MATH4105 Assignment 1

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Question 2

For a coordinate transformation $x^i \to x'^i$, we begin with

$$x^r_i x'^i_s = \delta^r_s$$

Taking the derivative of each side with respect to some $\delta x'^m$ (that is, multiplying by $\partial_m = \partial/\partial x'^m = \partial x^k/\partial x'^m \cdot \partial/\partial x^k$), we obtain

$$x^r_i x^k_m x'^i_{ks} + x^r_{im} x'^i_s = 0$$

We can then relabel the i index on the right term to n, giving

$$x^r_i x^k_m x'^i_{ks} + x^r_{mn} x'^n_s = 0$$

Now, multiply by x'^i_r, and then x'^m_k after:

$$x^k_m x'^i_{ks} + x'^i_r x^r_{mn} x'^n_s = 0$$
$$x'^i_{ks} + x'^i_r x^r_{mn} x'^m_s x'^n_k = 0$$

Finally, relabel the index $s \to j$ and we achieve the desired identity

$$x'^i_{jk} + x'^i_r x^r_{mn} x'^m_j x'^n_k = 0$$

Question 3

i. The acceleration vector over a change of coordinates $x^r(t) \to x'^r(t)$ is given by

$$\begin{aligned}
a'^r(t) &= \left.\frac{d^2 x'^r}{dt^2}\right|_{t=t_0} \\
&= \frac{d}{dt}\left(\frac{dx'^r}{dt}\right)_{t=t_0} \\
&= \frac{d}{dt}\left(\frac{dx'^r}{dx^j}\frac{dx^j}{dt}\right)_{t=t_0} \\
&= \frac{d}{dt}\left(x'^r_j \; v^j(t)\right)_{t=t_0} \\
&= \left(\frac{d}{dt}x'^r_j\right) v^j(t) + x'^r_j a^j(t)
\end{aligned}$$

Due to the extra term here, this is not the transformation law of a contravariant vector and hence acceleration is not, in general, a contravariant vector (unless $d/dt\, x'^r_j = 0$).

ii. Suppose that a set of quantities, ${\Gamma_{ij}}^r$, transforms at each point on a curve as

$${\Gamma'_{jk}}^r = x'^r_i x^l_j x^m_k \Gamma^i{}_{lm} + x'^r_l x^l{}_{jk}$$

Now suppose we have some quantity,

$$h^r = \frac{d^2 x^r}{dt^2} + {\Gamma_{jk}}^r \frac{dx^j}{dt}\frac{dx^k}{dt}$$

which we want to prove is a contravariant vector. That is, under a change of coordinates $(x^r) \to (x'^r)$ we would expect the usual transformation laws to hold:

$$\begin{aligned}
h'^r &= \frac{d^2x'^r}{dt^2} + {\Gamma'}_{jk}{}^r \frac{dx'^j}{dt}\frac{dx'^k}{dt} \\
&= \frac{d^2x'^r}{dt^2} + \left(x_i'^r x_j^l x_k^m \Gamma^i{}_{lm} + x_l'^r x^l{}_{jk}\right)\frac{dx'^j}{dt}\frac{dx'^k}{dt} \\
&= \frac{d}{dt}\frac{\partial x'^r}{\partial x^j}\frac{dx^j}{dt} + \frac{\partial x'^r}{\partial x^j}\frac{d^2x^j}{dt^2} + \left(x_i'^r x_j^l x_k^m \Gamma^i{}_{lm} + x_l'^r x^l{}_{jk}\right)\frac{\partial x'^j}{\partial x^s}\frac{dx^s}{dt}\frac{\partial x'^k}{\partial x^m}\frac{dx^m}{dt} \\
&= \frac{\partial^2 x'^r}{\partial x^j \partial x^s}\frac{dx^2}{dt}\frac{dx^j}{dt} + x_j'^r\frac{d^2x^j}{dt^2} + \left(x_i'^n x_j^l x_k^m \Gamma^i_{lm} + x_l'^n x^l_{jk}\right) x_s'^j x_m'^k \frac{dx^s}{dt}\frac{dx^m}{dt} \\
&= x_{js}'^r\frac{dx^s}{dt}\frac{dx^j}{dt} + x_j'^r\frac{d^2x^j}{dt^2} + \frac{dx^s}{dt}\frac{dx^m}{dt}\left(\delta_s^l\delta_k^m x_i'^r\Gamma^i_{lm} + x_l'^r x^l_{jk}x_s'^j x_m'^k\right)
\end{aligned}$$

Now we relabel $j \to s$ in the first term, giving

$$h'^r = x_i'^r\Gamma^i_{sm}\frac{dx^s}{dt}\frac{dx^m}{dt} + x_j'^r\frac{d^2x^j}{dt^2} + \frac{dx^s}{dt}\frac{dx^m}{dt}\left(x_{sm}'^r + x_l'^r x^l_{jk}x_s'^j x_m'^k\right)$$

The parentheses term on the right is the identity proven from question 2, and hence is equal to 0. Now we relabel $j \to i$ in the second term, finally giving

$$\begin{aligned}
h'^r &= x_i'^r\Gamma^i_{sm}\frac{dx^s}{dt}\frac{dx^m}{dt} + x_i'^r\frac{d^2x^i}{dt^2} \\
&= x_i'^r\left(\frac{d^2x^i}{dt^2} + \Gamma^i_{sm}\frac{dx^s}{dt}\frac{dx^m}{dt}\right) \\
&= x_i'^r h^i
\end{aligned}$$

which is the transformation law for a contravariant vector. Therefore, our term h^r is indeed a contravariant vector.

Question 4

i. We have that the transformation from cartesian to parabolic cylindrical coordinates is as

$$x = uv\cos\theta; \qquad y = uv\sin\theta; \qquad z = \frac{1}{2}(u^2 - v^2)$$

If a_i are the components of a covariant vector for the cartesian coordinate system, we have that the analogue of this for the parabolic coordinate system are given by

$$\begin{aligned}
a_i' &= x_i^j a_j \\
&= \frac{\partial x^j}{\partial x'^i}a_j \\
\Rightarrow \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} &= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial \theta} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} v\cos\theta & u\cos\theta & -uv\sin\theta \\ v\sin\theta & u\sin\theta & uv\cos\theta \\ u & -v & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}
\end{aligned}$$

And so

$$\begin{aligned}
a_1' &= (v\cos\theta,\ u\cos\theta,\ -uv\sin\theta) \\
a_2' &= (v\sin\theta,\ u\sin\theta,\ uv\cos\theta) \\
a_3' &= (u, -v, 0)
\end{aligned}$$

ii. With the relationship between the coordinate systems given before, we now isolate u, v and θ in terms of purely cartesian variables. Firstly, θ:

$$\frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan\theta \Longrightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

Now, for u, and v:

$$\begin{aligned}
x^2+y^2+z^2 &= u^2v^2 + \tfrac{1}{4}u^4 - \tfrac{1}{2}u^2v^2 + \tfrac{1}{4}v^4 \\
&= \tfrac{1}{4}u^4 + \tfrac{1}{2}u^2v^2 + \tfrac{1}{4}v^4 \\
&= \left(\tfrac{1}{2}(u^2+v^2)\right)^2 \\
\Rightarrow \sqrt{x^2+y^2+z^2} &= \tfrac{1}{2}(u^2+v^2) \\
\Rightarrow u &= \sqrt{\tfrac{1}{2}(u^2+v^2) + \tfrac{1}{2}(u^2-v^2)} \\
&= \sqrt{\sqrt{x^2+y^2+z^2}+z} \\
\Rightarrow v &= \sqrt{\sqrt{x^2+y^2+z^2}-z}
\end{aligned}$$

Now, to transform the coordinates of a cartesian contravariant rank 2 tensor to parabolic cylindrical coordinates, we invoke the contravariant transformation law:

$$\begin{aligned}
T'^{ij} &= x'^i_r \, x'^j_s \, T^{rs} \\
&= X T^{rs} X^T
\end{aligned} \tag{1}$$

Where,

$$\begin{aligned}
X &= \frac{\partial x'^i}{\partial x^r} \\
&= \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \end{pmatrix} \\
&= \begin{pmatrix} \frac{v\cos\theta}{u^2+v^2} & \frac{v\sin\theta}{u^2+v^2} & \frac{u}{u^2+v^2} \\ \frac{u\cos\theta}{u^2+v^2} & \frac{u\sin\theta}{u^2+v^2} & -\frac{v}{u^2+v^2} \\ -\frac{\sin\theta}{uv} & \frac{\cos\theta}{uv} & 0 \end{pmatrix}
\end{aligned}$$

Finally, given a rank 2 contravariant tensor field everywhere given by (in cartesian coordinates)

$$T = \begin{pmatrix} 1 & 1/2 & -4 \\ 1/2 & 2 & 0 \\ 3 & -6 & 7 \end{pmatrix}$$

we're able to enlist the help of the python package `sympy` to compute equation (1) (`.py` file given in assignment submission) and yield the components T'^{ij} in the parabolic cylindrical coordinate system:

$$\begin{aligned}
T'^{11} &= \frac{7u^2 + 6uv\sin\theta - uv\cos\theta - \frac{\sqrt{2}}{2}v^2\cos(2\theta+\frac{\pi}{4}) + \frac{3}{2}v^2}{(u^2+v^2)^2} \\
T'^{12} &= \frac{-6u^2\sin\theta + 3u^2\cos\theta - \frac{\sqrt{2}}{2}uv\cos(2\theta+\frac{\pi}{4}) - \frac{11}{2}uv + 4v^2\cos\theta}{(u^2+v^2)^2} \\
T'^{13} &= \frac{-3u\sin\theta - 6u\cos\theta + \frac{\sqrt{2}}{2}v\sin(2\theta+\frac{\pi}{4})}{uv(u^2+v^2)} \\
T'^{21} &= \frac{-4u^2\cos\theta - \frac{\sqrt{2}}{2}uv\cos(2\theta+\frac{\pi}{4}) - \frac{11}{2}uv + 6v^2\sin\theta - 3v^2\cos\theta}{(u^2+v^2)^2} \\
T'^{22} &= \frac{-\frac{\sqrt{2}}{2}u^2\cos(2\theta+\frac{\pi}{4}) + \frac{3}{2}u^2 + 6uv\sin\theta + uv\cos\theta + 7v^2}{(u^2+v^2)^2} \\
T'^{23} &= \frac{\frac{\sqrt{2}}{2}u\sin(2\theta+\frac{\pi}{4}) + 3v\sin\theta + 6v\cos\theta}{uv(u^2+v^2)} \\
T'^{31} &= \frac{4u\sin\theta + \frac{\sqrt{2}}{2}v\sin(2\theta+\frac{\pi}{4})}{uv(u^2+v^2)} \\
T'^{32} &= \frac{\frac{\sqrt{2}}{2}u\sin(2\theta+\frac{\pi}{4}) - 4v\sin\theta}{uv(u^2+v^2)}
\end{aligned}$$

$$T'^{33} = \frac{\frac{\sqrt{2}}{2}\cos(2\theta + \frac{\pi}{4}) + \frac{3}{2}}{(uv)^2}$$

Question 6

Given a tensor S_{ijk} which is antisymmetric in its first two suffices, we want to find a tensor T_{ijk}, antisymmetric in its last two suffices, such that

$$T_{ijk} - T_{jik} = S_{ijk}$$

Beginning with the above relationship, we can add and simplify the expression to obtain a result for T_{ijk} in terms of purely our covariant rank 3 tensor S:

$$\begin{aligned}
S_{ijk} &= T_{ijk} - T_{jik} \\
S_{ijk} + S_{kji} &= T_{ijk} - T_{jik} + T_{kji} - T_{jki} \\
&= T_{ijk} + T_{kji} \\
S_{ijk} + S_{kji} - S_{ikj} &= T_{ijk} + T_{kji} - T_{ikj} + T_{kij} \\
&= 2T_{ijk} \\
\implies T_{ijk} &= \frac{1}{2}\left(S_{ijk} + S_{kji} - S_{ikj}\right)
\end{aligned}$$

which is sufficient to satisfy the required relation.

Question 8

Suppose that $\varphi_{ij\ldots k}$ is a tensor of rank P. We then have that

$$\varphi'_{rs\ldots t} = x'^{i}_{r}\, x'^{j}_{s} \cdots x'^{k}_{t}\, \varphi_{ij\ldots k}$$

and,

$$\begin{aligned}
\varphi'_{rs\ldots t} a'^{r} b'^{s} \cdots c'^{t} &= x'^{i}_{r}\, x'^{j}_{s} \cdots x'^{k}_{t}\, \varphi_{ij\ldots k} \times x'^{r}_{i}\, x'^{s}_{j} \cdots x'^{t}_{k} a^{i} b^{j} \cdots c^{k} \\
&= x'^{i}_{r} x'^{r}_{i} \cdot x'^{j}_{s} x'^{s}_{j} \cdots x'^{k}_{t} x'^{t}_{k} \cdot \varphi_{ij\ldots k} a^{i} b^{j} \cdots c^{k} \\
&= \delta^{r}_{i} \delta^{j}_{s} \cdots \delta^{k}_{t} \varphi_{ij\ldots k} a^{i} b^{j} \cdots c^{k} \\
&= \varphi_{rs\ldots t} a^{r} b^{s} \cdots c^{t}
\end{aligned}$$

and hence $\varphi_{ij\ldots k} a^{i} b^{j} \cdots c^{k}$ is invariant under a coordinate transform.
Similarly, if $\varphi_{ij\ldots k}$ is a covariant tensor of rank P,

$$\varphi_{ij\ldots k} = \mu_i \lambda_j \cdots \psi_k$$

via the tensor product, for P rank 1 covariant tensors. If we then introduce P arbitrary contravariant vectors as $a^{i} b^{j} \cdots c^{k}$, we have

$$\begin{aligned}
\varphi_{ij\ldots k} a^{i} b^{j} \cdots c^{k} &= \mu_i a^{i} \cdot \lambda_j b^{j} \cdots \psi_k c^{k} \\
&= s_1 \cdot s_2 \cdots s_P \\
&= S
\end{aligned}$$

where $s_1, \ldots, s_P$ and S are scalars which are, by definition, invariant under a coordinate transformation.
And so $\varphi_{ij\ldots k} a^{i} b^{j} \cdots c^{k}$ being invariant, for P arbitrary contravariant vectors, is a necessary and sufficient condition for $\varphi_{ij\ldots k}$ being a covariant tensor of rank P.

Question 9

i. Since $A_{(ijk)}$ is symmetric with respect to all index permutations, we arrive at the requirement that all permutations have an equal contribution to the value of $A_{(ijk)} = A_{(jki)} = \ldots$. As there are $3! = 6$ permutations,

$$A_{(ijk)} = \frac{1}{6}\left(A_{ijk} + A_{ikj} + A_{jik} + A_{jki} + A_{kij} + A_{kji}\right)$$

ii. As in part i., we can construct an anti-symmetric covariant tensor field, $T_{(ij,k)}(x)$ as

$$T_{(ij,k)}(x) = \frac{1}{6}\left(T_{ij,k}(x) + T_{ji,k}(x) + T_{ik,j}(x) + T_{ki,j}(x) + T_{jk,i}(x) + T_{kj,i}(x)\right) \tag{2}$$

where $T_{ij,k}(x) = \partial T_{ij}/\partial x^k$. Due to the anti-symmetry in the covariant indices of our tensor field $T(x)$, we can then express equation (2) as

$$\begin{aligned} T_{(ij,k)}(x) &= \frac{1}{6}\left(T_{ij,k}(x) - T_{ij,k}(x) + T_{ik,j}(x) - T_{ik,j}(x) + T_{jk,i}(x) - T_{jk,i}(x)\right) \\ &= \frac{1}{6}(0+0+0) = 0 \end{aligned}$$

and hence $T_{(ij,k)}(x) = 0$ in our coordinate system (x^i). If we change coordinate systems via $(x^i) \to (x'^i)$, we obtain

$$\begin{aligned} T'_{(ij,k)}(x') &= \frac{1}{6}\left(T'_{ij,k}(x') + T'_{ji,k}(x') + T'_{ik,j}(x') + T'_{ki,j}(x') + T'_{jk,i}(x') + T'_{kj,i}(x')\right) \\ &= \frac{1}{6}\left(x_i^m x_j^n T_{mn,p}(x) + x_j^n x_i^m T_{nm,p}(x) + x_k^p x_i^m T_{mp,n}(x) + x_i^m x_k^p T_{pm,n}(x) + x_j^n x_k^p T_{np,m}(x) + x_k^p x_j^n T_{pn,m}(x)\right) \end{aligned}$$

However, due to the commutativity of $x_b^a x_d^c = x_d^c x_b^a$ and the anti-symmetry of the covariant tensor field, we have

$$\begin{aligned} T'_{(ij,k)}(x') &= \frac{1}{6}\left(x_i^m x_j^n T_{mn,p}(x) - x_i^m x_j^n T_{mn,p}(x) + x_k^p x_i^m T_{mp,n}(x) - x_k^p x_i^m T_{mp,n}(x) + x_j^n x_k^p T_{np,m}(x) - x_j^n x_k^p T_{np,m}(x)\right) \\ &= \frac{1}{6}(0+0+0) = 0 \end{aligned}$$

which shows that $T'_{(ij,k)}(x') = 0$ in any arbitrary coordinate system (x'^i).

Question 11

Suppose we have some rank (2, 2) tensor, $T^{j\ \ r}_{\ kl}$. We then have the transformation law, under $x^i \to x'^i$,

$$T'^{mq}_{np} = x'^m_j x^k_n x^l_p x'^q_r T^{jr}_{kl}$$

If, on the right hand side, we relabel $l \to j$ (and so also $p \to m$), we obtain

$$\begin{aligned} T'^{mq}_{mn} &= x'^m_j x^k_n x^j_m x'^q_r T^{jr}_{jk} \\ &= \delta^m_j x^k_n x'^q_r T^{jr}_{jk} \\ &= x^k_n x'^q_r T^{jr}_{jk} = x^k_n x'^q_r T^{mr}_{mk} \end{aligned}$$

which is precisely the transformation law for a rank (1, 1) tensor. Since Einstein notation implies summation over like-indices, we could similarly represent this as $T'^q_n = a \times x^k_n x'^q_r T^r_k$, where a is some constant as a result over summation of indices.
As before, if we now relabel $r \to k$ (and so also $q \to n$), we obtain

$$\begin{aligned} T'^{mn}_{mn} &= x^k_n x'^n_k T^{jk}_{jk} \\ &= \delta^k_n T^{jk}_{jk} \\ &= T^{jk}_{jk} = T^{mn}_{mn} \end{aligned}$$

which shows clearly that the result is invariant under a coordinate transformation, and so is a scalar. Intuitively, via Einstein notation, summation is implied over like indices and so both species of indices would disappear over summation and only a scalar would remain afterwards.