MATH2100 Final Exam

Ryan White
s4499039

3rd of November 2020

Question 1 (20 marks)

For the system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3x(2y+1) \\ -y+\cos(x) \end{pmatrix}$$

for $0 \leq x \leq 8$

a) The stationary points are found as

$$\begin{pmatrix} \dot{x}^* \\ \dot{y}^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3x(2y+1) \\ -y+\cos(x) \end{pmatrix}$$

which gives the equations:

1. $0 = -3x(2y+1)$
2. $0 = -y+\cos(x)$

firstly, 1. is solved:

$$0 = -6x^*y^* - 3x^*$$
$$\Rightarrow 3x^* = -6x^*y^*$$
$$\Rightarrow y^* = -\tfrac{1}{2}$$

then, using $y^* = -\frac{1}{2}$, 2. is solved:

$$0 = -y^* + \cos(x^*)$$
$$\Rightarrow \cos(x^*) = -\tfrac{1}{2}$$
$$x^* = \cos^{-1}(-\tfrac{1}{2})$$

calculated on the domain $0 \leq x \leq 8$, this gives

$$x_1^* = \frac{2\pi}{3}, \quad x_2^* = \frac{4\pi}{3}$$

Now, at the critical points, the stability and nature is determined by the Jacobian:

$$Df(x^*,y^*) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}$$

$$\Rightarrow Df(x^*,y^*) = \begin{pmatrix} -6y-3 & -6x \\ -\sin(x) & -1 \end{pmatrix}$$

and so for the two stationary points,

$$Df\left(\frac{2\pi}{3}, -\frac{1}{2}\right) = \begin{pmatrix} 0 & -4\pi \\ -\frac{\sqrt{3}}{2} & -1 \end{pmatrix} \quad \text{and} \quad Df\left(\frac{4\pi}{3}, -\frac{1}{2}\right) = \begin{pmatrix} 0 & -8\pi \\ \frac{\sqrt{3}}{2} & -1 \end{pmatrix}$$

$\operatorname{tr} Df = -1$

$\det Df = -2\pi\sqrt{3}$ where $\det Df < 0$, so $(x^*, y^*) = (\frac{2\pi}{3}, -\frac{1}{2})$ is an unstable saddle node

$\operatorname{tr} Df = -1$

$\det Df = 4\pi\sqrt{3}$ with $(\operatorname{tr} Df)^2 < 4 \det Df$, therefore $(x^*, y^*) = (\frac{4\pi}{3}, -\frac{1}{2})$ is a stable focus node

b. Firstly, for the saddle node, the eigenvalues and eigenvectors must be found.

We have

$$\lambda_{1,2} = \frac{tr\, Df}{2} \pm \sqrt{\left(\frac{tr\, Df}{2}\right)^2 - \det Df}$$

$$= -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\pi\sqrt{3}}$$

$$\therefore \lambda_1 = \frac{-1+\sqrt{1+8\pi\sqrt{3}}}{2} \qquad \lambda_2 = \frac{-1-\sqrt{1+8\pi\sqrt{3}}}{2}$$

with eigenvectors

$$\begin{pmatrix} 0-\lambda_1 & -4\pi \\ -\frac{\sqrt{3}}{2} & -1-\lambda_1 \end{pmatrix} x^{(1)} = 0 \Rightarrow x^{(1)} = \begin{pmatrix} \frac{-1-\sqrt{1+16\pi\sqrt{3}}}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{-1-\sqrt{1+8\pi\sqrt{3}}}{\sqrt{3}} \\ 1 \end{pmatrix}$$

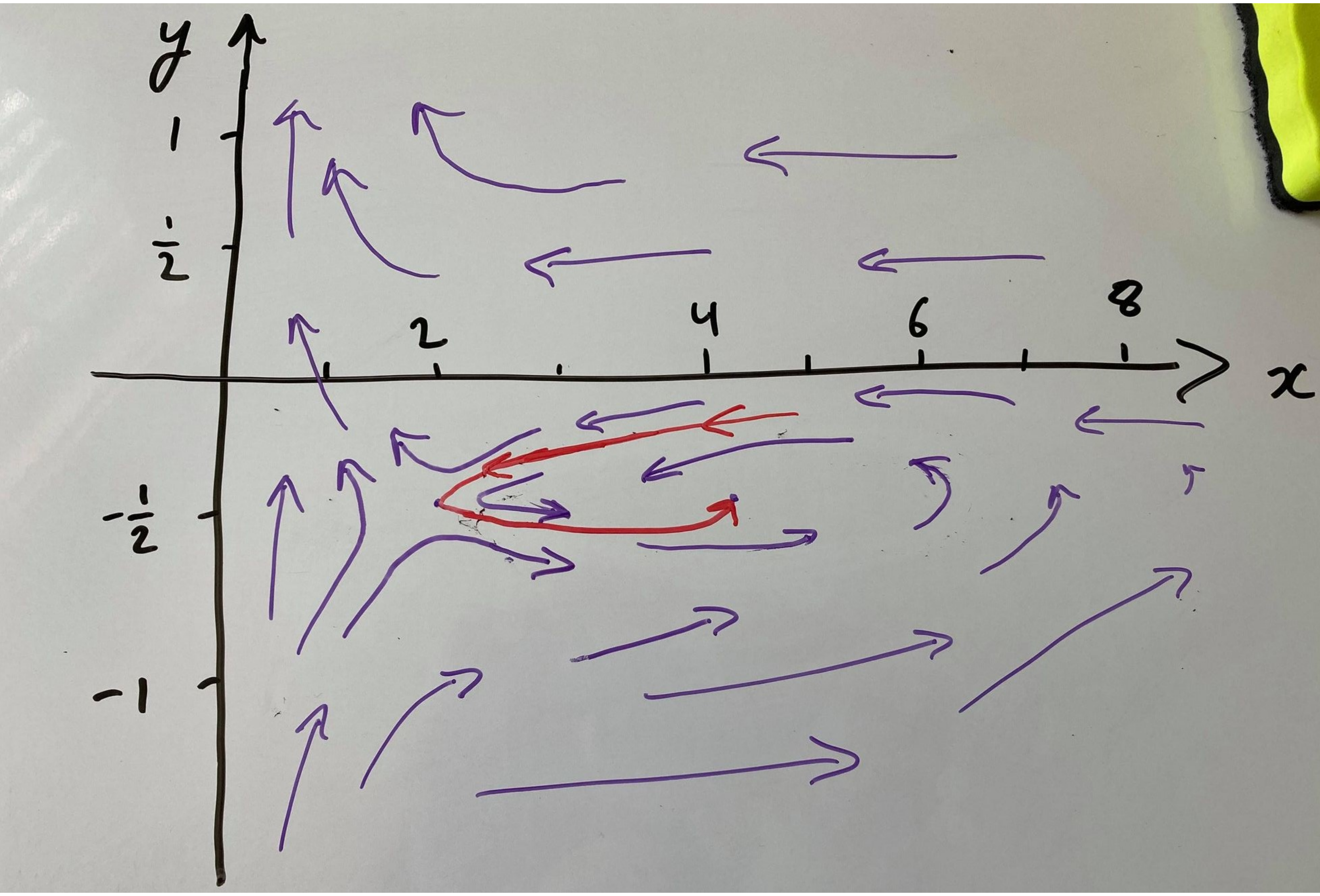
$$\begin{pmatrix} 0-\lambda_2 & -4\pi \\ -\frac{\sqrt{3}}{2} & -1-\lambda_2 \end{pmatrix} x^{(2)} = 0 \Rightarrow x^{(2)} = \begin{pmatrix} \frac{-1+\sqrt{1+16\pi\sqrt{3}}}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt{1+8\pi\sqrt{3}}}{\sqrt{3}} \\ 1 \end{pmatrix}$$

Then, for the stable focus node, the direction of flow is found by using $\dot{x} = -3x(2y+1)$

- Take the point $(\frac{4\pi}{3}, 0)$ which is just above the stationary point:

$$\dot{x}\left(\frac{4\pi}{3}, 0\right) = -4\pi \quad \text{(to the left)}$$

Therefore the spiral direction is anticlockwise.


y
x
1
1/2
-1/2
-1
2
4
6
8

Question 2:

Consider the system of equations:

$$\begin{cases} y''(t) - y(t) = t \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

a. For a case where $y(0) = 0$ and $y'(0) = 0$, the transfer function, $Q(s)$ is given by

$$Q(s) = \frac{1}{as^2 + bs + c}$$

where the coefficients are given by $ay''(t) + by'(t) + cy(t) = r(t)$

Thus, for this system, the transfer function is

$$Q(s) = \frac{1}{s^2 - 1} = \frac{1}{(s+1)(s-1)} = \frac{A}{(s+1)} + \frac{B}{(s-1)}$$

$$\Rightarrow 1 = A(s-1) + B(s+1)$$

when $s = 1$,

$1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$

when $s = -1$,

$1 = -2A \quad \Rightarrow \quad A = -\frac{1}{2}$

For systems of the form $F(s) = \frac{A}{s-a} + \frac{B}{s-b}$, the inverse laplace transform is

$$\mathcal{L}^{-1}(F(s)) = Ae^{at} + Be^{bt}$$

Therefore,

$$\mathcal{L}^{-1}(Q(s)) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{t} = \frac{e^{t}}{2} - \frac{e^{-t}}{2}$$

Question 3:

For the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2xy^3 \\ -y \end{pmatrix}$$

a. The system is stationary only when $y = 0$ (i.e. along the x-axis)

b. The solution curves, written in the form of $y(x)$ can be found by $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -y\frac{1}{-2xy^3}$$

$$= \frac{1}{2xy^2}$$

$$\Rightarrow \quad dy = \frac{1}{y^2}\int \frac{1}{2x}\,dx$$

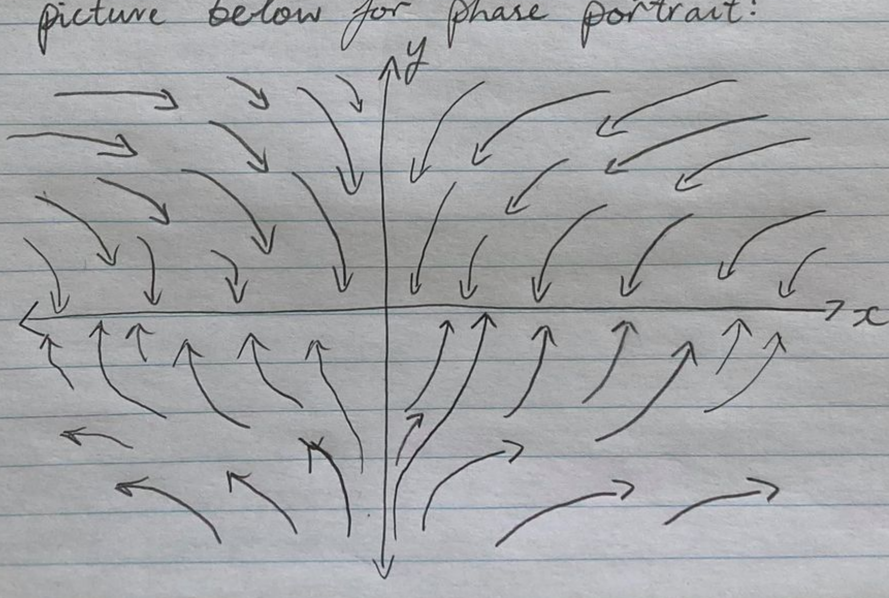
$$y = \frac{1}{y^2}\ln(2x) + c$$

$$y = \sqrt[3]{\ln(2x)} + C \qquad \text{where } C = \sqrt[3]{c}$$

c. For the four possible cases of initial conditions,

	$y(0) > 0$	$y(0) < 0$
$x(0) > 0$	$\dot{x} < 0 \rightarrow$ left dir $\dot{y} < 0 \rightarrow$ down dir	$\dot{x} > 0 \rightarrow$ right dir $\dot{y} > 0 \rightarrow$ up dir.
$x(0) < 0$	$\dot{x} > 0 \rightarrow$ right dir $\dot{y} < 0 \rightarrow$ down dir.	$\dot{x} < 0 \rightarrow$ left dir $\dot{y} > 0 \rightarrow$ up dir.

See picture below for phase portrait:



d. By the DE $\dot{y} = -y$, $\lim_{t \to \infty} y = 0$.
Then, using $y(x) = \sqrt[3]{\ln(2x)} + C$ at the starting conditions $(-1, 1)$, we have

$$1 = \sqrt[3]{\ln(-2)} + C$$
$$\Rightarrow C \simeq -0.33$$ (omitting imaginary component)

$$\therefore \lim_{t \to \infty} (x(t), y(t)) \simeq (0.33, 0)$$

Question 4:

As the function $f(x) = x^2$ is even, the fourier series will be of the form:

$$F(x) = A_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L} x\right)$$

where $A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$

$$= \frac{1}{2} \int_{-1}^{1} x^2\, dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_{-1}^{1} = \frac{1}{3}$$

and $C_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) dx$

for $n = 1$, $\sum_{n=1}^{1} C_n \cos\left(\frac{n\pi}{L} x\right) = -\frac{4\cos(\pi x)}{\pi^2}$

$n = 2$, $\sum_{n=1}^{2} C_n \cos\left(\frac{n\pi}{L} x\right) = \frac{\cos(2\pi x) - 4\cos(\pi x)}{\pi^2}$

$n = 3$, $\sum_{n=1}^{3} C_n \cos\left(\frac{n\pi}{L} x\right) = \frac{9\cos(2\pi x) - 4\cos(3\pi x) - 36(\pi x)}{9\pi^2}$

(these were calulated with Wolfram Alpha)

See Below picture for plot.


y
1
-1
1
x
f(x)
F(x)_{n=1}
F(x)_{n=2}
F(x)_{n=3}

Question 5:

Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

with $u(0,x) = e^{-x^2} = f(x)$ and $\frac{\partial u}{\partial t}(0,x) = \sin x = g(x)$

and $c = 1$

The general solution of the wave equation on an infinite domain is

$$u(x,t) = \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

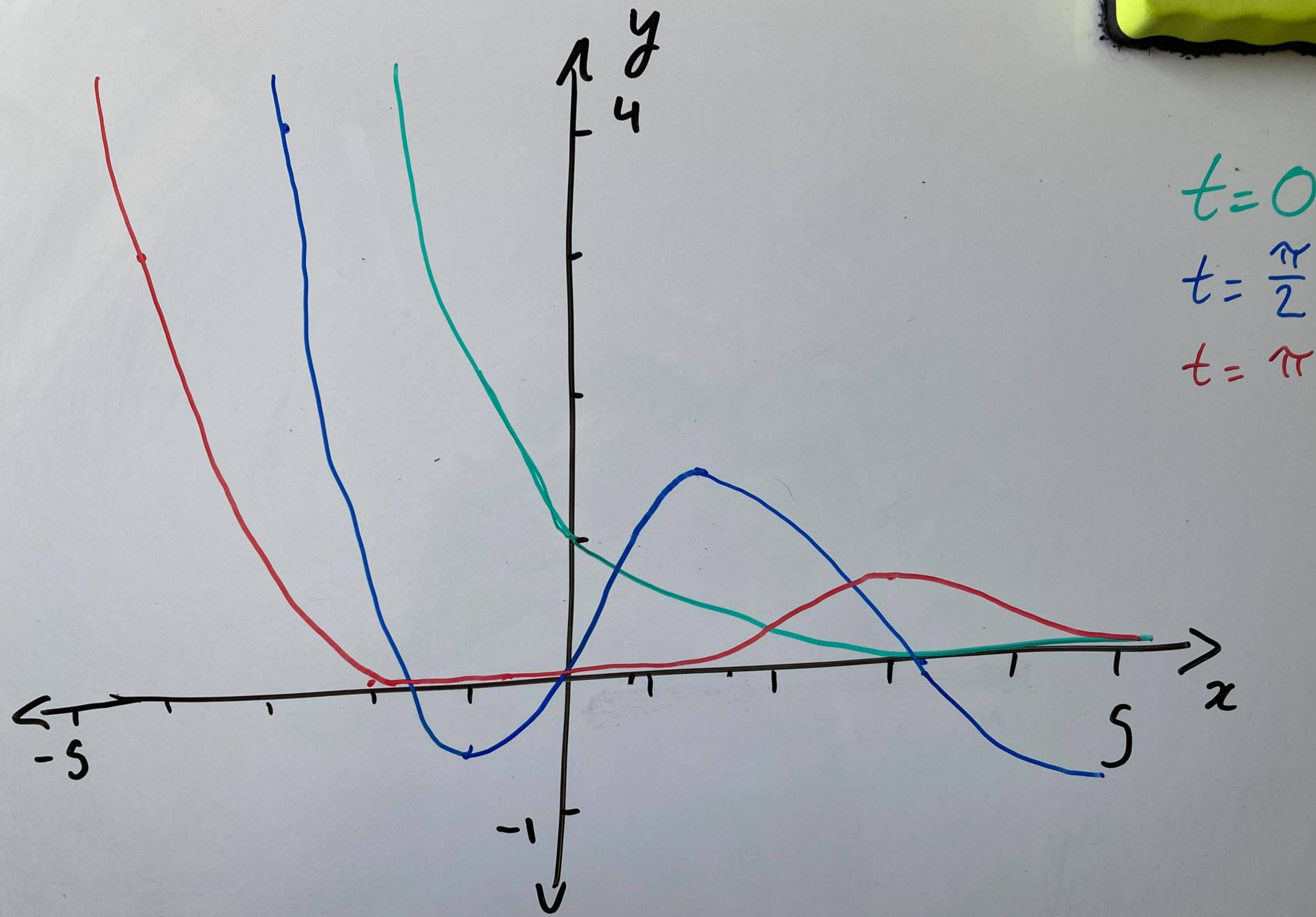
where $\frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz = \frac{1}{2}\int_{x-t}^{x+t} \sin(z)\,dz$

$$= \frac{1}{2}\left(\cos(x-t) - \cos(x+t)\right)$$

therefore, the solution is

$$u(x,t) = \frac{1}{2}\left(e^{-(x+t)^2} + e^{-(x-t)^2} + \cos(x-t) - \cos(x+t)\right)$$

See below for the plot on the domain $-5 < x < 5$



y
4
x
-5
5
-1
t=0
t= π/2
t= π

Question 6:

a. The system may be represented as:

$$\frac{\partial T}{\partial t} = D \frac{\partial T^2}{\partial x^2}$$

with boundary conditions

$$T(x,0) = 100°C \quad , \quad T(1,t) = 0°C$$

b. The series formula for the temperature distribution $u(x,t)$ is shown by the general solution to the heat equation:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2 D}{L^2}t} \times \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

given that $u(x,0) = f(x) = 100°C$

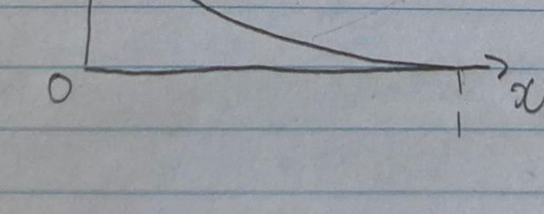
$L = 1m$, $D = 2m^2/s$, the formula becomes

$$u(x,t) = \sum_{n=1}^{\infty} 2\sin(n\pi x)e^{-2n^2\pi^2 t}\int_0^1 100\sin(n\pi x)\,dx$$

$$= \sum_{n=1}^{\infty} -\frac{100}{n\pi}\left(2\sin(n\pi x)\cos(n\pi)e^{-2n^2\pi^2 t} - 1\right)$$

c. Assuming that the ice-cube does not melt or change temperature, and that the bar is perfectly insulated except next to the ice cube, the equilibrium temperature at $t \to \infty$ is $u(x,\infty) = 0°C$. However, on finite timescales, it would never actually reach this temperature, as the temperature rate of change is decreased as the temperature decreases.

d. The temperature distribution before equilibrium is an inverse exponential function, whose slope approaches 0 as $t \to \infty$.



where the T-axis is infinitely short
As time goes on, the temperature at all points except $x = 1m$ gets proportionally smaller.