Question 1:

Part A:

Let $x \in \mathbb{R}$, prove that $x > 0$ implies $-x < 0$, and vice versa if $x < 0$ then $-x > 0$.

P/ Let $x \in \mathbb{R}$, and $x > 0$. By (A_4), $x + (-x) = 0$

Let $a = x$, $b = 0$, and $c = -x$. By (O_3), if $a > b$, then $a + c > b + c$ (*)

Substituting values for a, b, and c gives

$$x + (-x) > 0 + (-x)$$
$$\Rightarrow \quad -x < 0$$

$\therefore$ if $x > 0$, $-x < 0$

If $x < 0$, let $a = 0$, $b = x$ and $c = -x$ in (*). This satisfies $a > b$.

Substituting in values for a, b, and c gives

$$0 + (-x) > x + (-x)$$
$$\Rightarrow -x > 0$$

$\therefore$ if $x < 0$, $-x > 0$ QED

Part B:

Let $x \in \mathbb{R}$. Then, $x^2 \geq 0$.

P/ 3 Possible cases: $x > 0$, $x = 0$, or $x < 0$.

Case 1: $x > 0$

$$x \cdot x > 0 \qquad (O_4)$$
$$x^2 > 0$$

$\therefore$ if $x > 0$, $x^2 > 0$

Case 2: $x = 0$

By (M_2), $x \cdot x = x \cdot x$

$\Rightarrow x^2 = 0 \cdot 0$

$x^2 = 0$

$\therefore$ if $x = 0$, $x^2 = 0$.

Case 3: $x < 0$

Claim: if $x < 0$ and $y < z \ \forall y, z \in \mathbb{R}$, then $xy > xz$.

P/ if $x < 0$, $-x > 0$ by proof in Part A.

By (A_4) and (O_3), $0 = y + (-y) < z - y$

$\Rightarrow -x \cdot (z - y) > 0$ (O_4)

$xy - xz > 0$ (D)

$xy > xz$ (A_3)(A_4)($)

$\therefore$ if $x < 0$ and $y < z$, $xy > xz$ QED

Let $y = x$ and $z = 0$ in ($)

Since $y = x < 0$, this satisfies $y < z$

$\Rightarrow x \cdot x > x \cdot 0$

$x^2 > 0$

$\therefore$ if $x < 0$, $x^2 > 0$

Thus, it has been shown that if $x \in \mathbb{R}$, then $x^2 \geq 0$ QED

Question 2:

Part A:

$$S = \left\{\frac{n-1}{n+1} | n \in \{1, 2, 3, 4, ...\}\right\};$$

Claim: S is bounded above, with $sup(S) = 1$

P/ Firstly, a set is bounded above if $\exists b \in \mathbb{R}$ such that $a \leq b \quad \forall a \in A$
Set S can be seen to be strictly monotone increasing, as each term is larger than the last. As n gets larger, the term for S_n approaches 1.
Therefore, take $b = 1$, which satisfies $b \in \mathbb{R}$. Since $n + 1$ is always larger than $n - 1$, S_n will always be smaller than 1, as a large denominator dividing a small numerator is always less than 1. Thus, $b = 1$ satisfies $a \leq 1 \ \forall a \in S$, and 1 is an upper bound.

Now, a number s is the supremum of a set A, if for any $\epsilon > 0$, $s - \epsilon < a$, where $a \in A$. As was established, S is monotone increasing, and approaches, but does not reach, 1 as n increases. Thus, for $a \in S$, $a < 1$. That means that $s - \epsilon < a < 1$. Since ϵ was defined as being greater than 0, and s was claimed to equal 1, the statement $s - \epsilon < 1$ holds, and $s = sup(S) = 1$. QED

Claim: S is bounded below, with $inf(S) = 0$

P/ Firstly, a set is bounded below if $\exists b \in \mathbb{R}$ such that $b \leq a \ \forall a \in A$.
S_1 is the smallest term in the set, as it is the first term and it was determined that S is strictly monotone increasing. It was calculated that $S_1 = 0$, and we can take $b = 0 \leq a = 0$ for $a \in S$. Thus, 0 is a lower bound for S.

A number w is the infimum of a set A if $\forall \epsilon > 0$, $w + \epsilon > a$, where $a \in A$.
Take $w = 0$ and $a = 0 = S_1 \in S$. Thus, $0 + \epsilon > 0$, and $\epsilon > 0$ which is true by definition of ϵ. Therefore, $w = inf(S) = 0$. QED

Therefore, it has been shown that S is bounded above and below, with $sup(S) = 1$ and $inf(S) = 0$.

Part B:

$$T = \bigcup_{n=1}^{\infty} [n^2, n^2 + 1]$$

Claim: T is bounded below, with $inf(T) = 1$

P/ It is clear that T is strictly monotone increasing, so T_1 will be the smallest term in the set. From the definition in the previous part, we can take $b = T_1 = 1 \leq a = 1$ for $a \in T$. Therefore, T is bounded below, with 1 as a possible lower bound.

From the epsilon definition of infimum in Part A, we can take $w = 1$, and $a = 1 = T_1 \in T$. For any $\epsilon > 0$, we have $w + \epsilon > a$. Substituting in values gives, $1 + \epsilon > 1$, which is true by definition of ϵ. So, $w = inf(T) = 1$. QED

Claim: T is not bounded above, and so T has no supremum.

P/ Will prove with a contradiction.
Suppose T is bounded above by some least upper bound $b \in \mathbb{R}$. That is, $\forall a \in T, a \leq b$. Suppose $a = T_n = n^2 + 1 \leq b$. Thus, $n^2 \leq b - 1$, and $b - 1$ is an upper bound for T. However, b was defined as being the least upper bound for T, which would mean that $b \leq b - 1$, so a contradiction is found and T has no least upper bound (and consequently no upper bound). QED

Therefore, it has been shown that T is bounded only below, with $inf(T) = 1$.

Question 3:

Using the $\epsilon - N$ definition of a limit, prove that

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0$$

P/ Let $\epsilon > 0$. It can be seen that

$$0 \leq \left|\frac{\sin n}{n}\right| \leq \left|\frac{1}{n}\right|$$

By the Archimedean Property, $\exists N \in \mathbb{N}$ such that $N \cdot \epsilon > 1$. So, $\epsilon > \frac{1}{N} \geq \frac{1}{n} \ \forall n \geq N$

Relating the above equations shows that $\epsilon > \left|\frac{1}{n}\right| \geq \left|\frac{\sin n}{n}\right| = \left|\frac{\sin n}{n} - 0\right|$ (since $\left|\frac{1}{n}\right| = \frac{1}{n}$ as n is always positive). i.e. $\left|\frac{\sin n}{n} - 0\right| < \epsilon$, which is of the form of the definition of a limit. Thus, it has been shown that $L = 0$, and

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0$$

QED

Question 4:

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences such that $\{y_n\}_{n=1}^{\infty}$ converges to 0. Suppose that for all positive integers k and m with $m \geq k$, we have

$$|x_m - x_k| \leq y_k$$

Prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

P/ Since $\{y_n\}_{n=1}^{\infty}$ is convergent to 0, we have that $\exists N \in \mathbb{N}$ such that $|y_k - 0| < \epsilon$ for any $\epsilon > 0$, or $|y_k| < \epsilon \ \ \forall k > N$. It is also clear that $y_k \leq |y_k|$, so we have that $|x_m - x_k| \leq y_k \leq |y_k| < \epsilon \ \ \forall m \geq k > N$, or more simply, $|x_m - x_k| < \epsilon$. This final equation is in the form of the definition of a Cauchy sequence, and thus it has been shown that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy Sequence.

QED