Exam information	
Course code and title	PHYS2100 Dynamics and Relativity
Semester	Semester 2, 2021
Exam type	Online, non-invigilated, final examination
Exam technology	File upload to Blackboard Assignment
Exam date and time	Your examination will begin at the time specified in your personal examination timetable. If you commence your examination after this time, the end for your examination does NOT change. The total time for your examination from the scheduled starting time will be: 2 hours 10 minutes (including 10 minutes reading time during which you should read the exam paper and plan your responses to the questions). A 15-minute submission period is available for submitting your examination after the allowed time shown above. If your examination is submitted after this period, late penalties will be applied unless you can demonstrate that there were problems with the system and/or process that were beyond your control.
Exam window	You must commence your exam at the time listed in your personalised timetable. You have from the start date/time to the end date/time listed in which you must complete your exam.
Permitted materials	This is a closed book exam – only specified materials are permitted. Permitted materials are any lecture and contact class notes related to the course. This can include the course workbook, your own handwritten or typed notes from lectures or contact classes, and annotated pdfs of in-class notes, but does not include assignment solutions or past exam solutions. You may not make use of any other material. This includes websites, books, or any other material or software.
Recommended materials	Ensure the following materials are available during the exam: UQ approved calculator; bilingual dictionary; phone/camera/scanner
Instructions	You will need to download the question paper included within the Blackboard Test. Once you have completed the exam, upload the completed exam answers file to the Blackboard assignment submission link. You may submit multiple times, but only the last uploaded file will be graded. You can print the question paper and write on that paper or write your answers on blank paper (clearly label your solutions so that it is clear which problem it is a solution to) or annotate an electronic file on a suitable device.
Who to contact	Given the nature of this examination, responding to student queries and/or relaying corrections to exam content during the exam may not be feasible. If you have any concerns or queries about a particular question or need to make any assumptions to answer the question, state these at the start of your solution to that question. You may also include queries you may have made with respect to a particular question, should you have been able to 'raise your hand' in an examination-type setting. If you experience any interruptions to your examination, please collect evidence of the interruption (e.g. photographs, screenshots or emails).

	If you experience any issues during the examination, contact ONLY the Library AskUs service for advice as soon as practicable: **Chat:** support.my.uq.edu.au/app/chat/chat_launch_lib **Phone:** +61 7 3506 2615 **Email:** examsupport@library.uq.edu.au You should also ask for an email documenting the advice provided so you can provide this as evidence for a late submission.
Late or incomplete submissions	In the event of a **late submission**, you will be required to submit evidence that you completed the assessment in the time allowed. This will also apply if there is an **error in your submission** (e.g. corrupt file, missing pages, poor quality scan). We **strongly recommend** you use a phone camera to take time-stamped photos (or a video) of every page of your paper during the time allowed (even if you submit on time). If you submit your paper after the due time, then you should send details to SMP Exams (exams.smp@uq.edu.au) as soon as possible after the end of the time allowed. Include an explanation of why you submitted late (with any evidence of technical issues) AND time-stamped images of every page of your paper (eg screen shot from your phone showing both the image and the time at which it was taken).
Important exam condition information	Academic integrity is a core value of the UQ community and as such the highest standards of academic integrity apply to all examinations, whether undertaken in-person or online. This means: • You are permitted to refer to the allowed resources for this exam, but you cannot cut-and-paste material other than your own work as answers. • You are not permitted to consult any other person – whether directly, online, or through any other means – about any aspect of this examination during the period that it is available. • If it is found that you have given or sought outside assistance with this examination, then that will be deemed to be cheating. If you submit your online exam after the end of your specified reading time, duration, and 15 minutes submission time, the following penalties will be applied to your final examination score for late submission: • Less than 5 minutes – 5% penalty • From 5 minutes to less than 15 minutes – 20% penalty • More than 15 minutes – 100% penalty These penalties will be applied to all online exams unless there is sufficient evidence of problems with the system and/or process that were beyond your control. Undertaking this online exam deems your commitment to UQ's academic integrity pledge as summarised in the following declaration: *"I certify that I have completed this examination in an honest, fair and trustworthy manner, that my submitted answers are entirely my own work, and that I have neither given nor received any unauthorised assistance on this examination".*

Part A: Dynamics — 24 marks

1. A particle moves in one dimension with velocity v given by

$$v = f(x),$$

where x is the coordinate of the particle, and f is a differentiable function. Show that the motion is conservative with a potential given by

$$V(x) = -\frac{1}{2}m[f(x)]^2.$$

[3 marks]

2. Suppose that the vertical coordinate y is used as the generalised coordinate for a simple pendulum.

   (i) Calculate the generalised force as a function of y.

   [3 marks]

   (ii) Derive an expression for the kinetic energy $T(y, \dot{y})$.

   [3 marks]

   (iii) Write down the Lagrangian.

   [2 marks]

3. Consider the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(5\dot{x}^2 + \dot{y}^2 - 4\dot{x}\dot{y}) + C(2x - y),$$

where C is a constant.

   (i) How many degrees of freedom are there?

   [1 mark]

   (ii) Use Lagrange's equations to determine expressions for $\ddot{x}$ and $\ddot{y}$.

   [4 marks]

   (iii) Show that the Hamiltonian

$$H = p_x\dot{x} + p_y\dot{y} - L,$$

   where

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}}, \qquad \alpha = x,\ y,$$

   has the form

$$H = \frac{1}{2}(p_x^2 + 4p_xp_y + 5p_y^2) - C(2x - y).$$

   [4 marks]

   (iv) Determine Hamilton's equations of motion and show that they are equivalent to your result for item (ii).

   [4 marks]

Part B begins over page.

Part B: Chaos — 13 marks

4. For a real constant parameter a, consider the following Hamiltonian

$$H = \frac{p^2}{2} + V(q)\,, \qquad V(q) = -\frac{1}{2}q^2 + \frac{a}{4}q^4\,.$$

(a) Derive Hamilton's equations, find the fixed points for all values of a, and compute the value of the energy at the fixed points.

[3 marks]

(b) Sketch by hand a graph of (i) the potential $V(q)$ for two cases: $a = -1$ and $a = 1$ and (ii) the phase portrait of the system. Make sure that you identify the type of fixed points, identify the separatrices, and sketch the direction of the vector field flow on orbits.

[5 marks]

5. (a) Explain what the Lyapunov exponent is and for which values chaotic behaviour is expected in a dynamical system. For simplicity consider a one-dimensional dynamical system and/or a one dimensional logistic map.

[2 marks]

(b) Consider a generic one-dimensional map defined by the sequence

$$x_{n+1} = f(x_n)\,, \qquad 0 \leq x_i \leq 1\,.$$

Assume that the following limits exist

$$\lambda := \lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{\mathrm{d}f^n(x)}{\mathrm{d}x}\Big|_{x=x_0} \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|\,,$$

where $f^n(x) = f(f(f(\cdots f(x))))$ is the n-th iteration of the map f. λ defines a Lyapunov exponent of the map f.

Given a constant parameter $0 \leq r \leq 2$, consider the map, defined by

$$f(x) = \begin{cases} 2rx\,, & 0 \leq x \leq \frac{1}{4} \\ 2r\left(\frac{1}{2} - x\right), & \frac{1}{4} \leq x \leq \frac{1}{2} \\ 2r\left(x - \frac{1}{2}\right), & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2r(1-x)\,, & \frac{3}{4} \leq x \leq 1 \end{cases}$$

Compute the Lyapunov exponent for this map and indicate for which value of r one expects chaotic behaviour.

[3 marks]

Part C begins over page.

Part C: Special Relativity — 13 marks

In the following questions, we are using units for which $c = 1$.

6. The space-time interval between infinitesimally separated events in special relativity is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{1}$$

(i) Consider the space-time interval between two events on the trajectory of a massive particle. Is the space-time interval positive or negative? Give a physical interpretation of the magnitude of the space-time interval in this case. Justify your answer using Eq. (1).

(ii) The Lorentz transformations between reference frame K and reference frame K' which is boosted by velocity v in the x-direction are

$$\begin{aligned} \Delta x' &= \gamma(\Delta x - v\Delta t), \\ \Delta t' &= \gamma(\Delta t - v\Delta x), \end{aligned} \tag{2}$$

where $\gamma = (1 - v^2)^{-1/2}$.

A stationary rod in the K frame has length $2m$. Use Eqs (2) to derive its length in reference frame K' assuming $v = 0.8$. Show all working.

[4 marks]

7. The components of the 4-velocity of a particle in a particular reference frame are given by: $u^\alpha = (F(\tau), G(\tau), 0, 0)$.
(i) Find the 3-velocity of the particle in this reference frame in terms of $F(\tau)$.
(ii) Calculate the components of the 4-acceleration for this particle in terms of $F(\tau)$ and its derivative.
(iii) Is the 4-acceleration a time-like, null, or space-like vector? Justify your answer.

[4 marks]

8. Two balls of putty move directly towards each other and collide. The first ball of putty has mass m and velocity v. The second ball of putty has mass $3m$ and velocity $-v$. After the collision the balls are stuck together and the combined lump has mass M' and velocity v'. All velocities are purely in the x-direction.

(i) Find the mass, M' of the combined lump after the collision (Hint: it may be useful to compare the magnitudes of the 4-momenta before and after the collision).

(ii) Find the speed, v', of the combined lump after the collision (Hint: it may be useful to compare the components of the 4-momenta before and after the collision).

[5 marks]

END OF QUESTIONS.

Useful formulae

Dynamics

Acceleration in one dimension:

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx}.$$

Newtonian gravitation in one dimension:

$$F = -\frac{GMm}{r^2}.$$

Arclength:

$$\ell = \int_C ds$$

Standard cylindrical coordinates:

$$x = r\cos\theta, \;\; y = r\sin\theta, \;\; z = z.$$

Standard spherical coordinates:

$$x = r\cos\theta\sin\phi, \;\; y = r\sin\theta\sin\phi, \;\; z = r\cos\phi.$$

Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.$$

Hamiltonian Dynamics and Chaos

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}.$$

$$\{f, g\}_{(q,p)} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

$$I = \frac{1}{2\pi}\oint p(q, H)dq \;\; = \;\; -\frac{1}{2\pi}\oint q(p, H)dp.$$

$$\theta = \frac{\partial}{\partial I}\int_0^q p(q', I)dq'.$$

$$\omega = \frac{\partial H}{\partial I}.$$

$$\dim(S)_{\text{box}} = \lim_{\epsilon\to 0}\frac{\ln N(S,\epsilon)}{\ln(1/\epsilon)},$$

where $N(S, \epsilon)$ is the number of boxes of side length ϵ needed to cover the set S.