MATH3403 Assignment 4

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Question 1

a. The 3D heat equation in spherical coordinates is

$$\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{k}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial u}{\partial\phi}\right) + \frac{k}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$$

b. If there is spherical symmetry, there is no change in u across the surface of some sphere of radius r, and so the differential ϕ and θ terms are zero. The spherical heat equation is then

$$\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$$

c. We now have the system

$$\begin{cases} u_t = \frac{k}{r^2}\partial_r(r^2 u_r) \\ u(r,\,0) = u_0 \qquad r < a,\, t > 0 \\ u(a,\,t) = 0 \qquad t > 0 \end{cases}$$

Suppose the solution is a multiplication of two functions of each variable, $u(r,t) = X(r)T(t)$

$$\begin{aligned} \Rightarrow X(r)T_t(t) &= \frac{k}{r^2}\partial_r(r^2 X_r(r)T(t)) \\ &= \frac{k(2rX_rT + r^2X_{rr}T)}{r^2} \\ &= \frac{2k}{r}X_rT + kX_{rr}T \\ \Rightarrow \frac{T_t}{kT} &= \frac{2}{r}\frac{X_r}{X} + \frac{X_{rr}}{X} \end{aligned}$$

Now, we want to find a λ such that

$$\frac{T_t}{kT} = -\lambda = \frac{2}{r}\frac{X_r}{X} + \frac{X_{rr}}{X}$$

Beginning the right side,

$$\begin{aligned} \frac{X_{rr}}{X} + \frac{2}{r}\frac{X_r}{X} &= -\lambda \\ \Rightarrow X_{rr} + \frac{2}{r}X_r + \lambda X &= 0 \end{aligned}$$

Let $X(r) = v(r)w(r)$, where $w(r) = \exp(\int -1/r \, dr) = 1/r$

$$\begin{aligned} \Rightarrow \left(\frac{v}{r}\right)'' + \frac{2}{r}\left(\frac{v}{r}\right)' + \lambda\frac{v}{r} &= 0 \\ \left(\frac{v_r}{r} - \frac{v}{r^2}\right)' + \frac{2}{r}\left(\frac{v_r}{r} - \frac{v}{r^2}\right) + \lambda\frac{v}{r} &= 0 \\ \frac{v_{rr}}{r} - \frac{v_r}{r^2} - \frac{v_r}{r^2} + \frac{2v}{r^3} + \frac{2v_r}{r^2} - \frac{2v}{r^3} + \lambda\frac{v}{r} &= 0 \\ \frac{v_{rr}}{r} + \lambda\frac{v}{r} &= 0 \\ \Rightarrow v_{rr} + \lambda v &= 0 \end{aligned}$$

Which has general solution

$$v(r) = A\cos\left(\sqrt{|\lambda|}r\right) + B\sin\left(\sqrt{|\lambda|}r\right)$$

and so

$$X(r) = \frac{A\cos\left(\sqrt{|\lambda|}r\right) + B\sin\left(\sqrt{|\lambda|}r\right)}{r}$$

$u(r,t)$ must be bounded at $r = 0$, so $A = 0$ to remove the cos term which is non-zero at $r = 0$. B is non-zero at this boundary condition since $\sin(r) \to 0$ as $r \to 0$. And so

$$X(r) = \frac{B}{r}\sin\left(\sqrt{|\lambda|}r\right)$$

But $X(a) = 0 \Rightarrow 0 = B/a \sin\left(\sqrt{|\lambda|}a\right)$. So either $B = 0$ (trivial), or $\sin\left(\sqrt{|\lambda|}a\right) = 0$

$$\begin{aligned} \Rightarrow \sqrt{|\lambda_n|}a &= n\pi \qquad n \in \mathbb{N} \\ |\lambda_n| &= \frac{n^2\pi^2}{a^2} \\ \lambda_n &= -\frac{n^2\pi^2}{a^2} \end{aligned}$$

With the RHS of the last term negative due to λ being negative (from the solution for $X(r)$ being of complex exponents, or trigonometric functions).

$$\begin{aligned} \Rightarrow X_n(r) &= \frac{B_n}{r}\sin\left(\sqrt{\left|-\frac{n^2\pi^2}{a^2}\right|}r\right) \\ &= \frac{B_n}{r}\sin\left(\frac{n\pi}{a}r\right) \end{aligned}$$

Now, looking at $T(t)$:

$$\begin{aligned} \frac{T_t}{kt} &= -\lambda \\ \Rightarrow T_t &= -k\lambda T \end{aligned}$$

Which has solution

$$T(t) = e^{-kt}$$

But $\lambda_n = -\frac{n^2\pi^2}{a^2}$

$$\begin{aligned}
\Rightarrow T_n(t) &= e^{\left(\frac{n\pi}{a}\right)kt} \\
\Rightarrow u_n(r,\, t) &= T_n(t)X_n(r) \\
&= e^{\left(\frac{n\pi}{a}\right)kt}\frac{B_n}{r}\sin\left(\frac{n\pi}{a}r\right) \\
\Rightarrow u(r,\, t) &= \sum^n \frac{B_n}{r}\sin\left(\frac{n\pi}{a}r\right)e^{\left(\frac{n\pi}{a}\right)kt}
\end{aligned}$$

We now need a B_n such that

$$u(r,0) = u_0 = \sum^n \frac{B_n}{r}\sin\left(\frac{n\pi}{a}r\right)$$

Integrate this against $\int_0^a r\sin(m\pi r/a)dr$,

$$\begin{aligned}
\Rightarrow u_0\int_0^a r\sin\left(\frac{m\pi}{a}r\right)dr &= \sum^n B_n\int_0^a \sin\left(\frac{n\pi}{a}r\right)\sin\left(\frac{m\pi}{a}r\right)dr \\
(-1)^{n+1}\frac{u_0a^2}{n\pi} &= \sum^n \frac{a}{2}B_m\delta_{mn}
\end{aligned}$$

which is only non-zero when $m = n$,

$$\begin{aligned}
\Rightarrow \frac{a}{2}B_n &= (-1)^{n+1}\frac{u_0a^2}{n\pi} \\
B_n &= (-1)^{n+1}\frac{2u_0a}{n\pi}
\end{aligned}$$

And so the solution is

$$u(r,\, t) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2u_0a}{rn\pi}\sin\left(\frac{n\pi}{a}r\right)e^{\left(\frac{n\pi}{a}\right)kt}$$

Question 2

First assume that $u_i(x_i, t)$ are solutions to the heat equation. For a function $u(x_1, x_2, \ldots, x_n)$,

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$$

For $n = 2$, this reduces to

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

Now, suppose $u(x_1, x_2, \ldots, x_n, t) = u_1(x_1, t)\cdot u_2(x_2, t)\ldots u_n(x_n, t)$ with

$$\frac{\partial u_j}{\partial x_i} = 0 \qquad \forall\, i \neq j \text{ and } \forall x_i$$

Then, for $n = 2$,

$$\begin{aligned}
\partial_{xx}u &= \partial_{xx}(u_1(x_1, t)\cdot u_2(x_2, t)) \\
&= \left(u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_1} + u_1\frac{\partial u_2}{\partial x_2} + u_2\frac{\partial u_1}{\partial x_2}\right)_x \\
&= \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_1}{\partial x_2} + u_1\frac{\partial^2 u_2}{\partial x_2^2} \\
&= u_2\frac{\partial^2 u}{\partial x_1^2} + u_1\frac{\partial^2 u_2}{\partial x_2^2}
\end{aligned}$$

and

$$\begin{aligned}\partial_t u &= \partial_t(u_1 u_2)\\ &= u_1\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_1}{\partial t}\end{aligned} \tag{1}$$

Given that u_1 satisfies

$$\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial x_1^2}$$

(and similarly for all other u_n), equation (1) becomes

$$\begin{aligned}\partial_t u &= u_1\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_1}{\partial t}\\ &= u_1\frac{\partial^2 u_2}{\partial x_2^2} + u_2\frac{\partial^2 u}{\partial x_1^2}\\ &= \partial_{xx} u\end{aligned}$$

Which satisfies the 2D heat equation. Since the product rule for n functions is

$$\frac{d}{dx}\prod_{i=1}^{k} f_i(x) = \sum_{i=1}^{k}\left(\left(\frac{d}{dx}f_i(x)\right)\prod_{j=1\neq i}^{k} f_j(x)\right)$$

for $x = (x_1, x_2, \ldots, x_n)$ and $\frac{\partial^2 f_i}{\partial x_j^2} = 0$ for $i \neq j$, this translates to

$$\frac{\partial^2 u}{\partial x^2} = \sum_{i=1}^{n}\left(\frac{\partial^2 u_i}{\partial x_i^2}\prod_{j=1\neq i}^{n} u_j\right)$$

and

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial t}\prod_{j=1\neq i}^{n} u_j\right)$$

Now, assume that $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ is true for all n. Then, for $n+1$,

$$\begin{aligned}\sum_{i=1}^{n+1}\left(\frac{\partial u_i}{\partial t}\prod_{j=1\neq i}^{n+1} u_j\right) &= \sum_{i=1}^{n+1}\left(\frac{\partial^2 u_i}{\partial x_i^2}\prod_{j=1\neq i}^{n+1} u_j\right)\\ \Rightarrow \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial t}\prod_{j=1\neq i}^{n+1} u_j\right) + \frac{\partial u_{n+1}}{\partial t}\prod_{i=1}^{n} u_i &= \sum_{i=1}^{n}\left(\frac{\partial^2 u_i}{\partial x_i^2}\prod_{j=1\neq i}^{n+1} u_j\right) + \frac{\partial^2 u_{n+1}}{\partial x_{n+1}^2}\prod_{i=1}^{n} u_i\end{aligned}$$

But $\frac{\partial u_{n+1}}{\partial t} = \frac{\partial^2 u_{n+1}}{\partial x_{n+1}^2}$ by assumption, and so

$$\begin{aligned}\Rightarrow \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial t}\prod_{j=1\neq i}^{n+1} u_j\right) &= \sum_{i=1}^{n}\left(\frac{\partial^2 u_i}{\partial x_i^2}\prod_{j=1\neq i}^{n+1} u_j\right)\\ \Rightarrow u_{n+1}\sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial t}\prod_{j=1\neq i}^{n} u_j\right) &= u_{n+1}\sum_{i=1}^{n}\left(\frac{\partial^2 u_i}{\partial x_i^2}\prod_{j=1\neq i}^{n} u_j\right)\\ \Rightarrow \sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial t}\prod_{j=1\neq i}^{n} u_j\right) &= \sum_{i=1}^{n}\left(\frac{\partial^2 u_i}{\partial x_i^2}\prod_{j=1\neq i}^{n} u_j\right)\end{aligned}$$

which proves that

$$u(x_1, x_2, \dots, x_n, t) = u_1(x_1, t) \cdot u_2(x_2, t) \dots u_n(x_n, t)$$

is a solution to the heat equation in $\mathbb{R}^n$. QED

Question 3

The system is

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x, 0) = \delta(x+1) - 2\delta(x) + \delta(x-1) \end{cases}$$

where δ is the dirac delta. Using the fundamental solution to the heat equation, this is

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} \Phi(x-y,t)\left(\delta(y+1) - 2\delta(y) + \delta(y-1)\right) dy \\ &= \int_{\mathbb{R}} \Phi(x-y,t)\delta(y+1)dy - 2\int_{\mathbb{R}} \Phi(x-y,t)\delta(y)dy + \int_{\mathbb{R}} \Phi(x-y,t)\delta(y-1)dy \end{aligned}$$

Since

$$\int_{\mathbb{R}} \Phi(x-y,t)\delta(y+y_0)dy = \Phi(x+y_0,t)$$

The solution becomes

$$\begin{aligned} u(x,t) &= \Phi(x+1,t) - 2\Phi(x,t) + \Phi(x-1,t) \\ &= \frac{1}{\sqrt{4\pi t}}\left(e^{\frac{-(x+1)^2}{4t}} - 2e^{\frac{-x^2}{4t}} + e^{\frac{-(x-1)^2}{4t}}\right) \end{aligned}$$

Which is the solution to the 1D heat equation with initial condition $u(x,0) = \delta(x+1) - 2\delta(x) + \delta(x-1)$.

Question 4

$$\begin{cases} u_t - \Delta u = 0 \\ u(x_1, x_2, 0) = f(x_1, x_2) \\ \Delta u(x_1, 0, t) = 0 \end{cases}$$

First, reflect initial condition along the boundary $x_2 = 0$ so that, for a solution to the IVBP $v(x_1, x_2, t)$ (with f_0 initial condition),

$$f_0 = \begin{cases} f(x_1, x_2) & x_2 \geq 0 \\ -f(x_1, x_2) & x_2 < 0 \end{cases}$$

Since the initial condition of v is an odd function, its first derivative is even. The second derivative of v is consequently odd. As such, the laplacian is the sum of two odd functions and it itself odd. Along the plane $x_2 = 0$, an odd function is zero (to satisfy reflective properties) and so $\Delta v(x_1, 0, t) = 0$. The system is now

$$\begin{cases} v_t - \Delta v = 0 \\ v(x_1, x_2, 0) = f_0(x_1, x_2) \\ \Delta v(x_1, 0, t) = 0 \end{cases}$$

with $u(x_1, x_2, t) = v(x_1, x_2, t)\big|_{x_2 \geq 0}$
Since an n dimensional solution is equivalent to the product of n solutions,

$$\begin{aligned}
v(x_1, x_2, t) &= v_1(x_1, t) \cdot v_2(x_2, t) \\
&= \int_{\mathbb{R}^2} \Phi(x_1 - y_1, t)\Phi(x_2 - y_2, t) f_0(y_1, y_2)\ dy_2\ dy_1 \\
&= \int_{\mathbb{R}} \left(\int_0^{\infty} \Phi(x_1 - y_1, t)\Phi(x_2 - y_2, t) f_0(y_1, y_2) - \int_{-\infty}^{0} \Phi(x_1 - y_1, t)\Phi(x_2 + y_2, t) f_0(y_1, -y_2) \right)\ dy_1
\end{aligned}$$

But v is odd and therefore reflected, so $\int_{-\infty}^{0} \dots = -\int_0^{\infty} \dots$ and,

$$v(x_1, x_2, t) = \int_{\mathbb{R}} \int_0^{\infty} \Phi(x_1 - y_1, t) \left(\Phi(x_2 - y_2, t) f_0(y_1, y_2) - \Phi(x_2 + y_2) f_0(y_1, y_2) \right)\ dy_2\ dy_1$$

But $u(x_1, x_2, t) = v(x_1, x_2, t)$ and $f = f_0$ for $x_2 \geq 0$, so the solution is therefore

$$u(x_1, x_2, t) = \frac{1}{4\pi t} \int_{\mathbb{R}} \int_0^{\infty} e^{\frac{-(x_1 - y_1)^2}{4t}} f(y_1, y_2) \left(e^{\frac{-(x_2 - y_2)^2}{4t}} - e^{\frac{-(x_2 + y_2)^2}{4t}} \right)\ dy_2\ dy_1$$