Assignment 4

Tuesday, 21 September 2021 10:54 AM

Problem 7.3:

a. Use $\hat{H}$ to show that

$$|\psi(t)\rangle = \sum_n c_n |\chi_n\rangle e^{-i\frac{E_n}{\hbar}t}$$

Shrodinger Equation is

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

In terms of state vectors,

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

$$= \sum_n E_n |\chi_n\rangle\langle\chi_n||\psi(t)\rangle$$

From result from 7.1, $\langle\chi_n|\psi(t)\rangle = c_n$

$$\Rightarrow \quad i\hbar \frac{d}{dt}|\psi(t)\rangle = \sum_n c_n E_n |\chi_n\rangle$$

$$\frac{d}{dt}|\psi(t)\rangle = \sum_n c_n - i\frac{E_n}{\hbar}|\chi_n\rangle$$

but $|\psi\rangle = c_n|\chi_n\rangle$

$$\Rightarrow \frac{d}{dt}|\psi(t)\rangle = \sum_n -i\frac{E_n}{\hbar}|\psi\rangle$$

This ODE has solution

$$\frac{d}{dt} b e^{at} = ab e^{at}$$

with $a = -i\frac{E_n}{\hbar}$

$$\frac{d}{dt}|\psi(t)\rangle = \sum_n -i\frac{E_n}{\hbar}|\psi\rangle e^{-i\frac{E_n}{\hbar}t}$$

$$\Rightarrow |\psi(t)\rangle = \sum_n |\psi\rangle e^{-i\frac{E_n}{\hbar}t}$$

$$= \sum_n c_n|\chi_n\rangle e^{-i\frac{E_n}{\hbar}t} \qquad \text{QED}$$

b. This gives $c_6 = \frac{1}{\sqrt{6}}$, $c_{17} = -\frac{i}{\sqrt{2}}$ and $c_{271} = \frac{1}{\sqrt{3}}$

$$\Rightarrow \quad |\psi(t)\rangle = \frac{1}{\sqrt{6}}|\chi_6\rangle e^{-i\frac{E_6}{\hbar}t} - \frac{i}{\sqrt{2}}|\chi_{17}\rangle e^{-i\frac{E_{17}}{\hbar}t} + \frac{1}{\sqrt{3}}|\chi_{271}\rangle e^{-i\frac{E_{271}}{\hbar}t}$$

Problem 7.4:

Show that an arbitrary operator, $\hat{A}$, can be written as

$$\hat{A} = \sum \sum A_{mn} |\chi_m\rangle\langle\chi_n|$$

Show that an arbitrary operator, $\hat{A}$, can be written as

$$\hat{A} = \sum_m \sum_n A_{m,n} |\psi_m\rangle\langle\psi_n|$$

We have that $\hat{1} = \sum_n |\psi_n\rangle\langle\psi_n| = \sum_m |\psi_m\rangle\langle\psi_m|$

And $\hat{A}\hat{1} = \hat{1}\hat{A} = \hat{A}$, so

$$\hat{A} = \hat{1}\hat{A}\hat{1} = \sum_m |\psi_m\rangle\langle\psi_m|\hat{A}\hat{1}$$

$$= \sum_m \sum_n |\psi_m\rangle \langle\psi_m|\hat{A}|\psi_n\rangle\langle\psi_n|$$

Since $\langle\psi_m|\hat{A}|\psi_n\rangle = A_{m,n}$,

$$\hat{A} = \sum_m \sum_n A_{m,n} |\psi_m\rangle\langle\psi_n| \qquad \text{QED}$$

Problem 8.4:

a. $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

$$\hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A})$$

$$= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A}$$

$$\hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A}$$

b. We have

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p})$$

and want to find $\hat{a}_+^\dagger$:

$$(\hat{a}_+)^\dagger = \left(\frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p})\right)^\dagger$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p})$$

$$= \hat{a}_-$$

c. The criteria for a hermitian operator is $\hat{Q} = \hat{Q}^\dagger$. Since $\hat{a}_+ \neq \hat{a}_+^\dagger$, it is not hermitian.

Look at the operator $\hat{X} = \hat{a}_+ + \hat{a}_-$

$$\Rightarrow (\hat{X})^\dagger = (\hat{a}_+ + \hat{a}_-)^\dagger$$

By result in part b, $\hat{a}_+^\dagger = \hat{a}_-$, and it is clear to see that $\hat{a}_-^\dagger = \hat{a}_+$

$$\Rightarrow (\hat{X})^\dagger = (\hat{a}_+ + \hat{a}_-)^\dagger$$

$$= \hat{a}_+^\dagger + \hat{a}_-^\dagger$$

$$= \hat{a}_- + \hat{a}_+ = \hat{a}_+ + \hat{a}_-$$

$$= \hat{X}$$

$$= \hat{a}_+^{\dagger} + \hat{a}_-^{\dagger}$$
$$= \hat{a}_- + \hat{a}_+ = \hat{a}_+ + \hat{a}_-$$
$$= \hat{X}$$

$\therefore \hat{X}$ is hermitian by the aforementioned criteria.

Now consider $\hat{n} = a_+ a_-$

$$\Rightarrow \quad (\hat{n})^{\dagger} = (a_+ a_-)^{\dagger}$$
$$= (a_-)^{\dagger} (a_+)^{-}$$
$$= a_+ a_-$$
$$= \hat{n}$$

and so $\hat{n}$ is hermitian :)

Problem 8.5:

a. The Heisenberg equation is

$$\frac{d\hat{Q}(t)}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{Q}(t)]$$
$$= \frac{i}{\hbar}\hat{H}\hat{Q} - \frac{i}{\hbar}\hat{Q}\hat{H}$$

Since the Hamiltonian is
$$\hat{H} = \hbar\omega(\hat{a}_+\hat{a}_- + \tfrac{1}{2})$$
then,
$$\frac{d\hat{Q}(t)}{dt} = i\omega(\hat{a}_+\hat{a}_- + \tfrac{1}{2})\hat{Q} - i\omega\hat{Q}(\hat{a}_+\hat{a}_- + \tfrac{1}{2})$$
$$= i\omega\hat{a}_+\hat{a}\ \hat{Q} + \frac{i\omega}{2}\hat{Q} - i\omega\hat{Q}\hat{a}_+\hat{a}_- - \frac{i\omega}{2}\hat{Q}$$
$$= i\omega\hat{a}_+\hat{a}_-\hat{Q} - i\omega\hat{Q}\hat{a}_+\hat{a}_-$$

The equation of motion for a_+ is found by setting $\hat{Q} = \hat{a}_+$

$$\Rightarrow \frac{d}{dt}\hat{a}_+ = i\omega\hat{a}_+\hat{a}_-\hat{a}_+ - i\omega\hat{a}_+\hat{a}_+\hat{a}_-$$
$$= i\omega\hat{a}_+[\hat{a}_-, \hat{a}_+]$$
$$= i\omega\hat{a}_+ \qquad (\text{since } [\hat{a}_-, \hat{a}_+] = 1)$$

This has solution

$$\frac{d}{dt}\hat{a}_+ = i\omega\hat{a}_+(0) \Rightarrow \hat{a}_+(t) = \hat{a}_+(0)e^{i\omega t}$$

Similarly for $\hat{a}_-$, set $\hat{Q} = \hat{a}_-$

$$\Rightarrow \frac{d}{dt}\hat{a}_- = i\omega\hat{a}_+\hat{a}_-\hat{a}_- - i\omega\hat{a}_-\hat{a}_+\hat{a}_-$$
$$= i\omega[\hat{a}_+, \hat{a}_-]\hat{a}_-$$
$$= -i\omega[\hat{a}_-, \hat{a}_+]\hat{a}_-$$
$$= -i\omega\hat{a}_- \qquad (\text{since } [\hat{a}_-, \hat{a}_+] = 1)$$

Similarly to before, this has general solution

$$\hat{a}_-(t) = a_-(0)e^{-i\omega t}$$

In both cases, $t = 0 \Rightarrow e^{\pm i\omega t} = 1$

$$\Rightarrow \hat{a}_+(0) = \hat{a}_+(0) \text{ and } \hat{a}_-(0) = \hat{a}_-(0)$$

So this solution holds.

$\Rightarrow \quad \hat{a}_+(0) = \hat{a}_+(0)$ and $\hat{a}_-(0) = \hat{a}_-(0)$

So this solution holds.

b. $\hat{H}$ and $\hat{n}$ commute if $[\hat{H}, \hat{n}] = 0$

$$
\begin{aligned}
[\hat{H}, \hat{n}] &= \hat{H}\hat{n} - \hat{n}\hat{H} \\
&= \hbar\omega(\hat{a}_+\hat{a}_- + \tfrac{1}{2})\hat{a}_+\hat{a}_- - \hat{a}_+\hat{a}_- \hbar\omega(\hat{a}_+\hat{a}_- + \tfrac{1}{2}) \\
&= \hbar\omega\hat{a}_+\hat{a}_-\hat{a}_+\hat{a}_- + \frac{\hbar\omega}{2}a_+a_- - \hbar\omega\hat{a}_+\hat{a}_-\hat{a}_+\hat{a}_- - \frac{\hbar\omega}{2}a_+a_- \\
&= 0
\end{aligned}
$$

$\therefore$ $\hat{H}$ and $\hat{n}$ commute

The eigenvalue of $\hat{n}$ is then

$$
\begin{aligned}
\lambda|n\rangle &= \hat{n}|n\rangle \\
&= \hat{a}_+\hat{a}_-|n\rangle \\
&= \hat{a}_+\sqrt{n}\,|n-1\rangle \\
&= \sqrt{n}\,\hat{a}_+|n-1\rangle \\
&= \sqrt{n}\sqrt{n+1}\,|n\rangle \\
&= \sqrt{n^2+n}\,|n\rangle
\end{aligned}
$$

$$\Rightarrow \lambda = \sqrt{n^2+n}$$

The energy eigenstates are

$$
\begin{aligned}
\hat{H}|n\rangle &= \hbar\omega(a_+a_- + \tfrac{1}{2})|n\rangle \\
&= \hbar\omega(\hat{n} + \tfrac{1}{2})|n\rangle \\
&= \hbar\omega\hat{n}|n\rangle + \frac{\hbar\omega}{2}|n\rangle
\end{aligned}
$$

so the eigenstates of $\hat{H}$ are scaled eigenstates of $\hat{n}$