PHYS2100 Final Exam - Ryan White s4499039

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Question 1:

A force, which drives the velocity, $v = f(x)$, is conservative if there exists a potential, V, such that

$$\vec{F}\cdot\vec{v} = -\frac{dV}{dt}$$

Take the potential $\quad V(x) = -\frac{1}{2}m\left(f(x)\right)^2$

$$\Rightarrow \quad \vec{F}\cdot\frac{dx}{dt} = -\frac{dV}{dx}\frac{dx}{dt}$$

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$$m\ddot{a} = \frac{1}{2}m\cdot\frac{d\,f(x)^2}{dx}$$

$$\frac{dv}{dt} = \frac{1}{2}\cdot 2\, f'(x)$$

$$dv = \int f'(x)\,dt$$

$$v = f(x) + C$$

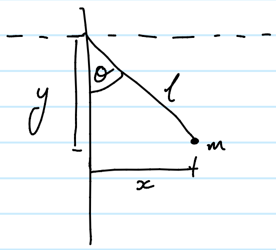
Alternatively, the force (and so the potential) is conservative if the Jacobi matrix, defined by

$$J_{\vec{F}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is symmetric.

Since this is in one dimension, there is only one entry in the Jacobi matrix, and is trivially symmetric $\Rightarrow$ conservative.

Question 2:



i) Normally,

$$\underline{r} = l\sin\theta\,\underline{i} - l\cos\theta\,\underline{j}$$

But only y is to be used.

$$x^2+y^2 = l^2 \Rightarrow x = \sqrt{l^2-y^2}$$

$$\Rightarrow \quad \underline{r} = \sqrt{l^2-y^2}\,\underline{i} - y\,\underline{j}$$

$$\Rightarrow \quad \frac{d\underline{r}}{dy} = \frac{d}{dy}\sqrt{l^2-y^2}\,\underline{i} - 1\,\underline{j}$$

$$= \frac{1}{2\sqrt{l^2-y^2}}\cdot -2y\,\underline{i} - 1\,\underline{j}$$

$$= -\frac{y}{\sqrt{l^2-y^2}}\,\underline{i} - 1\,\underline{j}$$

The force in this instance is due to gravity and tension:

$$\vec{F} = -mg\,\underline{j} - T\,\hat{\underline{r}}$$

The generalised force is then

$$Q_y = F\cdot\frac{dr}{dy} = -mg\cdot -1$$

The generalised force is then

$$Q_y = \underline{F} \cdot \frac{dr}{dy} = -mg \cdot -1 = mg$$

ii) The kinetic energy is given by

$$T = \frac{1}{2} m v^2 \quad \text{with} \quad \frac{dT}{dt} = -\frac{dV}{dt}$$

and $Q_i = -\frac{\partial V}{\partial q_i} \Rightarrow Q_y = -\frac{\partial V}{\partial y}$

$$\Rightarrow V = -\int Q_y \, dy = -mgy + c_1$$

$$\Rightarrow \frac{\partial T}{\partial t} = -\frac{\partial V}{\partial t} = mg\dot{y}$$

$$\Rightarrow T = mg\dot{y}t + c_2$$

iii) The Lagrangian is given by

$$L = T - V$$

$$\Rightarrow L = mg\dot{y}t + c_2 - (-mgy + c_1) = mg(\dot{y}t - y) + C$$

where $C = c_2 - c_1$ (I'm so sorry about this question)

Question 3:

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(5\dot{x}^2 + \dot{y}^2 - 4\dot{x}\dot{y}) + C(2x - y)$$

i) There are two degrees of freedom, one for each of x and y.

ii) Lagrange's Equation (for a conservative system) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

We can then compute:

$$\frac{\partial L}{\partial x} = 2C \qquad \frac{\partial L}{\partial y} = -C$$

$$\frac{\partial L}{\partial \dot{x}} = 5\dot{x} - 2\dot{y} \qquad \frac{\partial L}{\partial \dot{y}} = \dot{y} - 2\dot{x}$$

Then, for x, Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}(5\dot{x} - 2\dot{y}) - 2C = 0$$

$$5\ddot{x} - 2\ddot{y} = 2C$$

$$\Rightarrow 5\ddot{x} = 2C + 2\ddot{y}$$

For y,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\Rightarrow \frac{d}{dt}(\dot{y} - 2\dot{x}) - -C = 0$$

$$\ddot{y} - 2\ddot{x} = -C$$

$$\Rightarrow \ddot{y} = 2\ddot{x} - C$$

$$\ddot{y} - 2\ddot{x} = -C$$

$$\Rightarrow \quad \ddot{y} = 2\ddot{x} - C$$

$$\Rightarrow \quad 5\ddot{x} = 2C + 2(2\ddot{x} - C)$$

$$\Rightarrow \quad \ddot{x} = 0$$

$$\Rightarrow \quad \ddot{y} = -C$$

iii) $H = p_x \dot{x} + p_y \dot{y} - L$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} \qquad \alpha = x, y$$

as before, $\frac{\partial L}{\partial \dot{x}} = 5\dot{x} - 2\dot{y} = p_x$

$$\frac{\partial L}{\partial \dot{y}} = \dot{y} - 2\dot{x} = p_y$$

To show that the two forms are equivalent, equate them:

$$p_x \dot{x} + p_y \dot{y} - L = \frac{1}{2}\left(p_x^2 + p_x p_y + 5p_y^2\right) - C(2x - y)$$

$$\Rightarrow 5\dot{x}^2 - 2\dot{x}\dot{y} + \dot{y}^2 - 2\dot{x}\dot{y} - \left(\frac{1}{2}\left(5\dot{x}^2 + \dot{y}^2 - 4\dot{x}\dot{y}\right) + C(2x - y)\right)$$

$$= \frac{1}{2}\left((5\dot{x} - 2\dot{y})^2 + 4(5\dot{x} - 2\dot{y})(\dot{y} - 2\dot{x}) + 5(\dot{y} - 2\dot{x})^2\right) - C(2x - y)$$

$$\Rightarrow \frac{1}{2} 5\dot{x}^2 - 4\dot{x}\dot{y} + 2\dot{x}\dot{y} + \frac{1}{2}\dot{y}^2 - C(2x - y)$$

$$= \frac{1}{2}\left(25\dot{x}^2 - 20\dot{x}\dot{y} + 4\dot{y}^2 + 20\dot{x}\dot{y} - 40\dot{x}^2 - 8\dot{y}^2 + 16\dot{x}\dot{y} + 5\dot{y}^2 - 20\dot{x}\dot{y} + 20\dot{x}^2\right) - C(2x - y)$$

$$\Rightarrow \frac{1}{2}\left(5\dot{x}^2 + \dot{y}^2 - 4\dot{x}\dot{y}\right) - C(2x - y)$$

$$= \frac{1}{2}\left(5\dot{x}^2 + \dot{y}^2 - 4\dot{x}\dot{y}\right) - C(2x - y)$$

and the two sides are equivalent, and so are the two forms of the Hamiltonian.

iv) Hamilton's equations of motion are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

So,

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x + 2p_y$$

but $p_x = 5\dot{x} - 2\dot{y}$ and $p_y = \dot{y} - 2\dot{x}$

$$\Rightarrow \quad \dot{x} = 5\dot{x} - 2\dot{y} + 2\dot{y} - 4\dot{x}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = 2p_x + 5p_y$$

$$= 10\dot{x} - 4\dot{y} + 10\dot{y} - 20\dot{x}$$

$$= -10\dot{x} + 6\dot{y}$$

$$\Rightarrow \quad -5\dot{y} = -10\dot{x}$$

$$\Rightarrow \quad \dot{y} = 2\dot{x}$$

This is equivalent to a constant (missing the constant C)

Question 4:

$$H = \frac{p^2}{2} + V(q) \qquad V(q) = -\frac{1}{2}q^2 + \frac{a}{4}q^4$$

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a. Hamiltons equations are

$$\dot{q} = \frac{\partial H}{\partial p} = p \qquad \dot{p} = -\frac{\partial H}{\partial q} = q - aq^3$$

There are fixed points when $(\dot{q}, \dot{p}) = (0,0)$

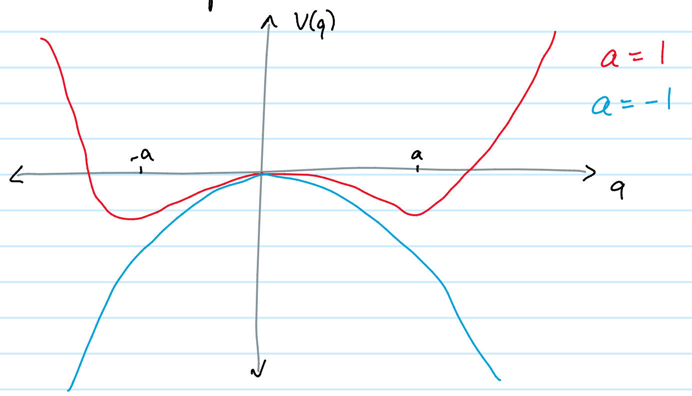
Therefore there is a fixed point at $p=0$ and when

$$0 = q - aq^3$$
$$\Rightarrow q = aq^3$$
$$\Rightarrow q^2 = \frac{1}{a}$$
$$\Rightarrow q = \pm\sqrt{\frac{1}{a}}$$

Therefore there are two fixed points at $(q,p) = (\sqrt{\frac{1}{a}}, 0)$ and $(-\sqrt{\frac{1}{a}}, 0)$

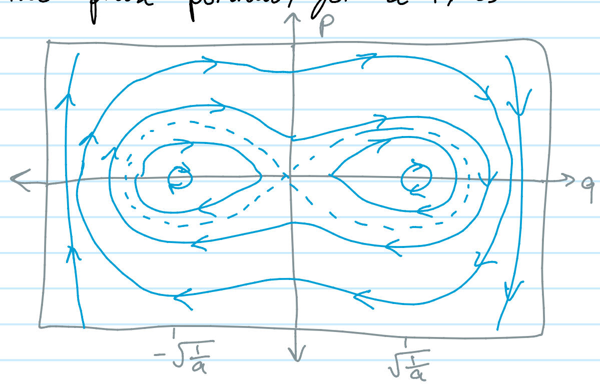
b. i) The potential is $V(q) = -\frac{1}{2}q^2 + \frac{a}{4}q^4$

This corresponds to



Notice that $a = -1$ corresponds to only one unstable fixed point at $q = 0$

ii) The phase portrait, for $a=1$, is



$q = \pm\sqrt{\frac{1}{a}}$ corresponds to stable potential minima

The separatrices have $H = 0$

$$\Rightarrow \frac{p^2}{2} = \frac{1}{2}q^2 - \frac{a}{4}q^4$$
$$p = \pm q\sqrt{1 - \frac{a}{2}q^2}$$

The separatrices are shown by the dotted line in the phase portrait above.

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The directions are given by:

$$\left.\begin{array}{lll} +p & \Rightarrow & +\text{ve } \dot{q} \\ -p & \Rightarrow & -\text{ve } \dot{q} \end{array}\right\} - \text{since } \dot{q} = p$$

Question 5:

a. A Lyapunov exponent roughly classifies how chaotic a system is. Specifically, it gives the average rate of divergence of the vector field.
If $\lambda > 0$, then trajectories in phase space diverge $\Rightarrow$ the system is chaotic.

b.
$$f(x) = \begin{cases} 2rx & 0 \le x \le \frac{1}{4} \\ 2r(\frac{1}{2} - x) & \frac{1}{4} \le x \le \frac{1}{2} \\ 2r(x - \frac{1}{2}) & \frac{1}{2} \le x \le \frac{3}{4} \\ 2r(1-x) & \frac{3}{4} \le x \le 1 \end{cases}$$

The Lyapunov exponent is

$$\lambda := \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_0)|$$

for all $0 \le x \le 1$, $|f'(x_0)| = 2r$:

$$f'(0 \le x_0 \le \tfrac{1}{4}) = 2r \Rightarrow |f'(x_0)| = 2r$$
$$f'(\tfrac{1}{4} \le x_0 \le \tfrac{1}{2}) = -2r \Rightarrow |f'(x_0)| = 2r$$
$$f(\tfrac{1}{2} \le x_0 \le \tfrac{3}{4}) = 2r \Rightarrow |f'(x_0)| = 2r$$
$$f'(\tfrac{3}{4} \le x_0 \le 1) = -2r \Rightarrow |f'(x_0)| = 2r$$

$$\Rightarrow \lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln(2r)$$
$$= \lim_{n\to\infty} \frac{(n-1)\ln(2r)}{n}$$
$$= \ln(2r) - \lim_{n\to\infty} \frac{1}{n}\ln(2r)$$
$$= \ln(2r)$$

One expects chaotic behaviour when $\lambda > 0$

i.e. when $\ln(2r) > 0$

$$2r > e^0 = 1$$
$$\Rightarrow r > \tfrac{1}{2}$$

$\therefore$ For the 1D map defined by $f(x)$, chaotic behaviour occurs for $r > \frac{1}{2}$

Question 6:

i) If two events occur on the trajectory of a massive particle (which rules out null worldlines), then the two events must be timelike separated which corresponds to a negative spacetime interval.
Since $c = 1$ in the spacetime interval, if $dx^2 + dy^2 + dz^2 > dt^2$ for the two events, then a particle travelling at c could never experience both events. Since the massive particle (which has $v < c$) experiences both.

then a" particle travelling at c could never experience both events. Since the massive particle (which has $v<c$) experiences both, it must have $dt^2 > dx^2 + dy^2 + dz^2$

$\Rightarrow \quad ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 < 0$

which means the spacetime interval is timelike separated.

ii) In K, $\Delta x = 2m$

In K', $\Delta x'$ is obtained by finding the distance between events at each end of the rod in K'

$\Rightarrow \quad \Delta x' = \gamma(\Delta x - V\Delta t) = \Delta x (\gamma)^{-1}$

$$\Rightarrow \quad \Delta x' = \Delta x \sqrt{1-v^2} = 2m\sqrt{1-0.8^2} = 1.2\,m$$

Question 7:

i. $u^\alpha = (F(\tau), G(\tau), 0, 0) = (dt, dx, dy, dz)$

$$V = \frac{(dx, dy, dz)}{dt} = \frac{G(\tau)}{F(\tau)}$$

ii. $$\vec{a} = \frac{d\vec{v}}{d\tau} = d\frac{G(\tau)}{F(\tau)} \Big/ d\tau = G(\tau)\frac{d(F(\tau)^{-1})}{d\tau} + \frac{1}{F(\tau)} \cdot \frac{dG(\tau)}{d\tau}$$

iii. The 4-acceleration is spacelike, since it's orthogonal to the 4-velocity at all points on the worldline, and 4-velocity is timelike.

Question 8:

a. Due to conservation of 4-momentum:

$$p_i = p_f$$

$$\sum_{n=1}^{k} p_{n,i} = \sum_{n=1}^{k} p_{n,f}$$

$$\Rightarrow \quad mv - 3mv = M'v'$$

$$-2Mv = M'v'$$

Since mass cannot be negative, the combined lump is travelling in the $-x$ direction after collision.