# MATH2400 Assignment 4

Ryan White
44990392

26th of May, 2021

## Question 1

Fix an interval $[a, b]$. Let $\mathcal{C}[a, b]$ be the set of continuous functions on $[a, b]$. For $f, g \in \mathcal{C}[a, b]$, define a dot product and norm by

$$f \cdot g := \int_a^b f(x)g(x)\,dx, \qquad \|f\|_2 := \sqrt{f \cdot f} = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}$$

(note the absolute value is actually not necessary). The dot product is clearly bilinear and symmetric (you do not need to show this or that $\cdot$ defines a dot product). Show that $\|\cdot\|_2$ is a norm on $\mathcal{C}[a, b]$.

A function $\|\cdot\| : X \to \mathbb{R}$ is defined as being a norm if the following properties hold (each will be proven for $\|\cdot\|_2 : \mathcal{C}[a, b] \to \mathbb{R}$ under the respective property):

a. $\|f\| \geq 0$, and $\|f\| = 0$ iff $f = 0$.

Firstly, take a function $f \in \mathcal{C}[a,\, b]$, where $a \neq b$. By construction, $f$ is a continuous function on all $[a,\, b]$. By proof in Assignment 3, Question 5, $\int_a^b |f(x)|\,dx = 0$ if and only if $f(x) = 0$ on all $[a,\, b]$. It follows that $\int_a^b |f(x)|^2\,dx = 0$ and $\left(\int_a^b |f(x)|^2\,dx\right)^{1/2} = 0$ for all $f(x) = 0$. If any value of $f(x) > 0$, $f \in [a,\, b]$, then it follows that the integral (and it's square root) are greater than 0. Since this is of the form of the definition of the norm, $\|f\|_2 \geq 0$ for all $f(x) \in [a,\, b]$.

b. $\|cf\| = |c|\,\|f\|$ for all $c \in \mathbb{R}$ and $f \in X$.

Proof is trivial for $c = 0$ or $f = 0$. Assume that $c \neq 0$ and $f \neq 0$, $f \in \mathcal{C}[a,\, b]$. Then,

$$\begin{aligned}
\|cf\|_2 &= \sqrt{cf \cdot cf} \\
&= \left(\int_a^b cf(x) \times cf(x)\,dx\right)^{1/2} \\
&= \left(\int_a^b c^2|f(x)|^2\,dx\right)^{1/2} \\
&= \left(c^2\int_a^b |f(x)|^2\,dx\right)^{1/2} \\
&= |c|\left(\int_a^b |f(x)|^2\,dx\right)^{1/2} = |c|\,\|f\|_2
\end{aligned}$$

And so the second property of norms has been shown for $\|\cdot\|_2$ on $\mathcal{C}[a, b]$.

c. $\|f + g\| \le \|f\| + \|g\|$ for all $f, g \in X$.

If $f = 0$ and/or $g = 0$, then the property would hold via previous proven properties. Take $\|f + g\|_2{}^2$, where $f, g \in \mathcal{C}[a, b]$. Then,

$$\begin{aligned} \|f + g\|^2 &= \|f + g\| \, \|f + g\| \\ &= f \cdot f + g \cdot g + 2(f \cdot g) \end{aligned}$$

By Cauchy-Schwarz inequality (Theorem 8.2.2 in Lebl II), $(f \cdot g) \le \|f\| \, \|g\|$. So,

$$\begin{aligned} \|f + g\|^2 &\le f \cdot f + g \cdot g + 2(\|f\| \, \|g\|) \\ &= \|f\|^2 + \|g\|^2 + 2(\|f\| \, \|g\|) \\ &= (\|f\| + \|g\|)^2 \end{aligned}$$

Taking the square root of each side,

$$\|f + g\| \le \|f\| + \|g\|$$

And so the third property (and all others) has been shown for $\|\cdot\|_2$ being a norm on $\mathcal{C}[a, b]$.

## Question 2

Consider the sequence of functions $f_n \colon [0, 1] \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} 1 - nx & \text{if } 0 \le x \le 1/n, \\ 1 & \text{otherwise}, \end{cases}$$

for $n > 0$, which converges pointwise to $f(x) = 1$ as $n \to \infty$. Show that $\{f_n\}_{n=1}^{\infty}$ does not converge to $f$ in the uniform norm, but it does converge using the norm defined in Problem (1). (As a consequence, for infinite dimensional vector spaces, there are norms that are not equivalent.)

A sequence of bounded functions converges uniformly if and only if

$$\lim_{n\to\infty} \|f_n - f\|_u = 0$$

The question states that $f$ converges to 1, so the sequence of bounded functions converges uniformly if and only if

$$\lim_{n\to\infty} \|f_n - 1\|_u = 0$$

where the uniform norm is defined by $\|f\|_u = \sup\{|f(x)| : x \in S\}$. Thus,

$$\begin{aligned} \lim_{n\to\infty} \|f_n - f\|_u &= \lim_{n\to\infty} \sup\{|f_n - f| : 0 \le x \le 1/n, \, x \in [0, 1]\} \\ &= \lim_{n\to\infty} \sup\{|1 - nx - 1| : 0 \le x \le 1/n, \, x \in [0, 1]\} \\ &= \lim_{n\to\infty} \sup\{nx : 0 \le x \le 1/n, \, x \in [0, 1]\} \\ &\le \lim_{n\to\infty} \sup\{1\} \\ &= 1 \end{aligned}$$

with the third last step having the relation that $\frac{nx}{n} \le 1, \forall x \in [0, 1]$. Therefore $\{f_n\}_{n=1}^{\infty}$ does not converge to $f$ in the uniform norm.

Take instead the definition of convergence of a sequence of bound functions, but with the definition of the norm defined in Question 1, as

$$\lim_{n\to\infty} \|f_n - f\| = \lim_{n\to\infty} \sqrt{f_n \cdot f_n - f \cdot f} = \lim_{n\to\infty} \left( \int_a^b |f_n(x)|^2 - |f(x)|^2 \, dx \right)^{1/2}$$

DNF

## Question 3

Show that the function defined by

$$f(x,y) = \begin{cases} x & \text{if } y = x^2, \\ 0 & \text{otherwise,} \end{cases}$$

is continuous at 0 with all directional derivatives defined at 0 but $f$ is not differentiable at 0.

Firstly, the function is given by $f\colon\ \mathbb{R}^2 \to \mathbb{R}$. To satisfy the question, all three of the following criteria must be proven:

i. $f(x,\ y)$ is continuous at $(0,\ 0)$.

   $f(x,\ y)$ is continuous at 0 if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

   It is clear to see that if $y = x^2$, as $y \to 0$, $x \to 0$ and the converse being true also. It follows that, in this case where $y = x^2$, $f(x,y) \to 0$ as $y,\ x \to 0$ since $f(x,y) = x$ in this situation. If $y \neq x^2$, then $f(x,y) = 0$. Therefore, $f(x,y)$ is continuous at 0.

ii. All directional derivatives of $f$ exist at 0.

   The directional derivative of a multivariate function at point $(x,\ y)$ is given by

$$\frac{\partial}{\partial u} f(x,y) = \lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}$$

   where $\vec{u}$ is defined as $\vec{u} = \{a, b\}$ where $a,\ b \in \mathbb{R}$. Taking the directional derivative of $f(x,y)$ at 0 along the curve $y = x^2 \Rightarrow bh = (ah)^2$,

$$\begin{aligned} \frac{\partial}{\partial u} f(0,0) &= \lim_{h\to 0} \frac{f(ah, bh)}{h} \\ &= \lim_{h\to 0} \frac{ah}{h} = \lim_{h\to 0} a = a \end{aligned}$$

   And so the directional derivative exists at 0, with $\vec{u} = (a, 0)$.

iii. $f$ is not differentiable at 0.

   $f$ is differentiable if all of it's partial derivatives are continuous. Firstly, assume that all of the partial derivatives of $f$ exist at 0. Take $f(x,y)$ along the curve $y = x^2$. Along this curve, $f(x,y) = x$ at every point. It is analogous to show the value as $f(x,y) = \sqrt{y}$. A contradiction is immediately found for showing continuity of the partial derivative with respect to $y$ at 0:

$$\begin{aligned} \lim_{(x,y)\to 0} \frac{\partial}{\partial y} f(x,y) &= \lim_{(x,y)\to 0} \frac{\partial}{\partial y} \sqrt{y} \\ &= \lim_{(x,y)\to 0} \frac{1}{\sqrt{y}} \end{aligned}$$

This limit is not defined, and so the partial derivative of $f$ with respect to $y$ is not continuous, meaning that $f$ is not differentiable at 0.

# Question 4

Using the definition of the derivative and limit, compute the derivative of the determinant function on $2 \times 2$ matrices at the identity (which we consider as a subset of $\mathbb{R}^4$ under the Euclidean norm).
*Hint*: For a matrix $H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$, consider it close to 0 if $|h_{ij}| < \epsilon$ for all $i, j = 1, 2$.

Firstly, define a matrix $H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$ such that $\|H\|$ is $\epsilon$ close to 0, $\epsilon > 0$. For $A$ to be a derivative of $\det : \mathbb{R}^4 \to \mathbb{R}$ at the identity $I$, the following must be true

$$\lim_{H \to 0} \frac{\|\det(I+H) - \det(I) - AH\|}{\|H\|} = 0$$

By properties of determinants, $\det(I) = 1$. $det(I+H)$ can be computed, with first calculating

$$\begin{aligned} I + H &= \begin{bmatrix} 1 + h_{11} & 0 + h_{12} \\ 0 + h_{21} & 1 + h_{22} \end{bmatrix} \\ &= \begin{bmatrix} 1 + h_{11} & h_{12} \\ h_{21} & 1 + h_{22} \end{bmatrix} \\ \Rightarrow \det(I+H) &= (1+h_{11})(1+h_{22}) - h_{12}h_{21} = 1 + h_{11} + h_{22} + h_{11}h_{22} - h_{12}h_{21} \\ \Rightarrow \det(I+H) - \det(I) &= 1 + h_{11} + h_{22} + h_{11}h_{22} - h_{12}h_{21} - 1 \\ &= h_{11} + h_{22} + h_{11}h_{22} - h_{12}h_{21} \end{aligned}$$

Now, expanding on the norms in the above limit gives

$$\begin{aligned} \lim_{H \to 0} \frac{\sqrt{(\det(I+H) - \det(I) - AH)^2}}{\sqrt{{h_{11}}^2 + {h_{12}}^2 + {h_{21}}^2 + {h_{22}}^2}} &= 0 \\ \Rightarrow \lim_{H \to 0} \frac{\sqrt{(h_{11} + h_{22} + h_{11}h_{22} - h_{12}h_{21} - AH)^2}}{\sqrt{{h_{11}}^2 + {h_{12}}^2 + {h_{21}}^2 + {h_{22}}^2}} &= 0 \end{aligned}$$

For the left hand side to satisfy being zero, take

$$\begin{aligned} 0 &= h_{11} + h_{22} + h_{11}h_{22} - h_{12}h_{21} - AH \\ \Rightarrow AH &= h_{11} + h_{22} + h_{11}h_{22} - h_{12}h_{21} \end{aligned}$$

So for some example $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the linear operator (derivative of the determinant function at the identity) $A$ would correspond to the 'function,'

$$A = a + d + ad - bc$$

## Question 5

Let $S$ denote the set of sequences whose series are absolutely convergent. We define two norms on $S$ by

$$\|\{a_n\}_{n=0}^{\infty}\|_1 = \sum_{n=0}^{\infty} |a_n|, \qquad \|\{a_n\}_{n=0}^{\infty}\|_{\text{sup}} = \sup\{|a_n|\}_{n=0}^{\infty}.$$

(Note that $S$ is the set of sequences such that $\|a\|_1 < \infty$. The sup-norm is sometimes called the $\infty$-norm.) Define a linear operator $\Sigma\colon S \to \mathbb{R}$ by

$$\Sigma(\{a_n\}_{n=0}^{\infty}) = \sum_{n=0}^{\infty} a_n$$

(i) Compute the operator norm of $\Sigma$ using $\|\cdot\|_1$.

By the definition of the operator norm,

$$||A|| = \sup\{||Ax|| \,|\, x \in X \text{ s.t. } ||x|| = 1\}$$

For the operator $\sum$ on $S$,

$$\begin{aligned}\left\|\sum\right\|_S &= \sup\left\{\left\|\sum_{n=1}^{\infty} a_n\right\| \,|\, a_n \in a \in S,\ \|a_n\| = 1\right\}\\ &= \sup\left\{\frac{\|\sum\{a_n\}\|}{\|\{a_n\}\|}\right\}\\ &= \sup\left\{\frac{|\sum\{a_n\}|}{\sum|\{a_n\}|}\right\}\end{aligned}$$

Due to a proposition in absolute convergence of series, $|\sum a_n| \leq \sum |a_n|$ due to the possibility of negative values of $a_n$. Therefore,

$$\begin{aligned}\left\|\sum\right\|_S &= \sup\left\{\frac{|\sum\{a_n\}|}{\sum|\{a_n\}|}\right\} \leq \sup\{1\}\\ &\leq 1\end{aligned}$$

(ii) Show that the operator norm of $\Sigma$ using $\|\cdot\|_{\text{sup}}$ is unbounded.

$$\begin{aligned}\left\|\sum\right\|_{\text{sup}} &= \sup\left\{\left\|\sum_{n=0}^{\infty} a_n\right\|_{\text{sup}} \,|\, a_n \in a \in S,\ ||a_n|| = 1\right\}\\ &= \sup\left\{\frac{\sup|\sum_{n=0}^{\infty}\{a_n\}|}{\sup\{|a_n|\}}\right\}\end{aligned}$$

Since $\{a_n\}$ is convergent, $\sup\{|a_n|\} = A \in \mathbb{R}^+$, and

$$\begin{aligned}\left\|\sum\right\|_{\text{sup}} &= \sup\left\{\frac{\sup|\sum_{n=0}^{\infty}\{a_n\}|}{A}\right\}\\ &= \frac{\sup|\sum_{n=0}^{\infty}\{a_n\}|}{A}\\ &= \frac{nA}{A} = n\end{aligned}$$

But since $n \to \infty$, the operator norm using $\|\cdot\|_{\text{sup}}$ is unbounded.