Exam information

Course code and name	PHYS2041, Introduction to Quantum Mechanics
Semester	Semester 2, 2020
Exam type	Online, non-invigilated
Exam date and time	Please refer to your personalised timetable
Exam duration	Working time: 120 minutes + additional online allowance: 30 minutes. TOTAL exam duration: 2 hrs 30 minutes from the advertised exam commencement time
Additional time	30 minutes additional time has been incorporated in recognition of the online environment and the different circumstances that students face in their home environments. This includes time for download and upload, and allowances for network or connection issues.
Reading time	Reading time has not been formally allocated for online exams, however students are encouraged to review and plan their approach for the exam before they start. The total exam time should be sufficient to do this.
Exam window	You must commence your exam at the time listed in your personalised timetable. The exam will remain open **only** for the duration of the exam.
Weighting	This exam is weighted at 70% of your total mark for this course.
Permitted materials	This is an open book exam – all materials permitted
Instructions	Answer all questions Write your answers on blank paper (clearly label your solutions so that it is clear which problem it is a solution to), or annotate an electronic file on a suitable device. You must submit your answers as a single electronic file through Blackboard before the end of the allowed time. You should include your **name and student number** on the first page of the file that you submit.
Who to contact	If you have any concerns or queries about a particular question, or need to make any assumptions to answer the question, state these at the start of your solution to that question. You may also include queries you may have made with respect to a particular question, should you have been able to 'raise your hand' in an examination room. If you experience any **technical difficulties** during the exam, contact the Library AskUs (https://web.library.uq.edu.au/contact-us) service for advice (open 7am–10pm, 7 days a week, Brisbane time): Chat: https://support.my.uq.edu.au/app/chat/chat_launch_lib/p/45 Phone: +61 7 3506 2615 Email: examsupport@library.uq.edu.au You should also ask for an email documenting the advice provided so you can provide this on request. In the event of a **late submission**, you will be required to submit evidence that you completed the exam in the time allowed. We recommend you use a phone camera to take photos (or a video) of every page of your exam. Ensure that the photos are time-stamped. If you submit your exam after the due time then you should send details (including any evidence) to SMP Exams (exams.smp@uq.edu.au) as soon as possible after the end of the exam.

Important exam condition information	The normal academic integrity rules apply. You cannot cut-and-paste material other than your own work as answers. You are not permitted to consult any other person – whether directly, online, or through any other means – about any aspect of this assessment during the period that this assessment is available. If it is found that you have given or sought outside assistance with this assessment then that will be deemed to be cheating and will result in disciplinary action. By undertaking this online assessment, you will be deemed to have acknowledged UQ's academic integrity pledge by making the following declaration: *"I certify that my submitted answers are entirely my own work and that I have neither given nor received any unauthorised assistance on this assessment item"*.

Question 1 [10 points]

(a) **[2 points]** What are the units of a quantum mechanical wave function in two spatial dimensions? Briefly explain your answer.

(b) **[2 points]** Which one of these expressions is the expectation value of the kinetic energy for a particle of mass m in one dimension?

$$(1)\ \langle \hat{T} \rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \Psi^* \frac{\partial^2 \Psi}{\partial x^2}\, dx$$

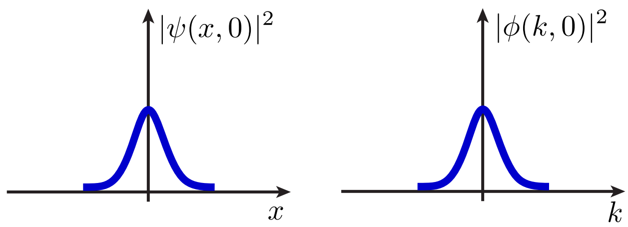
$$(2)\ \langle \hat{T} \rangle = \int_{-\infty}^{+\infty} \Psi^* \left(\frac{1}{2}m\left(\frac{\partial x}{\partial t}\right)^2\right) \Psi\, dx$$

$$(3)\ \langle \hat{T} \rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \Psi^* \left(\frac{\partial \Psi}{\partial x}\right)^2 dx$$

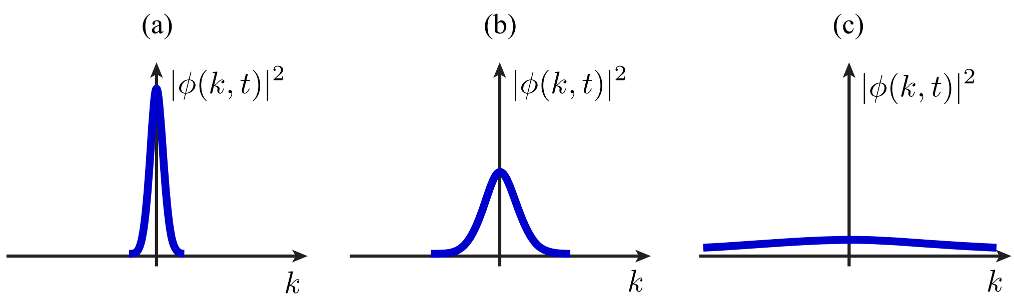
$$(4)\ \langle \hat{T} \rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \frac{\partial^2}{\partial x^2}|\Psi|^2\, dx$$

Briefly explain your answer.

(c) **[2 points]** For a free particle, if the position-space probability density $|\psi(x,0)|^2$ and the momentum-space probability density $|\phi(k,0)|^2$ look like these initially



and then I measure the position, what does the momentum-space probability density $|\phi(k,t)|^2$ look like immediately after the measurement? Choose your answer from one the options below and briefly explain why. [Assume that the graphs are on the same scale.]

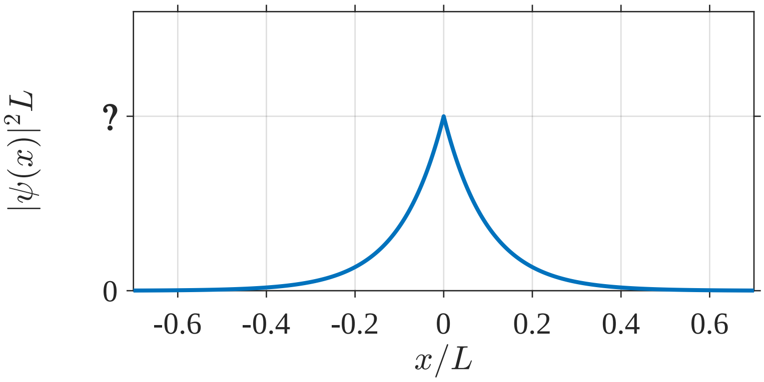


(d) **[4 points]** Show that the commutator between the momentum operator $\hat{p}$ and any function $f(x)$ of the coordinate x is given by

$$[\hat{p}, f(x)] = -i\hbar \frac{df}{dx}.$$

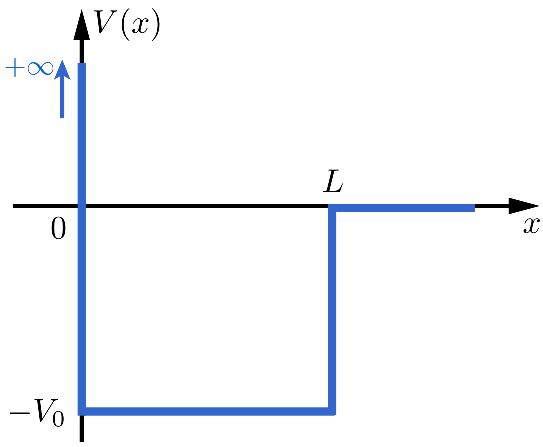
Question 2 [10 points]

(a) [**2 points**] Imagine you are a lecturer in 2nd year Quantum Mechanics, and one of your students shows you a plot (shown below) of the probability density for a quantum mechanical particle described by a wavefunction $\psi(x)$ that they have calculated for a certain problem. The student used a certain characteristic length-scale L to rescale the horizontal and vertical



axes. While the student is confident about the labels on the horizontal axes, they are unsure about the numerical value they have against the tick mark on the vertical axis (shown here by the question mark). Which of the following numerical values would make physical sense to you: 0.1; 0.2; 0.5; 1; 2; 5; 10; 20; 50. Briefly explain your answer.

(b) [**5 points**] A particle of mass m is confined by the trapping potential $V(x)$ shown below and defined via: $V(x)=+\infty$ for $x\leq 0$, $V(x)=-V_0$ for $0<x<L$, and $V(x)=0$ for $x\geq L$.



Assuming that the potential well is sufficiently deep and wide so that it supports a large ($\gg$1) number of bound states, and without solving any Schödinger equation, sketch—to the best of your ability and the knowledge of quantum mechanics—the three lowest energy eigenfunctions of the particle in this potential. Sketch them on three separate graphs, adhering approximately to the same scale as the above figure, and depicting sufficient amount of detail to demonstrate your understanding of the key features of these wavefunctions. You can supply brief annotations of the features that you are trying to show in your drawing.

(c) [**3 points**] For the problem in part (b), write down—again without solving any Schödinger equation, but only using your knowledge of other related quantum mechanical problems—your best estimate for the approximate energy eigenvalues of these three lowest energy eigenstates. Explain your answer.

Question 3 [10 points]

Consider a particle of mass m prepared in the ground state $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$ of a harmonic trapping potential $V(x) = \frac{1}{2}m\omega^2 x^2$, where ω is the trap frequency. At time $t = 0$, the shape of the trapping potential is instantaneously changed to $V'(x) = \frac{1}{2}m(2\omega)^2 x^2$, i.e., to have twice as large a frequency, while still remaining quadratic.

(a) [**6 points**] How will the wave function of the particle evolve with time in the new trap potential? [Write down all required expressions (but do not evaluate any integrals), including the relevant wave functions in terms of the Hermite-Gauss polynomials, and clearly define all quantities that you may introduce, so that if your expressions were used to write a computer code, the code would correctly model the evolution.]

(b) [**2 points**] If the energy of the particle is measured in the new potential at some later time t, what are the possible outcomes?

(c) [**2 points**] What is the probability of each outcome of the energy measurement?

Question 4 [10 points]

A two-level system is spanned by the orthonormal basis states denoted via

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The Hamiltonian for this particular system is given by

$$\hat{H} = \hbar\omega\big(|1\rangle\langle 2| + |2\rangle\langle 1|\big).$$

(a) **[2 points]** By evaluating the matrix elements $\hat{H}_{ij} = \langle i|\hat{H}|j\rangle$ $(i, j = 1, 2)$, write down the Hamiltonian in a matrix form, and show that the states

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

are the eigenstates of the Hamiltonian.

(b) **[2 points]** If you measure the energy of the system, what are the possible outcomes?

(c) **[3 points]** Assume that the system is prepared in the state $|\Psi\rangle = \frac{1}{\sqrt{2}}\big(|1\rangle + i|2\rangle\big)$. Calculate the uncertainty in the energy (the standard deviation σ_E) of the system in this state. Simplify your result as much as possible.

(d) **[3 points]** Consider another observable of interest

$$\hat{Q} = \alpha\big(|2\rangle\langle 2| - |1\rangle\langle 1|\big),$$

where α is a real number. Are $\hat{H}$ and $\hat{Q}$ compatible observables? Show why.

END OF EXAMINATION

Useful formulae

1. Time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x)}{\partial x^2} + V(x,t)\Psi(x,t).$$

2. Time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x),$$

where E (the energy, i.e., the expectation value of the Hamiltonian) plays the role of the separation constant for separable (and hence stationary) states, $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$.

3. General solution of the time-dependent Schrödinger equation as a superposition of separable solutions:

$$\Psi(x,t) = \sum_n c_n\Psi_n(x,t) = \sum_n c_n\psi_n(x)e^{-iE_nt/\hbar}.$$

4. Energy eigenvalues and energy eigenstates for a particle of mass m in an infinite square well potential, $V(x) = 0$ if $0 \le x \le a$, and $V(x) = \infty$ otherwise:

$$E_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 n^2,$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right),$$

where $n = 1, 2, 3, \dots$.

5. Energy eigenvalues and energy eigenstates for a particle of mass m in a harmonic potential $V(x) = \frac{1}{2}m\omega^2x^2$ of frequency ω:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \qquad n = 0, 1, 2, 3, \dots,$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)\,e^{-\xi^2/2}, \quad \xi \equiv \sqrt{\frac{m\omega}{\hbar}}x,$$

where $H_n(\xi) = (-1)^n e^{\xi^2}\left(\frac{d}{d\xi}\right)^n e^{-\xi^2}$ are the Hermite polinomials, with the first few of them given by

$$H_0 = 1, H_1 = 2\xi, H_2 = 4\xi^2 - 2, H_3 = 8\xi^3 - 12\xi, \dots .$$

6. Ladder operators for a quantum harmonic oscillator:

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}} \left(-i\hat{p} + m\omega\hat{x}\right),$$
$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}} \left(i\hat{p} + m\omega\hat{x}\right),$$

7. Expansion over a complete orthonormal basis, $\psi_n(x)$, for discrete spectrum:

$$f(x) = \sum_n c_n \psi_n(x),$$

$$c_n = \int \psi_n^*(x) f(x)\, dx.$$

8. Continuous Fourier transforms:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k)\, e^{ikx}\, dk,$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx.$$

9. Dirac delta function and some of its properties:

$$\delta(x) = \begin{cases} \infty, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}, \text{with } \int_{-\infty}^{+\infty} \delta(x)\, dx = 1, \tag{1}$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx}\, dk,$$
$$f(x)\delta(x-a) = f(a)\delta(x-a),$$
$$\delta(-x) = \delta(x),$$
$$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x),$$
$$\int_{a-\varepsilon}^{a+\varepsilon} f(x)\, \delta(x-a)\, dx = f(a), \quad \varepsilon > 0,$$

$$\delta(x) = \frac{d\theta(x)}{dx}, \text{ where } \theta(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{1}{2}, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases} \tag{2}$$

10. Some operators and commutators:

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x},\ [\hat{x},\hat{p}_x] = i\hbar,\ \hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y,\ \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z,\ \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x.$$

11. Heisenberg uncertainty principle:

$$\sigma_A\,\sigma_B \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|.$$

12. Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

13. Useful trigonometric identities:

$$\sin^2\alpha + \cos^2\alpha = 1,$$

$$2\sin\alpha\,\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta),$$

$$\sin(\alpha-\pi) = \sin(\alpha+\pi),$$

$$\sin(\alpha\pm\beta) = \sin\alpha\,\cos\beta \pm \cos\alpha\,\sin\beta,$$

$$\cos(\alpha\pm\beta) = \cos\alpha\,\cos\beta \mp \sin\alpha\,\sin\beta,$$

14. Integration by parts:

$$\int_a^b f\,\frac{dg}{dx}\,dx = (fg)\Big|_a^b - \int_a^b \frac{df}{dx}\,g\,dx.$$

15. Volume element in spherical coordinates:

$$d^3\mathbf{r} = r^2\,\sin\theta\,dr\,d\theta\,d\phi.$$

16. Useful integrals:

$$\int e^{ax} = \frac{1}{a}e^{ax},$$

$$\int_0^1 e^{-\xi^2}d\xi = \frac{\sqrt{\pi}}{2}\mathrm{erf}(1) \approx 0.7468$$

$$\int_1^{+\infty} e^{-\xi^2}d\xi = \frac{\sqrt{\pi}}{2}\mathrm{erfc}(1) \approx 0.1394$$

$$\int \cos y\, dy = \sin y,$$
$$\int \sin y\, dy = -\cos y,$$
$$\int_0^\pi \sin^2 y\, dy = \frac{\pi}{2},$$
$$\int \frac{dx}{x} = \ln|x|,$$
$$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x},$$
$$\int_{-\infty}^{\infty} e^{-iax} e^{-\frac{x^2}{b^2}}\, dx = b\sqrt{\pi} e^{-\frac{1}{4}a^2b^2}, \quad [a > 0,\ b > 0],$$
$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}, \quad [a > 0;\ n = 1, 2, 3, ...],$$
$$\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad [a > 0],$$
$$\int_0^\infty x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n}\sqrt{\frac{\pi}{a}}, \quad [a > 0;\ n = 1, 2, 3, ...],$$
$$\int_0^\infty x e^{-ax^2} dx = \frac{1}{2a}, \quad [a > 0],$$
$$\int_0^\infty x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}, \quad [a > 0;\ n = 1, 2, 3, ...],$$
$$\int_0^\infty e^{-ax}\, dx = \frac{1}{a}, \quad [a > 0],$$
$$\int_0^\infty x e^{-ax}\, dx = \frac{1}{a^2}, \quad [a > 0],$$
$$\int_0^\infty x^2 e^{-ax}\, dx = \frac{2}{a^3}, \quad [a > 0].$$

17. Geometric series:
$$\sum_{n=0}^{M} x^n = \frac{1 - x^{M+1}}{1 - x}, \quad (x \neq 1).$$
As M goes to infinity, one must have $|x| < 1$ for the series to converge:
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}, \quad (|x| < 1).$$

18. Hyperbolic functions:

$$\begin{aligned} \sinh x &= \frac{1}{2}(e^x - e^{-x}), \\ \cosh x &= \frac{1}{2}(e^x + e^{-x}), \\ \tanh x &= \frac{\sinh x}{\cosh x}. \end{aligned}$$

19. Constants:

vacuum speed of light	c	$= 2.998 \times 10^8$ m/s
electron mass	m_e	$= 9.109 \times 10^{-31}$ kg
neutron mass	m_n	$= 1.675 \times 10^{-27}$ kg
proton mass	m_p	$= 1.673 \times 10^{-27}$ kg
atomic mass unit	u	$= 1.6606 \times 10^{-27}$ kg
proton charge	e	$= 1.602 \times 10^{-19}$ C
Boltzmann constant	k_B	$= 1.381 \times 10^{-23}$ J/K
Planck constant h	h	$= 6.626 \times 10^{-34}$ J s
Reduced Planck constant $\hbar = h/2\pi$,	$\hbar$	$= 1.054 \times 10^{-34}$ J s/rad
gravitational constant	G	$= 6.670 \times 10^{-11}$ N m^2/kg^2

20. Unit conversions:

1 erg $= 10^{-7}$ J

1 eV $= 1.602 \times 10^{-19}$ J $= 1.602 \times 10^{-12}$ erg

1 Å $= 10^{-10}$ m