MATH2001 Assignment 1

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Question 1

a. To test for exactness, the functions P and Q can be differentiated with respect to y and x respectively. If they are equal as in equation (1), the parent separable equation is exact.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{1}$$

For the separable equation of the form (2), the functions P and Q are comprised of the x and y variables respectively only, as $P(x)$ and $Q(y)$:

$$P(x) + Q(y)\frac{dy}{dx} = 0 \tag{2}$$

When differentiated according to equation (1), each sides yields a result of 0 due to the absence of the differentiated variable within the functions.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0$$

And since they are equal, any separable function of the form (3) is exact per the test for exactness.

b. Consider the equation

$$4x^3 - 2y\frac{dy}{dx} = 0 \tag{3}$$

This is of the form of equation (2), with $P(x) = 4x^3$, so

$$\frac{\partial f}{\partial x} = 4x^3 \Rightarrow f = \int 4x^3 = x^4 + g(y) \tag{4}$$

and $Q(y) = -2y$, but

$$\frac{\partial f}{\partial y} = -2y + g'(y) = Q(y) = -2y$$

So $g'(y) = 0$, hence $g(y)$ must be a constant. Let $g(y) = k$.

$$\Rightarrow f = \int -2y = -y^2 + c \Rightarrow f(x,\, y) = x^4 - y^2 + k + c$$

But $f(x,\, y) =$ const. Let $f(x, y) - k - c = A$. Then,

$$A = x^4 - y^2$$
$$\Rightarrow y = \sqrt{x^4 - A}$$

Which satisfies equation (1).

Question 2

Consider the linear, non-homogeneous, second order differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin(e^x)$$

The general solution of which is of the form $y = y_H + y_P$ where y_H is the homogeneous solution and y_P is the particular solution. For y_H, the equation is expressed as a homogeneous equation in terms of lambda, with

$$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow (\lambda + 1)(\lambda + 2) = 0$$

This yields the roots $\lambda_1 = -1$ and $\lambda_2 = -2$. Since the roots are not equal, the homogeneous solution is of the form $y_H = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$. Substituting in the values for the roots gives the homogeneous solution

$$y_H = Ae^{-x} + Be^{-2x} \tag{5}$$

Let $y_1 = e^{-x}$ and $y_2 = e^{-2x}$. The Wronskian is defined by $W = y_1 y_2' - y_1' y_2$. Substituting in the values for y_1 and y_2 and expanding gives

$$\begin{aligned} W &= y_1 y_2' - y_1' y_2 \\ &= e^{-x} \cdot -2e^{-2x} - -e^{-x} e^{-2x} \\ &= -e^{-3x} \end{aligned}$$

The particular solution can be expressed as $y_P = u(x)y_1 + v(x)y_2$ where $u(x)$ and $v(x)$ are functions of the Wronskian:

$$u(x) = -\int \frac{y_2 r}{W} dx \qquad v(x) = \int \frac{y_1 r}{W} dx$$

where $r(x)$ is the non-homogeneous part of the differential equation, $r = \sin(e^x)$. Substituting all values into each,

$$\begin{aligned} u(x) &= -\int \frac{e^{-2x}\sin(e^x)}{-e^{-3x}} dx \\ &= \int e^x \sin(e^x) dx \\ u(x) &= -\cos(e^x) \end{aligned}$$

$$\begin{aligned} v(x) &= \int \frac{e^{-x}\sin(e^x)}{-e^{-3x}} dx \\ &= -\int e^{2x} \sin(e^x) dx \\ v(x) &= e^x \cos(e^x) - \sin(e^x) \end{aligned}$$

The particular solution can then be calculated,

$$\begin{aligned} y_P &= uy_1 + vy_2 \\ &= -e^{-x}\cos e^x + e^{-x}\cos(e^x) - e^{-2x}\sin(e^x) \\ &= -e^{-2x}\sin(e^x) \end{aligned}$$

And finally the general solution,

$$\begin{aligned} y &= y_H + y_P \\ &= Ae^{-x} + Be^{-2x} - e^{-2x}\sin(e^x) \\ y &= Ae^{-x} + e^{-2x}(B - \sin(e^x)) \end{aligned}$$

Question 3

Consider the matrix A, defined by

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 & -2 \\ -3 & 3 & -3 & -6 & 9 \\ 2 & 6 & -2 & 0 & 2 \\ 2 & 0 & 4 & 6 & -4 \\ 1 & -3 & 4 & 5 & -5 \end{pmatrix}$$

To get A in row-echelon-form, linear and scalar matrix row operations were used:

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 & -2 \\ -3 & 3 & -3 & -6 & 9 \\ 2 & 6 & -2 & 0 & 2 \\ 2 & 0 & 4 & 6 & -4 \\ 1 & -3 & 4 & 5 & -5 \end{pmatrix}$$

$$\begin{array}{r} \xrightarrow{R_2 \to R_2 + 3R_2} \\ \xrightarrow{R_3 \to R_3 - 2R_1} \\ \xrightarrow{R_4 \to R_4 - 2R_1} \\ \xrightarrow{R_5 \to R_5 - R_1} \end{array} \begin{pmatrix} 1 & 0 & 2 & 3 & -2 \\ 0 & 3 & 3 & 3 & 3 \\ 0 & 6 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 2 & 2 & -3 \end{pmatrix}$$

$$\begin{array}{r} \xrightarrow{R_3 \to R_3 - 2R_2} \\ \\ \xrightarrow{R_5 \to R_5 + R_2} \end{array} \begin{pmatrix} 1 & 0 & 2 & 3 & -2 \\ 0 & 3 & 3 & 3 & 3 \\ 0 & 0 & -12 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 5 & 0 \end{pmatrix}$$

$$\xrightarrow{R_5 \to R_5 + \frac{5}{12} R_3} \begin{pmatrix} 1 & 0 & 2 & 3 & -2 \\ 0 & 3 & 3 & 3 & 3 \\ 0 & 0 & -12 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(r.e.f.)}$$

Since matrix A is in row echelon form, $\text{rank}(A) = 3$, and the column space is comprised of the first 3 columns. The basis for the column space of A is thus

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 6 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -2 \\ 4 \\ 4 \end{bmatrix} \right\}$$

Question 4

Consider the inner product space $M_{2,2}(\mathbb{R})$ with inner product

$$\langle A, B\rangle = \operatorname{tr}(B^T A), \qquad A, B \in M_{2,2}(\mathbb{R})$$

Let

$$\beta = \left\{ \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{55}} & -\frac{1}{\sqrt{55}} \\ \frac{5}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix} \right\}$$

a. β is orthonormal if $\langle \beta_i, \beta_j\rangle = \delta_{ij}$ where

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

If β is orthonormal, expect $\langle \underline{\beta}_1, \underline{\beta}_1\rangle = \langle \underline{\beta}_2, \underline{\beta}_2\rangle = 1$, and $\langle \underline{\beta}_1, \underline{\beta}_2\rangle = 0$. Calculating each,

$$\begin{aligned}
\left\langle \underline{\beta}_1, \underline{\beta}_1 \right\rangle &= \operatorname{tr}\left(\beta_1{}^T \beta_1\right) \\
&= \operatorname{tr}\left(\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix} \right) \\
&= \operatorname{tr}\left(\begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix} \right) \\
&= \operatorname{tr}\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} \\
&= \frac{1}{5} + \frac{4}{5} = 1
\end{aligned}$$

$$\begin{aligned}
\left\langle \underline{\beta}_2, \underline{\beta}_2 \right\rangle &= \operatorname{tr}\left(\beta_2{}^T \beta_2\right) \\
&= \operatorname{tr}\left(\begin{pmatrix} \frac{2}{\sqrt{55}} & -\frac{1}{\sqrt{55}} \\ \frac{5}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix}^T \begin{pmatrix} \frac{2}{\sqrt{55}} & -\frac{1}{\sqrt{55}} \\ \frac{5}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix} \right) \\
&= \operatorname{tr}\left(\begin{pmatrix} \frac{2}{\sqrt{55}} & \frac{5}{\sqrt{55}} \\ -\frac{1}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{55}} & -\frac{1}{\sqrt{55}} \\ \frac{5}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix} \right) \\
&= \operatorname{tr}\begin{pmatrix} \frac{29}{55} & -\frac{27}{55} \\ -\frac{27}{55} & \frac{26}{55} \end{pmatrix} \\
&= \frac{29}{55} + \frac{26}{55} = 1
\end{aligned}$$

$$\begin{aligned}
\left\langle \underline{\beta}_1, \underline{\beta}_2 \right\rangle &= \operatorname{tr}\left(\beta_2{}^T \beta_1\right) \\
&= \operatorname{tr}\left(\begin{pmatrix} \frac{2}{\sqrt{55}} & -\frac{1}{\sqrt{55}} \\ \frac{5}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix}^T \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix} \right) \\
&= \operatorname{tr}\left(\begin{pmatrix} \frac{2}{\sqrt{55}} & \frac{5}{\sqrt{55}} \\ -\frac{1}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix} \right) \\
&= \operatorname{tr}\begin{pmatrix} \frac{2}{5\sqrt{11}} & \frac{4}{5\sqrt{11}} \\ -\frac{1}{5\sqrt{11}} & -\frac{2}{5\sqrt{11}} \end{pmatrix} \\
&= \frac{2}{5\sqrt{11}} - \frac{2}{5\sqrt{11}} = 0
\end{aligned}$$

And so, as expected, β is orthonormal.

b. The elements of β form an orthonormal basis, as they are each linearly independent and span β. So, the orthogonal projection of the identity matrix in $M_{2,2}(\mathbb{R})$ onto span(β) is

$$\begin{aligned}
\mathrm{Proj}_\beta(I) &= \langle I,\, \beta_1\rangle\, \beta_1 + \langle I,\, \beta_2\rangle\, \beta_2 \\
&= \mathrm{tr}({\beta_1}^T I)\beta_1 + \mathrm{tr}({\beta_2}^T I)\beta_2 \\
&= \mathrm{tr}\left(\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix} + \mathrm{tr}\left(\begin{pmatrix} \frac{2}{\sqrt{55}} & -\frac{1}{\sqrt{55}} \\ \frac{5}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} \frac{2}{\sqrt{55}} & -\frac{1}{\sqrt{55}} \\ \frac{5}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix} \\
&= \mathrm{tr}\left(\begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix} + \mathrm{tr}\left(\begin{pmatrix} \frac{2}{\sqrt{55}} & \frac{5}{\sqrt{55}} \\ -\frac{1}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} \frac{2}{\sqrt{55}} & -\frac{1}{\sqrt{55}} \\ \frac{5}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix} \\
&= \mathrm{tr}\begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix} + \mathrm{tr}\begin{pmatrix} \frac{2}{\sqrt{55}} & \frac{5}{\sqrt{55}} \\ -\frac{1}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{55}} & -\frac{1}{\sqrt{55}} \\ \frac{5}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & 0 \end{pmatrix} + \frac{-3}{\sqrt{55}} \begin{pmatrix} \frac{2}{\sqrt{55}} & -\frac{1}{\sqrt{55}} \\ \frac{5}{\sqrt{55}} & -\frac{5}{\sqrt{55}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{6}{55} & \frac{3}{55} \\ -\frac{15}{55} & \frac{15}{55} \end{pmatrix} \\
\mathrm{Proj}_\beta(I) &= \begin{pmatrix} \frac{5}{55} & \frac{25}{55} \\ -\frac{15}{55} & \frac{15}{55} \end{pmatrix} = \begin{pmatrix} \frac{1}{11} & \frac{5}{11} \\ -\frac{3}{11} & \frac{3}{11} \end{pmatrix}
\end{aligned}$$