Assignment 2

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Question 1:

$$y'' + y + \epsilon y^3 = 0 \quad ; \quad y(0) = 0, \; y'(0) = 1$$

for $t > 0$.

a. Want to find two term expansion for $\epsilon \ll 1$.
Want a solution of the form

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \ldots$$

$$\Rightarrow (y_0 + \epsilon y_1 + \ldots)'' + (y_0 + \epsilon y_1 + \ldots) + \epsilon (y_0 + \epsilon y_1 + \ldots)^3 = 0$$

$O(\epsilon^0)$ terms:

$$y_0'' + y_0 = 0 ; \qquad y_0(0) = 0 ; \; y_0'(0) = 1$$

$$\Rightarrow y_0(t) = A\sin(t) + B\cos(t)$$

$$y_0(0) = 0 \Rightarrow B = 0$$

$$y_0(t) = A\sin(t) \Rightarrow y_0'(t) = A\cos(t)$$

$$y_0'(0) = 1 \Rightarrow A = 1 \Rightarrow y_0(t) = \sin(t)$$

$O(\epsilon^1)$ terms:

$$y_1'' + y_1 + y_0^3 = 0$$

$$\Rightarrow y_1'' + y_1 + \sin^3(t) = 0$$

which has solution

$$y_1(t) = \left(\frac{3t}{8} + A\right)\cos(t) + B\sin(t) - \frac{1}{32}\sin(3t)$$

$$y_1(0) = 0 \Rightarrow A = 0$$

$$\Rightarrow y_1(t) = \frac{3t}{8}\cos(t) + B\sin(t) - \frac{1}{32}\sin(3t)$$

$$y_1'(t) = \frac{3}{8}\cos(t) - \frac{3t}{8}\sin(t) + B\cos(t) - \frac{3}{32}\cos(3t)$$

$$y_1'(0) = 0 \Rightarrow B = \frac{3}{32} - \frac{3}{8} = -\frac{9}{32}$$

$$\Rightarrow y_1(t) = \frac{1}{32}\left(12t\cos(t) - 9\sin(t) - \sin(3t)\right)$$

$\therefore$ two term solution is

$$y(t) = \sin(t) + \frac{\epsilon}{32}\left[12t\cos(t) - 9\sin(t) - \sin(3t)\right]$$

b. Now want to use method of multiple scales.
Introduce time variables

$$t_1 = t, \quad t_2 = \epsilon^\alpha t \qquad \Rightarrow y(t) = Y(t_1, t_2)$$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t_1} + \epsilon^\alpha \frac{\partial}{\partial t_2} \quad \Rightarrow \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t_1^2} + 2\epsilon^\alpha \frac{\partial^2}{\partial t_1 \partial t_2} + \epsilon^{2\alpha}\frac{\partial^2}{\partial t_2^2}$$

Hence rewrite Duffing eq. as

$$\frac{\partial^2 Y}{\partial t_1^2} + 2\epsilon^\alpha \frac{\partial^2 Y}{\partial t_1 \partial t_2} + \epsilon^{2\alpha}\frac{\partial^2 Y}{\partial t_2^2} + Y + \epsilon Y^3 = 0$$

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Initial conditions:

$$Y(0,0) = 0 \; ; \; \frac{\partial Y}{\partial t_1} + \epsilon^{\alpha}\frac{\partial Y}{\partial t_2}\Big|_{(t_1,t_2)=(0,0)} = 1$$

Now, substitute

$$Y(t_1,t_2) = Y_0(t_1,t_2) + \epsilon Y_1(t_1,t_2) + \dots$$

$O(\epsilon^0)$ terms:

$$\frac{\partial^2 Y_0}{\partial t_1^2} + Y_0 \; ; \quad Y_0(0,0) = 0 \, , \; \frac{\partial Y}{\partial t_1}(0,0) = 1$$

$$\Rightarrow Y_0(t_1,t_2) = A(t_2)\cos(t_1 + B(t_2))$$

A, B unknown functions with $A(0) = 1$, $B(0) = -\pi/2$

Choose $\alpha = 1$,

$O(\epsilon^1)$ terms:

$$\frac{\partial^2 Y_1}{\partial t_1^2} + 2\frac{\partial^2 Y_0}{\partial t_1 \partial t_2} + Y_1 + Y_0^3 = 0$$

$$\Rightarrow \frac{\partial^2 Y_1}{\partial t_1^2} + Y_1 = -2\frac{\partial^2 Y_0}{\partial t_1 \partial t_2} - Y_0^3$$

$$= -2\left[-A'\sin(t_1+B) - AB'\cos(t_1+B)\right] - A^3\cos^3(t_1+B)$$

Take $A' = 0$, and so the RHS can be rewritten as

$$2AB'\cos(t_1+B) - \frac{1}{4}A^3\left(3\cos(t_1+B) + \cos(3(t_1+B))\right)$$

Since $A(0) = 1$, and $A' = 0$, we must have $A = 1$ so RHS becomes

$$2B'\cos(t_1+B) - \frac{1}{4}\left(3\cos(t_1+B) + \cos(3(t_1+B))\right)$$

So, to eliminate $\cos(t_1+B)$, take

$$2B' = \frac{3}{4} \quad \Rightarrow B' = \frac{3}{8} \quad \Rightarrow B = \frac{3t_2}{8} + c$$

By initial conditions $B(0) = -\pi/2$, we get that

$$B = \frac{3t_2}{8} - \pi/2$$

So, $Y_0(t_1,t_2) = \cos\left(t_1 + \frac{3t_2}{8} - \pi/2\right)$

converting back to original variables, we get

$$Y(t) \simeq \cos\left(t + \frac{3\epsilon t}{8} - \pi/2\right)$$

which has initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = 1$$

as required (for $\epsilon \ll 1$).

Question 2:

$$y(t) = \frac{1}{2}t(2-t) + \frac{1}{12}\epsilon t^3(4-t)$$

$$y(t) = \frac{1}{2}t(2-t) + \frac{1}{12}\epsilon t^3(4-t)$$

a. Want to find non-dimensional time at maximum height. i.e. t such that $\dot{y}(t) = 0$

$$\frac{dy}{dt} = 1 - t + \epsilon\left(t^2 - \frac{1}{3}t^3\right)$$
$$= 1 - t + \frac{1}{3}\epsilon t^2(3-t)$$
$$\Rightarrow (1-t) + \frac{1}{3}\epsilon t^2(3-t) = 0$$

We want a root near $t=1$, since this is the dominant parameter for $\epsilon \ll 1$.

Define q_m as the time when the particle reaches maximum. We want a solution of the form

$$q_m(\epsilon) = q_0 + \epsilon q_1 + \dots$$

We want to make the substitution $t = q_m(\epsilon)$,

$$\Rightarrow (1 - q_0 - \epsilon q_1 - \dots) + \frac{1}{3}\epsilon(q_0 + \epsilon q_1 + \dots)^2(3 - q_0 - \epsilon q_1 - \dots) = 0$$

$O(\epsilon^0)$ terms:

$$1 - q_0 = 0 \quad \Rightarrow q_0 = 1$$

$O(\epsilon^1)$ terms:

$$-q_1 + \frac{1}{3}(q_0^2)(3 - q_0) = 0$$
$$\Rightarrow q_1 = \frac{2}{3}$$

Hence the two term perturbation series approximation for the time at maximum height is

$$q_m(\epsilon) = 1 + \frac{2}{3}\epsilon$$

And so the maximum height, y_m, is found by substituting $t = q_m(\epsilon)$ into $y(t)$,

$$\Rightarrow \quad y_m = \frac{1}{2}\left(1 + \frac{2}{3}\epsilon\right)\left(1 - \frac{2}{3}\epsilon\right) + \frac{1}{12}\epsilon\left(1 + \frac{2}{3}\epsilon\right)^3\left(3 - \frac{2}{3}\epsilon\right)$$
$$y_m = \frac{1}{2}\left(1 + \frac{4}{9}\epsilon^2\right) + \left(1 + \frac{2}{3}\epsilon\right)^3\left(\frac{\epsilon}{4} - \frac{\epsilon^2}{18}\right)$$

b. The time to hit the ground is when $y(t) = 0$

$$\Rightarrow \quad 0 = \frac{1}{2}t(2-t) + \frac{1}{12}\epsilon t^2(4-t)$$

So, we want to examine times near the root $t = 2$, as this is the dominant term for $\epsilon \ll 1$, and $t = 0$ is trivial.

We expect the time to hit the ground relies on ϵ, so define this time as

$$t = \tau_E(\epsilon)$$

which we want to have the form

$$\tau_E(\epsilon) = \tau_0 + \epsilon\tau_1 + \dots$$

Make this substitution:

$$0 = \frac{1}{2}(\tau_0 + \epsilon\tau_1 + \dots)(2 - \tau_0 - \epsilon\tau_1 - \dots) + \frac{1}{12}\epsilon(\tau_0 + \epsilon\tau_1 + \dots)^3(4 - \tau_0 - \epsilon\tau_1 - \dots)$$

$O(\epsilon^0)$ terms:

$$\frac{1}{2}\tau_0(2 - \tau_0) = 0$$

$\therefore$ sols for $\tau_0 = 0$, or $\tau_0 = 2$

$\frac{}{2}\tau_0(2-\tau_0) = 0$

$\therefore$ sols for $\tau_0 = 0$, or $\tau_0 = 2$

as before, we want the non-trivial case, so take $\tau_0 = 2$

$O(\epsilon^1)$ terms:

$$-\frac{1}{2}\tau_0\tau_1 + \frac{1}{2}\tau_1(2-\tau_0) + \frac{1}{12}\tau_0^3(4-\tau_0) = 0$$

$$-\tau_1 + \frac{2}{3}\times 2 = 0 \Rightarrow \tau_1 = \frac{4}{3}$$

Hence, the two-term approximation for the time for the projectile to hit the ground is

$$\tau_E(\epsilon) = 2 + \frac{4}{3}\epsilon$$

Question 3:

$$\rho = \frac{1}{a+bv+cv^2}$$

a. Want $a+bv+cv^2$ to have units $[\rho^{-1}]$

$[\rho]$ has units of inverse distance (for traffic), so we want $a+bv+cv^2$ to have units of distance,

or
$$[a] = L^1$$
$$[b] = L^1\cdot[v]^{-1}$$
$$[c] = L^1\cdot[v]^{-2}$$

but velocity has units of distance over time, $L^1 \cdot T^{-1}$, so,

$$[a] = L^1$$
$$[b] = T^1$$
$$[c] = T^2 . L^{-1}$$

b. Now we want to solve for v in terms of ρ.

$$\rho = \frac{1}{a+bv+cv^2} \Rightarrow a+bv+cv^2 = \frac{1}{\rho},$$

or,

$$cv^2+bv+(a+\frac{1}{\rho}) = 0$$

$\therefore$ expect two solutions, with

$$v_{1,2} = \frac{-b \pm \sqrt{b^2-4(a+\frac{1}{\rho})c}}{2c}$$

Since the constants are positive, we want a positive velocity (since we assume traffic moves left to right), and so we take the positive root, giving

$$v(\rho) = -\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2-4c(a+\frac{1}{\rho})}$$

which is the resultant constitutive law for this problem.

Question 4:

a. Solve

$$\frac{\partial\rho}{\partial t} + 2\frac{\partial\rho}{\partial x} = 1$$

5. Solve

$$\frac{\partial \rho}{\partial t} + 2\frac{\partial \rho}{\partial x} = 1$$

with

$$\rho(x,0) = (1+x^2)^{-1}$$

We can rewrite this as

$$\left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x}\right)\rho = 1$$

and define a new parameter r such that

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x}$$

so that

$$\frac{\partial \rho}{\partial r} = 1$$

Let $x = x(r,s)$ and $t = t(r,s)$ for a new variable s. The chain rule gives

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial t}{\partial r}\frac{\partial}{\partial t}$$

So, we require $\partial x/\partial r = 2$ and $\partial t/\partial r = 1$

Integrating, we get $x = 2r + q(s)$ and $t = r + p(s)$.

Choose variables such that the x-axis at $t=0$ maps to the s-axis (with $r=0$).

i.e. with $r=0$ and $t=0$, $x = q(s)$ and $p(s) = 0$

Hence, $r=t$, and

$$x = 2t + s \quad \Rightarrow \quad s = x - 2t$$

Back to the original equation, we can integrate, giving

$$\frac{\partial \rho}{\partial r} = 1 \quad \Rightarrow \quad \rho = r + \rho(s,0)$$

hence,

$$\rho(x,t) = t + \frac{1}{1+(x-2t)^2}$$

using the initial condition with $s = x - 2t$.

6. Want to solve

$$\frac{\partial \rho}{\partial t} - 6\frac{\partial \rho}{\partial x} = \rho$$

with initial condition $\rho(x,0) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

As before, we want to transform this into

$$\frac{\partial \rho}{\partial r} = \rho,$$

with

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial t}{\partial r}\frac{\partial}{\partial t}$$

$$\Rightarrow \quad \frac{\partial x}{\partial r} = -6 \quad \Rightarrow \quad x = -6r + q(s)$$

$$\frac{\partial t}{\partial r} = 1 \quad \Rightarrow \quad t = r + p(s)$$

Choose r,s such that the x-axis (i.e. $t=0$) maps to the s-axis, and $r=t$

$$t = r \Rightarrow p(s) = 0$$

$$\Rightarrow x = -6t + s$$

$$\Rightarrow \quad s = x + 6t$$

Back to the original equation, we have

$$\frac{\partial \rho}{\partial r} = \rho$$

Which has solution

$$\rho(s,r) = \rho(s,0)e^{r}$$

Transforming back to variables x and t, we get

$$\rho(x,t) = \begin{cases} e^{t} & \text{if } 0 < x + 6t < 1 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently,

$$\rho(x,t) = \begin{cases} e^{t} & \text{if } -6t < x < 1 - 6t \\ 0 & \text{otherwise.} \end{cases}$$