General Relativity: part 2
Course Outline

1. Introduction to gravitational physics
   Newton's theory, Special and General Relativity theory, When to use relativistic gravity

2. Special Relativity - Part I: Coordinates and Frames
   Inertial Reference Frames, the principle of relativity, speed of light invariance, events, spacetime diagrams and worldlines, the spacetime interval, proper time, spacetime separation, causality and light cones, Lorentz boosts and velocity addition

3. Special relativity - Part II: Kinematics and Dynamics
   4-vectors and the Minkowski metric, kinematics and dynamics, variational principle for free particle motion, geodesics, photons (null geodesics), observers and observations

4. Newtonian gravity
   Analogy between Newtonian gravity and electrostatics, the gravitational potential, dynamics, the equivalence of gravitational mass and inertial mass

5. Motion of particles in general relativity: the geodesic equation
   Gravity as a geometric property of spacetime, the geodesic equation and its derivation, Christoffel symbols

6. Newtonian limit of the geodesic equation and Einstein's field equation
   Newtonian gravity, recovering the force equation in the Newtonian limit, recovering the Poisson equation in the Newtonian limit

7. Solving the geodesic equation
   Conserved quantities, symmetries and Killing vectors

8. The Schwarzschild spacetime geometry: Basic properties
   The Schwarzschild metric, Killing vectors, Physical coordinates, GPS

9. The Schwarzschild spacetime geometry: Timelike geodesics
   Conserved quantities, effective potential and the radial equation, orbits, escape velocity

10. Light rays in General Relativity
    Affine parameter, light ray orbits in Swarzschild geometry, gravitational lensing, gravitational redshift, slow light

11. Black Holes
    Formation by gravitational collapse, escape velocity, light cone tilting, Eddington-Finkelstein coordinates

12. Gravitational Waves
    Linearized Einstein vacuum equations, linearized gravitational wave metric, detecting gravitational waves

13. Cosmology
    The cosmological principle and Robertson-Walker spacetime geometries, matter, radiation, and vacuum contributions to the energy density, the Friedmann equation, cosmological redshift

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Introduction to gravitational physics

Summary

- Newton's theory
- Special and General Relativity theory
- When to use relativistic gravity

From Newton's theory of gravity to general relativity

The first mathematical theory of gravity was developed by Isaac Newton in the 17th century, and is known as Newton's theory of gravitation or **Newtonian gravity.** The basis of this is the gravitational force law, which states two bodies of masses m_1 and m_2 that are separated by a distance r will attract each other by a force (the gravitational force) whose magnitude is given by

$$F_{\text{grav}} = \frac{Gm_1m_2}{r^2}\,, \tag{1}$$

where G is Newton's gravitational constant.

Gravity is one of the four fundamental forces in nature (the others are the electromagnetic force, and the weak and strong nuclear forces). It is interesting to compare Newton's gravitational force with these other forces, especially with the electromagnetic force, which is another force that is familiar to us from everyday life.

- **Newton's gravitational force is always attractive**, which is a consequence of masses always being positive. In contrast, forces of electrical origin can have either sign because there are both positive and negative electric charges.
- **Newton's gravitational force reduces with distance as the power law** $1/r^2$, whereas electrical charges of opposite sign will tend to screen each other, so that the "effective" electromagnetic force due to some excess charge will in many situations decay quickly (e.g. exponentially, as inside a metal) with distance.
- **Gravity is the dominant force on the largest scales**. Due to the electromagnetic forces, opposite charges attract each other, and so tend to form electrically neutral clumps of matter. This explains why gravity is the dominant force on large scales, despite the fact that the gravitational force between elementary particles is much less

than the electrical (Coulomb) force. As an example, the ratio of the gravitational force to the electric force for two protons is

$$\frac{F_{\rm grav}}{F_{\rm elec}} = \frac{Gm_P^2/r^2}{e^2/(4\pi\epsilon_0 r^2)} = \frac{Gm_P^2}{e^2/(4\pi\epsilon_0)} \sim 10^{-36} \tag{2}$$

where m_P is the proton mass and e its charge.

One final thing that Newton's law of gravity explained, which no previous explanation had managed, was why object of different mass accelerate at the same rate when they fall under gravity (as had been demonstrated by Galileo dropping weights from the leaning tower of Pisa). This is due to the fact that the inertial mass in Newton's force law, and the gravitational mass in Newton's gravitational force law are the same, and so cancel each other out when the acceleration is calculated. The fact that gravity is such an 'Inertial Force' implies something special about it, in contrast to the other forces, which are not.

Although Newton's theory of gravity can account well for a large range of phenomena, it is not consistent with the **Special Theory of Relativity** (aka **Special Relativity**) **(SR)** developed by Einstein in 1905. In Newton's theory, the gravitational force between two bodies is instantaneous in time, i.e. the force on either mass depends on the position of the other mass at the same time. The phrase "at the same time" indicates a problem with Newton's theory, since in SR simultaneity is not an absolute notion but depends on the observer. Another problem is that in SR no particle or signal can travel faster than the universal speed limit, the speed of light in vacuum, c, and thus an "instantaneous" gravitational force corresponding to an infinite propagation speed is impossible. Thus the development of SR made it clear that Newton's theory of gravity could not be correct in principle and had to be replaced by a different theory of gravity that would be consistent with the physical principles that had led to SR. (In contrast, Maxwell's equations for electromagnetism are fully consistent with SR.) On the other hand, as already noted, Newton's theory of gravity does give excellent agreement with experiments and observations in many important cases, and thus the new theory, whatever it would turn out to be, had to reduce to Newton's theory in those cases.

The new theory of gravity that came to replace Newton's was developed by Einstein in the period $\sim 1905 - 1915$ and is called the **General Theory of Relativity** (aka **General Relativity**) **(GR)**. This theory introduces a completely new picture of what gravity fundamentally is. While in Newton's theory, gravity had been a **force** between massive objects, in GR gravity is instead manifested as the curvature of 4-dimensional spacetime. Thus gravity is a geometric property of spacetime. The essence of GR is quite nicely captured by the following quote (originally due to Wheeler):

Matter tells spacetime how to curve, spacetime tells matter how to move.

The first part of this quote refers to Einstein's field equation in GR which relates curvature properties of spacetime to the energy-momentum tensor representing "matter". Einstein's equation is a tensor equation that corresponds to a set of partial differential equations for the metric tensor $g_{\mu\nu}$. The second part of the quote refers to the fact that according to GR, particles that are only subjected to gravity move along *geodesics* in the curved spacetime. These geodesics are the "straightest possible" paths in spacetime; their analogues in ordinary flat Euclidean space are straight lines. What Newton would have described as the motion of a particle acting under the gravitational *force* is in GR described as the motion of a *free*[1] particle moving in a *curved* spacetime.

Which problems require a ***relativistic*** treatment of gravity?

For gravitational effects involving a body with mass M and a characteristic size R, those effects for which a relativistic description of gravity is important are characterized by a ratio

$$GM/(Rc^2) \tag{3}$$

being a significant fraction of unity. Note however that even in situations where this ratio is small, so that relativistic effects are also small, such effects may still be important to take into account in various **precision measurements/calculations**. As an example, to ensure sufficient accuracy of the Global Positioning System (GPS), it is crucial to take into account GR (as well as SR) effects.

To get an idea of why a ratio like $GM/(Rc^2)$ should be important, let us consider some estimates based on naively applying Newtonian gravity to situations involving the speed-of-light velocity c. For example we could ask when the Newtonian gravitational force due to a mass M is enough to keep a test mass m with velocity c in a circular orbit at radius R. The answer is given by equating the gravitational force GMm/R^2 to $m(v^2/R)$ with $v = c$. This gives the ratio $GM/(Rc^2) = 1$.

Another example is to consider the escape velocity v_{esc} of a particle with test mass m sitting on an object with mass M and radius R. The escape velocity is found from the condition that the particle should have have total energy (kinetic + potential) equal to zero, so that it has zero kinetic energy infinitely far away from the mass M. Thus the condition is

$$\frac{1}{2}mv_{\text{esc}}^2 - \frac{GMm}{R} = 0. \tag{4}$$

Setting $v = c$ gives the ratio $GM/(Rc^2) = 1/2$. So in both examples we see that the given ratio is a number of order 1. (The second example here may strike you as relevant to black holes, and in fact this naive calculation does give the correct expression for the radius of a Schwarzschild black hole.)

[1]By definition, 'free' means 'not subjected to any forces.'

Special Relativity - Part I
Coordinates and Frames

These notes give a discussion of special relativity (SR). In preparing these notes I have leaned heavily on Hartle (Ch. 4) and Landau & Lifshitz (§1-§5).

Summary

- Inertial Reference Frames
- The principle of relativity, speed of light invariance
- Events, spacetime diagrams and worldlines
- The spacetime interval, proper time
- Spacetime separation, causality and light cones
- Lorentz boosts and velocity addition

Inertial reference frames

To describe physical phenomena, one needs to introduce a reference frame to label positions and times. For this purpose we will consider an important type of reference frame called an **inertial reference frame**, or just **inertial frame** for short. To set up an inertial frame we can in principle do as follows. Pick a **free** particle, i.e. a particle that (by definition of 'free') is not acted upon by any external forces.[1] We define the origin of the inertial frame to coincide with the position of the free particle. Define three rigid perpendicular axes x, y, and z emanating from this origin along which positions can be measured. We imagine that the spatial coordinate system is attached to the free particle so if the particle is moving the coordinate system will move too. At each spatial point (x, y, z) in this coordinate system we put a clock to measure the time t. We synchronize all the clocks so that they show the same time.[2] Obviously, an infinite number of different inertial frames can be constructed in this way.

When viewed from an inertial reference frame, any free particle will be seen to move in a straight line and with a constant velocity. Thus any free particle will in the inertial frame be described by the equations

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 0, \quad \frac{d^2z}{dt^2} = 0 \tag{1}$$

[1]Since we're not doing GR yet, gravity is still considered to be a force.

[2]The synchronization can e.g. be done as follows: Set the time on each clock to an initial value $t = D/c$ where D is the distance from the clock to the origin. Arrange for each clock to start ticking from its initial value once a light ray goes by. Then send light from the origin in all directions.

which is Newton's 1st law. Thus in an inertial frame, Newton's 1st law holds in the form (1). In contrast, in a non-inertial frame (e.g. a rotating frame) a free particle would appear to be accelerated.

The principle of relativity and the invariance of the speed of light

What was said in the previous section actually applies both to Newtonian mechanics and SR. Another feature common to both theories is the **principle of relativity**, which states that **the laws of physics should take the same form in all inertial frames**. However, SR differs in the additional postulate that **the speed of light c takes the *same finite value* in all inertial frames**.[3] Einstein introduced this postulate motivated by experimental observations (especially the Michelson-Morley experiment) but also by theoretical arguments based on applying the principle of relativity to Maxwell's equations for electromagnetic waves. The invariance of c can be seen to imply that the notion of an absolute time from Newtonian mechanics must be abandoned in SR. Thus different inertial frames S and S' will generally be associated with *different* times t and t' due to the clocks in the two frames ticking at different rates. Furthermore, the transformations connecting the coordinates of different inertial frames (i.e. the Lorentz transformations) generally mix the space and time coordinates of the two frames. Therefore in SR one cannot treat space and time as disconnected entities; instead one needs to introduce the concept of **spacetime**, a 4-dimensional union of (3-dimensional) space and (1-dimensional) time. One profound consequence of SR is that the notion of absolute simultaneity must be abandoned: in SR, two events that are simultaneous in one inertial frame will generally not be simultaneous in another inertial frame. Other well-known consequences are time dilation, length contraction, and a modified rule for "adding" velocities.

Events. Spacetime diagrams. Worldlines

An **event** is simply something that happens somewhere in space, at some time, i.e. it happens at a point in spacetime. It is useful to introduce the **spacetime diagram** for visualizing and analyzing (collections of) events. An example is shown in Fig. 1. The two perpendicular axes shown are the x and t axes for a particular inertial frame. It is an SR convention that the time axis is the vertical one. The time axis actually shows ct, not t (alternatively, we can work in units in which $c = 1$) so that both axes have units of length. In this diagram the y and z axes have been suppressed. At most one more of these axes could be included in addition to x and ct, because we are unable to visualize 4 dimensions.... An event P with coordinates (ct_0, x_0) (dropping the y and z coords) becomes the point P in this diagram.

A particle describes a curve in the spacetime diagram called a **world line**. It corresponds to the curve $x(t)$ describing the position x of the particle at the time t, for all possible times t. In SR nothing can move faster than c, and the velocity $v = dx/dt$ of particles with mass $m > 0$ must always be less than c. (In the following the word "particle" will be taken to stand for a particle with mass $m > 0$; the only zero mass objects we will consider are photons (light rays)). Thus the slope $d(ct)/dx$ of a particle's worldline must always satisfy $|d(ct)/dx| = |c/v| > 1$. Free particles move at constant speed so their world lines are straight lines. A special case of this is a particle sitting still in the chosen inertial frame ($v = 0$), whose worldline is vertical. The worldlines of light rays (which have velocity $v = \pm c$) have

[3] More precisely, the speed of light in vacuum, which takes the value $c \approx 3\times 10^8$ m/s. Newtonian mechanics can be recovered from SR by formally letting $c \to \infty$.

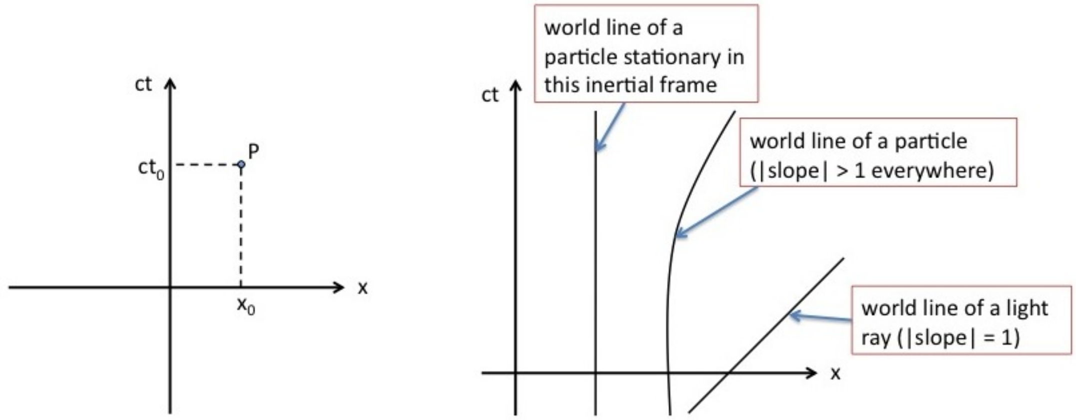

Figure 1: Spacetime diagrams. Left: An event P. Right: Examples of worldlines.

slope ± 1 and are thus oriented at 45 degree angles in the spacetime diagram. These things are illustrated in Fig. 1.

The spacetime interval and its invariant nature

Consider two events labeled 1 and 2, which in our chosen inertial frame K have coordinates (ct_1, x_1, y_1, z_1) and (ct_2, x_2, y_2, z_2). Define $\Delta t = t_2 - t_1$, and similarly for the other coordinates. The **spacetime interval** between the two events is denoted[4] Δs and is defined through the relation

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2. \tag{2}$$

One could equally well have chosen to use a different inertial frame K' in which the two events have coordinates $(ct'_1, x'_1, y'_1, z'_1)$ and $(ct'_2, x'_2, y'_2, z'_2)$. Defining $\Delta t' = t'_2 - t'_1$ and so on, and calculating the spacetime interval as defined in this frame, i.e. $(\Delta s')^2 \equiv -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2$, it is the case that

$$(\Delta s)^2 = (\Delta s')^2 \tag{3}$$

i.e. the spacetime interval between two events takes the same value in all inertial frames. Thus although observers in different inertial frames will generally not agree on the time difference and spatial separation between the events, they will always agree on the value of the spacetime interval. We say that the spacetime interval is **invariant** under a change of inertial frame.

If the two events are infinitesimally close together, we get the infinitesimal version of (2):

$$ds^2 = -(c\,dt)^2 + dx^2 + dy^2 + dz^2. \tag{4}$$

Here $ds^2 \equiv (ds)^2$ etc. The infinitesimal spacetime interval ds as defined here is also called the **line element** of **flat spacetime** (which is the spacetime geometry of SR).

The invariance of the spacetime interval (3) is easy to prove if the two events are points on the world line of a light ray, as then the postulate that the speed of light is the same in in all inertial frames implies that $\Delta s^2 = \Delta s'^2 = 0$. By using this fact and also invoking homogeneity of space and time as well as isotropy of space, it is shown in [LL] that $ds^2 = ds'^2$ is true in general. We won't discuss this proof here but will simply take the invariance of

[4] Some textbooks use the name "spacetime interval" about $(\Delta s)^2$ instead of Δs.

ds^2 under a change of inertial frame (i.e. change of inertial coordinate system) as a defining geometric property of SR spacetime. The invariance of the spacetime interval (3) between two arbitrary events that are not necessarily close together can then e.g. be derived as follows. Consider the straight line connecting events 1 and 2 in the spacetime diagram. Divide the line into N sections. The components of each section on the x and ct axes are $\Delta x/N$ and $c\Delta t/N$, respectively. So we can write

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 = -(N \cdot c\Delta t/N)^2 + (N \cdot \Delta x/N)^2 \tag{5}$$

$$= N^2\Big[-(c\Delta t/N)^2 + (\Delta x/N)^2\Big]. \tag{6}$$

In the limit $N \to \infty$ we define $dx = \Delta x/N$ and $dt = \Delta t/N$. Thus $(\Delta s)^2 = N^2[-(c\,dt)^2 + dx^2] = N^2 ds^2$, where ds is the line element between two infinitesimally close events on the straight line connecting events 1 and 2. It follows that we can write

$$\Delta s = \int_1^2 ds \tag{7}$$

where the integral goes along this straight line. Thus since we have expressed Δs as a sum of infinitesimals ds, each of which satisfies the invariance property $ds^2 = ds'^2$, it follows that the invariance holds also for Δs^2.

Types of separation between events. Light cones

It is useful to characterize pairs of events according to the sign of the square of the spacetime interval between them, $(\Delta s)^2$. Two events are said to be

$$\text{spacelike separated if } (\Delta s)^2 > 0 \tag{8}$$

$$\text{null separated if } (\Delta s)^2 = 0 \tag{9}$$

$$\text{timelike separated if } (\Delta s)^2 < 0. \tag{10}$$

Note however that the connection between the sign of $(\Delta s)^2$ and whether the interval is timelike or spacelike depends on the convention used for the overall sign in the definition of $(\Delta s)^2$ (i.e. on the overall sign of the metric). Thus rather than remembering the sign it is safer to think of timelike and spacelike separation in terms of whether the space part or the time part of the interval is the biggest one: If the time part is bigger the separation is timelike, if the space part is bigger the separation is spacelike.

Using the invariance of $(\Delta s)^2$ under a change of inertial frame one can show that (exercise): (i) if and only if the separation between two events is spacelike is it possible to find an inertial frame in which the two events happen at the same time; (ii) if and only if the separation between two events is timelike is it possible to find an inertial frame in which the two events happen at the same place.

Since in SR the speed of a particle of nonzero mass is always less than c, it follows that events on the worldline of a particle are timelike separated: $ds^2 < 0$ for any two infinitesimally close points on the particle's worldline. events that are null separated can be connected by the worldline of a light ray going between the two events. Thus light rays move on worldlines characterized by $ds^2 = 0$ for any two infinitesimally close points on the worldline.

Consider an event P. In Newtonian mechanics, any other event O either happens before P, after P, or at the same time as P. In SR the structure is not quite as simple, but causality (the property that "cause comes before effect") is still valid. To investigate this structure it is helpful to again consider spacetime diagrams and introduce the very important concept of **light cones**. Here you should read the subsection "Light cones" in Hartle (*hand-out*).

Proper time. Time dilation. The twin "paradox". Straight lines and longest distances

Consider a particle moving along its worldline. Two neighbouring points on this worldline are timelike separated, i.e. $ds^2 < 0$. In this case let us define

$$d\tau^2 \equiv -\frac{1}{c^2} ds^2. \tag{11}$$

The quantity $d\tau$ has units of time. Since $ds^2 < 0$, $d\tau$ is real and can be taken positive. The quantity $d\tau$ is called the **proper time**. It is the time difference that would be measured by a clock carried by the particle as the particle moves between the neighbouring points.[5] To see this, note that the motion between the infinitesimally separated points can be considered as uniform (i.e. moving with a constant velocity with respect to an inertial frame K), so the particle's instantaneous motion defines another inertial frame, call it K'. Calculating the spacetime interval between the neighbouring points in that inertial frame, one has $ds^2 = -(cdt')^2 + dx'^2 + dy'^2 + dz'^2 = -(c\,dt')^2$ since the two points are at the same place in space in that reference frame. Thus $dt' = d\tau$, the proper time. In contrast, the time dt between the two points as measured in the inertial frame K is larger. To see this we use $ds^2 = -(c\,dt)^2 + dx^2 + dy^2 + dz^2$ which gives $d\tau^2 = dt^2[1 - c^{-2}(dx^2 + dy^2 + dz^2)/dt^2] = dt^2[1 - \vec{v}^2/c^2]$ where $\vec{v}^2$ is the square of the particle's velocity as seen from the inertial frame K, i.e. $\vec{v}^2 = (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2$. Thus

$$d\tau = dt\,\sqrt{1 - \vec{v}^2/c^2}. \tag{12}$$

This relationship is called **time dilation**.

The time recorded by a clock following the particle as it moves along a finite section of its worldline between two events A and B is obtained by integrating the expression (12) along that section; this gives the proper time τ_{AB} between the points A and B for that worldline. That is, we divide the finite section into infinitesimal sections, on each of which the motion is uniform with some velocity v that may vary from one infinitesimal section to the next, and add together the proper times for each infinitesimal section. Thus

$$\tau_{AB} = \int_A^B d\tau = \int_{t_A}^{t_B} dt\sqrt{1 - \vec{v}(t)^2/c^2}. \tag{13}$$

Note that τ_{AB} does not just depend on the endpoints A and B but also on the shape of the worldline connecting them. The worldline isn't restricted to be straight (representing a free particle in unaccelerated motion) but can be an arbitrary worldline for a particle/observer that is being accelerated in some arbitrary manner.

[5] The meaning of the word 'proper' in this context is as far as I know related to the word 'proprietary' and refers to the fact that the proper time 'belongs to' the clock carried by the moving particle (in German, proper time is called Eigenzeit).

Suppose that the particle under consideration is moving at uniform velocity. It can then be considered to be an observer in the inertial frame K' introduced above, which moves at constant velocity $|\vec{v}|$ with respect to the inertial frame K. Thus the square root in (13) is a constant and can be taken outside the integral, giving

$$\Delta t' = \sqrt{1 - \vec{v}^2/c^2}\,\Delta t \tag{14}$$

for the relationship between time differences in K and K'. This time dilation effect is often summarized as "moving clocks run slow", which is however a somewhat imprecise statement since from the point of view of K' it is K that is moving. To understand better what is meant let us emphasize how the time differences Δt and $\Delta t'$ in (14) are obtained. The time difference $\Delta t'$ is a proper time obtained by measuring two times *on the same clock* in K', since the two events are at the same position in K'. On the other hand, the time difference Δt requires the measurement by *two different clocks* in K, since the two events are at different positions in K. So the "slowing of a clock" belonging to one inertial frame is what is observed when the time difference measured on that *one* clock is compared to the time difference obtained from measuring times with *different* clocks in the other inertial frame.

A famous example in SR is the so-called "twin paradox." Consider two twins Alice and Bob who are initially at the same spacetime point, but then part ways. While Bob moves like a free particle and thus can be used to define an inertial frame K, Alice jumps on a spaceship to take a journey in space before eventually returning to the place in K where Bob has been all along. Using the inertial frame K to draw a spacetime diagram of the situation, Bob's worldline is straight and vertical while Alice's worldline must change direction during the journey in order to eventually return to Bob's position. Then using the inertial frame K to calculate the proper times for Alice and Bob as given by (13) we find that $\tau_{\text{Alice}} < \tau_{\text{Bob}}$ since Bob's velocity $\vec{v}_{\text{Bob}}(t)$ has been zero throughout while Alices's velocity $\vec{v}_{\text{Alice}}(t)$ was not. So when Alice returns, the twins will find that Bob has aged more than Alice. The "paradox" comes from instead considering the situation from Alice's point of view in which Bob would be the one who is moving, in which case (it is claimed) one would reach the opposite conclusion about who had aged more. The claim is incorrect because Alice's motion is not inertial (i.e. not that of a free particle) since it will require forces to change the direction of her worldline, so it is impossible to associate with her a single inertial frame. Hence there is no paradox.

The proper time (13) is a convenient measure of the spacetime distance along timelike worldlines. In ordinary Euclidean space we know that the shortest line between two points is a straight line. In contrast, the twin paradox example showed that Bob's straight worldline between the two endpoints is longer than Alice's nonstraight one. From (13) one sees that Bob's straight worldline is indeed longer than any non-straight worldline Alice could have taken between the same endpoints, because in Bob's case the the factor $\sqrt{1 - \vec{v}(t)^2/c^2}$ equals 1 everywhere along his worldline while in Alice's case this factor will be smaller than 1 along at least some of her worldline. This is a general property of SR: the straight line path is the *longest* distance (longest proper time) between two timelike separated points in flat four-dimensional spacetime. (This is a consequence of the **non-Euclidean** nature of the spacetime geometry of SR, which arises from the relative minus sign between the space and time terms in the formula (4) for the infinitesimal line element.) This can be shown also for the case of a straight line path that is not vertical, by first transforming to another inertial frame in which it is vertical, and then using the above reasoning for a vertical straight line. (Since the straight line path is supposed to connect spacetime points that are timelike separated, it is possible to find an inertial frame in which the two points are at the same place in space (as discussed earlier), and in this new inertial frame the straight line path becomes vertical. It doesn't matter which inertial frame we use to do the calculation of the proper time since the proper time is invariant (i.e. it is a scalar) under such transformations.)

Lorentz boosts. Addition of velocities

Two inertial frames can differ from one another by displacements, rotations (in space), or uniform motions (or combinations of two or more of these). Displacements and rotations aren't very interesting. On the other hand, inertial frames that differ by uniform motion correspond to more abstract 'rotations' that mix space and time coordinates in the pseudo-Euclidean spacetime of SR. These transformations are called **Lorentz boosts** and are the generalizations to SR of the Galilean transformations in Newtonian mechanics.

For concreteness we will consider boosts that mix the ct and x coordinates while leaving the y and z coordinates unchanged. Much like rotations in a 2D Euclidean space leave the line element defined by $dS^2 = dx^2 + dy^2$ invariant, the "rotations" in the pseudo-Euclidean spacetime of SR will leave the corresponding line element $ds^2 = -(c\,dt)^2 + dx^2$ invariant. While rotations in Euclidean space involve trigonometric functions of the rotation angle, the relative minus sign in the definition of the SR line element implies that hyperbolic functions of a different parameter (call it θ) are involved. (This is related to the fact that while the trigonometric functions satisfy $\cos^2 + \sin^2 = 1$, the analogous relation for hyperbolic functions involves a relative minus sign: $\cosh^2 - \sinh^2 = 1$.) The Lorentz boost relating (ct, x) in one frame to (ct', x') in the other is given by

$$\begin{aligned} ct' &= ct\cosh\theta - x\sinh\theta, && (15) \\ x' &= -ct\sinh\theta + x\cosh\theta && (16) \end{aligned}$$

(and $y' = y$, $z' = z$). It is easy to verify that $-(c\,dt)^2 + dx^2 = -(c\,dt')^2 + x'^2$ as required. To see that this transformation indeed is between two inertial frames that move at uniform relative velocity with respect to each other, we consider the point $x' = 0$ which in the other frame is seen to be given by $0 = -ct\sinh\theta + x\cosh\theta$, i.e. $x = ct\,\tanh\theta$. So the point $x' = 0$ in the primed frame moves at speed $V = x/t = c\tanh\theta$ in the unprimed frame. It follows that the primed frame is moving to the right with velocity V with respect to the unprimed frame. If the Lorentz boost is written in terms of V rather than θ it becomes

$$\begin{aligned} ct' &= \gamma(ct - Vx/c), && (17) \\ x' &= \gamma(x - Vt), && (18) \end{aligned}$$

where $\gamma = (1 - V^2/c^2)^{-1/2}$. Using this transformation formula it is straightforward to deduce time dilation, the relativity of simultaneity, and length (Lorentz) contraction. In connection with length contraction one defines the **proper length** of an object as its length in the frame in which it is at rest. In an inertial frame that moves in the x direction at some velocity V with respect to this rest frame, the length of the object in the direction of motion will be a factor $\sqrt{1 - V^2/c^2} < 1$ of the proper length.

From the expressions for the Lorentz boost one can deduce the relationship between the velocity of a particle as measured in the two inertial frames. Here we only consider the simplest example of a particle moving in the x direction, i.e. $v^y = v^z = 0$. Using that by definition $v^x = dx/dt$ and $v^{x'} = dx'/dt'$ one has (with V the relative velocity of the two frames)

$$v^{x'} = \frac{dx'}{dt'} = \frac{\gamma(dx - V\,dt)}{\gamma(dt - V\,dx/c^2)} = \frac{v^x - V}{1 - Vv^x/c^2}. \qquad (19)$$

Two interesting special cases are: (i) $V/c \ll 1$ which gives $v^{x'} \approx v^x - V$ which is the non-relativistic law of addition of velocities, and (ii) $v^x = c$ which gives $v^{x'} = c$ as well, as required by the invariance of the speed of light.

Special relativity - Part II
Kinematics and Dynamics

This is the second part of the lecture notes on special relativity (SR). We follow Ch. 5 in Hartle closely. In these notes we set $c = 1$.

Summary

- 4-vectors and the Minkowski metric
- Kinematics and dynamics
- Variational principle for free particle motion, geodesics
- Photons (null geodesics)
- Observers and observations

4-vectors in SR

4-vectors will be written in boldface, e.g. $\boldsymbol{a}$ (when writing on the board I instead use a wiggly underscore for 4-vectors). In a given inertial frame we can introduce basis 4-vectors $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ pointing along the x^0, x^1, x^2, and x^3 axes (here $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$). An arbitrary 4-vector $\boldsymbol{a}$ can be expanded in terms of these basis vectors as (note summation convention used)

$$\boldsymbol{a} = a^\alpha \boldsymbol{e}_\alpha. \tag{1}$$

where the numbers a^α ($\alpha = 0, 1, 2, 3$) are the *contravariant* components of $\boldsymbol{a}$ in this inertial frame. We will often list these components as

$$a^\alpha = (a^0, \vec{a}) \tag{2}$$

where $\vec{a}$ represents the 3-vector with components a^1, a^2, and a^3 along the x^1, x^2 and x^3 coordinate axes.

The **scalar product** of two 4-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined in the usual way:

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a^\alpha \boldsymbol{e}_\alpha) \cdot (b^\beta \boldsymbol{e}_\beta) = (\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta) a^\alpha b^\beta. \tag{3}$$

The scalar products of the basis vectors,

$$\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta \equiv \eta_{\alpha\beta} \tag{4}$$

constitute the **metric**. Thus

$$\boldsymbol{a} \cdot \boldsymbol{b} = \eta_{\alpha\beta} a^\alpha b^\beta. \tag{5}$$

The metric $\eta_{\alpha\beta}$ of SR can be determined from the requirement that the spacetime interval $(\Delta s)^2$ between two spacetime events A and B is defined as the scalar product $\Delta\boldsymbol{x}\cdot\Delta\boldsymbol{x}$, where $\Delta\boldsymbol{x}$ is the displacement vector between the two events, which has coordinates $\Delta x^\alpha = x^\alpha_B - x^\alpha_A$. In other words, we require

$$(\Delta s)^2 = -(\Delta x^0)^2 + (\Delta\vec{x})^2 = \Delta\boldsymbol{x}\cdot\Delta\boldsymbol{x} = \eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta. \tag{6}$$

This gives $-\eta_{00} = \eta_{11} = \eta_{22} = \eta_{33} = 1$ and $\eta_{\alpha\beta} = 0$ for $\alpha \neq \beta$, which is often written on matrix form as

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{7}$$

where the matrix elements not explicitly shown are all equal to 0. This is called the metric of flat spacetime, or the **Minkowski metric**. It follows that

$$\boldsymbol{a}\cdot\boldsymbol{b} = -a^0 b^0 + \vec{a}\cdot\vec{b}. \tag{8}$$

In particular, the line element (spacetime interval for infinitesimally separated events) ds can be written

$$ds^2 = d\boldsymbol{x}\cdot d\boldsymbol{x} = \eta_{\alpha\beta}dx^\alpha dx^\beta = -(dx^0)^2 + (d\vec{x})^2. \tag{9}$$

An arbitrary 4-vector $\boldsymbol{a}$ is called spacelike if $\boldsymbol{a}\cdot\boldsymbol{a} > 0$, timelike if $\boldsymbol{a}\cdot\boldsymbol{a} < 0$, and null if $\boldsymbol{a}\cdot\boldsymbol{a} = 0$.

So far we have worked in one particular inertial frame, call it K. In a different inertial frame K' the components of a given 4-vector $\boldsymbol{a}$ will be different, since the basis vectors are different in that frame, while the 4-vector $\boldsymbol{a}$ is of course the same. Under a transformation from K to K' the components of a 4-vector are defined to transform in the same way as the components of displacement vectors $d\boldsymbol{x}$. Writing the transformation law for the latter as

$$dx^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} dx^\beta \tag{10}$$

we get from the required invariance of the line element under such transformations (i.e. $d\boldsymbol{x}\cdot d\boldsymbol{x}$ should be the same whether calculated in K or K')

$$\eta_{\alpha'\beta'}dx^{\alpha'}dx^{\beta'} = \eta_{\alpha'\beta'}\Lambda^{\alpha'}{}_{\gamma}\Lambda^{\beta'}{}_{\delta}dx^\gamma dx^\delta = \eta_{\gamma\delta}dx^\gamma dx^\delta \tag{11}$$

and therefore (since this must be true for arbitrary dx^α)

$$\eta_{\alpha'\beta'}\Lambda^{\alpha'}{}_{\gamma}\Lambda^{\beta'}{}_{\delta} = \eta_{\gamma\delta}. \tag{12}$$

The contravariant components of an arbitrary 4-vector $\boldsymbol{a}$ should by definition transform in the same way as those of a displacement vector, i.e.

$$a^{\alpha'} = \Lambda^{\alpha'}{}_{\beta}a^\beta. \tag{13}$$

It can then be verified that we get the same answer for the scalar product of two arbitrary 4-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ regardless of which frame we use to calculate it:

$$\boldsymbol{a}\cdot\boldsymbol{b} = \eta_{\alpha'\beta'}a^{\alpha'}b^{\beta'} = \eta_{\alpha'\beta'}\Lambda^{\alpha'}{}_{\gamma}\Lambda^{\beta'}{}_{\delta}a^\gamma b^\delta = \eta_{\gamma\delta}a^\gamma b^\delta. \tag{14}$$

Thus the scalar product is invariant under transformations between different inertial frames.

Kinematics and dynamics in SR

In this section we will introduce various 4-vectors relevant to the motion of particles in SR (4-velocity, 4-acceleration, 4-momentum, and 4-force) and discuss the relationships between them. See the next section for an example illustrating these concepts.

Particles move along timelike worldlines in spacetime.[1] The motion of a particle along its wordline can be specified by giving its coordinates x^α in a particular inertial frame as functions $x^\alpha(\sigma)$ of some parameter σ which varies along the worldline. Often it is convenient to choose the parameter to be the proper time τ along the worldline (with the zero of the proper time chosen to be at some specific point on the worldline).

4-velocity. The 4-velocity $\boldsymbol{u}$ can be defined in terms of its components in a particular inertial frame K:

$$u^\alpha = \frac{dx^\alpha}{d\tau}. \tag{15}$$

Thus in an infinitesimal proper time interval $d\tau$ the components of the particle change by $dx^\alpha = u^\alpha d\tau$. It follows from this that $\boldsymbol{u}$ is tangent to the particle's worldline at each point. Using the time dilation formula $d\tau = dt\sqrt{1-\vec{v}^2}$ where $\vec{v}$ is the particle's 3-velocity, we get

$$u^0 = \frac{dx^0}{d\tau} = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\vec{v}^2}} = \gamma, \tag{16}$$

$$\vec{u} = \frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{x}}{dt} = \frac{\vec{v}}{\sqrt{1-\vec{v}^2}} = \gamma\vec{v}. \tag{17}$$

Thus we have expressed the components of the particle's 4-velocity $\boldsymbol{u}$ in terms of the components of the particle's 3-velocity $\vec{v}$ in the same frame:

$$u^\alpha = (\gamma, \gamma\vec{v}). \tag{18}$$

This formula is valid at any point on the particle's worldline (unless the particle is free, $\boldsymbol{u}$ and $\vec{v}$ will vary from point to point). Furthermore,

$$\boldsymbol{u}\cdot\boldsymbol{u} = -\gamma^2 + \gamma^2\vec{v}^2 = -1. \tag{19}$$

Thus $\boldsymbol{u}$ is a timelike unit vector (it's timelike because $\boldsymbol{u}\cdot\boldsymbol{u} < 0$, and a unit vector because $|\boldsymbol{u}\cdot\boldsymbol{u}| = 1$). An alternative proof of the same result is

$$\boldsymbol{u}\cdot\boldsymbol{u} = \eta_{\alpha\beta}u^\alpha u^\beta = \eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = \frac{ds^2}{d\tau^2} = -1. \tag{20}$$

The property $\boldsymbol{u}\cdot\boldsymbol{u} = -1$ is called the **normalization** of the 4-velocity.

4-acceleration. The 4-acceleration $\boldsymbol{a}$ is the derivative of the 4-velocity with respect to the proper time:

$$\boldsymbol{a} = \frac{d\boldsymbol{u}}{d\tau}. \tag{21}$$

The 4-acceleration is orthogonal to the 4-velocity at every point of the worldline, in the sense that

$$\boldsymbol{a}\cdot\boldsymbol{u} = 0. \tag{22}$$

[1] We only consider particles with nonzero mass here. Photons, which move along null worldlines, will be considered in a later section.

This follows from differentiating both sides of $\boldsymbol{u}\cdot\boldsymbol{u}=-1$ with respect to τ and using the definition (21).

4-momentum. The 4-momentum $\boldsymbol{p}$ is defined as

$$\boldsymbol{p}=m\boldsymbol{u} \tag{23}$$

where m is the particle's mass. From (18) it follows that

$$p^\alpha=(m\gamma, m\gamma\vec{v}). \tag{24}$$

The nonrelativistic ($|\vec{v}|\ll 1$) limit of these expressions is obtained by expanding the square root in γ, which gives, to leading order in $\vec{v}$,

$$p^0 \approx m+\frac{1}{2}m\vec{v}^2, \tag{25}$$

$$\vec{p} \approx m\vec{v}. \tag{26}$$

The rhs of (25) equals the rest energy m (i.e. mc^2 if reinstating c) plus the nonrelativistic kinetic energy $m\vec{v}^2/2$. Thus p^0 is also called the energy and written $p^0=E$. We recognize the rhs of (26) as the nonrelativistic expression for the 3-momentum. For these reasons $\boldsymbol{p}$ is alternatively known as the **energy-momentum** 4-vector. Now, on the one hand, $\boldsymbol{p}\cdot\boldsymbol{p}=-E^2+\vec{p}^2$, and on the other hand, using (19), $\boldsymbol{p}\cdot\boldsymbol{p}=m^2\boldsymbol{u}\cdot\boldsymbol{u}=-m^2$. Thus

$$E=\sqrt{m^2+\vec{p}^2}. \tag{27}$$

Thus if $\vec{p}=0$, $E=m$ (i.e. $E=mc^2$ if reinstating c).

4-force and equation of motion. The SR equation of motion is

$$\boldsymbol{f}=m\boldsymbol{a} \tag{28}$$

where $\boldsymbol{f}$ is the 4-force. This equation is the SR analogue of Newton's 2nd law in nonrelativistic classical mechanics. Equivalent forms are $\boldsymbol{f}=d\boldsymbol{p}/d\tau$ and $\boldsymbol{f}=m\,d\boldsymbol{u}/d\tau$. Note that although this is a 4-vector equation, and thus represents 4 separate equations, only 3 of them are independent (as in Newtonian mechanics), since (22) and (28) imply the constraint

$$\boldsymbol{f}\cdot\boldsymbol{u}=0 \tag{29}$$

Next, define the 3-force $\vec{F}$ by

$$\frac{d\vec{p}}{dt}\equiv\vec{F}. \tag{30}$$

This equation has the same form as Newton's 2nd law except that $\vec{p}$ takes the relativistic form $m\gamma\vec{v}$. We have

$$\vec{f}=\frac{d\vec{p}}{d\tau}=\frac{dt}{d\tau}\frac{d\vec{p}}{dt}=\gamma\vec{F}. \tag{31}$$

Next, using (29) and (18) gives

$$0=\boldsymbol{f}\cdot\boldsymbol{u}=-f^0u^0+\vec{f}\cdot\vec{u}=\gamma(-f^0+\vec{f}\cdot\vec{v}), \tag{32}$$

and so

$$f^0=\vec{f}\cdot\vec{v}=\gamma\vec{F}\cdot\vec{v}. \tag{33}$$

Thus we have expressed the 4-force $\boldsymbol{f}$ in terms of the 3-force $\vec{F}$:

$$f^\alpha = (\gamma \vec{F} \cdot \vec{v}, \gamma \vec{F}). \tag{34}$$

The time component of $\boldsymbol{f}$ can also be written

$$f^0 = \frac{dp^0}{d\tau} = \frac{dE}{d\tau} = \gamma \frac{dE}{dt}. \tag{35}$$

Comparing with (33) then gives

$$\frac{dE}{dt} = \vec{F} \cdot \vec{v}. \tag{36}$$

In conclusion, we have rewritten the relativistic equation of motion (28) as Eqs. (30) and (36) (to derive the latter we invoked the constraint (29)). We have seen these equations before in Newtonian mechanics, but note that here the relativistic expressions for E and $\vec{p}$ have to be used. However, in the limit $|\vec{v}| \ll 1$ the equations reduce to the Newtonian versions.

Example: Worldline of constant 4-acceleration

Consider a particle worldline defined parametrically by

$$\begin{aligned} t(\sigma) &= a^{-1} \sinh \sigma, & (37) \\ x(\sigma) &= a^{-1} \cosh \sigma & (38) \end{aligned}$$

where a is some constant whose meaning we will soon identify. As the parameter σ changes from $-\infty$ to $+\infty$, the particle traces out a hyperbola given by

$$x^2 - t^2 = a^{-2}. \tag{39}$$

This is the motion of an accelerated particle because the worldline is not straight. The particle comes in from $x = +\infty$, moving leftward until it arrives at $x = a^{-1}$ when it turns around and moves right again.

We want to find the particle's 4-velocity and 4-acceleration. For this purpose we first need to parameterize the particle's motion in terms of its proper time τ. For an infinitesimal section of the worldline,

$$\begin{aligned} d\tau^2 &= dt^2 - dx^2 = (a^{-1} \cosh \sigma \, d\sigma)^2 - (a^{-1} \sinh \sigma \, d\sigma)^2 \\ &= (a^{-1} d\sigma)^2, & (40) \end{aligned}$$

i.e. $d\tau = a^{-1} d\sigma$. Choosing $\tau = 0$ for $\sigma = 0$ gives $\tau = a^{-1}\sigma$. So we can parameterize the motion as

$$\begin{aligned} t(\tau) &= a^{-1} \sinh(a\tau), & (41) \\ x(\tau) &= a^{-1} \cosh(a\tau). & (42) \end{aligned}$$

The components of the particle's 4-velocity are

$$\begin{aligned} u^t &= \frac{dt}{d\tau} = \cosh(a\tau), & (43) \\ u^x &= \frac{dx}{d\tau} = \sinh(a\tau). & (44) \end{aligned}$$

This has the correct normalization (cf. (19)):

$$\boldsymbol{u}\cdot\boldsymbol{u} = -(u^t)^2 + (u^x)^2 = -\cosh^2(a\tau) + \sinh^2(a\tau) = -1. \tag{45}$$

If one draws $\boldsymbol{u}$ one sees it is tangent to the worldline at each point. Note however that the fact that $\boldsymbol{u}$ is a unit vector does not imply that its *apparent* length as seen on a figure is constant and equal to 1 at each point. In fact, both $|u^t|$ and $|u^x|$ approach ∞ as $\tau \to \pm\infty$, but in such a way that the difference $(u^x)^2 - (u^t)^2 = -1$ always. In conclusion, do not confuse the 4-vector's *apparent* (Euclidean) length as seen on a plot with its *true* length in the non-Euclidean (pseudo-Euclidean) flat spacetime.

The particle's 3-velocity is

$$v^x = \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} = \frac{\sinh(a\tau)}{\cosh(a\tau)} = \tanh(a\tau). \tag{46}$$

Since the tanh function always has absolute value less than 1, $|v^x| < 1$ always, but $|v^x|$ approaches 1 (i.e. the speed of light) for $\tau \to \pm\infty$. As expected $v^x < 0$ for $\tau < 0$ and > 0 for $\tau > 0$, and vanishes at the turning point.

The components of the 4-acceleration $\boldsymbol{a}$ are

$$a^t = \frac{du^t}{d\tau} = a\sinh(a\tau), \tag{47}$$

$$a^x = \frac{du^x}{d\tau} = a\cosh(a\tau). \tag{48}$$

Thus

$$\boldsymbol{a}\cdot\boldsymbol{a} = -(a^t)^2 + (a^x)^2 = a^2. \tag{49}$$

Thus the particle has a constant-magnitude 4-acceleration given by the constant a. The 4-force needed to accelerate the particle along its worldline is $\boldsymbol{f} = m\boldsymbol{a}$ where m is the particle's mass. We can check that $\boldsymbol{u}$ and $\boldsymbol{a}$ are orthogonal as required by (22):

$$\boldsymbol{a}\cdot\boldsymbol{u} = -a\sinh(a\tau)\cosh(a\tau) + a\cosh(a\tau)\sinh(a\tau) = 0. \tag{50}$$

Again it's worth noting that this does not imply that $\boldsymbol{a}$ and $\boldsymbol{u}$ *look* orthogonal when drawn on the plane.

Variational principle for free-particle motion, geodesics

A free particle experiences no force, in which case the equation of motion (28) reduces to $\boldsymbol{a} = 0$, i.e.

$$\frac{d^2x^\alpha}{d\tau^2} = 0. \tag{51}$$

This is equivalent to Newton's 1st law, i.e. that the 3-velocity $\vec{v} = d\vec{x}/dt$ of the particle is constant in any inertial frame (this can be seen by writing (51) as $d\boldsymbol{u}/d\tau = 0$ and using (18)). The worldline of particle free from all external, non-gravitational force is called a **geodesic**. We will now show that (51) also can be derived from the following **variational principle** for free-particle motion:

The worldline of a free particle between two timelike separated points extremizes the proper time between them.

We have already seen an example of this principle in the twin paradox discussion. To prove it, we consider an arbitrary timelike worldline connecting two timelike separated points A and B. We parameterize the wordline with a parameter σ that takes the value $\sigma = 0$ at A and $\sigma = 1$ at B. Thus the proper time for the worldline is

$$\begin{aligned}\tau_{AB} &= \int_A^B d\tau = \int_A^B [dt^2 - dx^2 - dy^2 - dz^2]^{1/2} \\ &= \int_0^1 d\sigma \left[-\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta\right]^{1/2} \equiv \int_0^1 d\sigma\, L = \int_0^1 d\sigma \frac{d\tau}{d\sigma}\end{aligned} \tag{52}$$

where we have defined $\dot{x}^\alpha \equiv dx^\alpha/d\sigma$. The condition for an extremum is that the change in τ_{AB} vanishes to first order when the worldline between the two points is varied, i.e.

$$\delta\tau_{AB} = 0. \tag{53}$$

Mathematically, the problem is identical to finding the extremum of an action, with σ playing the role of a time coordinate and the x^α being position coordinates. Thus the solution must satisfy the Euler-Lagrange equations for the coordinates, i.e.

$$\frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = 0. \tag{54}$$

In general, L is a function of the x^α and their derivatives $\dot{x}^\alpha$. However, in the case studied here L doesn't depend on the x^α, so the Euler-Lagrange equations reduce to

$$\frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\alpha} = 0. \tag{55}$$

We get

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}^\lambda} &= \frac{\partial}{\partial \dot{x}^\lambda}\left[-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right]^{1/2} = \frac{1}{2L}\frac{\partial}{\partial \dot{x}^\lambda}\left[-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right] \\ &= -\frac{1}{2L}(\eta_{\lambda\beta}\dot{x}^\beta + \eta_{\alpha\lambda}\dot{x}^\alpha) = -\frac{1}{L}\eta_{\lambda\beta}\dot{x}^\beta = -\frac{1}{L}\eta_{\lambda\lambda}\dot{x}^\lambda\end{aligned} \tag{56}$$

where we used that $\eta_{\alpha\beta}$ is symmetric and diagonal (note that there is no sum over λ in the last expression as there is more than two identical indices). Inserting this result into (55) and using that $\eta_{\alpha\alpha}$ is just a constant gives

$$\frac{d}{d\sigma}\left[\frac{1}{L}\frac{dx^\alpha}{d\sigma}\right] = 0. \tag{57}$$

Now we use $L = d\tau/d\sigma$ (see (52)) which implies that the expression inside square brackets is just $dx^\alpha/d\tau$. Multiplying (57) with $d\sigma/d\tau$ then turns it into (51). QED.

Photons (null geodesics)

Worldline parametrizations

Massless particles (i.e. particles with mass $m = 0$) move at the speed of light ($V = 1$). Their worldlines are null worldlines [2], i.e. $ds^2 = 0$ along them. Although the following discussion is valid for any massless particle, in practice we'll only consider photons in this course.

[2] Actually null geodesics, since no force can influence a photon's trajectory, unless it is absorbed or emitted by a charged particle.

Since the proper time τ doesn't change along null worldlines ($d\tau^2 = -ds^2 = 0$), it can not be used to parametrize the motion along such worldlines. Thus we need to use some other parameter, say λ, so $x^\alpha = x^\alpha(\lambda)$. Given such a parametrization the tangent vector to a point on the worldline is $\boldsymbol{u}$ with components defined as

$$u^\alpha = \frac{dx^\alpha}{d\lambda}. \tag{58}$$

Consider the null worldline

$$x = t + \text{const.} \tag{59}$$

representing a photon moving in the positive x direction. This can be written parametrically as

$$x^\alpha = u^\alpha \lambda + \text{const.} \tag{60}$$

where λ is the parameter and $u^\alpha = (1, 1, 0, 0)$. Note that $\boldsymbol{u}$ is a null vector:

$$\boldsymbol{u} \cdot \boldsymbol{u} = \eta_{\alpha\beta} u^\alpha u^\beta = -1^2 + 1^2 + 0^2 + 0^2 = 0. \tag{61}$$

This contrasts with the tangent vector $\boldsymbol{u} = d\boldsymbol{x}/d\tau$ for massive particles, which was (unit) timelike ($\boldsymbol{u} \cdot \boldsymbol{u} = -1$). Since with the parametrization (60) the tangent vector u^α is indep. of λ the equation of motion for the worldline can be written

$$\frac{d\boldsymbol{u}}{d\lambda} = 0 \tag{62}$$

which takes the same form as the equation of motion for a free massive particle ($d\boldsymbol{u}/d\tau = 0$). Parameters for which the equation of motion of massless particles take the same form as for massive particles are called *affine* parameters. Affine parameters are not unique: If λ is affine, then so is $\lambda' = k\lambda$ where k is an arbitrary constant.[3]

Energy, momentum, frequency, wavevector

In any inertial frame, the following relations hold:

$$E = \hbar\omega, \tag{63}$$
$$\vec{p} = \hbar\vec{k}, \tag{64}$$
$$\omega = |\vec{k}|, \tag{65}$$

where E, $\vec{p}$, and $\vec{k}$ are the photon's energy, 3-momentum, and wave 3-vector, respectively, in the inertial frame. From this we get

$$E = \hbar\omega = \hbar|\vec{k}| = |\vec{p}|. \tag{66}$$

The photon's 4-momentum $\boldsymbol{p}$ has components $p^\alpha = (E, \vec{p}) = (\hbar\omega, \hbar\vec{k}) \equiv \hbar k^\alpha$, where k^α are the components of the photon's wave 4-vector $\boldsymbol{k}$. Thus

$$\boldsymbol{p} = \hbar\boldsymbol{k}. \tag{67}$$

[3] As an example of a parameter that is not affine, consider the parametrization of (59) as $x^\alpha = d^\alpha\sigma^3$+const. where σ is the parameter and $d^\alpha = (1, 1, 0, 0)$. Then the tangent vector would be $u^\alpha = dx^\alpha/d\sigma = 3d^\alpha\sigma^2$. While this is still a null vector, the equation of motion would be $d\boldsymbol{u}/d\sigma = 6\boldsymbol{d}\sigma \neq 0$, not of the same form as a free massive particle.

We have

$$\boldsymbol{p} \cdot \boldsymbol{p} = -E^2 + \vec{p}^2 = -(\hbar\omega)^2 + (\hbar\vec{k})^2 = 0, \tag{68}$$

so $\boldsymbol{p}$ is a null vector (and thus so is $\boldsymbol{k}$). Since we found earlier that the 4-momentum of a massive particle with mass m satisfies $\boldsymbol{p} \cdot \boldsymbol{p} = -m^2$ we can interpret (68) as saying that the photon has zero mass. Both $\boldsymbol{p}$ and $\boldsymbol{k}$ are tangent to the world line of the photon (e.g. for the worldline $t = +x + \text{const.}$ we have $p^0 = +p^1$, $k^0 = +k^1$). So by rescaling the affine parameter appropriately the tangent vector $\boldsymbol{u}$ could be chosen to coincide with $\boldsymbol{p}$ or $\boldsymbol{k}$.

Doppler shift

Consider a photon source at rest in an inertial frame S. The source emits photons in all directions. Let the frequency of the photon emitted be ω in S. Now consider another frame S' that moves wrt S with velocity $-V$ along the x axis. Thus the photon source is moving with velocity $+V$ along the x' direction in S'. In S', what is the frequency ω' of photon emitted at an angle α' relative to the direction of motion of the source? This problem can be solved by using the Lorentz transformation to transform the components of the photon's wave 4-vector from S to S'. In S the components are $(\omega, \vec{k})$ and in S' they are $(\omega', \vec{k}')$. The Lorentz transformation gives

$$\omega = \gamma(\omega' - V k^{x'}) \tag{69}$$

Since in S' the photon's direction is at an angle α' to the x' direction, we get $\cos\alpha' = k^{x'}/\omega'$. Inserting this into (69) and solving for ω' gives

$$\omega' = \omega \frac{\sqrt{1 - V^2}}{1 - V\cos\alpha'}. \tag{70}$$

This is called the relativistic Doppler shift. To get some further insight, assume for concreteness that $V > 0$ and consider the following two cases:

$$\begin{aligned}
&\alpha' = 0 \quad \text{(photon emitted in same direction as source moving):} \\
&\omega' \;=\; \omega\sqrt{\frac{1+V}{1-V}} > \omega \quad \text{(blueshift)} \qquad (71) \\
&\alpha' = \pi \quad \text{(photon emitted in opposite direction as source moving):} \\
&\omega' \;=\; \omega\sqrt{\frac{1-V}{1+V}} < \omega \quad \text{(redshift).} \qquad (72)
\end{aligned}$$

The nonrelativistic limit $|V| \ll 1$ can be studied as usual by Taylor-expanding in V.

Observers and observations

See handout from Hartle.

Newtonian gravity, the equivalence principle and the Rindler metric

Summary

- Newtonian gravity compared to electrostatics
- The gravitational potential
- Dynamics and the equivalence principle
- Uniform acceleration

Analogy between Newtonian gravity and electrostatics. The gravitational potential.

The mathematical structure of Newtonian gravity is the same as for electrostatics. The reason is that both Newton's gravitational force and the electrostatic (Coulomb) force vary like $1/r^2$ and both forces are proportional to the relevant "charges" of the objects ("sources") involved (the "gravitational charge" of an object is just its mass while the electric charge is what you think it is). The only difference between Newtonian gravity and electrostatics is that electric charges can have either positive or negative sign while gravitational charges (i.e. masses) can only have positive sign. Thus the electrostatic force is either repulsive or attractive depending on the relative sign of the electric charges involved, while the gravitational force is always attractive (as there is an overall minus sign difference in the definition of the forces; cf. table below).

The analogue of the electrostatic potential Φ_e is the gravitational potential Φ_g. Furthermore, ρ_e is the electric charge density and ρ_g is the gravitational charge (i.e. mass) density. The electrostatic analogue of Newton's gravitational constant G is the constant $k = 1/(4\pi\varepsilon_0)$. The following table illustrates the analogies in detail (note that the field equation for the electric potential is just the Poisson equation; its right hand side is more commonly rewritten as $-4\pi k\rho_e = -\rho_e/\varepsilon_0$).

Dynamics. The equivalence of gravitational mass and inertial mass.

For an object that experiences only a gravitational force, the total force $\vec{F}_t$ on the body is given by

$$\vec{F}_t = -m\nabla\Phi \tag{1}$$

(here and in the following we drop the subscript g on gravitational quantities) while from Newton's 2nd law,

$$\vec{F}_t = m\vec{a}. \tag{2}$$

Concept	Newtonian gravity	Electrostatics
Force between two sources	$\vec{F}_g = -\frac{GmM}{r^2}\hat{e}_r$	$\vec{F}_e = +\frac{kqQ}{r^2}\hat{e}_r$
Force from potential	$\vec{F}_g = -m\nabla\Phi_g$	$\vec{F}_e = -q\nabla\Phi_e$
Potential due to continuous distribution of sources	$\Phi_g(\vec{r}) = -G\int d^3r'\frac{\rho_g(\vec{r}')}{\lvert\vec{r}-\vec{r}'\rvert}$	$\Phi_e(\vec{r}) = k\int d^3r'\frac{\rho_e(\vec{r}')}{\lvert\vec{r}-\vec{r}'\rvert}$
Potential due to single source at $\vec{r}' = 0$	$\Phi_g(\vec{r}) = -\frac{GM}{r}$	$\Phi_e(\vec{r}) = \frac{kQ}{r}$
Field equation for the potential	$\nabla^2\Phi_g = 4\pi G\rho_g$	$\nabla^2\Phi_e = -4\pi k\rho_e$

Thus the acceleration is given by

$$\vec{a} = -\nabla\Phi. \tag{3}$$

Note that the mass m cancels out. Thus the acceleration of an object due to gravity is independent of the mass of the object. This differs from the acceleration due to a Coulomb force because the latter is independent of the mass:

$$\vec{F} = -q\nabla\Phi_e \quad \Rightarrow \quad \vec{a} = -\frac{q}{m}\nabla\Phi_e. \tag{4}$$

The result (3), although innocuous-looking, is a consequence of a profound property of our world, namely that

$$\text{gravitational mass } m_G = \text{ inertial mass } m_I. \tag{5}$$

To understand what is meant by this, rewrite Newton's gravitational force law and Newton's 2nd law for a body as

$$\vec{F}_g = -m_G\nabla\Phi, \tag{6}$$

$$\vec{F}_t = m_I\vec{a}. \tag{7}$$

Here m_G, the gravitational mass (which we called gravitational charge earlier), affects the strength of one particular force, the gravitational force $\vec{F}_g$, acting on the body, while m_I, the inertial mass, determines the inertia of the body when subjected to a *total* force $\vec{F}_t$, *of whatever origin(s)*. Within Newtonian mechanics there seems to be no theoretical reason to expect m_G and m_I to be related to each other, since they represent very different things. However, within experimental accuracy[1] one finds empirically that they are equal. Therefore one writes $m_G = m_I \equiv m$ which is just called the mass of the body.

The result (3) means that all bodies, regardless of their mass, fall with the same acceleration in a gravitational field. Thus any two bodies that have the same initial conditions (i.e. same initial position and same initial velocity) will follow exactly the same trajectory in spacetime when they are not influenced by any other forces than gravity. This opens up the possibility that this trajectory can be a property of the geometry of spacetime rather than being due to a force acting on the body. This is the reinterpretation of gravity that is embodied in Einstein's general theory of relativity.

[1]According to Hartle (2003), the most accurate experiments involve measurements of the accelerations of the Earth and the Moon due to the gravitational field of the Sun, for which the accelerations are found to be equal to an accuracy of $\sim 10^{-13}$.

The Equivalence Principle.

Consider an observer falling freely in a uniform gravitational field. Other objects nearby that experience the same gravitational field will be at rest relative to the observer or move at a constant velocity relative to the observer. But this would also be the case if the gravitational field was absent! This leads to the 1st formulation of the equivalence principle: *no (local) experiment can distinguish between free-fall and the absence of a gravitational field.* The gravitational field is "eliminated" by falling freely; observers in free fall feel weightless.

Now consider a different situation in which observers carry out experiments inside a laboratory accelerating upwards with acceleration a = g in empty space. Objects dropped inside the laboratory would appear to accelerate at a rate g towards the bottom of the laboratory. In fact dropped objects would behave in the same way as for observers carrying out experiments inside a stationary laboratory on Earth (gravitational field of strength g downwards). No such (local) experiment can tell the difference between between these two situations. This leads us to the 2nd formulation of the equivalence principle: *no (local) experiment can distinguish a uniform acceleration from a uniform gravitational field.*

We emphasise the inclusion of the word "local" here. The equivalence principle applies between uniform gravitational fields and uniform acceleration. But in nature no gravitational field is uniform globally, thus if we move far enough or wait long enough we will observe the non-uniformity of the field. Never-the-less, over local regions of space-time the field can be approximately uniform - under these conditions the equivalence principle applies.

Uniform Acceleration

The equivalence principle motivates us to study a non-inertial frame of reference based on uniform acceleration. This will lead us to our first non-inertial metric – the Rindler metric. Although strictly this scenario is physically unrealizable, it is quite instructive in demonstrating a number of basic issues and concepts.

Consider the following parametrisation of the Minkowski coordinates (t, x, y, z) in terms of what are called Rindler coordinates (τ, ξ):

$$t = \frac{1}{a} e^{a\xi} \sinh(a\tau), \qquad x = \frac{1}{a} e^{a\xi} \cosh(a\tau), \tag{8}$$

with y and z unchanged. For constant ξ, these coordinates parametrise uniform acceleration along the x-axis, as can be seen by making the substitutions $a' = ae^{-a\xi}$ and $\tau' = \tau e^{a\xi}$ and comparing with Eqs 40 - 41 on page 18 of the notes I. Hence the coordinate τ is the proper time experienced by an observer uniformly accelerating with acceleration a, or equivalently, an observer "stationary" at $\xi = 0$. The transformation in Eq.8 is invertible, giving:

$$\tau = \frac{1}{a} \mathsf{arctanh}(t/x), \qquad \xi = \frac{1}{2a} \ln[a^2(x^2 - t^2)]. \tag{9}$$

The (τ, ξ) coordinates take values $-\infty < \tau, \xi < \infty$ and cover the region $x > |t|$, referred to as the right Rindler wedge (see Figure).

The Minkowski metric can be rewritten in these new coordinates by recognizing that $dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$, with the primed coordinates being $x'^\nu = (\tau, \xi, y, z)$; this gives:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = e^{2\kappa\xi} \left(-d\varsigma^2 + d\xi^2 \right) + dy^2 + dz^2. \tag{10}$$

This is the Rindler metric and is our first example of a non-inertial metric. Through the equivalence principle we can associate it with a uniform gravitational field.

The Rindler metric exhibits a number of generic features of non-inertial metrics:

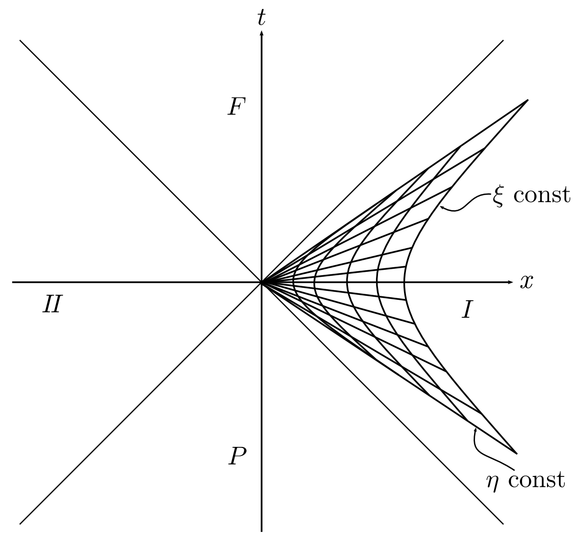

Figure 1: The Rindler coordinate system in the right Rindler wedge; the coordinates y and z are suppressed. Worldlines of uniformly accelerating observers are the lines of constant ξ. Lines of constant $\eta \equiv \tau$ are equal time slices according to a clock with constant $\xi = 0$.

- locally the metric can appear flat for some choice of ξ – in particular $\xi = 0$;
- all observers locally measure $c = 1$, but this is not necessarily true globally;
- clocks run slower as the local acceleration increases, i.e. the proper time τ' for an observer stationary at ξ' (where the local acceleration is $a' = ae^{-\xi' a}$) is $\tau' = \tau e^{\xi' a}$.

How does a "dropped" object behave? Let's calculate the relative velocity between an object released at $\tau_0 = 0$ from $\xi_0 = 0$ and an observer, stationary at $\xi = -|\xi_{obs}|$. We can use the relationship $|V| = \sqrt{1 - (\tilde{\mu}.\tilde{u}_{obs})^{-2}}$ (see page 9 of notes II), where $|V|$ is the relative velocity and $\tilde{\mu}$ is the 4-velocity of the object and $\tilde{u}_{obs}$ is the 4-velocity of the stationary observer. Observers located at constant $(\xi, y, z) = (\xi_0, y_0, z_0)$ follow a trajectory given by the tangent vector

$$x^\mu(\tau) = \frac{1}{a} e^{a\xi_0} \left[\sinh(a\tau), \cosh(a\tau), y_0, z_0\right] \tag{11}$$

from which the 4-velocities can be calculated in Minkowski coordinates. For our situation

$$\tilde{\mu}.\tilde{u}_{obs} = (1, 0, 0, 0).(\cosh(a'\tau'), \sinh(a'\tau'), 0, 0) = \cosh(a'\tau'), \tag{12}$$

where we have used that an object released at $\tau_0 = 0$ will be stationary in Minkowski coordinates. Hence we find

$$|V| = \tanh(a'\tau') = \tanh(a\tau) \tag{13}$$

with $\tau \geq 0$. For $\tau << 1$, $|V| = a\tau$, the Newtonian limit. However as $\tau \to \infty$, $|V| \to 1$, the relative velocity asymptotes to the speed of light, even as the local acceleration of the stationary observers diverges.

Notice that according to the Rindler coordinates observers can never cross the *horizon* at $x = t$. Reaching the horizon would require both ξ and τ to go to infinity. However, if we go back to Minkowski coordinates it is clear that our dropped object will cross this horizon at $t = 1/a$ because the object is stationary at $x = 1/a$ in Minkowski coordinates.

Motion of particles in general relativity: the geodesic equation

Summary

- Gravity as a geometric property of spacetime
- The geodesic equation and its derivation
- Christoffel symbols

Particles[1] move on timelike worldlines that can be specified parametrically by 4 functions $x^\alpha(\tau)$ of the proper time τ along them. The proper time along such a worldline connecting two timelike separated spacetime points A and B is given by

$$\tau_{AB} = \int_A^B \sqrt{-g_{\alpha\beta}dx^\alpha dx^\beta} \tag{1}$$

where the integral is taken along the worldline. Compared to special relativity (SR), the only change is that the metric $\eta_{\alpha\beta}$ describing flat spacetime has been replaced by the metric $g_{\alpha\beta}$ describing the curved spacetime in general relativity (GR). The proper time τ_{AB} is the time elapsed on a clock carried along the given worldline from A to B.

In GR gravity is not a force but a geometric property of spacetime. Particles that are only affected by gravity are therefore called **free** particles. Furthermore, a **test** particle is a particle that has such a small (nonzero) mass that its own effect on the curvature of spacetime can be neglected.

In Special Relativity, we found that there is a **variational principle for the motion of free test particles**:

The worldline of a free test particle between two timelike separated spacetime points extremizes the proper time between them.

We now generalise this for any metric, which is the **geodesic equation**:

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \tag{2}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the **Christoffel symbols** given by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\lambda}(\partial_\beta g_{\gamma\lambda} + \partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\beta\gamma}). \tag{3}$$

[1] As usual, the word "particle" is used in a rather broad sense to refer to an object with nonzero mass, e.g. an observer, a planet, a spaceship, a satellite etc. Note that light rays are not included in this definition since photons have zero mass.

There are several ways to derive the geodesic equation from the variational principle. Hartle discusses one method, in which the worldline is parametrized as $x^\alpha(\sigma)$ where σ is some parameter s.t. $\sigma = 0$ at point A and $\sigma = 1$ at point B, giving

$$\tau_{AB} = \int d\tau = \int_0^1 d\sigma\, \frac{d\tau}{d\sigma} = \int_0^1 d\sigma \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}} \equiv \int d\sigma\, L. \tag{4}$$

Thus the Lagrangian is

$$L = \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}} \tag{5}$$

and the variational principle leads to the standard Euler-Lagrange equations

$$\frac{d}{d\sigma}\frac{\partial L}{\partial(dx^\alpha/d\sigma)} - \frac{\partial L}{\partial x^\alpha} = 0. \tag{6}$$

When differentiating the Lagrangian, one can use the chain rule to first differentiate the square root, giving a factor $1/(2\sqrt{\ldots}) = 1/(2L)$. Then, using that $L = d\tau/d\sigma$ one can rewrite derivatives wrt σ in terms of derivatives wrt τ, i.e. $(1/L)df/d\sigma = (d\sigma/d\tau)(df/d\sigma) = df/d\tau$. This was the method we used when we considered the variational principle in special relativity and used it to derive the geodesic equation in that case, namely $d^2x^\alpha/d\tau^2 = 0$.

Another, perhaps more convenient method uses proper time itself to parametrize the worldlines. Thus one writes

$$\tau_{AB} = \int d\tau = \int d\tau\, \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}} \equiv \int d\tau\, L. \tag{7}$$

Thus the Lagrangian in this case is

$$L = \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}. \tag{8}$$

The Euler-Lagrange equations again take their standard form[2]

$$\frac{d}{d\tau}\frac{\partial L}{\partial(dx^\alpha/d\tau)} - \frac{\partial L}{\partial x^\alpha} = 0. \tag{9}$$

However, one can instead work with an alternative set of Euler-Lagrange equations for a simpler Lagrangian $L' = L^2$:

$$L' = -g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}. \tag{10}$$

To see this, we consider the Euler-Lagrange equation (9). First use the chain rule to write

$$\frac{\partial L}{\partial(dx^\alpha/d\tau)} = \frac{1}{2L}\frac{\partial L'}{\partial(dx^\alpha/d\tau)} \tag{11}$$

[2] However, in this case there is a subtlety involved in the derivation. Recall that in the standard derivation of the Euler-Lagrange equations in non-relativistic classical mechanics, the integration variable is the time t and the integration limits are two *fixed* times t_A and t_B, the same for all paths being varied over. In the variational principle considered here, the role of time is played by the parameter that parametrizes the worldlines. When this parameter is chosen to be the proper time τ, different worldlines connecting A and B will generally take different proper times to traverse and thus in this case it is not possible to fix both integration limits.

Now, since $L = d\tau/d\tau = 1$ for all τ, $d/d\tau$ goes straight through L:

$$\frac{d}{d\tau}\frac{\partial L}{\partial(dx^\alpha/d\tau)} = \frac{d}{d\tau}\frac{1}{2L}\frac{\partial L'}{\partial(dx^\alpha/d\tau)} = \frac{1}{2L}\frac{d}{d\tau}\frac{\partial L'}{\partial(dx^\alpha/d\tau)}. \tag{12}$$

Furthermore,

$$\frac{\partial L}{\partial x^\alpha} = \frac{1}{2L}\frac{\partial L'}{\partial x^\alpha}. \tag{13}$$

Thus the factor $1/(2L)$ can be cancelled in the original Euler-Lagrange equations, leaving

$$\frac{d}{d\tau}\frac{\partial L'}{\partial(dx^\alpha/d\tau)} - \frac{\partial L'}{\partial x^\alpha} = 0, \tag{14}$$

which are just the Euler-Lagrange equation for the alternative Lagrangian L', which is simpler to work with than L due to the absence of a square root.

Appendix: Deriving the geodesic equation from Euler-Lagrange equation

Remember that the equation for the **Christoffel symbols** $\Gamma^\alpha_{\beta\gamma}$ is

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\lambda}(\partial_\beta g_{\gamma\lambda} + \partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\beta\gamma}). \tag{15}$$

The Lagrangian we have defined is:

$$L = \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}} \tag{16}$$

and the variational principle leads to the standard Euler-Lagrange equations

$$\frac{d}{d\sigma}\frac{\partial L}{\partial(dx^\alpha/d\sigma)} - \frac{\partial L}{\partial x^\alpha} = 0. \tag{17}$$

However, one can instead work with an alternative set of Euler-Lagrange equations for a simpler Lagrangian $L' = L^2$:

$$L' = -g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{18}$$

where by chain rule we see that

$$\frac{\partial L}{\partial(dx^\alpha)/d\tau)} = \frac{1}{2L}\frac{\partial L'}{\partial(dx^\alpha/d\tau)}, \tag{19}$$

and

$$\frac{\partial L}{\partial x^\alpha} = \frac{1}{2L}\frac{\partial L'}{\partial x^\alpha}. \tag{20}$$

Inserting the Lagrangian we have chosen into this, we get

$$\frac{d}{d\tau}\left[\frac{1}{2L}\frac{-\partial(g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau})}{\partial dx^\mu/d\tau}\right] = \tfrac{1}{2L}\frac{\partial g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}{\partial x^\mu} \tag{21}$$

$$\frac{d}{d\tau}\left[\frac{1}{2L}\left(g_{\mu\beta}\frac{dx^\beta}{d\tau} + g_{\alpha\mu}\frac{dx^\alpha}{d\tau}\right)\right] = \frac{1}{2L}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{22}$$

$$\frac{d}{d\tau}\left(\frac{1}{L}g_{\mu\alpha}\frac{dx^\alpha}{d\tau}\right) = \frac{1}{2L}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{23}$$

$$\frac{1}{L}\frac{\partial g_{\mu\alpha}}{d\tau}\frac{dx^\alpha}{d\tau} + \frac{1}{L}g_{\mu\alpha}\frac{d^2x^\alpha}{d\tau} = \frac{1}{2L}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{24}$$

$$\frac{1}{L}\frac{\partial g_{\mu\alpha}}{dx^\beta}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau} + \frac{1}{L}g_{\mu\alpha}\frac{d^2x^\alpha}{d\tau} = \frac{1}{2L}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{25}$$

$$\frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{dx^\beta} + \frac{\partial g_{\mu\beta}}{dx^\alpha}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + g_{\mu\alpha}\frac{d^2x^\alpha}{d\tau} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{26}$$

This finally gives us

$$g_{\mu\alpha}\frac{d^2x^\alpha}{d\tau^2} = -\frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{dx^\beta} + \frac{\partial g_{\mu\beta}}{dx^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \tag{27}$$

or similarly

$$\frac{d^2x^\alpha}{d\tau^2} = -\frac{1}{2}g^{\mu\alpha}\left(\frac{\partial g_{\alpha\mu}}{dx^\beta} + \frac{\partial g_{\mu\beta}}{dx^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\,, \tag{28}$$

which is equivalent to the geodesic equation.

Newtonian limit of the geodesic equation and Einstein's field equation

Summary

- Newtonian gravity
- Recovering the force equation in the Newtonian limit
- Recovering the Poisson equation in the Newtonian limit

Before delving into concrete applications of GR it is instructive to first see how GR reduces to the familiar Newtonian theory of gravity in the so-called Newtonian limit. The discussion that follows is mainly based on Carroll, Secs. 4.1-4.2.

The Newtonian counterpart of the geodesic equation is the force equation

$$\vec{a} = -\nabla\Phi \tag{1}$$

where $\vec{a}$ is the acceleration of the particle under consideration and Φ is the gravitational potential at the position of the particle. The Newtonian counterpart of Einstein's field equation is the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho \tag{2}$$

where $\rho = \rho(\vec{r})$ is the mass density. Our goal here is to show that the geodesic equation and Einstein's field equation lead to respectively (1) and (2) in the **Newtonian limit**, which is defined by the following three requirements:

- the particles are **moving slowly** compared to the speed of light
- the gravitational field is **weak**
- the gravitational field is **static** (i.e. time-independent)

Newtonian limit of the geodesic equation

The geodesic equation is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \tag{3}$$

where the Christoffel symbol is

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta}). \tag{4}$$

We use coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$. In the following, Greek indices take values 0, 1, 2, 3 as usual, while Latin indices will be used for spatial coordinates and thus only take values 1, 2, 3.

The requirement that the particle under consideration moves slowly compared to the speed of light means that

$$\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \left(= \frac{dt}{d\tau} \right). \tag{5}$$

Therefore, in the second term in the geodesic equation we approximate the sum over ρ and σ by keeping only the dominant term having $\rho = \sigma = 0$. Thus the geodesic equation reduces to

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00} \left(\frac{dt}{d\tau} \right)^2 = 0. \tag{6}$$

Furthermore, the requirement that the gravitational field be static means that $\partial_0 g_{\alpha\beta} = 0$. Thus

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\lambda} (\underbrace{\partial_0 g_{0\lambda}}_{=0} + \underbrace{\partial_0 g_{\lambda 0}}_{=0} - \partial_\lambda g_{00}) = -\frac{1}{2} g^{\mu\lambda} \partial_\lambda g_{00}. \tag{7}$$

Next we need to consider the form of the metric tensor. The requirement that the gravitational field be weak means spacetime curvature is small, so one can write $g_{\mu\nu}$ as the sum of the metric $\eta_{\mu\nu}$ of flat spacetime plus a small perturbation $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{8}$$

Since $\eta_{\mu\nu}$ takes the form $\mathrm{diag}(-1, 1, 1, 1)$ in our coordinates, the requirement that $h_{\mu\nu}$ be small in comparison means $|h_{\mu\nu}| \ll 1$. Due to this smallness condition it is sufficient to work to first order in the perturbation $h_{\mu\nu}$. We also need $g^{\mu\nu}$ which by definition of being the inverse of the metric satisfies

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu{}_\sigma. \tag{9}$$

To first order in h, $g^{\mu\nu}$ is given by

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{10}$$

where we have defined $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. (Note that in this definition we have used the flat-spacetime (inverse) metric $\eta^{\mu\nu}$ to raise indices. This is accurate enough, because if we instead (as we really should) used the exact (inverse) metric $g^{\mu\nu}$ to raise indices, we'd just get corrections that are higher order in h, which we neglect anyway.) This gives

$$\begin{aligned} g^{\mu\nu} g_{\nu\sigma} &= (\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\nu\sigma} + h_{\nu\sigma}) = \eta^{\mu\nu}\eta_{\nu\sigma} - h^{\mu\nu}\eta_{\nu\sigma} + \eta^{\mu\nu}h_{\nu\sigma} + O(h^2) \\ &= \delta^\mu{}_\sigma - h^\mu{}_\sigma + h^\mu{}_\sigma + O(h^2) = \delta^\mu{}_\sigma + O(h^2) \end{aligned} \tag{11}$$

which confirms the correctness of (10). Thus

$$\Gamma^\mu_{00} = -\frac{1}{2} g^{\mu\lambda} \partial_\lambda g_{00} = -\frac{1}{2} (\eta^{\mu\lambda} - h^{\mu\lambda}) \partial_\lambda (\eta_{00} + h_{00}) = -\frac{1}{2} \eta^{\mu\lambda} \partial_\lambda h_{00} \tag{12}$$

to first order in h (here we also used that $\partial_\lambda \eta_{\mu\nu} = 0$). Thus the geodesic equation simplifies further to

$$\frac{d^2x^\mu}{d\tau^2} - \frac{1}{2} \eta^{\mu\lambda} \partial_\lambda h_{00} \left(\frac{dt}{d\tau} \right)^2 = 0. \tag{13}$$

Now consider the time ($\mu = 0$) component of this equation. Since $\eta^{0\lambda} = -1$ for $\lambda = 0$ and 0 otherwise, and since $\partial_0 h_{00} = 0$ because the field is static, the second term vanishes in this case, so the $\mu = 0$ component is simply

$$\frac{d^2 t}{d\tau^2} = 0 \tag{14}$$

(which can be integrated to give $dt/d\tau =$ constant). Next consider the spatial components of (13). Since $\eta^{i\lambda} = 1$ if $\lambda = i$ and 0 otherwise, we get for $\mu = i$ in (13):

$$\frac{d^2 x^i}{d\tau^2} - \frac{1}{2}\partial_i h_{00}\left(\frac{dt}{d\tau}\right)^2 = 0. \tag{15}$$

Next, we use that

$$\frac{d^2 x^i}{d\tau^2} = \frac{d^2 t}{d\tau^2}\frac{dx^i}{dt} + \left(\frac{dt}{d\tau}\right)^2 \frac{d^2 x^i}{dt^2} = \left(\frac{dt}{d\tau}\right)^2 \frac{d^2 x^i}{dt^2}. \tag{16}$$

The first equality follows from the chain rule and is proven in Appendix A. In the second equality we used (14). Inserting this into (15) and cancelling the common factor $(dt/d\tau)^2$ gives

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}. \tag{17}$$

Let us compare this to the Newtonian analogue (1) which on component form reads

$$\frac{d^2 x^i}{dt^2} = -\partial_i \Phi. \tag{18}$$

We see that the two equations agree if

$$h_{00} = -2\Phi \tag{19}$$

and thus in the Newtonian limit, g_{00} is related to Newton's gravitational potential Φ by

$$g_{00} = -(1 + 2\Phi). \tag{20}$$

In $c \neq 1$ units this instead becomes $g_{00} = -(1 + 2\Phi/c^2)$ (this is obtained by making the replacement $t \to ct$ in (17)). Thus the weak-field condition can be written $|\Phi|/c^2 \ll 1$ (we already encountered this condition in the discussion of gravitational time dilation based on the equivalence principle).

Newtonian limit of Einstein's field equation

Einstein's field equation is a tensor equation that is usually written

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{21}$$

Here $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar (aka the scalar curvature), $T_{\mu\nu}$ is the energy-momentum tensor, and $\kappa = 8\pi G$. For our purposes, however, it is more convenient to instead start from an equivalent form of Einstein's equation (see Appendix B for a derivation), namely

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}) \tag{22}$$

where $T \equiv T^\lambda{}_\lambda = g^{\lambda\mu}T_{\mu\lambda}$. In the following we also take κ to be an unspecified constant; we will see that the value of κ can be deduced from demanding agreement between GR and Newtonian gravity in the Newtonian limit.

We consider a massive body, like e.g. the Sun or the Earth, to be the source of the energy-momentum tensor $T_{\mu\nu}$. We will work in the rest frame of this body. We assume that the mass density (energy density) ρ of this body is small so that the resulting curved spacetime (described by the metric tensor $g_{\mu\nu}$) will be only weakly perturbed away from flat spacetime (described by $\eta_{\mu\nu}$). Under these conditions it can be shown that

$$T_{00} \approx \rho \tag{23}$$

and all other components of $T_{\mu\nu}$ are small in comparison to T_{00} (see Appendix C). Thus we focus on the 00 component of Einstein's field equation since the other components are much smaller. We have

$$T = T^\lambda{}_\lambda = g^{\lambda\mu}T_{\mu\lambda} \approx g^{00}T_{00} \approx (\eta^{00} - h^{00})T_{00} \approx (\eta^{00} - h^{00})\rho \approx \eta^{00}\rho = -\rho. \tag{24}$$

Here we neglected the term $-h^{00}\rho$ since we expect the perturbation h^{00} to be proportional to the small quantity ρ and thus $h^{00}\rho$ is of second order in ρ, so it can be neglected in comparison with the first order term $-\rho$. Inserting this result for T into the 00 component of (22)

$$R_{00} = \kappa(T_{00} - \frac{1}{2}Tg_{00}) \approx \kappa(\rho - \frac{1}{2}(-\rho)(-1)) = \frac{1}{2}\kappa\rho. \tag{25}$$

Here, since the rhs already contained a factor ρ we could again approximate $g_{00} \approx \eta_{00}$ to leading order.

Next we consider R_{00}. The Ricci tensor $R_{\mu\nu}$ is given by

$$R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu} \tag{26}$$

where

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{27}$$

is the Riemann curvature tensor. Here we're only interested in R_{00}, given by

$$R_{00} = R^\lambda{}_{0\lambda0} = R^0{}_{000} + R^i{}_{0i0} = R^i{}_{0i0} \tag{28}$$

since it follows from (27) that $R^0{}_{000} = 0$ due to pairwise cancellation of terms. So consider

$$R^i_{0j0} = \partial_j\Gamma^i_{00} - \partial_0\Gamma^i_{j0} + \Gamma^i_{j\lambda}\Gamma^\lambda_{00} - \Gamma^i_{0\lambda}\Gamma^\lambda_{j0}. \tag{29}$$

The second term on the rhs vanishes because we're dealing with a time-independent gravitational field. In the third and fourth terms, each of the two factors Γ involves derivatives $\partial g = \partial h$, so both terms are second order in h and are therefore neglected. Thus $R^i{}_{0j0} \approx \partial_j\Gamma^i_{00}$. Then, using also Eq. (12), we get

$$R_{00} = R^i{}_{0i0} \approx \partial_i\Gamma^i_{00} = -\partial_i\frac{1}{2}\eta^{i\lambda}\partial_\lambda h_{00}\left(= -\frac{1}{2}\partial_i\partial^i h_{00}\right) = -\frac{1}{2}\nabla^2 h_{00}. \tag{30}$$

It follows that the 00 component of Einstein's field equation can be written

$$\nabla^2 h_{00} = -\kappa\rho. \tag{31}$$

This becomes, upon writing $h_{00} = -2\Phi$ as found in Eq. (19),

$$\nabla^2 \Phi = \frac{\kappa}{2}\rho. \tag{32}$$

Comparing this to the Newtonian equivalent, Eq. (2), we get agreement by taking

$$\kappa = 8\pi G. \tag{33}$$

Appendix A. Derivation of the first equality in Eq. (16)

This follows from successive applications of the chain rule. Consider a function $F(u)$ where $u = u(x)$. Using the chain rule we have

$$\frac{dF}{dx} = \frac{du}{dx}\frac{dF}{du} \tag{34}$$

and

$$\frac{d^2F}{dx^2} = \frac{d}{dx}\frac{dF}{dx} = \frac{d}{dx}\left(\frac{du}{dx}\frac{dF}{du}\right) = \frac{d^2u}{dx^2}\frac{dF}{du} + \frac{du}{dx}\frac{d}{dx}\frac{dF}{du} = \frac{d^2u}{dx^2}\frac{dF}{du} + \frac{du}{dx}\frac{du}{dx}\frac{d^2F}{du^2} = \frac{d^2u}{dx^2}\frac{dF}{du} + \left(\frac{du}{dx}\right)^2\frac{d^2F}{du^2}. \tag{35}$$

Here we used that $\frac{d}{dx}\frac{dF}{du}$ is on the form (34) with dF/du playing the role of F. If we now take $F = x^i$, $x = \tau$, $u = t$ the first equality in Eq. (16) follows.

Appendix B. Derivation of Eq. (22)

Start from the Einstein equation on the form (21). Raising an index gives

$$g^{\lambda\mu}R_{\mu\nu} - \frac{1}{2}Rg^{\lambda\mu}g_{\mu\nu} = \kappa g^{\lambda\mu}T_{\mu\nu}, \tag{36}$$

i.e.

$$R^{\lambda}{}_{\nu} - \frac{1}{2}R\delta^{\lambda}{}_{\nu} = \kappa T^{\lambda}{}_{\nu}. \tag{37}$$

Contracting this equation (i.e. summing by setting the upper and lower index equal), and using that $\delta^{\lambda}{}_{\lambda} = 4$ as well as the definitions $R^{\lambda}{}_{\lambda} \equiv R$, $T^{\lambda}{}_{\lambda} \equiv T$, gives

$$R - \frac{1}{2}R \cdot 4 = \kappa T, \tag{38}$$

i.e.

$$R = -\kappa T. \tag{39}$$

Inserting this expression for R in Eq. (21) and rearranging then gives Eq. (22).

Appendix C. Derivation of Eq. (23)

In order to establish the relationship between the momentum 4-vector and the stress-energy tensor we need to introduce the number-current 4-vector:

$$\tilde{N} = n\tilde{\mu}, \tag{40}$$

where n is the number density in the rest frame. The time component of $\tilde{N}$ is the number density whilst the spatial components are the number current density. Now a 3-D volume is a 3-surface in 4-D space, $\tilde{n}\upsilon$, where the normal 4-vector $\tilde{n}$ defines the the orientation of the surface. In particular $n^\alpha = (1,0,0,0)$ defines a spatial volume at a constant time (i.e. a space-like hyper-surface). Hence we can define the number of particles in the 3-volume $\tilde{n}\upsilon$ as

$$\aleph = N^\alpha n_\alpha \upsilon. \tag{41}$$

Similarly we can define the energy-momentum in a 3-volume as

$$p^\alpha = T^{\alpha\beta} n_\beta \upsilon. \tag{42}$$

Taking $n^\alpha = (1,0,0,0)$ we then find the mass-energy density

$$\rho = \frac{\triangle p^0}{\upsilon} = T^{00}. \tag{43}$$

This derivation is for flat-space – if the curvature is weak then this relationship will still be approximately true as per Eq. (23).

Solving the geodesic equation: conserved quantities, symmetries, and Killing vectors

Summary

- Conserved quantities
- Metric symmetries
- Killing vectors

In this note we discuss some concepts that are useful for analyzing the timelike geodesics of free particles in general relativity.[1]

Consider the timelike worldline (not necessarily a geodesic) of a massive particle. The worldline is parametrized by the proper time τ along it, with coordinates given by $x^\alpha(\tau)$. The 4-velocity $\boldsymbol{u}$ at any point on the worldline is normalized to -1. This was also true in SR, and the proof is essentially identical. The components of the 4-velocity are $u^\alpha = dx^\alpha/d\tau$. Thus

$$\boldsymbol{u} \cdot \boldsymbol{u} = g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = \frac{ds^2}{d\tau^2} = -1. \tag{1}$$

Thus $\boldsymbol{u} \cdot \boldsymbol{u} = -1$ is a conserved quantity along the worldline.

For a geodesic there are additional conserved quantities if the metric in question possesses **symmetries**. For example, suppose that the metric $g_{\alpha\beta}$ is independent of the coordinate x^1. By definition, this means that the metric $g_{\alpha\beta}$ is unchanged if we make the transformation $x^1 \to x^1 + \text{const.}$ where const. is any constant. We say that the metric is invariant under this transformation. We will show that this symmetry of the metric implies that

$$\boldsymbol{\xi} \cdot \boldsymbol{u} \text{ is conserved along a geodesic} \tag{2}$$

where $\boldsymbol{\xi}$ is the **Killing vector** associated with this symmetry, its components in this case being

$$\xi^\alpha = (0, 1, 0, 0). \tag{3}$$

That is, all components are 0 except the component in the x^1 direction (the direction associated with the symmetry) which is 1.

Let us now prove (2). Our starting point will be that the geodesic equation is equivalent to the Euler-Lagrange equation resulting from the variational principle of free-particle motion. Suppose we use a parameter σ as the integration variable in the expression for the proper

[1] The relevant section in Hartle is 8.2.

time for the worldline (see the notes on the variational principle). Then the Euler-Lagrange equations read

$$\frac{d}{d\sigma}\frac{\partial L}{\partial(dx^\alpha/d\sigma)} = \frac{\partial L}{\partial x^\alpha} \tag{4}$$

where

$$L = \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}}. \tag{5}$$

If $g_{\alpha\beta}$ is independent of x^1, L is independent of x^1, i.e. $\partial L/\partial x^1 = 0$. Thus the Euler-Lagrange equation for $\alpha = 1$ simplifies to

$$\frac{d}{d\sigma}\frac{\partial L}{\partial(dx^1/d\sigma)} = 0 \tag{6}$$

and so, integrating this equation, we conclude that

$$\frac{\partial L}{\partial(dx^1/d\sigma)} = \text{const. along a geodesic.} \tag{7}$$

We can rewrite the lhs as follows:

$$\begin{aligned}\frac{\partial L}{\partial(dx^1/d\sigma)} &= \frac{1}{2L}\frac{\partial}{\partial(dx^1/d\sigma)}\left(-g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right) = -\frac{1}{2L}\left(g_{1\beta}\frac{dx^\beta}{d\sigma} + g_{\alpha 1}\frac{dx^\alpha}{d\sigma}\right) = -g_{1\beta}\frac{1}{L}\frac{dx^\beta}{d\sigma} \\ &= -g_{1\beta}\frac{d\sigma}{d\tau}\frac{dx^\beta}{d\sigma} = -g_{1\beta}\frac{dx^\beta}{d\tau} = -g_{\alpha\beta}\xi^\alpha u^\beta = -\boldsymbol{\xi}\cdot\boldsymbol{u},\end{aligned} \tag{8}$$

which proves (2). Here we used that $L = d\tau/d\sigma$ as discussed in the notes on the variational principle. Note that since the particle's 4-momentum $\boldsymbol{p}$ is given by $\boldsymbol{p} = m\boldsymbol{u}$ where m is the particle's (rest) mass (an invariant), an equivalent statement of (2) is that $\boldsymbol{\xi}\cdot\boldsymbol{p}$ is conserved along a geodesic.