Chaos Assignment 3

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1a

Box Counting Dimension	Example:
0	A fixed point attractor. This could be A stable star or spiral. An example is the Damped pendulum described by $\ddot{\theta} = -\frac{g}{a}\sin\theta - \alpha\dot{\theta}$
1	A limit cycle attractor. For example, the Van Der Pol oscillator: $\ddot{x} - \mu(1-x^2)\dot{x} + x = 0$
Non-Integer	A fractal attractor, or for example the Logistic map $f(x_n) = r x_n(1-x_n)$ Which has dimension ~0.55 above the Chaotic dimension

1b The van der Pol oscillator divides phase space into two connected regions:
- one inside the attractor boundary
- one outside the attractor boundary

This is shown in Figures 6.1 and 6.2 in the chaos course notes.

Q2 $f(x_n) = x_{n+1} = r x_n(1-x_n)$ for $0 \le x_n \le 1$ and $0 \le r \le 4$

$$\lambda = \prod_{i=1}^{p} f'(x_i)$$

a. If the function is of period 1, then

$$x_{1+1} = x_1$$

$$x_2 = r(rx_1(1-x_1))(1-rx_1(1-x_1)) = rx_1(1-x_1) = x_1$$

$$\Rightarrow \quad r(1-rx_1(1-x_1)) = 1$$

$$r + r^2x_1^2 - r^2x_1 - 1 = 0$$

This is of the form of the quadratic equation with

$a = r^2$

$b = -r^2$

$c = r-1$

And so the fixed points $(x=0)$ correspond to

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{r^2 \pm \sqrt{r^4 - 4r^2(r-1)}}{2r^2}$$

$$= \frac{1}{2} \pm \frac{1}{2r}\sqrt{r^2-4r+4}$$

$$= \frac{1}{2} \pm \frac{1}{2r}\sqrt{(r-2)^2}$$

$$= \frac{1}{2} \pm \frac{1}{2r}\sqrt{(r-2)^2}$$

$$= \frac{1}{2} \pm \frac{r-2}{2r} = \frac{1}{2} \pm \left(\frac{1}{2} - \frac{1}{r}\right)$$

$$\Rightarrow x_1 = 1 - \frac{1}{r} \qquad x_2 = \frac{1}{r}$$

Therefore there are fixed points for $p=1$ at $x = 1 - \frac{1}{r}$ for $1 \leq r \leq 4$ and $x = \frac{1}{r}$ for $1 \leq r \leq 4$ (there is also a trivial fixed point at $x = 0$ for any $0 \leq r \leq 4$)

The stability can be assessed by

$$f(x_n) = r x_n - r x_n^2$$

$$f'(x_n) = r - 2 r x_n$$

$$\Rightarrow \lambda = \prod_{i=1}^{p} f'(x_n)$$

$$= r - 2 r x_n \qquad (p=1)$$

The fixed point $x = \frac{1}{r}$ is stable only when $\lambda \leq 0$,

$$\Rightarrow \lambda = r - \frac{2r}{r}$$

$$= r - 2$$

And so $x = \frac{1}{r}$ is only stable when $1 \leq r \leq 2$, and is superstable when $r = 2$.

For the fixed point $x = 0$,

$$\lambda = r - 0$$

$$= r$$

and so $x = 0$ is only stable when $r = 0$

For the fixed point $x = 1 - \frac{1}{r}$

$$\lambda = r - (2r - 2)$$

$$= -r + 2$$

and so $x = 1 - \frac{1}{r}$ is only stable when $2 \leq r \leq 4$ and superstable when $r = 2$.

The point just after $r = 3.2$ in Figure 1 likely corresponds to a stable fixed point. Since the Lyapunov exponent implies the level of chaotic behaviour of a system, a more negative exponent corresponds to a more stable system and since $r \simeq 3.2$ is a local minimum, this is likely a stable fixed point.

6. We know that

$$\lambda = \prod_{i=1}^{p} f'(x) = (r - 2 r x_1)(r - 2 r x_2) \ldots (r - 2 r x_p)$$

for $x = \frac{1}{2}$, $r - 2 r x_1 = r - r = 0$

$$\Rightarrow \lambda = 0 \cdot (r - 2 r x_2) \cdot \ldots \cdot (r - 2 r x_p)$$

$$= 0$$

and so, by definition, $x = \frac{1}{2}$ corresponds to a super stable point.

Q3

a. The number of faces, f, at n number of iterations is:

$$F_n = 5F_{n-1} = 4\cdot 5^{n-1} \quad (n \geq 1)$$

This is because the n iteration splits one face into 5.

The number of 'squares' at each iteration is

$$N_n = 5N_{n-1} = \tfrac{1}{4}F_{n-1} = 5^{n-1} \quad (n \geq 1)$$

But the area of the added square reduces at each iteration.

The area for each added square for iteration n is

$$a_n = f^{2n} \qquad \text{for} \quad 0 < f < \tfrac{1}{3}$$

So the total area is

$$1 + 4f^2 + 20f^4 + 100f^6 + \dots$$

And the area addition at each step is

$$A_n = 4\cdot 5^{n-1} f^{2n} = \frac{4}{5}(5f^2)^n \quad \text{for } n \geq 1$$

since $A_0 = 1$, the total area is

$$\sum_{i=0}^{n} A_n = 1 + 4f^2 + 20f^4 + 100f^6 + \dots$$

so this formula suits.

Since $0 \leq f \leq \frac{1}{3}$, $f = \frac{1}{3}$ is an upper bound for side length (and also the added square area).

Substituting this into A_n:

$$A_n = \frac{4}{5}\left(5\cdot\left(\frac{1}{3}\right)^2\right)^n$$

$$= \frac{4}{5}\left(\frac{5}{9}\right)^n$$

And so as $n \to \infty$

$$\left(\frac{5}{9}\right)^n \to 0 \Rightarrow \frac{4}{5}\left(\frac{5}{9}\right)^n \to 0$$

Therefore the sequence converges, and

$$\sum_{n=0}^{\infty} \frac{4}{5}\left(\frac{5}{9}\right)^n = \frac{4}{5}\sum_{n=0}^{\infty}\left(\frac{5}{9}\right)^n$$

$$= \frac{4}{5}\cdot\frac{1}{1-\frac{5}{9}}$$

where $\displaystyle\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \quad 0 < a < 1$

was used.

This means that the total area as $n \to \infty$ is

$$A = 1 + \frac{4}{5}\cdot\frac{1}{1-\frac{5}{9}}$$

$$= 1 + \frac{4}{5}\cdot\frac{9}{4}$$

$$= 1 + \frac{9}{5} = 2.8 \text{ units squared.} \quad (f = \tfrac{1}{3})$$

or more generally, $A = 1 + \frac{4}{5}\cdot\frac{1}{1-5f^2}$ for $0 < f < \frac{1}{3}$

b. The total number of faces at n iterations was found in part a as

$$F_n = 4 \cdot 5^{n-1} = \frac{4}{5} \cdot 5^n \quad (n \geq 1)$$

with side length f^{n-1} $(n \geq 1)$

And so the change in perimeter at iteration n is

$$\Delta P_n = \frac{8}{5}(5f)^n$$

$$\Rightarrow P = 4 + 8f + 40f^2 + 200f^3 + \dots$$

Then the perimeter after n iterations is

$$P_n = \frac{12}{5} + \sum_{k=0}^{n} \frac{8}{5}(5f)^k$$

and for $n \to \infty$,

$$P = \frac{12}{5} + \frac{8}{5}\sum_{k=0}^{\infty}(5f)^k$$

and so if $f \geq \frac{1}{5}$, $P \to +\infty$

since $5f \geq 1 \Rightarrow \sum_{k=0}^{\infty}(5f)^k \to +\infty$

but if $f < \frac{1}{5}$, then

$$P = \frac{12}{5} + \frac{8}{5}\sum_{k=0}^{\infty}(5f)^k$$

$$= \frac{12}{5} + \frac{8}{5} \cdot \frac{1}{1-5f}$$

$$= \frac{12}{5} + \frac{8}{5-25f}$$

and so P is either bounded or unbounded depending on the choice of f.

c.

The length scale at each $n \geq 0$ is $\epsilon_n = f^n = \left(\frac{1}{L}\right)^n$

where $L \in (3, \infty)$ since $0 < f < \frac{1}{3}$

The number of boxes along the perimeter at this length scale is

$$N_n(\epsilon) = 4 \cdot 5^{n-1} \quad (n \geq 1)$$

$$= 4 \cdot 5^n \quad (n \geq 0)$$

And so the box-counting dimension is

$$D_0 = \lim_{n\to\infty} \frac{\ln(N(\epsilon))}{\ln(\frac{1}{\epsilon})}$$

$$= \lim_{n\to\infty} \frac{\ln(4 \cdot 5^n)}{\ln((\frac{1}{f})^n)}$$

$$= \lim_{n\to\infty} \frac{\ln 4}{n\ln(\frac{1}{f})} + \frac{n \ln 5}{n \ln(\frac{1}{f})}$$

$$= 0 + \frac{\ln 5}{\ln(\frac{1}{f})}$$

$$= 0 + \frac{\ln 5}{\ln\left(\frac{1}{\gamma}\right)}$$

$$= \frac{\ln 5}{\ln\left(\frac{1}{\gamma}\right)}$$

and so, for $L = \frac{1}{\gamma} = 5$, $D_0 = 1$
for $L < 5$, $D_0 > 1$ and $L > 5$, $D_0 < 1$.