PHYS2041 Notes

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1 Wave Function

Schrodinger eq.,

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

$|\Psi(x,t)|^2$ gives the probability of finding the particle at (x,t).
Wave function is complex, so $|\Psi|^2 = \Psi^*\Psi$ where Ψ^* is the complex conjugate - let $i = -i$.

1.1 Probability

For some quantity, j, the mean is $\langle j \rangle$.
The distance from the mean is

$$\Delta j = j - \langle j \rangle$$

Standard deviation is

$$\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$$

1.2 Normalization

We require total probability to be 1, so

Normalization,

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, \mathrm{d}\, x = 1$$

1.3 Momentum

For a particle in state Ψ, the expected value of x is

$$\langle x \rangle = \int_{-\infty}^{\infty} x \, |\Psi(x,t)|^2 \, \mathrm{d}\, x$$

$$\langle v \rangle = \frac{\mathrm{d}\langle x \rangle}{\mathrm{d}\, t} = -\frac{i\hbar}{m}\int \Psi * \frac{\partial \Psi}{\partial x}\mathrm{d}\, x$$

Customary to work instead with momentum,

$$\langle p \rangle = m\frac{\mathrm{d}\langle x \rangle}{\mathrm{d}\, t} = -i\hbar\int \Psi * \frac{\partial \Psi}{\partial x}\mathrm{d}\, x$$

or, equivalently,

Expected position and momentum,

$$\langle x \rangle = \int \Psi^*(x)\Psi \mathrm{d}\, x$$
$$\langle p \rangle = \int \Psi^* \left(\frac{h}{i}\frac{\partial}{\partial x}\right)\Psi \mathrm{d}\, x$$

The 'sandwhich'ed bits ((x) and then $\left(\frac{h}{i}\frac{\partial}{\partial x}\right)$) are called *operators*.
Every other dynamical variable can be expressed in terms of p and x. Kinetic energy,

$$T = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

Angular momentum,

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = \mathbf{r} \times \mathbf{p}$$

To calculate expectation value, replace p with the operator $\left(\frac{h}{i}\frac{\partial}{\partial x}\right)$, insert, and integrate.

$$\langle Q(x,p) \rangle = \int \Psi^* Q\left(x, \frac{h}{i}\frac{\partial}{\partial x}\right)\Psi \mathrm{d}\, x$$

1.4 Uncertainty Principle

de Broglie formula,

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}$$

Heisenberg **uncertainty principle**,

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

2 Time Independent Schrodinger Equation

2.1 Stationary States

Recalling the Schrodinger equation, if we assume the solution is of the form

$$\Psi(x,t) = \psi(x)\varphi(t)$$

We obtain system of ODEs,

$$\frac{d\varphi}{dt} = -\frac{iE}{\hbar}\varphi$$
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

The first is generally solved by $\varphi(t) = e^{-iEt/\hbar}$. The second equation is referred to as the

time-independent Schrodinger eq.,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

which we cannot solve without $V(x)$.
Solutions of this form make up a small subset of the possible solutions (that is, most solutions aren't separable). However, we can do some useful things with the solution of the form

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$$

Probability density,

$$|\Psi(x,t)|^2 = |\psi(x)|^2$$

Energy is constant, so given the hamiltonian,

$$H(x,p) = \frac{p^2}{2m} + V(x)$$

we have the **Hamiltonian operator**,

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

and the time independent Schrodinger eq. becomes

$$\hat{H}\psi = E\psi$$

with expectation $\langle H \rangle = E$, and 0 variance.
Finally, the general solution will be a linear combination of separable solutions, so

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

Or, more traditionally,

> Solution to Schrodinger equation,
>
> $$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} = \sum_{n=1}^{\infty} c_n \Psi_n(x,t)$$
>
> with
> $$\Psi_n(x,t) = \psi_n(x) e^{-iE_n t/\hbar}$$

2.2 Infinite Square Well

$$V(x) = \begin{cases} 0, & \text{if } 0 \le x \le a, \\ \infty, & \text{otherwise} \end{cases}$$

Inside the well,

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -k^2\psi, \quad \text{where } k \equiv \frac{\sqrt{2mE}}{\hbar}$$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

General solution is then

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) e^{-i\left(n^2\pi^2\hbar/2ma^2\right)}$$

with coefficients

$$c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0) dx$$

2.3 Harmonic Oscillator

Classically, we have **Hooke's law**,

$$F = -kx = m\frac{\mathrm{d}\,x^2}{\mathrm{d}\,t^2}$$

with solution

$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$

where $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency. Finally, we have potential

$$V(x) = \frac{1}{2}kx^2$$

Hooke's law is not perfect - springs are not always elastic, and eventually break, etc. However, the potential is parabolic around the local minimum. Taylor expansion around minimum,

$$V(x) \approxeq \frac{1}{2}V''(x_0)(x - x_0)^2$$

The *quantum* problem is to solve Schrodinger equation for potential

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

2.3.1 Analytical Solution

Introduce

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$$

Solution is

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

with the first few **Hermite polynomials**,

$$\begin{aligned} H_0 &= 1 \\ H_1 &= 2\xi \\ H_2 &= 4\xi^2 - 2 \\ H_3 &= 8\xi^3 - 12\xi \\ H_4 &= 16\xi^4 - 48\xi^2 + 12 \\ H_5 &= 32\xi^5 - 160\xi^3 + 120\xi \end{aligned}$$

2.3.2 Ladder Operators

The Schrodinger equation may be re-written

$$\frac{1}{2m}\left[p^2 + (m\omega x)^2\right]\psi = E\psi$$

Define the

> Ladder operator,
>
> $$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$$

and the

> **commutator** of operators A and B is
>
> $$[A, B] = AB - BA$$

For example the **canonical commutation relation**,

$$[x, p] = i\hbar$$

and so the Schrodinger equation becomes

$$\hbar\omega\left(a_\pm a_\mp \pm \frac{1}{2}\right)\psi = E\psi$$

Crucially,

> If ψ satisfies the Schrodinger equation with energy E, then $a_+\psi$ *also* satisfies with energy $(E + \hbar\omega)$. Similarly, $a_-\psi$ satisfies with energy $(E - \hbar\omega)$

We call

> a_+ the **raising operator**, and
> a_- the **lowering operator**.

What if we apply the lowering operator repeatedly? We'd eventually have negative energy - this would still be a solution, but not normalisable. Denote ψ_0 as the lowest energy normalisable solution,

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

with energy

$$E_0 = \frac{1}{2}\hbar\omega$$

Increasing the energy each step,

$$\psi_n(x) = A_n\left(a_+\right)^n\psi_0(x), \quad \text{with } E_n = \left(n+\frac{1}{2}\right)\hbar\omega$$

or

$$\psi_n = \frac{1}{\sqrt{n!}}(a_+)^n\psi_0$$

2.4 The Free Particle

$V(x) = 0$ for all x. Expect solution of form

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

No boundary conditions (unlike infinite square well), so cannot restrict k. Tacking on standard time dependence, $\exp(-iEt/\hbar)$,

$$\Psi(x,t) = Ae^{ik\left(x-\frac{\hbar k}{2m}t\right)} + Be^{-ik\left(x+\frac{\hbar k}{2m}t\right)}$$

Or, setting

$$k \equiv \pm\frac{\sqrt{2mE}}{\hbar} \quad \text{with} \begin{cases} k>0 \Rightarrow & \text{traveling to the right,} \\ k<0 \Rightarrow & \text{traveling to the left.}\end{cases}$$

we have

$$\Psi_k(x,t) = Ae^{i\left(kx-\frac{\hbar k^2}{2m}t\right)}$$

with $p = \hbar k$ and $v_{\text{quantum}} = \frac{\hbar|k|}{2m} = \sqrt{\frac{E}{2m}}$.
Not normalisable - no such thing as a free particle with a definite energy. General solution is

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\phi(k)e^{i\left(kx-\frac{\hbar k^2}{2m}t\right)}dk$$

We call this a **wave packet** - carries a range of k's and range of energies and speeds.

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Psi(x.0)e^{-ikx}dx$$

2.5 Delta-function Potential

$$\begin{cases} E<0 \Rightarrow & \text{bound state} \\ E>0 \Rightarrow & \text{scattering state}\end{cases}$$

The **Dirac delta function**,

$$\delta(x) = \left\{\begin{array}{ll} 0, & \text{if } x\neq 0 \\ \infty, & \text{if } x=0\end{array}\right\}, \text{ with } \int_{-\infty}^{+\infty}\delta(x)dx = 1$$

We consider a potential of the form

$$V(x) = -\alpha\delta(x)$$

and the bound state, with $E<0$, and the Schrodinger eq. becomes

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \kappa^2\psi, \quad \text{where } \kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

with solution

$$\psi(x) = \begin{cases} Be^{\kappa x}, & x\leq 0 \\ Be^{-\kappa x}, & x\geq 0\end{cases}$$

Normalization of ψ gives $B = \sqrt{\kappa}$, with energy

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^2}; \quad E = -\frac{m\alpha^2}{2\hbar^2}$$

What about scattering states, $E>0$? Schrodinger eq. becomes

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -k^2\psi, \quad k = \frac{\sqrt{2mE}}{\hbar}$$

with general solution

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

for $x<0$, and

$$\psi(x) = Fe^{ikx} + Ge^{-ikx}$$

for $x>0$. Continuity of $\psi(x)$ and $x=0$ requires that $F+G = A+B$ and that the derivatives are equal, which implies

$$F-G = A(1+2i\beta) - B(1-2i\beta), \quad \text{where } \beta \equiv \frac{m\alpha}{\hbar^2 k}$$

1. A is the amplitude of the **incident** wave,
2. B is the amplitude of the **reflected** wave, and
3. F is the amplitude of the **transmitted** wave.

Solving for B, F,

$$B = \frac{i\beta}{1-i\beta}A, \quad F = \frac{1}{1-i\beta}A$$

We define the **reflection coefficient**,

$$R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}$$

and **transmission coefficient**,

$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2}$$

and $R+T = 1$. Otherwise, we may express these as

> Reflection R and transmission T coefficients,
>
> $$R = \frac{1}{1+(2\hbar^2E/m\alpha^2)}, \quad T = \frac{1}{1+(m\alpha^2/2\hbar^2E)}$$

2.6 Finite Square Well

Finally, consider a potential

$$V(x) = \begin{cases} -V_0, & \text{for } -a<x<a \\ 0, & \text{for } |x|>a\end{cases}$$

with solutions of the form

$$\psi(x) = \begin{cases} Fe^{-\kappa x}, & \text{for } (x>a) \\ D\cos(lx), & \text{for } (0<x<a) \\ \psi(-x), & \text{for } (x<0)\end{cases}$$

where

$$l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

3 Formalism

3.1 Hilbert Space

In N dimensional space, we represent a vector, $|\alpha\rangle$, by the N-tuple of it's components,

$$|\alpha\rangle \to \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

The **inner product** of two vectors,

$$\langle\alpha|\beta\rangle = a_1^* b_1 + a_2^* b_2 + \cdots + a_N^* b_N$$

Linear transformations, T, are represented by matrices,

$$|\beta\rangle = T|\alpha\rangle \to \mathbf{b} = \mathbf{Ta} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1N} \\ t_{21} & t_{22} & \cdots & t_{2N} \\ \vdots & \vdots & & \vdots \\ t_{N1} & t_{N2} & \cdots & t_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

For quantum mechanics in particularly, we define

The **Hilbert space** is the vector space of

$$f(x) \text{ such that } \int_a^b |f(x)|^2 \,\mathrm{d}x < \infty$$

Then, *wave functions live in Hilbert space.*

Inner product of two function, $f(x)$ and $g(x)$ as

$$\langle f|g\rangle \equiv \int_a^b f(x)^* g(x) \,\mathrm{d}x$$

Notice,

$$\langle g|f\rangle = \langle f|g\rangle^*$$

and

$$\langle f|f\rangle = \int_a^b |f(x)|^2 \,\mathrm{d}x$$

Finally, a set of functions is *complete* if any other function (in Hilbert space) can be expressed as a linear combination of them,

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

If the functions $\{f_n(x)\}$ are orthogonal, coefficients are given by Fourier's trick,

$$c_n = \langle f_n|f\rangle$$

3.2 Dirac Notation

Let $|\mathcal{S}\rangle$ represent the state of a system in QM,

$$\Psi(x,t) = \langle x|\mathcal{S}(t)\rangle$$

with $|x\rangle$ standing for the eigenfunction of $\hat{x}$ with eigenvalue x. Likewise, Momentum space,

$$\Phi(p,t) = \langle p|\mathcal{S}(t)\rangle$$

Or, in terms of the energy eigenfunctions,

$$c_n(t) = \langle n|\mathcal{S}(t)\rangle$$

These are just different ways of expression the exact same vector in Hilbert space, but with different basis. Operators are linear transformations,

$$|\beta\rangle = \hat{Q}\,|\alpha\rangle$$

Operators are represented by matrix elements,

$$\left\langle e_m \middle| \hat{Q} e_n \right\rangle = Q_{mn}$$

and with the linear transformations notation,

$$\sum_n b_n\,|e_n\rangle = \sum_n a_n \hat{Q}\,|e_n\rangle$$

3.2.1 Dirac Notation

Dirac proposed to split the inner product notation $\langle\alpha|\beta\rangle$, into the **bra** $\langle\alpha|$ and **ket** $|\beta\rangle$. The latter is a column vector. The former is a *linear function* of vectors. In function space,

$$\langle f| = \int f^* \,(\ldots)\,\mathrm{d}x$$

with $(\ldots)$ to be determined by the ket it encounters.

In finite-dimensional vector space,

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

and

$$\langle\alpha| = (a_1^* a_2^* \ldots a_n^*)$$

3.3 Observables

3.3.1 Hermitian Operators

The expectation value of an observable $Q(x,p)$ may be expressed in inner-product notation,

$$\langle Q\rangle = \int \Psi^* \hat{Q} \Psi dx = \langle \Psi \mid \hat{Q}\Psi\rangle$$

We call an operator **hermitian** if

$$\langle f \mid \hat{Q} f\rangle = \langle \hat{Q} f \mid f\rangle \quad \text{for all } f(x)$$

3.3.2 Determinate States

For determinate states, where every measurement of Q should return the same value, q, then

$$\hat{Q}\Psi = q\Psi$$

This is the **eigenvalue equation** for $\hat{Q}$; Ψ is an **eigenfunction** of $\hat{Q}$, and q is the corresponding **eigenvalue**.

3.4 Generalised Statistical Interpretation

If I measure $Q(x,p)$, on a particle with state $\Psi(x,t)$, I will get one of the eigenvalues of hermitian operator $\hat{Q}(x, -\frac{ih\mathrm{d}}{\mathrm{d}x})$. If spectrum of $\hat{Q}$ is discrete, probability of getting q_n with eigenfunction f_n is

$$|c_n|^2 \quad \text{where} \quad c_n = \langle f_n|\Psi\rangle$$

If the spectrum continuous, eigenvalues $q(z)$ and eigenfunctions $f_z(x)$, probability of result in range $\mathrm{d}z$ is

$$|c(z)|^2 dz \quad \text{where} \quad c(z) = \langle f_z \mid \Psi\rangle$$

Eigenfunctions of an observable operator are complete, so

$$\Psi(x,t) = \sum_n c_n f_n(x)$$

with coefficients

$$c_n = \langle f_n|\Psi\rangle = \int f_n(x)^* \Psi(x,t) \mathrm{d}\, x$$
$$\sum_n |c_n|^2 = 1$$

Similarly, expectation value of Q should be the sum of all possible outcomes/eigenvalues, times the probability of that eigenvalue,

$$\langle Q|Q\rangle = \sum_n q_n |c_n|^2$$

Momentum space wave function,

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,t) dx$$
$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \Phi(p,t) dp$$

3.5 Uncertainty Principle

Shwarz inequality,

$$\langle f|f\rangle \langle g|g\rangle \geq |\langle f|g\rangle|^2$$

Commutator between operators $\hat{A}$ and $\hat{B}$,

$$[\hat{A}.\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

Energy-time uncertainty principle,

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

4 Three-Dimensions

4.1 Schrodinger eq.

In 3D, the Schrodinger eq.,

3D Schrodinger eq.,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi$$

where *Laplacian*,

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Normalisation,

$$\int |\Psi|^2 d^3\mathbf{r} = 1$$

Stationary states,

$$\Psi_n(\mathbf{r},t) = \psi_n(\mathbf{r}) e^{-iE_n t/\hbar}$$

where ψ_n satisfies the

time-independent Schrodinger eq.,

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi$$

General solution,

$$\Psi(\mathbf{r},t) = \sum c_n \psi_n(\mathbf{r}) e^{-iE_n t/\hbar}$$

4.1.1 Spherical coordinates

Of form (r, θ, ϕ).
Solution to *something* is

$$\Theta(\theta) = AP_l'''(\cos\theta)$$

where P_l^m is **Legendre function**,

$$P_l^m(x) \equiv \left(1 - x^2\right)^{|m|/2} \left(\frac{d}{dx}\right)^{|n|} P_l(x)$$

and $P_l(x)$ is the lth **Legendre polynomial**, defined by **Rodrigues formula**,

$$P_l(x) \equiv \frac{1}{2^l l!} \left(\frac{d}{dx}\right)^l \left(x^2 - 1\right)^l$$

Normalised angular wave function are called

spherical harmonics

$$Y_l^m(\theta,\phi) = \epsilon \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} e^{im\phi} P_l^m(\cos\theta)$$

l is called the **azimuthal quantum number**, and m the **magnetic quantum number**.

Radial equation,

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left[V + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}\right] u = Eu$$

4.2 Hydrogen Atom

Potential energy

$$V(r) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r}$$

Principal quantum number, $n = j_{\max} + l + 1$.
Allowed energies are **bohr formula**,

$$E_n = -\left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right] \frac{1}{n^2} = \frac{E_1}{n^2}, \quad n = 1.2.3, \ldots$$

and **Bohr radius**,

$$a \equiv \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.529 \times 10^{-10} \text{ m}$$

Ground state, $n = 1$,

$$E_1 = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right] = -13.6\text{eV}$$

Noramlised hydrogen wave functions are

$$\psi_{nlm} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} e^{-r/na} \cdot \left(\frac{2r}{na}\right)^l \left[L^{2l+1}_{n-l-1}(2r/na)\right] Y^m_l(\theta, \phi)$$

TABLE 4.5: The first few Laguerre polynomials, $L_q(x)$.

$L_0 = 1$
$L_1 = -x + 1$
$L_2 = x^2 - 4x + 2$
$L_3 = -x^3 + 9x^2 - 18x + 6$
$L_4 = x^4 - 16x^3 + 72x^2 - 96x + 24$
$L_5 = -x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120$
$L_6 = x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720$

TABLE 4.6: Some associated Laguerre polynomials, $L^p_{q-p}(x)$.

$L^0_0 = 1$	$L^2_0 = 2$
$L^0_1 = -x + 1$	$L^2_1 = -6x + 18$
$L^0_2 = x^2 - 4x + 2$	$L^2_2 = 12x^2 - 96x + 144$
$L^1_0 = 1$	$L^3_0 = 6$
$L^1_1 = -2x + 4$	$L^3_1 = -24x + 96$
$L^1_2 = 3x^2 - 18x + 18$	$L^3_2 = 60x^2 - 600x + 1200$

4.2.1 Spectrum

Energy of transitions are equal to differences between initial and final states.
Planck formula, energy of photon proportional to its frequency, $E_\gamma = hv$. Wavelength $\lambda = c/v$
Rydberg formula,

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

and **Rydberg constant** for Hydrogen,

$$R \equiv \frac{m}{4\pi c\hbar^3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = 1.097 \times 10^7 \text{ m}^{-1}$$

4.3 Angular Momentum

Classically, angular momentum,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

or

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x$$

Quantum operators defined $p_x \to -\frac{\hbar\partial}{\partial x}$ and so on.

4.3.1 Eigenvalues

Fundamental commutation relations for angular momentum,

$$[L_x, L_y] = i\hbar L_z; \quad [L_y, L_z] = i\hbar L_x; \quad [L_z, L_x] = i\hbar L_y$$

$$\left[L^2, L_x\right] = 0, \quad \left[L^2, L_y\right] = 0, \quad \left[L^2, L_z\right] = 0$$

Define the ladder operator,

$$L_\pm \equiv L_x \pm iL_y$$

So

$$[L_z, L_\pm] = \pm\hbar L_\pm, \text{ and } \left[L^2, L_\pm\right] = 0$$

Eigenfunctions,

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \quad L_z f_l^m = \hbar m f_l^m$$

where $l = 0, 1/2, 1, 3/2, \ldots$ and $m = -l, -l+1, \ldots l-1, l$.

4.3.2 Eigenfunctions

Gradient, in spherical coordinates,

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

implies

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

and

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

4.4 Spin

Good luck!

5 Identical Particles

Ran out of time. GL!

6 Fundamental Constants

Planck's constant, $\hbar = 1.05457 \times 10^{-34}\,\text{J s}$
Speed of light, $c = 2.99 \times 10^8\,\text{m s}^{-1}$
Mass of election, $m_e = 9.109 \times 10^{-31}\text{kg}$
Mass of proton, $m_p = 1.67 \times 10^{-27}\text{kg}$
Charge of proton, $e = 1.602 \times 10^{-19}\text{C}$
Charge of electron, $-e$
Permittivity of space, $\varepsilon_0 = 8.85 \times 10^{-12}\text{C}^2\,\text{J}^{-1}\,\text{m}$
Boltzmann constant, $k_B = 1.38 \times 10^{-23}\text{J K}^{-1}$