Ass 3

Monday, 1 May 2023 9:02 AM

Q1

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

a. $\Delta t = 0$

$$\Delta t' = t_2' - t_1'$$

$$= \gamma(ct_2 - Vx_2/c) - \gamma(ct_1 - Vx_1/c)$$

$$= \gamma c(t_2 - t_1) + \gamma V/c\,(x_1 - x_2)$$

$$= \gamma c\Delta t - \gamma V/c\,\Delta x = -\gamma V/c\,\Delta x$$

Since we have no spatial information about the events, we take the absolute value of this, and set $c=1$ as per convention to yield:

$$\Delta t' = \gamma V \Delta x$$

We don't have enough information to say if Δx is the proper length between events (as we don't know if the two events are at rest in frame K). If there was some stationary rod in frame K, however, whose endpoints were at the spatial coords of events 1 and 2, Δx would be the proper length of this rod.

b. $x = r\cos\phi$
$y = r\sin\phi$
$z = z$

$$dx = \frac{dx}{dr}dr + \frac{dx}{\partial\phi}d\phi$$

$$= \cos\phi\, dr + r\sin\phi\, d\phi$$

$$\Rightarrow dx^2 = \cos^2\phi\, dr^2 + r^2\sin^2\phi\, d\phi + 2r\sin\phi\cos\phi\, dr\, d\phi$$

$$\Rightarrow dy^2 = \sin^2\phi\, dr^2 + r^2\cos^2\phi\, d\phi - 2r\sin\phi\cos\phi\, dr\, d\phi$$

$$dz^2 = dz^2\ ;\quad dt^2 = dt^2$$

And so, Minkowski spacetime in cylindrical coords is

$$ds^2 = -dt^2 + (\cos^2\phi + \sin^2\phi)dr^2 + r^2(\sin^2\phi + \cos^2\phi)d\phi + dz^2$$

$$= -dt^2 + dr^2 + r^2 d\phi^2 + dz^2$$

Q2

We have that $\Delta t' = 10$ years, with $V = 0.96c$. Hence,

$$\Delta t_1' = \Delta t_1\sqrt{1 - V^2/c^2}$$

$$\Rightarrow \Delta t_1 = \frac{\Delta t_1'}{} = \frac{10 \text{ years}}{} \simeq 35.7 \text{ years}$$

$$\Delta t_1' = \Delta t_1 \sqrt{1 - v^2/c^2}$$

$$\Rightarrow \Delta t_1 = \frac{\Delta t_1'}{\sqrt{1 - v^2/c^2}} = \frac{10 \text{ years}}{\sqrt{1 - 0.96^2}} \simeq 35.7 \text{ years}$$

And so, according to Bob, Alice travels

$$\Delta t_1 \times v \simeq 34.3 \text{ light years}$$

in her initial journey. She then reverses and travels at $0.32c$, hence taking

$$\Delta t_2 = \frac{d}{v} = \frac{34.3 \text{ ly}}{0.32\, c}$$

$$\simeq 107.2 \text{ years}$$

for the second stretch of the journey (according to Bob). Hence, Bob has aged a total of $107.2 + 35.7 = 142.9$ years across his perception of Alice's journey.

Meanwhile, the second stretch of the journey according to Alice took

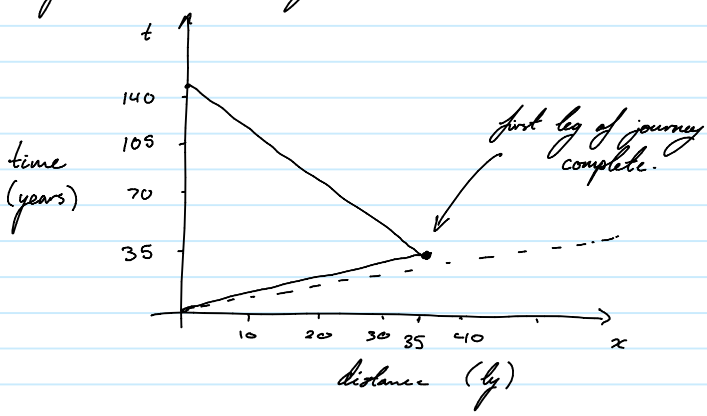
$$\Delta t_2' = \frac{\Delta x_2'}{v}$$

$$= \frac{\sqrt{1 - v^2/c^2}\ \Delta x_2}{v}$$

$$= \frac{\sqrt{1 - 0.32^2} \cdot 34.3 \text{ ly}}{0.32\, c}$$

$$\simeq 101.6 \text{ years}$$

And so, according to Alice, her entire journey took 111.6 years. Therefore, at their reunion, we'd expect an age difference of about 31.3 years between Alice and Bob



Q3.

$$\cos\alpha' = \frac{\cos\alpha + v}{1 + v\cos\alpha}$$

$$\omega = \omega' \gamma (1 - v\cos\alpha')$$

$$\Rightarrow \omega' = \frac{\omega\sqrt{1 - v^2/c^2}}{1 - \frac{v\cos\alpha + v^2}{1 + v\cos\alpha}}$$

for $\alpha = \pi/2$, and $v = -0.8$

$$\omega' = \omega \frac{\sqrt{1 - (-0.8)^2}}{1 - \frac{-0.8 \times 0 + (-0.8)^2}{\ldots}}$$

$$\omega' = \omega \frac{\sqrt{1-(-0.8)^2}}{1 - \frac{-0.8\times 0 + (-0.8)^2}{1-0.8\times 0}}$$

$$= \omega \frac{\sqrt{1-0.64}}{1-0.64}$$

$$= \frac{5}{3}\omega$$

And so the source is blueshifted.

Q4

$$u^\alpha = (\cosh a\tau, \sinh a\tau, 0, 0)$$

i. $u^\alpha = (\gamma, \gamma\vec{v})$

$$\Rightarrow \gamma = \cosh a\tau$$

$$\gamma\vec{v} = \sinh a\tau$$

$$\vec{v} = \frac{\sinh a\tau}{\cosh a\tau} = \tanh a\tau$$

$$a^\alpha = \frac{du^\alpha}{d\tau} = \left(\frac{d}{d\tau}\gamma, \frac{d}{d\tau}\gamma\vec{v}\right)$$

but $\frac{d}{d\tau}\gamma = 0$

$$\Rightarrow a^\alpha = \left(0, \frac{d}{d\tau}\gamma\vec{v} + \gamma\frac{d}{d\tau}\vec{v}\right)$$

$$= \left(0, 0 + \gamma\left(\frac{dt}{d\tau}\frac{d\vec{v}}{dt}\right)\right)$$

$$= \left(0, \gamma\left(\frac{dt}{d\tau}\right)\left(\frac{d\vec{v}}{dt}\right)\right)$$

$$= (0, \gamma^2\vec{a})$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\tanh a\tau)$$

$$= \frac{d\tau}{dt}\frac{d}{d\tau}\tanh a\tau$$

$$= \gamma^{-1} a \operatorname{sech}^2 a\tau$$

$$= a \operatorname{sech}^3 a\tau$$

ii. For $|\tau| \ll 1$, we can effectively approximate $\tau \simeq 0$, hence

$$\vec{a} \simeq a \operatorname{sech}^3(0)$$
$$= a$$

iii. Since the proper time is the same in all inertial frames, τ is unchanged when we boost into the frame co-moving with the particle. Since its comoving at $\tau = \tau_0$, our boost velocity is

$$V = \vec{v}(\tau_0) = \tanh a\tau_0$$

Now,

$$t' = \gamma(t - Vx) \Rightarrow dt' = \gamma(dt - V dx)$$
$$x' = \gamma(x - Vt) \Rightarrow dx' = \gamma(dx - V dt)$$

and so

$$v' = \frac{dx'}{dt'} = \frac{\gamma(dx - Vdt)}{\gamma(dt - Vdx)}$$

and so

$$v' = \frac{dx'}{dt'} = \frac{\gamma(dx - Vdt)}{\gamma(dt - Vdx)}$$

$$= \frac{dx - Vdt}{dt - Vdx}$$

$$= \frac{dx/dt - V}{1 - V\,dx/dt}$$

but $dx/dt = \tanh a\tau$

$$\Rightarrow \quad v' = \frac{\tanh a\tau - \tanh a\tau_0}{1 - \tanh a\tau_0 \tanh a\tau}$$

$$= \frac{\tanh a\tau + \tanh(-a\tau_0)}{1 + \tanh a\tau \tanh(-a\tau_0)}$$

$$= \tanh a(\tau - \tau_0)$$

Now we want the 3-acceleration in this boosted frame:

$$\vec{a} = \frac{d^2 x}{dt^2} = \frac{d}{dt} \tanh a(\tau - \tau_0)$$

$$= \frac{d\tau}{dt}\frac{d}{d\tau} \tanh a(\tau - \tau_0)$$

$$= \gamma^{-1} a \,\mathrm{sech}^2 a(\tau - \tau_0)$$

$$= a \,\mathrm{sech}^3 a(\tau - \tau_0)$$

As in part ii), as $|\tau - \tau_0| \ll 1$, take limit as $|\tau - \tau_0| \to 0$

$$\Rightarrow \lim_{|\tau-\tau_0| \ll 1} \vec{a} \simeq a \times 1$$

$$= a$$

And so the local acceleration felt by the particle is always a, since we could boost into any co-moving frame and the acceleration would still be equal to a.

Q5.

$$ds^2 = -(1-\omega^2 r^2)dt^2 + 2\omega r^2 d\phi dt + r^2 d\phi^2 + dr^2 + dz^2$$

$$u^\alpha = \left(\frac{dt}{d\tau}, \frac{d\phi}{d\tau}, \frac{dr}{d\tau}, \frac{dz}{d\tau}\right)$$

$$= \left(\gamma, \gamma\left(\frac{v}{r} - \omega\right), 0, 0\right)$$

$$g_{\alpha\beta} = \begin{pmatrix} -(1-\omega^2 r^2) & 2\omega r^2 & 0 & 0 \\ 2\omega r^2 & r^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underline{u} \cdot \underline{u} = g_{\alpha\beta} u^\alpha u^\beta$$

$$= -(1-\omega^2 r^2)\gamma^2 + 2\omega r^2 \gamma^2 \left(\frac{v}{r} - \omega\right)^2 + r^2 \gamma^2 \left(\frac{v}{r} - \omega\right)^2$$

$$= \gamma^2\left(-1 + \omega^2 r^2 + 2\omega r^2\left(\frac{v^2}{r^2} - \frac{2v\omega}{r} + \omega^2\right) + r^2\left(\frac{v^2}{r^2} - \frac{2v\omega}{r} + \omega^2\right)\right)$$

$$= \gamma^2\left(-1 + \omega^2 r^2 + 2\omega v^2 - 4\omega^2 r v + 2\omega^3 r^2 + v^2 - 2v\omega r + r^2\omega^2\right)$$

$$= \gamma^2\left(-1 + \omega^2 r^2 + 2\omega v^2 - 4\omega^2 r v + 2\omega^3 r^2 + v^2 - 2v\omega r + r^2\omega^2\right)$$

$v = \omega r$ (circular motion)

$$\underline{u}\cdot\underline{u} = \gamma^2\left(-1 + v^2 + 2\omega v^2 - 4\omega v^2 + 2\omega v^2 + v^2 - 2v^2 + v^2\right)$$

$$= \gamma^2(-1 + v^2)$$

$$= -\frac{1}{1-v^2} + \frac{v^2}{1-v^2}$$

$$= -\frac{1}{1-v^2} + \frac{1}{\frac{1}{v^2}-1}$$

$$= \frac{-\left(\frac{1}{v^2}-1\right) + (1-v^2)}{(1-v^2)\left(\frac{1}{v^2}-1\right)}$$

$$= \frac{2 - \frac{1}{v^2} - v^2}{\frac{1}{v^2} - 1 - 1 + v^2}$$

$$= \frac{2 - \frac{1}{v^2} - v^2}{-2 + \frac{1}{v^2} + v^2}$$

$$= -1$$

And so $\underline{u}\cdot\underline{u}$ has the correct normalisation. This scenario represents uniform circular motion in two dimensions, with $v = \omega r$.

Q6

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$\Rightarrow \quad g_{\alpha\beta} = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

a. We have

$$\tau_{AB} = \int_A^B \sqrt{-g_{\alpha\beta}\, dx^\alpha dx^\beta}$$

To find the Lagrangian, we need to solve for the extremal case $\tau_{AB} = 0$. Parametrising the worldline with respect to τ, we get

$$\tau_{AB} = \int_A^B d\tau \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}$$

So, for $\tau_{AB} = 0$, we must have

$$\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}} = 0 = \mathcal{L}'$$

Notice that we could just as easily have

$$\mathcal{L} = \mathcal{L}'^2 = -g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

In terms of our 4-D Schwarzschild metric, this becomes

$$\mathcal{L} = \left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2$$

b. The associated Euler-Lagrange equations for this Lagrangian are

$$\frac{d}{d\tau}\left(\frac{-\partial\mathcal{L}}{\partial(dx^\mu/d\tau)}\right) + \frac{\partial\mathcal{L}}{\partial x^\mu} = 0$$

$$\frac{d}{d\tau}\left(\frac{-\partial \mathcal{L}}{\partial(dx^\mu/d\tau)}\right)+\frac{\partial \mathcal{L}}{\partial x^\mu}=0$$
$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial(dx^\mu/d\tau)}\right)=\frac{\partial \mathcal{L}}{\partial x^\mu}$$

for $x^\mu = t$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial(dt/d\tau)}\right)-\frac{\partial \mathcal{L}}{\partial t}=0$$
$$0=\frac{d}{d\tau}\left(2\left[1-\frac{2M}{r}\right]\frac{dt}{d\tau}\right)-0$$
$$=2\left[1-\frac{2M}{r}\right]\frac{d^2t}{d\tau^2}+2\frac{dt}{d\tau}\frac{d}{d\tau}\left[1-\frac{2M}{r}\right]$$
$$\Rightarrow \frac{d^2t}{d\tau}=\frac{-1}{\left(1-\frac{2M}{r}\right)}\frac{dt}{d\tau}\left(\frac{2M}{r^2}\frac{dr}{d\tau}\right)$$
$$=-\frac{2M}{r(r-2M)}\frac{dt}{d\tau}\frac{dr}{d\tau}$$

for $x^\mu = r$,

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial(dr/d\tau)}\right)=\frac{\partial \mathcal{L}}{\partial r}$$
$$\frac{d}{d\tau}\left(-2\left(1-\frac{2M}{r}\right)^{-1}\frac{dr}{d\tau}\right)=\frac{2M}{r^2}\frac{dt^2}{d\tau^2}+\frac{2M}{(r-2M)^2}\frac{dr^2}{d\tau^2}-2r\frac{d\theta^2}{d\tau^2}-2r\sin^2\theta\frac{d\phi^2}{d\tau^2}$$
$$-2\left(1-\frac{2M}{r}\right)^{-1}\frac{d^2r}{d\tau^2}-2\frac{dr}{d\tau}\frac{d}{d\tau}\left(1-\frac{2M}{r}\right)^{-1}=\frac{2M}{r^2}\frac{dt^2}{d\tau^2}+\frac{2M}{(r-2M)^2}\frac{dr^2}{d\tau^2}-2r\frac{d\theta^2}{d\tau^2}-2r\sin^2\theta\frac{d\phi^2}{d\tau^2}$$
$$\Rightarrow \frac{d^2r}{d\tau^2}=\left(1-\frac{2M}{r}\right)\left[\frac{2M}{(r-2M)^2}\frac{dr^2}{dt^2}-\frac{M}{(r-2M)^2}\frac{dr^2}{dt^2}-\frac{M}{r^2}\frac{dt^2}{d\tau^2}+r\frac{d\theta^2}{d\tau^2}+r\sin^2\theta\frac{d\phi^2}{d\tau^2}\right]$$
$$=-\frac{M(r-2M)}{r^3}\frac{dt^2}{d\tau^2}+\left(1-\frac{2M}{r}\right)\frac{M}{(r-2M)^2}\frac{dr^2}{d\tau^2}+(r-2M)\frac{d\theta^2}{d\tau^2}+(r-2M)\sin^2\theta\frac{d\phi^2}{d\tau^2}$$
$$=\frac{-M(r-2M)}{r^3}\frac{dt^2}{d\tau^2}+\frac{M}{r(r-2M)}\frac{dr^2}{d\tau^2}+(r-2M)\frac{d\theta^2}{d\tau^2}+(r-2M)\sin^2\theta\frac{d\phi^2}{d\tau^2}$$

for $x^\mu = \theta$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial(d\theta/d\tau)}\right)-\frac{\partial \mathcal{L}}{\partial \theta}=0$$
$$0=\frac{d}{d\tau}\left(-2r^2\frac{d\theta}{d\tau}\right)+2r^2\sin\theta\cos\theta\frac{d\phi^2}{d\tau^2}$$
$$=-\left(2r^2\frac{d^2\theta}{d\tau^2}+2\frac{d\theta}{d\tau}\frac{d}{d\tau}r^2\right)+2r^2\sin\theta\cos\theta\frac{d\phi^2}{d\tau^2}$$
$$=-r^2\frac{d^2\theta}{d\tau^2}-2r\frac{d\theta}{d\tau}\frac{dr}{d\tau}+r^2\sin\theta\cos\theta\frac{d\phi^2}{d\tau^2}$$
$$\Rightarrow \frac{d^2\theta}{d\tau^2}=-\frac{2}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau}+\sin\theta\cos\theta\frac{d\phi^2}{d\tau^2}$$

finally for $x^\mu = \phi$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial(d\phi/d\tau)}\right)-\frac{\partial \mathcal{L}}{\partial \phi}=0$$
$$0=\frac{d}{d\tau}\left(-2r^2\sin^2\theta\frac{d\phi}{d\tau}\right)-0$$
$$=-2r^2\sin^2\theta\frac{d^2\phi}{d\tau^2}-2r^2\frac{d\phi}{d\tau}\frac{d}{d\tau}\left(\sin^2\theta\right)-2\sin^2\theta\frac{d\phi}{d\tau}\frac{d}{d\tau}r^2$$
$$r^2\sin^2\theta\frac{d^2\phi}{d\tau^2}=-2r^2\sin\theta\cos\theta\frac{d\phi}{d\tau}\frac{d\theta}{d\tau}-2r\sin^2\theta\frac{d\phi}{d\tau}\frac{dr}{d\tau}$$
$$\Rightarrow \frac{d^2\phi}{d\tau^2}=-\frac{2\cos\theta}{\sin\theta}\frac{d\phi}{d\tau}\frac{d\theta}{d\tau}-\frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau}$$

c. We have that

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\sigma}\frac{dx^\beta}{d\tau}\frac{dx^\sigma}{d\tau}$$

Hence, from part b, we have

$$\Gamma^t_{rt} = \Gamma^t_{tr} = \frac{M}{r(r-2M)}$$

$$\Gamma^r_{tt} = \frac{M(r-2M)}{r^3} \qquad \Gamma^r_{\theta\theta} = -(r-2M)$$

$$\Gamma^r_{rr} = -\frac{M}{r(r-2M)} \qquad \Gamma^r_{\phi\phi} = -(r-2M)\sin^2\theta$$

$$\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \qquad \Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}$$

d. I didn't get time to attempt this question, but I'll still comment on $R_{\alpha\beta} = 0$.

If the Ricci curvature $R_{\alpha\beta}$ vanishes, the Einstein field equation becomes

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -8\pi G T_{\alpha\beta}$$

$$\Rightarrow \quad 0 - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu}R_{\mu\nu} = -8\pi G T_{\alpha\beta}$$

$$\Rightarrow \quad T_{\alpha\beta} = \underline{0}$$

i.e. the energy momentum tensor is all zero. That is, the Schwarzschild metric is a solution to the vacuum field equation, and that it completely describes spacetime due to gravity.