MATH3070 Natural Resource Mathematics

Lecture Notes: Quantifying Parameter Uncertainty

1 Posterior Simulation

Let's discuss a few ways to compute the posterior distribution for a parameter. Recall that we'd like to compute $p(\theta|\mathbf{x})$ (for a discrete parameter) or $f(\theta|\mathbf{x})$, for a continuous parameter. We showed that sometimes you can calculate this distribution analytically for given specific conjugate priors for θ, but usually this is not possible.

Your first option for computing $p(\theta|\mathbf{x})$, if you do not have an analytic expression, is to actually compute the proportionality constant numerically, using **numerical integration** (Riemann sums back from calculus). This works pretty well when there is only a single parameter that takes on values on a closed interval, but quickly becomes computationally intractable in several dimensions.

1.1 Rejection Sampling

Even if you have an analytic formula for the probability distribution of interest, this does not tell you how to draw a random sample of parameter values from the distribution. One way to do this is rejection sampling.

In such a case you need a positive function $g(\theta)$ defined for all θ where $p(\theta|x) > 0$, with g having a finite integral, and such that there is a known constant $M > 0$ for which $p(\theta|x)/g(\theta) \leq M$ for all θ. We also need to be able to draw random samples from a probability density proportional to g. If one can write down such a function g, to generate random draws from the target distribution $p(\theta|x) \leq M$,

1. Sample θ randomly from the probability density proportional to g,
2. With probability $p(\theta|x)/[Mg(\theta)]$ accept your draw of θ, as a draw from p. If the draw is rejected return to step 1,
3. repeat (return to step 1).

The above algorithm was written with having an analytic expression for $p(\theta|x)$, but this is not required. Simply having an analytic expression for a distribution proportional to $p(\theta|x)$ suffices. You only need to replace that distribution proportional to $p(\theta|x)$ in all the spots where you see $p(\theta|x)$, and you will achieve draws from the desired normalised distribution. Choosing g can sometimes be tricky, usually we choose a distribution that is easily simulated on the computer (e.g. uniform distribution, for which numbers can be drawn using a pseudo random number generator).

While easy to do in one dimension, it can be difficult to choose a g in multiple dimensions such that sampling is efficient.

2 Markov Chain Monte Carlo

The best solution for multiple dimensions (e.g. a set of parameters, such as r, k, and q) is Markov Chain Monte Carlo Algorithms. These algorithms include the Metropolis algorithm, Metropolis-Hastings algorithm, and the Gibbs sampling algorithm.

We will focus our attention on the Metropolis algorithm because it is the simplest. The idea is to sample parameter space, as a random walk, with an acceptance/rejection rule that allows for the stationary distribution of the random walk to converge to the target posterior distribution for the parameter. The algorithm goes as follows:

1. Draw starting point for the random walk, θ^0, for which $p(\theta^0|x) > 0$ randomly from the probability density proportional to g
2. For $t = 1, 2, ...$ sample a proposed value for θ^* from a proposal distribution (also called a jumping distribution) $J_t(\theta^*|\theta^{t-1})$, where J is symmetric, meaning $J_t(a|b) = J_t(b|a)$
3. Calculate the ratio of densities,
$$r = \frac{p(\theta^*|x)}{p(\theta^{t-1}|x)}. \tag{1}$$
4. Set $\theta^t = \theta^*$ with probability $\min(r, 1)$, $\theta^t = \theta^{t-1}$ otherwise.

A symmetric choice for J_t is often in practice a normal distribution, with mean equal to θ^{t-1}.

3 Optimal Decisions Under Parameter Uncertainty

One type of problem we are interested in is maximising the expected yield given some harvest restriction. You have already studied this problem in a deterministic setting. But what happens when the parameters are uncertain. The best harvest decision given one set of parameter values will not be the best quota given other values for the parameters. One might naively compute the best fitting parameter values to the data, and use those parameter values to determine the quota. But what if there is a slightly less likely set of parameter values for which that quota rule would cause the population to go extinct. For example consider one parameter set θ_0 and other parameter set θ_1, $p(\theta_0|x) = 0.75$ and $p(\theta_1|x) = 0.25$, if the parameter values are equal to θ_0 or θ_1, then it is optimal to choose quota Q_0 and Q_1, respectively. Assume the yields are $Y(Q_0, \theta_0) = 10$ and $Y(Q_1, \theta_0) = 8$. If $= \theta_1$ the yields under the two quotas are $Y(Q_0, \theta_1) = 0$ and $Y(Q_1, \theta_1) = 9$. Which strategy is optimal?

In general, if we have a full distribution describing how likely different parameter values are, such as the ones we generated using Bayesian Statistics in the previous lecture, and a formula for the objective function given the parameter values and management decision, we can calculate the expected value of the objective under any management strategy. See lecture for coding and abstract formulation.