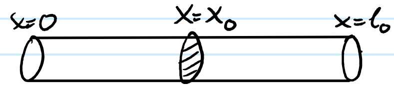
Assignment 4

Saturday, 23 September 2023 11:08 PM

Question 1:

a.



$t=0$

b. At the interface $x=x_0$, we have the displacement of the left side material equaling that of the right side material.
That is

$$U(x_0,t) = U_L(x_0,t) = U_R(x_0,t)$$

Similarly, the stress must also be equal between the sides at the interface. Since this is a linearly elastic material, this means

$$T = E\,\frac{\partial U}{\partial x}$$

$$\Rightarrow T = T_L = T_R$$

$$= E_L \frac{\partial U_L(x_0,t)}{\partial x} = E_R \frac{\partial U_R(x_0,t)}{E_L}$$

c. We take that the entire bar is stretched to length l at time t. The left end is held fixed: $U(0,t)=0$, and there are no body forces, i.e. $F=0$.
We aim to find a steady state solution, i.e. $\partial_t U = \partial_{tt} U = 0$
The momentum equation then reads

$$0 = \partial_x T = E_L \partial_{xx} U_L = E_R \partial_{xx} U_R$$

$$= \partial_{xx} U_L = \partial_{xx} U_R$$

we then have a piecewise system for the material sections,

$$\begin{cases} \begin{cases} 0 = \partial_{xx} U_L \\ U_L(0,t) = 0 \\ U_L(x_0,t) = U_R(x_0,t) \end{cases} & \text{if } 0 \le x \le x_0 \\ \\ \begin{cases} 0 = \partial_{xx} U_R \\ U_R(l_0,t) = l - l_0 \\ U_R(x_0,t) = U_L(x_0,t) \end{cases} & \text{if } x_0 < x \le l_0 \end{cases}$$

looking at the first material, we see that

$$0 = \partial_{xx} U_L \Rightarrow U_L(x,t) = \alpha x + \beta$$

then,

$$U_L(0,t) = 0 \Rightarrow \beta = 0$$

so, $U_L(x,t) = \alpha x$

Now, for the second system,

$$0 = \partial_{xx} U_R \Rightarrow U_R(x,t) = \sigma X + \gamma$$

then,

$$U_R(l_0,t) = l - l_0 \Rightarrow l - l_0 = \sigma l_0 + \gamma$$

$$\gamma = l - l_0(1+\sigma)$$

$$\Rightarrow U_R(x,t) = \sigma X + l - l_0(1+\sigma)$$

We have the interface boundary, requiring

$$U_L(x_0,t) = U_R(x_0,t)$$

$$\Rightarrow \alpha X_0 = \sigma X_0 + l - l_0(1+\sigma)$$

$$\Rightarrow \alpha = \sigma + \frac{l - l_0(1+\sigma)}{x_0} = \sigma\left(1 - \frac{l_0}{x_0}\right) + \frac{l-l_0}{x_0}$$

and so $U_L(X,t) = \left(\sigma\left(1-\frac{l_0}{x_0}\right) + \frac{l-l_0}{x_0}\right)X$

We can determine the final parameter using the fact that

$$T_L = T_R$$

$$\Rightarrow E_L \partial_x U_L = E_R \partial_x U_R$$

$$\Rightarrow E_L\left(\sigma\left(1-\frac{l_0}{x_0}\right) + \frac{l-l_0}{x_0}\right) = E_R \sigma$$

and so

$$\sigma\left(E_R - E_L\left(1 - \frac{l_0}{x_0}\right)\right) = E_L\left(\frac{l-l_0}{x_0}\right)$$

$$\Rightarrow \sigma = \frac{E_L}{E_R - E_L\left(1-\frac{l_0}{x_0}\right)}\left(\frac{l-l_0}{x_0}\right)$$

and so, finally, for $0 \le X \le X_0$,

$$U_L(x,t) = \left(\frac{E_L}{E_R - E_L\left(1-\frac{l_0}{x_0}\right)}\left(\frac{l-l_0}{x_0}\right)\left(1-\frac{l_0}{x_0}\right) + \frac{l-l_0}{x_0}\right)X$$

$$\Rightarrow T_L(x,t) = E_L \partial_x U_L$$

$$= E_L\left(\frac{E_L}{E_R - E_L\left(1-\frac{l_0}{x_0}\right)}\left(\frac{l-l_0}{x_0}\right)\left(1-\frac{l_0}{x_0}\right) + \frac{l-l_0}{x_0}\right)$$

$$\Rightarrow R_L(x,t) = \frac{R_L}{1 + \partial_x U_L}$$

$$= \frac{R_L}{1 + \left(\frac{E_L}{E_R - E_L\left(1-\frac{l_0}{x_0}\right)}\left(\frac{l-l_0}{x_0}\right)\left(1-\frac{l_0}{x_0}\right) + \frac{l-l_0}{x_0}\right)}$$

and for $X_0 < X \le l_0$, we have

and for $X_0 < X \le l_0$, we have

$$U_R(X,t) = \sigma X + l - l_0(1+\sigma)$$

$$= \frac{E_L}{E_R - E_L\left(1-\frac{l_0}{X_0}\right)}\left(\frac{l-l_0}{X_0}\right)X + l - l_0\left(1 + \frac{E_L}{E_R - E_L\left(1-\frac{l_0}{X_0}\right)}\left(\frac{l-l_0}{X_0}\right)\right)$$

$$\Rightarrow T_R(X,t) = E_R\, \partial_X U_R$$

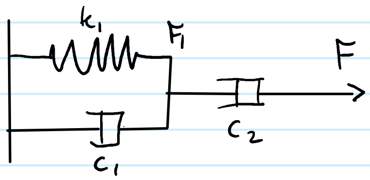
$$= \frac{E_L E_R}{E_R - E_L\left(1-\frac{l_0}{X_0}\right)}\left(\frac{l-l_0}{X_0}\right)$$

and finally

$$R_R(X,t) = \frac{R_R}{1 + \partial_X U_R}$$

$$= \frac{R_R}{1 + \frac{E_L}{E_R - E_L\left(1-\frac{l_0}{X_0}\right)}\left(\frac{l-l_0}{X_0}\right)}$$

Question 2:



a. We have a Kelvin-Voigt element and dashpot in series.

From the Kelvin-Voigt contribution, we have

$$F_1 = k_1 u_1 + c_1 \dot{u}_1 \Rightarrow u_1 = \frac{F_1 - c_1\dot{u}_1}{k_1} \Rightarrow \dot{u}_1 = \frac{\dot{F}_1 - c_1\ddot{u}_1}{k_1}$$

From the extra dashpot, we have $F_2 = c_2 \dot{u}_2 \Rightarrow \dot{u}_2 = \frac{F_2}{c_2} \Rightarrow \ddot{u}_2 = \frac{\dot{F}_2}{c_2}$

Due to being in series, the forces are equal:

$$F = F_1 = F_2$$

and the displacements add up

$$u = u_1 + u_2 \Rightarrow \dot{u} = \dot{u}_1 + \dot{u}_2 \quad (\Rightarrow \ddot{u} = \ddot{u}_1 + \ddot{u}_2)$$

$$= \frac{\dot{F}_1 - c_1\ddot{u}_1}{k_1} + \frac{F_2}{c_2}$$

$$= \frac{\dot{F} - c_1(\ddot{u} - \ddot{u}_2)}{k_1} + \frac{F}{c_2}$$

$$= \frac{\dot{F} - c_1(\ddot{u} - \dot{F}/c_2)}{k_1} + \frac{F}{c_2}$$

$$\Rightarrow F + \frac{c_2}{k_1}\left(\dot{F} - c_1\ddot{u} + \frac{c_1}{c_2}\dot{F}\right) = c_2 \dot{u}$$

$$\Rightarrow \quad F + \frac{c_2}{k_1}\left(\dot{F} - c_1\ddot{u} + \frac{c_1}{c_2}\dot{F}\right) = c_2\dot{u}$$

$$\Rightarrow \quad F + \frac{c_1+c_2}{k_1}\dot{F} = c_2\dot{u} + \frac{c_1c_2}{k_1}\ddot{u} \qquad ①$$

as required.

b. We know that $F \sim T$ and $u \sim \varepsilon$, and so ① becomes

$$T + \frac{c_1+c_2}{k_1}\frac{\partial T}{\partial t} = c_2\frac{\partial \varepsilon}{\partial t} + \frac{c_1c_2}{k_1}\frac{\partial^2 \varepsilon}{\partial t^2}$$

Now make the substitutions

$$\tau_0 = \frac{c_1+c_2}{k_1}, \quad \tau_1 = c_2 \quad \text{and} \quad \tau_2 = \frac{c_1c_2}{k_1}$$

since c_1, c_2, and k_1 are all >0, we have that

$$\tau_0\tau_1 = c_2\left(\frac{c_1+c_2}{k_1}\right) = \frac{c_1c_2+c_2^2}{k_1} = \tau_2 + \frac{c_2^2}{k_1} > \tau_2.$$

and so

$$T + \tau_0\frac{\partial T}{\partial t} = \tau_1\frac{\partial \varepsilon}{\partial t} + \tau_2\frac{\partial^2 \varepsilon}{\partial t}$$

c. Take the Laplace transform of both sides of ② to get

$$\mathcal{L}(T) + \tau_0\mathcal{L}\left(\frac{\partial T}{\partial t}\right) = \tau_1\mathcal{L}\left(\frac{\partial \varepsilon}{\partial t}\right) + \tau_2\mathcal{L}\left(\frac{\partial^2 \varepsilon}{\partial t^2}\right)$$

$$\mathcal{L}(T) + \tau_0\left(s\mathcal{L}(T) - T(0)\right) = \tau_1\mathcal{L}(\varepsilon') + \tau_2\left(s\mathcal{L}(\varepsilon') - \varepsilon'(0)\right)$$

We assume $T(0)=0$ and $\varepsilon'(0)=0$, so

$$\mathcal{L}(T) + \tau_0 s\mathcal{L}(T) = \tau_1\mathcal{L}(\varepsilon') + \tau_2 s\mathcal{L}(\varepsilon')$$

$$\mathcal{L}(T)(1+\tau_0 s) = (\tau_1 + \tau_2 s)\mathcal{L}(\varepsilon')$$

$$\Rightarrow \quad \mathcal{L}(T) = \frac{\tau_1+\tau_2 s}{1+\tau_0 s}\mathcal{L}(\varepsilon')$$

$$= \frac{\tau_1+\tau_2 s}{s(1+\tau_0 s)}s\mathcal{L}(\varepsilon')$$

We have $\mathcal{L}^{-1}\left(\frac{\tau_1+\tau_2 s}{s(1+\tau_0 s)}s\mathcal{L}(\varepsilon')\right)$

$$= \int_0^t \mathcal{L}^{-1}\left(\frac{\tau_1+\tau_2 s}{s(1+\tau_0 s)}\right)\mathcal{L}^{-1}\left(s\mathcal{L}(\varepsilon')\right)ds$$

with $\mathcal{L}^{-1}(s\mathcal{L}(\varepsilon')) = \varepsilon' - \varepsilon(0) = \varepsilon'$ (assuming $\varepsilon=0$ at $t=0$), and

$$\mathcal{L}^{-1}\left(\frac{\tau_1+\tau_2 s}{s(1+\tau_0 s)}\right) = \mathcal{L}^{-1}\left(\frac{\tau_1+\tau_2 s}{s(1/\tau_0+s)}\right)$$

$$= \tau_0\left[\left(\tau_2/\tau_0 - \tau_1\right)e^{-t/\tau_0} + \tau_1\right]$$

so,

$$T = \int_0^t \tau_0\left[\left(\tau_2/\tau_0 - \tau_1\right)e^{-(t-s)/\tau_0} + \tau_1\right]\varepsilon'\,ds$$

so,

$$T = \int_0^t \tau_0\left[(\tau_2/\tau_0 - \tau_1)e^{-(t-s)/\tau_0} + \tau_1\right]\varepsilon'\,ds$$

We can set

$$K(t) = \tau_0\left[(\tau_2/\tau_0 - \tau_1)e^{-t/\tau_0} + \tau_1\right]$$

$$\Rightarrow\ K(0) = \tau_2 \ = K_1$$

$$\Rightarrow\ G(t) = \frac{\partial K(t)}{\partial t} = \left(\tau_1 - \tau_2/\tau_0\right)e^{-t/\tau_0}$$

$$= K_2 e^{-t/\lambda}$$

with $K_2 = (\tau_1 - \tau_2/\tau_0)$ and $\lambda = \tau_0$

and so, with integration by parts, we get

$$T = K_1\varepsilon' + \int_0^t G(t-s)\varepsilon'(s)\,ds$$

We have $K_1 = \tau_2 > 0$, and $\lambda = \tau_0 > 0$.

We also have $\tau_0 K_2 = \tau_0\tau_1 - \tau_2$, but $\tau_0\tau_1 > \tau_2$ and $\tau_0 > 0 \Rightarrow \tau_0 K_2 > 0 \Rightarrow K_2 > 0/\tau_0 = 0$

$$\Rightarrow K_2 > 0$$