MATH2400 Assignment 1

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17th of March 2021

Question 1

Using the axioms of the real numbers, and indicating which axioms you used in each step of the argument, prove the following statements (you may also use auxiliary results seen in class):

a. Let $x \in \mathbb{R}$. Prove that $x > 0$ implies $-x < 0$, and viceversa, if $x < 0$ then $-x > 0$.

Let $x \in \mathbb{R}$ and $x > 0$. We have that $x + (-x) = 0$ (A5)
Let $a = x$, $b = 0$ and $c = -x$. By the definition of ordered fields, if $a > b$, then $a + c > b + c$. (*)
Substituting in the chosen values gives

$$\begin{aligned} a + c &> b + c \\ x + (-x) &> 0 + (-x) \\ 0 &> -x \end{aligned}$$

And so it has been shown that if $x > 0$, then $-x < 0$. On the contrary, let $x < 0$ and $a = 0$, $b = x$, and $c = -x$ in (*). This satisfies $a > b$.
Substituting the chosen values gives

$$\begin{aligned} a + c &> b + c \\ 0 + (-x) &> x + (-x) \\ -x &> 0 \end{aligned}$$

And so if $x < 0$, then $-x > 0$. QED

b. Let $x \in \mathbb{R}$. Then, $x^2 \geq 0$ (that is, $x^2 > 0$ or $x^2 = 0$).

There are three possible cases: either $x > 0$, $x = 0$, or $x < 0$.
For $x > 0$, $x \cdot x > x \cdot 0 \Rightarrow x^2 > 0$ (via result proven in L02).

For $x = 0$, $x \cdot x = 0 \cdot 0 \Rightarrow x^2 = 0$ (via same proof in L02).

For $x < 0$, claim that if $x < 0$ and $y < z \, \forall y, z \in \mathbb{R}$, then $xy > xz$ (1)
Let $y = x$ and $z = 0$. Since $y = x < 0$, this satisfies $y < z$. Thus, $x \cdot x > x \cdot 0 \Rightarrow x^2 > 0$, and it has been shown that $x^2 \geq 0$ for all possible values of $x \in \mathbb{R}$. QED
Proof of (1): If $x < 0$, $-x > 0$ via proof in a.
By the definition of ordering of fields, $0 = y + (-y) < z - y$ (A5)
$\Rightarrow -x \cdot (z - y) > 0$ via result in L02. Then, $xy - xz > 0 \Rightarrow xy > xz$. (D) QED

Question 2

Determine if the following sets are bounded above and/or below, and whenever applicable find their infimum and/or supremum:

a.

$$S = \left\{ \frac{n-1}{n+1} \mid n \in \{1, 2, 3, 4, \ldots\} \right\};$$

Consider the first few terms of the set S:

$$S_1 = 0; \qquad S_2 = \frac{1}{3}; \qquad S_3 = \frac{1}{2}; \qquad S_4 = \frac{3}{5}; \qquad \ldots$$

Since the set's values are strictly monotone increasing, the first value represents the greatest lower bound. Thus, $\inf S = 0$.
The definition of the sequence S has the numerator smaller than the denominator for all n ($n - 1 < n + 1\, \forall n \in \mathbb{N}$). As such, the value for S_n will never reach 1 for a finite n ($\lim_{n\to\infty} S_n = 1$). As the set's values are strictly monotone increasing up to (but not including) 1 for any finite number of values, the least upper bound for S is 1 ($\sup S = 1$). In summary, S is a bounded set, with an infimum of 0 and a supremum of 1.

b.

$$T = \bigcup_{n=1}^{\infty} [n^2, n^2 + 1].$$

Consider the first few terms of the set T:

$$T_1 = 1; \qquad T_2 = 2; \qquad T_3 = 4; \qquad T_4 = 5; \qquad T_5 = 9; \qquad \ldots$$

As in part a, the set is strictly monotone increasing with a $T_1 = 1$. Thus, the set is bounded below with $\inf T = 1$.
Since the set it strictly monotone increasing with no subtractions or divisors in the definition of the value T_{n_k} for some $n \in \mathbb{N}$ and $k \in [1, 2]$, the set is unbounded above. Suppose that there was in fact some upper bound $b \in \mathbb{R}$, such that for all $T_{n_k} \in T$, $T_{n_2} \leq b$ (define $T_{n_1} = n^2$; $T_{n_2} = n^2 + 1 \Rightarrow T_{n_2} > T_{n_1}\, \forall n \in \mathbb{N}$). Expanded, that is $n^2 + 1 \leq b \Rightarrow n^2 \leq b - 1$ and so $b - 1$ is an upper bound for $T_{n_1} \in T$. But, $T_{n_2} > T_{n_1}$ and so $b \leq b - 1$ and a contradiction is found, cementing that there is no upper bound for T.

Question 3

Using the $\epsilon - m$ definition of a limit, prove that

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

It can be seen that, due to the oscillatory nature of sine between 1 and -1,

$$0 \leq \left| \frac{\sin n}{n} \right| \leq \frac{1}{n}$$

Let $\epsilon > 0$. By the Archimedean Property, $\exists N \in \mathbb{N}$ such that $N \cdot \epsilon > 1$.
Rearranging gives $\epsilon > 1/N \geq 1/n\ \forall n \geq N$. So,

$$\left| \frac{\sin n}{n} \right| = \left| \frac{\sin n}{n} - 0 \right| \leq \frac{1}{n} < \epsilon$$

Which is of the form of the definition of a limit, with $\lim_{n\to\infty} \frac{\sin n}{n} = 0$. QED

Question 4

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences such that $\{y_n\}_{n=1}^{\infty}$ converges to zero. Suppose that for all positive integers k and l with $l \geq k$, we have

$$|x_l - x_k| \leq y_k.$$

Prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Since $\{y_n\}$ is convergent to 0, there exists an $N \in \mathbb{N}$ such that $|y_k - 0| < \epsilon$ for any $\epsilon > 0$, or rather, $|y_k| < \epsilon \ \forall k \geq N$. By the definition of absolute values, $y_k \leq |y_k| < \epsilon$. Thus,

$$|x_l - x_k| \leq y_k < \epsilon \qquad \forall l \geq k > N$$

The above is of the form of a Cauchy sequence, and so $\{x_n\}$ is Cauchy. QED

Question 5

Let a be a positive real number. Define the sequence $\{x_n\}_{n=1}^{\infty}$ by $x_1 = a$ and for $n > 1$

$$x_{n+1} = x_n \left(x_n + \frac{1}{n} \right).$$

a. Suppose that a is such that the sequence $\{x_n\}_{n=1}^{\infty}$ is monotone increasing and bounded. Find the value of $\lim_{n\to\infty} x_n$. (You do not have to show that such an a exists (it does).)

Since the sequence is bounded and monotone increasing, it will converge to some point equal to or less than 1. Consider the expansion of the sequence definition above by $x_{n+1} = {x_n}^2 + x_n/n$. If the positive real value of $a = x_1$ is greater than or equal to 1, the squared x_n term would quickly balloon into an infinite number (for increasing values of n), and x_{n+1} would be divergent (see part c). Since this isn't the case, $0 < a < 1$. Now, since the series is monotone increasing and convergent, higher and higher values of n would give $x_{n+1} \approx x_n$, and since $\{x_{n+1}\}$ is the 1-tail of $\{x_n\}$, each sequence will converge to the same limit. Define $x := \lim x_n$

$$x_{n+1} = {x_n}^2 + \frac{x_n}{n} \Rightarrow x = x^2 + \frac{x}{n}$$
$$x^2 - x = -\frac{x_n}{n}$$

As n grows large, $x_n/n \to 0$. So,

$$x^2 - x = 0 \qquad \Rightarrow \qquad x_1 = 0 \qquad x_2 = 1$$

However, since $a > 0$ and x_n is monotone increasing, $x_n \geq a > 0$ and so $\lim x_n > 0$. Therefore $\lim x_n = 1$.

b. Show that there exists a such that the sequence $\{x_n\}_{n=1}^{\infty}$ is not monotone.

Take $a = 3/10$. That is, $x_1 = 3/10$, $x_2 = 39/100 > x_1$, and $x_3 = 3471/10000 < x_2$. As the value of x_n rises and then falls again, $a = 3/10$ gives a sequence x_n that is not monotone.

c. Show that there exists a such that the sequence $\{x_n\}_{n=1}^{\infty}$ is unbounded.

Define a sequence $b_n = n^2$, and choose $a = x_1 = 2$. Then,

$$x_{n+1} = {x_n}^2 + \frac{x_n}{2} \Rightarrow x_n = {x_{n-1}}^2 + \frac{x_{n-1}}{n-1}$$

Take the 1-tail of each b_n and x_n and start at $n = 1$:

$$\begin{aligned} b_1 &= 1^2 = 1 & x_1 &= 2 \\ b_2 &= 2^2 = 4 & x_2 &= 2^2 + \frac{2}{1} = 6 \\ b_3 &= 3^2 = 9 & x_3 &= 6^2 + \frac{6}{2} = 39 \\ b_4 &= 4^2 = 16 & x_4 &= 39^2 + \frac{39}{3} = \dots \end{aligned}$$

Clearly, $n^2 < {x_{n-1}}^2$ for all values of $n \geq 1$, so $b_n < x_n$. Now, to prove that b_n is unbounded. A sequence $\{c_n\}$ is bounded if there exists a $C \in \mathbb{R}$ such that $|c_n| \leq C$ for all $n \in \mathbb{N}$. Propose that such a C exists for the sequence $b_n = n^2$ for $n \in \mathbb{N}$. Since $n^2 \in \mathbb{N}$, the absolute value can be dropped. Take $b_{n+C} = (n + C)^2 = n^2 + C^2 > C$. Thus a contradiction is found, and the sequence b_n is unbounded. By the earlier statement that $b_n < x_n \, \forall n \in \mathbb{N}$, x_n is also unbounded.