

# Math 492 Quantum Physics Problem

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## 1 Problem Statement

We have set out to analyze how the parameters of quantum systems govern how the state of the system changes over time.

## 2 Axioms and Definitions

Definition 2.1) A complex  $A$  matrix is **skew hermitian** iff its transpose conjugate is equal to  $-A$ . Therefore, all  $2 \times 2$  skew hermitian matrices fit the form

$$\begin{pmatrix} ai & b + ci \\ -b + ci & di \end{pmatrix} \text{ or alternatively, } \begin{pmatrix} ai & \beta \\ -\bar{\beta} & di \end{pmatrix}$$

Axiom 2.2) The state of a quantum system can be described entirely by  $\begin{pmatrix} x \\ y \end{pmatrix}$ , where  $x, y$  are complex numbers.

Axiom 2.3) Two quantum states are considered the same if one is a multiple of another, i.e.  $\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} v \\ w \end{pmatrix}$  for any  $x, y, \alpha, v, w \in \mathbb{C}$ , means both states are the same. Therefore any state  $\begin{pmatrix} x \\ y \end{pmatrix}$  can be represented as a ratio  $z = x/y$ .

Axiom 2.4) The function describing the state of a quantum system at time  $t$  is the solution to the ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

where  $A$  is a skew hermitian matrix.

## 3 Initial Results

Combining Axiom 2.4 with the general form of a  $2 \times 2$  skew hermitian matrix gives

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} ai & \beta \\ -\bar{\beta} & di \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ or } \begin{cases} \dot{x} = aix + \beta y \\ \dot{y} = -\bar{\beta}x + diy \end{cases}$$

So if you consider  $z = y/x$ , and its derivative, then we get the following new equation

$$\begin{aligned} \dot{z} &= \frac{\dot{y}x - y\dot{x}}{x^2} \\ \dot{z} &= \frac{(-\bar{\beta}x + diy)x - y(aix + \beta y)}{x^2} \\ \dot{z} &= \frac{-\bar{\beta}x^2 + diyx - aixy - \beta y^2}{x^2} \\ \dot{z} &= -\bar{\beta} + diz - aiz - \beta z^2 \end{aligned}$$

From here we can make assumptions to make this easier to solve

#### 4 Assume $\beta = 0, d = -a, a \neq 0$

Adding these assumptions to our initial result gives:

$$\dot{z} = -2aiz$$

Now we can define  $f, g$  as the real and complex parts of  $z$  respectively

$$\begin{aligned}\dot{f} + i\dot{g} &= -2ai(f + ig) \\ \dot{f} + i\dot{g} &= -2aif + 2ag\end{aligned}$$

Giving the system:

$$\begin{aligned}\begin{cases} \dot{f} = 2ag \\ \dot{g} = -2af \end{cases} \\ \therefore \ddot{f} &= 2a\dot{g} \\ \ddot{f} &= -4a^2f \\ \ddot{f} + 4a^2f &= 0 \\ \lambda^2 + 4a^2 &= 0 \\ \lambda^2 &= -4a^2 \\ \lambda &= \pm\sqrt{-4a^2} = \pm 2ai\end{aligned}$$

So the solution is

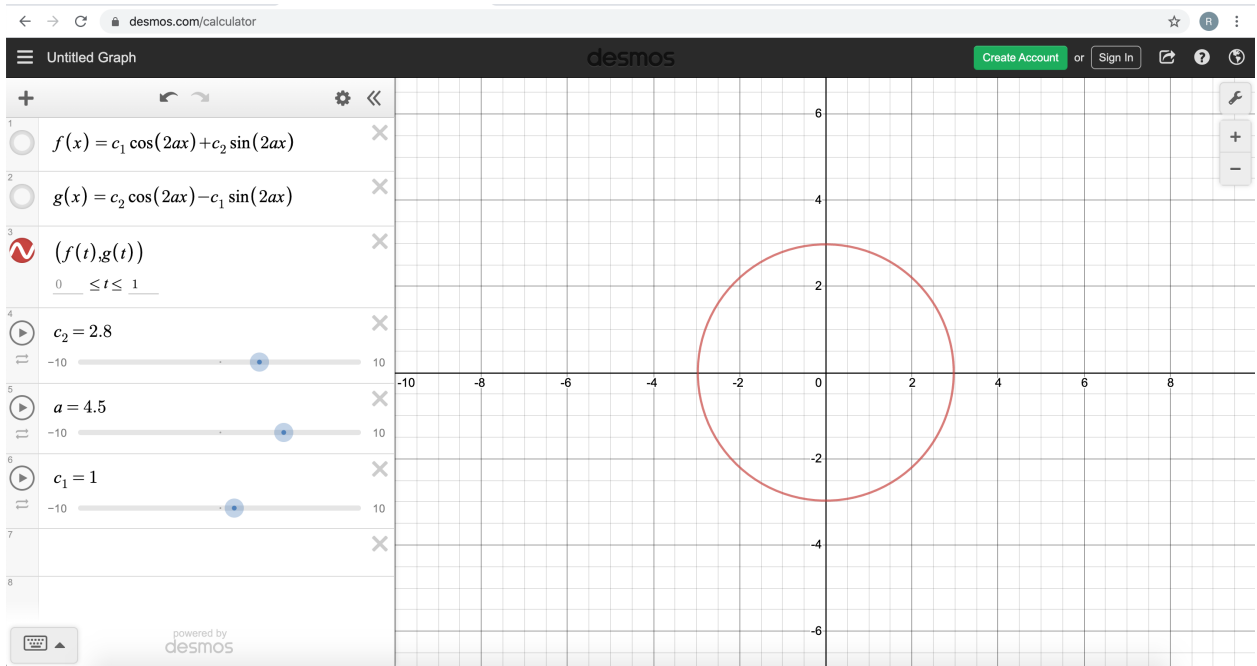
$$\begin{aligned}f &= c_1 e^0 \cos(2at) + c_2 e^0 \sin(2at) \\ f &= c_1 \cos(2at) + c_2 \sin(2at) \\ \therefore \dot{f} &= -c_1 2a \sin(2at) + c_2 2a \cos(2at)\end{aligned}$$

plugging  $f$  into our equation for  $\dot{f}$  gives  $g$ :

$$\begin{aligned}\dot{f} &= 2ag \\ -c_1 2a \sin(2at) + c_2 2a \cos(2at) &= 2ag \\ -c_1 \sin(2at) + c_2 \cos(2at) &= g\end{aligned}$$

As a vector gives:

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} c_1 \cos(2at) + c_2 \sin(2at) \\ -c_1 \sin(2at) + c_2 \cos(2at) \end{pmatrix}$$



Changing  $c_1$  and  $c_2$  affects the size of the circle. Changing  $a$  only affects the parameterization.

Now let's begin an analysis of the radius of the circle. Notice that  $z\bar{z} = (f + g)(f - gi) = f^2 + g^2$ , which is the radius of our circle. Since this is a constant with respect to time, we should expect  $\frac{d}{dt}(z\bar{z}) = 0$ . Let's check this assumption:

$$\begin{aligned}
 \frac{d}{dt}(z\bar{z}) &= \dot{z}\bar{z} + z\dot{\bar{z}} \\
 &= (-2aiz)\bar{z} + z\overline{(-2aiz)} \\
 &= (-2aiz)\bar{z} + z\overline{(-2ai)}\bar{z} \\
 &= (-2aiz)\bar{z} + z(2ai)\bar{z} \\
 &= 0
 \end{aligned}$$

## 5 Assuming $\beta$ is real, $a=d=0$

This gives us the equation

$$\begin{aligned}
 \dot{z} &= -\bar{\beta} + diz - aiz - \beta z^2 \\
 &= -\bar{\beta} - \beta z^2 \\
 &= -\beta(1 + z^2)
 \end{aligned}$$

Which is separable and has the solution  $z = \tan(-\beta t + C)$