MATH3403 Assignment 1

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Question 1 (5 Marks)

To show that the operator $\mathcal{L}[u] = \partial_t u + u\partial_x u$ is not linear, consider the same operator under the variable $\alpha f + \beta g$:

$$\begin{aligned}\mathcal{L}[\alpha f + \beta g] &= \partial_t(\alpha f + \beta g) + (\alpha f + \beta g)\partial_x(\alpha f + \beta g) \\ &= \alpha\partial_t f + \beta\partial_t + (\alpha^2 f\partial_x f + \alpha\beta f\partial_x g + \alpha\beta g\partial_x f + \beta^2 g\partial_x g) \\ &= \alpha(\partial_t f + \alpha f\partial_x f + \beta f\partial_x g + \beta g\partial_x f) + \beta(\partial_t g + \beta g\partial_x g) \\ &\neq \alpha(\partial_t f + f\partial_x f) + \beta(\partial_t g + g\partial_x g) = \alpha\mathcal{L}[f] + \beta\mathcal{L}[g]\end{aligned}$$

Hence the condition of linearity was not met and the operator is not linear. On inspection, the u term before $\partial_x u$ in $\mathcal{L}[u]$ shows from the start that the operator is not linear.

Question 2 (5 Marks)

The equation

$$\begin{aligned}\partial_{tt} f - 2\partial_{xx} f + \frac{1}{3}\partial_{xy} f - \epsilon\partial_{yy} f &= -x^2 \\ \partial_{tt} f &= 2\partial_{xx} f - \frac{1}{3}\partial_{xy} f + \epsilon\partial_{yy} f - x^2\end{aligned}$$

is a hyperbolic PDE if the determinant of its coefficient matrix of $\mathcal{L}[u] = 2\partial_{xx} f - \frac{1}{3}\partial_{xy} f + \epsilon\partial_{yy} f$ is greater than 0. Plugging in the numbers gives

$$\begin{aligned}\det\begin{pmatrix} 2 & -\frac{1}{6} \\ -\frac{1}{6} & \epsilon \end{pmatrix} &= 2\epsilon - \left(-\frac{1}{6}\right)^2 > 0 \\ &\Rightarrow 2\epsilon > \frac{1}{36} \\ &\Rightarrow \epsilon > \frac{1}{72}\end{aligned}$$

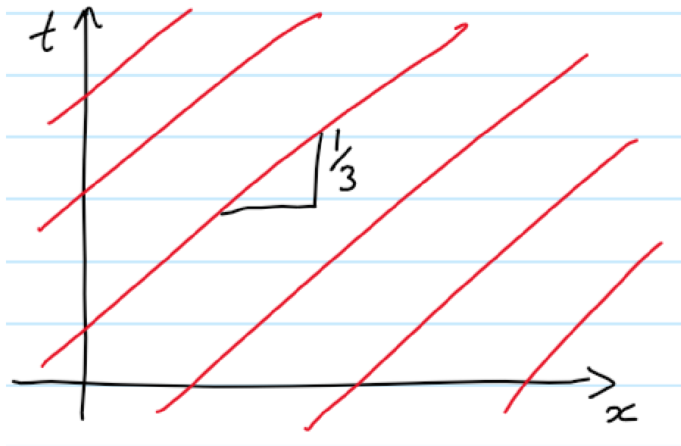
Therefore, for the aforementioned PDE to be hyperbolic, epsilon must be greater than 1/72.

Question 3 (10 Marks)

For the system

$$\begin{cases} \partial_t u + 3\partial_x u + u^2 = 0 & t \geq 0,\, x \geq 0 \\ u(t=0,x) = \cos(x) \end{cases}$$

Define $v(t,\xi)$ and $\xi = x - ct$, where $c = 3 \Rightarrow \xi = x - 3t$ such that $u(t,x) = v(t,\xi)$. The system has characteristic lines



Using the chain rule from multivariate calculus (as per the Peter Olver - Intro to PDEs):

$$\partial_t u = \partial_t v - c\partial_\xi v \qquad \partial_x u = \partial_\xi v$$

Applying this to the system in question gives

$$\begin{aligned} \partial_t u &= \partial_t v - 3\partial_\xi v \\ \partial_x u &= \partial_\xi v \end{aligned}$$

Substituting these back into the original PDE gives

$$\begin{aligned} \partial_t v - 3\partial_\xi v + 3\partial_\xi + v^2 = 0 \\ \Rightarrow \partial_t v = -v^2 \end{aligned}$$

Which gives a new system,

$$\begin{cases} \partial_t v = -v^2 \\ v(t=0,\xi) = \cos(\xi) \end{cases}$$

Which can then be solved for v, and hence u,

$$\begin{aligned}
-\frac{dv}{v^2} &= dt \\
-\int_{v(0,\xi)}^{v(t,\xi)} \frac{1}{v^2} &= \int_0^t dt \\
\frac{1}{v}\bigg|_{v(0,\xi)}^{v(t,\xi)} &= t \\
\frac{1}{v(t,\xi)} - \frac{1}{v(0,\xi)} &= t \\
\frac{1}{v} &= t + \frac{1}{\cos(\xi)} \\
\Rightarrow v &= \frac{1}{t + \frac{1}{\cos(\xi)}} \\
&= \frac{\cos(\xi)}{t\cos(\xi) + 1}
\end{aligned}$$

Since $u(t,x) = v(t,\xi)$ and $\xi = x - 3t$,

$$u(t,x) = \frac{\cos(x-3t)}{t\cos(x-3t)+1}$$

Question 4 (15 Marks)

The system,

$$\begin{cases} \partial_x u + \sin(x)\partial_y u = y \qquad x,\, y \in \mathbb{R} \\ u(x=0,y) = 0 \end{cases}$$

has vectors

$$\vec{V} = \begin{pmatrix} 1 \\ \sin(x) \end{pmatrix} \qquad \vec{\Gamma} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

Performing a change of variables $(x,y) \mapsto (s,r)$,

$$\left.\begin{cases} x_s = 1 \\ \\ x(0,r) = 0 \end{cases} \Rightarrow \begin{array}{c} x(s,r) = s + A(r) \\ \\ x(0,r) = 0 = A(r) \end{array}\right\} \Rightarrow x(s,r) = s$$

$$\left.\begin{cases} y_s = \sin(s) \\ \\ y(0,r) = r \end{cases} \Rightarrow \begin{array}{c} y(s,r) = -\cos(s) + B(r) \\ \\ y(0,r) = r = B(r) \end{array}\right\} \Rightarrow y(s,r) = r - \cos(s)$$

With $x(s,r) = s$ and $y(s,r) = r - \cos(s) \Rightarrow r = y + \cos(x)$
Define $\tilde{u}(s,r) = u(x(s,r), y(s,r))$, which is expanded to yield

$$\begin{aligned} \tilde{u}_s &= \partial_x u \frac{\partial x}{\partial s} + \partial_y u \frac{\partial y}{\partial s} \\ &= y = r - \cos(s) \end{aligned}$$

This can then be solved to solve the initial problem for $u(x,y)$:

$$\left.\begin{cases} \tilde{u}_s = r - \cos(s) \\ \\ \tilde{u}(s=0,r) = 0 \end{cases} \Rightarrow \begin{array}{c} \tilde{u}_s = rs - \sin(s) + c \\ \\ \tilde{u}(s=0,r) = 0 = c \end{array}\right\} \Rightarrow \tilde{u}(s,r) = rs - \sin(s)$$

$$\begin{aligned} u(x,y) &= \tilde{u}(s(x,y), r(x,y)) \\ &= \tilde{u}(x, y + \cos(x)) \\ &= (y + \cos(x))x - \sin(x) \\ &= xy + x\cos(x) - \sin(x) \end{aligned}$$

And so the initial PDE has the solution $u(x,y) = xy + x\cos(x) - \sin(x)$