PHYS3051 Content Notes

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Semester 1, 2023

1 Week 1

1.2 – The Photon (1900 - 1924)

Electromagnetic radiation is ***quantized***, coming in little packages of energy

$$E = h\nu \tag{1.1}$$

where ν is the frequency of the radiation and $h = 6.626 \times 10^{-27}$ erg.s is Planck's constant.

Photoelectric effect: when an incoming photon hits an electron in metal, giving up its energy of $h\nu$, an excited electron breaks through the metal surface, losing in the process an energy w (the so called *work function* of the material). The electron emerges (free) with an energy of

$$E \leq h\nu - w \tag{1.3}$$

1.4 – Antiparticles (1930 - 1956)

For every kind of particle there must exist a corresponding ***antiparticle***, with the same mass but opposite electric charge. The standard notation for antiparticles is an overbar. For example, p denotes a proton and $\overline{p}$ an antiproton. In some cases, it is customary simply to specify the charge. Most people write e^+ for the positron (not $\overline{e}$) and μ^+ for the antimuon. Some neutral particles are their *own* antiparticles. For example, the photon: $\overline{\gamma} \equiv \gamma$.

The antineutron differs from a neutron only in quantum number. Although the net charge of a neutron is zero, the neutron *does* have a charge structure (positive at the center and near the surface, and negative in between) and a magnetic dipole moment. These have opposite signs for $\overline{n}$.

There is a general principle in particle physics that goes under the name of ***crossing symmetry.*** Suppose that a reaction of the form

$$A + B \to C + D$$

is known to occur. Any of these particles can be 'crossed' over to the other side of the equation, provided it is turned into its antiparticle, and the resulting interaction will also be allowed. For example,

$$A \to \overline{B} + C + D$$
$$A + \overline{C} \to \overline{B} + D$$

In addition, the *reverse* reaction occurs: $C + D \to A + B$, but technically this derives from the principle of *detailed balance* rather than from crossing symmetry. However, there is one important caveat in all this: conservation of energy may veto a reaction that is otherwise permissible. e.g. a minimum kinetic energy may need to occur to make up for some mass difference in the reaction.

2 Week 2

1.6 – Strange Particles (1947-1960)

A new particle, with at least twice the mass of the pion was discovered, called the kaon, K^0. This decays via

$$K^0 \to \pi^+ + \pi^- \tag{1.22}$$

A positively charged kaon was also discovered, with

$$K^+ \to \pi^+ + \pi^+ + \pi^- \tag{1.23}$$

Soon after, the lambda, Λ, was discovered, which decays via

$$\Lambda \to p^+ + \pi^- \tag{1.24}$$

which means that the lambda is substantially heavier than the kaon. The lambda belongs with the proton and neutron in the baryon family.

What stops the proton from spontaneously decaying? A law of *conservation of baryon number* was proposed: assign to all baryons a 'baryon' number $A = +1$, and to the antibaryons $A = -1$ – the total baryon number is conserved in any physical process. Since the proton is the lightest baryon, it has nothing to decay down to (while still maintaining baryon number), and so it remains stable.

Unlike leptons and baryons, there is *no* conservation of mesons. In pion decay ($\pi^- \to \mu^- + \overline{\nu}_\mu$) a meson disappears, and in lambda decay a meson is created.

Strange particles are produced by the strong force (the same that holds atomic nuclei together), but they decay by the weak force (the one that accounts for beta decay and all other neutrino processes). Strange particles are also produced in pairs (so called ***associated production***). Each particle was assigned a new property called *strangeness* that (like charge, lepton number, and baryon number) is conserved in any strong interaction. But, unlike those others, is *not* conserved in a weak interaction. Leptons and photons don't experience strong forces at all, so strangeness does not apply to them.

1.8 – The Quark Model (1964)

The quarks come in three types (or 'flavours'): u (for 'up') quark which carries a charge of $\frac{2}{3}$ and a strangeness of 0; the d ('down') quark which carries a charge of $-\frac{1}{3}$ and $S = 0$; the s ('strange') quark which carries a charge of $-\frac{1}{3}$ and $S = -1$. To each quark (q) there corresponds an antiquark ($\overline{q}$), with the opposite charge and strangeness. There are two ***composition rules***:

1. Every baryon is composed of three quarks (and every antibaryon is composed of three antiquarks).
2. Every meson is composed of a quark and an antiquark.

The Pauli exclusion principle states that no two electrons can carry the same state. It was later realised that the same rule applies to all particles of half-integer spin. In particular, the exclusion principle should apply to quarks, which, as we shall see, must carry spin $\frac{1}{2}$. Now the Δ^{++}, for instance, is supposed to consist of three identical u quarks in the same state - appearing to be inconsistent with the Pauli principle. To avoid this, it was proposed that quarks not only come in three flavours, but each of these also come in three *colours* (red, green, and blue, say). To make a baryon, we simply take one quark of each colour, then the three u's in Δ^{++} are no longer identical.

Redness, blueness, and greenness are simply *labels* used to denote three new properties that, in addition to charge and strangeness, the quarks possess. A *red* quark carries one unit of redness, zero blueness, and zero greenness; its antiparticle carries minus one unit of redness, and so on.
The colour terminology has one especially nice feature: it suggests a delightfully simple characterisation of the particular quark combinations that are found in nature:

All naturally occurring particles are colourless.

That is, *either* the total amount of each colour is zero, *or* all three colours are present in equal amounts. This explains why you can't find isolated quarks, and why you can't make a particle out of two, or four quarks (for example). The only colourless combinations you can make are $q\bar{q}$ (the mesons), qqq (the baryons), and $\overline{qqq}$ (the antibaryons).

1.9 – The November Revolution

A newly discovered ψ meson was an electrically neutral, extremely heavy neutron – more than three times the weight of a proton. This particle has an extraordinarily long lifetime (on the order of 10^{-20}s opposed to 10^{-23}s – $\sim$ 1000 times longer). The ψ is a bound state of a new (fourth) quark, the c (for *charm*) and its antiquark, $\psi = (c\bar{c})$.
Eventually, fifth and sixth quarks were identified (b – bottom; t – top, respectively) along with a new heavy meson (the *upsilon*) and a new lepton (the tau, which has its own neutrino).

1.10 – Intermediate Vector Bosons (1983)

The process of beta decay is actually mediated by some exchange particle (instead of it being from a contact interaction). This mediator became known by the name *intermediate vector boson*. There are in fact *three* intermediate vector bosons, two of them charged ($W^{\pm}$) and one neutral (Z).

1.11 – The Standard Model

In the current view, all matter is made out of three kinds of elementary particles: leptons, quarks, and mediators. There are also antileptons, with all the signs reversed. The positron, for example, carries a charge of +1 and an electron number −1.

l	Q	L_e	L_μ	L_τ
e	−1	1	0	0
ν_e	0	1	0	0
μ	−1	0	1	0
ν_μ	0	0	1	0
τ	−1	0	0	1
ν_τ	0	0	0	1

Table 1: Lepton Classification

Similarly, there are six 'flavours' of quarks, classified by charge, strangeness (S), charm (C), beauty (B), and truth (T) as well as 'upness', U, and 'downness', D.

q	Q	D	U	S	C	B	T
d	−1/3	−1	0	0	0	0	0
u	2/3	0	1	0	0	0	0
s	−1/3	0	0	−1	0	0	0
c	2/3	0	0	0	1	0	0
b	−1/3	0	0	0	0	−1	0
t	2/3	0	0	0	0	0	1

Table 2: Quark Classification

Again, all signs would be reversed on the table of antiquarks. Meanwhile, each quark and antiquark comes in three colours, so there are 36 of them in all.

Finally, every interaction has its mediator – the photon for the electromagnetic force, two W's and a Z for the weak force, the graviton (presumably) for gravity, and now the ***gluon*** for the strong force (which is exchanged between quarks in a strong process). Gluons carry colour and so should not exist as isolated particles.

3 Week 3

2.1 – The Four Forces

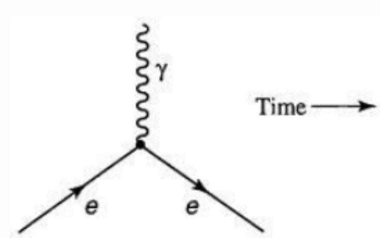
As far as we know, there are just four fundamental forces in nature:

Force	Strength	Theory	Mediator
Strong	10	Chromodynamics	Gluon
Electromagnetic	10^{-2}	Electrodynamics	Photon
Weak	10^{-13}	Flavordynamics	W and Z
Gravitational	10^{-42}	Geometrodynamics	Graviton

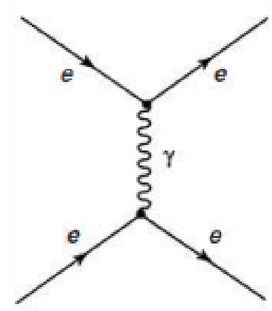
Table 3: The Four Forces

2.2 – Quantum Electrodynamics (QED)

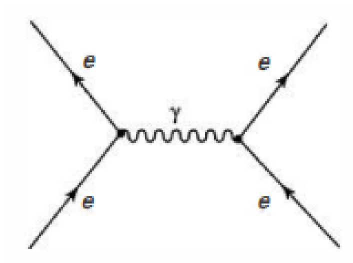
All electromagnetic phenomena are ultimately reducible to the following elementary process



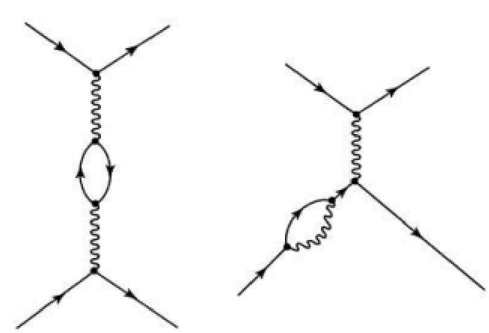
In these figures, time flows horizontally to the right (often represented as time flowing vertically up, too). So, the above diagram reads: a charged particle, e, enters, emits (or absorbs) a photon, γ, and exits.



In the above diagram, two electrons enter, a photon passes between them (it doesn't matter which emits and which absorbs; the diagram represents both orderings), and the two exit. In QED, this process is called ***Moller scattering***; we say that the interaction is mediated by the exchange of a photon.



A particle line running 'backward in time' (an arrow pointing to the left) is interpreted as the corresponding *antiparticle* going forward (the photon is its own antiparticle, hence no arrow is needed). In the above process, an electron and a positron annihilate to form a photon, which in turn produces a new electron-positron pair.

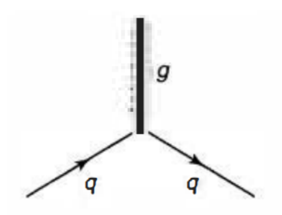


In each of the above figures, two electrons went in and two electrons came out. The 'innards' of the diagram are irrelevant as far as the observed process is concerned. Internal lines (those which begin and end within the diagram) represent particles which are not observed – indeed, that *cannot* be observed without entirely changing the process. We call them ***virtual particles***. Only the *external* lines (those that enter or leave the diagram) represent 'real' (observable) particles. The external lines, then, tell you what physical process is occurring; the internal lines describe the *mechanism* involved.

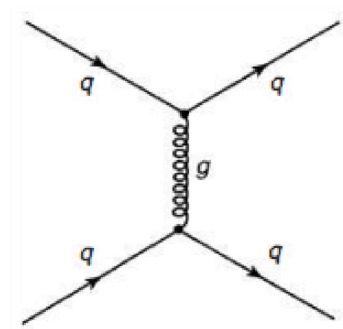
The Feynman rules enforce conservation of energy and momentum at each vertex, and hence for the diagram as a whole. As such, a pair annihilation will always produce *two* photons due to conservation of momentum (as photons have momentum, and the particle and antiparticle pair contact with opposite and equal momenta).

2.3 – Quantum Chromodynamics (QCD)

In chromodynamics, *colour* plays the role of charge, and the fundamental process (analogous to $e \to e + \gamma$) is quark $\to$ quark plus gluon ($q \to q + g$):



As before, we combine two or more such 'primitive vertices' to represent more complicated processes. For example, the force between two quarks (which is responsible in the first instance for binding quarks together to make hadrons) is described in lowest order by the diagram:



We say that the force between two quarks is 'mediated' by the exchange of gluons.

At this level, chromodynamics is similar to electrodynamics. However, there is only one kind of electric charge, but there are *three* kinds of colour. In the fundamental process $q \to q + g$, the colour of the quark (but not its flavour) may change. In this case, the gluon must carry away the difference in colour. Gluons, then, are 'bicoloured', carrying one positive unit of colour (taken from a quark) and one negative unit (where the negative value is the colour given to the quark). Since the gluons themselves carry colour, they couple directly to other gluons, and hence in addition to the fundamental quark-gluon vertx, we also have primitive gluon-gluon vertices.

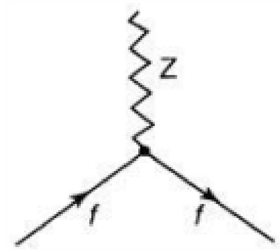
2.4 – Weak Interactions

There is no particular name for the 'stuff' that produces weak forces, in the sense that electric charge produces electromagnetic forces and colour produces strong forces. Some call it

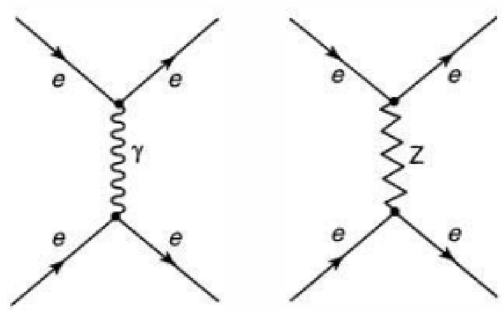
'weak charge', but whatever word you use, all quarks and leptons carry it.
Leptons have no colour (and so don't experience the strong force), neutrinos have no charge (and so don't experience electromagnetic forces), but all of them experience weak interactions. There are two kinds of weak interactions: *charged* (mediated by the Ws) and *neutral* (mediated by the Z).

2.4.1 - Neutral

The fundamental neutral vertex is



where f can be any lepton or quark.
Any process mediated by the photon could also be mediated by the Z - for example, electron-electron scattering:



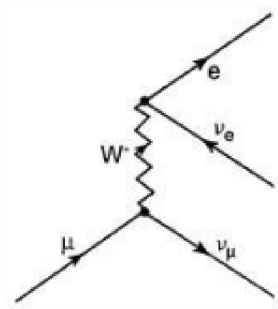
although photon-mediated processes overwhelmingly dominate.

2.4.2 - Charged

The primitive vertices for strong, electromagnetic, and neutral weak interactions all share the feature that the *same* quark or lepton comes out as went in (at least the same flavour) – accompanied, of course, by a gluon, photon, or Z, as the case may be. The charged weak interactions are the only ones that change flavour, and in this sense they are the only ones capable of causing a 'true' decay (as opposed to a mere repackaging of the quarks, or a hidden pair production/annihilation).

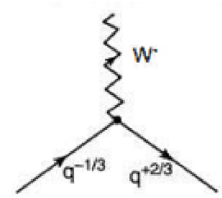
2.4.2.1 Leptons

The fundamental charged vertex is described by a negative lepton (it could be e^-, μ^-, or τ^-) converting into the corresponding neutrino, with emission of a W^- (or absorption of a W^+): $l^- \to \nu_l + W^-$.
As always, we combine the primitive vertices to generate more complicated reactions. Since Ws are charged, they must have a direction in their arrows. For example, the decay of a muon $\mu^- \to e^- + \nu_\mu + \overline{\nu_e}$:



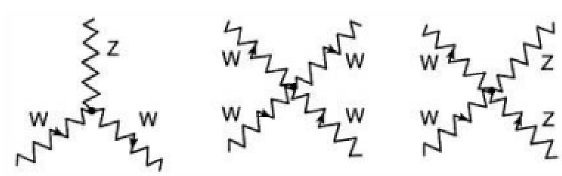
2.4.2.2 Quarks

Notice that the leptonic weak vertices connect members of the *same* generation: e^- converts to ν_e (with an emission of W^-), etc. In this way, the theory enforces the conservation of electron number, muon number, and tau number. It is tempting to suppose that the same rule applies to the quarks, so that the fundamental charged vertex is

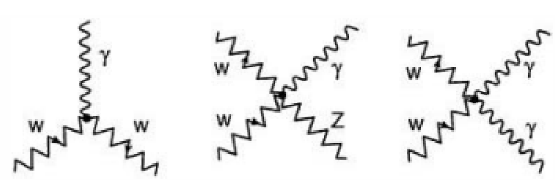


A quark with the charge $-\frac{1}{3}$ converts into the corresponding quark with charge $+\frac{2}{3}$ with the emission of a W^-. The outgoing quark carries the same colour as the ingoing one, but with a different flavour. The far end of the W line can couple to leptons (a 'semileptonic' process), or to other quarks (a purely hadronic process).

2.4.2.3 Weak and Electromagnetic Couplings of W and Z There are also direct couplings of W and Z to one another (just as there are direct gluon-gluon couplings in QCD):



Moreover, because the W is charged, it couples to the photon:



2.5 – Decays and Conservation Laws

One of the most striking general properties of elementary particles is their tendency to disintegrate; we might also call it a universal principle that *every particle decays into lighter* *particles, unless prevented from doing so by some conservation law.* The photon is stable (having zero mass); the electron is stable (it's the lightest charged particle, so conservation of charge prevents decay); the proton is presumably stable (it's the lightest baryon, and the conservation of baryon number saves it) – likewise, conservation of lepton number protects the lightest of the neutrinos. But, *most* particles spontaneously disintegrate – even the neutron (although it is stable in the environment of many atomic nuclei). In practice, our world is populated by these stable particles; more exotic things are created now and then (by collisions), but often don't last long. Each unstable species has a characteristic lifetime of some fraction of a second. In fact, most particles exhibit several decay modes, where some proportion of the same species of particle will decay into one group of products, and another proportion into another group of products – called ***branching ratios***.

4 Week 4 - Relativistic Kinematics

3.1 – Lorentz Transformations

According to special relativity, the laws of physics are equally valid in all inertial reference systems. An inertial system is one in which Newton's first law is obeyed: objects keep moving in straight lines at constant speeds unless acted upon by some force.
The Lorentz Transformations, which handle a change of coordinates in spacetime from one inertial reference frame S to another S', have a number of consequences:

1. ***The Relativity of Simultaneity***: if two events occur at the same time in S, but at different locations, then they do *not* occur at the same time in S'. Specifically, if $t_A = t_B$, then

$$t'_A = t'_B + \frac{\gamma v}{c^2}(x_B - x_A) \tag{3.4}$$

   That is, events that are simultaneous in one inertial system are not simultaneous in others.

2. ***Lorentz Contraction***: A *moving object* is shortened by a factor of γ, as compared with its length in the system in which it is at rest. Lorentz contraction only applies to lengths along the direction of motion; perpendicular dimensions are not affected.

3. ***Time Dilation***: Moving clocks *run slow* by a factor of γ. That is, for a moving inertial frame S' observed from a 'stationary' S, some Δt in S will correspond to $\gamma\Delta t$ in S'.

4. ***Velocity Addition:*** Suppose a particle is moving in the x direction at speed u', with respect to S' (which is moving at v w.r.t S) - what is its speed, u, with respect to S?

$$u = \frac{u' + v}{1 + (u'v/c^2)} \tag{3.5}$$

   Notice that if $u' = c$, then $u = c$ also: *the speed of light is the same in all inertial systems.*

It can be confusing to remember which numbers should be primed and what signs attach to the velocities. Three rules help with this:

- Moving sticks are short (by a factor of γ)
- Moving clocks are slow (by a factor of γ)
- If v_{AB} is the velocity of A with respect to B, then

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)} \tag{3.6}$$

Simply by remembering those rules, we can put the γ factor (which is *greater* than 1) on whichever side of the equation to achieve those results.

4.2 – Four-Vectors

We define the position-time four-vector as

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \tag{3.7}$$

In terms of x^μ, the Lorentz transformations are

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 - \beta x^1); \qquad x^{1\prime} = \gamma(x^1 - \beta x^0) \\ x^{2\prime} &= x^2; \qquad x^{3\prime} = x^3 \end{aligned} \tag{3.8}$$

where $\beta \equiv v/c$.
More compactly, we can write

$$x^{\mu\prime} = \sum_{v=0}^{3} \Lambda^\mu_\nu x^\nu \tag{3.10}$$

The coefficients Λ^μ_ν are then given by

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.11}$$

To streamline things, we can rewrite equation (3.10) using Einstein notation, where summation over like contravariant and covariant indices is implied:

$$x^{\mu\prime} = \Lambda^\mu_\nu x^\nu \tag{3.12}$$

Although the individual coordinates of an event change in a reference frame transformation, there is a particular combination of them that remains the same

$$I \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 \tag{3.13}$$

Such a quantity, which is the same in any inertial system, is called an ***invariant***. We'd like to write this invariant in the form of a sum, but first we need to introduce the ***metric***, $g_{\mu\nu}$, to deal with the minus signs:

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.14}$$

Hence the invariant can be written then as

$$I = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}x^\mu x^\nu = g_{\mu\nu}x^\mu x^\nu \tag{3.15}$$

We can then define the ***covariant*** four-vector x_μ as

$$x_\mu \equiv g_{\mu\nu}x^\nu \tag{3.16}$$

Where the 'original' four-vector, x^μ, is a ***contravariant*** four-vector. The invariant I can then be written as

$$I = x_\mu x^\mu \tag{3.17}$$

We define a four-vector, a^μ, as a four-component object that transforms from one coordinate system to another as

$$a'^\mu = \Lambda^\mu_\nu a^\nu \tag{3.18}$$

For each covariant vector, we also have a contravariant vector defined as

$$a_\mu = g_{\mu\nu}a^\nu \tag{3.19}$$

Where we can go back to a contravariant vector via

$$a^\mu = g^{\mu\nu}a_\nu \tag{3.20}$$

where $g^{\mu\nu}$ are technically the elements of g^{-1}, but out metric is its own inverse and so $g^{\mu\nu} = g_{\mu\nu}$.
Given any two four-vectors, a^μ and b^μ, we have that

$$a^\mu b_\mu = a_\mu b^\mu = a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3 \tag{3.21}$$

is invariant, and called the ***scalar product*** of a and b. We can also use dot notation for the scalar product

$$a \cdot b \equiv a_\mu b^\mu = a^0b^0 - \mathbf{a}\cdot\mathbf{b} \Longrightarrow a^2 \equiv a\cdot a = (a^0)^2 - \mathbf{a}^2 \tag{3.22 $\rightarrow$ 3.24}$$

We see that a^2 need not be positive, and so we can classify four-vectors according to the sign of a^2:

$$\begin{aligned} &\text{If } a^2 > 0, && a^\mu \text{ is called } \textit{timelike} \\ &\text{If } a^2 < 0, && a^\mu \text{ is called } \textit{spacelike} \\ &\text{If } a^2 = 0, && a^\mu \text{ is called } \textit{lightlike} \end{aligned} \tag{3.25}$$

All four-vectors are tensors of rank 1.

3.3 – Energy and Momentum

It is customary to work with ***proper time*** in relativistic physics, as it is invariant across coordinate transforms. For a particle moving at $\sim c$, the proper time τ is defined by

$$d\tau = \frac{dt}{\gamma} \tag{3.29}$$

with respect to a 'stationary' observer whose clock increments by dt time. *All* observers can read the particle's 'watch', and so they must agree on its value at any given moment even though their own clocks may differ from it and from one another.

With the introduction of proper time, we now also introduced ***proper velocity***

$$\eta \equiv \frac{dx}{d\tau} = \gamma v \tag{3.31 \& 3.32}$$

where $v = dx/dt$ as we'd expect. Proper velocity is part of a four-vector

$$\eta^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, v_x, v_y, v_z) \tag{3.33 \& 3.35}$$

Also, $\eta_\mu\eta^\mu = c^2$ and so is invariant.
We can redefine momentum in special relativity with respect to the proper velocity:

$$p = m\eta = \gamma m v \tag{3.37 \& 3.39}$$

We define relativistic energy, E as

$$E \equiv \gamma mc^2 \tag{3.41}$$

and so our ***energy-momentum four-vector*** (or four-momentum) is

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right) \tag{3.42}$$

We then have that $p_\mu p^\mu = m^2c^2$.
We then define the rest energy and ***relativistic kinetic energy*** respectively as

$$R \equiv mc^2 \tag{3.45}$$
$$T \equiv mc^2(\gamma - 1) \tag{3.46}$$

In classical mechanics, there's no such thing as a massless particle with momentum. However, when accounting for relativity, we can allow an $m = 0$ particle provided that the particle *travels at the speed of light*, where

$$v = c; \quad E = |\mathbf{p}|\,c; \quad E = h\nu \tag{3.48 \& 3.49}$$

3.4.2 – Relativistic Collisions

In a relativistic collision, energy and momentum are always conserved. In other words, all four component of the energy-momentum four-vector are conserved. As in the classical case, kinetic energy may or may not be conserved. We can classify collisions as sticky, explosive, or elastic:

1. *Sticky* (kinetic energy decreases): rest energy and mass increse.
2. *Explosive* (kinetic energy increases): rest energy and mass decrease.
3. *Elastic* (kinetic energy is conserved): rest energy and mass are conserved.

If mass is conserved, then the collision is elastic. In elementary particle physics, there is only one way this ever happens: the same particles come out as went in.

5 Week 5 – Scattering

6.1 – Decays and Scattering

We have three experimental probes of elementary particle interactions: bound states, decays, and scattering. Non-relativistic quantum mechanics is particularly well adapted to handle bound states. By contrast, the relativistic theory is especially well suited to describe decays and scattering.

6.1.1 – Decay Rates

In the case of decays, the item of greatest interest is the lifetime of the particle in question. Since decay is an inherently random process, we cannot hope to calculate the lifetime of any particular particle, but rather the average lifetime of some species of particle in a large sample.
Elementary particles have no memories, so the probability of a given particle decaying in the next microsecond is independent of how long ago that particle was created. The critical parameter, then, is the ***decay rate***, Γ; the probability per unit time that any given particle will disintegrate. If we had a large collection of particles, say $N(t)$, at time t, then $N\Gamma dt$ of them would decay in the next instant dt. This would, of course, decrease the number of particles remaining by $dN = -\Gamma N dt$, and so it follows that

$$N(t) = N(0)e^{-\Gamma t} \tag{6.2}$$

Evidently, the number of particles decreases exponentially with time, where the mean lifetime is the reciprocal of the decay rate:

$$\tau = \frac{1}{\Gamma} \tag{6.3}$$

In actuality, most particles can decay by several different routes. In such circumstances, the *total* decay rate is the sum of the individual decay rates:

$$\Gamma_{\text{tot}} = \sum_{i=1}^{n} \Gamma_i \Longrightarrow \tau = \frac{1}{\Gamma_{\text{tot}}} \tag{6.4 & 6.5}$$

In addition to τ, we want to calculate the various ***branching ratios***, that is, the fraction of all particles of the given type that decay by each mode. Branching ratios are determined by the decay rates:

$$\text{branching ratio for the ith decay mode } = \Gamma_i/\Gamma_{\text{tot}} \tag{6.6}$$

6.1.2 – Cross Sections

The greatest item of interest for scattering is the *effective* cross-sectional area, σ, a particle has. The effective cross-section also is dependent on the type of incident particle, as well as the structure of the target particle, as different interactions are involved. It depends, too, on the *outgoing* particles; if the energy is high enough, we can have not only *elastic* scattering ($e{+}p \to e{+}p$), but also a variety of *inelastic* processes (such as $e+p \to e+p+\gamma$). Each one of these has its own ('exclusive') scattering cross section, σ_i (for process i). In some experiments, however, the final products are not examined and we are interested only in the *total* ('inclusive') cross section:

$$\sigma_{\text{tot}} = \sum_{i=1}^{n} \sigma_i \tag{6.7}$$

Finally, each cross section typically depends on the *velocity* of the incident particle. At the most naive level, we might expect the cross section to be proportional to the amount of time the incident particle spends in the vicinity of the target, which is to say that σ should be inversely proportional to v. This behaviour is dramatically altered in the neighbourhood of a 'resonance' - a special energy at which the particles involved 'like' to interact, forming a short-lived semibound state before breaking apart. Such 'bumps' in the graph of σ vs v (or equivalently σ vs E) are in fact the principal means by which short-lived particles are discovered.

Now, for what we mean by a 'cross section' when the target is 'soft'. Suppose a particle comes along, encounters some kind of potential, and scatters off at an angle θ. This *scattering angle* is a function of the ***impact parameter***, b, the distance by which the incident particle would have missed the ***scattering center*** had it continued on its original trajectory (Fig 6.1). If the particle comes in with an

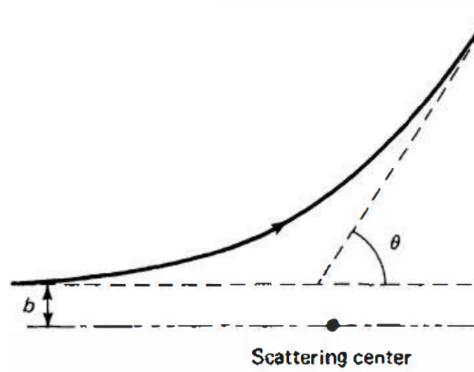


Figure 6.1: Scattering from a fixed potential: θ is the scattering angle and b is the impact parameter.

impact parameter between b and $b + db$, it will emerge with a scattering angle between θ and $\theta + d\theta$. More generally, if it passes through an infinitesimal area $d\sigma$, it will scatter onto a corresponding ***solid angle*** $d\Omega$ (Fig 6.3). Naturally, the larger we make $d\sigma$, the larger $d\Omega$ will be. The proportionality factor is called the ***differential (scattering) cross section***, D:

$$d\sigma = D(\theta)d\Omega \tag{6.8}$$

where

$$D(\theta) = \frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin\theta}\left(\frac{db}{d\theta}\right)\right| \tag{6.10}$$

AZ 21.1 – Introduction

An incident electromagnetic wave is said to *scatter* or *diffract* from a sample of matter when the field produced by the sample cannot be described using Fresnel's theory of reflection and refraction from a flat interface. We focus here

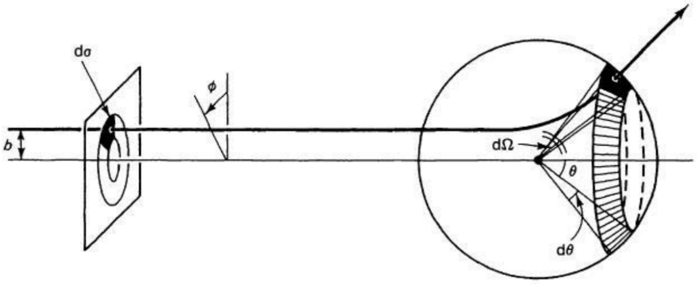


Figure 6.3: Particle incident in area $d\sigma$ scatters into solid angle $d\Omega$.

where this occurs because the wavelength of the incident monochromatic field is not small compared to the curvature of a material boundary. From a Fresnel point of view, the total field of these cases results from the interference of many different "reflected" and "refracted" waves propagating in different directions.

The physics which produces scattering and diffraction is identical to the physics which produces the Fresnel equations. An incident electromagnetic wave sets the charged particles of a medium into motion. Each accelerated charge produces a retarded field which is felt by, and thus affects the motion of, every other charge in the medium. The sum of the fields produced by all the particles of the medium is called the ***scattered field***, and the total field at any point is the sum of the incident field and the scattered field:

$$\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{scatt}} \tag{21.1}$$

AZ 21.2 – The Scattering Cross Section

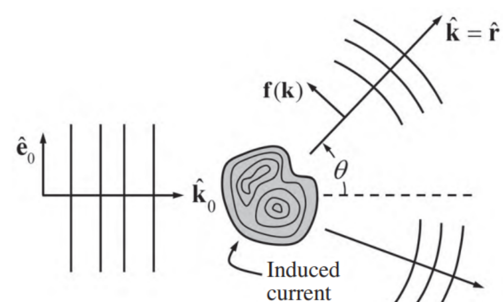


Figure 21.2: A plane wave with polarization vector $\hat{\mathbf{e}}_0$ propagating in the $\hat{\mathbf{k}}_0$ direction induces currents in a sample of matter. The scattering amplitude $\mathbf{f}(\mathbf{k})$ transverse to the local plane wave propagation direction $\hat{\mathbf{k}}$ characterises the strength of the field radiated in that direction.

The Figure 21.2 shows a conducting or dielectric object which scatters an incident plane wave with polarization vector $\hat{\mathbf{e}}_0$ and propagation vector $\hat{\mathbf{k}}_0$. The figure of merit for this process is called the differential cross section for scattering. Up to a factor of r^2, this is the radial component of the time-averaged Poynting vector of the radiation field $\mathbf{E}_{\text{rad}}$ (the long-distance part of $\mathbf{E}_{\text{scatt}}$) divided by the magnitude of the time-averaged Poynting vector of the incident field $\mathbf{E}_{\text{inc}}$:

$$\begin{aligned}\frac{d\sigma_{\text{scatt}}}{d\Omega} &= \frac{\text{scattered power radiated into a unit solid angle}}{\text{incident power per unit area}} \\ &= \frac{r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{\text{rad}} \rangle}{|\langle \mathbf{S}_{\text{inc}} \rangle|}\end{aligned} \tag{21.2}$$

A convenient form for Eq (21.2) follows from the general characteristics of $\mathbf{E}_{\text{rad}}$. First, it is an outgoing spherical wave with radial dependence $\exp(ikr)/r$. Second, the spherical wave front flattens out (locally) into a plane wave front with propagation vector $\mathbf{k} = k\hat{\mathbf{r}}$. Third, the field direction is transverse to the propagation direction. A ***scattering amplitude*** $\mathbf{f}(\mathbf{k}) \perp \mathbf{k}$ is commonly used to describe the vector and angular behaviour of the scattered radiation field. Combining all this information permits us to write the asymptotic ($r \to \infty$) total field from Eq (21.1) in the form

$$\lim_{r\to\infty} \mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{rad}} = E_0 \left[\hat{\mathbf{e}}_0 e^{i\mathbf{k}_0 \cdot \mathbf{r}} + \frac{e^{ikr}}{r} \mathbf{f}(\mathbf{k}) \right] e^{-i\omega t} \tag{21.3}$$

All the wave fields in Eq (21.3) propagate in vacuum, so $k = k_0 = \omega/c$.

Week 6 - Scattering

AZ 21.3 – Thomson Scattering

Thomson scattering occurs when an electromagnetic plane wave interacts with a single free electron. The Thomson scattering cross section is given by

$$\frac{d\sigma_{\text{Thom}}}{d\Omega} = \left(\frac{e^2}{4\pi\epsilon_0 mc^2} \right)^2 |\hat{\mathbf{k}} \times \hat{\mathbf{e}}_0|^2 \equiv r_e^2 \left(1 - |\hat{\mathbf{k}} \cdot \hat{\mathbf{e}}_0|^2 \right) \tag{21.11}$$

This formula is valid for all choices of $\hat{\mathbf{e}}_0$, whether real (for linear polarisation) or complex (for circular/elliptical polarisation).
The magnitude of the frequency-independent Thomson cross section is set by a length called the ***classical electron radius***,

$$r_e = \frac{e^2}{4\pi\epsilon_0 mc^2} \approx 2.82 \times 10^{-15} \text{ m} \tag{21.12}$$

This is the radius of a charged sphere whose Coulomb self-energy is equal to the rest energy of the electron.
The absence of scattering along the direction of the electric field ($\hat{\mathbf{k}} \parallel \hat{\mathbf{e}}_0$) and the angular dependence of Eq (21.11) are reminiscent of the behaviour of dipole radiation. The motion of the oscillating point charge (electron) produces an electric dipole moment, which in turn produces an electric dipole radiation field.

Equation (21.11) is the cross section for scattering when the incident plane wave has fixed polarisation $\hat{\mathbf{e}}_0$. To find the cross section for an *unpolarised* incident wave (a random mixture of waves with any two orthogonal polarisation vectors, $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$), we perform a statistical average of

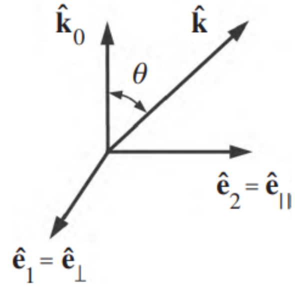


Figure 21.3: Orthogonal linear polarisation vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$. The latter lies in the scattering plane defined by $\hat{\mathbf{k}}_0$ and $\hat{\mathbf{k}}$.

(21.11) over the two polarisations:

$$\left.\frac{d\sigma_{\text{Thom}}}{d\Omega}\right|_{\text{unpol}} = \frac{1}{2}\sum_{m=1}^{2} r_e^2(1 - |\hat{\mathbf{k}} \cdot \hat{\mathbf{e}}_m|^2) \tag{21.15}$$

Using the polarization vectors defined in Figure 21.3 to evaluate the contributions to (21.15), we find immediately that

$$\frac{d\sigma_{\perp}}{d\Omega} = r_e^2 \qquad \text{and} \qquad \frac{d\sigma_{\parallel}}{d\Omega} = r_e^2 \cos^2\theta \tag{21.16}$$

Therefore,

$$\left.\frac{d\sigma_{\text{Thom}}}{d\Omega}\right|_{\text{unpol}} = \frac{1}{2}\left[\frac{d\sigma_{\perp}}{d\Omega} + \frac{d\sigma_{\parallel}}{d\Omega}\right] = \frac{1}{2}r_e^2(1 + \cos^2\theta) \tag{21.17}$$

Unlike Eq (21.11), the cross section (21.17) for scattering unpolarised waves is non-zero at every scattering angle. The total Thomson cross section is the integral of (21.17) over all these angles:

$$\sigma_{\text{Thom}} = \int \left.\frac{d\sigma_{\text{Thom}}}{d\Omega}\right|_{\text{unpol}} d\Omega = \frac{8\pi}{3}r_e^2 \tag{21.18}$$

The information in Eq (21.16) also leads naturally to a definition for the ***degree of polarisation*** of the scattered radiation. This is

$$\Pi(\theta) = \frac{\frac{d\sigma_{\perp}}{d\Omega} - \frac{d\sigma_{\parallel}}{d\Omega}}{\frac{d\sigma_{\perp}}{d\Omega} + \frac{d\sigma_{\parallel}}{d\Omega}} = \frac{\sin^2\theta}{1 + \cos^2\theta} \tag{21.19}$$

The electric dipole scattering of unpolarised waves peaks in the forward ($\theta = 0$) and backward ($\theta = \pi$) directions, and that the radiation is 100% linearly polarised for scattering at right angles ($\theta = \pi/2$) to the direction of incidence.

AZ 21.4 – Rayleigh Scattering

Rayleigh scattering occurs when an electromagnetic plane wave impinges on a small dielectric or conducting object. By small, we mean that all characteristic linear dimensions of the object are small compared to the wavelength, $\lambda = 2\pi/k_0$, of the plane wave. The incident **E** and **B** fields are nearly constant over the object's volume and the far-zone field is dominated by the radiation produced by the time-harmonic electric and magnetic dipole moments induced in the object. We compute the cross section as

$$\frac{d\sigma_{\text{Ray}}}{d\Omega} = \left(\frac{k_0^2}{4\pi\epsilon_0 E_0 c}\right)^2 \left|\hat{\mathbf{k}} \times \mathbf{m} + \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times c\mathbf{p})\right|^2 \tag{21.20}$$

The simplest example is a non-magnetic dielectric object with electric polarisability α where the electric dipole moment is

$$\mathbf{p} = \alpha\epsilon_0 E_0 \hat{\mathbf{e}}_0 \tag{21.21}$$

The corresponding cross section is

$$\frac{d\sigma_{\text{Ray}}}{d\Omega} = \left(\frac{k_0^2\alpha}{4\pi}\right)^2 \left(1 - \left|\hat{\mathbf{k}} \cdot \hat{\mathbf{e}}_0\right|\right) \tag{21.22}$$

On dimensional grounds alone, the polarisability scales with the object volume, which is of the order of a^3. Therefore, the total Rayleigh scattering cross section, σ_{Ray}, obtained by integrating Eq (21.22) over all angles, is much smaller than geometric cross section of the object:

$$\sigma_{\text{Ray}} \sim a^2(k_0\, a)^4 \ll a^2 \tag{21.23}$$

The electric dipole Rayleigh cross section (21.22) and the Thomson cross section (21.11) have exactly the same angular dependence. Therefore, the discussion leading to the $(1 + \cos^2\theta)$ dependence in Eq (21.20) for an unpolarised incident wave remains valid for electric dipole Rayleigh scattering. The same is true for the discussion leading to the degree of polarization in Eq (21.19).

21.4.1 – Atmospheric Colour

The striking λ^{-4} dependence of Eq (21.22) is the origin of the blue colour of the daylight sky, and the red colour of the setting Sun. Because $\lambda_{\text{blue}} < \lambda_{\text{red}}$, Eq (21.23) implies that blue light is the dominant component of the scattered light. We see red light looking directly at the Sun at sunset because the blue light has been scattered out of our line of sight. This explanation, however, glosses over some subtleties.