

Math 492 Quantum Physics Problem

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1 Problem Statement

We have set out to analyze how the parameters of quantum systems govern how the state of the system changes over time.

2 Axioms and Definitions

Definition 2.1) A complex A matrix is **skew hermitian** iff its transpose conjugate is equal to $-A$. Therefore, all 2×2 skew hermitian matrices fit the form

$$\begin{pmatrix} ai & b + ci \\ -b + ci & di \end{pmatrix} \text{ or alternatively, } \begin{pmatrix} ai & \beta \\ -\bar{\beta} & di \end{pmatrix}$$

Axiom 2.2) The state of a quantum system can be described entirely by $\begin{pmatrix} x \\ y \end{pmatrix}$, where x, y are complex numbers.

Axiom 2.3) Two quantum states are considered the same if one is a multiple of another, i.e. $\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} v \\ w \end{pmatrix}$ for any $x, y, \alpha, v, w \in \mathbb{C}$, means both states are the same. Therefore any state $\begin{pmatrix} x \\ y \end{pmatrix}$ can be represented as a ratio $z = x/y$.

Axiom 2.4) The function describing the state of a quantum system at time t is the solution to the ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

where A is a skew hermitian matrix.

3 Initial Results

Combining Axiom 2.4 with the general form of a 2×2 skew hermitian matrix gives

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} ai & \beta \\ -\bar{\beta} & di \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ or } \begin{cases} \dot{x} = aix + \beta y \\ \dot{y} = -\bar{\beta}x + diy \end{cases}$$

So if you consider $z = y/x$, and its derivative, then we get the following new equation

$$\begin{aligned} \dot{z} &= \frac{\dot{y}x - y\dot{x}}{x^2} \\ \dot{z} &= \frac{(-\bar{\beta}x + diy)x - y(aix + \beta y)}{x^2} \\ \dot{z} &= \frac{-\bar{\beta}x^2 + diyx - aixy - \beta y^2}{x^2} \\ \dot{z} &= -\bar{\beta} + diz - aiz - \beta z^2 \end{aligned}$$

From here we can make assumptions to make this easier to solve

4 Assume $\beta = 0, d = -a, a \neq 0$

Adding these assumptions to our initial result gives:

$$\dot{z} = -2aiz$$

Now we can define f, g as the real and complex parts of z respectively

$$\begin{aligned}\dot{f} + i\dot{g} &= -2ai(f + ig) \\ \dot{f} + i\dot{g} &= -2aif + 2ag\end{aligned}$$

Giving the system:

$$\begin{aligned}\begin{cases} \dot{f} = 2ag \\ \dot{g} = -2af \end{cases} \\ \therefore \ddot{f} &= 2a\dot{g} \\ \ddot{f} &= -4a^2f \\ \ddot{f} + 4a^2f &= 0 \\ \lambda^2 + 4a^2 &= 0 \\ \lambda^2 &= -4a^2 \\ \lambda &= \pm\sqrt{-4a^2} = \pm 2ai\end{aligned}$$

So the solution is

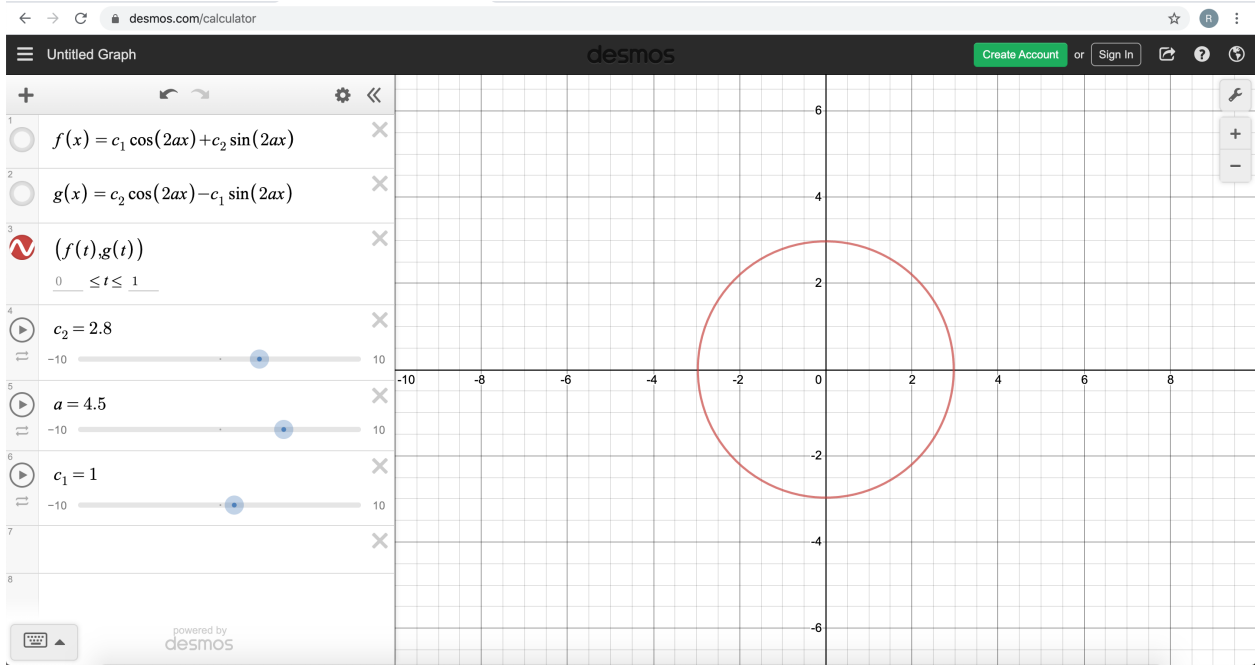
$$\begin{aligned}f &= c_1 e^0 \cos(2at) + c_2 e^0 \sin(2at) \\ f &= c_1 \cos(2at) + c_2 \sin(2at) \\ \therefore \dot{f} &= -c_1 2a \sin(2at) + c_2 2a \cos(2at)\end{aligned}$$

plugging f into our equation for \dot{f} gives g :

$$\begin{aligned}\dot{f} &= 2ag \\ -c_1 2a \sin(2at) + c_2 2a \cos(2at) &= 2ag \\ -c_1 \sin(2at) + c_2 \cos(2at) &= g\end{aligned}$$

As a vector gives:

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} c_1 \cos(2at) + c_2 \sin(2at) \\ -c_1 \sin(2at) + c_2 \cos(2at) \end{pmatrix}$$



Changing c_1 and c_2 affects the size of the circle. Changing a only affects the parameterization.

Now let's begin an analysis of the radius of the circle. Notice that $z\bar{z} = (f + g)(f - gi) = f^2 + g^2$, which is the radius of our circle. Since this is a constant with respect to time, we should expect $\frac{d}{dt}(z\bar{z}) = 0$. Let's check this assumption:

$$\begin{aligned}
 \frac{d}{dt}(z\bar{z}) &= \dot{z}\bar{z} + z\dot{\bar{z}} \\
 &= (-2aiz)\bar{z} + z\overline{(-2aiz)} \\
 &= (-2aiz)\bar{z} + z\overline{(-2ai)}\bar{z} \\
 &= (-2aiz)\bar{z} + z(2ai)\bar{z} \\
 &= 0
 \end{aligned}$$

So if the radius of the circle never changes it is just equal to the initial value $z(0)\bar{z}(0)$.

5 Assuming β is real

Let $\beta = b \in \mathbb{R}$. Recall our ODE.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

Here we are going to take a different approach than what we did in section 4. As we saw when $\beta = 0$ the matrix:

$$A = \begin{pmatrix} ai & 0 \\ 0 & -ai \end{pmatrix}$$

results in a situation where $z = \text{constant}$ since $\frac{d}{dt}z\bar{z} = 0$. For our current assumptions we have:

$$A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

and it follows that we could diagonalize this matrix to get the same situation as the previous case where $b = 0$. To diagonalize the matrix above we need to find a matrix T such that:

$$T \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} T^{-1}$$

is diagonal. Furthermore we need to modify the original system of equations we started with.

$$\begin{aligned}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ T \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= T \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ T \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= T \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} T^{-1} T \begin{pmatrix} x \\ y \end{pmatrix}\end{aligned}$$

Now we have the original system in terms of the diagonal matrix that we want. To simplify, let:

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}$$

Then

$$\begin{aligned}u &= T_{1,1}x + T_{1,2}y \\ v &= T_{2,1}x + T_{2,2}y\end{aligned}$$

To find T we need to find the eigenvalues and their corresponding eigenvectors for the matrix:

$$A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

Which are

$$\begin{aligned}\lambda_1 &= ib, v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \\ \lambda_2 &= -ib, v_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}\end{aligned}$$

Thus

$$\begin{aligned}T &= \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\ T^{-1} &= \begin{pmatrix} \frac{1}{2} & \frac{-i}{2} \\ \frac{-i}{2} & \frac{1}{2} \end{pmatrix}\end{aligned}$$

Plugging the values of T into the previous system involving u and v we get

$$\begin{aligned}u &= 1x + iy \\ v &= ix + 1y\end{aligned}$$

Now we want to consider $|vu^{-1}|$ similarly to when we considered $z = yx^{-1}$ which gives us

$$vu^{-1} = \frac{ix+y}{x+iy}$$

We can write this in terms of z with some manipulation as follows

$$vu^{-1} = \frac{ix+y}{x+iy} \cdot \frac{x^{-1}}{x^{-1}} = \frac{i+yx^{-1}}{1+iyx^{-1}} = \frac{i+z}{1+iz}$$

Therefore

$$|vu^{-1}| = \frac{|i+z|}{|1+iz|} = c$$

where $c \in \mathbb{R}$ is a constant. Letting $z = f + ig$ we get.

$$\frac{|f+(1+g)i|}{|(1-g)+if|} = c$$

Now we can solve this for f to get the shape of this equation.

$$\frac{\sqrt{f^2+(1+g)^2}}{\sqrt{(1-g)^2+f^2}} = c$$

$$\frac{f^2+(1+g)^2}{(1-g)^2+f^2} = c^2$$

$$f^2 + (1+g)^2 = c^2((1-g)^2 + f^2)$$

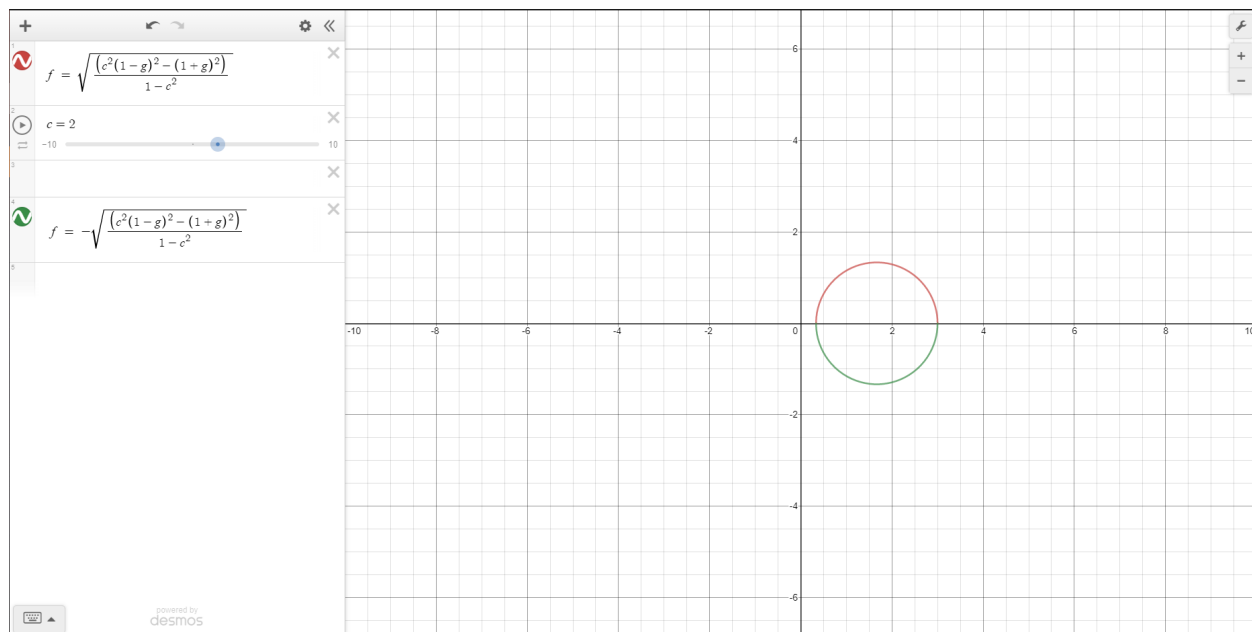
$$f^2 + (1+g)^2 = c^2(1-g)^2 + c^2f^2$$

$$f^2 + c^2f^2 = c^2(1-g)^2 - (1+g)^2$$

$$f^2(1-c^2) = c^2(1-g)^2 - (1+g)^2$$

$$f = \sqrt{\frac{c^2(1-g)^2 - (1+g)^2}{1-c^2}}$$

Using the same tool as in section 4 we can graph this equation as well to see its shape. Note that since f represents the real part of z while g represents the imaginary part the axes are inverted.



Here, c controls both the center of the circle and the radius, where $c \neq 1$

6 Assuming β is real, $a=d=0$

This gives us the equation

$$\begin{aligned}\dot{z} &= -\bar{\beta} + d iz - a iz - \beta z^2 \\ &= -\bar{\beta} - \beta z^2 \\ &= -\beta(1 + z^2)\end{aligned}$$

Which is separable and has the solution $z = \tan(-\beta t + C)$