PHYS2100 Assignment 3

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Question 1

a. If the motion due to a force is conservative, there is some potential function, $V(x)$, such that

$$\mathbf{F} \cdot \mathbf{v} = -\frac{dV}{dt} \tag{1}$$

Given the *central* force (acting towards to the origin, and hence is negative for positive x) in the description, $\mathbf{v} = dx/dt$, and so equation 1 can be solved for $V(x)$:

$$\begin{aligned}
\mathbf{F} \cdot \frac{dx}{dt} &= -\frac{dV}{dt} \\
\int dV &= -\int \mathbf{F}\, dx \\
V &= -\int -m\mu \left(x + \frac{a^4}{x^3}\right) dx \\
&= m\mu \left(\int x\, dx + a^4 \int \frac{1}{x^3}\, dx\right) \\
&= m\mu \left(\frac{x^2}{2} - \frac{a^4}{2x^2} + c\right) \\
V(x) &= m\mu \left(\frac{x^4 - a^4}{2x^2}\right) + c
\end{aligned}$$

And so the potential function due to the force is that shown above, where c is due to some initial conditions (where c isn't relevant to the force due to the time derivative in equation 1, which would destroy the constant term once differentiated).

b. The velocity at some x due to the force can be found from newton's second law $F = ma = m\ddot{x}$:

$$\begin{aligned}
m\ddot{x} &= -m\mu \left(x + \frac{a^4}{x^3}\right) \\
\ddot{x} &= -\mu \left(x + \frac{a^4}{x^3}\right)
\end{aligned}$$

but $\ddot{x} = dv/dt = \frac{dv}{dx}\frac{dx}{dt}$, with $v = dx/dt$. So,

$$\begin{aligned}
\Rightarrow \frac{dv}{dx}\frac{dx}{dt} &= -\mu\left(x + \frac{a^4}{x^3}\right) \\
\int_0^v v\,dv &= -\mu\left(\int_a^x x\,dx + a^4\int_a^x \frac{1}{x^3}\,dx\right) \\
\frac{v^2}{2} &= -\mu\left(\left[\frac{x^2}{2}\right]_a^x + a^4\left[-\frac{1}{2x^2}\right]_a^x\right) \\
&= -\mu\left(\frac{x^2 - a^2}{2} + \frac{a^4}{2a^2} - \frac{a^4}{2x^2}\right) \\
&= \mu\left(\frac{a^4 - x^4}{2x^2}\right) \\
v^2 &= \mu\left(\frac{a^4 - x^4}{x^2}\right) \\
v &= \pm\sqrt{\mu\left(\frac{a^4 - x^4}{x^2}\right)}
\end{aligned}$$

However, since the force is 'pulling' the particle to the origin, and the particle begins at rest, the velocity will be negative at any x smaller than a, and so

$$v = -\sqrt{\mu\left(\frac{a^4 - x^4}{x^2}\right)}$$

Question 2

a. Newton's second law states that $F = ma$. In terms of each coordinate axis component, $F = m(\ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}})$. Substituting in each of the double derivatives of the axis components gives

$$\begin{aligned}
F &= m(-a\omega^2\cos(\omega t)\hat{\mathbf{i}} - a\omega^2\sin(\omega t)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}) \\
&= m\omega^2(-x\hat{\mathbf{i}} - y\hat{\mathbf{j}} - z\hat{\mathbf{k}} + z\hat{\mathbf{k}}) \\
&= m\omega^2(z\hat{\mathbf{k}} - \mathbf{r})
\end{aligned}$$

Thus, for the position vector in this scenario, the law of force is that shown above.

b. Given that $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and $d\mathbf{p}/dt = \mathbf{F}$,

$$\begin{aligned}
\frac{d}{dt}\mathbf{L} &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \\
&= \mathbf{r} \times (\frac{d}{dt}\mathbf{p}) \\
&= \mathbf{r} \times \mathbf{F}
\end{aligned}$$

where $\mathbf{r} = a\cos(\omega t)\hat{\mathbf{i}} + a\sin(\omega t)\hat{\mathbf{j}} + ct\hat{\mathbf{k}}$ and $\mathbf{F} = m\omega^2(z\hat{\mathbf{k}} - \mathbf{r}) = -am\omega^2\cos(\omega t)\hat{\mathbf{i}} - am\omega^2\sin(\omega t)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$

Thus,

$$
\begin{aligned}
\dot{\mathbf{L}} &= \mathbf{r} \times \mathbf{F} \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a\cos(\omega t) & a\sin(\omega t) & ct \\ -am\omega^2\cos(\omega t) & -am\omega^2\sin(\omega t) & 0 \end{vmatrix} \\
&= (a\sin(\omega t)\cdot 0 - ct\cdot -am\omega^2\sin(\omega t))\hat{\mathbf{i}} \\
&+ (ct\cdot -am\omega^2\cos(\omega t) - a\cos(\omega t)\cdot 0)\hat{\mathbf{j}} \\
&+ (a\cos(\omega t)\cdot -am\omega^2\sin(\omega t)) - a\sin(\omega t)\cdot -am\omega^2\cos(\omega t))\hat{\mathbf{k}} \\
&= ctm\omega^2 a\sin(\omega t)\hat{\mathbf{i}} - ctm\omega^2 a\cos(\omega t)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}
\end{aligned}
$$

Since the z-component of the *change* in angular momentum is zero, the z-component of the angular momentum is therefore constant.

Question 3

a. The force acting upon a particle is

$$
F = \begin{cases} -\frac{\mu m x}{a^4} & x \le a \\ -\frac{\mu m}{x^2} & x > a \end{cases}
$$

Let $F_1 x$ correspond to the case $x \le a$, and $F_2(x)$ correspond to when $x > a$. The potential functions corresponding to each piecewise force can be found by

$$
\begin{aligned}
F_1 &= -V_1'(x) \\
\Rightarrow V_1(x) &= -\int F_1(x)\,dx \\
&= -\int -\frac{\mu m x}{a^4}\,dx \\
&= \frac{\mu m x^2}{2a^4} + c
\end{aligned}
\qquad
\begin{aligned}
F_2 &= -V_2'(x) \\
\Rightarrow V_2(x) &= -\int F_2(x)\,dx \\
&= -\int -\frac{\mu m}{x^3}\,dx \\
V_2(x) &= -\frac{\mu m}{2x^2}
\end{aligned}
$$

As per the description, the potential function is continuous at $x = a$. With this condition, the value of the constant can be found by equating $V_1(x)$ and $V_2(x)$ at $x = a$:

$$
\begin{aligned}
\frac{\mu m a^2}{2a^4} + c &= -\frac{\mu m}{2a^2} \\
c &= -\frac{\mu m}{2a^2} - \frac{\mu m}{2a^2} \\
&= -\frac{\mu m}{a^2} \\
\Rightarrow V_1(x) &= \frac{\mu m x^2}{2a^4} + -\frac{\mu m}{a^2}
\end{aligned}
$$

And so, each of the piecewise potential functions were calculated. Both hand-drawn and calculated graphs of the functions were created:

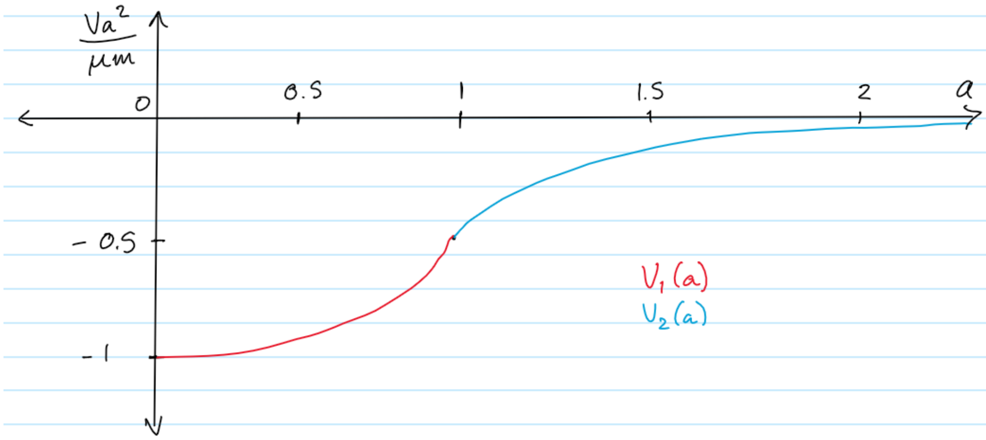

Figure 1: Hand Drawn Graph of the Potential Functions

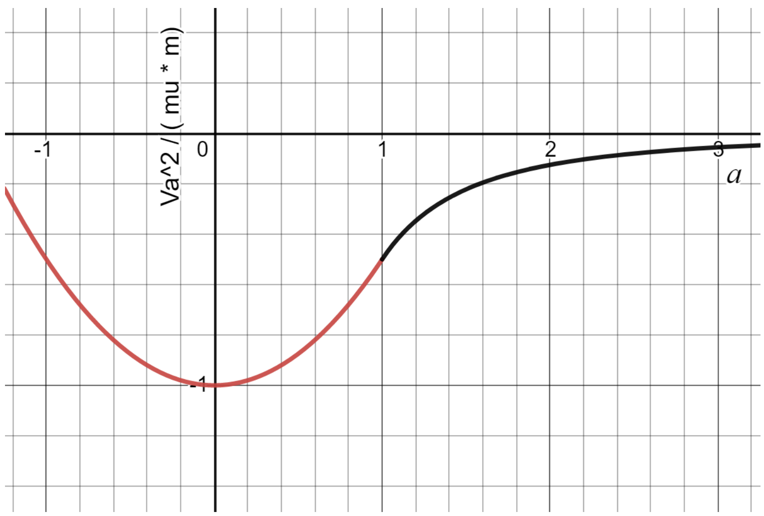

Figure 2: Desmos Graph of Potential Functions
Red corresponds to $V_1(a)$, and black to $V_2(a)$

b. The energy of a particle is $E = 1/2mv^2 + V(x)$. To calculate the speed needed to escape the potential bound, take the change in energy of the potential from $0 \to \infty$:

$$\begin{aligned}\Delta V &= [V_1(x)]_0^a + [V_2(x)]_a^\infty \\ &= -\frac{\mu m}{2a^2} - 0 + 0 - \frac{\mu m}{2a^2} \\ &= -\frac{\mu m}{a^2}\end{aligned}$$

Then, the energy needed to *escape* the potential bound (i.e. go to infinity) is $E > 0$. Now, to find that in terms of initial speed u, set $v = u$ in the energy equation, and $V(x) = \Delta V$:

$$\begin{aligned} E > 0 \Rightarrow \frac{1}{2}mu^2 - \frac{\mu m}{a^2} &> 0 \\ \Rightarrow \frac{1}{2}mu^2 &> \frac{\mu m}{a^2} \\ u^2 &> \frac{2\mu}{a^2} \\ \Rightarrow a^2u^2 &> 2\mu \end{aligned}$$

And so the particle will move to infinity if the above relation holds.