Math 492 Quantum Physics Problem

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1 Problem Statement

We have set out to analyze how the parameters of quantam systems govern how the state of the system changes over time.

2 Axioms and Definitions

Definition 2.1) A complex A matrix is **skew hermitian** iff its transpose conjugate is equal to $-A$. Therefore, all 2×2 skew hermitian matricies fit the form

$$\begin{pmatrix} ai & b+ci \\ -b+ci & di \end{pmatrix} \text{ or alternatively, } \begin{pmatrix} ai & \beta \\ -\bar{\beta} & di \end{pmatrix}$$

Axiom 2.2) The state of a quantum system can be described entirely by $\begin{pmatrix} x \\ y \end{pmatrix}$, where x, y are complex numbers.

Axiom 2.3) Two quantum states are considered the same if one is a multiple of another, i.e. $\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} v \\ w \end{pmatrix}$ for any $x, y, \alpha, v, w \in \mathbb{C}$, means both states are the same. Therefore any state $\begin{pmatrix} x \\ y \end{pmatrix}$ can be represented as a ratio $z = x/y$.

Axiom 2.4) The function describing the state of a quantum system at time t is the solution to the ODE

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

where A is a skew hermitian matrix.

3 Initial Results

Combining Axiom 2.4 with the general form of a 2×2 skew hermitian matrix gives

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} ai & \beta \\ -\bar{\beta} & di \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ or } \begin{cases} \dot{x} = aix + \beta y \\ \dot{y} = -\bar{\beta}x + diy \end{cases}$$

So if you consider $z = y/x$, and its derivative, then we get the following new equation

$$\begin{aligned}
\dot{z} &= \frac{\dot{y}x - y\dot{x}}{x^2} \\
\dot{z} &= \frac{(-\bar{\beta}x + diy)x - y(aix + \beta y)}{x^2} \\
\dot{z} &= \frac{-\bar{\beta}x^2 + diyx - aixy - \beta y^2}{x^2} \\
\dot{z} &= -\bar{\beta} + diz - aiz - \beta z^2
\end{aligned}$$

From here we can make assumptions to make this easier to solve

4 Assume $\beta = 0, d = -a, a \neq 0$

Adding these assumptions to our initial result gives:

$$\dot{z} = -2aiz$$

Now we can define f, g as the real and complex parts of z respectively

$$\begin{aligned}
\dot{f} + i\dot{g} &= -2ai(f + ig) \\
\dot{f} + i\dot{g} &= -2aif + 2ag
\end{aligned}$$

Giving the system:

$$\begin{aligned}
&\begin{cases} \dot{f} = 2ag \\ \dot{g} = -2af \end{cases} \\
\therefore \ddot{f} &= 2a\dot{g} \\
\ddot{f} &= -4a^2 f \\
\ddot{f} + 4a^2 f &= 0 \\
\lambda^2 + 4a^2 &= 0 \\
\lambda^2 &= -4a^2 \\
\lambda = \pm\sqrt{-4a^2} &= \pm 2ai
\end{aligned}$$

So the solution is

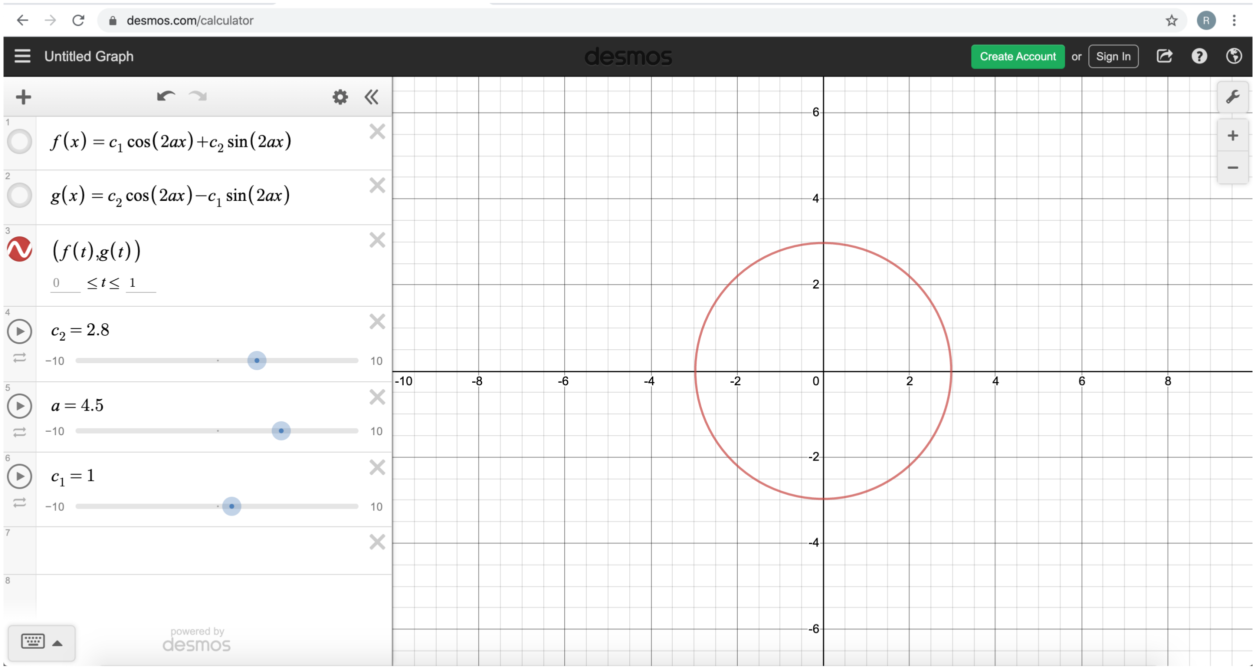
$$\begin{aligned}
f &= c_1 e^0 \cos(2at) + c_2 e^0 \sin(2at) \\
f &= c_1 \cos(2at) + c_2 \sin(2at) \\
\therefore \dot{f} &= -c_1 2a \sin(2at) + c_2 2a \cos(2at)
\end{aligned}$$

plugging f into our equation for $\dot{f}$ gives g:

$$\begin{aligned}
\dot{f} &= 2ag \\
-c_1 2a \sin(2at) + c_2 2a \cos(2at) &= 2ag \\
-c_1 \sin(2at) + c_2 \cos(2at) &= g
\end{aligned}$$

As a vector gives:

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} c_1 \cos(2at) + c_2 \sin(2at) \\ -c_1 \sin(2at) + c_2 \cos(2at) \end{pmatrix}$$



Changing c_1 and c_2 affects the size of the circle. Changing a only affects the paramaterization.

Now let's begin an analysis of the radius of the circle. Notice that $z\bar{z} = (f+g)(f-gi) = f^2 + g^2$, which is the radius of our circle. Since this is a constant with respect to time, we should expect $\frac{d}{dt}(z\bar{z}) = 0$. Let's check this assumption:

$$\begin{aligned}
\frac{d}{dt}(z\bar{z}) &= \dot{z}\bar{z} + z\dot{\bar{z}} \\
&= (-2aiz)\bar{z} + z\overline{(-2aiz)} \\
&= (-2aiz)\bar{z} + z\overline{(-2ai)}\bar{z} \\
&= (-2aiz)\bar{z} + z(2ai)\bar{z} \\
&= 0
\end{aligned}$$

So if the radius of the circle never changes it is just equal to the initial value $z(0)\bar{z}(0)$.

5 Assuming β is real, a=d=0

Let $\beta = b \in \mathbb{R}$. Recall our ODE.

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = A\begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

Here we are going to take a different approach than what we did in section 4. As we saw when $\beta = 0$ the matrix:

$$A = \begin{pmatrix} ai & 0 \\ 0 & -ai \end{pmatrix}$$

results in a situation where $z = constant$ since $\frac{d}{dt} z\bar{z} = 0$. For our current assumptions we have:

$$A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

and it follows that we could diagonalize this matrix to get the same situation as the previous case where $b = 0$. To diagonalize the matrix above we need to find a matrix T such that:

$$T\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} T^{-1}$$

is diagonal. Furthermore we need to modify the original system of equations we started with.

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$T\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = T\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$T\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = T\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} T^{-1}T \begin{pmatrix} x \\ y \end{pmatrix}$$

Now we have the original system in terms of the diagonal matrix that we want. To simplify, let:

$$\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix}$$

Then

$$u = T_{1,1}x + T_{1,2}y$$
$$v = T_{2,1}x + T_{2,2}y$$

To find T we need to find the eigenvalues and their corresponding eigenvectors for the matrix:

$$A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

Which are

$$\lambda_1 = ib,\ v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$
$$\lambda_2 = -ib,\ v_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Thus

$$T = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$
$$T^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{-i}{2} \\ \frac{-i}{2} & \frac{1}{2} \end{pmatrix}$$

Plugging the values of T into the previous system involving u and v we get

$$u = 1x + iy$$
$$v = ix + 1y$$

Now we want to consider $|vu^{-1}|$ similarly to when we considered $z = yx^{-1}$ which gives us

$$vu^{-1} = \frac{ix+y}{x+iy}$$

We can write this in terms of z with some manipulation as follows

$$vu^{-1} = \frac{ix+y}{x+iy} \cdot \frac{x^{-1}}{x^{-1}} = \frac{i+yx^{-1}}{1+iyx^{-1}} = \frac{i+z}{1+iz}$$

Therefore

$$|vu^{-1}| = \frac{|i+z|}{|1+iz|} = c$$

where $c \in \mathbb{R}$ is a constant. Letting $z = f + ig$ we get.

$$\frac{|f+(1+g)i|}{|(1-g)+if|} = c$$

Now we can solve this for f to get the shape of this equation.

$$\frac{\sqrt{f^2+(1+g)^2}}{\sqrt{(1-g)^2+f^2}} = c$$

$$\frac{f^2+(1+g)^2}{(1-g)^2+f^2} = c^2$$

$$f^2 + (1+g)^2 = c^2((1-g)^2 + f^2)$$

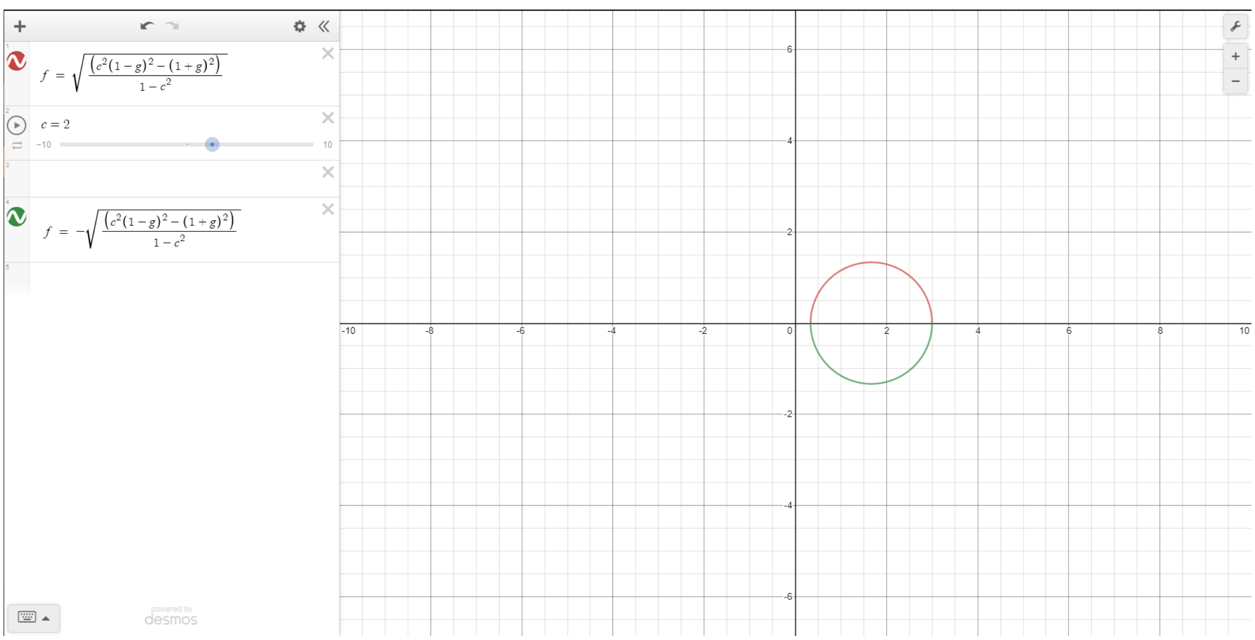
$$f^2 + (1+g)^2 = c^2(1-g)^2 + c^2 f^2$$

$$f^2 + c^2 f^2 = c^2(1-g)^2 - (1+g)^2$$

$$f^2(1-c^2) = c^2(1-g)^2 - (1+g)^2$$

$$f = \sqrt{\frac{c^2(1-g)^2-(1+g)^2}{1-c^2}}$$

Using the same tool as in section 4 we can graph this equation as well to see its shape. Note that since f represents the real part of z while g represents the imaginary part the axes are inverted.



Here, c controls both the center of the circle and the radius, where $c \neq 1$

6 Assuming β is purely imaginary, a=d=0

Let $\beta = bi$ where $b \in \mathbb{R}$ then our matrix look like:

$$A = \begin{pmatrix} 0 & bi \\ bi & 0 \end{pmatrix}$$

We can take the same approach here as described in section 5. The eigenvalues and eigenvectors of A are:

$$\lambda_1 = bi, v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \; \lambda_1 = -bi, v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Thus the matrix T we are looking for is:

$$T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Plugging the values of T into the system with u and v gives us:

$$u = 1x + -1y = x - y$$
$$v = 1x + 1y = x + y$$

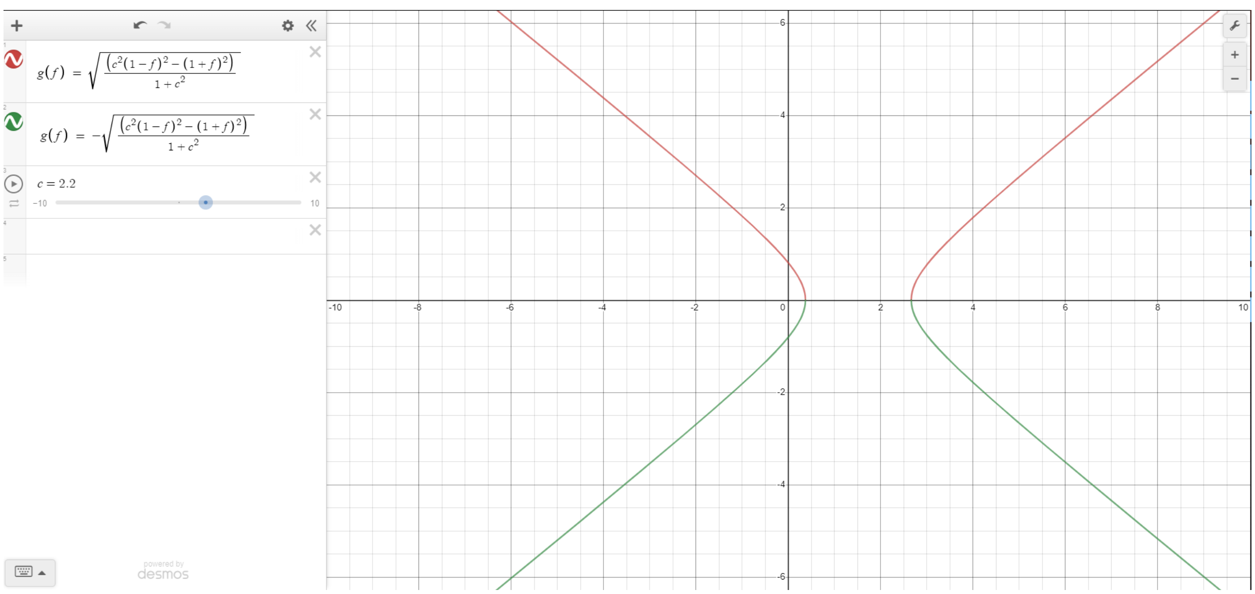
Then:

$$|vu^{-1}| = \frac{x+y}{x-y} \cdot \frac{x^{-1}}{x^{-1}} = \frac{1+yx^{-1}}{1-yx^{-1}} = \frac{1+z}{1-z} = c$$

For some $c \in \mathbb{R}$. Letting $z = f + gi$ for $f, g \in \mathbb{R}$, and solving for g results in:

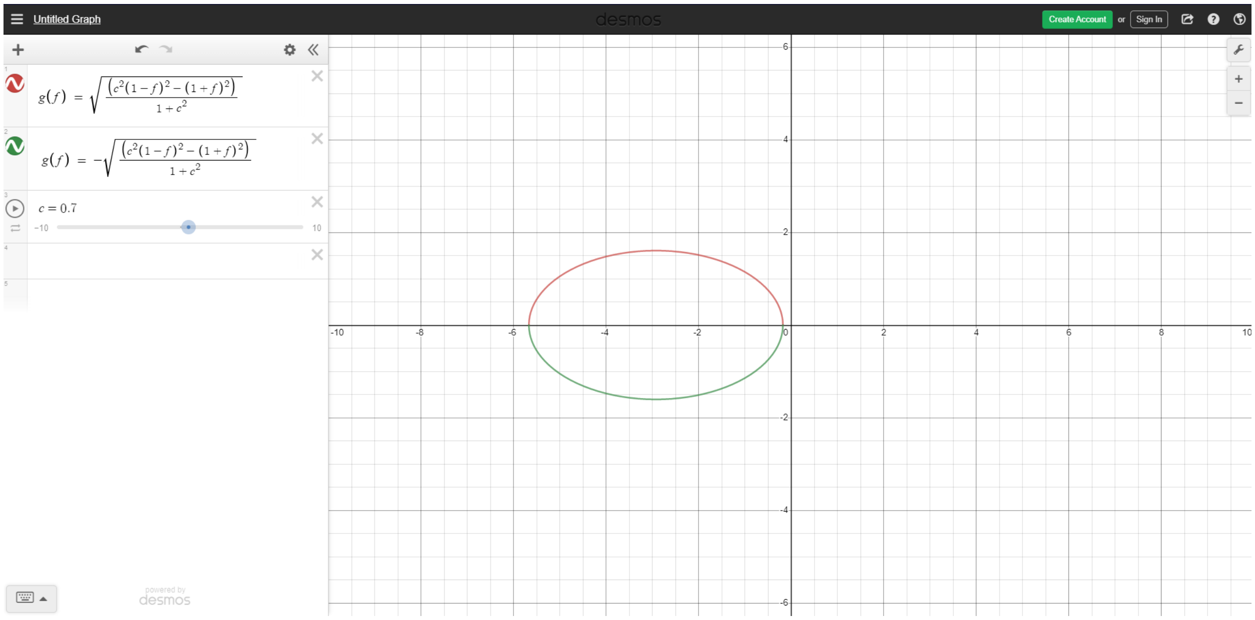
$$g = \sqrt{\frac{c^2(1-f)^2-(1+f)^2}{1+c^2}}$$

The equation above results in a few possible shapes depending on the value of c.
If $c > 1$ or $c < -1$:



When $c > 1$ or $c < -1$ we get a hyperbola whose center and focal points change as c changes.

If $-1 \leq c \leq 1$:



When $-1 \le c \le 1$ we get an ellipse or circle whose center and focal points change as c changes between this interval.