MATH3403 Assignment 3

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Question 1

Assume that there are two solutions, u and v. Take $w = u - v$, which is a solution in itself (by the superposition principle). The system is now

$$\begin{cases} w_{tt} - c^2 w_{xx} + \alpha w_t = 0 \qquad (*) \\ w_x(t,\, x=0) = w_x(t,\, x=l) = 0 \\ w(t=0,\, x) = 0 \\ w_t(t=0,\, x) = 0 \end{cases}$$

Start by integrating $(*)$ against w_t with respect to x:

$$\begin{aligned} 0 &= \int_0^l (w_{tt} - c^2 w_{xx} + \alpha w_t)\partial_t w dx \\ &= \int_0^l w_{tt}\partial_t w dx - c^2 \int_0^l w_{xx}\partial_t w dx + \alpha \int_0^l {w_t}^2 dx \\ &= \int_0^l w_{tt}\partial_t w dx - \left[c^2 w_x w_t\right]_0^l + c^2 \int_0^l w_x w_{xt} dx + \alpha \int_0^l {w_t}^2 dx \end{aligned}$$

But since the boundary conditions have $w_x\big|_0^l = 0$,

$$\begin{aligned} \Rightarrow 0 &= \int_0^l w_{tt}\partial_t w dx + c^2 \int_0^l w_x w_{xt} dx + \alpha \int_0^l {w_t}^2 dx \\ &= \frac{d}{dt}\int_0^l \frac{1}{2}(w_t)^2 dx + c^2 \frac{d}{dt}\int_0^l \frac{1}{2}(w_x)^2 + \alpha \int_0^l (w_t)^2 dx \\ &= \frac{d}{dt}\left(\int_0^l \frac{1}{2}(w_t)^2 dx + c^2 \int_0^l \frac{1}{2}(w_x)^2\right) + \alpha \int_0^l (w_t)^2 dx \\ &= \frac{d}{dt}\varepsilon[w](t) + \alpha \int_0^l (w_t)^2 dx \\ \Rightarrow \frac{d}{dt}\varepsilon[w](t) &= -\alpha \int_0^l (w_t)^2 dx \end{aligned}$$

where $\varepsilon[w](t) = \frac{1}{2}\int_0^l [(w_t)^2 dx + c^2 (w_x)^2] dx$. But we have initial conditions

$$\partial_t w(t=0,\, x) = 0 \qquad \text{and} \qquad w(t=0,\, x) = 0$$
$$\Rightarrow w_x(t=0,\, x) = 0$$

$$\Rightarrow \varepsilon[w](t=0) = 0$$
$$\frac{d}{dt}\varepsilon[w](t=0) = -\alpha \int_0^l (0)^2 dx = 0$$

And so
$$\left.\begin{matrix} \frac{d}{dt}\varepsilon[w](t=0) = 0 \\ \varepsilon[w](t=0) = 0 \end{matrix}\right\} \Rightarrow \varepsilon[w](t) = 0 = \frac{1}{2}\int_0^l [(w_t)^2 dx + c^2(w_x)^2] dx$$

Since both $(w_t)^2$ and $(w_x)^2$ are continuous and positive, for $\varepsilon[w](t)$ to be zero, both $w_t = 0$ and $w_x = 0$. Using $w(0,\, x) = 0$ and $w_t = 0$, $w = 0$ $\forall t \geq 0$.

$$\Rightarrow 0 = u - v \Rightarrow u = v$$

Therefore there is only one solution to the initial problem, and is unique.

Question 2

We have the system
$$\begin{cases} \partial_t u - \partial_{xx} u + 2\partial_x u = 0 \\ u(t,\, x=0) = u(t,\, x=1) = 0 \\ u(t=0,\, x) = x(1-x) \end{cases}$$

This can be solved by the separation of variables method, so set $u(t,x) = T(t)X(x)$;

$$\Rightarrow \partial_t T(t)X(x) - \partial_{xx} T(t)X(x) + 2\partial_x T(t)X(x) = 0$$
$$\Rightarrow T_t(t)X(x) = T(t)X_{xx}(x) - 2T(t)X_x(x)$$

Dividing both sides by $T(t)X(x)$ yields
$$\frac{T_t}{T} = \frac{X_{xx}}{X} - 2\frac{X_x}{X}$$

The next step is to find a λ such that
$$\frac{T_t}{T} = \lambda = \frac{X_{xx}}{X} - 2\frac{X_x}{X}$$

Begin with the right hand side (case for $X(x)$), remembering the initial conditions $X(0) = X(1) = 0$:

$$\frac{X_{xx}}{X} - 2\frac{X_x}{X} = \lambda \Rightarrow X_{xx} - 2X_x - \lambda X = 0$$

This is a second order ODE. Guess that the solution is $X(x) = e^{\alpha x}$. Then,

$$\Rightarrow X_x = \alpha e^{\alpha x} \qquad \text{and} \qquad X_{xx} = \alpha^2 e^{\alpha x}$$
$$\Rightarrow \alpha^2 e^{\alpha x} - 2\alpha e^{\alpha x} - \lambda e^{\alpha x} = 0$$

Dividing by $e^{\alpha x}$ and solving for α:

$$\alpha^2 - 2\alpha - \lambda = 0$$
$$\Rightarrow \alpha_{1,2} = \frac{2 \pm \sqrt{4 + 4\lambda}}{2} = 1 \pm \sqrt{1+\lambda}$$

Now we're left with three possible cases for the value of λ: $\lambda = -1$, $\lambda > -1$, and $\lambda < -1$:

a. $\lambda = -1 \Rightarrow \alpha_1 = \alpha_2 = 1$ and the general solution becomes

$$\begin{aligned}
\Rightarrow X(x) &= (A + Bx)e^x \\
X(0) &= 0 = Ae^0 = A \\
\Rightarrow X(x) &= Bxe^x \\
X(1) &= 0 = Be^1 \Rightarrow B = 0
\end{aligned}$$

Therefore only the trivial solution applies for $\lambda = -1$.

b. $\lambda > -1 \Rightarrow \alpha_1 = 1 + \beta \;\; \alpha_2 = 1 - \beta$ where $\beta = \sqrt{1+\lambda} \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned}
\Rightarrow X(x) &= Ae^{(1+\beta)x} + Be^{(1-\beta)x} \\
X(0) &= 0 = A + B \Rightarrow B = -A \\
\Rightarrow X(x) &= A\left(e^{(1+\beta)x} - e^{(1-\beta)x}\right) \\
X(1) &= 0 = A\left(e^{(1+\beta)} - e^{(1-\beta)}\right) \\
&\Rightarrow A = 0 \text{ (trivial) or} \\
e^{(1+\beta)} - e^{(1-\beta)} &= 0 \\
1 + \beta - (1 - \beta) &= 0 \Rightarrow 2\beta = 0
\end{aligned}$$

but $\beta = \sqrt{1+\lambda} \neq 0$ (since $1 + \lambda > 0$). Therefore only the trivial solution applies in this case too.

c. $\lambda < -1 \Rightarrow \beta = \sqrt{1+\lambda} = i\sqrt{|\lambda+1|}$
$\Rightarrow \alpha_1 = 1 + i\sqrt{|\lambda+1|} \qquad \alpha_2 = 1 - i\sqrt{|\lambda+1|} \Rightarrow \alpha_{1,2} = r \pm is$ where $r = 1$ and $s = \sqrt{|\lambda+1|}$. The general solution is then

$$\begin{aligned}
X(x) &= e^{rx}(A\cos(sx) + B\sin(sx)) \\
&= e^x\left(A\cos\left(\sqrt{|\lambda+1|}x\right) + B\sin\left(\sqrt{|\lambda+1|}x\right)\right) \\
X(0) &= 0 = e^0 A \Rightarrow A = 0 \\
\Rightarrow X(x) &= Be^x \sin\left(\sqrt{|\lambda+1|}x\right) \\
X(1) &= 0 = Be\sin\left(\sqrt{|\lambda+1|}\right) \\
&\Rightarrow B = 0 \text{ (trivial) or} \\
\sin\left(\sqrt{|\lambda+1|}\right) &= 0 \\
\Rightarrow \sqrt{|\lambda+1|} &= n\pi \\
|\lambda+1| &= n^2\pi^2
\end{aligned}$$

Since $\lambda + 1 < 0$,

$$\begin{aligned}
\lambda + 1 &= -n^2\pi^2 \\
\Rightarrow \lambda &= -n^2\pi^2 - 1 \\
\Rightarrow X_n(x) &= B_n e^x \sin(n\pi x)
\end{aligned}$$

which is a suitable solution for the eigenfunctions of $X(x)$.

Now to examine the case for $T(t): \frac{T_t}{T} = \lambda \Rightarrow T_t = \lambda T$.
Guess the solution as $T = e^{\lambda t} \Rightarrow T_t = \lambda e^{\lambda t} = \lambda T$. Therefore the eigenfunctions of T are $T_n(t) = e^{(-n^2\pi^2-1)t}$,

and the general solution of $u(t,\, x)$ is

$$u(t,x) = \sum_{n=1}^{\infty} e^{(-n^2\pi^2-1)t} B_n e^x \sin(n\pi x)$$

Now need to find a B_n that satisfies the initial condition

$$u(t=0,x) = x(1-x) = \sum_{n=1}^{\infty} B_n e^x \sin(n\pi x) \qquad (*)$$

Integrate $(*)$ against $\sin(m\pi x)$ with respect to x:

$$\begin{aligned}
\sum_{n=1}^{\infty} B_n \int_0^1 e^x \sin(n\pi x)\sin(m\pi x)\ dx &= \int_0^1 x(1-x)\sin(m\pi x)\ dx \\
\Rightarrow \sum_{n=1}^{\infty} B_n \int_0^1 \sin(n\pi x)\sin(m\pi x)\ dx &= \int_0^1 \frac{x(1-x)}{e^x}\sin(m\pi x)\ dx
\end{aligned}$$

From result in lectures, $\int_0^l \sin(n\pi x)\sin(m\pi x)\ dx = \frac{l}{2}\delta_{mn}$ which is only non-trivial when $m=n$,

$$\Rightarrow \frac{1}{2}B_m = \int_0^1 \frac{x(1-x)}{e^x}\sin(m\pi x)\ dx$$

Using wolfram alpha (pro computation time) to solve this yields

$$\frac{1}{2}B_m = \frac{4e\pi m(\pi^2m^2-1) - (\pi^4m^4+6\pi^2m^2-3)\sin(\pi m) + 8\pi m\cos(\pi m)}{e(\pi^2m^2+1)^3}$$

Given that $m \in \mathbb{N}$, $\sin(\pi m) = 0$ and so

$$B_m = \frac{8e\pi m(\pi^2m^2-1) - 0 + 16\pi m\cos(\pi m)}{e(\pi^2m^2+1)^3}$$

But $\cos(\pi m)$ switches between -1 and 1, beginning at -1 for $m=1$:

$$\begin{aligned}
B_m &= \frac{8e\pi m(\pi^2m^2-1) + 16\pi m(-1)^m}{e(\pi^2m^2+1)^3} \\
&= \frac{8\pi m\left(e(\pi^2m^2-1) + 2(-1)^m\right)}{e(\pi^2m^2+1)^3}
\end{aligned}$$

As this is only valid for cases where $n=m$ (since $\delta_{mn}=1$), $B_m = B_n$ and so the n eigenfunction for $u(t,x)$ is

$$u_n(t,\, x) = \frac{8\pi n\left(e(\pi^2n^2-1) + 2(-1)^n\right)}{e(\pi^2n^2+1)^3} e^x \sin(n\pi x) e^{-(n^2\pi^2+1)t}$$

with solution

$$u(t,\, x) = \sum_{n=1}^{\infty} \frac{8\pi n\left(e(\pi^2n^2-1) + 2(-1)^n\right)}{e(\pi^2n^2+1)^3} e^x \sin(n\pi x) e^{-(n^2\pi^2+1)t}$$

Question 3

Take the SL system

$$\begin{cases} x^2y'' + \lambda y = 0 \\ y(1) = y(e) = 0 \end{cases}$$

Guess a solution $y = x^n \Rightarrow y' = nx^{n-1} \Rightarrow y'' = n(n-1)x^{n-2}$
The original ODE is then

$$\begin{aligned} x^2n(n-1)x^{n-2} + \lambda x^n &= 0 \\ n(n-1)x^n + \lambda x^n &= 0 \end{aligned}$$

Dividing by x^n gives

$$\begin{aligned} n^2 - n + \lambda = 0 \\ \Rightarrow n_{1,2} = \frac{1 \pm \sqrt{1-4\lambda}}{2} \end{aligned}$$

Now there are three possible cases:

a. $\lambda = 1/4 \Rightarrow n_1 = n_2 = 1/2$
   The general solution in this case is

$$\begin{aligned} y(x) &= Ax^{1/2}\ln(x) + Bx^{1/2} \\ y(1) &= 0 \Rightarrow A\ln(1) + B = 0 \Rightarrow B = 0 \\ \Rightarrow y(x) &= Ax^{1/2}\ln(x) \\ y(e) &= 0 = Ae^{1/2}\ln(e) \Rightarrow A = 0 \end{aligned}$$

   Therefore there is only the trivial solution for $\lambda = 1/4$.

b. $\lambda < 1/4 \Rightarrow$ Define $\beta = \frac{1}{2}\sqrt{1-4\lambda} \in \mathbb{R} \setminus \{0\}$

$$\Rightarrow n_1 = 1/2 + \beta \qquad n_2 = 1/2 - \beta$$

   This case has general solution

$$\begin{aligned} y(x) &= Ax^{1/2}x^{\beta} + Bx^{1/2}x^{-\beta} = x^{1/2}\left(Ax^{\beta} + Bx^{-\beta}\right) \\ y(1) &= 0 = A + B \Rightarrow B = -A \\ \Rightarrow y(x) &= Ax^{1/2}\left(x^{\beta} - x^{-\beta}\right) \\ y(e) &= 0 = Ae^{1/2}\left(e^{\beta} - e^{-\beta}\right) \\ \Rightarrow A &= 0 \text{ (trivial) or} \\ e^{\beta} - e^{-\beta} &= 0 \Rightarrow \beta + \beta = 0 \Rightarrow \beta = 0 \end{aligned}$$

   But $\beta \neq 0$ since $\lambda < 1/4 \Rightarrow \frac{1}{2}\sqrt{1-4\lambda} > 0$. This means that only the trivial solution is valid for this case. This leaves:

c. $\lambda > 1/4 \;\Rightarrow\; n_1 = 1/2 + i\sqrt{|1-4\lambda|}; \;\; n_2 = 1/2 - i\sqrt{|1-4\lambda|}$

This has general solution

$$
\begin{aligned}
y(x) &= Ax^{1/2}\cos\left(\sqrt{|1-4\lambda|}\ln(x)\right) + Bx^{1/2}\sin\left(\sqrt{|1-4\lambda|}\ln(x)\right)\\
y(1) &= 0 = A\cos\left(\sqrt{|1-4\lambda}\ln(1)\right) + B\sin\left(\sqrt{|1-4\lambda|}\ln(1)\right) \Rightarrow A = 0\\
\Rightarrow y(x) &= Bx^{1/2}\sin\left(\sqrt{|1-4\lambda|}\ln(x)\right)\\
y(e) &= 0 = Be^{1/2}\sin\left(\sqrt{|1-4\lambda|}\ln(e)\right) \Rightarrow Be^{1/2}\sin\left(\sqrt{|1-4\lambda|}\right) = 0\\
&\Rightarrow B = 0 \text{ (trivial) or}
\end{aligned}
$$

$$
\begin{aligned}
\sin\left(\sqrt{|1-4\lambda|}\right) &= 0\\
\Rightarrow \sqrt{|1-4\lambda|} &= n\pi\\
|1-4\lambda| &= n^2\pi^2\\
1-4\lambda &= -n^2\pi^2\\
\Rightarrow \lambda_n &= \frac{n^2\pi^2+1}{4} \Rightarrow \sqrt{|1-4\lambda|} = n\pi
\end{aligned}
$$

Which is an appropriate form for the eigenvalues.

And so the solution eigenfunctions (for eigenvalues λ_n) are

$$y_n(x) = B_n x^{1/2}\sin\left(n\pi\ln(x)\right)$$

Now, to find the associated scalar product, express the original ODE as

$$x^2y'' = -\lambda y$$

which is of the form of an SL system with

$$A(x) = x^2 \qquad B(x) = C(x) = 0$$

So

$$
\begin{aligned}
p(x) &= e^{\int \frac{B(x)}{A(x)}\,dx} = e^0 = e\\
\Rightarrow \sigma(x) &= \frac{p(x)}{A(x)} = \frac{e}{x^2}
\end{aligned}
$$

From the lectures, the scalar product for a general SL system, for two arbitrary functions of x, f and g, on the domain $[a,\, b]$ is

$$\langle f,\, g\rangle = \int_a^b f(x)g(x)\sigma(x)\ dx$$

Substituting in the weight function $\sigma(x)$ on the domain $[1,\, e]$ gives

$$\langle f,\, g\rangle = e\int_1^e f(x)g(x)\frac{1}{x^2}\ dx$$

The scalar product for $y_n(x)$ against itself is thus

$$\begin{aligned}
\langle y_n,\, y_n \rangle &= e{B_n}^2 \int_1^e \frac{x \sin^2(n\pi \ln(x))}{x^2}\, dx \\
&= e{B_n}^2 \int_1^e \frac{\sin^2(n\pi \ln(x))}{x}\, dx \\
&= e{B_n}^2 \left[\frac{\ln(x)}{2} - \frac{\sin(2n\pi \ln(x))}{4n\pi} \right]_1^e \\
&= e{B_n}^2 \left[\frac{1}{2} - \frac{\sin(2n\pi)}{4n\pi} - 0 + \frac{\sin(0)}{4n\pi} \right]
\end{aligned}$$

But since $n \in \mathbb{N}$, $\sin(2n\pi) = 0$ and

$$\langle y_n,\, y_n \rangle = \frac{e}{2}{B_n}^2$$