THE UNIVERSITY OF QUEENSLAND
SCHOOL OF MATHEMATICS AND PHYSICS
PHYS2041 – Quantum Physics

Tutorial 10 Solutions

Problem 10.1

(a) This means it is in a state of well define total orbital angular momentum squared. i.e. an eigenfunction of total orbital angular momentum squared. l is the quantum number associated with orbital angular momentum squared. The magnitude of L is $\sqrt{l(l+1)}\hbar$.

(b) The possible values for L_z are given by its eigenvalues $m_l\hbar$ where m_l span from $-l$ to l in units of 1. So for $l = 1$, possible L_z are $-\hbar, 0, \hbar$.

(c) This is because all the components of OAM commute with total OAM squared, however the components themselves don't commute. So we can only choose one, which is conventionally the *z*-component to be compatible observables.

(d) The ket's that define the OAM states of an electron are: $|l, m_l\rangle$, l is the quantum number associated with orbital angular momentum squared and the m_l are the quantum numbers associated with L_z .

(e) To carry out a calculation of the electron's average L_z one could use any basis. However it is easiest to use the eigenkets of L_z , since we know they must form an orthonormal basis and we also know the observable of L_z for each eigenket.

(f) The grad student has made a mistake – typical! L_z does not come in half units of hbar!

(g) The ket that represents this state is:

$$|\Psi\rangle = \frac{2}{\sqrt{10}}|1,-1\rangle + \frac{2}{\sqrt{10}}|1,0\rangle + \frac{\sqrt{2}}{\sqrt{10}}|1,1\rangle$$

(h) The state in (g) is an eigenfunction of $\hat{L}^2$ so it has a definite value of total OAM squared – which will also be its expectation value.

$$\langle\Psi|\hat{L}^2|\Psi\rangle = l(l+1)\hbar^2\langle\Psi|\Psi\rangle = l(l+1)\hbar^2$$

Here $l = 1$

$$\langle\Psi|\hat{L}^2|\Psi\rangle = 2\hbar^2$$

(i) To determine uncertainty we need to calculate $\left(\sigma_{L_z}\right)^2 = \left\langle\left(\hat{L}_z\right)^2\right\rangle - \left\langle\hat{L}_z\right\rangle^2$, start by calculating the mean:

$\langle\Psi|\hat{L}_z|\Psi\rangle$

$$=\left(\frac{2}{\sqrt{10}}\langle -1,1|+\frac{2}{\sqrt{10}}\langle 0,1|+\frac{\sqrt{2}}{\sqrt{10}}\langle 1,1|\right)\hat{L}_z\left(\frac{2}{\sqrt{10}}|1,-1\rangle+\frac{2}{\sqrt{10}}|1,0\rangle+\frac{\sqrt{2}}{\sqrt{10}}|1,1\rangle\right)$$

$$=\left(\frac{2}{\sqrt{10}}\langle -1,1|+\frac{2}{\sqrt{10}}\langle 0,1|+\frac{\sqrt{2}}{\sqrt{10}}\langle 1,1|\right)\left(-\frac{2}{\sqrt{10}}\hbar|1,-1\rangle+\frac{\sqrt{2}}{\sqrt{10}}\hbar|1,1\rangle\right)$$

$$=-\frac{4}{10}\hbar+\frac{2}{10}\hbar=-\frac{2}{10}\hbar$$

Now calculate the first term:

$\langle\Psi|\left(\hat{L}_z\right)^2|\Psi\rangle$

$$=\left(\frac{2}{\sqrt{10}}\langle -1,1|+\frac{2}{\sqrt{10}}\langle 0,1|+\frac{\sqrt{2}}{\sqrt{10}}\langle 1,1|\right)\left(\hat{L}_z\right)^2\left(\frac{2}{\sqrt{10}}|1,-1\rangle+\frac{2}{\sqrt{10}}|1,0\rangle+\frac{\sqrt{2}}{\sqrt{10}}|1,1\rangle\right)$$

$$=\left(\frac{2}{\sqrt{10}}\langle -1,1|+\frac{2}{\sqrt{10}}\langle 0,1|+\frac{\sqrt{2}}{\sqrt{10}}\langle 1,1|\right)\left(\frac{2}{\sqrt{10}}\hbar^2|1,-1\rangle+\frac{\sqrt{2}}{\sqrt{10}}\hbar^2|1,1\rangle\right)$$

$$=\frac{4}{10}\hbar^2+\frac{2}{10}\hbar^2=\frac{6}{10}\hbar^2$$

Thus: $\left(\sigma_{L_z}\right)=\sqrt{\left\langle\left(\hat{L}_z\right)^2\right\rangle-\left\langle\hat{L}_z\right\rangle^2}=\sqrt{\frac{6}{10}\hbar^2-\frac{4}{100}\hbar^2}=\frac{\sqrt{56}}{10}\hbar$

(j) To work out the matrix L_z, I would choose an orthonormal basis and then in that basis I would invoke: $\left(\hat{L}_z\right)_{ij}=\langle\Psi_i|\hat{L}_z|\Psi_j\rangle$. The best basis to use is the eigenket basis, because I know the action of L_z on these vectors. In this basis the L_z would be diagonal, with its eigenvalues on the diagonal.

<u>Problem 10.2</u> [FOR ASSIGNMENT 5; max 10 points]

(a) The components of the angular momentum operator $\hat{\mathbf{L}}=(\hat{L}_x,\hat{L}_y,\hat{L}_z)$ are given by $\hat{\mathbf{L}}=\hat{\mathbf{r}}\times\hat{\mathbf{p}}$,

$$\hat{L}_x = y\hat{p}_z - z\hat{p}_y \tag{1}$$

$$\hat{L}_y = z\hat{p}_x - x\hat{p}_z \tag{2}$$

$$\hat{L}_z = x\hat{p}_y - y\hat{p}_x. \tag{3}$$

We also know that

$$[r_i,\hat{p}_j] = -[\hat{p}_j, r_i] = i\hbar\delta_{ij}. \tag{4}$$

Using the commutator identity $[\hat{A}+\hat{B},\hat{C}+\hat{D}]=[\hat{A},\hat{C}]+[\hat{B},\hat{C}]+[\hat{A},\hat{D}]+[\hat{B},\hat{D}]$ (check this for yourself) we can write

$$\left[\hat{L}_x,\hat{L}_y\right] = [y\hat{p}_z - z\hat{p}_y, z\hat{p}_x - x\hat{p}_z] \tag{5}$$

$$= [y\hat{p}_z, z\hat{p}_x] - [y\hat{p}_z, x\hat{p}_z] - [z\hat{p}_y, z\hat{p}_x] + [z\hat{p}_y, x\hat{p}_z] \tag{6}$$

Let's just look at the first term: $[y\hat{p}_z, z\hat{p}_x]$. We know that y and $\hat{p}_x$ commute with every other term inside the commutator. It's only $\hat{p}_z$ and z don't commute, so we are free to pull y and $\hat{p}_x$ outside the commutator (if you're unsure, write out the full commutator and check),

$$\begin{aligned} [y\hat{p}_z, z\hat{p}_x] &= y\hat{p}_x [\hat{p}_z, z] & (7) \\ &= -i\hbar y\hat{p}_x & (8) \end{aligned}$$

using Eq. (4) in the second line. Similarly, $[y\hat{p}_z, x\hat{p}_z] = 0$ since all four of these operators commute (Eq. (4) again). Returning to $\left[\hat{L}_x, \hat{L}_y\right]$ we have

$$\begin{aligned} \left[\hat{L}_x, \hat{L}_y\right] &= [y\hat{p}_z, z\hat{p}_x] - [y\hat{p}_z, x\hat{p}_z] - [z\hat{p}_y, z\hat{p}_x] + [z\hat{p}_y, x\hat{p}_z] & (9) \\ &= y\hat{p}_x [\hat{p}_z, z] - \cancel{[y\hat{p}_z, x\hat{p}_z]} - \cancel{[z\hat{p}_y, z\hat{p}_x]} + \hat{p}_y x [z, \hat{p}_z] & (10) \\ &= -i\hbar y\hat{p}_x + i\hbar\hat{p}_y x & (11) \\ &= i\hbar\hat{L}_z, & (12) \end{aligned}$$

using Eq. (3) in the final line. Following the same reasoning,

$$\begin{aligned} \left[\hat{L}_y, \hat{L}_z\right] &= [z\hat{p}_x, x\hat{p}_y] - [z\hat{p}_x, y\hat{p}_x] - [x\hat{p}_z, x\hat{p}_y] + [x\hat{p}_z, y\hat{p}_x] & (13) \\ &= -i\hbar z\hat{p}_y + i\hbar\hat{p}_z y = i\hbar\hat{L}_x & (14) \end{aligned}$$

and

$$\begin{aligned} \left[\hat{L}_z, \hat{L}_x\right] &= [x\hat{p}_y, y\hat{p}_z] - [x\hat{p}_y, z\hat{p}_y] - [y\hat{p}_x, y\hat{p}_z] + [y\hat{p}_x, z\hat{p}_y] & (15) \\ &= -i\hbar x\hat{p}_z + i\hbar\hat{p}_x z = i\hbar\hat{L}_y & (16) \end{aligned}$$

Operators that do not commute must have an uncertainty trade-off, recall the generalised uncertainty principle between any operators $\hat{A}$ and $\hat{B}$ is

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle\right)^2. \tag{17}$$

Thus we have shown that it is *impossible* for a quantum particle to have a well defined angular momentum vector. If you were to measure the z component of the angular momentum, subsequent measurements of the x or y components would destroy this knowledge. This is quite remarkable. Compare angular momentum to linear momentum, the operators $\hat{p}_x$, $\hat{p}_y$ and $\hat{p}_z$ all commute, so a quantum particle can have a perfectly well defined momentum vector.

(b) The operator $\hat{L}^2$ is defined

$$\hat{L}^2 = \hat{\mathbf{L}} \cdot \hat{\mathbf{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \tag{18}$$

and is interpreted as the magnitude of the angular momentum (squared, in this case). The commutator $[\hat{L}^2, \hat{L}_x]$ is

$$\begin{aligned} [\hat{L}^2, \hat{L}_x] &= [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_x] & (19) \\ &= \cancel{[\hat{L}_x^2, \hat{L}_x]} + [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x] & (20) \\ &= [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x] & (21) \end{aligned}$$

Using the commutator identity $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$ we can write this as

$$
\begin{aligned}
[\hat{L}^2, \hat{L}_x] &= [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x] && (22)\\
&= \hat{L}_y[\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x]\hat{L}_y + \hat{L}_z[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x]\hat{L}_z && (23)\\
&= \hat{L}_y(-i\hbar\hat{L}_z) + (-i\hbar\hat{L}_z)\hat{L}_y + \hat{L}_z(i\hbar\hat{L}_y) + (i\hbar\hat{L}_y)\hat{L}_z && (24)\\
&= -i\hbar\hat{L}_y\hat{L}_z - i\hbar\hat{L}_z\hat{L}_y + i\hbar\hat{L}_z\hat{L}_y + i\hbar\hat{L}_y\hat{L}_z && (25)\\
&= 0. && (26)
\end{aligned}
$$

Following a similar procedure, you should also find that $[\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0$.

Although we previously found that the all three components of a quantum particle's angular momentum cannot be known simultaneously, (i.e. its direction is ill-defined), there is no uncertainty trade-off between the *magnitude* of the angular momentum vector and any of its components. In practical terms, if I were to measure the magnitude of particle's angular momentum, subsequent measurements on any of the components would not change this result.

(c) Using the components of the angular momentum vector Eq. (1-3) and the canonical commutation relations Eq. (4) we can immediately write

$$
\begin{aligned}
[\hat{L}_z, x] &= [x\hat{p}_y - y\hat{p}_x, x] = \cancel{[x\hat{p}_y, x]} - [y\hat{p}_x, x] = -y[\hat{p}_x, x] = i\hbar y && (27)\\
[\hat{L}_z, y] &= [x\hat{p}_y - y\hat{p}_x, y] = [x\hat{p}_y, y] - \cancel{[y\hat{p}_x, y]} = x[\hat{p}_y, y]] = -i\hbar x && (28)\\
[\hat{L}_z, z] &= [x\hat{p}_y - y\hat{p}_x, z] = \cancel{[x\hat{p}_y, z]} - \cancel{[y\hat{p}_x, z]} = 0 && (29)\\
[\hat{L}_z, \hat{p}_x] &= [x\hat{p}_y - y\hat{p}_x, \hat{p}_x] = [x\hat{p}_y, \hat{p}_x] - \cancel{[y\hat{p}_x, \hat{p}_x]} = \hat{p}_y[x, \hat{p}_x] = i\hbar\hat{p}_y && (30)\\
[\hat{L}_z, \hat{p}_y] &= [x\hat{p}_y - y\hat{p}_x, \hat{p}_y] = \cancel{[x\hat{p}_y, \hat{p}_y]} - [y\hat{p}_x, \hat{p}_y] = -\hat{p}_x[y, \hat{p}_y] = -i\hbar\hat{p}_x && (31)\\
[\hat{L}_z, \hat{p}_z] &= [x\hat{p}_y - y\hat{p}_x, \hat{p}_z] = \cancel{[x\hat{p}_y, \hat{p}_z]} - \cancel{[y\hat{p}_x, \hat{p}_z]} = 0 && (32)
\end{aligned}
$$

Notice that the z component of the angular momentum commutes with the position and momentum operators along the z axis (and the same is true for $\hat{L}_x$, $\hat{L}_y$). The position, momentum and angular momentum in the same direction are all compatible observables!

Problem 10.3 [FOR ASSIGNMENT 5; max 10 points]

First, from lectures and readings you should know that the eigenvalues of each are

$$
\begin{aligned}
\hat{L}^2 f_l^m &= \hbar^2 l(l+1) f_l^m && (33)\\
\hat{L}_z f_l^m &= \hbar m f_l^m. && (34)
\end{aligned}
$$

In the previous question we found that different components of angular momentum don't commute, but that the total angular momentum $\hat{L}^2$ commutes with each. For this reason we arbitrarily choose to work in the eigenbasis of $\hat{L}_z$, and denote the simultaneous eigenstates of $\hat{L}_z$ and $\hat{L}^2$ as f_l^m (just by the way, it actually turns out any operators that commute share a set of eigenstates).

First we will show that $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y = (\hat{L}_\mp)^\dagger$,

$$\begin{aligned}
(\hat{L}_\mp)^\dagger &= (\hat{L}_x \mp i\hat{L}_y)^\dagger && (35)\\
&= \hat{L}_x^\dagger \mp (i\hat{L}_y)^\dagger && (36)\\
&= \hat{L}_x \pm i\hat{L}_y \quad (\hat{L}_x, \hat{L}_y \text{ are Hermitian}) && (37)\\
&= \hat{L}_\pm. && (38)
\end{aligned}$$

We don't need to use this right away, but it be useful in a moment.

Next, we'll use $\hat{L}_\mp\hat{L}_\pm = \hat{L}^2 - \hat{L}_z^2 \mp \hbar\hat{L}_z$ (Eq. 4.112 in Griffifths). Using Dirac notation we can write

$$\begin{aligned}
\langle f_l^m|\hat{L}_\mp\hat{L}_\pm f_l^m\rangle &= \langle f_l^m|\left(\hat{L}^2 - \hat{L}_z^2 \mp \hbar\hat{L}_z\right) f_l^m\rangle && (39)\\
&= \langle f_l^m|\left(\hbar^2 l(l+1) - \hbar^2 m^2 \mp \hbar^2 m\right) f_l^m\rangle && (40)\\
&= \hbar^2\left[l(l+1) - m(m\pm 1)\right]\langle f_l^m|f_l^m\rangle && (41)\\
&= \hbar^2\left[l(l+1) - m(m\pm 1)\right] && (42)
\end{aligned}$$

We can also evaluate $\langle f_l^m|\hat{L}_\pm\hat{L}_\mp f_l^m\rangle$ directly

$$\begin{aligned}
\langle f_l^m|\hat{L}_\mp\hat{L}_\pm f_l^m\rangle &= \langle(\hat{L}_\mp)^\dagger f_l^m|\hat{L}_\pm f_l^m\rangle && (43)\\
&= \langle\hat{L}_\pm f_l^m|\hat{L}_\pm f_l^m\rangle \text{ (using Eq. (38))} && (44)\\
&= |A_l^m|^2\langle f_l^{m\pm 1}|f_l^{m\pm 1}\rangle && (45)\\
&= |A_l^m|^2 && (46)
\end{aligned}$$

Equating this to Eq. (42) gives the result,

$$|A_l^m| = \hbar\sqrt{l(l+1) - m(m\pm 1)} \quad (47)$$

These coefficients ensure the angular momentum eigenstates remain normalised if we either raise or lower one of them. As such, the phase is unimportant (it will always cancel in the normalisation integral), so we can just choose A_l^m to be real, $|A_l^m| = A_l^m$ (evidently A_l^m is positive).

At the either the top or bottom of the "ladder", we have $m = \pm l$, which gives

$$\begin{aligned}
A_l^{\pm l} &= \hbar\sqrt{l(l+1) \mp l(\pm l \pm 1)} && (48)\\
&= \hbar\sqrt{l^2 + l - l^2 - l} && (49)\\
&= 0. && (50)
\end{aligned}$$

Applying the lowering operator to the ground-state, or the raising operator to the maximally excited eigenstate returns zero. This ensures the Hilbert space for angular momentum is finite dimensional, with $2l+1$ non-zero vectors.