THE UNIVERSITY OF QUEENSLAND
SCHOOL OF MATHEMATICS AND PHYSICS
PHYS2041 – Quantum Physics

Tutorial 5 Solutions

Problem 5.1

In lectures we have seen the "regular" x and $\hat{p} = -i\hbar\partial/\partial x$ operators can be defined in terms of the *ladder* operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_+ + \hat{a}_-\right), \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}_+ - \hat{a}_-\right) \tag{1}$$

which raise and lower the simple harmonic oscillator (SHO) eigenstates,

$$\hat{a}_-\psi_n = \sqrt{n}\psi_{n-1} \tag{2}$$
$$\hat{a}_+\psi_n = \sqrt{n+1}\psi_{n+1}. \tag{3}$$

To find $\langle\hat{x}\rangle$ rather than integrate $\int x|\psi_n|^2 dx$, we use ladder operators and thus evaluate

$$\langle\hat{x}\rangle = \int_{-\infty}^{\infty}\psi_n^* x\psi_n \mathrm{d}x = \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{\infty}\psi_n^*(a_+ + a_-)\psi_n \mathrm{d}x \tag{4}$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{\infty}\left(\sqrt{n+1}\psi_n^*\psi_{n+1} + \sqrt{n}\psi_n^*\psi_{n-1}\right)\mathrm{d}x \tag{5}$$
$$= 0 \tag{6}$$

This is zero by orthonormality. Following the same reasoning we can deduce that $\langle\hat{p}\rangle = 0$ also.

The mean square values are not 0 however. When expanding out $(a_+ + a_-)^2$ we must be careful with the ordering as $\hat{a}_\pm$ do not commute

$$\langle\hat{x}^2\rangle = \int_{-\infty}^{\infty}\psi_n^* x^2\psi_n \mathrm{d}x = \frac{\hbar}{2m\omega}\int_{-\infty}^{\infty}\psi_n^*(a_+ + a_-)^2\psi_n \mathrm{d}x \tag{7}$$
$$= \frac{\hbar}{2m\omega}\int_{-\infty}^{\infty}\psi_n^*(a_+a_+ + a_-a_+ + a_+a_- + a_-a_-)\psi_n \mathrm{d}x \tag{8}$$
$$\tag{9}$$

Here we can see that the terms with two raising operators or two lowering operators will not contribute to the answer since raising/lowering ψ_n twice will give $\psi_{n\pm2}$. The integral of these terms will go to zero due to orthonormality. The remaining terms give

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\int_{-\infty}^{\infty}\psi_n^*\left(a_-a_+\psi_n + a_+a_-\psi_n\right)\mathrm{d}x \tag{10}$$
$$= \frac{\hbar}{2m\omega}\int_{-\infty}^{\infty}\psi_n^*\left((n+1)\psi_n + n\psi_n\right)\mathrm{d}x \tag{11}$$
$$= \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right). \tag{12}$$

Similarly,

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \psi_n^* p^2 \psi_n \mathrm{d}x = -\frac{\hbar m\omega}{2}\int_{-\infty}^{\infty} \psi_n^*(a_+ - a_-)^2\psi_n \mathrm{d}x \tag{13}$$

$$= -\frac{\hbar m\omega}{2}\int_{-\infty}^{\infty} \psi_n^*(a_+a_+ - a_-a_+ - a_+a_- + a_-a_-)\psi_n \mathrm{d}x \tag{14}$$

$$= \frac{\hbar m\omega}{2}\int_{-\infty}^{\infty} \psi_n^*\left(a_-a_+\psi_n + a_+a_-\psi_n\right)\mathrm{d}x \tag{15}$$

$$= \frac{\hbar m\omega}{2}\int_{-\infty}^{\infty} \psi_n^*\left((n+1)\psi_n + n\psi_n\right)\mathrm{d}x \tag{16}$$

$$= \hbar m\omega\left(n + \frac{1}{2}\right) \tag{17}$$

So we have $\langle x\rangle = 0$, $\langle p\rangle = 0$, $\langle x^2\rangle = \frac{\hbar}{2m\omega}(n+1/2)$ and $\langle p^2\rangle = \hbar m\omega(n+1/2)$.

The standard deviations are,

$$\sigma_x = \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)} \tag{18}$$

$$\sigma_p = \sqrt{\hbar m\omega\left(n+\frac{1}{2}\right)} \tag{19}$$

$$\sigma_x\sigma_p = \hbar\left(n+\frac{1}{2}\right) \geq \hbar/2 \quad \forall n = 0,1,2,.... \tag{20}$$

Problem 5.2

The energy of the ground state is

$$E_0 = \frac{1}{2}\hbar\omega \tag{21}$$

A classical particle has maximum amplitude when the kinetic energy is 0. If the total energy is the ground state energy E_0 a classical particle has maximum amplitude l_{ho} ,

$$\frac{1}{2}m\omega^2 l_{\text{ho}}^2 = \frac{1}{2}\hbar\omega \tag{22}$$

$$l_{\text{ho}} = \sqrt{\frac{\hbar}{m\omega}} \tag{23}$$

A classical particle with energy E_0 oscillates between $-l_{ho}$ to $+l_{ho}$. But a quantum particle can be outside of this region with probability

$$
\begin{aligned}
P_{\text{outside}} &= \int_{-\infty}^{-l_{\text{ho}}} |\psi_0(x)|^2 dx + \int_{+l_{\text{ho}}}^{+\infty} |\psi_0(x)|^2 dx && (24)\\
&= 2\int_{+l_{\text{ho}}}^{+\infty} |\psi_0(x)|^2 dx \qquad \text{because the wavefunction is even} && (25)\\
&= 2\int_{+l_{\text{ho}}}^{+\infty} \frac{1}{\pi^{1/2} l_{\text{ho}}} e^{-x^2/l_{\text{ho}}^2} dx && (26)\\
&= \frac{2}{\sqrt{\pi}} \int_{1}^{+\infty} e^{-y^2} dy \qquad \text{where} \qquad y = \frac{x}{l_{\text{ho}}} && (27)\\
P_{\text{outside}} &= 1 - \text{erf}(1) = \text{erfc}(1) = 0.157 && (28)
\end{aligned}
$$

<u>Problem 5.3</u> [FOR ASSIGNMENT 3; max 10 points]

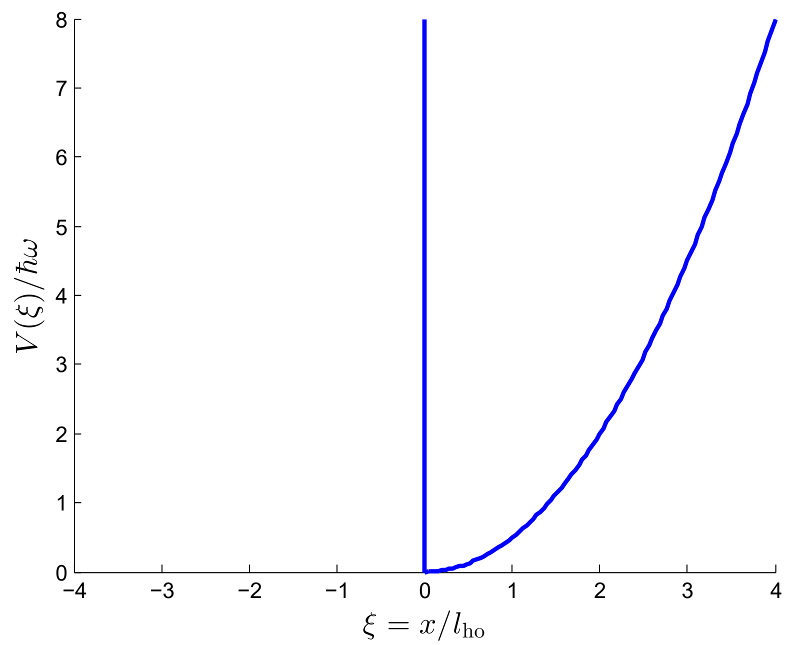

Figure 1: Sketch of the potential (in energy units $\hbar\omega$) in terms of the dimensionless length $\xi = x/l_{\text{ho}}$.

Figure 1 is a plot of the potential. The allowed energies are simply the SHO eigenstates which fit the new boundary condition that $\psi_n(0) = 0$.

Recall that in terms of the dimensionless length variable $\xi = x/\mathrm{l}_{\text{ho}}$ the SHO eigenstates are

$$
\psi_n(\xi) = \frac{1}{\pi^{1/4}\sqrt{l_{\text{ho}}}} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \tag{29}
$$

where $H_n(\xi)$ are the Hermite polynomials. The first few are

$$H_0 = 1 \tag{30}$$
$$H_1 = 2\xi \tag{31}$$
$$H_2 = 4\xi^2 - 2 \tag{32}$$
$$H_3 = 8\xi^3 - 12\xi \tag{33}$$
$$H_4 = 16\xi^4 - 48\xi^2 + 12 \tag{34}$$
$$\tag{35}$$

Examining ψ_n we can deduce that when n is odd, $\psi_n(x)$ is odd, and all odd functions go to 0 at $x = 0$. When n is even $\psi_n(x)$ is even, but the even wavefunctions do not satisfy $\psi_n(0) = 0$, which means they are not valid solutions for the half potential.

The allowed energies are simply the *odd* SHO eigenstates, which have energies

$$E_k = \hbar\omega\left(2k + \frac{3}{2}\right) \tag{36}$$

where $n = 2k + 1$ and $k = 0, 1, 2, 3....$

Problem 5.4 [FOR ASSIGNMENT 3; max 10 points]

(a) First the state must be normalised (remember the eigenstates of *any* Hamiltonian are orthonormal),

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = A^2 \int_{-\infty}^{\infty} \left(\sqrt{2}\psi_0^* + \psi_1^*\right)\left(\sqrt{2}\psi_0 + \psi_1\right) dx \tag{37}$$
$$= A^2 \int_{-\infty}^{\infty} \left(2|\psi_0|^2 + \psi_0^*\psi_1 + \psi_0\psi_1^* + |\psi_1|^2\right) dx \tag{38}$$
$$= A^2(2 + 0 + 0 + 1) = 1 \tag{39}$$

and so $A = 1/\sqrt{3}$. The normalised state at $t = 0$ is

$$\Psi(x,0) = \sqrt{\frac{2}{3}}\psi_0 + \sqrt{\frac{1}{3}}\psi_1, \tag{40}$$

i.e. the state is a superposition of the ground and first excited energy eigenstates of the simple harmonic oscillator (SHO). If we measured the energy we would get $E_0 = \hbar\omega/2$ with probability $P_0 = 2/3$ or $E_1 = 3\hbar\omega/2$ with probability $P_1 = 1/3$.

The expectation value of the energy is just the weighted average of these outcomes,

$$\langle\hat{H}\rangle = \frac{2}{3}E_0 + \frac{1}{3}E_1 = \frac{5}{6}\hbar\omega \tag{41}$$

(b) The state at later times is

$$\Psi(x,t) = \sqrt{\frac{2}{3}}\psi_0 e^{-i\omega t/2} + \sqrt{\frac{1}{3}}\psi_1 e^{-i3\omega t/2} \tag{42}$$

which means the probability density is

$$|\Psi(x,t)|^2 = \left(\sqrt{\frac{2}{3}}\psi_0^* e^{i\omega t/2} + \sqrt{\frac{1}{3}}\psi_1^* e^{i3\omega t/2}\right)\left(\sqrt{\frac{2}{3}}\psi_0 e^{-i\omega t/2} + \sqrt{\frac{1}{3}}\psi_1 e^{-i3\omega t/2}\right) \tag{43}$$

$$= \frac{2}{3}|\psi_0|^2 + \frac{1}{3}|\psi_1|^2 + \frac{\sqrt{2}}{3}\left(\psi_0^*\psi_1 e^{-i\omega t} + \psi_1^*\psi_0 e^{i\omega t}\right) \tag{44}$$

$$= \frac{2}{3}\psi_0^2 + \frac{1}{3}\psi_1^2 + \frac{2\sqrt{2}}{3}\psi_0\psi_1\cos(\omega t) \tag{45}$$

where the last line follows because the eigenstates of the SHO Hamiltonian are real.

To plot the density we need to know the ground and first excited states of the SHO. In terms of the dimensionless length $\xi = x/l_{\rm ho}$ they are

$$\psi_0(\xi) = \frac{1}{\pi^{1/4}\sqrt{l_{\rm ho}}}e^{-\xi^2/2}, \qquad \psi_1(\xi) = \frac{\sqrt{2}}{\pi^{1/4}\sqrt{l_{\rm ho}}}\xi e^{-\xi^2/2} \tag{46}$$

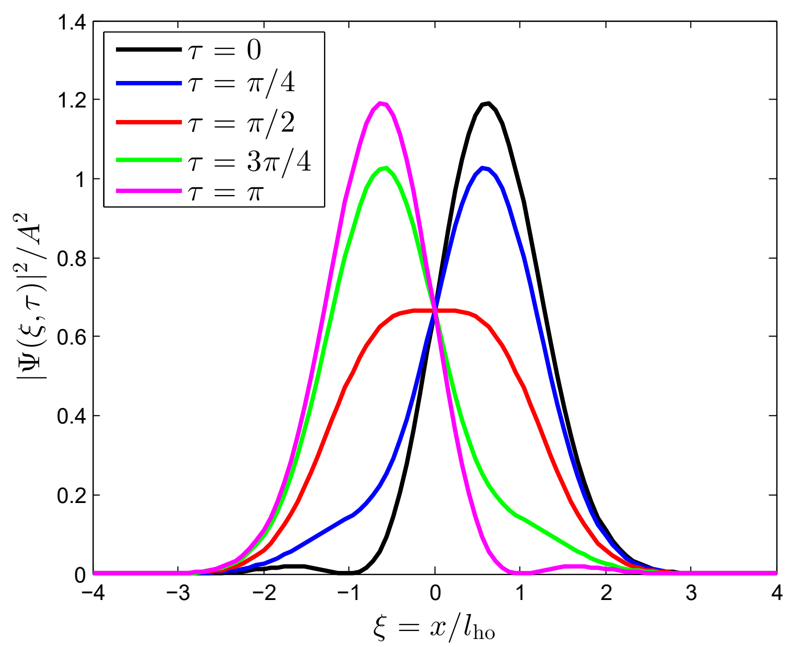

Figure 2: Probability density at different times $\tau = \omega t$, with $A^2 = 1/\sqrt{\pi}l_{\rm ho}$. The mean position $\langle x\rangle$ starts at $2l_{\rm ho}/3$ and appears to oscillate sinusoidally at frequency ω, which is consistent with our answer to part (c).

(c)

Calculating $\langle x \rangle$ Using Hermite-Gauss polynomials

To begin with we will calculate $\langle x \rangle$ the "old way ", by directly integrating the probability density, which isn't too bad since we're only dealing with ψ_0 and ψ_1. We've already calculated the probability density [Eq. (45)], so we simply need to evaluate the integral $\int x|\Psi|^2 dx$,

$$\langle x(t) \rangle = \frac{2}{3}\int_{-\infty}^{\infty} x\psi_0^2 dx + \frac{1}{3}\int_{-\infty}^{\infty} x\psi_1^2 dx + \frac{2\sqrt{2}}{3}\cos(\omega t)\int_{-\infty}^{\infty} x\psi_0\psi_1 dx. \tag{47}$$

We know that ψ_0^2 and ψ_1^2 are *even* (in fact, the square of all SHO eigenstates is even), which means $x\psi_0^2$ and $x\psi_1^2$ are *odd*, and so we can immediately conclude

$$\int_{-\infty}^{\infty} x\psi_0^2 dx = \int_{-\infty}^{\infty} x\psi_1^2 dx = 0. \tag{48}$$

However the final term has even integrand, which is straight forward to evaluate using the supplied integrals:

$$\begin{aligned} \int_{-\infty}^{\infty} x\psi_0\psi_1 dx &= 2\int_0^{\infty} x\psi_0\psi_1 dx & (49)\\ &= \frac{2\sqrt{2}}{\sqrt{\pi} l_{\text{ho}}^2}\int_0^{\infty} x^2 e^{-(x/l_{\text{ho}})^2} dx & (50)\\ &= \frac{2\sqrt{2}}{\sqrt{\pi} l_{\text{ho}}^2} \times 2\sqrt{\pi}\frac{l_{\text{ho}}^3}{2^3} = \frac{l_{\text{ho}}}{\sqrt{2}} & (51) \end{aligned}$$

which gives

$$\langle x(t) \rangle = \frac{2l_{\text{ho}}}{3}\cos(\omega t) = \frac{2}{3}\sqrt{\frac{\hbar}{m\omega}}\cos(\omega t) \tag{52}$$

Calculating $\langle x \rangle$ Using Ladder Operators

When using ladder operators it is useful to label the position with a hat $\hat{x}$ as we do for energy and momentum, to remind us that we need to be careful with ordering. The expectation value is $\langle \hat{x} \rangle = \int \Psi^* \hat{x} \Psi dx$. The wavefunction is given by Eq. (42), and so the integral is

$$\langle \hat{x}(t) \rangle = \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{3}}\psi_0 e^{i\omega t/2} + \sqrt{\frac{1}{3}}\psi_1 e^{i3\omega t/2}\right)(\hat{a}_+ + \hat{a}_-)\left(\sqrt{\frac{2}{3}}\psi_0 e^{-i\omega t/2} + \sqrt{\frac{1}{3}}\psi_1 e^{-i3\omega t/2}\right) dx \tag{53}$$

If we were to expand this out we'd get eight terms, however all but two are zero. For instance, $\hat{a}_-\psi_0 = 0$ [the energy is bounded from below, enforced by the property Eq. (3)]. Additionally, by orthogonality, any term with a ladder operator "sandwitched" between two eigenstates $\int \psi_m \hat{a}_\pm \psi_n dx = 0$ unless $m = n$ *after* the eigenstate ψ_n has been raised or lowered.

The two non-zero terms are

$$\int_{-\infty}^{\infty} \psi_0 \hat{a}_- \psi_1 dx = \int_{-\infty}^{\infty} \psi_0 \psi_0 dx = 1 \tag{54}$$

$$\int_{-\infty}^{\infty} \psi_1 \hat{a}_+ \psi_0 dx = \int_{-\infty}^{\infty} \psi_1 \psi_1 dx = 1. \tag{55}$$

Putting this back in the integral (being careful to keep track of the complex exponentials when expanding out the brackets) gives

$$\langle \hat{x}(t) \rangle = \frac{\sqrt{2}}{3}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{i\omega t} + e^{-i\omega t}\right) \tag{56}$$

$$= \frac{2}{3}\sqrt{\frac{\hbar}{m\omega}}\cos(\omega t) \tag{57}$$

which agrees with the answer we already have. Notice that this method did not require any integrals over Hermite-Gauss polynomials, which is generally quite tedious.

Calculating $\langle \hat{p} \rangle$ Using Ladder Operators

Ordinarily to find the mean momentum we'd have to differentiate Ψ, which is tedious for the Hermite-Gauss polynomial SHO eigenstates. Using ladder operators the mean momentum is just

$$\langle \hat{p}(t) \rangle = i\sqrt{\frac{\hbar m\omega}{2}}\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{3}}\psi_0 e^{i\omega t/2} + \sqrt{\frac{1}{3}}\psi_1 e^{i3\omega t/2}\right)(\hat{a}_+ - \hat{a}_-)\left(\sqrt{\frac{2}{3}}\psi_0 e^{-i\omega t/2} + \sqrt{\frac{1}{3}}\psi_1 e^{-i3\omega t/2}\right) dx \tag{58}$$

As with $\langle \hat{x} \rangle$ the only two non-zero terms when we expand out the brackets are $\int \psi_0 \hat{a}_- \psi_1 dx = 1$ and $\int \psi_1 \hat{a}_+ \psi_0 dx = 1$. The result is

$$\langle \hat{p}(t) \rangle = i\frac{\sqrt{2}}{3}\sqrt{\frac{\hbar m\omega}{2}}\left(e^{i\omega t} - e^{-i\omega t}\right) \tag{59}$$

$$= -\frac{2}{3}\sqrt{\hbar m\omega}\sin(\omega t) \tag{60}$$

which also agrees with $\langle \hat{p}(t) \rangle = md\langle \hat{x}(t) \rangle / dt$.

(d) At later times, the mean energy could be calculated by evaluating $\langle \hat{H} \rangle = \int \Psi^*(x,t)\hat{H}\Psi(x,t)dx$. However, the Hamiltonian $\hat{H} = -\hbar^2/2m\partial^2/\partial x^2 + V(x)$ is independent of time, so the measurement outcomes and mean energy are the same as at $t = 0$.

Thus, the answer at time t is what we calculated in part (a) for $t = 0$. Measurements would yield E_0 with $P_0 = 2/3$ and E_1 with $P_1 = 1/3$ and so the mean is $\langle \hat{H} \rangle = 5\hbar\omega/6$.

Problem 5.5

(a) $\hat{H}(\hat{a}_-\psi_n) = \underbrace{\hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2})}_{\hat{H}}(\hat{a}_-\psi_n)$

We want to move $\hat{a}_-$ all the way to the left of $\hat{H}$ (to obtain something like $\hat{a}_-\hat{H}\psi_n$ + extra term(s) from the commutator $[\hat{a}_-,\hat{a}_+]=1$ when we swap $\hat{a}_+\hat{a}_-$ to $\hat{a}_-\hat{a}_+$). Then, in $\hat{a}_-\hat{H}\psi_n$ we can use $\hat{H}\psi_n = E_n\psi_n$

$\hookrightarrow E_n = \hbar\omega(n+\frac{1}{2})$

So:

$$H\hat{a}_-\psi_n = \hbar\omega(\hat{a}_+\hat{a}_- + \tfrac{1}{2})\hat{a}_-\psi_n = \hbar\omega\underbrace{\hat{a}_+\hat{a}_-}\hat{a}_-\psi_n + \tfrac{1}{2}\hbar\omega\hat{a}_-\psi_n$$

use

$$[\hat{a}_-,\hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = 1$$

$$\therefore \hat{a}_+\hat{a}_- = \hat{a}_-\hat{a}_+ - 1$$

$$= \hbar\omega(\hat{a}_-\hat{a}_+ - 1)\hat{a}_-\psi_n + \tfrac{1}{2}\hbar\omega\hat{a}_-\psi_n$$

$$= \underline{\hbar\omega\hat{a}_-\hat{a}_+\hat{a}_-\psi_n} - \hbar\omega\hat{a}_-\psi_n + \underline{\hat{a}_-\tfrac{1}{2}\hbar\omega\psi_n}$$

$$= \hat{a}_-\underbrace{\hbar\omega(\hat{a}_+\hat{a}_- + \tfrac{1}{2})}_{=\hat{H}}\psi_n - \hbar\omega\hat{a}_-\psi_n$$

$$= \hat{a}_-\hat{H}\psi_n - \hbar\omega\hat{a}_-\psi_n = \hat{a}_-E_n\psi_n - \hbar\omega\hat{a}_-\psi_n$$

$$= E_n\hat{a}_-\psi_n - \hbar\omega\hat{a}_-\psi_n = (E_n - \hbar\omega)(\hat{a}_-\psi_n)$$

Thus $\boxed{\hat{H}(\hat{a}_-\psi_n) = (E_n - \hbar\omega)(\hat{a}_-\psi_n)}$

(b) $\hat{a}_- \psi_0(x) = 0$

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}\left(i\hat{p} + m\omega\hat{x}\right)$$

In coordinate representation $\hat{x} = x$, $\hat{p} = -i\hbar\frac{d}{dx}$

$$\therefore\ \hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}\left(i(-i)\hbar\frac{d}{dx} + m\omega x\right) = \frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)$$

$\therefore\ \hat{a}_0 \psi_0(x) = 0$ is equivalent to

$$\frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\psi_0(x) = 0$$

or $\boxed{\frac{d\psi_0(x)}{dx} + \frac{m\omega}{\hbar}x\psi_0 = 0}$ 1st order differential equation.

To solve it:

$$\frac{d\psi_0(x)}{dx} = -\frac{m\omega}{\hbar}x\psi_0(x)$$

or $$\frac{d\psi_0(x)}{\psi_0(x)} = -\frac{m\omega}{\hbar}x\,dx$$

Integrate (indefinite integrals)

$$\int\frac{d\psi_0(x)}{\psi_0(x)} = -\frac{m\omega}{\hbar}\int x\,dx$$

$$\ln\psi_0(x) = -\frac{m\omega}{2\hbar}x^2 + \text{constant}(C)$$

exponentiate

$$\psi_0(x) = e^{-\frac{m\omega}{2\hbar}x^2 + C} = \underbrace{e^C}_{\text{def}\equiv A} e^{-\frac{m\omega}{2\hbar}x^2} = Ae^{-\frac{m\omega}{2\hbar}x^2}$$

— new constant

Thus $\boxed{\psi_0(x) = Ae^{-\frac{m\omega}{2\hbar}x^2}}$

Normalize $\psi_0(x)$ to find A :

$$1 = \int_{-\infty}^{+\infty} |\psi_0(x)|^2 \, dx = |A|^2 \int_{-\infty}^{+\infty} \underbrace{e^{-\frac{\cancel{2} m\omega}{\cancel{2}\hbar} x^2}}_{\text{even fn}} \, dx$$

$$= |A|^2 \, 2 \int_0^{\infty} e^{-\frac{m\omega}{\hbar} x^2} \, dx = |A|^2 \sqrt{\frac{\pi\hbar}{m\omega}}$$

$$\left(\text{use } \int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\right)$$

$$\therefore \; |A|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}$$

$$\therefore \; |A| = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

→ can choose A to be real

$A = |A| e^{i\varphi}$ → phase does not matter here

$$\therefore \; \boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}}$$

(c) $\hat{a}_+ \psi_0(x)$ — ?

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{p} + m\omega\hat{x}\right)$$ — definition

In coordinate representation $\left(\hat{p} = -i\hbar\frac{d}{dx}\right)$

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}\left(-i(-i\hbar)\frac{d}{dx} + m\omega x\right)$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)$$

$$\therefore \hat{a}_+\psi_0(x) = \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\hbar\frac{d}{dx}\left(e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x e^{-\frac{m\omega}{2\hbar}x^2}\right)$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left((-1)\hbar\left(-\frac{m\omega}{\cancel{2}\hbar}\right)\cancel{2}x e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x e^{-\frac{m\omega}{2\hbar}x^2}\right)$$

$$= \frac{2m\omega}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} x e^{-\frac{m\omega}{2\hbar}x^2}$$

Thus $\psi_1(x) = \hat{a}_+\psi_0(x) =$ (from $\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_n$ for $n=0$)

$$= \sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} x e^{-\frac{m\omega}{2\hbar}x^2}$$

Same as $\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2}}H_1(\xi)e^{-\xi^2/2}$, where

$H_1(\xi) = 2\xi$, with $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$