PHYS3071 Assignment 4

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1 Part A

We devised a root finding algorithm that combined both bisection and newton's method in order to find the roots of (reasonably) simple functions with known derivatives. Bisection is the process of halving a possible solution space (a domain between a known upper and lower bound on a root solution) until a possible solution is within some tolerance of the analytic solution (an algorithmic example of this is shown in Algorithm 1). Newton's method involves using the derivative of function in order to quickly approach the root, in a iterated process given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{1}$$

which successively converges to a root (provided that the problem is 'nice').
For non-linear equations, there are often stationary points in a desired domain, where the functions derivative goes to zero and Newton's method yields a solution that 'escapes to infinity'. In the event of that happening, our algorithm instead employed bisection (specifically for 10 steps) until a closer solution to the root could be found. After these 10 steps, Newton's method would be applied again to ideally achieve an accurate root in less time. 10 steps was chosen somewhat arbitrarily in order to have sufficient chance of escaping a danger region (where Newton's method would sink into a stationary point) while also being small enough so that computation time wasn't unnecessarily wasted (since bisection typically converges to a root much slower than Newton's method, for example).

As a test case, the algorithm was applied to a range of simple to somewhat challenging problems, with the test parameters and results given in table 1.

Function	$f(x)$	Derivative ($f'(x)$)	Tested a	Possible Range of a	Actual Value at Test a	Found Value	Valid Range (Tested)
1	$\sin(x)/x - a$	$\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}$	0.5	$[-0.217,\ 1]$	± 1.89549	± 1.895	$[-200,\ 200]$
2	$(x-1)e^{-x^2} - a$	$e^{-x^2}\left(-2x^2+2x+1\right)$	0	$[-1.195,\ 0.057]$	1	1	$[-200,\ 200]$
3	$1/x - a$	$-1/x^2$	1	$\{\mathbb{R}\backslash 0\}$	1	1	$[1/100,\ 2]$
4	$x^2 - a$	$2x$	0.5	$\{\mathbb{R}^+, 0\}$	± 0.707	± 0.707	$[-200,\ 200]$

Table 1: Tested values and root-finding results for each of the specified functions. The tested ranges were found by incrementing the initial guess by 1 (in the case of functions 1, 2, and 4) and by 1/100 (in the case of function 3). Function 3 in particular is extremely sensitive to both the initial guess, and the offset value a.

We note that the algorithm seems to handle stationary points extremely well, but can struggle with singularities in a function (as well as discontinuities). For example, the upper bound in function 3 was extremely sensitive, and if placed too high for any given a, the guess would irreparably escape to infinity. The lower bound in this case wasn't as sensitive, as long as its magnitude was greater than zero and the sign was the same as the analytic root.

2 Part B

In order to calculate the needed velocity for a circular orbit of some radius, we employed a bisection algorithm with an initial guess of a tangential velocity $|\vec{\mathbf{r}}| = 0.4$ at a radius of $|\vec{r}| = 1$ (with $GM = 1$). The system of equations needed to simulate an orbit at each time step was given by Euler's method:

$$\vec{r}(t_2) = \vec{r}(t_1) + \vec{v}(t_1) \cdot dt \tag{2}$$
$$\vec{v}(t_2) = \vec{v}(t_1) + \vec{a}(t_1) \cdot dt \tag{3}$$

$$(4)$$

where

$$\vec{a}(t) = \frac{d^2\vec{r}(t)}{dt^2} = -\frac{GM}{|\vec{r}(t)|^3}\vec{r}(t) \tag{5}$$

Of course, we could arbitrarily choose the initial coordinates of particle such that it was at the correct $|\vec{r}(0)| = 1$. We chose $\vec{r}(0) = (x, y) = (1, 0)$, and so the velocity vector initially was purely in the y direction (with an initial guess strength of 0.4).

In order to assess the validity of each velocity, we calculated the required time for a circular orbit at the given velocity by

$$\Delta t = \frac{2\pi||\vec{r}||}{||\vec{v}||} \tag{6}$$

and simulated an orbit over that duration. The goodness of fit was calculated as

$$\text{Goodness of fit } = \epsilon = ||\vec{r}(\Delta t)|| - ||\vec{r}(0)|| \tag{7}$$

such that an $\epsilon = 0$ corresponds to a perfect circular orbit (and hence ideal velocity). If the initial guess gave an $\epsilon < 0$, then the upper bound velocity was set as the escape velocity ($||\vec{v}|| = \sqrt{2GM/||\vec{r}||}$), else if $\epsilon > 0$, the velocity was set to a very low value ($||\vec{v}|| = 0.0001$ in this case). After bisecting with a tolerance of 0.001, we arrived at velocity values of

$$\vec{v} = (0,\ 1.00111) \text{ units/time} \qquad \Longrightarrow \qquad ||\vec{v}|| = 1.00111 \text{ units/time} \tag{8}$$

which is practically within tolerance error of the analytic solution of $||\vec{v}|| = 1$ (as per $v_{\text{circ}} = \sqrt{GM/||\vec{r}||}$). The resultant orbit with this velocity and radius is shown in Figure 1.

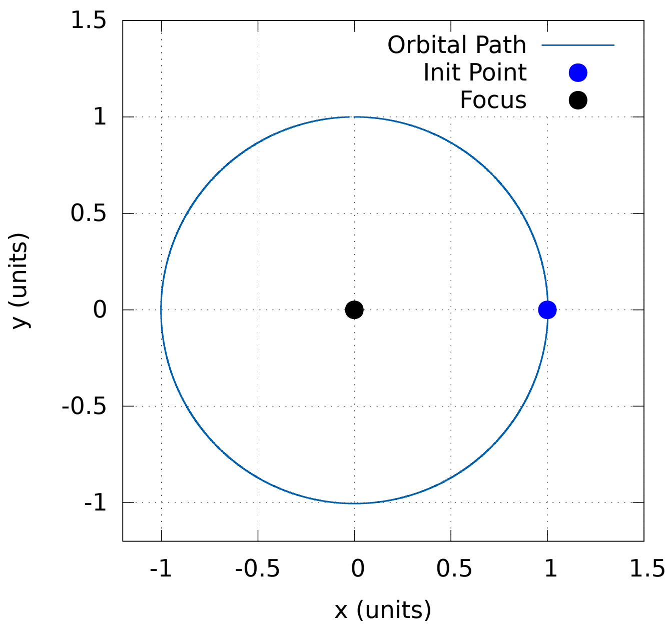

Figure 1: Calculated circular orbit, given the initial radius with respect to the focal point.

3 Part C

The problem of finding an orbit based on two observed positions and a time between the observations is called Lambert's problem. In this case, we're given a little less information in that we only have initial and final radii ($r_1 = 1$ and $r_2 = 1/2$), and the separation between them ($||\vec{r_2} - \vec{r_1}|| = 0.884$). This problem becomes much easier when we have specific coordinates that we can work with. Since we're given no set coordinate frame, we can arbitrarily set the first point to be at $\vec{r_1} = (x_1, y_1) = (0, 1)$, which leaves the coordinates of the second point to be found. Given that

$$||\vec{r_2} - \vec{r_1}|| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{9}$$

we can solve this quite simply with the known coordinates of $\vec{r_1}$ and the known separation. We also know that

$$||\vec{r_2}|| = \frac{1}{2} = \sqrt{x_2^2 + y_2^2} \Rightarrow y_2 = \pm\sqrt{\frac{1}{4} - x_2^2} \tag{10}$$

Hence,

$$\begin{align}
0.884 &= \sqrt{x_2^2 + (y_2 - 1)^2} \tag{11}\\
\Rightarrow x_2^2 &= 0.884^2 - (y_2 - 1)^2 \tag{12}\\
&= 0.781 - y_2^2 + 2y_2 - 1 \tag{13}\\
&= -0.219 - \tfrac{1}{4} + x_2^2 + 2\sqrt{\tfrac{1}{4} - x_2^2} \tag{14}\\
\frac{0.469}{2} &= \sqrt{\frac{1}{4} - x_2^2} \tag{15}\\
\Rightarrow x_2 &= \pm\ 0.4416 \tag{16}
\end{align}$$

Substituting this into equation (10) then gives $y_2 = 0.2345$ (strictly positive to maintain the chord of 0.884). Since there are obviously two possible points that this corresponds to (one for each of the plus/minus of x_2), we have chosen both positive values. In truth, these should correspond to two valid orbits (two orbits of two opposing velocity vectors), and so they correspond to coordinate axis flips and are both the same, unique orbit.

To solve Lambert's problem with these two known points, we adapted a simplified algorithm proposed by [1], [3], and [2]. The algorithm entails optimising for the time of flight between the first and second point, given by

$$\Delta t = \sqrt{\frac{a^3}{GM}}\,(\alpha - \beta - (\sin\alpha - \sin\beta)) \tag{17}$$

where,

$$c = ||\vec{r_2} - \vec{r_1}||; \qquad s = \frac{c + r_1 + r_2}{2}; \qquad a_{\text{min}} = \frac{r_1 + r_2 + c}{4} = \frac{s}{2} \tag{18}$$

$$\alpha = 2\arcsin\left(\sqrt{\frac{s}{2a}}\right); \qquad \beta = 2\arcsin\left(\sqrt{\frac{s-c}{2a}}\right) \tag{19}$$

We apply root-finding algorithms to the $\epsilon = \Delta t - \Delta t_{\text{observed}}$, such that it is as close to 0 as possible (to within a desired tolerance). Once the best-fit semi-major axis is found, we can then find the velocity vector at the initial point by

$$\vec{v}(t_1) = (B + A)\vec{u_c} + (B - A)\vec{u_1} \tag{20}$$

where

$$A = \sqrt{\frac{GM}{4a}}\cot\left(\frac{\alpha}{2}\right); \qquad B = \sqrt{\frac{GM}{4a}}\cot\left(\frac{\beta}{2}\right) \tag{21}$$

$$\vec{u_1} = \frac{\vec{r}(t_1)}{r_1}; \qquad \vec{u_c} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{c} \tag{22}$$

The resultant algorithm used is shown in Algorithm 1. With this implementation in C++, we obtained the orbit shown in Figure 2.

Algorithm 1 Semi-major axis determination via Lambert's equation and a bisection algorithm

```
lower ← a_min
upper ← r_1
while ε_upper × ε_lower > 0 do                ▷ so that we find semi-major axis guesses either side of root
    upper ← upper × 2
end while
previous ← ∞
current ← (upper + lower)/2                   ▷ Current best guess
while |Δt_previous − Δt_current| > tol do      ▷ Initiate the bisection root-finding
    previous ← current                        ▷ Previous best guess
    if ε_current == 0 then                    ▷ Perfect guess
        return current
    else if ε_current > 0 then                ▷ The current guess is too big
        upper ← current
    else                                      ▷ Current guess is too small
        lower ← current
    end if
    current ← (upper + lower)/2               ▷ Bisection step
end while
return current
```

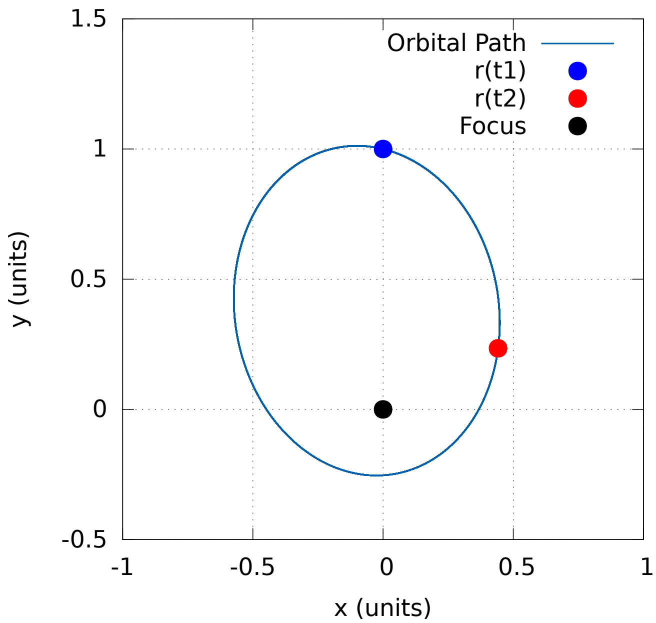

Figure 2: Best fit orbit of the comet over its orbital period. We obtained a semi-major axis of $a \simeq 0.636242$ units, with an eccentricity of $e \simeq 0.604201$. The found perigee and apogee of the orbit are $r_{\text{min}} \simeq 0.251748$ and $r_{\text{max}} \simeq 1.02035$ units respectively.

To obtain values of perigee and apogee, the orbit was computed numerically (with initial point of $(x, y) = (0, 1)$, and initial velocity given by equation (20)) according to the found semi-major axis and the acceleration equation (eq. (5). Then, the perigee and apogee were taken as the minimum and maximum radii from the focus (at $(x, y) = (0, 0)$) respectively. The eccentricity was then found via

$$e = \frac{r_{\text{max}} - r_{\text{min}}}{r_{\text{max}} + r_{\text{min}}} \tag{23}$$

References

[1] Dario Izzo. "Revisiting Lambert's problem". In: *Celestial Mechanics and Dynamical Astronomy* 121.1 (Jan. 2015), pp. 1–15. ISSN: 1572-9478. DOI: `10.1007/s10569-014-9587-y`. URL: `https://doi.org/10.1007/s10569-014-9587-y`.

[2] Ulrich Walter. *Astronautics - The Physics of Space Flight. Third Edition.* Springer, 2018. ISBN: 978-3-319-74372-1. DOI: `10.1007/978-3-319-74373-8`.

[3] Gang Zhang et al. "Covariance analysis of Lambert's problem via Lagrange's transfer-time formulation". In: *Aerospace Science and Technology* 77 (2018), pp. 765–773. ISSN: 1270-9638. DOI: `https://doi.org/10.1016/j.ast.2018.03.039`. URL: `https://www.sciencedirect.com/science/article/pii/S1270963817311999`.