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General Physics/Math

For matrix multiplication, the value in the cell (ij) (i.e. row i, column j) is equal to $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. If A is an $m \times n$ size matrix, and B is $n \times p$, then C is $m \times p$ in size.

$$E = \hbar\omega; \quad E^2 = \hbar^2\omega^2 = p^2c^2 + m^2c^4; \quad m = \left(\frac{m_A c}{\hbar}\right)^2$$

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}; \quad \nabla \cdot (\nabla \times A) = 0$$

The Taylor series for some useful functions is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \quad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Wave Equation

The wave equation (in one dimension) is

$$\frac{\partial^2}{\partial t^2} u(x,t) = v^2 \frac{\partial^2}{\partial x^2} u(x,t)$$

If we consider $u(x,t) = u_0 e^{i(kx - \omega t)}$ as a solution, we get

$$-\omega^2 u(x,t) = -v^2 k^2 u(x,t) \quad \Rightarrow \quad \omega = vk$$

where the relationship between ω and k is the **dispersion relation**.

Principle of Least Action

Generalised momenta and generalised forces are given by:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}; \quad F_i = \frac{\partial L}{\partial q_i}$$

And the Euler-Lagrange equations are

$$L = T - V \quad \Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

And in 1 + 1D, for $L = L(\eta, d\eta/dx, d\eta/dt, x, t)$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \frac{d\eta}{dt}}\right) + \frac{d}{dx}\left(\frac{\partial L}{\partial \frac{d\eta}{dx}}\right) - \frac{\partial L}{\partial \eta} = 0$$

Regardless of the number of components in L, in relativistic notation we get,

$$\partial_\nu \left(\frac{\partial L}{\partial(\partial_\nu \eta_\rho)}\right) - \frac{\partial L}{\partial \eta_\rho} = 0$$

Relativistic Notation

The flat metric is:

$$g_{\alpha\beta} = g^{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$$

To convert between covariant and contravariant vectors:

$$x_\alpha = g_{\alpha\beta} x^\beta; \quad x^\alpha = g^{\alpha\beta} x_\beta$$

Derivatives are:

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha}; \quad \partial^\alpha = g^{\alpha\beta}\partial_\beta = \frac{\partial}{\partial x_\alpha}$$

$$\Box^2 = \partial^\alpha \partial_\alpha = g^{\alpha\beta}\partial_\beta \partial_\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

Klein-Gordon Equation

The Klein-Gordon equation is the relativistic generalisation of the Schrodinger equation for scalar (spin-0) particles. The Klein-Gordon equation and Lagrangian density are:

$$\left[\partial_\mu \partial^\mu + \left(\frac{mc}{\hbar}\right)^2\right]\phi = 0; \quad L = (\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2c^2}{\hbar^2}\phi\phi^*$$

The Euler-Lagrange equations for this are:

$$\partial_\nu\left(\frac{\partial L}{\partial(\partial_\nu \phi)}\right) - \frac{\partial L}{\partial \phi} = 0 = \partial_\nu\left(\frac{\partial L}{\partial(\partial_\nu \phi^*)}\right) - \frac{\partial L}{\partial \phi^*}$$

Electromagnetism

Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}; \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0; \qquad \nabla \times \mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

In a vacuum, $\rho = 0$, and $\mathbf{J} = \mathbf{0}$. For a gauge with $\phi = 0$:

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

$$\mathbf{B} = \nabla \times \mathbf{A}; \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}; \quad \nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}$$

The electromagnetic field tensor is defined as

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

This is antisymmetric ($F_{\alpha\beta} = -F_{\beta\alpha}$), and so $F_{\alpha\alpha} = 0$ for all α.

$$F^{\alpha\beta} = g^{\alpha\sigma} F_{\sigma\rho} g^{\rho\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

And so Maxwell's equations are

$$\partial_\alpha F^{\alpha\beta} = J^\beta \quad \Rightarrow \quad \partial_\beta F^{\alpha\beta} = -J^\alpha$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\beta\alpha} = 0$$

In the quantum theory of spin ices, there are electric monopoles (charges) as well as magnetic monopoles. Hence Maxwell's equations become

$$\nabla \cdot \mathbf{E} = \rho_e; \qquad \nabla \times \mathbf{E} = -\mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = \rho_m; \qquad \nabla \times \mathbf{B} = \mathbf{J}_e + \frac{\partial \mathbf{E}}{\partial t}$$

where ρ_m is the magnetic charge density, and $\mathbf{J}_m$ is the magnetic current.

Proca Lagrangian

For massive vector fields (i.e. for spin-1 particles), we use the Proca Lagrangian:

$$L = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{m_A c}{\hbar}\right)^2 A^\mu A_\mu =$$
$$-\frac{1}{16\pi}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + \frac{1}{8\pi}\left(\frac{m_A c}{\hbar}\right)^2 A^\mu A_\mu$$

where

$$\frac{\partial F_{\mu\nu}}{\partial(\partial_\alpha A_\beta)} = \frac{\partial}{\partial(\partial_\alpha A_\beta)}\left[\partial_\mu A_\nu - \partial_\nu A_\mu\right] = \delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu$$
$$\frac{\partial F^{\mu\nu}}{\partial(\partial_\alpha A_\beta)} = \frac{\partial\left[\partial^\mu A^\nu - \partial^\nu A^\mu\right]}{\partial(\partial_\alpha A_\beta)} = g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}$$
$$\frac{\partial(F^{\mu\nu}F_{\mu\nu})}{\partial(\partial_\alpha A_\beta)} = F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial(\partial_\alpha A_\beta)} + F_{\mu\nu}\frac{\partial F^{\mu\nu}}{\partial(\partial_\alpha A_\beta)} = 4F^{\alpha\beta}$$

and

$$\frac{\partial}{\partial A_\beta}A^\mu A_\mu = A^\mu\frac{\partial}{\partial A_\beta}A_\mu + A_\mu\frac{\partial}{\partial A_\beta}A^\mu$$

By setting $m_A = 0$, the Proca Lagrangian gives the Maxwell Lagrangian (i.e. that for electromagnetism).

Dirac Lagrangian

This describes spinor fields (i.e. spin 1/2 particles). It is

$$L = i\hbar c\overline{\psi}\gamma^\mu\partial_\mu\psi - mc^2\overline{\psi}\psi$$

where $\overline{\psi} = \psi^\dagger\gamma_0$ where the dagger is conjugate transpose. The Dirac matrices, γ^μ, are

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} \mathbb{0} & \sigma^i \\ -\sigma^i & \mathbb{0} \end{pmatrix}$$

where $\mathbb{0}$ and $\mathbb{1}$ represent zeros and the 2-identity respectively. The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

ψ is a '4-spinor' field - it has 4 components, which correspond to spin-up matter (e.g. electrons), spin-down matter, spin-up antimatter (e.g. positrons), and spin-down antimatter respectively.

Gauge Invariance

For some $\theta \to \theta + \delta\theta$ in the wave function $\phi = |\phi|e^{i\theta}$, the perturbation in the Lagrangian is then

$$\delta L = \sum\frac{\partial L}{\partial\phi_\alpha}\delta\phi_\alpha = \sum_A\left(\frac{\partial L}{\partial\phi_A}\delta\phi_A + \frac{\partial L}{\partial(\partial_\mu\phi_A)}\partial_\mu\delta\phi_A\right)$$

where ϕ_α is all instances of ϕ and the operations done onto it.
If $\delta L = 0$, then the Lagrangian is invariant under the gauge transformation. For example, the perturbation on some Lagrangian might be

$$L = \frac{\hbar^2}{2m}\nabla\psi\cdot\nabla\psi^* + \frac{\hbar}{2i}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right)$$
$$\Rightarrow \delta L = \frac{\partial L}{\partial\psi}\delta\psi + \frac{\partial L}{\partial\psi^*}\partial\psi^* + \frac{\partial L}{\partial\nabla\psi}\delta\nabla\psi + \frac{\partial L}{\partial\nabla\psi^*}\delta\nabla\psi^* + \frac{\partial L}{\partial\dot{\psi}}\delta\dot{\psi} + \frac{\partial L}{\partial\dot{\psi}^*}\delta\dot{\psi}^*$$

Noether's Theorem

Noether's theorem says that if we have a system (Lagrangian) with a continuous symmetry, then there is a conserved quantity due to this symmetry. Translational invariance $\to$ conservation of momentum; time invariance $\to$ conservation of energy; rotational invariance $\to$ conservation of angular momentum.
For example, for translational invariance, let $q(s) = q + s$. i.e. varying s just translates the system. Then,

$$\frac{d\,q(s)}{ds} = 1$$

and so any quantity multiplied by this is conserved. The Noether current is defined as

$$J^\mu_N = \sum_A\left(\frac{\partial L}{\partial(\partial_\mu\phi_A)}\right)\frac{\partial\phi_A}{\partial\lambda} - W^\mu$$

where λ is some parametrisation quantity, and W^μ is a general quantity satisfying $\delta L = (\partial_\mu W^\mu)\delta\lambda$. For a complex field $\phi = \phi_1 + i\phi_2$, the conserved current is defined as

$$J^\mu = i\left[\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi\right]$$

where

$$\partial_\mu J^\mu = 0 = \partial_\mu J^\mu_N$$

The divergence theorem says that $\partial_i J^i_N = 0$.

Free Energy

The free energy is required to be real, and so cannot have odd powers of a complex valued function.

Nambu-Goldstone Modes

Generally, whenever a continuous symmetry is spontaneously broken there is a new massless mode in the broken symmetry phase. The massless mode is called the Nambu-Goldstone boson.

Meissner Effect

The Meissner effect is the expulsion of a magnetic field from the interior of a superconductor when it transitions into its superconducting state, which occurs below a critical temperature.
Higgs field causes spontaneous symmetry breaking – this triggers Higgs mechanism causing bosons it interacts with to have mass. Higgs mechanism” refers specifically to the generation of masses for the W±, and Z weak gauge bosons through electroweak symmetry breaking.
In summary, while the Meissner effect and the Higgs mechanism share the concept of a phase transition and the emergence of a new physical state, they operate in different domains of physics and have distinct underlying principles. The Meissner effect is a phenomenon observed in superconductors, involving the expulsion of magnetic fields and zero electrical resistance. On the other hand, the Higgs mechanism is a theoretical framework in particle physics, explaining the acquisition of mass by bosons through the interaction with the Higgs field.

Standard Model