Math 492 Quantam Physics Problem

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1 Problem Statement

We have set out to analyze how the parameters of quantam systems govern how the state of the system changes over time.

2 Axioms and Definitions

Definition 2.1) A complex A matrix is **skew hermitian** iff its transpose conjugate is equal to $-A$. Therefore, all 2×2 skew hermitian matricies fit the form

$$\begin{pmatrix} ai & b+ci \\ -b+ci & di \end{pmatrix} \text{ or alternatively, } \begin{pmatrix} ai & \beta \\ -\bar{\beta} & di \end{pmatrix}$$

Axiom 2.2) The state of a quantam system can be described entirely by $\begin{pmatrix} x \\ y \end{pmatrix}$, where x, y are complex numbers.

Axiom 2.3) Two quantam states are considered the same if one is a multiple of another, i.e. $\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} v \\ w \end{pmatrix}$ for any $x, y, \alpha, v, w \in \mathbb{C}$, means both states are the same. Therefore any state $\begin{pmatrix} x \\ y \end{pmatrix}$ can be represented as a ratio $z = x/y$.

Axiom 2.4) The function describing the state of a quantam system at time t is the solution to the ODE

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

where A is a skew hermation matrix.

3 Initial Results

Combining Axiom 2.4 with the general form of a 2×2 skew hermation matrix gives

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} ai & \beta \\ -\bar{\beta} & di \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ or } \begin{cases} \dot{x} = aix + \beta y \\ \dot{y} = -\bar{\beta}x + diy \end{cases}$$

So if you consider $z = y/x$, and its derivative, then we get the following new equation

$$\begin{aligned}
\dot{z} &= \frac{\dot{y}x - y\dot{x}}{x^2} \\
\dot{z} &= \frac{(-\bar{\beta}x + diy)x - y(aix + \beta y)}{x^2} \\
\dot{z} &= \frac{-\bar{\beta}x^2 + diyx - aixy - \beta y^2}{x^2} \\
\dot{z} &= -\bar{\beta} + diz - aiz - \beta z^2
\end{aligned}$$

From here we can make assumptions to make this easier to solve

4 Assume $\beta = 0, d = -a, a \neq 0$

Adding these assumptions to our initial result gives:

$$\dot{z} = -2aiz$$

Now we can define f, g as the real and complex parts of z respectively

$$\begin{aligned} \dot{f} + i\dot{g} &= -2ai(f + ig) \\ \dot{f} + i\dot{g} &= -2aif + 2ag \end{aligned}$$

Giving the system:

$$\begin{aligned} &\begin{cases} \dot{f} = 2ag \\ \dot{g} = -2af \end{cases} \\ \therefore \ddot{f} &= 2a\dot{g} \\ \ddot{f} &= -4a^2 f \\ \ddot{f} + 4a^2 f &= 0 \\ \lambda^2 + 4a^2 &= 0 \\ \lambda^2 &= -4a^2 \\ \lambda = \pm\sqrt{-4a^2} &= \pm 2ai \end{aligned}$$

So the solution is

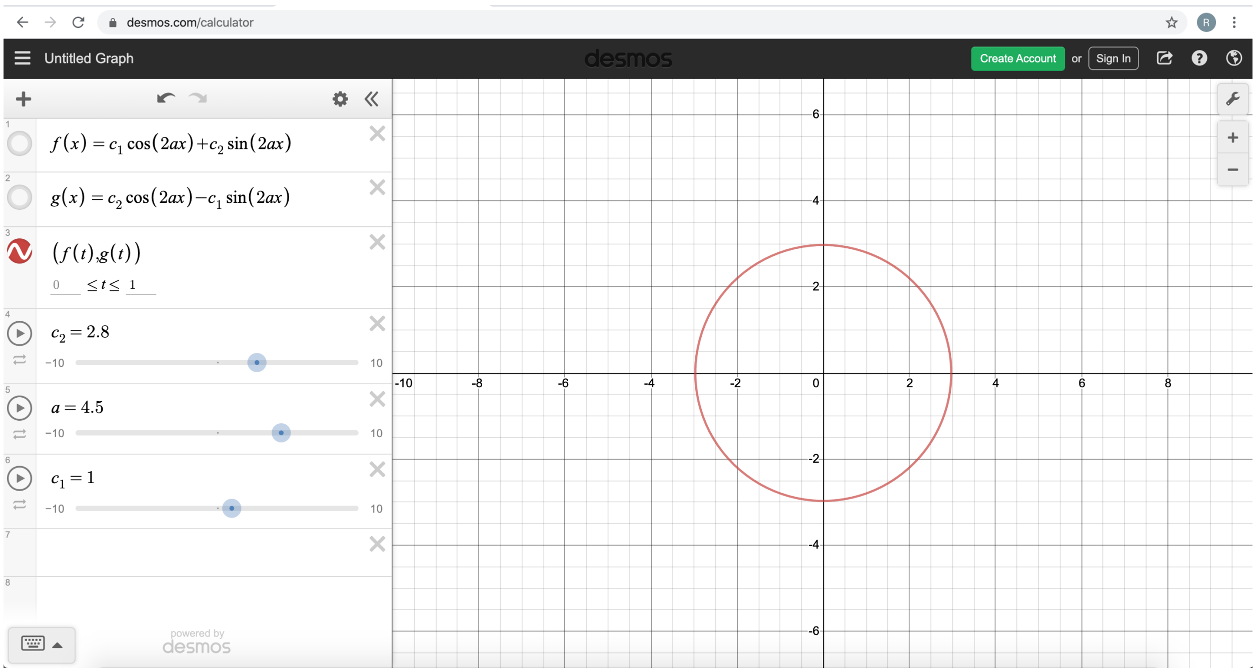
$$\begin{aligned} f &= c_1 e^0 \cos(2at) + c_2 e^0 \sin(2at) \\ f &= c_1 \cos(2at) + c_2 \sin(2at) \\ \therefore \dot{f} &= -c_1 2a \sin(2at) + c_2 2a \cos(2at) \end{aligned}$$

plugging f into our equation for $\dot{f}$ gives g:

$$\begin{aligned} \dot{f} &= 2ag \\ -c_1 2a \sin(2at) + c_2 2a \cos(2at) &= 2ag \\ -c_1 \sin(2at) + c_2 \cos(2at) &= g \end{aligned}$$

As a vector gives:

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} c_1 \cos(2at) + c_2 \sin(2at) \\ -c_1 \sin(2at) + c_2 \cos(2at) \end{pmatrix}$$



Changing c_1 and c_2 affects the size of the circle. Changing a only affects the paramaterization.

Now let's begin an analysis of the radius of the circle. Notice that $z\bar{z} = (f+g)(f-gi) = f^2 + g^2$, which is the radius of our circle. Since this is a constant with respect to time, we should expect $\frac{d}{dt}(z\bar{z}) = 0$. Let's check this assumption:

$$\begin{aligned}
\frac{d}{dt}(z\bar{z}) &= \dot{z}\bar{z} + z\dot{\bar{z}} \\
&= (-2aiz)\bar{z} + z\overline{(-2aiz)} \\
&= (-2aiz)\bar{z} + z\overline{(-2ai)}\bar{z} \\
&= (-2aiz)\bar{z} + z(2ai)\bar{z} \\
&= 0
\end{aligned}$$

So if the radius of the circle never changes it is just equal to the initial value $z(0)\bar{z}(0)$.

5 Assuming β is real

Let $\beta = b \in \mathbb{R}$. Recall our ODE.

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = A\begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

The eivenvalues of A are $\lambda_1 = i\beta$, $\lambda_2 = -i\beta$. The eigenvectors corresponding to each λ are $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} i \\ 1 \end{pmatrix}$ respectively. So there exists R and R^{-1} so $A = R^{-1}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}R$.

$$R\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = RAR^{-1}R\begin{pmatrix} x \\ y \end{pmatrix}$$

1:

$$\frac{c-i}{1-ci} = \frac{(c-i)(1+ci)}{1+c^2} = \frac{-i+c^2i+2c}{1+c^2}$$

2:

$$\frac{-c-i}{1+ci} = \frac{-(c+i)(1-ci)}{1+c^2} = \frac{-i+c^2i-2c}{1+c^2}$$

So in General

$$\frac{-i + c^2 i \pm 2c}{1 + c^2}$$

6 Assuming β is real, a=d=0

This gives us the equation

$$\begin{aligned}\dot{z} &= -\bar{\beta} + diz - aiz - \beta z^2 \\ &= -\bar{\beta} - \beta z^2 \\ &= -\beta(1 + z^2)\end{aligned}$$

Which is separable and has the solution $z = \tan(-\beta t + C)$