Assignment 1

Wednesday, 17 August 2022 11:06 AM

a. $X_1, \dots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Where the pdf of any single random variable is

$$f_x(x;\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where $\theta = (\mu, \sigma^2)^T$

Therefore the joint pdf, $f_{x_1 \dots x_n}(x_1 \dots x_n; \theta)$ is given by

$$f_{x_1 \dots x_n}(x_1 \dots x_n; \theta) = \prod_{i=1}^{n} f_{x_i}(x_i;\theta)$$

$$= \prod_{i=1}^{n} (2\pi)^{-\frac{1}{2}} \sigma^{-1} \exp\left(-\frac{1}{2} \cdot \frac{(x_i-\mu)^2}{\sigma^2}\right)$$

$$= (2\pi)^{-\frac{n}{2}} \sigma^{-n} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\sigma^2}\right)$$

$$= (2\pi)^{-\frac{n}{2}} (\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\sigma^2}\right)$$

$$= \mathcal{L}(x_1 \dots x_n; \theta)$$

$$\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{n} [x_i^2 - 2x_i\mu + \mu^2]$$

$$= \frac{1}{\sigma^2}\left[\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \mu^2 - \sum_{i=1}^{n} 2x_i\mu\right]$$

$$= \frac{1}{\sigma^2} \sum_{i=1}^{n} x_i^2 - \frac{2\mu}{\sigma^2} \sum_{i=1}^{n} x_i + \frac{n\mu^2}{\sigma^2}$$

$$\Rightarrow \mathcal{L}(\theta) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\left(\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2 - \frac{2\mu}{\sigma^2}\sum_{i=1}^{n} x_i + \frac{n\mu^2}{\sigma^2}\right)\right)$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2 + \frac{\mu}{\sigma^2}\sum_{i=1}^{n} x_i\right) \exp\left(-\frac{n\mu^2}{\sigma^2}\right)$$

Set $a(\theta) = (2\pi\sigma^2)^{\frac{n}{2}} \exp\left(\frac{n\mu^2}{\sigma^2}\right)$

$$\underset{\sim}{c}(\theta) = \begin{pmatrix} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{pmatrix}$$

$$\underset{\sim}{T}(x_1 \dots x_n) = \begin{pmatrix} \sum^n x_i \\ \sum^n x_i^2 \end{pmatrix}$$

$$\Rightarrow \mathcal{L}(\theta) = f(x_1 \dots x_n; \theta) = \frac{\exp\left(\underset{\sim}{c}(\theta)^T \underset{\sim}{T}(x_1 \dots x_n)\right)}{a(\theta)}$$

which is of the form of the two parameter regular exponential family.

ii. We have that $\underset{\sim}{T}(x_1 \dots x_n) = \begin{pmatrix} \sum^n x_i \\ \sum^n x_i^2 \end{pmatrix}$

As per the lectures, the ML estimate of $\underset{\sim}{\theta}$, $\hat{\underset{\sim}{\theta}}$, is the value of $\underset{\sim}{\theta}$ that satisfies

$$E(T(X_1, \dots, X_n))_{\underset{\sim}{\theta} = \hat{\underset{\sim}{\theta}}} = \underset{\sim}{T}(x_1 \dots x_n)$$

Firstly, $E(\underset{\sim}{T}_1(x_1 \dots x_n)) = E\left(\sum^n x_i\right)$

Firstly, $\mathbb{E}(\underset{\sim}{T}_1(x_1, \ldots x_n)) = \mathbb{E}(\sum^{n} x_i)$
$= n\bar{x}$
$= n\hat{\mu}$

$\Rightarrow \quad \hat{\mu} = \sum_{i=1}^{n} \frac{x_i}{n} = \underset{\sim}{\theta}_1$

Next, $\mathbb{E}(\underset{\sim}{T}_2(x_1, \ldots x_n)) = \mathbb{E}(\sum^{n} x_i^2)$.

We have that $\sigma^2 = var(\underset{\sim}{X}) = \mathbb{E}(\underset{\sim}{X}^2) - \mathbb{E}(\underset{\sim}{X})^2$

$$\Rightarrow \quad \hat{\sigma}^2 = \mathbb{E}(\underset{\sim}{X}^2) - \mathbb{E}(\underset{\sim}{X})^2$$
$$= \mathbb{E}(\sum^{n} x_i^2) - \mathbb{E}(\sum^{n} x_i)^2$$
$$= n\left(\sum^{n} \frac{x_i^2}{n} - \mu^2\right)$$

Hence $\quad \underset{\sim}{\theta} = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} = \begin{pmatrix} \sum^{n} \frac{x_i}{n} \\ n\left(\sum^{n} \frac{x_i^2}{n} - \mu^2\right) \end{pmatrix}$

iii.
$$F(x;\underset{\sim}{\theta}) = \frac{1}{2}\left(1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)$$

We want to find x such that $F(x;\theta) = 0.75$

$$0.75 = \frac{1}{2}\left(1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)$$
$$\frac{1}{2} = erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)$$
$$\Rightarrow x - \mu = \sqrt{2}\,\sigma\, erf^{-1}(1/2)$$

define $g(\underset{\sim}{\theta}) = x$ as the third quantile of the normal distribution. Then,

$$g(\underset{\sim}{\theta}) = \sqrt{2}\,\sigma\, erf^{-1}(1/2) + \mu$$
$$\simeq \mu + 0.675\sigma \quad = \mu + 0.675\sqrt{\sigma^2}$$

iv. The maximum likelihood estimate of $g(\underset{\sim}{\theta})$ is given by

$$\widehat{g(\underset{\sim}{\theta})} = g(\hat{\underset{\sim}{\theta}}) \simeq \hat{\mu} + 0.675\sqrt{\hat{\sigma}^2}$$

v. The bias of $g(\hat{\underset{\sim}{\theta}})$ is given as

$$bias(g(\hat{\underset{\sim}{\theta}})) = \mathbb{E}(g(\hat{\underset{\sim}{\theta}})) - g(\underset{\sim}{\theta})$$
$$= \mathbb{E}(\hat{\mu} + 0.675\hat{\sigma}) - g(\underset{\sim}{\theta})$$
$$= \mathbb{E}(\hat{\mu}) + 0.675\,\mathbb{E}(\hat{\sigma}) - g(\underset{\sim}{\theta})$$

We have that $\mathbb{E}(\hat{\mu}) = \mu$, so,

$$bias(g(\hat{\underset{\sim}{\theta}})) = 0.675\,\mathbb{E}(\hat{\sigma}) + \mu - g(\underset{\sim}{\theta})$$

but $g(\underset{\sim}{\theta}) = \mu + 0.675\sigma$

$$\Rightarrow bias(g(\hat{\underset{\sim}{\theta}})) = 0.675\,\mathbb{E}(\hat{\sigma}) - 0.675\sigma$$

Now, define a random variable $Y \sim \chi^2_n$, so

$$\frac{\hat{\sigma}^2 n}{\sigma^2} = y \sim Y$$

And take

$$h(y) = \sqrt{y\frac{\sigma^2}{n}} \quad \left(= \sqrt{\frac{\hat{\sigma}^2 n}{\sigma^2}\frac{\sigma^2}{n}} = \hat{\sigma}\right)$$

So,

$$\mathbb{E}(\hat{\sigma}) = \mathbb{E}(h(y)) = \int h(y) f(y)\, dy$$

but $f(y) = \frac{1}{2^{n/2}\Gamma(n/2)} y^{n/2-1} e^{-y/2}$

$$\Rightarrow h(y) f(y) = \frac{1}{2^{n/2}\Gamma(n/2)} \cdot \frac{\sigma}{\sqrt{n}} \cdot y^{\frac{(n-1)}{2}} e^{-y/2}$$

$$\Rightarrow \mathbb{E}(\hat{\sigma}) = \frac{1}{2^{n/2}\Gamma(n/2)} \cdot \frac{\sigma}{\sqrt{n}} \int_0^\infty y^{\frac{(n-1)}{2}} e^{-y/2}\, dy$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \cdot \frac{\sigma}{\sqrt{n}} \cdot 2^{\frac{(n+1)}{2}} \cdot \Gamma\left(\frac{n+1}{2}\right)$$

$$= \sqrt{\frac{2}{n}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)}\sigma$$

So finally,

$$bias(g(\hat{\underset{\sim}{\theta}})) = 0.675\sigma\left(\sqrt{\frac{2}{n}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)} - 1\right)$$

And so a bias corrected estimator of $g(\underset{\sim}{\theta})$ is given by

$$\tilde{g}(\hat{\underset{\sim}{\theta}}) = g(\hat{\underset{\sim}{\theta}}) - bias(g(\hat{\underset{\sim}{\theta}}))$$

$$= \hat{\mu} + 0.675\sqrt{\hat{\sigma}^2} - 0.675\sigma\left(\sqrt{\frac{2}{n}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - 1\right)$$

vi. We know that the standard deviation is given by

$$s.d.(\underset{\sim}{X}) = \sqrt{var(\underset{\sim}{X})}$$

So,

$$var(\tilde{g}) = var\left(\hat{\mu} + 0.675\sqrt{\hat{\sigma}^2} - 0.675\sigma\left(\sqrt{\frac{2}{n}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)} - 1\right)\right)$$

$$= var(\hat{\mu}) + 0.675^2 var(\hat{\sigma}) + \left(0.675\left[\sqrt{\frac{2}{n}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n/2)} - 1\right]\right)^2 var(\sigma)$$

where any covariances that would be present are just zero since $\hat{\mu}$, $\hat{\sigma}$, and σ are distributed independently.

$$\begin{aligned}
var(\hat{g}) &= \mathbb{E}(\hat{\mu}^2) - \mathbb{E}(\hat{\mu})^2 + 0.675^2(\mathbb{E}(\hat{\sigma}^2) - \mathbb{E}(\hat{\sigma})^2) \\
&\quad + \left(0.675\left[\sqrt{\tfrac{2}{n}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(n/2)} - 1\right]\right)^2 var(\sigma) \\
&= \mu^2 - \mu^2 + 0.675^2(\sigma^2 - \sigma^2) \\
&\quad + \left(0.675\left[\sqrt{\tfrac{2}{n}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(n/2)} - 1\right]\right)^2 var(\sigma) \\
&= \left(0.675\left[\sqrt{\tfrac{2}{n}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(n/2)} - 1\right]\right)^2 var(\sigma)
\end{aligned}$$

And so

$$s.d.(\hat{g}) = 0.675\left(\sqrt{\tfrac{2}{n}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(n/2)} - 1\right)\sqrt{var(\sigma)}$$

The standard error is given by

$$\begin{aligned}
s.e(\hat{g}) &= \widehat{s.d(\hat{g})} \\
&= s.d.(\hat{\hat{g}}) \\
&= 0.675\left(\sqrt{\tfrac{2}{n}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(n/2)} - 1\right)\sqrt{var(\hat{\sigma})}
\end{aligned}$$