MATH2400 Assignment 2

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Question 1

Determine whether or not each of the following series converge. Justify your claims adequately, either using the definitions, or quoting appropriate results from the lectures.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^3}{3^n}\ ; \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{n+1}{n^2}\ ; \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{(-1)^n\, n!}{2^n}\ .$$

(*Hint:* For (a), show that there exists a constant $C > 0$ such that $n^3 \leq C \cdot 3^{n/2}$ for all $n \in \mathbb{N}$; you may use without proving it that $x^3/3^{x/2} \to 0$ as $x \to \infty$.)

a. We have the series,

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n} = \sum_{n=1}^{\infty} a_n$$

The ratio test for series whose sequences are strictly positive is formulated as

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$$

If $r < 1$, the series $\sum_{n=1}^{\infty} a_n$ is convergent. Otherwise, the series may diverge. Applying the ratio test to the sequence a_n gives

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{(n+1)^3}{3^{n+1}} \times \frac{3^n}{n^3} \right) \\
&= \lim_{n\to\infty} \left(\frac{3^n (n+1)^3}{3n^3 \cdot 3^n} \right) \\
&= \lim_{n\to\infty} \left(\frac{n^3 + 3n^2 + 4n + 1}{3n^3} \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{n} + \frac{4}{3n^2} + \frac{1}{3n^3} \right)
\end{aligned}$$

Since all of the terms with n in the denominator converge to 0 as $n \to \infty$, the ratio converges to $r = 1/3 < 1$, and, by definition of the ratio test for series, the series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ is convergent.

b. We have the series,

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2} = \sum_{n=1}^{\infty} b_n$$

Slightly expanding, we have

$$b_n = \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2}$$

For any $n \in \mathbb{N}$,

$$\frac{1}{n} < \frac{1}{n} + \frac{1}{n^2} = b_n$$

It is known that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so since b_n is greater than the harmonic series for any $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} b_n$ will also be divergent (from results stated in lectures).

c. We have the series,

$$\sum_{n=1}^{\infty} \frac{(-1)^n \, n!}{2^n} = \sum_{n=1}^{\infty} c_n$$

The ratio test for series whose terms are non-zero is given by

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r$$

As stated before, if $r < 1$, the series converges and if $r > 1$ (or is not defined), the series diverges. Applying the test to the sequence c_n gives

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \, (n+1)!}{2^{n+1}} \times \frac{2^n}{(-1)^n \, n!} \right| \\
&= \lim_{n\to\infty} \left| - \frac{(-1)^n \, (n+1)n!}{2 \cdot 2^n} \times \frac{2^n}{(-1)^n \, n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)}{2} \right|
\end{aligned}$$

Therefore, the limit of the ratio is not defined, and by definition, the series $\sum_{n=1}^{\infty} \frac{(-1)^n \, n!}{2^n}$ diverges.

Question 2

Using the $\epsilon - \delta$ definition, show that the function $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$, is continuous. (*Hint:* you may use without proof the fact that $\ln x \leq x - 1$, for all $x \in (0, \infty)$.)

For f to be continuous across it's domain, a $\delta > 0$ needs to be found such that, for all $\epsilon > 0$, $x, a \in (0, \infty)$,

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

Expanding on the right hand side, and assuming that there is some δ such that $|x - a| < \delta \ \forall x, a \in (0, \infty)$

$$\begin{aligned} |f(x) - f(a)| &= |\ln x - \ln a| \\ &\leq |x - 1 - (a - 1)| \\ &= |x - a| \\ &< \delta \end{aligned}$$

Choose $\delta = \epsilon$. Therefore, omitting some of the steps in the previous arithmetic,

$$|f(x) - f(a)| < |x - a| < \delta = \epsilon$$

Therefore, by the $\epsilon - \delta$ definition of continuity, the function f is continuous on it's stated domain. QED

Question 3

Prove that any cubic polynomial with real coefficients $a_3x^3+a_2x^2+a_1x+a_0$, $a_3 \neq 0$, has at least one root in $\mathbb{R}$.

Any polynomial function is assumed to be continuous on $\mathbb{R}$, and a root of said polynomial is defined to be a point at which $f(x) = y = 0$. The question may be rephrased to say "Prove that for any cubic polynomial, $f : \mathbb{R} \to \mathbb{R}$, $a_3 \neq 0$, there exists at least one point, $c \in \mathbb{R}$, at which $f(c) = 0$."

Consider the limits of $f(x)$ as x goes to $\pm\infty$. As x^3 goes to $\pm\infty$ faster than any other term, the whole function will follow. That is,

$$\lim_{x \to \infty} f(x) = \infty$$

and,

$$\lim_{x \to -\infty} f(x) = -\infty$$

Therefore, as $f(x)$ is continuous across $\mathbb{R}$, there must be some point along x, c, at which $f(c) = 0$ (by the Intermediate Value Theorem - $f(-\infty) < 0 < f(\infty)$, then $\exists\, c \in \mathbb{R}$ such that $f(c) = 0$), and at this point there will be a root by definition. QED

Question 4

Prove that the function $f : [1, \infty) \to \mathbb{R}$, $f(x) = x^{-1}$, is uniformly continuous.

If $f : [1, \infty) \to \mathbb{R}$, $f(x) = x^{-1}$ is uniformly continuous, then for all $\epsilon > 0$, $\exists\, \delta > 0$ such that wherever $x, y \in [1, \infty)$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Firstly, x and y may be chosen to be at least $\delta > 0$ close, and $x \leq y$. Therefore, $|x - y| < \delta$ holds. Then,

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{x} - \frac{1}{y} \right| \\ &\leq |x - y| \\ &< \delta \end{aligned}$$

Thus, δ may be chosen to be equal to ϵ, and,

$$|f(x) - f(y)| \leq |x - y| < \delta = \epsilon$$

By definition of uniform continuity, the function $f(x) = x^{-1}$ on the domain $[1, \infty)$ is uniformly continuous. QED

Question 5

Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^{1/3}$, is *not* a differentiable function.

Recall the definition of differentiability: $f : (a, b) \to \mathbb{R}$ is differentiable at the point x if the limit

$$\lim_{h \to 0} \frac{f(x + h) - f(h)}{h}$$

or, if $x = x_0 + h$,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. That is, by the law of contraposition, a function is *not* differentiable if the limit at this point does not exist, and if the function is not differentiable at one point in it's domain, it is defined as not being differentiable. With this in mind, take $f(x) = x^{1/3}$ with $f : \mathbb{R} \to \mathbb{R}$. Thus, for a point $x_0 = 0 \in \mathbb{R}$, let $x = x_0 + h$,

$$\begin{aligned} \frac{d}{dx} f(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ \frac{d}{dx} f(0) &= \lim_{x \to 0} \frac{x^{1/3} - 0}{x - 0} \\ &= \lim_{x \to 0} \frac{1}{x} \cdot x^{1/3} \\ &= \left(\lim_{x \to 0} \frac{1}{x} \right) \cdot \left(\lim_{x \to 0} x^{1/3} \right) \end{aligned}$$

However, $\frac{1}{x}$ is not defined as $x \to 0$, and so the derivative of $f(x) = x^{1/3}$ is not defined at $0 \in \mathbb{R}$. Therefore, f is not differentiable on it's domain. QED