Chaos Assignment 1

Tuesday, 12 October 2021 8:40 AM

Q1 $H = \frac{p^2}{2} + \frac{a}{2}q^2 - \frac{1}{4}q^4$

a. $\dot{q} = \frac{\partial H}{\partial p} = p$

$\dot{p} = -\frac{\partial H}{\partial q} = q^3 - aq$

b. There are fixed points at $(\dot{q}, \dot{p}) = (0,0)$

$\Rightarrow 0 = \dot{q} = p \quad \Rightarrow p = 0$

$0 = \dot{p} = q^3 - aq \Rightarrow q^3 = aq$
or $q^2 = a \Rightarrow q = \pm\sqrt{a}$

Therefore, there are fixed points at $(\sqrt{a}, 0)$, $(-\sqrt{a}, 0)$ and $(0,0)$

The energy at these fixed points is

$H(\sqrt{a}, 0) = 0 + \frac{a}{2}(\sqrt{a})^2 - \frac{1}{4}(\sqrt{a})^4$
$= \frac{a^2}{2} - \frac{1}{4}a^2 = \frac{1}{4}a^2$

$H(-\sqrt{a}, 0) = \frac{1}{4}a^2$ (as above)

$H(0,0) = 0$

These points can be classified by the double derivative test:

$(\sqrt{a}, 0)$: $\left(\frac{\partial^2 H}{\partial q^2}, \frac{\partial^2 H}{\partial p^2}\right) = (-2a, 1)$

$\Rightarrow (\sqrt{a}, 0)$ is a saddle since $a > 0$ (for $\sqrt{a}$ to be real)

$(-\sqrt{a}, 0)$: $\left(\frac{\partial^2 H}{\partial q^2}, \frac{\partial^2 H}{\partial p^2}\right) = (-2a, 1)$

$\Rightarrow (-\sqrt{a}, 0)$ is analogous to $(\sqrt{a}, 0)$

$(0,0)$: $\left(\frac{\partial^2 H}{\partial q^2}, \frac{\partial^2 H}{\partial p^2}\right) = (a, 1)$

$\Rightarrow (0,0)$ is a minimum if $a > 0$
max if $a < 0$

But a must be positive for real critical points [$(\sqrt{a}, 0)$ and $(-\sqrt{a}, 0)$], so $(0,0)$ is a minimum.

c. Separatrices have $H = \frac{1}{4}a^2$ (for unstable crit points)

$\Rightarrow \frac{1}{4}a^2 = \frac{1}{2}p^2 + \frac{a}{2}q^2 - \frac{1}{4}q^4$

$p^2 = \frac{1}{2}q^4 - aq^2 + \frac{1}{2}a^2$

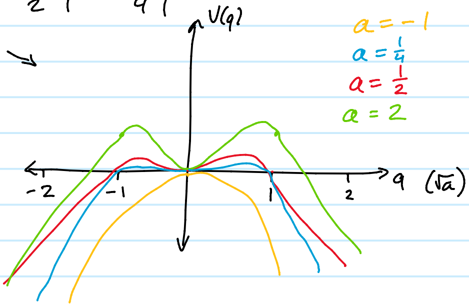
$p = \pm\sqrt{\frac{1}{2}q^4 - aq^2 + \frac{1}{2}a^2}$

Close to $q = 0$, this is approximately a

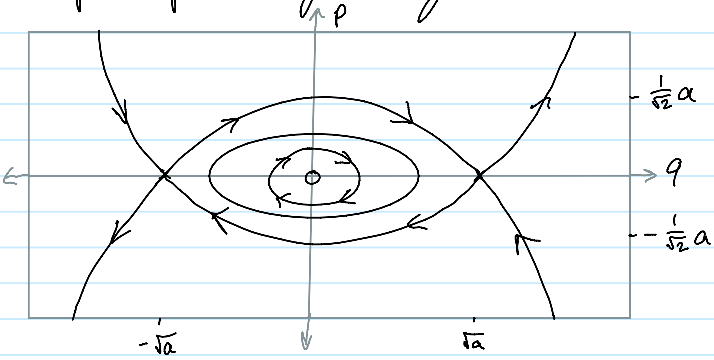
Close to $q=0$, this is approximately a straight line. ($p \simeq \pm \frac{1}{\sqrt{2}} a$)
As $|q| \to \infty$, $p \to \pm\infty$ due to the q term out the front.

d. Since a must be positive, the potential
$$V(q) = \frac{a}{2} q^2 - \frac{1}{4} q^4$$
has graph for different values of a →

Working:
$V''(q) = a - 3q$



The phase portrait of the system is then



Recall $\dot{q} = p$ and $\dot{p} = q^3 - aq$
The direction at each region is

Region	$\dot{q}$	$\dot{p}$	Direction $(\dot{q}, \dot{p})$
$(q \le -\sqrt{a}, p)$	>0	<0	$(+,-)$
$(-\sqrt{a} \le q \le 0, p)$	>0	>0	$(+,+)$
$(q \le -\sqrt{a}, -p)$	<0	<0	$(-,-)$
$(-\sqrt{a} \le q \le 0, -p)$	<0	>0	$(-,+)$
$(0 \le q \le \sqrt{a}, p)$	>0	<0	$(+,-)$
$(\sqrt{a} \le q, p)$	>0	>0	$(+,+)$
$(0 \le q \le \sqrt{a}, -p)$	<0	<0	$(-,-)$
$(\sqrt{a} \le q, -p)$	<0	>0	$(-,+)$

For all a, the saddle points $[(q,p) = (\pm\sqrt{a}, 0)]$ will move further away from the origin as a increases
If a<0, there would be a saddle point at (0,0), since d^2 H /dp^2 = 1, and d^2 H/dq^2 = a < 0 => double derivative is (-, +) => saddle point. For a<0, there is only one fixed point: the saddle at (0,0).

e. Recall that the potential function is
$$V(q) = \frac{a}{2} q^2 - \frac{1}{4} q^4$$
with
$$V'(q) = aq - q^3$$
$$V''(q) = a - 3q^2$$

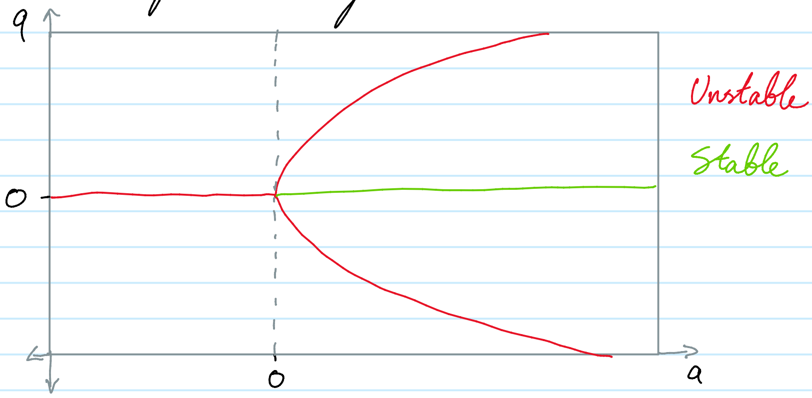
This has fixed points at $V'(q) = 0$

$\Rightarrow \quad a = q^2$

With solutions $q=0$ and $q = \pm\sqrt{a}$

For $a<0$, there is one real fixed point at $q=0$
For $a>0$, there are three real fixed points (as above)
As from the potential diagram in part d,
$q=0$ is unstable for $a<0$, and stable for $a>0$. $q=\pm\sqrt{a}$ is unstable for $a>0$

The bifurcation diagram is then



Q2

$$H = mc^2\sqrt{1+\left(\frac{p}{mc}\right)^2} + \frac{1}{2}m\omega^2 q^2$$

a. Hamiltons equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{mc^2}{2}\left(1+\frac{p^2}{m^2c^2}\right)^{-\frac{1}{2}} \cdot \frac{2}{m^2c^2}p$$

$$= \frac{p}{m\sqrt{1+\left(\frac{p}{mc}\right)^2}}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q$$

The above is for the relativistic case. Below shows the non-relativistic case:

$$H_{nr} = mc^2 + \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

$$\Rightarrow \quad \dot{q}_{nr} = \frac{\partial H_{nr}}{\partial p} = \frac{p}{m}$$

$$\dot{p}_{nr} = -\frac{\partial H_{nr}}{\partial q} = -m\omega^2 q$$

Clearly, $\dot{p}_{nr} = \dot{p}$ indicating that change in momentum is independent of relativistic effects.
$\dot{q}_{nr}$ and $\dot{q}$ differ by a factor of $1/\sqrt{1+\left(\frac{p}{mc}\right)^2}$ which corresponds to the relativistic lorentz factor γ:

$$\frac{1}{\sqrt{1+\frac{p^2}{m^2c^2}}} = \frac{1}{\sqrt{1+\frac{v^2}{c^2}}} = \gamma$$

The physical interpretation of this is that as $|p| \to \infty$, $v \to c$, and since $v = \dot{q}$, $\dot{q} \to c$.
So while momentum can approach infinity, velocity approaches only c.

b. Fixed point when $(\dot{q}, \dot{p}) = (0,0)$

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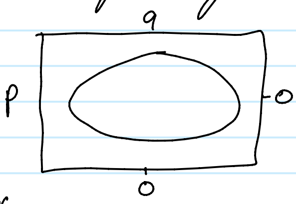
$$\dot{q} = 0 = \frac{p}{m\sqrt{1+\left(\frac{p}{mc}\right)^2}} \Rightarrow p = 0$$

$$\dot{p} = 0 = -m\omega^2 q \Rightarrow q = 0$$

Therefore there is a fixed point at $(q,p) = (0,0)$

c. For a physical (bounded by $\dot{q} < c$) orbit, the maximum value of $|p|$, p_{max}, can be found by setting $q = 0$. Mathematically, the q term will be negated in order to eventually find p, so the q term should be minimised. Intuitively, p_{max} occurs at $q=0$ as per the 'phase' portrait for a stable orbit above.



At $q=0$ and $p = p_{max}$,

$$H = mc^2\sqrt{1 + \frac{p_{max}^2}{m^2c^2}} + 0$$

$$\Rightarrow 1 + \frac{p_{max}^2}{m^2c^2} = \frac{H^2}{m^2c^4}$$

$$p_{max}^2 = \frac{H^2}{c^2} - m^2c^2$$

$$= \frac{H^2 - m^2c^4}{c^2}$$

$$= \frac{(H+mc^2)(H-mc^2)}{c^2}$$

$$= \frac{(H+mc^2)\Delta H}{c^2}$$

$$= \frac{(H - mc^2 + 2mc^2)\Delta H}{c^2}$$

$$= \frac{\Delta H(\Delta H + 2mc^2)}{c^2}$$

$$\Rightarrow p_{max} = \pm\sqrt{\frac{\Delta H(\Delta H + 2mc^2)}{c^2}}$$

Taking the positive of the above (since we want a <u>maximum</u> value) gives

$$p_{max} = \sqrt{\frac{\Delta H(\Delta H + 2mc^2)}{c^2}} = \sqrt{\left(\frac{\Delta H}{c}\right)^2 + 2m\Delta H}$$

For the non-relativistic oscillator, take $q=0$ as before,

$$H_{nr} = mc^2 + \frac{p_{max}^2}{2m}$$

$$p_{max}^2 = 2m\Delta H_{nr}$$

$$p_{max} = \sqrt{2m\Delta H_{nr}} \quad \text{(positive taken as before)}$$

Clearly the relativistic case has an extra term $\left(\frac{\Delta H}{c}\right)^2$ inside the square root, indicating that the relativistic p_{max} is larger for some ΔH.

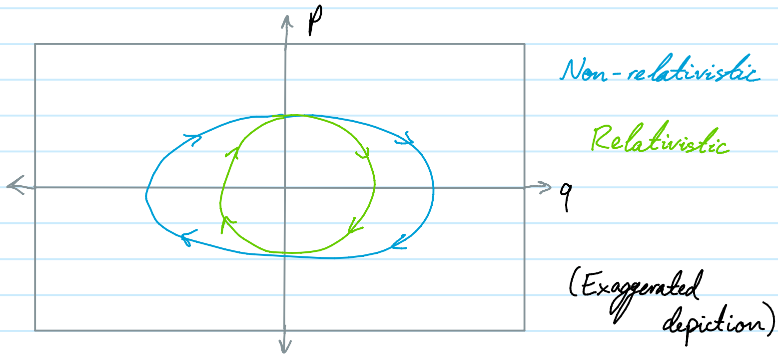
d. In the region $p > 0$, $\dot{q} > 0$
$p < 0$, $\dot{q} < 0$
so therefore an orbit rotates about the origin clockwise (in the non-relativistic too, since the sign is the same in each case).

$\dot{q}$ and $\dot{p}$ differ by a factor of $1/\sqrt{1+(p/\ldots)^2}$ (...

"clockwise" (in the non-relativistic too, since the sign is the same in each case).

$\dot{q}$ and $\dot{q}_{nr}$ differ by a factor of $1/\sqrt{1+(\frac{p}{mc})^2}$ (present in the relativistic case) so $\dot{q} < \dot{q}_{nr}$ for the same p and so the relativistic phase portrait will be 'squished' in the q axis.

The qualitative phase portrait is then



e. Recall $\dot{q} = \dfrac{p}{m\sqrt{1+(\frac{p}{mc})^2}}$ and $\dot{p} = -m\omega^2 q$

As $p \to +\infty$, $\dot{q} \to c$ by postulate of special relativity

$\Rightarrow q(t) = \int \dot{q}\, dt = \int c\, dt = ct + q_0$

where q_0 is some initial position.

Substituting this into $\dot{p} = -m\omega^2 q$ gives

$$\dot{p} = -m\omega^2 (ct + q_0)$$

$$\Rightarrow p = -m\omega^2 \int ct + q_0 \, dt$$

$$p(t) = -m\omega^2 \left(\tfrac{1}{2}ct^2 + q_0 t + k\right)$$

where k is some constant relating to the initial momentum at $t=0$.