MATH3403 Assignment 2

Ryan White s4499039

26th of August 2021

Question 1

We have the PDE
$$3u_{xx} - 7u_{xy} + 2u_{yy} + 5u_x - 10u_y = 0$$

$$\begin{aligned}
\Rightarrow \Delta &= B^2 - 4AC \\
&= 49 - 4 \cdot 3 \cdot 2 = 49 - 24 \\
&= 25 > 0
\end{aligned}$$

Therefore the PDE is hyperbolic, and the $w_{\xi\xi}$ and $w_{\eta\eta}$ terms in the canonical form are both 0. Set $u(x, y) = w(\xi(x, y), \eta(x, y))$, with

$$\begin{aligned}
y'_{1,2} &= \frac{B \pm \sqrt{\Delta}}{2A} \\
\Rightarrow y_1(x) &= \frac{-7+5}{6}x + \xi = -\frac{1}{3}x + \xi \\
\Rightarrow \xi &= y + \frac{1}{3}x \\
\Rightarrow y_2(x) &= \frac{-7-5}{6}x + \eta = -2x + \eta \\
\Rightarrow \eta &= y + 2x
\end{aligned}$$

With these definitions of ξ and η,

$$\begin{aligned}
\xi_x &= \frac{1}{3} \qquad & \xi_y &= 1 \qquad & \xi_{xx} &= \xi_{xy} = \xi_{yy} = 0 \\
\eta_x &= 2 \qquad & \eta_y &= 1 \qquad & \eta_{xx} &= \eta_{xy} = \eta_{yy} = 0
\end{aligned}$$

The canonical form of the PDE is then

$$\begin{aligned}
0 &= w_{\xi\eta}(2A\xi_x\eta_x + 2C\eta_y\xi_y + B(\eta_y\xi_x + \xi_y\eta_x)) + 0w_{\xi\xi} + 0w_{\eta\eta} \\
&\quad + w_\xi(0 + 0 + 0 + D\xi_x + E\xi_y) + w_\eta(0 + 0 + 0 + D\eta_x + E\eta_y) \\
&= w_{\xi\eta}(2 \cdot 3 \cdot 1/3 \cdot 2 + 2 \cdot 2 \cdot 1 \cdot 1 - 7(1 \cdot 1/3 + 1 \cdot 2) \\
&\quad + w_\xi(5 \cdot 1/3 - 10 \cdot 1) + w_\eta(5 \cdot 2 - 10 \cdot 1) \\
&= -25/3(w_{\xi\eta} + w_\xi) \\
0 &= w_{\xi\eta} + w_\xi \\
&= (w_\eta + w)_\xi
\end{aligned}$$

Integrating both sides by ξ then gives
$$A(\eta) = w_\eta + w$$

Now multiply both sides by e^η,

$$\begin{aligned} e^\eta A(\eta) &= e^\eta w_\eta + e^\eta w \\ A(\eta) &= (e^\eta w)_\eta \end{aligned}$$

(The last line due to the product rule of differentiation). Integrating both sides with respect to η gives

$$\begin{aligned} e^\eta w &= F(\xi) + G(\eta) \qquad \text{where } G(\eta) = \int A(\eta) d\eta \\ \Rightarrow w &= G(\eta) + e^{-\eta} F(\xi) \end{aligned}$$

(In the last line, $e^{-\eta}$ is absorbed into $G(\eta)$). Substituting in the definition of ξ and η, and given that $u(x, y) = w(\xi(x, y), \eta(x, y))$, we finally have

$$u(x, y) = G(y + 2x) + e^{-(y+2x)} F\left(y + \frac{1}{3}x\right)$$

where F and G are arbitrary functions of (x, y).

Question 2

We have the PDE

$$\begin{aligned} u_{xx} - 4u_{xy} + 4u_{yy} &= y - 2x \\ \Rightarrow \Delta = B^2 - 4AC &= 16 - 16 = 0 \end{aligned}$$

Therefore the PDE is parabolic, and the canonical form's $w_{\xi\xi}$ and $w_{\xi\eta}$ coefficients are 0. The PDE has one characteristic variable:

$$\begin{aligned} \frac{dy}{dx} &= \frac{-B}{2A} = \frac{4}{2} = 2 \\ \Rightarrow y(x) &= 2x + \xi \\ \Rightarrow \xi &= y - 2x \end{aligned}$$

Choose another characteristic variable, $\eta = x$. Then,

$$\begin{aligned} \xi_x &= -2 \qquad \xi_y = 1 \qquad \xi_{xx} = \xi_{xy} = \xi_{yy} = 0 \\ \eta_x &= 1 \qquad \eta_y = 0 \qquad \eta_{xx} = \eta_{xy} = \eta_{yy} = 0 \end{aligned}$$

Check the determinant of the Jacobian, $\det \mathcal{J} = -2 \cdot 0 - 1 \cdot 1 = -1 \neq 0$. Now, the inverse transform is $\eta = x$ and $y = 2x + \xi = 2\eta + \xi$. Since there are no lower order terms in the original PDE, and all of the double derivatives of the characteristic variables are 0, the canonical form consists only of higher derivatives of w. Since the PDE is parabolic, only the $w_{\eta\eta}$ term remains:

$$\begin{aligned} w_{\eta\eta}(A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2) &= y - 2x \\ \Rightarrow w_{\eta\eta}(1^2 - 4 \cdot 1 \cdot 0 + 4 \cdot 0^2) &= 2\eta + \xi - 2\eta \\ w_{\eta\eta} &= \xi \end{aligned}$$

Integrating both sides with respect to η twice gives

$$\begin{aligned} w_\eta &= \xi\eta + f(\xi) \\ w &= \frac{1}{2}\xi\eta^2 + \eta F(\xi) + G(\xi) \end{aligned}$$

Since $u(x,y) = w(\xi(x,y), \eta(x,y))$, $\eta = x$ and $\xi = y - 2x$,

$$\begin{aligned} u(x,y) &= \frac{1}{2}(y-2x)x^2 + xF(y-2x) + G(y-2x) \\ \Rightarrow u(x,y) &= \frac{x^2 y}{2} - x^3 + xF(y-2x) + G(y-2x) \end{aligned}$$

Question 3

We have the inhomogeneous wave equation with initial conditions,

$$\begin{cases} \partial_{tt} u - c^2 \partial_{xx} u = c^2(x+ct) \\ u(t=0,x) = 0 \qquad\qquad = f(x) \\ \partial_t u(t=0,x) = 0 \qquad\quad = g(x) \end{cases}$$

$$\Delta = B^2 - 4AC = 0^2 - 4 \cdot -c^2 = 4c^2 > 0$$

Therefore the PDE is hyperbolic, with canonical terms $w_{\xi\xi}$ and $w_{\eta\eta}$ both being 0. The characteristic variables are then

$$\begin{aligned} x'_{1,2}(t) &= \frac{-B \pm \sqrt{\Delta}}{2A} = \frac{\pm 2c}{2} = \pm c \\ \frac{dx_1}{dt} &= c \Rightarrow x = ct + \xi \Rightarrow \xi = x - ct \\ \frac{dx_2}{dt} &= -c \Rightarrow x = -ct + \eta \Rightarrow \eta = x + ct \end{aligned}$$

The partial derivatives of these characteristic variables are then

$$\begin{aligned} \xi_t &= -c \qquad \xi_x = 1 \qquad \xi_{tt} = \xi_{tx} = \xi xx = 0 \\ \eta_t &= c \qquad \eta_x = 1 \qquad \eta_{tt} = \eta_{tx} = \eta_{xx} = 0 \end{aligned}$$

Since all of the second derivatives are 0 and the original PDE has no lower order terms, the canonical form consists only of the $w_{\xi\eta}$ coefficient:

$$\begin{aligned} w_{\xi\eta}(2A\xi_t\eta_t + 2C\xi_x\eta_x) &= c^2(x+ct) \\ w_{\xi\eta}(2 \cdot 1 \cdot -c \cdot c + -2c^2 \cdot 1 \cdot 1) &= c^2\eta \\ -4c^2 w_{\xi\eta} &= c^2\eta \\ \Rightarrow w_{\xi\eta} &= -\frac{1}{4}\eta \end{aligned}$$

Integrating first with respect to η and then secondly to ξ,

$$\begin{aligned} w_\xi &= -\frac{1}{8}\eta^2 + A(\xi) \\ w &= -\frac{1}{8}\xi\eta^2 + F(\xi) + G(\eta) \end{aligned}$$

where $F(\xi) = \int A(\xi) d\xi$. Given that $u(x,y) = w(\xi(x,y), \eta(x,y))$, $\xi = x - ct$ and $\eta = x + ct$,

$$u(x,y) = -\frac{1}{8}(x-ct)(x+ct)^2 + F(x-ct) + G(x+ct)$$

Now, using the initial conditions,

$$\begin{aligned} f(x) = u(0,x) &= 0 - -\frac{1}{8}(x)(x)^2 + F(x) + G(x) \\ &= -\frac{1}{8}x^3 + F(x) + G(x) \end{aligned}$$

$$\begin{aligned} g(x) = u_t(0,x) = u_t(t,x)\Big|^0 &= d/dt\left(\frac{1}{8}(x^3 + ctx^2 - c^2t^2x - c^3t^3) + F(x-ct) + G(x+ct)\right)\Bigg|^0 \\ &= \frac{1}{8}(cx^2 - 2c^2tx - 3c^3t^2) - cF'(x-ct) + cG'(x+ct)\Big|^0 \\ 0 &= \frac{c}{8}x^2 - cF'(x) + cG'(x) \\ &= \frac{1}{8}x^2 - F'(x) + G'(x) \end{aligned}$$

Integrating gives

$$\frac{1}{24}x^3 - F(x) + G(x) = \int_{x_0}^{x} g(\tilde{x}) + A$$

but $g(x) = 0$, so

$$0 = \frac{1}{24}x^3 - F(x) + G(x) + A$$

where A is some constant of integration. From manipulating the two sets of initial conditions, we now have the system

$$\begin{cases} F(x) + G(x) = \frac{1}{8}x^3 \\ -F(x) + G(x) = -\frac{1}{24}x^3 - A \end{cases}$$

$$\begin{aligned} \Rightarrow 2G(x) &= \frac{1}{8}x^3 - \frac{1}{24}x^3 - A = \frac{1}{12}x^3 - A \\ \hookrightarrow G(x) &= \frac{1}{24}x^3 - \frac{1}{2}A \\ \Rightarrow 2F(x) &= \frac{1}{8}x^3 + \frac{1}{24}x^3 + A = \frac{1}{6}x^3 + A \\ \hookrightarrow F(x) &= \frac{1}{12}x^3 + \frac{1}{2}A \end{aligned}$$

Substituting these into $u(x,y)$ gives

$$\begin{aligned} u(x,y) &= -\frac{1}{8}(x-ct)(x+ct)^2 + \frac{1}{12}(x-ct)^3 + \frac{1}{2}A + \frac{1}{24}(x+ct)^3 - \frac{1}{2}A \\ &= -\frac{1}{8}(x-ct)(x+ct)^2 + \frac{2(x-ct)^3 + (x+ct)^3}{24} \\ \Rightarrow u(x,y) &= \frac{2(x-ct)^3 + (x+ct)^3 - 3(x-ct)(x+ct)^2}{24} \end{aligned}$$