THE UNIVERSITY OF QUEENSLAND
SCHOOL OF MATHEMATICS AND PHYSICS
PHYS2041 – Quantum Physics

Tutorial 6 Solutions

Problem 6.1

Here we will use the wavenumber $k = 2\pi/\lambda$, which is related to the momentum p according to the de-Broglie relation $p = \hbar k$. The momentum-space wavefunction is given by the Fourier transform of the position-space wavefunction,

$$\begin{aligned}
\phi_0(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0(x) e^{-ikx} \, dx && (1)\\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\pi^{1/4} l_{\text{ho}}^{1/2}} \int_{-\infty}^{\infty} e^{-x^2/2l_{\text{ho}}^2} e^{-ikx} \, dx && (2)\\
&= \frac{1}{\pi^{3/4}\sqrt{2 l_{\text{ho}}}} \cdot \sqrt{2} l_{\text{ho}} \sqrt{\pi} e^{-\frac{2}{4}k^2 l_{\text{ho}}^2} \quad \text{from supplied integral(8)} && (3)\\
&= \frac{\sqrt{l_{\text{ho}}}}{\pi^{1/4}} e^{-\frac{1}{2}k^2 l_{\text{ho}}^2} && (4)
\end{aligned}$$

The momentum-space wavefunction is

$$\phi_0(k) = \frac{\sqrt{l_{\text{ho}}}}{\pi^{1/4}} e^{-\frac{l_{\text{ho}}^2}{2}k^2}. \tag{5}$$

The probability distribution is

$$P(k) = |\phi(k)|^2 = \frac{l_{\text{ho}}}{\sqrt{\pi}} e^{-k^2 l_{\text{ho}}^2}. \tag{6}$$

Notice that the width of the position-space distribution is l_{ho}, whereas the width of the momentum distribution is $1/l_{\text{ho}}$, i.e. a large uncertainty in position results in a small uncertainty in momentum, and visa versa. This is an example of the Heisenberg uncertainty principle!

Problem 6.2 [FOR ASSIGNMENT 3; max 10 points]

(a) The initial state at $t = 0$ is prepared the ground state of a Harmonic potential with frequency ω, but with a momentum kick $p_0 = \hbar k_0$. However the potential is $V(x) = 0$ everywhere during the evolution at $t > 0$, corresponding to a free particle. Since the particle is free, the energy eigenstates are plane waves,

$$\psi_k(x) = e^{ikx} \tag{7}$$

with allowed energies

$$E_k = \frac{\hbar^2 k^2}{2m} \tag{8}$$

which have *continuous* eigenvalues with index k which can be any real number. For discrete eigenstates, we calculate time evolution using $\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar}$. To generalise this to continuous systems, we replace the n with k and the sum with an integral over k, so for a free particle the time evolved state is

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} e^{-i\frac{\hbar k^2}{2m}t} dk \tag{9}$$

where $\phi(k)$ is the momentum wavefunction, which is the Fourier transform of the initial state. For

$$\Psi(x,0) = \frac{1}{\pi^{1/4}\sqrt{l_{\text{ho}}}} e^{x^2/2l_{\text{ho}}^2} e^{ik_0 x} \tag{10}$$

with harmonic oscillator length $l_{\text{ho}} = \sqrt{\hbar/m\omega}$. The momentum wavefunction can be evaluated using the supplied integral (5),

$$\begin{aligned}
\phi(k) &= \frac{1}{\sqrt{2\pi}} \frac{1}{\pi^{1/4}\sqrt{l_{\text{ho}}}} \int_{-\infty}^{\infty} e^{x^2/2l_{\text{ho}}^2} e^{ik_0 x} e^{-ikx} dx & (11)\\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\pi^{1/4}\sqrt{l_{\text{ho}}}} \int_{-\infty}^{\infty} e^{x^2/2l_{\text{ho}}^2} e^{-i(k-k_0)x} dx & (12)\\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\pi^{1/4}\sqrt{l_{\text{ho}}}} \sqrt{2} l_{\text{ho}} \sqrt{\pi} e^{-\frac{1}{4}(k-k_0)^2 2 l_{\text{ho}}^2} & (13)\\
&= \frac{\sqrt{l_{\text{ho}}}}{\pi^{1/4}} e^{-(k-k_0)^2 l_{\text{ho}}^2/2} & (14)
\end{aligned}$$

Momentum is conserved (the momentum operator commutes with the free particle Hamiltonian), which means $|\phi(k,t)|^2 = |\phi(k,0)|^2$, i.e. the momentum probability density does not evolve. However the position space wavefunction *does* evolve in time. Substituting $\phi(k)$ into the integral for $\Psi(x,t)$ [Eq. (9)] gives

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{l_{\text{ho}}}}{\pi^{1/4}} \int_{-\infty}^{\infty} e^{-(k-k_0)^2 l_{\text{ho}}^2/2} e^{ikx} e^{-i\frac{\hbar k^2}{2m}t} dk & (15)\\
&= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{l_{\text{ho}}}}{\pi^{1/4}} \int_{-\infty}^{\infty} e^{-k^2 l_{\text{ho}}^2/2} e^{kk_0 l_{\text{ho}}^2} e^{-k_0^2 l_{\text{ho}}^2/2} e^{ikx} e^{-i\frac{\hbar k^2}{2m}t} dk & (16)\\
&= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{l_{\text{ho}}}}{\pi^{1/4}} e^{-k_0^2 l_{\text{ho}}^2/2} \int_{-\infty}^{\infty} e^{-ik\left(ik_0 l_{\text{ho}}^2 - x\right)} e^{-k^2\left(l_{\text{ho}}^2/2 + i\frac{\hbar}{2m}t\right)} dk & (17)\\
& & (18)
\end{aligned}$$

which again using the supplied integral (6) evaluates to (also introducing $\theta(t) = \hbar t/m l_{\text{ho}}^2$)

$$\Psi(x,t) = \frac{1}{\pi^{1/4}\sqrt{l_{\text{ho}}}\sqrt{1+i\theta(t)}} \exp\left[-\frac{k_0^2 l_{\text{ho}}^2}{2} - \frac{\left(ik_0 l_{\text{ho}}^2 - x\right)^2}{2l_{\text{ho}}^2\left[1+i\theta(t)\right]}\right] \tag{19}$$

$$= \frac{1}{\pi^{1/4}\sqrt{l_{\text{ho}}}\sqrt{1+i\theta(t)}} \exp\left[-\left(\frac{k_0^2 l_{\text{ho}}^4\left[1+i\theta(t)\right] + \left(ik_0 l_{\text{ho}}^2 - x\right)^2}{2l_{\text{ho}}^2\left[1+i\theta(t)\right]}\right)\right] \tag{20}$$

$$= \frac{1}{\pi^{1/4}\sqrt{l_{\text{ho}}}\sqrt{1+i\theta(t)}} \exp\left[-\left(\frac{\cancel{k_0^2 l_{\text{ho}}^4} + i\theta(t)k_0^2 l_{\text{ho}}^4 - \cancel{k_0^2 l_{\text{ho}}^4} - 2ik_0 l_{\text{ho}}^2 x + x^2}{2l_{\text{ho}}^2\left[1+i\theta(t)\right]}\right)\right] \tag{21}$$

$$= \frac{1}{\pi^{1/4}\sqrt{l_{\text{ho}}}\sqrt{1+i\theta(t)}} \exp\left[-\left(\frac{\frac{1}{2}x^2 - iB(t)}{l_{\text{ho}}^2\left[1+i\theta(t)\right]}\right)\right], \tag{22}$$

where in the last line we have defined the real, time-dependent variable

$$B(t) = k_0^2 l_{\text{ho}}^4 \left(\frac{x}{k_0 l_{\text{ho}}^2} - \frac{\theta(t)}{2}\right). \tag{23}$$

Note that the first line [Eq. (19)] is a perfectly acceptable answer, however, the final line will be easier to work with when calculating $|\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t)$.

(b) The probability density is

$$|\Psi(x,t)|^2 = \frac{\exp\left[-\left(\frac{\frac{1}{2}x^2 - iB(t)}{l_{\text{ho}}^2[1+i\theta(t)]} + \frac{\frac{1}{2}x^2 + iB(t)}{l_{\text{ho}}^2[1-i\theta(t)]}\right)\right]}{\sqrt{\pi}\, l_{\text{ho}}\sqrt{[1+i\theta(t)][1-i\theta(t)]}}. \tag{24}$$

We can immediately tidy up the denominator $l_{\text{ho}}\sqrt{[1+i\theta(t)][1-i\theta(t)]} = l_{\text{ho}}\sqrt{1+\theta(t)^2} \equiv l_{\text{ho}}(t)$. The numerator will be a little more work,

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{\pi}\, l_{\text{ho}}(t)} \exp\left[-\left(\frac{\left[\frac{1}{2}x^2 - iB(t)\right]\left[1-i\theta(t)\right] + \left[\frac{1}{2}x^2 + iB(t)\right]\left[1+i\theta(t)\right]}{l_{\text{ho}}^2\left[1+\theta(t)^2\right]}\right)\right] \tag{25}$$

$$= \frac{1}{\sqrt{\pi}\, l_{\text{ho}}(t)} \exp\left[-\left(\frac{\frac{1}{2}x^2 - \cancel{\frac{1}{2}i\theta(t)x^2} - \cancel{iB(t)} - B(t)\theta(t) + \frac{1}{2}x^2 + \cancel{\frac{1}{2}i\theta(t)x^2} + \cancel{iB(t)} - B(t)\theta(t)}{l_{\text{ho}}^2\left[1+\theta(t)^2\right]}\right)\right] \tag{26}$$

$$= \frac{1}{\sqrt{\pi}\, l_{\text{ho}}(t)} \exp\left[-\left(\frac{x^2 - 2B(t)\theta(t)}{l_{\text{ho}}^2\left[1+\theta(t)^2\right]}\right)\right]. \tag{27}$$

Now we write $l_{\rm ho}(t)^2 = l_{\rm ho}^2\left[1+\theta(t)^2\right]$, and reinsert $B(t)$

$$|\Psi(x,t)|^2 = \frac{1}{\sqrt{\pi}l_{\rm ho}(t)}\exp\left[-\left(\frac{x^2 - 2\theta(t)k_0^2 l_{\rm ho}^4\left(\frac{x}{k_0 l_{\rm ho}^2} - \frac{\theta(t)}{2}\right)}{l_{\rm ho}(t)^2}\right)\right] \tag{28}$$

$$= \frac{1}{\sqrt{\pi}l_{\rm ho}(t)}\exp\left[-\left(\frac{x^2 - 2\theta(t)k_0 l_{\rm ho}^2 x + \theta(t)^2 k_0^2 l_{\rm ho}^4}{l_{\rm ho}(t)^2}\right)\right] \tag{29}$$

$$= \frac{1}{\sqrt{\pi}l_{\rm ho}(t)}\exp\left[-\left(\frac{x - \theta(t)k_0 l_{\rm ho}^2}{l_{\rm ho}(t)}\right)^2\right] \tag{30}$$

$$= \frac{1}{\sqrt{\pi}l_{\rm ho}(t)}\exp\left[-\left(\frac{x - x_0(t)}{l_{\rm ho}(t)}\right)^2\right], \tag{31}$$

$$\tag{32}$$

where

$$x_0(t) = \theta(t)k_0 l_{\rm ho}^2 = \hbar k_0 t/m. \tag{33}$$

Remember that a generic Gaussian distribution with mean x_0 and standard deviation σ is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right]. \tag{34}$$

Comparing this to $|\Psi(x,t)|^2$ reveals that the probability density remains Gaussian at all times, but with a mean $x_0(t) = v_0 t$ that moves to with constant velocity $v_0 = \hbar k_0/m$ (drift), and standard deviation $\sigma = l_{\rm ho}(t)/\sqrt{2}$ that increases with time (diffusion).

(c) We have already identified that

$$\langle\hat{x}(t)\rangle = x_0(t) = \frac{\hbar k_0}{m}t. \tag{35}$$

Of course this agrees with the integral,

$$\langle x(t)\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx \tag{36}$$

$$= \frac{1}{\sqrt{\pi}l_{\rm ho}(t)}\int_{-\infty}^{\infty} x\exp\left[-\left(\frac{x-x_0(t)}{l_{\rm ho}(t)}\right)^2\right]dx \tag{37}$$

which we could evaluate using the supplied integrals. The expectation value of momentum is

$$\langle\hat{p}(t)\rangle = m\frac{d\langle\hat{x}(t)\rangle}{dt} = \hbar k_0. \tag{38}$$

Comment: You didn't have to do this, but the position and momentum uncertainties are

$$\sigma_x(t)^2 = \frac{l_{\rm ho}(t)^2}{2} \tag{39}$$

$$\sigma_p(t)^2 = \frac{\hbar^2}{2l_{\rm ho}(0)^2}, \tag{40}$$

which we can see by inspecting $|\Psi(x,t)|^2$ and $|\phi(p,t)|^2$ (c.f. Eq. (34) - they are both Gaussians at all time). Remember that for a free particle $|\phi(p,t)|^2 = |\phi(p,0)|^2$, and $p = \hbar k$ ($\phi(k,0)$ is given in Eq. (14)). The uncertainty product is

$$\sigma_x(t)^2\sigma_p(t)^2 = \frac{\hbar^2}{4}\left(1+\theta(t)^2\right) \geq \frac{\hbar^2}{4} \tag{41}$$

Notice this is independent of k_0. You may recall that the ground-state of a SHO (which was the initial state) is a minimum uncertainty state. At $t = 0$ the harmonic trap is turned off, and the particle evolves under the free-particle Hamiltonian. Because the momentum of a free-particle is conserved (more precisely, the $\hat{p}$ operator commutes with $\hat{H}$), the initial momentum distribution is a stationary state, and so the momentum uncertainty is also stationary. However the same is not true for position - the wavefunction diffuses and is therefore no longer a minimum uncertainty state when $t > 0$.

<u>Problem 6.3</u> *

(a) To evaluate this integral we simply make use of the property

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{42}$$

and so

$$\int_{-\infty}^{\infty} e^{-|x|/L}\delta(x+a)dx = e^{-|-a|/L} = e^{-|a|/L} \tag{43}$$

(b) The difference is that now the integral is evaluated over a domain that *does not* include the point c. The delta function $\delta(x-c)$ is 0 everywhere except at $x = c$, and so this integral is 0.

(c) To make use of the property Eq. (42) we need to change variables, so let's define $y = \alpha x$. First check the case when $\alpha > 0$:

$$\int_{-\infty}^{\infty} f(x)\delta(\alpha x)dx = \frac{1}{\alpha}\int_{-\infty}^{\infty} f(y/\alpha)\delta(y)dy = \frac{1}{\alpha}f(0) \tag{44}$$

which is the same result we'd get if we used $\delta(\alpha x) = \delta(x)/\alpha$.

Now let's check the negative case, for clarity let's write $\alpha = -|\alpha|$. Thus our change of variables it $y = -|\alpha|x$, so the limit $x \to \infty$ corresponds to $y \to -\infty$ and visa versa. Thus we get

$$\int_{-\infty}^{\infty} f(x)\delta(\alpha x)dx = \frac{-1}{|\alpha|}\int_{\infty}^{-\infty} f(-y/|\alpha|)\delta(y)dy = \frac{1}{|\alpha|}\int_{-\infty}^{\infty} f(y/\alpha)\delta(y)dy = \frac{1}{|\alpha|}f(0) \tag{45}$$

Thus we have shown $\delta(\alpha x) = \delta(x)/|\alpha|$ for all real α, as required.

(d) The Heaviside step function and Dirac delta function aren't strictly functions, they're distributions. It makes more sense to work with them under an integral. Thus the starting point is integration by parts

$$\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = [f(x)\theta(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df}{dx}\theta dx. \tag{46}$$

Using the property that $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x > 0$ we get

$$\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = [f(x)]_{0}^{\infty} - \int_{0}^{\infty} \frac{df}{dx}dx \tag{47}$$

$$= f(\infty) - f(\infty) + f(0) = f(0) \tag{48}$$

So far we have shown that

$$\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = f(0). \tag{49}$$

By the property Eq. (42) with $a = 0$, $f(0)$ can be also obtained as

$$f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx \tag{50}$$

which implies

$$\frac{d\theta}{dx} = \delta(x), \tag{51}$$

as required. Now, the last statement does not imply that we used $\int_{-\infty}^{\infty} F(x) = \int_{-\infty}^{\infty} G(x)$ and concluded that $F(x) = G(x)$ (with $F(x) \equiv f(x)\frac{d\theta}{dx}$ and $G(x) \equiv f(x)\delta(x)$, hence $\frac{d\theta}{dx} = \delta(x)$); such a conclusion is not true in general. Instead, we view Eq. (49) as the normalisation condition for $\frac{d\theta}{dx}$ for the case $f(x) = 1$, hence $f(0) = 1$ and therefore $\int_{-\infty}^{\infty} \frac{d\theta}{dx}dx = 1$ which is the same normalisation as that of the delta function. Thus, we have that $\frac{d\theta(x)}{dx} = 0$ for $x < 0$ and $x > 0$, because in these regions the $\theta(x)$ function is continuous and differentiable, and we observe that the singularity at $x = 0$ (where $\frac{d\theta}{dx}dx = \infty$) has the same strength as the $\delta(x)$ function, because $\int_{-\infty}^{\infty} \frac{d\theta}{dx}dx = \int_{-\infty}^{\infty} \delta(x)dx = 1$. Therefore we have reproduced all properties of the delta function (including its normalisation) from $\frac{d\theta}{dx}$, and since the delta function is uniquely defined by these properties, we conclude that $\frac{d\theta}{dx} = \delta(x)$. (To appreciate what we mean by the strength of the delta function, consider $\alpha\delta(x)$ instead of $\delta(x)$ with the constant $\alpha \neq 1$; $\alpha\delta(x)$ is normalised to α, and not to 1.)

(e) The Fourier transform of the Dirac delta function is

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x)e^{-ikx}dx. \tag{52}$$

Again using the property Eq. (42) this is easy to evaluate, it is simply

$$F(k) = \frac{1}{\sqrt{2\pi}} \tag{53}$$

(f) The inverse Fourier transform of the constant $F(k) = 1$ is given by the integral

$$F^{-1}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 1 \times e^{ikx} dx \tag{54}$$

which is a little tricky to evaluate, but certainly possible. It's acceptable to simply look this up. However, it's far simpler to just use the result from the previous question. Since we know that the Fourier transform of a Delta function is a constant $F(k) = 1/\sqrt{2\pi}$, it must be true that the inverse Fourier transform $F(k) = 1$ should be

$$F^{-1}(x) = \sqrt{2\pi}\delta(x). \tag{55}$$

i.e. using the result of the previous question it's easy to see that the Fourier transform of $\sqrt{2\pi}\delta(x)$ is 1, so the inverse transform of 1 is $\sqrt{2\pi}\delta(x)$.

Problem 6.4

The solution to the TDSE for a delta function well is

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} = \frac{\sqrt{m\alpha}}{\hbar} \begin{cases} e^{-m\alpha x/\hbar^2} & x \geq 0 \\ e^{m\alpha x/\hbar^2} & x \leq 0 \end{cases} \tag{56}$$

Calculating $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

So $\langle x \rangle = 0$, since the integrand is a odd function. But $\langle x^2 \rangle$ is even, so we can make use of the supplied integrals:

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi|^2 dx = 2\frac{m\alpha}{\hbar^2} \int_0^{\infty} x^2 e^{-m\alpha x/\hbar^2} dx \tag{57}$$

$$= \frac{2m\alpha}{\hbar^2} 2 \left(\frac{\hbar^2}{2m\alpha} \right)^3 \tag{58}$$

$$= \frac{\hbar^4}{2m^2\alpha^2} \tag{59}$$

Therefore

$$\sigma_x = \frac{\hbar^2}{\sqrt{2}m\alpha} \tag{60}$$

Calculating $\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

The absolute value wavefunction has an odd derivative. To convince yourself this is true, use the piecewise definition and compare the derivative when $x > 0$ and $x < 0$. Because $\frac{d\psi}{dx}$ is an odd function and $\psi(x)$ is an even function (and is real), their product is an odd function and therefore $\langle p \rangle = \int_{-\infty}^{\infty} \psi(x)^* \Big(- i\hbar \frac{d}{dx} \Big) \psi(x) dx = 0$.

However $\langle \hat{p}^2 \rangle = \int_{-\infty}^{\infty} \psi(x)^* \Big(- \hbar^2 \frac{d^2}{dx^2} \Big) \psi(x) dx$ is not 0. The first derivative is straight-forward to evaluate piecewise,

$$\frac{d\psi}{dx} = \frac{\sqrt{m\alpha}}{\hbar} \begin{cases} -\frac{m\alpha}{\hbar^2} e^{-m\alpha x/\hbar^2} & x \geq 0 \\ \frac{m\alpha}{\hbar^2} e^{m\alpha x/\hbar^2} & x \leq 0 \end{cases} \tag{61}$$

$$= \left(\frac{\sqrt{m\alpha}}{\hbar} \right)^3 \left[-\theta(x) e^{-m\alpha x/\hbar^2} + \theta(-x) e^{m\alpha x/\hbar^2} \right]. \tag{62}$$

Writing the derivative in terms of step-functions enables us to use the results of the previous question to evaluate the second derivative. Note that $d\theta(-x)/dx = -\delta(x)$ (using the chain rule, and $\delta(-x) = \delta(x)$). We obtain

$$\frac{d^2\psi}{dx^2} = \left(\frac{\sqrt{m\alpha}}{\hbar} \right)^3 \left[-\delta(x) e^{-m\alpha x/\hbar^2} + \frac{m\alpha}{\hbar^2} \theta(x) e^{-m\alpha x/\hbar^2} - \delta(x) e^{m\alpha x/\hbar^2} + \frac{m\alpha}{\hbar^2} \theta(-x) e^{m\alpha x/\hbar^2} \right]. \tag{63}$$

However, because $\delta(x)$ is zero everywhere except at $x = 0$, we can use $f(x)\delta(x) = f(0)\delta(x)$ to tidy this up. Remember also that

$$\theta(x) e^{-m\alpha x/\hbar^2} + \theta(-x) e^{m\alpha x/\hbar^2} = e^{-m\alpha |x|/\hbar^2}, \tag{64}$$

so finally we can write the second derivative as

$$\frac{d^2\psi}{dx^2} = \left(\frac{\sqrt{m\alpha}}{\hbar} \right)^3 \left[\frac{m\alpha}{\hbar^2} e^{-m\alpha|x|/\hbar^2} - 2\delta(x) \right] \tag{65}$$

$$= \left(\frac{\sqrt{m\alpha}}{\hbar} \right)^3 \left[\frac{\sqrt{m\alpha}}{\hbar} \psi(x) - 2\delta(x) \right]. \tag{66}$$

Another way we could have obtained this result (instead of evaluating the second derivative by actually differentiating the given wave function twice) is by rearranging the time-independent Schrödinger equation for a delta function well $V = -\alpha\delta(x)$ (the energy is given in the question),

$$\frac{d^2\psi(x)}{dx^2} = \left(\frac{m^2\alpha^2}{\hbar^4} - \frac{2m\alpha}{\hbar^2} \delta(x) \right) \psi(x). \tag{67}$$

They look a little different, but agree if we use $\psi(x)\delta(x) = \psi(0)\delta(x)$, where $\frac{\sqrt{m\alpha}}{\hbar}$.

In either case, the mean square momentum is given by

$$\langle \hat{p}^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi^* \frac{d^2\psi}{dx^2} dx \tag{68}$$

$$= -\hbar^2 \left(\frac{\sqrt{m\alpha}}{\hbar} \right)^3 \left(\frac{\sqrt{m\alpha}}{\hbar} \int_{-\infty}^{\infty} \psi^2 dx - 2 \int_{-\infty}^{\infty} \delta(x) \psi dx \right) \tag{69}$$

where $\int_{-\infty}^{\infty} \psi^2 dx = 1$ by normalisation and $\int_{-\infty}^{\infty} \delta(x)\psi dx = \psi(0) = \sqrt{m\alpha}/\hbar$. Using these values we get

$$\langle \hat{p}^2 \rangle = \left(\frac{m\alpha}{\hbar}\right)^2 \tag{70}$$

The momentum uncertainty is

$$\sigma_p = \frac{m\alpha}{\hbar}, \tag{71}$$

which gives uncertainty product,

$$\sigma_x \sigma_p = \frac{\hbar^2}{\sqrt{2}m\alpha}\frac{m\alpha}{\hbar} = \sqrt{2}\frac{\hbar}{2} \geq \frac{\hbar}{2} \tag{72}$$

The Heisenberg uncertainty principle is satisfied, as required.

Problem 6.5 **[FOR ASSIGNMENT 3; max 10 points]**

(a) Figure 1(a) shows a plot of the potential and a sketch of the components of the wavefunction for $E < V_0$. The wavefunction is

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & \text{if } x \leq 0, \\ Fe^{-\kappa x}, & \text{if } x > 0, \end{cases} \tag{73}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar} \text{ and } \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar} > 0. \tag{74}$$

We write this because the incident part of the wavefunction, Ae^{ikx}, is an unbound state propagating to the right, the reflected part, Be^{-ikx}, is also unbound and is propagating to the left, the transmitted component, $Fe^{-\kappa x}$, is a bound state.

Continuity requirements on ψ and $d\psi/dx$ at $x = 0$ give us two simultaneous equations:

$$A + B = F \text{ for } \psi \tag{75}$$

$$ik(A - B) = -\kappa F \text{ for } \frac{d\psi}{dx}. \tag{76}$$

So we have

$$A + B = -\frac{ik}{\kappa}(A - B) \tag{77}$$

$$A\left(1 + \frac{ik}{\kappa}\right) = -B\left(1 - \frac{ik}{\kappa}\right). \tag{78}$$

The reflection coefficient is given by

$$R = \left|\frac{B}{A}\right|^2 = \frac{|(1 + ik/\kappa)|^2}{|(1 - ik/\kappa)|^2} = \frac{1 + (k/\kappa)^2}{1 + (k/\kappa)^2} = 1. \tag{79}$$

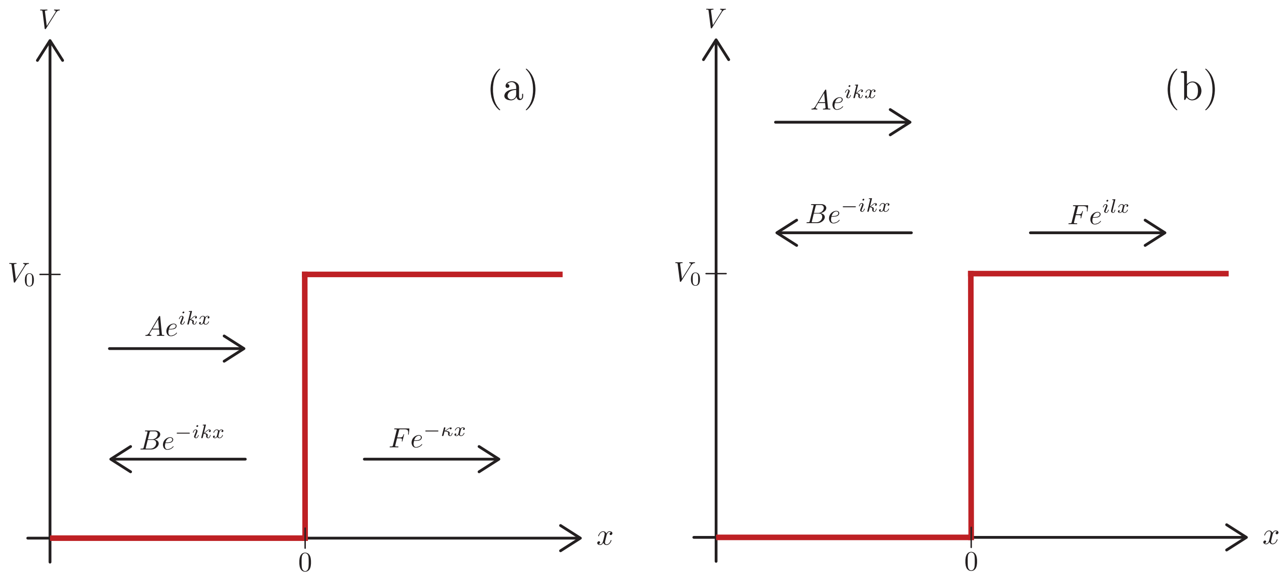

Figure 1: Wavefunction components for (a) $E < V_0$ and (b) $E > V_0$.

We can conclude that, although the wave function penetrates into the barrier, it is eventually *all* reflected.

(b) Figure 1(b) shows a sketch of the components of the wavefunction for the case $E > V_0$. In this case, all components are unbound, so we have

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & \text{if } x \leq 0, \\ Fe^{ilx}, & \text{if } x > 0, \end{cases} \tag{80}$$

where

$$k = \frac{\sqrt{2mE}}{\hbar} \text{ and } l = \frac{\sqrt{2m(E - V_0)}}{\hbar} > 0. \tag{81}$$

To preserve continuity at $x = 0$, we have

$$A + B = F \text{ for } \psi \tag{82}$$

$$ik(A - B) = ilF \text{ for } \frac{d\psi}{dx}. \tag{83}$$

So

$$A + B = \frac{k}{l}(A - B) \tag{84}$$

$$A\left(1 - \frac{k}{l}\right) = -B\left(1 + \frac{k}{l}\right) \tag{85}$$

and the reflection coefficient is

$$R = \left|\frac{B}{A}\right|^2 = \frac{(1-k/l)^2}{(1+k/l)^2} = \frac{(k-l)^2}{(k+l)^2} = \frac{(k-l)^4}{(k^2-l^2)^2} \tag{86}$$

$$= \frac{(\sqrt{E}-\sqrt{E-V_0})^4}{E-E+V_0} = \frac{(\sqrt{E}-\sqrt{E-V_0})^4}{V_0^2}. \tag{87}$$

(c) Since the transmitted wave travels at a different *speed*, we need to calculate the transmission coefficient with $T = J_t/J_i$ where t denotes the transmitted component of the wavefunction and i denotes the incident component. The probability current J is dP/dt so it's best to include time dependence. Using (6) from the problem set and (80) (adding in time dependence), we get

$$J_i = \frac{i\hbar}{2m}\left(Ae^{i(kx-\frac{E}{\hbar}t)}\frac{\partial}{\partial x}A^*e^{-i(kx-\frac{E}{\hbar}t)} - A^*e^{-i(kx-\frac{E}{\hbar}t)}\frac{\partial}{\partial x}Ae^{i(kx-\frac{E}{\hbar}t)}\right) \tag{88}$$

$$= \frac{i\hbar}{2m}|A|^2\left(e^{i(kx-\frac{E}{\hbar}t)}(-ik)e^{-i(kx-\frac{E}{\hbar}t)} - e^{-i(kx-\frac{E}{\hbar}t)}(ik)e^{i(kx-\frac{E}{\hbar}t)}\right) \tag{89}$$

$$= \frac{i\hbar}{2m}|A|^2(-2ik) \tag{90}$$

$$= \frac{\hbar k}{m}|A|^2. \tag{91}$$

J_t is very similar,

$$J_t = \frac{\hbar l}{m}|F|^2. \tag{92}$$

So the transmission coefficient is

$$T\frac{J_t}{J_i} = \frac{l}{k}\left|\frac{F}{A}\right|^2 = \sqrt{\frac{E-V_0}{E}}\frac{|F|^2}{|A|^2} \tag{93}$$

as required.

(d) From (82) and (83), the constraints on F and A are

$$F = A + B = A + A\frac{\left(\frac{k}{l}-1\right)}{\left(\frac{k}{l}+1\right)} = A\frac{2k/l}{\left(\frac{k}{l}+1\right)} = \frac{2k}{k+l}A. \tag{94}$$

So we have

$$T = \left|\frac{F}{A}\right|^2\frac{l}{k} = \left(\frac{2k}{k+l}\right)^2\frac{l}{k} = \frac{4kl}{(k+l)^2} = \frac{4kl(k-l)^2}{(k^2-l^2)^2} \tag{95}$$

$$= \frac{4\sqrt{E}\sqrt{E-V_0}\left(\sqrt{E}-\sqrt{E-V_0}\right)^2}{V_0^2}. \tag{96}$$

Calculating $R+T$, using R from (b), gives

$$
\begin{aligned}
R+T &= \frac{(k-l)^2}{(k+l)^2} + \frac{4kl}{(k+l)^2} && (97)\\
&= \frac{4kl + k^2 - 2kl + l^2}{(k+l)^2} && (98)\\
&= \frac{k^2 + 2kl + l^2}{(k+l)^2} && (99)\\
&= \frac{(k+l)^2}{(k+l)^2} && (100)\\
&= 1 && (101)
\end{aligned}
$$

as required.