 5 and 283 DF,  p-value: 0.0546
```

- The reference group for the factor is `Ethnic/regionalist Irredentist`.

## Dealing With Heteroscedascity From Group Effects

- A model with robust standard errors:

```
lapply(c("sandwich","lmtest","plm"),library, character.only=TRUE)
# GET FUNCTION FROM http://jeffgill.org/files/jeffgill/files/clx.r.txt
source("Class.Multilevel/Code/clx.R")
M <- length(table(tweed$attitude))
new.df <- tweed.lm$df / (tweed.lm$df - (M -1))
clx(tweed.lm, new.df, tweed$attitude)

                                             Estimate Std. Error t value Pr(>|t|)
(Intercept)                                  252.0696   112.4921    2.24    0.026
year                                          -0.1233     0.0566   -2.18    0.030
arrests                                        5.3585     6.6622    0.80    0.422
factor(attitude)Ethnic/regionalist Separatist -6.3725     0.3234  -19.70  < 2e-16
factor(attitude)Left wing extremist Other     -6.7882     0.7866   -8.63  4.5e-16
factor(attitude)Right wing extremist Other    -4.7096     0.0379 -124.22  < 2e-16
```

## Dealing With Heteroscedascity From Group Effects

- `Stata` calculates Huber-White standard errors differently by using the same coefficient estimates as the regular linear model results, but scaling the variance covariance matrix by the degrees of freedom:

$$VC^{\texttt{Stata}} = \frac{M}{(M-1)}\frac{(N-1)}{(N-K)} \times f_{\text{robust}}(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{U}'\mathbf{U})(\mathbf{X}'\mathbf{X})^{-1}$$

- This is easily obtained in `R` with the following:

```
library(sandwich)
hw.se <- sqrt(diag(vcovHC(tweed.lm,type="HC1")))
cbind(tweed.lm$coef,hw.se)

(Intercept)                                       252.06960 219.06530
year                                               -0.12333   0.11027
arrests                                             5.35855   6.04840
factor(attitude)Ethnic/regionalist Separatist      -6.37254   3.05351
factor(attitude)Left wing extremist Other          -6.78818   3.14835
factor(attitude)Right wing extremist Other         -4.70957   3.41273
```