PHYS4080 Astroparticle Project

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August 2023

Question 1

Here, we want to simplify some of the expressions given in the background document to find the thermally averaged annihilation cross-section for a stable relic particle under a few assumptions/simplifications. Firstly, this particle is taken as stable (such that it doesn't spontaneously decay) and that no other particles (SM or otherwise) decay into it. We're assuming that this hypothetical particle can only interact with itself or its own antiparticle, and hence the relative momentum between particle species i and j is simplified from $p_{ij} \to p_{11}$ with

$$\begin{aligned} p_{ij} &= \frac{1}{2\sqrt{s}}\left[s - (m_i + m_j)^2\right]^{\frac{1}{2}}\left[s - (m_i - m_j)^2\right]^{\frac{1}{2}} \\ \Rightarrow p_{11} &= \frac{1}{2\sqrt{s}}\left[s - (m_1 + m_1)^2\right]^{\frac{1}{2}}\left[s - (m_1 - m_1)^2\right]^{\frac{1}{2}} \\ &= \frac{1}{2\sqrt{s}}\left[s - 4{m_1}^2\right]^{\frac{1}{2}}\sqrt{s} \\ &= \frac{\sqrt{s - 4{m_1}^2}}{2} \end{aligned}$$

The derivative of this relative momentum with respect to the Mandelstam variable is then

$$\frac{dp_{11}}{ds} = \frac{1}{4\sqrt{s - 4{m_1}^2}} \quad \Longrightarrow \quad dp_{11} = \frac{ds}{4\sqrt{s - 4{m_1}^2}}$$

This will allow us to change the integration variable and bounds within the general interaction cross section expression:

$$\langle \sigma_{\mathrm{eff}} v \rangle \equiv \frac{\int_0^\infty p_{11}^2 W_{\mathrm{eff}} K_1\left(\frac{\sqrt{s}}{T}\right) dp_{11}}{{m_1}^4 T\left[\sum_i \frac{g_i}{g_1}\frac{{m_i}^2}{{m_1}^2}K_2\left(\frac{m_i}{T}\right)\right]^2} \tag{1}$$

when $p_{11} = 0$, we have that $s = 4{m_1}^2$ which gives a new integral bound when we change integration variables from $p_{11} \to s$. When $p_{11} \to \infty$, we also have that $s \to \infty$, and so our integration bounds are transformed from $[0, \infty) \to [4{m_1}^2, \infty)$. Also, we have that

$$p_{11}^2 = \frac{s - 4{m_1}^2}{4}$$

and so equation (1) becomes (under change of variables, and a further simplification that we're only dealing with one species of particle)

$$\langle \sigma_{\mathrm{eff}} v \rangle \equiv \int_{4{m_1}^2}^\infty \frac{\sqrt{s - 4{m_1}^2}\, W_{\mathrm{eff}} K_1\left(\frac{\sqrt{s}}{T}\right) ds}{16{m_1}^4 T\left[K_2\left(\frac{m_1}{T}\right)\right]^2} \tag{2}$$

From here we can expand and simplify the effective invariant annihilation rate, W_{eff}:

$$W_{\mathrm{eff}} \equiv 2\sum_{ij}\frac{p_{ij}}{p_{11}}\frac{g_i g_j}{g_1^2}(s - {m_i}^2 - {m_j}^2)\sigma_{ij}\, v_{ij,\mathrm{lab}}$$

As before, we're dealing with just one particle species and so the sum and the first two ratios disappear. We have that

$$v_{ij,\mathrm{lab}} = \frac{v_{ij,\mathrm{cms}}}{2(1 - 2m^2/s)}$$

and so

$$\begin{aligned} W_{\mathrm{eff}} &= \frac{2(s - 2m^2)\sigma_{11} v_{11,\mathrm{cms}}}{2(1 - 2m^2/s)} \\ &= s\sigma_{11} v_{11,\mathrm{cms}} \end{aligned}$$

Finally, substituting this into equation (2), as well as making the substitution $x = m/T$, we can finally restate equation (1) in the desired form of

$$\langle \sigma_{\mathrm{eff}} v \rangle(x) = \int_{4m^2}^\infty \frac{xs\sqrt{s - 4m^2}K_1\left(\frac{x\sqrt{s}}{m}\right)\sigma v_{\mathrm{cms}}(s)}{16m^5 K_2(x)^2}\, ds \tag{3}$$

Question 2

We wrote a program to compute the dimensionless abundance Y (and by extension the cosmological abundance Ωh^2) in Python, utilising the packages `numpy`, `scipy` and `matplotlib`. This program, including all code that generates the figures in this report, is attached along with the submission.

We note that we had to rescale the relevant equations (such as equation (3) in the task sheet) in order to shift the ODEs closer to order unity which is where numerical integration works best. Naturally, one might want to integrate the function from the beginning of time (i.e. $x = 0$) up to the present time ($x \sim 10^3$ for most particle masses of interest), but problems follow.

Since the cosmological epoch parameter x is related to the change in abundance via an inverse square law, overly small or large x values shift the ODE away from order unity where the numerical integration can no longer work with good accuracy. As such, we typically chose an initial x such that $1 \leq x \lesssim 10$ so that it was low enough that a particle had not yet frozen out, but high enough that we avoid most numerical integration errors. We usually didn't encounter numerical errors for the upper bound of $x \sim 10^3$, although we reduced it to $x \sim 10^2$ in the event of an obvious error.

Question 3

We present plots for parts (a) through (c) in Figures 1 to 3 respectively.

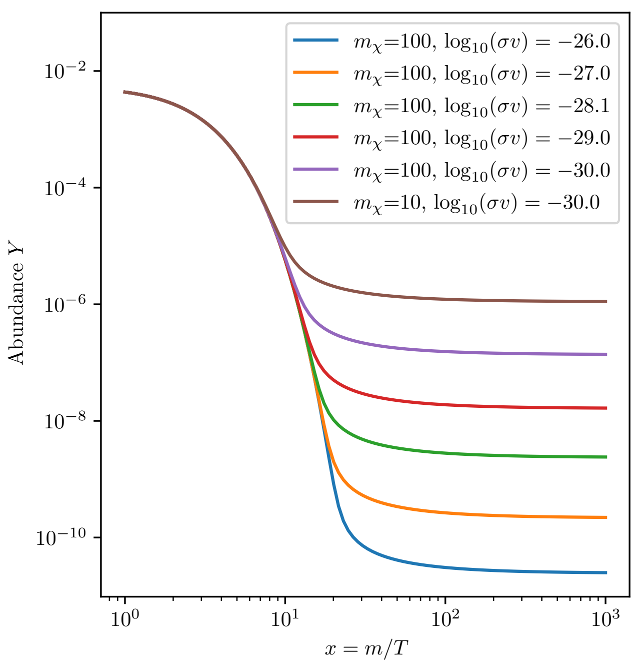

Figure. 1 The dimensionless comoving abundance Y is clearly dependent on the mass and interaction cross section of the associated particle. We see that all particle species seem to have the same abundance for $x \lesssim 10^1$, but are then roughly equally dependent on the order of magnitude of mass and cross section. The present day abundance (that is, Y in the high x domain) is affected more or less equally for an order of magnitude change of mass or cross section (at least for these values), where a decrease in mass or cross section corresponds to an increase in abundance and vice versa. The discrepancy between the curves at $x = (1 \text{ to } 3) \times 10^1$ is from each species' 'freeze out', where their production/annihilation practically ceases and their population stabilises.

Question 4

We present plots for parts (a), (b) and (c) in Figures 6, 5 and 7 respectively.
We used a cubic spline to interpolate/extrapolate the predicted decay width of the Higgs boson as a function of mass (data given in the task sheet). We fit this spline with the decay width on a log scale, which we believe increased the accuracy especially over the orders of magnitude that we're interested in. The fit of the spline to the data, as well as some linearly extrapolated region in the high mass regime, is shown in Figure 4.

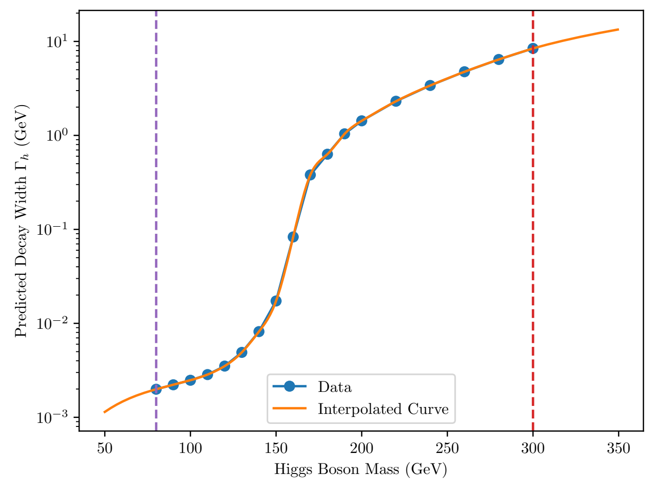

Figure. 4 The interpolation via a cubic spline fits the predicted decay width data remarkably well due to the high sampling and smooth nature of the curve. The region between the dashed lines describes the interpolation region. Masses to the left of the purple dashed line (left) use the interpolation function to extrapolate decay widths, and the masses to the right of the red dashed line (right) use a linear model $\Gamma_h = a + bQ_m$ to extrapolate decay widths [where a is the last data point from the sample, $b = 0.1$, and $Q_m = Q - 300$ is the mass over 300 GeV.

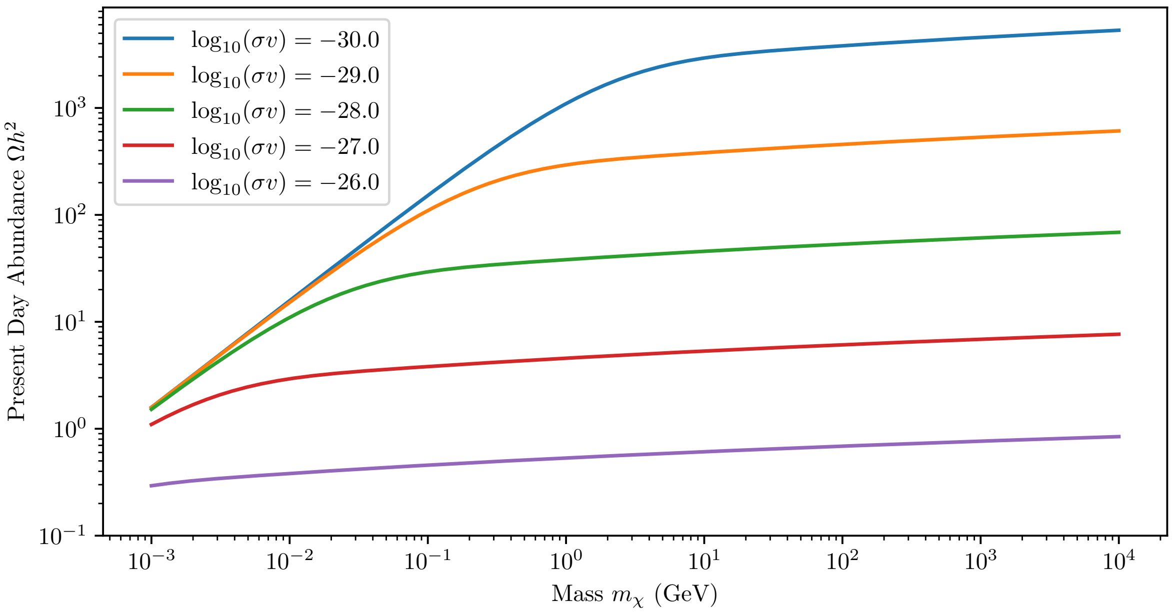

Figure. 2 The present day abundance Ωh^2 is approximately linear for a given cross section with increasing mass after some particular value. We can see that the abundance increases at a given mass with decreasing cross section, which agrees with the results in Figure 1. For each curve, we see a linear increase in abundance with mass but over two distinct regions – a steep increase in the low mass region, and a shallow increase for higher masses. We expect a linear change since the present day abundance is linearly proportional to mass (equation (2) in the task sheet). The region where the slope changes can be attributed to the freeze out of each particle species, since each 'present' day value was taken at $x = m/T = 10^3$ and so a low mass for the same x effectively corresponds to a low time.

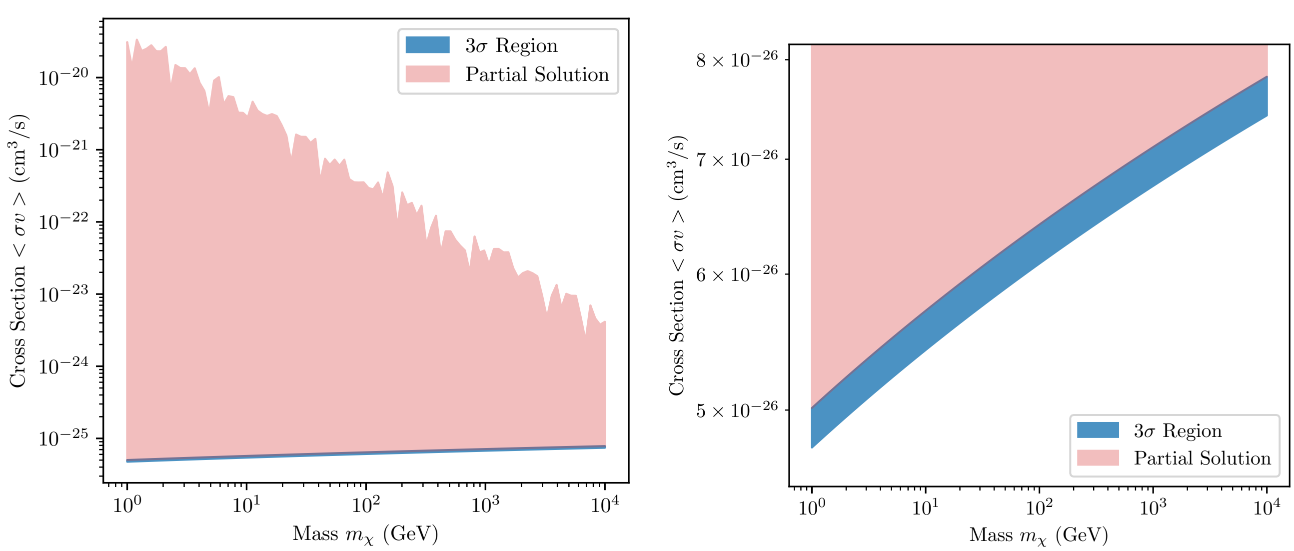

Figure. 3 *Left:* the entire cross-section/mass parameters space for which the Majorana fermion can explain some of the observed cosmological dark matter. The jagged region at the top is due to numerical error in finding a 0 root, but the solid region between the top and the blue 3σ region shows that there is more room in the parameter space for *some* description of observed dark matter from a lower mass Majorana fermion. *Right:* a cross-section zoomed area of the whole plot, showing the 3σ region in which all of the cosmological dark matter can be explained. We see that increasing mass corresponds to a lower interaction cross section; a more massive particle requires a larger cross section to 'annihilate' the right number of particles before freeze out to describe the observed present day abundance.

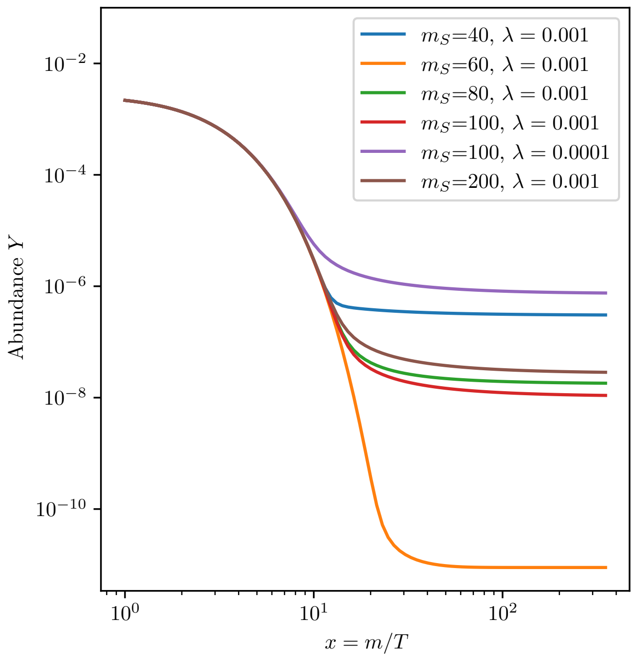

Figure. 5 In comparison with Figure 1, we see that the shape of the abundance curve over time is heavily reliant on the cross-section function. There a constant cross section (across mass) results in the same shaped abundance curve no matter the mass. Here, we see strong inconsistencies in the abundance of a particle after freeze out, where some particles plateau sooner than others (blue/yellow vs brown/red curves in the figure). On comparison with Figure 6, we note that these quick declines are from particle masses below $m_h/2$ and so they are subject to the broad-to-narrow peak change with increasing x. On a final note, we can see that the actual abundance as a function of mass does not behave in a linear way (on a log scale) like it does in Figure 1. Here we see the abundance start high for low mass, quickly decrease for medium mass (as it approaches the peak in Figure 6), and then rise again in the high mass regime. An order of magnitude change in λ_{hS} seems to correspond to a larger than order of magnitude change in Y, as the momentum-dependent cross-section is proportional to λ_{hS}^2.

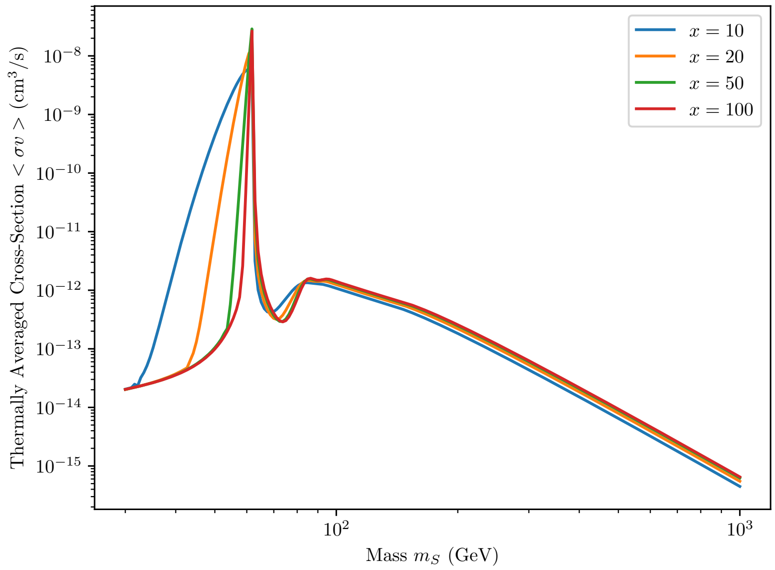

Figure. 6 The thermally averaged cross-section for scalar singlet dark matter (with $\lambda_{hS} = 10^{-3}$) is anything but constant as a function of mass or cosmological epoch, x. We see a large peak for all x at about $m_s \sim m_h/2 \approx 63$ GeV, which is sharper for larger x and broader for low x which we can attribute to the difference in temperature; for lower mass, the m^{-5} term in equation (3) doesn't entirely dominate and the cosmological epoch x has a larger contribution to the function. After about the vacuum expectation $v_0 = 246$ GeV, we see an approximately linear decline in the cross-section.

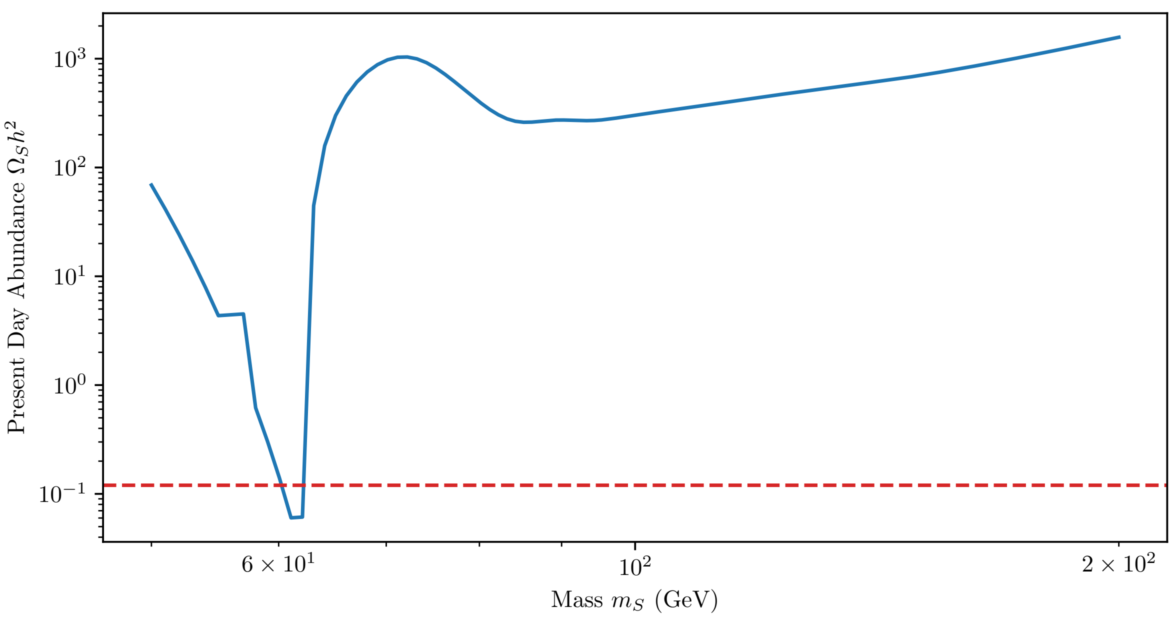

Figure. 7 Keeping λ_{hS} fixed at 10^{-3}, we see that the present day abundance $\Omega_S h^2$ of scalar singlet dark matter is not monotone changing with respect to the particle mass. Interestingly, we see that this plot is roughly an inversely-shaped curve of Figure 6, which shows the large effect of a non-constant interaction cross-section. The red dashed line corresponds to the Planck result of $\Omega_\chi h^2 = 0.12$, which means that only $60 \lesssim m_S \lesssim 63$ GeV scalar singlet dark matter aligns with the Planck results.