PHYS2100 Assignment 1

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Question 1

a. With $(ct_1,\, x_1) = (3, 7)$ and $(ct_2,\, x_2) = (4, 5)$, the spacetime interval between event 1 and event 2 is

$$\begin{aligned}(\Delta s)^2 &= -(\Delta ct)^2 + (\Delta x)^2 \\ &= -(3-4)^2 + (7-5)^2 \\ &= -1 + 4 = 3 > 0\end{aligned}$$

And so the two events are spacelike separated.

b. Transforming coordinates from $K \to K'$, $(ct_1, x_1) \to (ct_1', x_1')$ and $(ct_2, x_2) \to (ct_2', x_2')$, where the relative velocity between the frames is $V = -2 \times 10^8$ m/s $\approx -2/3c$. Firstly, calculate the lorentz transform

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{1}{\sqrt{1 - \left(\frac{2}{3}\right)^2}} = \frac{1}{\sqrt{1 - \frac{4}{9}}} = \frac{1}{\sqrt{\frac{5}{9}}} = \frac{\sqrt{9}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

Now, each of the respective transforms can be calculated:

$$\begin{aligned}ct_1' &= \gamma\left(ct_1 - V\frac{x_1}{c}\right) = \frac{3\sqrt{5}}{5}\left(3 + \frac{2c}{3}\frac{7}{c}\right) = \frac{23\sqrt{5}}{5} \\ ct_2' &= \gamma\left(ct_2 - V\frac{x_2}{c}\right) = \frac{3\sqrt{5}}{5}\left(4 + \frac{2c}{3}\frac{5}{c}\right) = \frac{22\sqrt{5}}{5} \\ x_1' &= \gamma(x_1 - Vt_1) = \frac{3\sqrt{5}}{5}\left(7 + \frac{2c}{3}\frac{3}{c}\right) = \frac{27\sqrt{5}}{5} \\ x_2' &= \gamma(x_2 - Vt_2) = \frac{3\sqrt{5}}{5}\left(5 + \frac{2c}{3}\frac{4}{c}\right) = \frac{23\sqrt{5}}{5}\end{aligned}$$

And with these transforms, the spacetime interval in K' can be calculated,

$$\begin{aligned}(\Delta s')^2 &= -(\Delta ct')^2 + (\Delta x')^2 \\ &= -\left(-\frac{22\sqrt{5}}{5} - \frac{23\sqrt{5}}{5}\right)^2 + \left(\frac{23\sqrt{5}}{5} - \frac{27\sqrt{5}}{5}\right)^2 \\ &= -\frac{1}{5} + \frac{16}{5} = 3 = (\Delta s)^2\end{aligned}$$

Therefore, the spacetime interval remains invariate across reference frame changes, as required by non-inertial frames.

c. By part b. above, event 1 (shown with the yellow light cone in K', and the red light cone in K) occurs *after* event 2 (shown with the purple light cone in K, and the green light cone in K') in reference frame K' (shown with the blue light cone). Figure 1 shows that even though event 1 happens before event 2 in K, it experiences event 2 first. K' experiences event 1 after it sees event 2 (shown with the green light cone in K' and the purple light cone in K) just like in K.

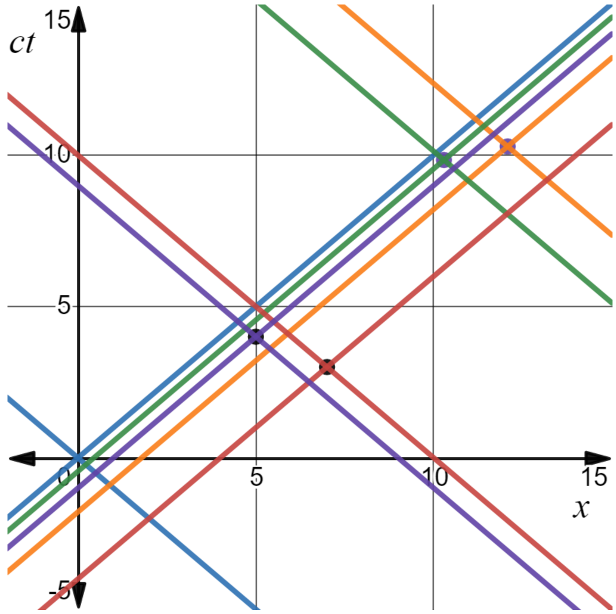

Figure 1: Lightcone of K' and Events in that Frame

That is, an observer in either frame would see event 2, and then event 1 shortly after. Since the events are spacelike separated, there is no logical progression between the two events as they can have no impact on each other and so this does not create a paradox.

Question 2

For this question, approximate each reference frame (the clock on the surface of the Earth, and the clock on the satellite orbiting Earth) as inertial. The velocity of the satellite at an altitude of 300km can be calculated using a known circular orbital velocity equation, and the radius and mass of the Earth, $r_\oplus = 6.38 \times 10^6$m and $M_\oplus = 5.97 \times 10^24$kg, where r is the distance from the center of mass of the planet and r_o is the orbital altitude:

$$
\begin{aligned}
v = \sqrt{\frac{GM_\oplus}{r}} &= \sqrt{\frac{GM_\oplus}{r_\oplus + r_o}} \\
&= \sqrt{\frac{6.67 \times 10^{-11} \cdot 5.97 \times 10^{24}}{6.38 \times 10^6 + 3 \times 10^5}} \\
&\approx 7.7 \times 10^3 \text{ m/s}
\end{aligned}
$$

Since the observer on Earth is also moving with some velocity, v_s, the *relative* velocity between the surface and the satellite in orbit must be found, denoted by V:

$$
V = v - v_s = 7.7 \times 10^3 - \frac{2\pi r_\oplus}{P} = 7.7 \times 10^3 - \frac{2\pi \cdot 6.38 \times 10^6}{86400} \approx 7.2 \times 10^6 \text{ m/s}
$$

Finally, the difference in time observed by the satellite in orbit is

$$\begin{aligned}
\delta t &= \Delta t' - \Delta t \\
&= \Delta t \sqrt{1 - \frac{V^2}{c^2}} - \Delta t \\
&= \Delta t \left(\sqrt{1 - \frac{V^2}{c^2}} - 1 \right) \\
&= 86400 \left(\sqrt{1 - \frac{(7200)^2}{c^2}} - 1 \right) \\
&\approx -2.5 \times 10^{-5}\,\mathrm{s} \\
&= -25 \text{ Microseconds}
\end{aligned}$$

And so the clock on the surface of Earth experiences 25 microseconds more time than the satellite in a 300km orbit.

Question 3

The spacetime diagram for Alice's journey, from Bob's perspective, is shown in Figure 2. In this figure, the yellow lines show Bob's light cone, the blue line shows Alice's first stretch of the journey, the green line her second, and the red line Bob's 'journey' through time.

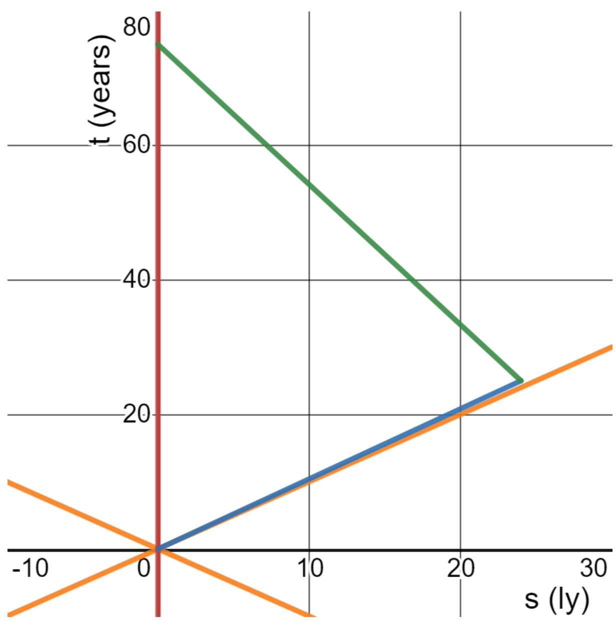

Figure 2: Spacetime Diagram of Alice's Journey from Bob's Perspective

Alice resides in a frame K', while Bob is in K. Let the apostrophe symbolise journey parameters according to Alice and K'. For the first stretch, $\Delta t_1' = 7$ years, and $V = 0.96c$. Δt_1 can then be calculated:

$$\Delta t_1 = \frac{\Delta t_1'}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{7}{\sqrt{1 - 0.96^2}} = 25 \text{ years}$$

From Alice's perspective, she is travelling $0.96c$ over 7 years, so $\Delta x_1' = 0.96c \cdot 7 = 6.72$ light years. With this, the proper distance she travelled is

$$\Delta x = \gamma \Delta x' = \frac{6.72}{\sqrt{1-0.96^2}} = 24 \text{ light years}$$

Now, for the journey back Alice travels at $V = 0.48c$ (half that of her first stretch of the journey) and at the same distance. The distance she perceives on the second stretch is

$$\Delta x_2' = \Delta x \gamma^{-1} = 24\sqrt{1-0.48^2} \approx 21.05 \text{ light years}$$

The time that Alice experiences on the way back is

$$\Delta t_2' = \frac{\Delta x_2'}{V} = \frac{21.05}{0.48} \approx 43.85 \text{ light years}$$

and Bob's perception of this second stretch back is

$$\Delta t_2 = \gamma \Delta t_2' = \frac{43.85}{\sqrt{1-0.48^2}} \approx 50 \text{ years}$$

So finally, the difference in the final ages can be calculated by summing the difference in their ages across each leg of the journey, $\delta t = \delta t_1 + \delta t_2$. For the first leg, $\delta t_1 = \Delta t_1 - \Delta t_1' = 25 - 7 = 18$ years. For the second leg, $\delta t_2 = \Delta t_2 - \Delta t_2' = 50 - 43.85 = 6.15$ years. Therefore, $\delta t = 18 + 6.15 = 24.15$ years. This means that Bob is 24.15 years older than Alice upon her return, with Bob experiencing 75 years and Alice experiencing about 51 years across the journey.

Question 4

To transform the particle location formula to a different reference frame, the velocity functions must first be subject to Lorentz boosts. First, note that

$$\begin{aligned} dx' &= \gamma(dx - V\,dt) \\ dt' &= \gamma(dt - v/c^2\,dx) \\ dy' &= dy \end{aligned}$$

With the last term being equal since there is no y component to the relative velocities of the reference frames. The velocity components in K' are then

$$\begin{aligned} v_{x'} = \frac{dx'}{dt'} &= \frac{\gamma(dx - V\,dt)}{\gamma(dt - V\,dx/c^2} \\ &= \frac{\frac{dx}{dt} - V}{1 - \frac{V}{c^2}\frac{dx}{dt}} \\ &= \frac{v_x - V}{1 - \frac{Vv_x}{c^2}} \end{aligned} \qquad \begin{aligned} v_{y'} = \frac{dy'}{dt'} &= \frac{dy}{\gamma(dt - V\,dx/c^2} \\ &= \frac{\frac{dy}{dt}}{\gamma\left(1 - \frac{V}{c^2}\frac{dx}{dt}\right)} \\ &= \frac{v_y}{\gamma\left(1 - \frac{Vv_x}{c^2}\right)} \end{aligned}$$

Substituting in these velocity functions into the original trajectory functions gives

$$x'(t') = \left(\frac{v_x - V}{1 - \frac{Vv_x}{c^2}}\right) t' + x_0 \qquad y'(t') = \left(\frac{v_y}{\gamma\left(1 - \frac{Vv_x}{c^2}\right)}\right) t'$$

Question 5

a. The special relativistic velocity addition law reads

$$v' = \frac{v - V}{1 - Vv/c^2}$$

Substituting in $v = \tanh\theta$, $V = \tanh\Theta$, and $c = 1$ gives

$$v' = \frac{\tanh\theta - \tanh\Theta}{1 - \tanh\theta\tanh\Theta}$$

however, tanh has the trigonometric identity

$$\frac{\tanh x - \tanh y}{1 - \tanh x \tanh y} = \tanh(x - y)$$

and so, $v' = \tanh(\theta - \Theta)$.

b. The information given can be set up as an boundary value problem,

$$\begin{cases} \frac{dx}{dt} = V \\ x(t = 0) = 0 \end{cases}$$

Therefore,

$$\begin{aligned} \int dx &= \int V\,dt \\ x &= Vt + c \end{aligned}$$

But when $t = 0$, $x = 0 \Rightarrow c = 0 \Rightarrow x(t) = Vt$. But, $t = \gamma\tau$,

$$\Rightarrow x(\tau) = \gamma V\tau$$

Question 6

a. To find the properties of $\boldsymbol{a}$ and $\boldsymbol{b}$, the dot product of themselves are calculated:

$$\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{a} &= -a_0a_0 + \vec{a}\cdot\vec{a} \\ &= -5\cdot 5 + \sqrt{5}^2 + 2^2 + 4^2 \\ &= -25 + 5 + 4 + 16 = 0 \\ \boldsymbol{b}\cdot\boldsymbol{b} &= -b_0b_0 + \vec{b}\cdot\vec{b} \\ &= -(-3)^2 + 0^2 + 0^2 + 2^2 \\ &= -9 + 4 = -5 \end{aligned}$$

Therefore, $\boldsymbol{a}$ is null, and $\boldsymbol{b}$ is timelike.

b. The components of $\boldsymbol{a} - 5\boldsymbol{b}$ are found by multiplying and subtracting $\boldsymbol{b}$ from $\boldsymbol{a}$ component wise,

$$\boldsymbol{a} - 5\boldsymbol{b} = (5 - 5\cdot -3, \sqrt{5} - 0, 2 - 0, 4 - 5\cdot 2) = (20, \sqrt{5}, 2, -6)$$

c. The dot-product of the two 4-vectors is

$$\boldsymbol{a}\cdot\boldsymbol{b} = -a_0b_0 + \vec{a}\cdot\vec{b} = -5\cdot -3 + \sqrt{5}\cdot 0 + 2\cdot 0 + 4\cdot 2 = 15 + 8 = 23$$

Therefore, the two 4-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are timelike separated.