MATH3102 Final

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1 Part 1

1.1 Non-Dimensionalisation

For variable x, chance $x \to x'$ by $x = x_0 x'$ where x_0 reduces number of parameters, x' non-dimensional, and also $dx = x_0 dx'$

1.2 Singular Perturbation

If $\varepsilon = 0$ changes the order of the equation, transform variables as

$$x = \varepsilon^\alpha z$$

and choose α such that the highest order term participates in the dominant part of the equation as $\varepsilon \to 0$

1.3 Boundary Layer MAE

Given boundary layer at $x = a$, solve normally to get $y_{\text{out}}(x) = y_0(x)$ for $y = y_0 + \varepsilon y_1$ using B.C. at $x \neq a$.
Let $x = \varepsilon^\alpha z$ so $y(x) \to Y(z)$, so $Y'(z) = y'(x)\varepsilon^\alpha$ and $Y''(z) = y''(x)\varepsilon^{2\alpha}$. Solve for $Y_0(z)$ and using $x = a$ B.C to rearrange for a single unknown constant. Change back from $Y(z) \to y_{BL}(x)$.
Match limits to find constant

$$\lim_{z\to\infty} Y_0 = \lim_{x\to a} y_{out}$$

Then final answer is

$$y = y_{out} + y_{BL} - \lim_{x\to a} y_{out}$$

1.4 Multiple Scales

For $t_1 = t$ and $t_2 = \varepsilon^\alpha t$

$$\begin{aligned} y(t) &\implies Y(t_1, t_2) \\ \frac{d}{dt} &= \frac{\partial}{\partial t_1} + \epsilon^\alpha \frac{\partial}{\partial t_2} \\ \frac{d^2}{dt^2} &= \frac{\partial^2}{\partial t_1^2} + 2\varepsilon^\alpha \frac{\partial^2}{\partial t_1 \partial t_2} + \varepsilon^{2\alpha} \frac{\partial^2}{\partial t_2^2} \end{aligned}$$

Sub into ODE to get new ODE for Y and solve $Y = Y_0 + \varepsilon Y_1$. Choose constants to **avoid resonances** in next term Y_1.

1.5 Balance Law

For ρ density and J flux

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x}$$

For traffic, $J = \rho v$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$

1.6 Greenshield's Constitutive Law

$$v(\rho) = v_m\left(1 - \frac{\rho}{\rho_m}\right)$$

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \qquad c(\rho) = v_m\left(1 - \frac{2\rho}{\rho_m}\right)$$

1.7 Constant Velocity Infinite Domain

For $v(\rho) = a \implies$

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + a\dfrac{\partial \rho}{\partial x} = 0 \\ \rho(x, 0) = f(x) \end{cases}$$

Solution is $\rho(x, t) = f(x - at)$

1.8 Constant Velocity Finite Domain

For $v(\rho) = a \implies$

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + a\dfrac{\partial \rho}{\partial x} = 0 \quad x \in [0, L] \\ \rho(x, 0) = f(x) \\ \rho(0, t) = g(t) \end{cases}$$

Then

$$\rho(x, t) = \begin{cases} f(x - at) & x > at \\ g(t - x/a) & x < at \end{cases}$$

1.9 Non-Constant Velocity

For

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \\ &\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \qquad c(\rho) = v(\rho) + \rho v'(\rho) \end{aligned}$$

For $\rho(x, 0) = f(x)$. Find starting point x_0 of characteristic line passing through (x_1, t_1)

$$x_1 = x_0 + c(f(x_0))t_1$$

Solve for $x_0 = x_0(x, t)$, then we have $\rho(x_1, t_1) = f(x_0)$, and

$$\rho(x, t) = f(x_0(x, t))$$

1.10 Shockwaves

Occurs if $c_L > c_R$

$$s'(t) = \frac{1}{\rho_R - \rho_L} \int_{\rho_L}^{\rho_R} c(\rho)\ d\rho$$

For Greenshield's

$$s'(t) = \frac{1}{2}(c_R + c_L)$$

Where $c_R = c(\rho_R)$ and $c_L = c(\rho_L)$

For

$$\rho(x,0) = \begin{cases} a & x < 0 \\ b & x > 0 \end{cases}$$

To solve, solve for characteristic lines $x_L = x_0 + c_L t$ and $x_R = x_0 + c_R t$. Solve for $s(t)$ using $s'(t)$ above, then

$$\rho(x,t) = \begin{cases} a & x < s(t) \\ b & x > s(t) \end{cases}$$

1.11 Expansion Fans

Occurs if $c_L < c_R$. For

$$\rho(x,0) = \begin{cases} a & x < 0 \\ b & x > 0 \end{cases}$$

Work out characteristic lines, $x_L = x_0 + c_L t$ and $x_R = x_0 + c_R t$. Then, solve

$$c(\rho) = \frac{x - x_0}{t}$$

for ρ. Define this as $\theta = \rho$ for the next line. Then

$$\rho(x,t) = \begin{cases} a & x \le x_L \\ \theta & x_L < x < x_R \\ b & x \ge x_R \end{cases}$$

2 Part 2

2.1 Material Derivative

For a spatial quantity $f(x,t)$ transformed to material coordinates $F(X,t)$ by

$$f(\mathcal{X}(X,t),t) = F(X,t)$$

The material derivative states

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$$

i.e.

$$\frac{\partial F}{\partial t} = \frac{Df}{Dt}$$

2.2 Momentum & Continuity

The Momentum and Continuity Equations in Material Coordinates

$$R(X,t) = \frac{R_0}{1+U_X}$$
$$R_0 U_{tt} = R_0 F + T_X$$

Or in Spatial Coordinates

$$\partial_t \rho + \partial_x(\rho v) = 0$$
$$\rho(\partial_t v + v\partial_x v) = f\rho + \partial_x \tau$$

Usually have $T = EU_X \implies T_X = EU_{XX}$ where E is Young/Elastic Modulus

2.3 Time-Dependent Solutions Infinite Domain

$$\begin{cases} U_{tt} - c^2 U_{XX} = 0 & \text{on } \mathbb{R} \\ U(t=0,X) = f(x) \\ U_t(t=0,X) = g(x) \end{cases}$$

Where $c^2 = E/R_0$. The solution is given by d'Alemberts

$$U(t,X) = \frac{1}{2}(f(X-ct) + f(X+ct)) + \frac{1}{2c}\int_{X-ct}^{X+ct} g(\hat{x})d\hat{x}$$

2.4 Time-Dependent Solutions Finite Domain

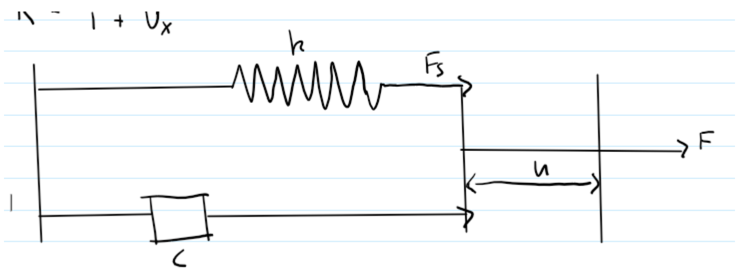
$$\begin{cases} U_{tt} - c^2 U_{XX} = 0 & \text{on } [0,l_0] \\ U(t,X=0) = 0 = U(t,X=l_0) \\ U(t=0,X) = f(x) \\ U_t(t=0,X) = g(x) \end{cases}$$

Where $c^2 = E/R_0$. The solution is given by d'Alemberts

$$U(t,X) = \frac{1}{2}\Big(\hat{f}(X-ct) + \hat{f}(X+ct)\Big) + \frac{1}{2c}\int_{X-ct}^{X+ct} \hat{g}(\hat{x})d\hat{x}$$

where $\hat{f}, \hat{g}$ are $2l_0$ periodic extensions of f, g.

2.5 Kelvin-Voigt Element



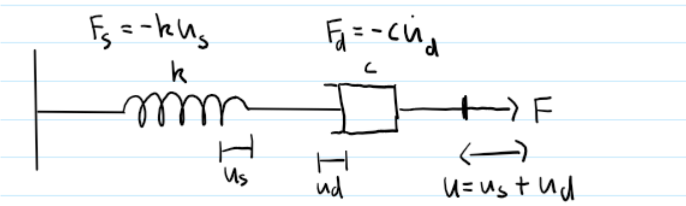
Has

$$\begin{cases} F = -(F_s + F_d) \\ u = u_s = u_d \end{cases}$$

For $F_s = -ku$ and $F_d = -c\dot{u}$

$$F = ku + c\dot{u} \implies T = k\varepsilon + c\dot{\varepsilon} = E(\varepsilon + \tau_1\dot{\varepsilon})$$

Maxwell Element



Has

$$\begin{cases} u = u_s + u_d \\ F = -F_s = -F_d \end{cases}$$

So

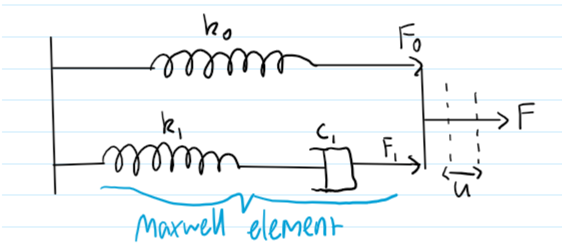
$$\begin{cases} F = -F_s = ku_s \implies u_s = F/k \implies \dot{u}_s = \dot{F}/k \\ F = -F_d = c\dot{u}_d \implies \dot{u}_d = F/c \\ u = u_s + u_d \implies \dot{u} = \dot{u}_s + \dot{u}_d = \dot{F}/k + F/c \end{cases}$$

Then we can get

$$T + \tau_0 \partial_t T = E\tau_1\dot{\varepsilon}$$

where $c = E\tau_1$ and $c/k = \tau_0$

2.6 Standard Linear Model



2.7 Relaxation Laws

Given an equation from one of the above scenarios, e.g.

$$T + \tau_0 \partial_t T = E\tau_1\dot{\varepsilon}$$

Solve for $T(t)$ = integrals and use integration by parts to ensure ε terms are the same (e.g. not ε and $\dot{\varepsilon}$) to get OTF (not exactly, bounds are chosen for example)

$$T = \int_0^t G(t-\tilde{t})\partial_t \varepsilon(\tilde{t})\ d\tilde{t}$$

where $G(t-\tilde{t})$ is the **relaxation function**

2.8 Quasi-Stationary Solutions

For trigonometric B.C's, e.g. the system

$$\begin{cases} R_0 U_{tt} = T_X \\ U(X=0,t) = a\sin(\omega t) \\ \lim\limits_{x\to\infty} U(X,t) = 0 \\ T + \tau_0 T_t = E(\varepsilon + \tau_1 \dot{\varepsilon}) = E(U_X + \tau_1 U_{Xt}) \end{cases}$$

Write the solution in complex coordinates (in this example, $U(X=0,t) = ae^{i\omega t}$) and look for **quasi-stationary solutions**

$$\begin{cases} U(X,t) = \bar{U}(X)e^{i\omega t} \\ T(X,t) = \bar{T}(X)e^{i\omega t} \end{cases}$$

2.9 3D Material Derivative

For $\mathbf{F}$ in material coordinates, $\mathbf{f}$ in spatial, with

$$\mathbf{F}(\mathbf{X},t) = \mathbf{f}(\mathcal{X}(\mathbf{X},t),t)$$

Then

$$\partial_t \mathbf{F} = (\partial_t + \mathbf{v}\cdot\nabla)\mathbf{f} := D_t \mathbf{f}$$

2.10 Deformation Gradient

For $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3)$ and $\mathbf{X} = (X_0, X_1, X_2)$, we have the **deformation gradient**

$$F = \nabla_X \mathcal{X} = \begin{pmatrix} \frac{\partial \mathcal{X}_1}{\partial X_1} & \frac{\partial \mathcal{X}_1}{\partial X_2} & \frac{\partial \mathcal{X}_1}{\partial X_3} \\ \frac{\partial \mathcal{X}_2}{\partial X_1} & \frac{\partial \mathcal{X}_2}{\partial X_2} & \frac{\partial \mathcal{X}_2}{\partial X_3} \\ \frac{\partial \mathcal{X}_3}{\partial X_1} & \frac{\partial \mathcal{X}_3}{\partial X_2} & \frac{\partial \mathcal{X}_3}{\partial X_3} \end{pmatrix}$$

We also need for the **impenetrability of matter assumption** that

$$J = \det(F) = \det(\nabla_X \mathcal{X}) \neq 0$$

2.11 3D Continuity

$$D_t\rho + \rho\nabla\cdot\mathbf{v} = 0$$
$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0$$

From **conservation of linear momentum** we get

$$\rho D_t\mathbf{v} = \nabla\cdot\sigma + \rho\mathbf{f}$$

where σ is the Cauchy Stress Tensor

$$\sigma = \begin{pmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{pmatrix}$$

so

$$\nabla\cdot\sigma = \begin{pmatrix}\partial_x\sigma_{11} + \partial_y\sigma_{12} + \partial_z\sigma_{13}\\ \partial_x\sigma_{21} + \partial_y\sigma_{22} + \partial_z\sigma_{23}\\ \partial_x\sigma_{31} + \partial_y\sigma_{32} + \partial_z\sigma_{33}\end{pmatrix}$$

From **conservation of angular momentum** we get σ is **symmetric**

2.12 Material Frame Indifference

Stress $\mathbf{t} = \sigma(t,\mathbf{x})\mathbf{n}$ and for $\sigma = G(R)$

$$G(R^*) = QG(R)Q^T \quad \forall R, Q$$

where $Q(t)$ is a rotation matrix ($QQ^T = I$ and $\det(Q(t)) = 1$)

2.13 Rivlin-Erickson Theorem

Assume R and σ are symmetric, and $\sigma = G(R)$ is MFI, then we can write

$$\sigma = \alpha_0\mathbf{I} + \alpha_1 R + \alpha_2 R^2$$

where

$$\alpha_i = \alpha_i(I_R, II_R, III_R)$$

and

$$\begin{cases} I_R = \mathrm{Tr}(R) = \lambda_1 + \lambda_2 + \lambda_3\\ II_R = \dfrac{1}{2}(\mathrm{Tr}(R)^2 - \mathrm{Tr}(R^2)) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\\ III_R = \det(R) = \lambda_1\lambda_2\lambda_3\end{cases}$$

2.14 Navier-Stokes

Compressible version

$$\begin{cases}\rho D_t\mathbf{v} = -\nabla\mathbf{p} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \mu\Delta\mathbf{v} + \rho\mathbf{f}\\ D_t\rho = \rho\nabla\cdot\mathbf{v} = 0\\ \mathbf{p} = \rho RT\end{cases}$$

Incompressible

$$\begin{cases} \rho D_t \mathbf{v} = -\nabla \mathbf{p} + \mu \Delta \mathbf{v} + \rho \mathbf{f} \\ \nabla \cdot \mathbf{v} = 0 \end{cases}$$

Notice that we also can substitute

$$D_t \mathbf{v} = \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}$$