PHYS3080 Problem Set 2

Ryan White   s4499039

Semester 1, 2022

Question 1

a. If a quasar is observed at redshift $z = 3$, calculating the scale factor at the time of emission is a simple task:

$$a(z) = \frac{1}{1+z} \tag{1}$$

$$a(3) = \frac{1}{1+3}$$

$$= \frac{1}{4}$$

   And so the light from the quasar was emitted when the universe was one quarter of its current scale.

b. An infinite cosmological redshift is possible, and corresponds to the light emitted at the very beginning of the universe. Note that equation (1) implies that as $z \to \infty$, $a \to 0$ since the denominator on the right hand size becomes infinitely large. Since the Big Bang occurred at $a = 0$, an infinite redshift corresponds to the light emitted during that process. In practice, this is not entirely possible to observe since the universe was largely opaque to photons for redshifts about $z \sim 1090$.

c. In principle, it shouldn't be possible to have an infinite blueshift, with $z = -\infty$. With this in mind, this possibility applies to both peculiar velocities and recession velocities. Taking the limit of a as $z \to -\infty$ in equation (1) would imply a negative scalefactor which is implausible and so an infinite blueshift is impossible in the context of recession velocities. In the context of peculiar velocities, however, an infinite blueshift would correspond to an object moving towards an observer at (effectively) the speed of light. Since mass would require an infinite amount of energy to be accelerated to c, this infinite blueshift could never happen. In practice, light could be increasingly blueshifted (approaching infinity) as its velocity towards the observer approaches c in increasingly appreciable fractions.

d. The observed redshift of a light source can be calculated from the cosmological redshift and peculiar redshift by

$$(1 + z_0) = (1 + \bar{z})(1 + z_p) \tag{2}$$

   where the peculiar redshift is calculated as

$$z_p = \gamma - 1 = \frac{1}{\sqrt{1 - v^2/c^2}} - 1 \tag{3}$$

   A galaxy with a peculiar velocity of 600km/s therefore has a peculiar redshift of

$$z_p = \frac{1}{\sqrt{1 - \frac{(600 \times 10^3)^2}{c^2}}} - 1$$

$$\simeq 2 \times 10^{-6}$$

   The observed redshift is then

$$z_0 = (1 + \bar{z})(1 + z_p) - 1$$

$$= (1 + 0.1)(1 + 2 \times 10^{-6}) - 1$$

$$\simeq 0.1000022 = 0.1 + 2.2 \times 10^{-6}$$

e. The redshift with respect to velocity is calculated as

$$z = \frac{v}{c} \tag{4}$$

With velocity being distance divided by time, we can find the time until Andromeda and the Milky Way collide given a distance of about 2.54×10^6 light years. Since we assume that it's moving directly towards us, the negative of the velocity (and hence the negative of either time or distance) must be taken. Rearranging and substituting in values gives

$$z = -\frac{v}{c} = -\frac{d}{ct}$$
$$\Rightarrow t = \frac{-d}{zc}$$
$$= \frac{-2.54 \times 10^6 \text{ lyr}}{-0.001001 \times c}$$
$$\simeq 2.537 \text{ Gyr} = 2.537 \times 10^9 \text{ years}$$

Of course, this number deviates from the typically quoted "4 to 5 billion years" until collision, since this blueshift only accounts for relative velocity of Andromeda relative to the Sun. In practice, the Sun is rotating about the center of the Milky Way somewhat in the direction of Andromeda, and so the blueshift appears larger than it would be with respect to the center of the Milky Way galaxy as a whole (and hence the time until collision appears lower).

Question 2

a. If the universe is flat, $\Omega_k = 1 - \Omega_m - \Omega_\Lambda - \Omega_r = 0$. If radiation density is negligible, $\Omega_r \simeq 0$ and so $\Omega_\Lambda = 1 - \Omega_m$. With a matter density parameter of $\Omega_m = 0.3$, Ω_Λ is clearly 0.7.

b. The Hubble distance refers to the region of spacetime which is expanding at (or faster than) the speed of light relative to our position. The extent of this distance can be calculated by the equation

$$v = HD \Rightarrow D = vH^{-1}$$

and by setting $v = c$, with $H = H_0$. Since $H(t)$ decreases with time, the Hubble distance is larger now than it was in the past (due to the Hubble parameter being an inverse term).

c. The critical density corresponds to the case of $K = 0$, i.e. that of a flat universe. In a flat universe, expansion will continue indefinitely but at a decreasing rate up to a point of essentially 0 expansion. The critical density more-or-less aligns with a balance between indefinite expansion and eventual recollapse, and values of $K < 0$ (open universe) will expand without constraint, while values of $K > 0$ (closed universe) will eventually recollapse.

d. The temperature of the CMB can be found at some redshift given its current observed temperature via the relation

$$T = T_0(1 + z) \tag{5}$$

If $T_0 = 2.728$K, and the redshift of the surface of last scattering was $z \sim 1090$, then substituting these values into equation (5) yields $T \simeq 2976$K. Given that the temperature of the surface of the Sun is $T = 5778$, this CMB initial temperature was about half of the Sun's current surface temperature. As such, the peak of the CMB emission spectrum at $z = 1090$ would have been at about 973nm (via Wien's displacement law) and so it would have been in the near infrared spectrum, similar to some low-mass red dwarf and large red giant stars.

Question 3

a. If the universe is flat, then $\sum^n \Omega_i = 1$. If the universe is dominated by the radiation density, then all other densities are negligible and so $\Omega_r \simeq 1$.

b. Beginning with Friedmann's equation, we can rearrange to eventually give a function for the scalefactor, a, in terms of time:

$$H(a) = \frac{\dot{a}}{a}$$
$$\Rightarrow \frac{da}{dt} = aH(a)$$

$$= aH_0E(a)$$
$$= aH_0\sqrt{\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda}$$

Since the radiation density is dominant, the other parameters are negligible and can be taken as 0. This gives:

$$\frac{da}{dt} = aH_0\sqrt{\Omega_r a^{-4}}$$
$$= aH_0\sqrt{\Omega_r}a^{-2}$$
$$= \frac{H_0\sqrt{\Omega_r}}{a}$$

But, $\Omega_r \simeq 1$ and so the function is simplified:

$$\frac{da}{dt} = \frac{H_0}{a}$$
$$\Rightarrow \int_0^a a\ da = H_0\int_0^t dt$$
$$\frac{a^2}{2} = H_0t$$
$$a = \sqrt{2H_0t} \tag{6}$$

c. The current age of the universe can be found by substituting in $a = 1$ into equation (6):

$$1 = \sqrt{2H_0t_0}$$
$$= 2H_0t_0$$
$$\Rightarrow t_0 = \frac{1}{2H_0}$$

Since $H_0 = 70$ km/s/Mpc, this is equivalently $H_0 \simeq 2.269 \times 10^{-18}$ s^{-1}. Substituting this value into the above gives

$$t_0 = \frac{10^{18}}{4.538}\ \text{s}\ \simeq 6.99\ \text{Gyr}$$

d. Recall that $a = 1/(1+z)$.

   i. To calculate the time light has been travelling from a galaxy at redshift $z = 1$,

$$a = \frac{1}{1+z} = \sqrt{2H_0t}$$
$$\frac{1}{2} = \sqrt{2H_0t}$$
$$\frac{1}{4} = 2H_0t$$
$$\Rightarrow t = \frac{1}{4}\cdot\frac{1}{2H_0}$$
$$= \frac{1}{4}t_0$$
$$\simeq 1.75\ \text{Gyrs}$$

And so the light emitted from a galaxy at redshift $z = 1$ has been travelling for about 1.75 billion years.

   ii. The time that light has travelled from the surface of last scattering at $z = 1090$ can be calculated in the same way as above:

$$a = \frac{1}{1+z} = \sqrt{2H_0t}$$
$$\frac{1}{1091} = \sqrt{2H_0t}$$
$$\Rightarrow t = \frac{1}{2H_0}\left(\frac{1}{1091}\right)^2$$

$$= t_0 \left(\frac{1}{1091} \right)^2$$
$$\simeq 5.87 \times 10^{-6} \text{ Gyr}$$
$$= 5.86 \times 10^3 \text{ years}$$

e. To calculate comoving distance in terms of redshift, Friedmann's equation must first be represented in terms of redshift:

$$H(a) = \frac{\dot{a}}{a}$$
$$= \frac{1}{a} \frac{da}{dt}$$

But $da/dt = H_0/a$ and so

$$H(a) = \frac{H_0}{a^2}$$
$$\Rightarrow H(z) = H_0(z+1)^2$$

Then, the comoving distance in terms of redshift is expressed as

$$R_0 \int d\chi = c \int \frac{1}{H(z)} \, dz$$
$$R_0\chi(z) = c \int_{z_2}^{z_1} \frac{1}{H_0(z+1)^2} \, dz$$
$$= \frac{c}{H_0} \int_0^z (z+1)^{-2} \, dz$$
$$= \frac{c}{H_0} \left(\left[-\frac{1}{z+1} \right]_0^z \right)$$
$$= \frac{c}{H_0} \left(1 - \frac{1}{z+1} \right)$$

f. Before calculating the comoving distance to each of the objects from part d, first note that

$$\frac{c}{H_0} = \frac{c}{2.269 \times 10^{-18} \text{ s}^{-1}} \simeq 4278 \text{ Mpc}$$

With this in mind,

i. The comoving distance to the galaxy at $z = 1$ is calculated as

$$R_0\chi(1) = \frac{c}{H_0} \left(1 - \frac{1}{2} \right)$$
$$\simeq 2139 \text{ Mpc}$$

and so a galaxy at redshift $z = 1$ is about 2.139 Gigaparsec from the observer.

ii. The comoving distance to the surface of last scattering at $z = 1090$ is calculated as

$$R_0\chi(1090) = \frac{c}{H_0} \left(1 - \frac{1}{1091} \right)$$
$$\simeq 4274 \text{ Mpc}$$

And so the surface of last scattering at $z = 1090$ is about 4.274 Gigaparsec away from the observer.

g. The proper distance at time of emission is calculated by

$$D(a) = aR_0\chi$$
$$\Rightarrow D(z) = \frac{R_0\chi}{z+1}$$

With this in mind, the comoving distance at time of emission for the objects described above are

i. For the galaxy at $z = 1$,

$$D(1) = \frac{2139}{2} \text{ Mpc} \simeq 1070 \text{ Mpc}$$

ii. and for the surface of last scattering at $z = 1090$,

$$D(1090) = \frac{4274}{1091} \text{ Mpc} \simeq 3.92 \text{ Mpc}$$

h. The proper distance at time of emission for the surface of last scattering is actually lower than that of the galaxy at $z = 1$ (by a factor of almost one thousand), despite being almost twice as far away in the present day. This is due to the expansion of the universe. When the photons were 'emitted' from the CMB, the universe was far smaller than it was today. The expansion of the universe necessitated that the photons had to travel further to reach us today than those of a galaxy which emitted light at a much later time.

Question 4

a. Given that the number density of some particle is proportional to its energy and the temperature by

$$n \propto \exp\left(-\frac{\epsilon}{k_B T}\right) \tag{7}$$

The ratio of neutrons to protons at the time of neutrino decoupling is

$$\frac{n_n}{n_p} = \frac{\exp\left(-\frac{\epsilon_n}{k_B T}\right)}{\exp\left(-\frac{\epsilon_p}{k_B T}\right)}$$

$$= \exp\left(-\frac{(\epsilon_n - \epsilon_p)}{k_B T}\right)$$

$$= \exp\left(-\frac{c^2 \Delta m}{k_B T}\right)$$

After substituting in $T \sim 10^{10}$K, the constants, and $\Delta m = 1.293$ MeV/c^2, the ratio is computed as

$$\frac{n_n}{n_p} \simeq 0.223$$

b. The relative proportion of neutrons to protons 3 minutes after the big bang (i.e. t = 179s after neutrino decoupling), and thus after some neutrons had decayed via their mean decay time of $\tau = 811$s is:

$$\left.\frac{n_n}{n_p}\right|_{t=179 \text{ s}} = \left.\frac{n_n}{n_p}\right|_{t=1 \text{ s}} e^{-t/\tau}$$

$$= 0.223 e^{-179/811}$$

$$= 0.179$$

c. The mass fraction of helium is calculated by the mass proportion of helium versus the mass proportion of everything (including the helium). Mathematically, that is:

$$Y = \frac{4n_{\text{He}}}{n_{\text{tot}}}$$

$$= \frac{4n_{\text{He}}}{4n_{\text{He}} + n_{\text{H}}}$$

Notice that the number of helium atoms is half of the number of neutrons, since we're assuming that all neutrons are bound in helium, and He^4 contains two neutrons. That is, $n_{\text{He}} = n_n/2$. Since two protons are needed in He^4 also, $n_{\text{H}} = n_p - n_n$. The mass fraction then becomes

$$Y = \frac{2n_n}{2n_n + (n_p - n_n)}$$
$$= \frac{2n_n}{n_n + n_p}$$
$$= \frac{2(n_n/n_p)}{1 + (n_n/n_p)}$$
$$\simeq 0.30$$

where the final step involved the substitution of the fraction of neutrons to protons found in part b. This means that helium made up approximately 30% of the mass fraction of baryonic matter just after Big Bang nucleosynthesis ceased.

Question 5

The thermal history of the universe describes primarily the explosive initial few seconds of the universe's existence.

An important first step is with the process of inflation, where the universe exponentially increased in size for a fraction of a second. In this inflationary period, the vacuum energy density was dominant which drove unconstrained growth but for only a short period. After this, Friedmann evolution of the universe began.

With the inflationary period over, electromagnetism and the weak magnetic force separate from each other and act differently over different scales. Soon after, quark-antiquark pairs combine to form baryon and anti-baryon pairs.

Since the temperature exponentially decreased over time, soon after the quark-hadron transition, neutrinos decoupled from the rest of the massive particles in what is called the neutrino-decoupling or neutrino freeze-out. This occurred at a temperature of about $\sim 10^{10}$K, or equivalently, one second after the Big Bang. From hereon out, the universe was essentially transparent to neutrinos and they were no longer subject to the thermal processes during the universe's initial thermal period.

Now, pair annihilation of electron-positron pairs becomes significant with their efficient production ceasing at around $\sim 5 \times 10^9$K. This annihilation injected energetic photons into the surrounding environment which increased its temperature relative to the decoupled neutrinos. The effects of this are still seen today with the CMB being a higher temperature than the predicted early-universe neutrinos. After this process ceases, the universe is more-or-less electrically neutral.

Next comes the nucleosynthesis of atomic nuclei. With temperatures and pressures sufficiently high (but not too high), protons and neutrons could fuse together to form basic nuclei of hydrogen (by default) and helium atoms (by first producing and then fusing two deuterium nuclei). This occurs just after the temperature in the universe drops below 8×10^8K, or roughly 3 minutes after the Big Bang.

After some time, the matter density begins to dominate the expansion of the universe. Once the universe has cooled to a temperature on the order of ~ 3000K, the majority of electrons have bounded to free protons to form neutral hydrogen atoms in a process called recombination. With the number of particles in the environment significantly reduced, the universe becomes essentially transparent to photons (which would have otherwise been absorbed by baryons) and they may propagate freely. This corresponds to the 'last scattering event' at an observed redshift of $z \sim 1000$. This photon scattering event is what we see today as the Cosmic Microwave Background, which has been significantly redshifted from its original near-infrared peak wavelength (corresponding to $T \sim 3000$K).

At some time with $z < 1$, the dark energy density begins to dominate, and we now see accelerated expansion of the universe as a result of this.