PHYS2100 Assignment 2

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Question 1

Set observer to be in reference frame where $\boldsymbol{a}$ is at rest, such that $a_\alpha = (a_0, 0, 0, 0)$ and $b_\alpha = (b_0, 0, 0, 0) \Rightarrow b'_\alpha = (\gamma b_0, 0, 0, 0)$. Thus,

$$\begin{aligned} a = (-\boldsymbol{a} \cdot \boldsymbol{a})^{1/2} &= (-(-a_0)(a_0))^{1/2} \\ &= a_0 \\ b = (-\boldsymbol{b} \cdot \boldsymbol{b})^{1/2} &= (-(-b_0)(b_0))^{1/2} \\ &= b_0 \end{aligned}$$

The general form for scalar products of 4 vectors (in the same frame) is:

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= -a_0 b_0 + \vec{a} \cdot \vec{b} \\ &= -a_0 \gamma b_0 + 0 + 0 + 0 \\ &= -a_0 b_0 \cosh \theta_{ab} \\ &= -ab \cosh \theta_{ab} \end{aligned}$$

where $\gamma = \cosh \theta_{ab}$.

Question 3

We have that

$$\boldsymbol{a} \equiv \frac{d\boldsymbol{u}}{d\tau}$$

Set the observer's frame such that $a_y = a_z = 0$ and $v_y = v_z = \text{const}$. Then,

$$\begin{aligned} a_\alpha &= \left(\frac{du_\alpha}{d\tau} \right) \\ &= \left(\frac{d\gamma}{d\tau}, \frac{d\gamma\vec{v}}{d\tau} \right) \\ &= \left(\frac{d}{d\tau} \frac{1}{\sqrt{1 - \vec{v}^2}}, \frac{d}{d\tau} \left(\frac{1}{\sqrt{1 - \vec{v}^2}} \vec{v} \right) \right) \end{aligned}$$

To integrate the components of $\boldsymbol{a}$, set $u = 1 - \vec{v}^2$,

$$\Rightarrow \frac{du}{d\tau} = \frac{d}{d\tau}(1) - \frac{d}{d\tau} \left(\frac{dx}{d\tau} \right)^2$$

Now set $w = dx/d\tau$,

$$\Rightarrow \frac{du}{d\tau} = -\frac{d}{dw} w^2 \frac{dw}{d\tau}$$

with $dw/d\tau = \vec{a}$, $d/dw\, w^2 = 2w$ and $dw/d\tau = \vec{v}$,

$$\frac{du}{d\tau} = -2\vec{v}\vec{a}$$

Now, with the original substitution of $u = 1 - \vec{v}^2$,

$$\begin{aligned}
\frac{d}{d\tau} \frac{1}{\sqrt{1 - \vec{v}^2}} &= \frac{d}{d\tau} \frac{1}{\sqrt{u}} \frac{du}{d\tau} \\
&= -\frac{1}{2(u)^{3/2}} \frac{du}{d\tau} \\
&= \frac{\vec{v}\vec{a}}{(1 - \vec{v}^2)^{3/2}} \\
&= \gamma^3 \vec{v}\vec{a}
\end{aligned}$$

For the second term in $\boldsymbol{a}$, set $u = \vec{v}$ and $w = 1/\sqrt{1 - \vec{v}^2}$

$$\begin{aligned}
\frac{d}{d\tau} \left(\frac{1}{\sqrt{1 - \vec{v}^2}} \vec{v} \right) &= w \frac{du}{d\tau} + u \frac{dw}{d\tau} \\
&= \gamma \vec{a} + \vec{v} \gamma^3 \vec{v}\vec{a} \\
&= \gamma \vec{a} \left(1 + \gamma^2 \vec{v}^2 \right) \\
&= \gamma \vec{a} \left(\frac{1 - \vec{v}^2 + \vec{v}^2}{1 - \vec{v}^2} \right) \\
&= \gamma^3 \vec{a}
\end{aligned}$$

Thus, $\boldsymbol{a} = (\gamma^3 \vec{v}\vec{a},\, \gamma^3 \vec{a})$ and $\boldsymbol{u} = (\gamma,\, \gamma\vec{v})$,

$$\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{u} &= -a_0 b_0 + \vec{a} \cdot \vec{u} \\
&= -\gamma^3 \vec{a}\vec{v}\gamma + \gamma^3 \vec{a}\vec{v}\gamma \\
&= 0
\end{aligned}$$

And so it has been shown that the scalar product between acceleration and velocity along a worldline is 0.

Question 4

a. We have that

$$
\begin{aligned}
u_\alpha &= (\gamma,\, \gamma\vec{v}) \\
\Rightarrow \gamma^{-1} &= \sqrt{1-v^2} \\
&= \sqrt{1-\frac{g^2t^2}{1+g^2t^2}} \\
&= \sqrt{\frac{1+g^2t^2-g^2t^2}{1+g^2t^2}} \\
&= \sqrt{\frac{1}{1+g^2t^2}} \\
\Rightarrow \gamma &= \sqrt{1+g^2t^2} \\
\Rightarrow \gamma\vec{v} &= \sqrt{1+g^2t^2}\frac{gt}{\sqrt{1+g^2t^2}} = gt
\end{aligned}
$$

And so $u_\alpha = (\sqrt{1+g^2t^2},\, gt,\, 0, 0)$.

Question 5

a. Choose frame such that $v_2 = 0$ (i.e. rest frame for the second ball of putty). Equating the second components of the 4-momenta for conservation of momentum,

$$
\begin{aligned}
p_{1,1} + p_{2,1} &= p_{a,1} \\
m\gamma v + 2m \cdot 0 &= M'\gamma' v' \\
\Rightarrow M' &= \frac{m\gamma v}{\gamma' v'} \qquad (1)
\end{aligned}
$$

Now, equating the first components gives

$$
\begin{aligned}
p_{1,0} + p_{2m,0} &= p_{a,0} \\
\gamma m + 2m &= \gamma' M' \\
\Rightarrow M' &= \frac{m(\gamma+2)}{\gamma'} \qquad (2)
\end{aligned}
$$

Equating (1) and (2) gives,

$$
\begin{aligned}
\frac{m\gamma v}{\gamma' v'} &= \frac{m(\gamma+2)}{\gamma'} \\
\Rightarrow v' &= \frac{\gamma v}{\gamma+2} \\
&= \frac{4}{11}
\end{aligned}
$$

Question 6

We have that

$$\cos \alpha' = \frac{\cos \alpha + V}{1 + V \cos \alpha}$$

and

$$\begin{aligned} \omega' &= \omega \frac{\sqrt{1 - V^2}}{1 - V \cos \alpha'} \\ &= \omega \frac{\sqrt{1 - V^2}}{1 - V \left(\frac{\cos \alpha + V}{1 + V \cos \alpha} \right)} \end{aligned}$$

Substituting in $V = -0.75$ and $\alpha = \pi/2$,

$$\begin{aligned} \omega' &= \omega \frac{\sqrt{1 - 0.75^2}}{1 + 0.75 \left(\frac{\cos \pi/2 - 0.75}{1 - 0.75 \cos \pi/2} \right)} \\ &= \frac{4\sqrt{7}}{7} \omega \\ &\approx 1.51 \omega \end{aligned}$$

Therefore the light frequency is blueshifted for the observer.