PHYS2055 Assignment 4

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Part A

Problem 4.1

Forty-one years ago Kermit sang the following lyrics:

Rainbows are visions
They're only illusions
And rainbows have nothing to hide

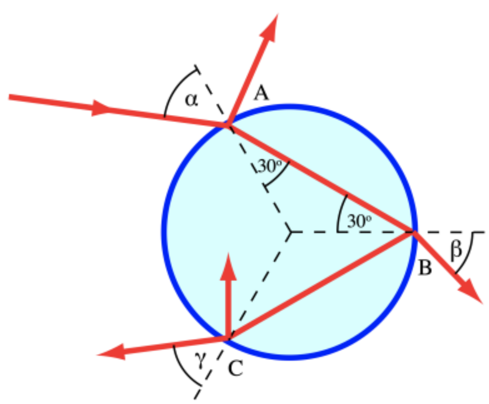

Figure 1: Sunlight refracting and reflecting through an airborne water droplet.

a. Briefly explain to Kermit how rainbows form. To help, you can use the drawing in Figure 1 that shows a single frequency of sunlight-say, violet light at 380 nm - refracting and reflecting through a water droplet.

   A rainbow is the result of light of a multitude of wavelengths (approximated as white light) passing from air through droplets of water, and then back through air again as the light travels to our eyes. As Figure 1 shows, the trajectory of light is 'bent' as it transverses the interface of media (in this case, from air to water, and then water to air). Since refractive index (how much the light bends as it passes through different media) is a function of the angular frequency of light, different wavelengths of light will be refracted to a different degree, resulting in the 'spreading' of different wavelengths against some wall behind the interface. Since white light is composed of the colours in the visible spectrum, this results in a display of the entire spectrum behind the interface (or the water droplet).

b. For violet light, calculate each of the angles in Figure 1. The refractive index of water at this wavelength is $n = 1.34$, and to make your life - and mine - easier you can neglect losses, be they due to absorption or surface irregularities.

Rounding the refractive index of air to 1, the calculations of the angles are relatively straight forward. For an incident and transmitted wave of light, the refractive indices and their angles of refraction are related by

$$n_i \sin(\theta_i) = n_t \sin(\theta_t)$$

At point A, the light is going from air to water, so substituting in values gives

$$\begin{aligned}\sin(\alpha) &= 1.34 \sin(30) \\ \alpha &= \arcsin(1.34 \sin(30)) \\ &\approx 42^{o}\end{aligned}$$

For point B, the situation is analogous to that at point A (the angle inside the droplet is the same, and the refractive indices are the same). Therefore the calculation is the same, and $\beta = \alpha \approx 42^{o}$. At point C, it can be shown that the situation is exactly the same as points A and B, with the angle of incidence to the interface being 30^{o}, as the angle of reflection from point B is defined as $\theta_i = \theta_r = 30^{o}$. Therefore, $\gamma = \beta = \alpha \approx 42^{o}$

c. Sunlight is randomly polarised. Calculate the polarisation of the light exiting the droplet, that is at C. There is evidence that some species in the order Anura - that's frogs and toads to you and me - have polarised vision. It turns out Kermit is such a frog, with eyes that polarised in the same orientation as polaroid sunglasses. What would Kermit see when looking at a rainbow? What if he turns his head 90^{o}?

Since sunlight is randomly polarised, there are equal parts of the light that is both p and s polarised. Thus, calculations (by the Fresnel Equations) must be performed for each polarisation state for the three relevant interface interactions A, B, and C. Firstly, for perpendicularly polarised light,

Point	n_i	n_t	θ_i	θ_t	$r_\perp$	$t_\perp$	Resultant Intensity (Prop. of incident intensity)
A	1	1.34	42	30	0.22	0.78	0.952
B	1.34	1	30	42	0.22	0.78	0.048
C	1.34	1	30	42	0.22	0.78	0.952
Final Wave							0.044

Table 1: Perpendicularly Polarised Violet Light Calculations of Reflection through Water Droplet

Similarly for parallel polarised light,

Point	n_i	n_t	θ_i	θ_t	$r_\parallel$	$t_\parallel$	Resultant Intensity (Prop. of incident intensity)
A	1	1.34	42	30	0.07	0.8	0.995
B	1.34	1	30	42	-0.07	1.25	0.005
C	1.34	1	30	42	-0.07	1.25	0.995
Final Wave							0.005

Table 2: Parallel Polarised Violet Light Calculations of Reflection through Water Droplet

And so, looking at the final waves in comparison to the incident intensity, it can be seen that perpendicularly polarised light is far more intense than parallel polarised light, although they are both present. Relating the above polarisations to horizontal-vertical reference frames, the parallel polarised light corresponds to vertical, whereas the perpendicularly polarised light correcponds to horizontal.

Given that polaroid sunglasses only allow vertically polarised light in, Kermit would only see a very faint rainbow, and would see a much more intense rainbow if his head was tilted 90°.

Problem 4.2

My microwave has interior dimensions of W × D × H of 35cm × 37cm × 21cm.

a. What are the resonant frequencies of the four lowest-frequency modes? (*n.b.* the linear frequencies not the angular frequencies).

   Since only one of *nml* can be zero in a cavity resonator, the four lowest modes are 011, 101, 110, and 102 (note: these lowest-frequency modes were found with trial-and-error substitution into the resonant frequency equation). Since the frequency of a wave is related to it's angular frequency by $\omega = 2\pi f$, the resonant (linear) frequency may be given by

$$f = \frac{c}{2}\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2 + \left(\frac{l}{d}\right)^2}$$

   Taking $a = W = 35\text{cm}$, $b = H = 21\text{cm}$, and $d = D = 37\text{cm}$, the resonant frequencies of the four lowest-frequency modes are

Mode	Frequency (MHz)
011	820.75
101	589.53
110	832.42
102	916.47

Table 3: Resonant Frequencies of Four Lowest-Frequency Modes

b. For the lowest-order mode, describe how the electric-field boundary conditions are satisfied at the oven walls.
   n.b. I clean my oven regularly, so assume the walls are metal as opposed to layers of congealed food.

   The boundary conditions described are $E_{\parallel} = 0$, and $B_{\perp} = 0$, where all waves are totally internally reflected. As shown in Table 3, the lowest-order mode is represented by a TE_{101} mode of electromagnetic wave. This corresponds to there being no transverse electric field in the vertical direction (z-axis) of the microwave oven, and there only being a field in the width and depth (x and y axes) dimensions. Since there are $n+1$ and $m+1$ nodes (zeros in the electric field) inside the oven, there are only 2 nodes on each of the width and depth dimensions. To satisfy the boundary conditions, this corresponds to the nodes being at each of the boundaries of the microwave oven. Since the boundaries are metal (as opposed to congealed food), there is negligible absorption of the field, and it is assumed to be perfectly conductive (reflective).

c. For the lowest-order mode, where is the highest intensity in my oven? Explain your reasoning.

   The lowest-order mode, as described in the previous part, corresponds to there being one maxima and 2 zero points on each of the width and depth axes of the microwave oven. As the zero points were determined to be along the boundaries, the only possible position of the (singular) maxima that satisfies the shape of cosine curve is in the middle of the boundaries, or rather in the middle of the microwave oven.

d. In a real oven, there are many more modes resonant than just the lowest-order one, or indeed four. Considering that, what can you now say about the distribution of high intensities? Please *briefly* explain your reasoning. (A calculation is not necessary).

Given that there are many modes resonant in the microwave oven, the higher nodes would mean that there are many more maxima and minima of the electric field throughout the oven, as there are n maxima and $n+1$ minima. Because of this, it's difficult to qualitatively determine where the maxima exactly are, but it can be said that there are multiple points throughout the oven at which maxima occur, as opposed to just in the middle.

Part B - Advanced

Problem 4.3

Microwaves are not only good for cooking, they're useful in point-to-point communications as well, for example from ship to ship at sea.
Consider a microwave transmitter with an angular frequency of $\omega = 21\pi$ GHz. The metal plate shown in Figure 2 is used to polarise the beam. In one orientation, the microwaves are almost completely reflected; however when rotated 90^o about an axis into the page the microwaves are transmitted.

Explain how the plate polarises and transmits the microwaves, and derive an expression for transmitted intensity as a function of the thickness of the plate. What thickness will block 95% of the microwaves when 'cross-polarised'?

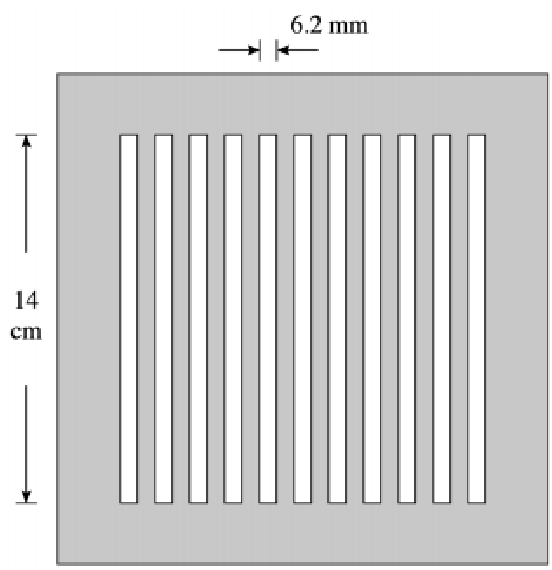

Figure 2: A polariser for microwaves. Slot dimensions are not to scale

Firstly, the microwaves in question have a linear frequency of $f = \omega/2\pi = 10.5$GHz (which corresponds to a wavelength of $\lambda = c/f \approx 2.86$cm). The cutoff frequency for a rectangular waveguide with a width of $a = 6.2$mm is equal to $\omega_{pc} = \pi c/a \approx 48.4\pi$GHz (with the subscript pc to indicate polarised). Comparing this value with the given angular frequency of 21πGHz, it can be seen that the microwaves transmitted are far below the required frequency to travel through the waveguide when the plate is oriented to the amplitude of the wave. As the transverse wave number is then imaginary, the amplitude of the wave gradually decreases, and the wave is gradually reflected back out of the waveguide. In the other orientation, the cutoff frequency is found with the width $a = 14$cm, $\omega_{tc} = \pi c/a \approx 2.14\pi$GHz (with the subscript tc indicated waves are transmitted). As the angular frequency of the microwaves are almost $10\times$ higher than this critical frequency, the waves travel freely through the plate in this orientation.

Now, to determine an expression for transmitted intensity as a function of the thickness of the plate. The wavenumber for an electromagnetic wave propagating in the z-direction may be given by

$$k_z = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2}$$

Substituting in the values of the problem gives $k_z \approx 456.43i$. Now, the electric field through a waveguide may be approximated by

$$E_y = E_0 e^{i(k_z z - \omega t)} \sin k_x x$$

Expanding this gives

$$E_y = E_0 e^{ik_z z} e^{-\omega t} \sin k_x x$$

Substituting in the calculated value for k_z gives

$$E_y = E_0 e^{-456.43z} e^{-\omega t} \sin k_x x$$

The intensity of the electromagnetic wave as a function of its distance through the waveguide is therefore

$$E = E_0 e^{-456.43z} = \frac{E_0}{e^{456.43z}}$$

The thickness of the waveguide that will result in the microwaves at 95% of the source intensity is thus

$$\begin{aligned}
0.95E_0 &= \frac{E_0}{e^{456.43z}} \\
e^{456.43z} &= \frac{E_0}{0.95E_0} \\
z &= \frac{\ln\left(\frac{1}{0.95}\right)}{456.43} \\
&\approx 1.12 \times 10^{-4}\,\mathrm{m}
\end{aligned}$$

Therefore, a waveguide thickness of approximately 0.12 millimetres would result in a reduction of intensity of 95% of any microwaves transmitted.