The Schwarzschild spacetime geometry: Basic properties

Summary

- The Schwarzschild metric
- Killing vectors
- Physical coordinates

The solution of Einstein's field equation describing the curved spacetime of empty space outside a spherically symmetric non-spinning[1] mass distribution is called the Schwarzschild spacetime geometry. This geometry is spherically symmetric and time-independent.[2] The spacetime geometry outside our own Sun is to an excellent approximation described by the Schwarzschild solution. It can also be used to describe the spacetime outside more compact objects like white dwarfs, neutron stars and black holes.

Many important aspects of the behaviour of material bodies and light in a Schwarzschild spacetime geometry are most naturally analyzed and described in terms of so-called **Schwarzschild coordinates** $x = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$ where t is a time coordinate and (r, θ, ϕ) are spatial coordinates: r is a radial coordinate, while θ and ϕ are angles defined in the same way as for standard spherical coordinates (θ is the polar angle and ϕ is the azimuthal angle). In these coordinates, the line element for the Schwarzschild spacetime is given by

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

The corresponding Schwarzschild metric $g_{\alpha\beta}(x)$ defined by $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ is easily extracted from this expression. The metric is diagonal ($g_{\alpha\beta} = 0$ if $\alpha \neq \beta$) with components $g_{tt} = -(1 - 2GM/(c^2 r))$, $g_{rr} = (1 - 2GM/(c^2 r))^{-1}$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2 \sin^2\theta$ (note that $-g_{tt} = g_{rr}^{-1}$). We now discuss some aspects of the Schwarzschild metric/spacetime:

- **Time independence.** The metric does not depend on the time coordinate t. This time independence is a symmetry of the metric, reflecting the invariance of the metric under time translations $t \to t + t_0$ where t_0 is an arbitrary constant. Associated to this symmetry is a Killing vector $\boldsymbol{\xi}$ with components

$$\xi^\alpha = (1, 0, 0, 0). \tag{2}$$

[1] For a spherically symmetric mass distribution that is **spinning** around some axis (i.e. has nonzero angular momentum) it turns out that spacetime only has an **axial** symmetry around the axis of rotation. This is the Kerr spacetime geometry, which we unfortunately won't have time to discuss in this course.

[2] Even for a spherically symmetric, non-spinning mass distribution that is not time-independent (e.g. describing a star whose radius is shrinking due to gravitational collapse), the curved spacetime outside it is described by the time-independent Schwarzschild solution.

- **Spherical symmetry.** If one considers the spacetime at constant t and r (i.e. setting $dt = 0$ and $dr = 0$) the line element reduces to

$$r^2(d\theta^2 + \sin^2\theta\; d\phi^2) \equiv d\Sigma^2. \tag{3}$$

  This is the line element for a spherical surface in flat 3D space. While the problem thus has the full rotational invariance of a sphere, the particular rotational symmetry that is most evident is the invariance under changes in the angle ϕ (rotations around the z axis), since the line element is independent of ϕ. The associated Killing vector $\boldsymbol{\eta}$ has components

$$\eta^\alpha = (0, 0, 0, 1). \tag{4}$$

- **The meaning of the coordinates r and t.** In GR, coordinates are not fundamental; instead they are to some extent "arbitrary". One often applies different coordinate systems in the analysis of different aspects of a given system, the choice of coordinate system simply coming down to what is useful and convenient. In contrast, quantities of physical significance are what is measured by observers; such physically measurable quantities are therefore invariant under transformations between different coordinate systems. In order to attribute physical meaning to coordinates one needs to relate them to physically measurable quantities built from proper times and proper lengths. A proper time interval is a time interval measured by a clock carried by an observer; proper times are built from timelike spacetime intervals. A proper length is a spatial distance measured by rulers; proper lengths are built from spacelike spacetime intervals.

  The radial coordinate r does not measure physical distance in the radial direction. To understand why, let us consider the physical spatial distance dR (a proper length) between two infinitesimally separated points A and B in space that have the same θ and ϕ coordinates but differ in the r coordinate: A has radial coordinate r while B has $r + dr$. Thus A and B lie on the same radial line emanating from the origin. We find the physical distance by measuring the spacetime separation between A and B at a fixed time ($dt = 0$). This is found from the line element ds^2 obtained by putting $dt = 0$ and also $d\theta = 0$ and $d\phi = 0$ since A and B have the same angular coordinates. Thus

$$dR^2 \equiv ds^2(dt = 0, dr, d\theta = 0, d\phi = 0) = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2. \tag{5}$$

  It follows that $dR = (1 - 2GM/c^2r)^{-1/2} dr > dr$. Thus the physical distance dR is different from the difference dr in radial coordinates due to the fact that $g_{rr} \neq 1$. If the two points A and B are separated by a finite radial coordinate difference, the physical distance between them is obtained by integrating: $\Delta R = \int_{r_A}^{r_B} \sqrt{g_{rr}} dr$. It follows that r is not the distance to the center of attraction. Given this, how can an observer "living" at a particular value of r deduce the value of r? By measuring a physical distance or area that is defined at a constant value of r, so that $dr = 0$ and thus g_{rr} does not enter the analysis. The simplest example is the circumference C of a great circle. For example the circumference around the "equator" can be defined by the sum of infinitesimal line elements connecting infinitesimally close points lying in the equatorial plane $\theta = \pi/2$. For each such line element we have $ds^2 = g_{\phi\phi} d\phi^2$, i.e. $ds = \sqrt{g_{\phi\phi}} d\phi = r \sin\theta\; d\phi = r\; d\phi$. So

$$C = \int ds = \int_0^{2\pi} r\; d\phi = r \int_0^{2\pi} d\phi = 2\pi r, \tag{6}$$

giving $r = C/(2\pi)$. Thus r can be deduced from a measurement of this circumference C. For this reason the radial coordinate r is sometimes called the **reduced circumference**. Alternatively, r can be obtained from a measurement of the area A of the surface of the sphere. This area is also a constant-r property, so again g_{rr} does not enter the analysis. One thus finds the standard relationship $A = 4\pi r^2$, and so $r = A/(4\pi^2)$.

The coordinate time t can be interpreted as the proper time measured by a clock carried by a stationary observer infinitely far away from the source of curvature. To see this, note that the proper time elapsed on the worldline of a stationary observer at radial coordinate r during a coordinate time interval dt is (since the observer is stationary, $dr = d\theta = d\phi = 0$)

$$d\tau^2 = -ds^2/c^2 = -g_{tt}dt^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 \quad \Rightarrow \quad d\tau = \sqrt{1 - 2GM/c^2 r}\, dt. \tag{7}$$

Thus if the observer is infinitely far away ($r \to \infty$) we see that $d\tau = dt$ in that case. Because of this meaning of the coordinate time t, it is sometimes referred to as the **far-away time**.

- **The interpretation of the parameter M as the total mass of the source of curvature.** If the parameter GM/c^2r is small, either because M is sufficiently small, or because r is very large (i.e. sufficiently far away from the source), or both, the metric can be expanded in this parameter (using $(1+x)^n \approx 1+nx$ for $|x| \ll 1$) to give

$$ds^2 \approx -(1 + 2\Phi/c^2)(c\, dt)^2 + (1 - 2\Phi/c^2)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{8}$$

where $\Phi = \Phi(r) = -GM/r$. This is exactly the same form as the static weak-field metric considered earlier in connection with our discussion of gravitational time dilation, where Φ is the standard Newtonian gravitational potential due to a body with mass M. This leads to the interpretation of the parameter M as the total mass of the body that is the source of the spacetime curvature.

- **Reduction to flat space for $M = 0$.** We expect that the Schwarzschild spacetime should reduce to flat spacetime if there is no mass to cause the curvature, and indeed this is verified by setting $M = 0$, which gives

$$ds^2 = -(c\, dt)^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{9}$$

which is just the line element for flat spacetime in spherical coordinates (transforming to Cartesian coordinates gives $dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) = dx^2 + dy^2 + dz^2$).

Although ordinarily mass is measured in the SI unit of kilograms, it will here be more convenient to measure it as a length, i.e. in meters, by defining

$$M_{\text{in meters}} = \frac{G}{c^2} M_{\text{in kilograms}} = 7.42 \times 10^{-28} M_{\text{in kilograms}}. \tag{10}$$

In the following the symbol M will denote $M_{\text{in meters}}$. This corresponds to setting $G = c = 1$ in the original formulas. These units are called *geometrized units*. To go back to "ordinary" units we simply make the replacement $M \to GM/c^2$.

The mass distribution causing the curvature may e.g. be our own Sun, a white dwarf star, a neutron star, or a black hole. For the first three types of objects, the Schwarzschild solution describes the empty space *outside* the surface of the object. Our Sun has a mass $M \approx 1.5$ km, while the radius of its surface is $R \approx 7 \times 10^5$ km, giving $R/M \sim 10^5$. White dwarfs and neutron stars have $R/M \sim 1000$ and 10, respectively. Thus for none of these objects can an orbiting particle have a radial coordinate as small as $r = 2M$. However, for a black hole the radius $r = 2M$ is the so-called **event horizon**, and the region $r < 2M$ is physically accessible in this case. To begin with, we will restrict our discussion to $r \geq 2M$, but later, when considering black holes, we also discuss the region $r < 2M$.

The Schwarzschild spacetime geometry: Timelike geodesics

Summary

- Conserved quantities
- Effective potential and the radial equation
- Orbits in the Schwarzschild geometry

(This section is taken from Hartle, section 9.3)

The study of geodesics in the Schwarzschild geometry is considerably aided by the laws of conservation of energy and angular momentum that hold because the metric is independent of time and spherically symmetric. The two Killing vectors are given by ξ and η, and their explicit forms are given as

$$e = -\boldsymbol{\xi}.\mathbf{u} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}\,, \tag{1}$$

and

$$\ell = \boldsymbol{\eta}.\mathbf{u} = r^2 \sin^2\theta \frac{d\phi}{d\tau}\,. \tag{2}$$

At large r the constant e becomes energy per unit rest mass because in flat space $E = mu^t = m(dt/d\tau)$, so we will continue to refer to it as such. The conserved quantity ℓ is the angular momentum per unit rest mass (at low velocities). So these Killing vectors correspond to conservation of energy and angular momentum.

Effective potential

The conservation of angular momentum implies that the orbits lie in a plane, as they do in Newtonian mechanics. Consider a particle with 3-velocity $\vec{u}$ at a particular instant. Orient the coordinates such that the angular velocity $d\phi/d\tau = 0$ at that instant and the angle $\phi = 0$. From equation (2) we see that $d\phi/d\tau = 0$ everywhere along the geodesic, and so the particle remains in the meridional "plane" $\phi = 0$. Having established this, it is simpler to continue work in coordinates where particles are fixed in the equatorial plane, where $\theta = \pi/2$ and $u^\theta = 0$.

The normalisation of the 4-velocity gives us another constraint equation for the geodesic equation in addition to those for energy (eqn. 1) and angular momentum (eqn. 2). This is given by

$$\mathbf{u}.\mathbf{u} = g_{\alpha\beta}u^\alpha u^\beta = -1$$

Since we have three non-zero elements of the 4-velocity, and three constraint equations, we should be able to express one set in terms of the other through some relation. Starting with the Schwarzschild metric, and working solely in the equatorial plane ($\theta = \pi/2$ and $u^\theta = 0$), we get

$$-\left(1 - \frac{2GM}{c^2 r}\right)(u^t)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}(u^r)^2 + r^2(u^\phi)^2 = -1\,.$$

Writing $u^t = dt/d\tau$, $u^r = dr/d\tau$ and $u^\phi = d\phi/d\tau$, using equations (1) and (2) to eliminate $dt/d\tau$ and $d\phi/d\tau$, we get

$$-\left(1 - \frac{2GM}{c^2 r}\right)^{-1} e^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2} = -1\,.$$

This then can be rewritten as

$$\frac{e^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) - 1\right]\,.$$

We have written the equation this way to show the correspondance with energy integral of Newtonian mechanics. We define the constant

$$\mathcal{E} \equiv (e^2 - 1)/2$$

and the effective potential

$$V_{\text{eff}}(r) = \frac{1}{2}\left[\left(1 - \frac{2GM}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right] = -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3} \tag{3}$$

then the correspondence becomes exact, and eqn. 3 becomes

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r)\,. \tag{4}$$

Thus, the techniques use for treating orbits by effective potnetials in Newtonian mechanics can be applied to orbits in the Schwarzschild geometry. Indeed the form of the effective potential (equation 3) differs from that of a $-GM/r$ Newtonian central potential by only the additional $-GM\ell^2/r^3$ term. That term, however, will have important consequences for orbits, as we explore shortly.

Greater insights into (4) can be obtained by consdiering the nonrelativistic limit. We put the factor of the speed of light c back into the equation, by replacing t with ct, τ with $c\tau$ and GM with GM/c^2. The angular momentum term ℓ is replaced with ℓ/c, as ℓ continues to have the definition $r^2(d\phi/d\tau)$. The effective potential becomes

$$V_{\text{eff}}(r) = \frac{1}{c^2}\left(-\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{c^2 r^3}\right)\,.$$

The dimensionless constant e is the total energy per unit rest mass. Anticipating a correspondence with the usual Newtonian energy, let us define E_{Newt} by

$$e \equiv \frac{mc^2 + E_{\text{Newt}}}{mc^2}\,.$$

Using the definition of $\mathcal{E}$, we get

$$E_{\text{Newt}} = \frac{m}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} - \frac{GML^2}{c^2 m r^3}$$

where $L = m\ell$. This has the same form as the energy integral in Newtonian gravity with an additional relativistic correction to the potential proportional to $1/r^3$. The Newtonian limit is recovered when this relativistic derivative is dropped and τ-derivatives can be replaced by t-derivatives.

Relativistic orbits

Returning to the analysis of the relativsitic orbits, consider the effective potential $V_{\text{eff}}(r)$. A few simple properties are immediate from its definition (3):

$$V_{\text{eff}}(r)\ \overrightarrow{r\to\infty}\ -\frac{M}{r}\,, \qquad V_{\text{eff}}(2GM)=\frac{1}{2}$$

For large values of R the potential is close to to the Newtonian effective potential for motion in a $1/r$ potential, as Figure 1 illustrates. That is because the first two terms in (3) are the same as in Newtonian theory. However, as r decreases, the attractive $1/r^3$ correction from General Relativity becomes increasingly important.

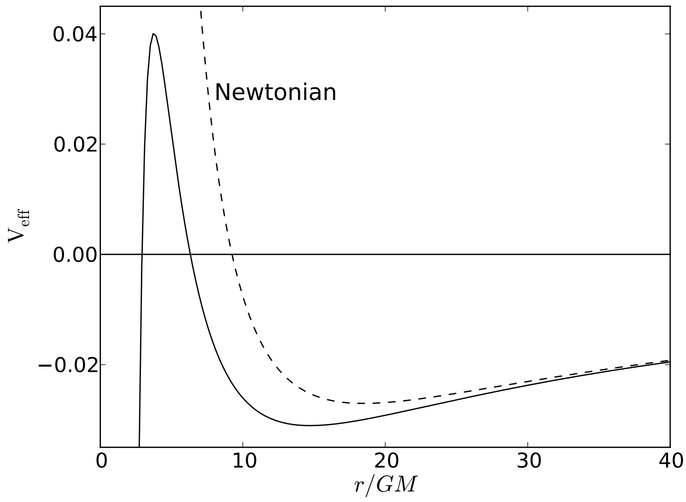

Figure 1: The relativistic and Newtonian effective potentials for radial motion compared for $\ell/GM = 4.3$. The relativistic effective potential $V_{\text{eff}}(r)$ is defined by equation 3 and we take the Newtonian effective potential to be the first two terms. The two are close for large r, but differ significantly for small r, where the $1/r^3$ term becomes important. In particular the infinite centrifugal barrier of Newtonian theory becomes a barrier of finite height. For the Earth in orbit around the Sun $\ell/M \sim 10^9$ and the difference between the Newtonian and relativistic potential over the orbit of the Earth are tiny but detectable in precise measurements.

The extrema of the effective potential can be found from solving $dV_{\text{eff}}/dr = 0$. There is one local minimum and one local maximum, whose radii r_{min} and r_{max} are

$$r_{\text{extrema}} = \frac{\ell^2}{2GM}\left[1 \pm \sqrt{1-12\left(\frac{GM}{\ell}\right)^2}\,\right].$$

Figure 2 is a plot of V_{eff} for various values of ℓ. If $\ell/GM < \sqrt{12} = 3.46$, there are no real extrema and the effective potential is negative for all values of r. If $\ell/GM > \sqrt{12}$ the effective potential has one maximum and one minimum. The maximum lies above $V_{\text{eff}} = 0$ if $\ell/GM > 4$ and otherwise lies below it. There is a centrifugal barrier, but it has a maximum height, on contrast to the one in Newtonian theory that has infinite height.

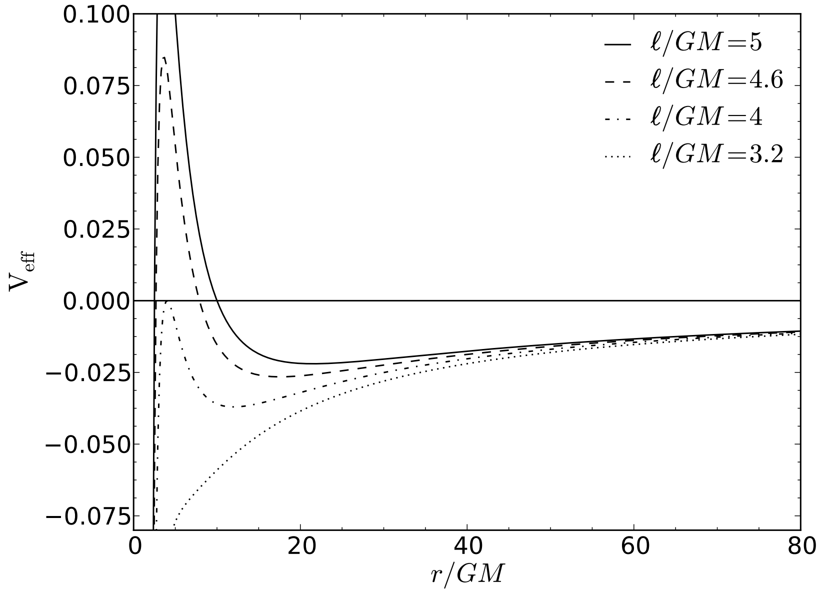

Figure 2: The effective potential V_{eff} for radial motion for several different values of ℓ.

The qualitative behaviour of an orbit depends on a the relationship between $\mathcal{E} \equiv (e^2 - 1)/2$ and the effective potential, just as in a Newtonian central force problem. Turning points occur at the radii r_{tp}, where $\mathcal{E} = V_{\mathrm{eff}}(r_{tp})$, because that's where the radial velocity vanishes. If $\ell/GM < \sqrt{12}$, there are no turning points for positive values $\mathcal{E}$. This is in contrast to Newtonian theory, where as long as $\ell \neq 0$ there is a positive centrifugal barrier that will reflect the particle (see Figure 1).

There are four types of orbits possible for values of $\ell/GM > \sqrt{12}$:

- Circular orbits, at the radii at which the effective potential has a maximum or a minimum. (The orbit at the maximum is unstable because a small increase in $\mathcal{E}$ will lead the object to escape to infinity or collapse to $r = 0$. The orbit at the minimum is stable.)

- Bound orbits, that oscillate between two turning points, where $\mathcal{E} < 0$. (The planets are moving in bound orbits in the spacetime geometry of the Sun to a good approximation.)

- Scattering orbits, where the object approaches from infinity, orbits the center of attraction, and then returns. These orbits have with positive $\mathcal{E}$ but less than the maximum of the effective potential.

- Radial plunge orbits, where objects with $\mathcal{E}$ greater than the maximum fall into the center of attraction.

Escape velocity in the Schwarzschild geometry

Summary

- Relative velocity in Schwarzschild
- Escape velocity

Relevant pages in Hartle: 199-200 (example 9.1).

Consider an observer stationary in Schwarzschild coordinates (r, θ, ϕ) at radial coordinate $r = R$ who launches a projectile radially outwards with velocity V, as measured in the observer's frame. How large must V be for the projectile to reach $r = \infty$ with zero velocity? This critical value of V is known as the escape velocity V_{escape}.

The energy E of the projectile, as measured by the observer, is given by

$$E = -\boldsymbol{p} \cdot \boldsymbol{u}_{\text{obs}} = -m\boldsymbol{u} \cdot \boldsymbol{u}_{\text{obs}}, \tag{1}$$

where $\boldsymbol{p}$ is the projectile's 4-momentum $\boldsymbol{p} = m\boldsymbol{u}$, with m the projectile's (rest) mass and $\boldsymbol{u}$ its 4-velocity. The relationship between the energy E and velocity V, both measured by the observer, is $E = m/\sqrt{1 - V^2}$. Thus

$$V = \sqrt{1 - (\boldsymbol{u} \cdot \boldsymbol{u}_{\text{obs}})^{-2}}. \tag{2}$$

(To derive these relationships in a convincing manner one first works in a **local inertial frame** at the spacetime point where the projectile is launched (in such a frame the metric at the given spacetime point takes the same form as in flat spacetime), choosing this frame such that the observer is at rest in it (i.e. the observer's 4-velocity has coordinates $(1, 0, 0, 0)$ in this frame), and then one rewrites the result in terms of a **scalar (dot) product** (here $\boldsymbol{u} \cdot \boldsymbol{u}_{\text{obs}}$), which makes the result valid in **any** coordinate system (frame) since the dot product is a **scalar** (i.e. invariant under arbitrary coordinate transformations).

We must find the 4-velocities $\boldsymbol{u}$ and $\boldsymbol{u}_{\text{obs}}$. First consider the 4-velocity of the projectile, which in Schwarzschild coordinates has components $u^\alpha = (dt/d\tau, dr/d\tau, d\theta/d\tau, d\phi/d\tau)$. Since the projectile is moving radially, $d\theta/d\tau = d\phi/d\tau = 0$. Furthermore, since the motion is along a geodesic,

$$\varepsilon = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r), \tag{3}$$

where $\varepsilon = (e^2 - 1)/2$ with $e = (1 - 2M/r)dt/d\tau$. Since e is constant along the geodesic, we can choose to evaluate it at $r = \infty$ where the projectile should be at rest by assumption, which implies $d\tau = dt$ there, and thus $e = 1$ and $\varepsilon = 0$. Furthermore, since $d\phi/d\tau = 0$, it follows that $\ell = r^2 \sin^2\theta(d\phi/d\tau) = 0$, and so $V_{\rm eff}(r)$ simplifies to $-M/r$. From (3) we then get $dr/d\tau = +\sqrt{2M/r}$ (the positive solution must be chosen since the projectile is moving outwards). Now $dt/d\tau$ can be found from the normalization of the 4-velocity:

$$
\begin{aligned}
-1 &= \boldsymbol{u}\cdot\boldsymbol{u} \\
&= g_{tt}(dt/d\tau)^2 + g_{rr}(dr/d\tau)^2 + 0 + 0 \\
&= -(1-2M/r)(dt/d\tau)^2 + (1-2M/r)^{-1}(2M/r) \\
&\Rightarrow\ dt/d\tau = (1-2M/r)^{-1}. \qquad (4)
\end{aligned}
$$

Thus the components of the projectile's 4-velocity at $r = R$ in Schwarzschild coordinates are given by

$$u^\alpha = ((1-2M/R)^{-1}, (2M/R)^{1/2}, 0, 0). \qquad (5)$$

Next we consider the corresponding 4-velocity $\boldsymbol{u}_{\rm obs}$ of the observer, given by $u^\alpha_{\rm obs} = ((dt/d\tau)_{\rm obs}, (dr/d\tau)_{\rm obs}$ where the subscript 'obs' means that all quantities involved (dt, dr, $d\theta$, $d\phi$, and $d\tau$) are for the observer. As the observer's values of r, θ, ϕ are stationary, this simplifies to $u^\alpha_{\rm obs} = ((dt/d\tau)_{\rm obs}, 0, 0, 0)$. Normalization of the 4-velocity then gives

$$
\begin{aligned}
-1 &= \boldsymbol{u}_{\rm obs}\cdot\boldsymbol{u}_{\rm obs} \\
&= g_{tt}(dt/d\tau)^2_{\rm obs} + 0 + 0 + 0 \\
&= -(1-2M/R)(dt/d\tau)^2_{\rm obs} \quad \Rightarrow (dt/d\tau)_{\rm obs} \\
&= (1-2M/R)^{-1/2}. \qquad (6)
\end{aligned}
$$

Thus the components of the observer's 4-velocity are

$$u^\alpha_{\rm obs} = ((1-2M/R)^{-1/2}, 0, 0, 0). \qquad (7)$$

Thus

$$
\begin{aligned}
\boldsymbol{u}\cdot\boldsymbol{u}_{\rm obs} &= g_{tt}u^t u^t_{\rm obs} + 0 + 0 + 0 \\
&= -(1-2M/R)(1-2M/R)^{-1}(1-2M/R)^{-1/2} \\
&= -(1-2M/R)^{-1/2} \qquad (8)
\end{aligned}
$$

and so

$$V_{\rm escape} = \sqrt{1 - [-(1-2M/R)^{-1/2}]^{-2}} = \sqrt{2M/R}. \qquad (9)$$

Amusingly, this is the same result as in Newtonian theory. Also note that as $R \to 2M$, the escape velocity approaches 1 (the speed of light). This latter fact relates to black holes, which we will discuss later.

The gravitational redshift (in the Schwarzschild geometry)

Summary

- Energy differences
- Gravitational redshift

Reference: Hartle Sec. 9.2.

Consider two stationary observers 1 and 2. Observer 1 has spatial coordinates (r_1, θ_1, ϕ_1), observer 2 has spatial coordinates (r_2, θ_2, ϕ_2). Observer 1 emits a light signal that is received by observer 2. The emitted light signal has frequency ω_1 as measured by observer 1. What is the frequency ω_2 of the received light signal as measured by observer 2?

We can work this out as follows. First, the relationship between the energy E and frequency ω of the photon, both measured by the same observer, is given by $E = \hbar\omega$. Next, the energy E of the photon measured by an observer with 4-velocity $\boldsymbol{u}_{\text{obs}}$ is $E = -\boldsymbol{p} \cdot \boldsymbol{u}_{\text{obs}}$ where $\boldsymbol{p}$ is the photon's 4-momentum. So we have

$$\hbar\omega_1 = -(\boldsymbol{p} \cdot \boldsymbol{u}_{\text{obs}})_1, \tag{1}$$
$$\hbar\omega_2 = -(\boldsymbol{p} \cdot \boldsymbol{u}_{\text{obs}})_2. \tag{2}$$

The 4-velocity of a stationary observer at radial coordinate r in the Schwarzschild geometry is given by (see the lecture notes on escape velocity)

$$\boldsymbol{u}_{\text{obs}} = \left(\left(1 - \frac{2GM}{r}\right)^{-1/2}, 0, 0, 0 \right) = \left(1 - \frac{2GM}{r}\right)^{-1/2} \boldsymbol{\xi}, \tag{3}$$

where in the second expression we introduced the Killing vector $\boldsymbol{\xi}$ associated with the time invariance of the Schwarzschild metric ($\xi^\alpha = (1, 0, 0, 0)$). Thus

$$\hbar\omega_1 = -\left(1 - \frac{2GM}{r_1}\right)^{-1/2} (\boldsymbol{\xi} \cdot \boldsymbol{p})_1, \tag{4}$$
$$\hbar\omega_2 = -\left(1 - \frac{2GM}{r_2}\right)^{-1/2} (\boldsymbol{\xi} \cdot \boldsymbol{p})_2. \tag{5}$$

The photon's worldline is a null geodesic, so the quantity $\boldsymbol{\xi} \cdot \boldsymbol{p}$ is conserved along it. Thus $(\boldsymbol{\xi} \cdot \boldsymbol{p})_1 = (\boldsymbol{\xi} \cdot \boldsymbol{p})_2$, so

$$\frac{\omega_2}{\omega_1} = \left(\frac{1 - 2GM/r_1}{1 - 2GM/r_2} \right)^{1/2}. \tag{6}$$

Now consider the special case that observer 2 is at infinity (i.e. $r_2 = \infty$). This gives (defining in this case $\omega_2 \equiv \omega_\infty$, $r_1 \equiv R$, and $\omega_1 \equiv \omega_*$)

$$\omega_\infty = \left(1 - \frac{2GM}{R}\right)^{1/2} \omega_* \tag{7}$$

Since $\omega_\infty < \omega_*$, the light received by the observer is redshifted. This effect is called the **gravitational redshift**. When $M/R \ll 1$, the expression (7) can be expanded in this small parameter, which gives

$$\omega_\infty \approx \left(1 - \frac{GM}{R}\right) \omega_* \stackrel{c \neq 1}{=} \left(1 - \frac{GM}{Rc^2}\right) \omega_* . \tag{8}$$

Light rays in GR: general aspects

Summary

- Null Geodesics
- Light ray orbits
- Deflection of light
- Gravitational lensing

Relevant sections in Hartle: 7.5 and 8.3.

Null Geodesics

In GR, just as in SR, light rays move along null worldlines in spacetime. These are curves along which $ds^2 = 0$. The family of null directions emerging from, or converging on, a point P in spacetime, span the **local** future and past light cones at P. The existence of **local inertial frames** at each spacetime point P implies that the *local* light cone structure in GR is the same as in SR (flat spacetime). Any point on the timelike worldline of a massive particle lies *within* the local light cone at that point. This implies that the particle moves at a speed that is less than the speed of light.

Since $ds^2 = 0$ along null worldlines, it follows that $d\tau = 0$ along such worldlines and thus the proper time can not be used to parameterize them. One must therefore use some other parameter; let's call it λ. Thus we write the coordinates along the null worldline as $x^\alpha(\lambda)$. Define the tangent vector $u^\alpha = dx^\alpha/d\lambda$. Then

$$\boldsymbol{u} \cdot \boldsymbol{u} = g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = \left(\frac{ds}{d\lambda}\right)^2 = 0 \tag{1}$$

since $ds^2 = 0$. So $\boldsymbol{u}$ is a **null** vector.

We know that the equation of motion describing the timelike worldline of a **free** massive particle in GR is given by the geodesic equation

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0. \tag{2}$$

We state here without proof that the corresponding equation of motion for light rays can be written on the same form, i.e. as the geodesic equation, but with τ replaced by λ:

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0. \tag{3}$$

Actually, this form presumes that λ is a so-called affine parameter. If λ were not affine, the equation of motion would take a different form than the geodesic equation (but (1) would still hold). Note that if λ is affine, then so is $\lambda' = a\lambda + b$ where a and b are arbitrary constants. So affine parameters are not unique.

Light ray orbits in Schwarzschild

The Schwarzschild metric is independent of time t and angle ϕ, and so there are two conserved quantities along geodesics (including null geodesics). These are given by the two Killing vectors ξ and η, given as

$$e = -\boldsymbol{\xi}.\mathbf{u} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda}\,, \tag{4}$$

and

$$\ell = \boldsymbol{\eta}.\mathbf{u} = r^2 \sin^2\theta \frac{d\phi}{d\lambda}\,. \tag{5}$$

However now, we have a new constraint equation

$$\mathbf{u}.\mathbf{u} = g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{6}$$

as these null geodesics are normalised to 0 (where they were normalised to -1 for timelike geodesics). Writing this out explicitly for Schwarzschild (and once working solely in the equatorial plane, $\theta = \pi/2$ and $u^\theta = 0$), we get

$$-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0\,.$$

And rewriting equations 4 and 5 as

$$\frac{dt}{d\lambda} = e\left(1 - \frac{2GM}{r}\right)^{-1}$$

and

$$\frac{d\phi}{d\lambda} = \frac{\ell}{r^2}\,,$$

we can then insert these into the constraint equation for the null geodesic to give us

$$\boxed{\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{\text{eff}}(r) \qquad (7)}$$

where

$$b \equiv \left|\frac{\ell}{e}\right|$$

and

$$W_{\text{eff}}(r) \equiv \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)$$

This equation is the analogue of the radial equation of motion for timelike geodesics.

Here we are free to rescale the affine parameter λ; this cannot affect the physics. However, such a rescaling with scale both ℓ and e in the same way. This the physics (here the properties of the light ray orbits) can only depend on the ratio of ℓ and e (i.e. b, the impact

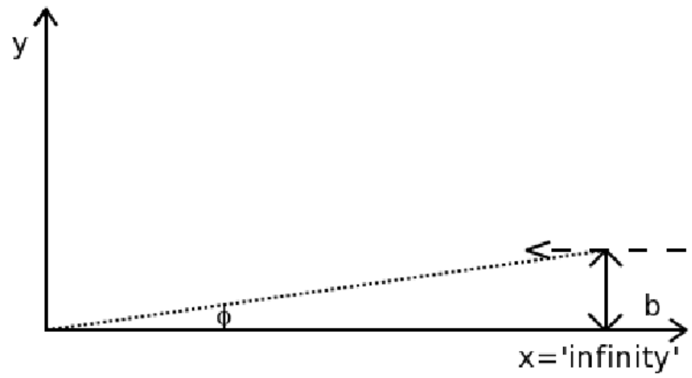

Figure 1: Definition of impact parameter

parameter) which is invariant under such a scaling.

The sign of ℓ gives the direction of the light ray orbit.

The physical meaning of the impact parameter b can be understood as the 'offset' between the direction of the photon and the gravitating mass. Consider orbits that reach infinity, when spacetime is flat. The flight ray will travel in an approximately straight line, and if spacetime were to remain flat, it would 'miss' the mass by a distance b.

The definition of the impact parameter at large distances is such that

$$\begin{aligned} b &= \left|\frac{\ell}{e}\right| = \left|\frac{r^2 d\phi/d\lambda}{\left(1-\frac{2GM}{r}\right)\frac{dt}{d\lambda}}\right| \\ &\underset{(r\to\infty)}{\simeq} \left|\frac{r^2 d\phi/d\lambda}{\frac{dt}{d\lambda}}\right| = r^2\left|\frac{d\phi}{dt}\right| \qquad (8) \\ &\simeq r^2\left|\frac{dr}{dt}\frac{d\phi}{dr}\right| \qquad (9) \end{aligned}$$

For very large r, $\phi \simeq b/r$, and also $\frac{d\phi}{dr} = -b/r^2$, and finally $dr/dt = -1$ (the minis sign is there as the light ray is moving to smaller values of r). Inserting this into the equation above give us

$$b = r^2\left|\frac{dr}{dt}\frac{d\phi}{dr}\right| = r^2\frac{b}{r^2} = b$$

A purely radial path as $\ell = 0$ and so the impact parameter $b = 0$.

Plotting $(GM)^2 W_{\text{eff}}(r)$ as a function of $r/(GM)$ (figure 2), we see that it goes to zero at $r/GM = 2$, has a maximum at $r/GM = 3$ (this maximum is $W_{\text{eff}}^{\text{max}} = 1/(27G^2M^2)$, and $W_{\text{eff}} \to 0$ as $r/GM \to \infty$. Circular orbits of radius $r = 3GM$ are possible, if $1/b^2 = W_{\text{eff}}^{\text{max}}$, i.e. $b^2 = 27G^2M^2$. However, these circular orbits are unstable, since a small change in b results in an orbit that moves away from $r = 3GM$ (since W_{eff} has a maximum at $r = 3GM$).

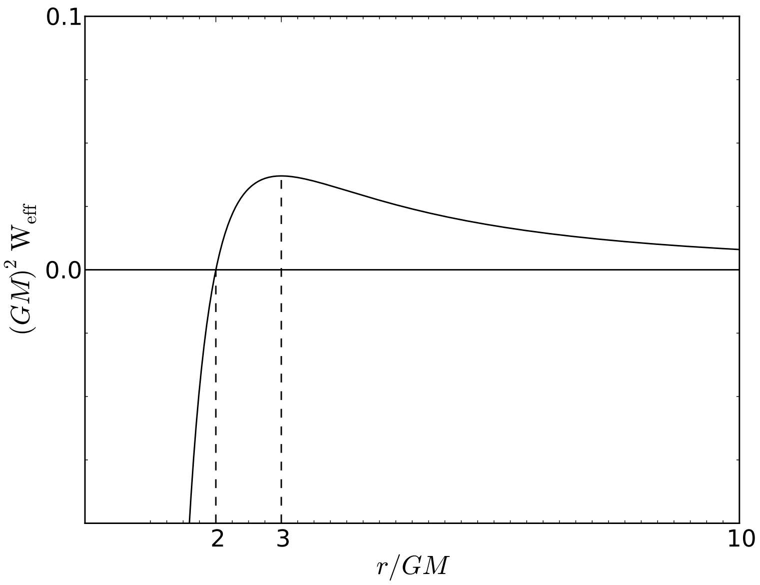

Figure 2: $W_{\rm eff}$ as a function of radius.

A circular light ray orbit is not possible around the Sun, since $R_{\rm Sun} = 10^5$km $\gg 3GM_{\rm Sun} \simeq$ 4.5km. But circular orbits at $r = 3GM$ are possible around black holes.

Now we consider orbits that start at infinity. These either scatter off the mass, if $1/b^2 < W_{\rm eff}^{\rm max}$, and escape back to infinity (see figure 3), or plunge into the center of attraction, if $1/b^2 > W_{\rm eff}^{\rm max}$ (see figure 4),.

Finally we consider orbits of photons that start from a radius r between $2GM$ and $3GM$ (if possible) and are emitted at an angle Ψ with respect to the radial direction. Whether this photon escapes is dependent on the impract parameter $1/b^2$, as if $1/b^2 > W_{\rm eff}^{\rm max}$, it will escape, but if it is not, then it will not. The impact parameter is defined as $b = |\ell/e|$, so the angular momentum ℓ will have to be very small, i.e. the photon will have to be aimed sufficiently close to the radial direction. As r approaches $2GM$, the photon will have to be aimed closer and closer to the radial direction to escape (since $e \propto (1-2GM/r) \to 0$ as $r \to 2GM$).

This critical escape angle $\Psi_{\rm crit}$ (such that the photon escapes if $\Psi < \Psi_{\rm crit}$), is given by (example 9.2 in Hartle)

$$\tan \Psi_{\rm crit} = \frac{1}{r}\left(1 - \frac{2GM}{r}\right)^{1/2}\left[\frac{1}{27G^2M^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)\right]^{-1/2}$$

If $r = 3GM$, the inverse square-root term in the square brackets goes to zero, and so $\tan \Psi_{\rm crit}$ goes to infinity and $\Psi_{\rm crit} = \pi/2$. This corresponds to the circular orbits, as the photon is emitted at exactly right-angles to the radial. Similarly, if $r = 2GM$, the square-root term in the round brackets goes to zero, and so $\Psi_{\rm crit} = 0$, i.e. only photons emitted exactly in the radial direction will escape.

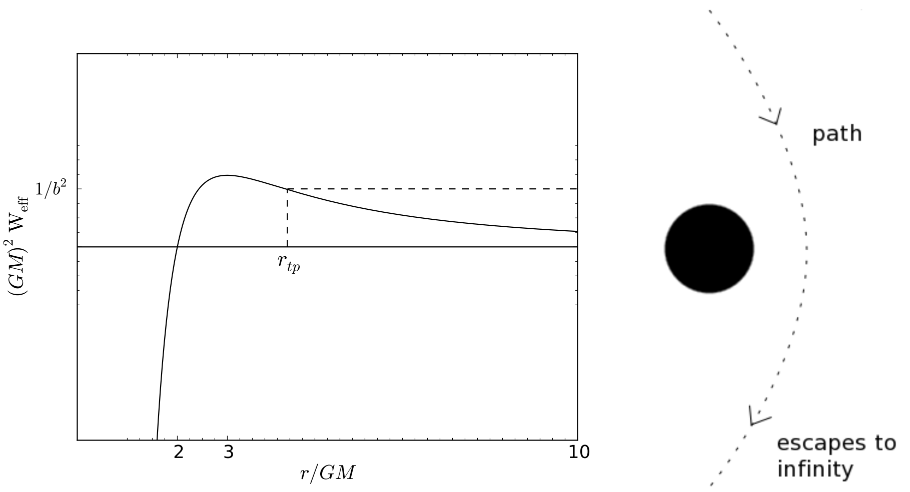

Figure 3: $1/b^2 < W_{\rm eff}^{\rm max}$, 'scattering' orbit. r_{tp} is the turning point, where $dr/d\lambda = 0$.

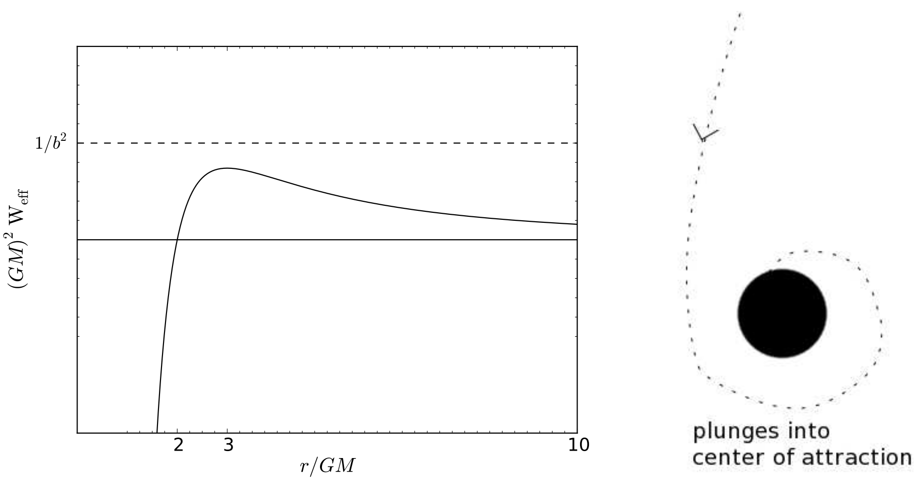

Figure 4: $1/b^2 > W_{\rm eff}^{\rm max}$, 'plunge' orbit.

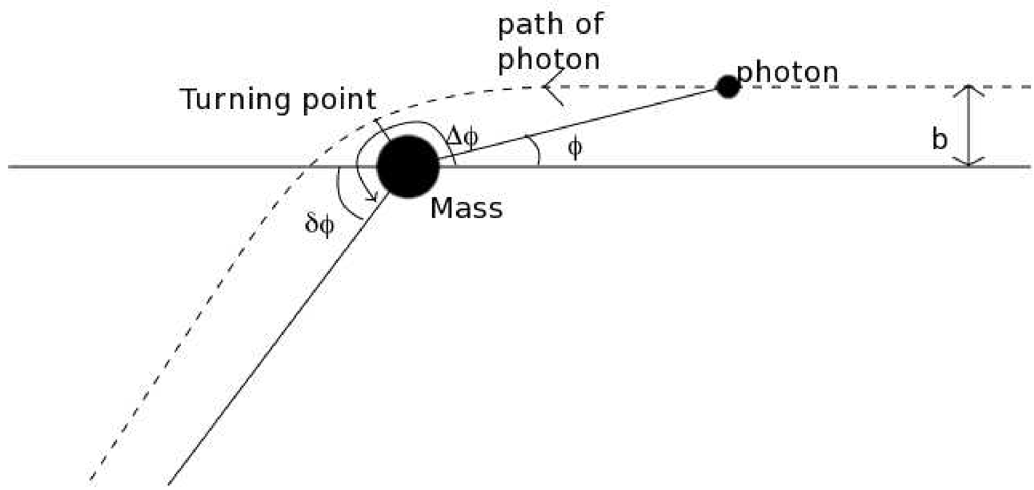

Figure 5: The deflection of the trajectory of a photon by a massive object.

Deflection of light

The presence of massive objects will warp spacetime, and so deflect the 'straight-line' path of photons. This is an important test of General Relativity, as it gives rise to an observable effect we call 'Gravitational Lensing'.

The deflection of light angle is defined as $\delta\phi_{\text{def}} \equiv \Delta\phi - \pi$, where $\Delta\phi$ is defined as the angle between the radial direction on the incoming trajectory and the radial direction on the outgoing trajectory. If there is no deflection then by definition $\Delta\phi = \pi$, and so $\delta\phi = 0$. The deflection is illustrated in figure 5.

We consider the trajectory of the photon by returning to the equation for the radial motion

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{\text{eff}}(r)$$

From this we see that

$$\frac{dr}{d\lambda} = \pm\ell\left(\frac{1}{b^2} - W_{\text{eff}}(r)\right)^{1/2}$$

Remembering the definition of $\ell = r^2 d\phi/d\lambda$, we derive the rate of change of the angle as a function of the radius

$$\frac{d\phi}{dr} = \frac{d\phi/d\lambda}{dr/d\lambda} = \pm\frac{1}{r^2}\left(\frac{1}{b^2} - W_{\text{eff}}(r)\right)^{-1/2}$$

(The plus or minus sign gives the direction of the trajectory, whether the photon is in-falling or outgoing. It should be obvious from the equation that the motion is completely symmetric around the turning point).

If we define the turning point to be at $r = r_1$, we see that $\Delta\phi$ is simply two times the angle swept out from $r = r_1$ up to $r = \infty$, i.e.

$$\begin{aligned}
\Delta\phi &= 2\int_{r=r_1}^{r=\infty} d\phi = 2\int_{r_1}^{\infty} dr \frac{d\phi}{dr} \\
&= 2\int_{r_1}^{\infty} \frac{1}{r^2}\left(\frac{1}{b^2} - W_{\text{eff}}(r)\right)^{-1/2} dr \\
&= 2\int_{r_1}^{\infty} \frac{1}{r^2}\left(\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)\right)^{-1/2} dr \qquad (10)
\end{aligned}$$

At $r = r_1$, $\frac{1}{b^2} = W_{\text{eff}}(r_1)$ (since by definition this must be the point where $dr/d\lambda = 0$). So we can introduce a new variable $r = b/w$, giving us

$$\Delta\phi = 2\int_0^{w_1} dw \left[1 - w^2\left(1 - \frac{2GM}{b}w\right)\right]^{-1/2}$$

where $w_1 = b/r_1$. We see that $\Delta\phi$ is a function of the ratio GM/b.

Consider the bending of light by the Sun, where $GM_{\text{Sun}} = 1.47$km. The largest bending will be those photons whose trajectories just graze the outer surface of the Sun, i.e. the impact parameter $b \simeq$ solar radius $R_{\text{Sun}} = 6.96 \times 10^5$km. So the ratio $2GM/b \sim 10^{-6} \ll 1$, a very small number.

However, this very small value means that we can expand the equation for $\Delta\phi$ to lowest order in GM/b. For this purpose, we rewrite $\Delta\phi$ as

$$\begin{aligned}
\Delta\phi &= 2\int_0^{w_1} dw \left[1 - w^2\left(1 - \frac{2GM}{b}w\right)\right]^{-1/2} \\
&\approx 2\int_0^{w_1} dw \left(1 - \frac{2GM}{b}w\right)^{-1/2}\left[\left(1 - \frac{2GM}{b}w\right)^{-1} - w^2\right]^{-1/2} \\
&\approx 2\int_0^{w_1} dw \frac{1 + \frac{GM}{b}w}{\left[1 + \frac{2GM}{b}w - w^2\right]^{1/2}} \qquad (11)
\end{aligned}$$

Solving for this integral using matlab or Wolfram Alpha (or similar), we find

$$\Delta\phi \approx \pi + \frac{4GM}{b}$$

and so

$$\delta\phi = \frac{4GM}{b} = \frac{4GM}{c^2 b}$$

For a photon trajectory grazing the edge of the sun, this gives us

$$\delta\phi = 1.7''.$$

Gravitational Lensing

Because of the warping of spacetime by mass, a photon travelling from a source to an observer can be deflected by the mass that it passes along the way. This can lead to the observer seeing multiple images of the distant source, as shown in Figure 6. As the intervening mass acts a lens, the phenomenal is known as **gravitational lensing**.

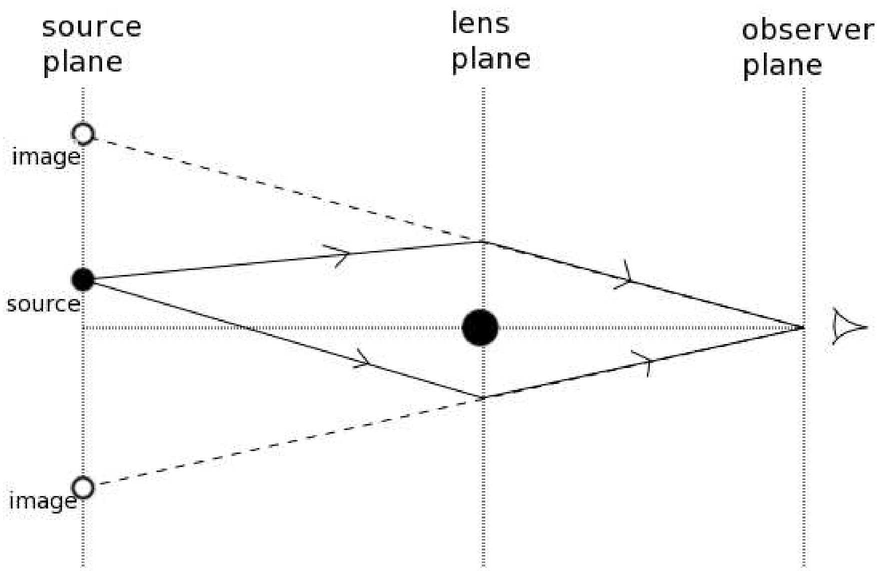

Figure 6: A schematic of the multiple images created by a gravitational lens.

Black holes

Summary

- Formation of a Black Hole
- Event horizons and coordinate breakdown
- Eddington-Finkelstein coordinates

The first two paragraphs below are not examinable.

Formation

In a star like the Sun, the pressure supporting the star against the contracting force of gravity comes from the heat produced by fusion of light nuclei into heavier nuclei. This process of thermonuclear burning has many stages. First hydrogen is burnt to helium. Then, when a significant fraction of the hydrogen is exhausted, the burning process becomes too weak to withstand gravity, and the star contracts. The compression causes the temperature to increase. When the temperature gets high enough, reactions which burn helium to heavier elements begin, which can again balance gravity at the new (smaller) radius. When the helium starts to get exhausted, the star again contracts, and yet another stage of the burning process begins when the temperature has gone up sufficiently. And so on. However, this thermonuclear burning process eventually comes to a stop. The heavier the nuclei, the less energy is gained by burning them and producing even heavier nuclei, and the gain eventually comes to a complete halt when Fe (iron) is reached.

Some stars can then use nonthermal sources of pressure to prevent complete gravitational collapse. In white dwarf stars ($R/M \sim 1000$) the pressure comes from the "Fermi pressure" of electrons, due to the Pauli exclusion principle, which forbids two fermions to be in the same quantum state. In contrast, neutron stars ($R/M \sim 10$) are supported by the Fermi pressure of neutrons and by nuclear forces. However, it is believed that if the mass of the star is sufficiently large ($\sim 2 - 4$ solar masses), these types of pressure will be unable to prevent complete gravitational collapse, thus leading to a black hole. In addition to black holes that are essentially the end result of the evolution of such stars, much heavier ("supermassive") black holes with masses $\sim 10^6 - 10^9$ solar masses are believed to exist at the centers of galaxies (there are also other possible types of black holes).

Schwarzschild Black Holes

While realistic black holes are expected to be spinning (i.e. possess nonzero angular momentum), we will in the following discuss the simplest type of black hole, the spherically symmetric non-spinning variety described by the Schwarzschild geometry.

The metric expressed in Schwarzschild coordinates has singularities at $r = 2GM$ and $r = 0$. At $r = 2GM$, g_{rr} diverges, while at $r = 0$ both g_{tt} and g_{rr} diverge. The singularity at $r = 2GM$ is actually not a **physical** singularity of the spacetime, but is instead just a **coordinate** singularity associated with the use of Schwarzschild coordinates. In other words, it is possible to find other coordinate systems in which the metric is not singular at $r = 2GM$ and the regions $r > 2GM$ and $r < 2GM$ are smoothly connected. Nevertheless, the surface $r = 2GM$ is still special. Referred to as the **event horizon**, it is a "one-way" surface: objects (e.g. observers, light rays) can go from $r > 2GM$ to $r < 2GM$, but not in the opposite direction. The singularity at $r = 0$, on the other hand, is a point of infinite spacetime curvature and is therefore a true (i.e. physical) singularity: there is no change of coordinates that can make this singularity go away.

To justify the above claims involving the event horizon $r = 2GM$, let us first consider an observer plunging radially inwards from infinity with zero start velocity. This example is similar to the one about the escape velocity discussed earlier, except that the motion is reversed. Thus one has the conserved quantity $e = 1$ and so $\varepsilon = (e^2 - 1)/2 = 0$. Since the motion is radial, we further have $\ell = 0$ and so $V_{\text{eff}} = -GM/r$. The "energy conservation" equation $\varepsilon = (1/2)(dr/d\tau)^2 + V_{\text{eff}}(r)$ thus becomes

$$0 = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r}, \tag{1}$$

which gives

$$dr\sqrt{\frac{r}{2GM}} = -d\tau \tag{2}$$

(we have taken the negative solution since the observer is moving inwards). Thus

$$\frac{1}{\sqrt{2GM}}\int_{r_0}^{r_1} dr\, r^{1/2} = -\int_{\tau_0}^{\tau_1} d\tau \tag{3}$$

Carrying out the integrals gives

$$\tau_1 = \tau_0 + \frac{4GM}{3}\left[\left(\frac{r_0}{2GM}\right)^{3/2} - \left(\frac{r_1}{2GM}\right)^{3/2}\right] \tag{4}$$

This expression relates the observer's proper time and radial coordinate. Clearly there is nothing singular about this expression. In particular, taking $r_0 > 2GM$ and $r_1 < 2GM$, the expression gives the change in proper time $\tau_1 - \tau_0$ during a segment of the journey that involves crossing the event horizon. We therefore conclude that an infalling observer will not experience anything physically singular (abrupt) at $r = 2GM$; the observer's proper time, measured by a clock carried by the observer, simply increases smoothly as the observer passes the event horizon. From this expression we can also see that the proper time interval between the observer being at the horizon $r_0 = 2GM$ and reaching the singularity $r_1 = 0$ is given by

$$\Delta\tau_{2GM\to 0} = \frac{4GM}{3} \tag{5}$$

Putting back c, we have $c\Delta\tau = 4GM/(3c^2)$, i.e. $\Delta\tau = 2R_s/(3c)$ where $R_s = 2GM/c^2$ is the Schwarzschild radius. As an example, taking $R_s \sim 30$ km (corresponding to a mass of about 10 solar masses) gives $\Delta\tau \sim 10^{-4}$ seconds, a finite time.

On the other hand, let us now describe the observer's motion in terms of the Schwarzschild coordinate time t as a function of the radial coordinate r. From the fact that $e = 1$ and $e = (1 - 2GM/r)dt/d\tau$ we get

$$dt = \left(1 - \frac{2GM}{r}\right)^{-1} d\tau. \tag{6}$$

Inserting for $d\tau$ from Eq. (2) gives

$$dt = -\frac{(r/2GM)^{3/2}}{r/2GM - 1} dr. \tag{7}$$

Integrating this gives

$$t = t_0 + k + 2GM\left[-\frac{2}{3}\left(\frac{r_1}{2GM}\right)^{3/2} - 2\left(\frac{r_1}{2GM}\right)^{1/2} + \log\left|\frac{1+\sqrt{r_1/2GM}}{1-\sqrt{r_1/2GM}}\right|\right] \tag{8}$$

where k is a term (unimportant for our discussion) that only depends on r_0. If we now consider the observer's motion as he/she approaches the event horizon (i.e. take $r_0 > 2GM$ and let r_1 approach $2GM$ from above) we see that the time interval $t_1 - t_0$ diverges because of the term $-\log|1 - \sqrt{r_1/2GM}|$. Thus it takes an infinite coordinate time interval for the observer to reach the horizon, despite the fact that this corresponds to a finite proper time interval for the observer. This shows that the Schwarzschild coordinate t is flawed when it comes to describing physical phenomena near $r = 2GM$.

As another example of this, let us consider radial light rays sent from a radial coordinate $r_0 > 2GM$. The fact that the rays are light rays implies $ds^2 = 0$, the fact that they are radial implies $d\theta = d\phi = 0$. Thus along such a light ray,

$$0 = ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -(1 - 2GM/r)dt^2 + (1 - 2GM/r)^{-1}dr^2, \tag{9}$$

i.e.

$$dt = \pm(1 - 2GM/r)^{-1}dr \tag{10}$$

where the two signs correspond to outgoing (towards $r = \infty$) or ingoing (towards $r = 2GM$) light rays. Integrating gives

$$t = \pm(r + 2GM\log|r - 2GM|) + \text{ const.} \tag{11}$$

where const. is an integration constant. For $r \gg 2GM$ the r term overwhelms the logarithm, so $t \approx \pm r +$ const., i.e. the light cones far away from the event horizon have approximately unit slope, as in flat spacetime. If, on the other hand, we let r approach $2GM$ from above, the light cones appear to "close up", as shown in Fig. 1, and so from this description it looks as if ingoing light rays will not reach the horizon. However, as in the previous example, this is just an illusion and a consequence of using the coordinate time t to describe the light ray's motion. (On the other hand, outside observers will never be able to see anything reaching

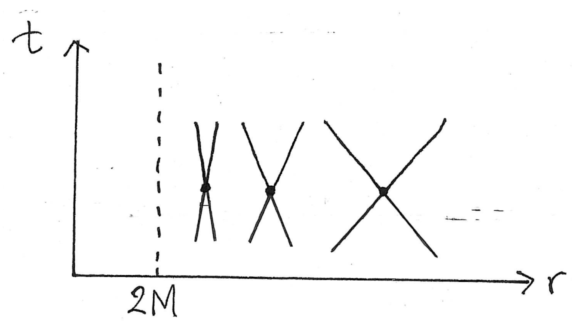

Figure 1: In Schwarzschild coordinates, light cones have slope $\approx \pm 1$ far away from the event horizon $r = 2GM$, but appear to "close up" as the event horizon is approached.

the event horizon, as we'll discuss in some detail later).

The above examples indicate that the event horizon $r = 2GM$ plays a special role in the physics of black holes. Other evidence of this includes the following results for what happens as $r \to 2GM$ from above (see formulas in previous lecture notes):

- the escape velocity of material particles approaches 1 (speed of light)
- the "opening" angle for escape of outgoing light rays approaches 0
- the gravitational redshift approaches ∞ (equivalently, the frequency ratio $\omega_2/\omega_1 \to 0$) implying that light sent from the horizon becomes undetectable to outside observers

Eddington-Finkelstein coordinates

As we have seen, it is not possible to use the Schwarzschild coordinates to describe massive objects or light rays crossing the horizon. The problem originates in the coordinate singularity of the Schwarzschild metric at $r = 2GM$ in these coordinates. Thus we'd like to find a new set of coordinates where this coordinate singularity is absent. One possibility is to replace the Schwarzschild coordinate time t with a new time coordinate $\tilde{t}$ defined as

$$\tilde{t} \equiv t + 2GM \log \left| \frac{r}{2GM} - 1 \right| . \tag{12}$$

This definition is motivated by the equation for ingoing radial light rays (the minus sign solution in Eq. (11)), because if we now express that equation in terms of $\tilde{t}$ it becomes

$$\tilde{t} = -r + \text{ const.}, \tag{13}$$

i.e. in the $r\tilde{t}$ plane it is a straight line of slope -1 for all r. Using Eq. (12) to eliminate t in favour of $\tilde{t}$, the line element written in terms of the new coordinate system $(\tilde{t}, r, \theta, \phi)$ becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) d\tilde{t}^2 + \frac{4GM}{r} d\tilde{t}\, dr + \left(1 + \frac{2GM}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{14}$$

Note that in these coordinates the metric has no singularity at $r = 2GM$. This explicitly demonstrates that the singularity at $r = 2GM$ in Schwarzschild coordinates was merely a coordinate singularity, not a physical singularity. On the other hand, the singularity at $r = 0$

exists also in these new coordinates, which is consistent with this being a true (i.e. physical) singularity. Note also that the metric is no longer diagonal due to the $d\tilde{t}\,dr$ term.

Actually, the metric can be simplified a little more by making yet another change of coordinates, by replacing $\tilde{t}$ with v defined as

$$v = \tilde{t} + r. \tag{15}$$

When expressed in terms of the coordinates (v, r, θ, ϕ), called **Eddington-Finkelstein** coordinates, the line element becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2dv\,dr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{16}$$

Again, the metric is not singular at $r = 2GM$ while the singularity at $r = 0$ is still there. The metric is still off-diagonal in these coordinates (due to the term coupling dv and dr) but there is now no dr^2 term.

Let us now describe radial light rays in these new coordinate systems. Their equation is

$$0 = ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2dv\,dr. \tag{17}$$

Clearly, one solution to this equation is $dv = 0$, i.e. $v =$ const, which gives the solution (13) already discussed that describes infalling light rays. The other solution is

$$-\left(1 - \frac{2GM}{r}\right) dv + 2dr = 0. \tag{18}$$

Integrating gives

$$v = 2\left(r + 2GM \log\left|\frac{r}{2GM} - 1\right|\right) + \text{ const.}, \tag{19}$$

i.e.

$$\tilde{t} = r + 4GM \log\left|\frac{r}{2GM} - 1\right| + \text{ const.} \tag{20}$$

If one plots the solutions (13) and (20) in the $r\tilde{t}$ plane one gets the left panel of Fig. 2 (this is Fig. 12.2 in Hartle). The light cone at each point (see figure) is bounded by these two solutions: the left edge of the light cones is given by Eq. (13) and the right edge is given by Eq. (20). The left edge of the light cones represents ingoing radial light rays. The right edge of the light cones are outgoing radial light rays for $r > 2GM$ but become ingoing for $r < 2GM$. Thus inside the event horizon ($r < 2GM$) all light rays are ingoing, moving towards the singularity at $r = 0$. That is, it is impossible for light that is emitted from inside the event horizon to escape out through the horizon. Furthermore, since massive objects move on timelike worldlines that must lie inside the local light cone at each spacetime point, it follows that massive objects inside the event horizon can not escape into the outside region $r > 2GM$ either. Thus any objects inside the event horizon, whether massive or massless, are "doomed" to move towards the physical singularity at $r = 0$ and no information can be sent to the outside region.

It follows, as a consequence of the light cone structure, that it is possible for observers to be stationary for $r > 2GM$, but not for $r < 2GM$. A slightly different way of seeing this is to

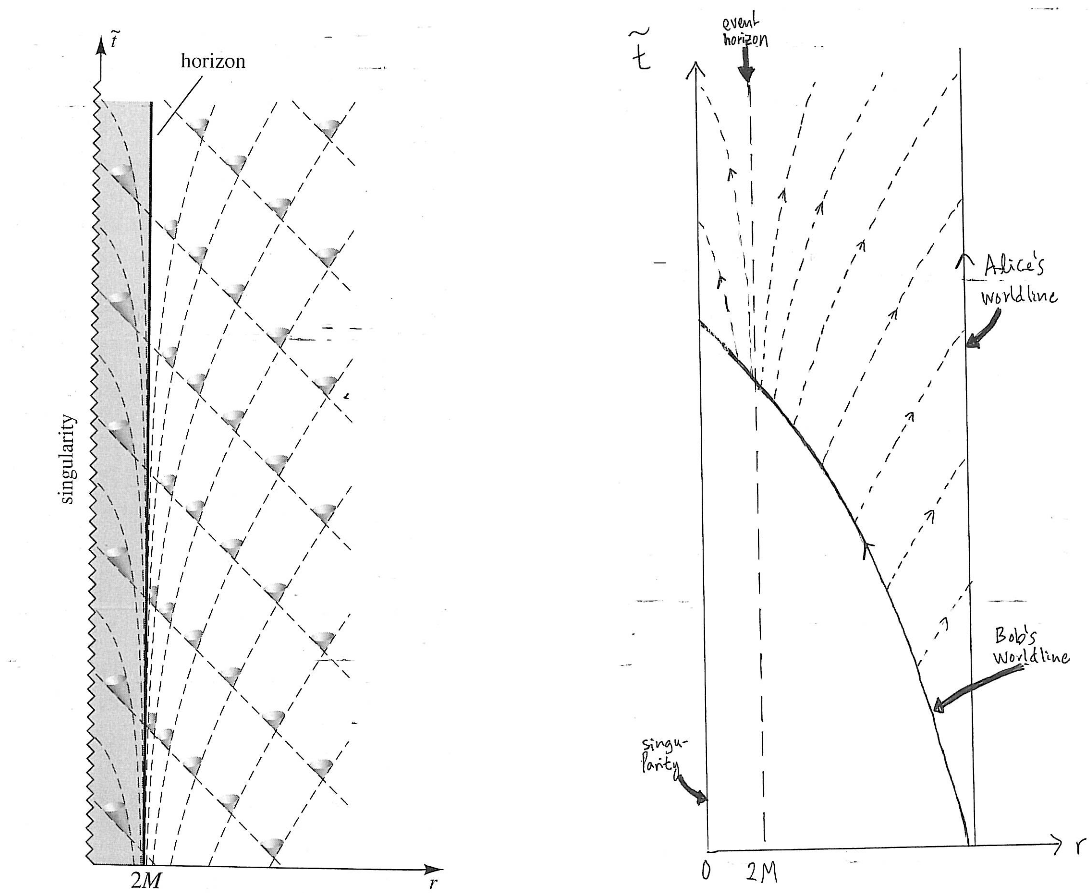

Figure 2: Left: Light cones in the Schwarzschild geometry (see text for details; the figure is from Hartle). Right: An experiment involving two observers Alice and Bob (see text for details). The figure shows the world lines of Alice and Bob (full lines) and some light rays emitted by Bob (curved dashed lines), which reach Alice if emitted from $r > 2GM$ but go towards the singularity otherwise.

assume stationarity (i.e. $dr = d\theta = d\phi = 0$) and insert this into the metric expressed in EF coordinates, which gives

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2. \tag{21}$$

If $r > 2GM$ the rhs is negative, i.e. the spacetime interval is timelike, which is what it should be for any observer, so having stationary observers is fine in this case. However, for $r < 2GM$ the rhs becomes positive, corresponding to a spacelike interval, which is impossible for an observer (or the worldline of any object, more generally). Thus stationary observers inside the event horizon are forbidden. Instead a timelike interval inside the horizon requires $dr < 0$, as seen from the line element (16). Thus the (negative) r direction becomes a time-like coordinate inside the event horizon. And consequently it can also be said that (Hartle, p. 262) "the $r = 0$ singularity in the Schwarzschild geometry is not a place in space; it is a moment in time."

Finally, let us consider the question of what an observer located outside the event horizon will see as another observer moves radially inwards, eventually going through the event horizon and then reaching the singularity at $r = 0$. Thus imagine two observers, Alice and Bob, each in their spaceship. Suppose Alice is stationary at some radial coordinate $r > 2GM$. Bob, on the other hand, decides to move inwards towards the singularity. Alice and Bob's worldlines are shown in Fig. 2 (right panel).[1]. Assume that as Bob embarks on his journey he also starts sending light signals to Alice. These light rays thus follow the right edge of the light cones in Fig. 2. As Bob emits signals from closer and closer to the event horizon, Alice's proper time interval between adjacent received signals as measured by her clock will get longer and longer in comparison to Bob's proper time interval between those signals as measured by his clock when they were emitted. As a result, Alice will never see Bob reach the horizon; instead he just appears to move more and more slowly as she sees him get closer to it (and the light from him appears more and more redshifted). Furthermore, any light signals that Bob emits after crossing the event horizon will not reach Alice, as they will instead move towards the singularity as shown by the light cones in this region.

[1]The worldlines of Alice and Bob must necessarily lie inside the local light cone at each point along the worldlines. Because I drew it by hand, Bob's worldline may perhaps look a little questionable for small r in this respect ...