PHYS3020 Module 1 Problem Set

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Q2

$$E_q = q\hbar\omega \qquad q \in \{\mathbb{Z}^+, 0\}$$

a. Two harmonic oscillators, each with ω and $E_q = q\hbar\omega$. Let

$$U_1 = n_1 \hbar\omega$$

be the total energy of the system.

Since the system is composed of two harmonic oscillators,

$$U_1 = E_i + E_j$$

$$= q_i\hbar\omega + q_j\hbar\omega$$
$$= (q_i + q_j)\hbar\omega$$
$$= n_1\hbar\omega$$
$$\Rightarrow \quad n_1 = q_i + q_j$$

Therefore there are $n_1 + 1$ possible microstates for the system of n_1 macrostates, as each of the harmonic oscillators could have any number of quanta, provided that their sum equals n_1, plus one to account for the case where one $q = 0$. Hence, $\Omega_1 = n_1 + 1$ But $U_1 = n_1\hbar\omega$, so

$$n_1 = \frac{U_1}{\hbar\omega}$$

$$\Rightarrow \quad \Omega_1 = \frac{U_1}{\hbar\omega} + 1$$

And since the entropy is given as

$$S_1 = k \ln \Omega_1$$

the entropy for the system (in terms of U_1) is

$$S_1 = k \ln\left(\frac{U_1}{\hbar\omega} + 1\right)$$

b. Two harmonic oscillators, each with 2ω. The total energy of the system is

$$U_2 = n_2\hbar\omega$$

Given that it's composed of two harmonic oscillators,

$$U_2 = E_i + E_j$$

$$= q_i\hbar 2\omega + q_j\hbar 2\omega$$

$$= 2(q_i + q_j)\hbar\ldots$$

$$= q_i \hbar 2\omega + q_j \hbar 2\omega$$

$$= 2(q_i + q_j)\hbar\omega$$

$$\Rightarrow n_2 = 2(q_i + q_j)$$

Therefore, there are $q_i + q_j + 1 = \frac{n_2}{2} + 1$ possible microstates for the system, as each harmonic oscillator could have any number of quanta, provided that twice their sum equals n_2, plus 1 to account for the 0 case.

Therefore, $\Omega_2 = \frac{n_2}{2} + 1$

$$= \frac{U_2}{2\hbar\omega} + 1$$

$$\Rightarrow S_2 = k \ln\left(\frac{U_2}{2\hbar\omega} + 1\right)$$

c. Considering both of these systems separated by an impassable divide, the total entropy is just the sum of individual energies:

$$S_{total} = S_1 + S_2$$

$$= k\ln\left(\frac{U_1}{\hbar\omega} + 1\right) + k\ln\left(\frac{U_2}{2\hbar\omega} + 1\right)$$

$$= k\ln\left(\left[\frac{U_1}{\hbar\omega} + 1\right]\left[\frac{U_2}{2\hbar\omega} + 1\right]\right)$$

Q4

a. $2s = N_\uparrow - N_\downarrow$ $\qquad N = N_\uparrow + N_\downarrow \Rightarrow N_\downarrow = N - N_\uparrow$

$$\Rightarrow 2s = N_\uparrow - N + N_\uparrow \qquad 2s = N - N_\downarrow - N_\downarrow$$

$$s = N_\uparrow - \tfrac{1}{2}N \qquad s = \tfrac{1}{2}N - N_\downarrow$$

$$\Rightarrow N_\uparrow = \tfrac{1}{2}N + s \qquad \Rightarrow N_\downarrow = \tfrac{1}{2}N - s$$

$$g(N,s) = \frac{N!}{N_\uparrow!\,N_\downarrow!} = \frac{N!}{(\frac{1}{2}N+s)!(\frac{1}{2}N-s)!}$$

Keep the $N_\uparrow, N_\downarrow$ notation. By Stirling's approximation,

$$N! \approx \sqrt{2\pi N}\, N^N e^{-N}$$

$$= \sqrt{2\pi}\, N^{N+\frac{1}{2}} e^{-N}$$

$$\Rightarrow \log N! \approx \tfrac{1}{2}\ln(2\pi) + (N + \tfrac{1}{2})\ln(N) - N$$

$$\Rightarrow \log N_\uparrow! \approx \tfrac{1}{2}\ln(2\pi) + (N_\uparrow + \tfrac{1}{2})\ln(N_\uparrow) - N_\uparrow$$

$$\log N_\downarrow! \approx \tfrac{1}{2}\ln(2\pi) + (N_\downarrow + \tfrac{1}{2})\ln(N_\downarrow) - N_\downarrow$$

But $N = N_\uparrow + N_\downarrow$, so

$$\ln N! = \tfrac{1}{2}\ln(2\pi) + (N_\uparrow + N_\downarrow + \tfrac{1}{2})\ln(N) - (N_\uparrow + N_\downarrow) + \tfrac{1}{2}\ln(N) - \tfrac{1}{2}\ln(N)$$

$$= \tfrac{1}{2}\ln\left(\tfrac{2\pi}{N}\right) + (N_\uparrow + \tfrac{1}{2} + N_\downarrow + \tfrac{1}{2})\ln(N) - (N_\uparrow + N_\downarrow)$$

Now,

$$\ln g = \ln N! - \ln N_\uparrow! - \ln N_\downarrow!$$

$$\approx \tfrac{1}{2}\ln\left(\tfrac{2\pi}{N}\right) - \ln(2\pi) + (N_\uparrow + \tfrac{1}{2})(\ln(N) - \ln(N_\uparrow))$$

$$+ (N_\downarrow + \tfrac{1}{2})(\ln(N) - \ln(N_\downarrow))$$

$$+\left(N_\downarrow+\tfrac{1}{2}\right)\left(\ln(N)-\ln(N_\downarrow)\right)$$

$$=\frac{1}{2}\ln\left(\frac{1}{2\pi N}\right)-\left(N_\uparrow+\tfrac{1}{2}\right)\ln\left(\frac{N_\uparrow}{N}\right)-\left(N_\downarrow+\tfrac{1}{2}\right)\ln\left(\frac{N_\downarrow}{N}\right)$$

Now, $\ln\left(\frac{N_\uparrow}{N}\right)=\ln\left(\frac{\frac{1}{2}N+s}{N}\right)=\ln\left(\frac{1}{2}+\frac{s}{N}\right)$

$$=\ln\left(\tfrac{1}{2}\left(1+\tfrac{2s}{N}\right)\right)$$

$$=\ln\left(\tfrac{1}{2}\right)+\ln\left(1+\tfrac{2s}{N}\right)$$

$$=-\ln(2)+\ln\left(1+\tfrac{2s}{N}\right)$$

Since $|s|<<N$, $\ln\left(1+\frac{2s}{N}\right)\approx\frac{2s}{N}-\frac{1}{2}\left(\frac{4s^2}{N^2}\right)=\frac{2s}{N}-\frac{2s^2}{N^2}$

$$\Rightarrow \ln\left(\frac{N_\uparrow}{N}\right)\approx-\ln(2)+\frac{2s}{N}-\frac{2s^2}{N^2}$$

Similarly, $\ln\left(\frac{N_\downarrow}{N}\right)\approx-\ln(2)-\frac{2s}{N}-\frac{2s^2}{N^2}$

$$\Rightarrow \ln g\approx\frac{1}{2}\ln\left(\frac{1}{2\pi N}\right)-\left(N_\uparrow+\tfrac{1}{2}\right)\left(-\ln(2)+\frac{2s}{N}-\frac{2s^2}{N^2}\right)$$
$$-\left(N_\downarrow+\tfrac{1}{2}\right)\left(-\ln(2)-\frac{2s}{N}-\frac{2s^2}{N^2}\right)$$
$$=\frac{1}{2}\ln\left(\frac{1}{2\pi N}\right)+\ln(2)\left(N_\uparrow+\tfrac{1}{2}+N_\downarrow+\tfrac{1}{2}\right)-\left(\frac{2s}{N}\right)\left(N_\uparrow+\tfrac{1}{2}-N_\downarrow-\tfrac{1}{2}\right)$$
$$+\left(\frac{2s^2}{N^2}\right)\left(N_\uparrow+\tfrac{1}{2}+N_\downarrow+\tfrac{1}{2}\right)$$

$$=\frac{1}{2}\ln\left(\frac{1}{2\pi N}\right)+\ln(2)\left(\tfrac{1}{2}N+s+\tfrac{1}{2}+\tfrac{1}{2}N-s+\tfrac{1}{2}\right)$$
$$-\left(\frac{2s}{N}\right)\left(\tfrac{1}{2}N+s+\tfrac{1}{2}-\tfrac{1}{2}N+s-\tfrac{1}{2}\right)$$
$$+\left(\frac{2s^2}{N^2}\right)\left(\tfrac{1}{2}N+s+\tfrac{1}{2}+\tfrac{1}{2}N-s+\tfrac{1}{2}\right)$$

$$=\frac{1}{2}\ln\left(\frac{1}{2\pi N}\right)+\ln(2)(N+1)-\left(\frac{2s}{N}\right)(2s)+\left(\frac{2s^2}{N^2}\right)(N+1)$$

$N+1\approx N$

$$\ln g\approx\frac{1}{2}\ln\left(\frac{1}{2\pi N}\right)+\ln(2)+N\ln(2)-\frac{4s^2}{N}+\frac{2s^2}{N}$$
$$=\frac{1}{2}\ln\left(\frac{4}{2\pi N}\right)+N\ln(2)-\frac{2s^2}{N}$$
$$=\ln\left(\sqrt{\frac{2}{\pi N}}\right)+N\ln(2)-\frac{2s^2}{N}$$

And finally,

$$g(N,s)\approx\sqrt{\frac{2}{\pi N}}\cdot 2^N e^{-\frac{2s^2}{N}}$$
$$=g(N,0)\,e^{-\frac{s^2}{N}}$$

where

$$g(N,0)=\sqrt{\frac{2}{\pi N}}\;2^N$$

b. $\langle x\rangle=\int_{-\infty}^{\infty}x\,f(x)\,dx$

$$\Rightarrow\langle s\rangle=\int_{-\infty}^{\infty}s\;2^N\sqrt{\frac{2}{\pi N}}\,e^{-\frac{2s^2}{N}}\,ds$$

Since this is an odd function $\langle s\rangle=0$

For the distribution to be normalised to unity,

$$A\int_{-\infty}^{\infty}2^N\sqrt{\frac{2}{\pi N}}\,e^{-\frac{2s^2}{N}}=1$$

Since this an even function, $\int_{-\infty}^{0}f(x)=\int_0^{\infty}f(x)$

$$\Rightarrow\int_{-\infty}^{\infty}f(x)=2\cdot\int_0^{\infty}f(x)$$

$\Rightarrow \frac{1}{\ldots}=2\;2^N\sqrt{2}\;\int^{\infty}e^{-\frac{2s^2}{\ldots}}\ldots$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) = 2\cdot\int_0^{\infty} f(x)$$

$$\Rightarrow \frac{1}{A} = 2\cdot 2^N \sqrt{\frac{2}{\pi N}} \int_0^{\infty} e^{\frac{-2s^2}{N}}\, ds$$

set $a = \sqrt{\frac{2}{N}}$ $\Rightarrow \int_0^{\infty} e^{\frac{-2s^2}{N}} = \int_0^{\infty} e^{-a^2s^2} = \frac{\sqrt{\pi}}{2a} = \frac{1}{2}\sqrt{\frac{\pi N}{2}}$

$$\Rightarrow \frac{1}{A} = 2^N \quad \Rightarrow \quad A = \frac{1}{2^N}$$

And so

$$\langle s^2 \rangle = \frac{2^N}{2^N}\sqrt{\frac{2}{\pi N}} \int_{-\infty}^{\infty} s^2 e^{\frac{-2s^2}{N}}\, ds$$

set $a = \sqrt{\frac{2}{N}}$ as before, and this an even function, so

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^{\infty} f(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} s^2 e^{-a^2s^2}\, ds = 2\int_0^{\infty} s^2 e^{-a^2s^2}\, ds = 2\cdot\frac{\sqrt{\pi}}{4a^3}$$

$$= \frac{1}{2}\sqrt{\pi}\cdot\sqrt{\frac{N^3}{2^3}} = \frac{1}{4}\sqrt{\frac{\pi N^3}{2}}$$

$$\Rightarrow \langle s^2 \rangle = \sqrt{\frac{2}{\pi N}}\cdot\frac{1}{4}\sqrt{\frac{\pi N^3}{2}} = \frac{N}{4}$$

Now, define the rms width as $\sigma = \sqrt{\langle s^2\rangle - \langle s\rangle^2}$

$$= \sqrt{\frac{N}{4} - 0}$$

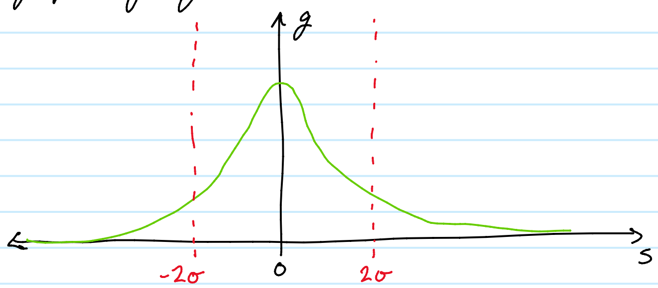
$$= \frac{\sqrt{N}}{2}$$

Define the fractional width as

$$S_w = \frac{2\sigma}{N}$$

$$= \frac{2}{N}\cdot\frac{\sqrt{N}}{2} = \frac{1}{\sqrt{N}}$$

A graph of $g(N,s)$ is shown below:



Q7

i. The number of possible five-card poker hands is given by $n = \binom{52}{5} = \frac{52!}{5!\cdot(52-5)!}$

$$= 2598960$$

ii. The probability of being dealt a royal flush on the first deal is

$$\mathbb{P}(\text{Royal Flush}) = \frac{n_{Ace}}{n_{cards}} \times \frac{n_{King}}{n_{cards}-1} \times \frac{n_{queen}}{n_{cards}-2} \times \frac{n_{Jack}}{n_{cards}-3} \times \frac{n_{10}}{n_{cards}-4}$$

$$= \frac{4}{52}\times\frac{4}{51}\times\frac{4}{50}\times\frac{4}{49}\times\frac{4}{48}$$

$$= \frac{8}{2436525} \simeq 3.28\times 10^{-6}$$

Q8

a. As from Q4a, the multiplicity of a system with spin

Q8

a. As from Q4a, the multiplicity of a system with spin excess $2s$ is

$$\Omega(N,s) = \frac{N!}{N_\uparrow!\,N_\downarrow}$$

$$\Rightarrow \ln\Omega = \ln N! - \ln N_\uparrow! - \ln N_\downarrow!$$
$$= \ln N! - \ln N_\uparrow! - \ln(N-N_\uparrow)!$$
(since $N_\downarrow = N - N_\uparrow$)

Using Stirling's approximation, this becomes

$$\ln\Omega \approx N\ln N - N - N_\uparrow\ln N_\uparrow + N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow) + (N-N_\uparrow)$$

$$= N\ln N - N_\uparrow\ln N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow)$$

$$S = k\ln\Omega = k\left(N\ln N - N_\uparrow\ln N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow)\right) \quad (1)$$

but $N_\uparrow = \frac{1}{2}N + s$

$$\Rightarrow S = k\left(N\ln(N) - (\tfrac{1}{2}N+s)\ln(\tfrac{1}{2}N+s) - (\tfrac{1}{2}N-s)\ln(\tfrac{1}{2}N-s)\right)$$

b.

Now, $2s = N_\uparrow - N_\downarrow = N_\uparrow - (N-N_\uparrow) = 2N_\uparrow - N$
and $U = -2smB = (N-2N_\uparrow)mB$

$$\Rightarrow N_\uparrow = \frac{N}{2} - \frac{U}{2mB}$$

And temperature is given by

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,V}$$

It is equivalent to write

$$\frac{\partial S}{\partial U} = \frac{\partial N_\uparrow}{\partial U}\frac{\partial S}{\partial N_\uparrow}$$

So, $$\frac{\partial N_\uparrow}{\partial U} = -\frac{1}{2mB}$$

and, looking at equation (1),

$$\frac{\partial S}{\partial N_\uparrow} = k\frac{\partial}{\partial N_\uparrow}\left(N\ln N - N_\uparrow\ln N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow)\right)$$

$$= k\left(-\ln N_\uparrow - 1 + \ln(N-N_\uparrow) + \frac{N-N_\uparrow}{N-N_\uparrow}\right)$$

$$= -k\ln\left(\frac{N_\uparrow}{N-N_\uparrow}\right)$$

$$= -k\ln\left(\frac{\frac{1}{2}N - U/2mB}{N - \frac{1}{2}N + U/2mB}\right)$$

$$= -k\ln\left(\frac{N - U/mB}{N + U/mB}\right)$$

$$\Rightarrow \frac{1}{T} = \frac{\partial N_\uparrow}{\partial U}\frac{\partial S}{\partial N_\uparrow}$$

$$= -\frac{1}{2mB}\cdot -k\ln\left(\frac{N-U/mB}{N+U/mB}\right)$$

$$= \frac{k}{2mB}\ln\left(\frac{N-U/mB}{N+U/mB}\right) \quad (2)$$

c. Rearrange (2) to yield

$$\frac{2mB}{kT} = \ln\left(\frac{N-U/mB}{N+U/mB}\right)$$

$$\frac{N-U/mB}{N+U/mB} = e^{\frac{2mB}{kT}}$$

$$\Rightarrow \frac{NmB-U}{NmB+U} = e^{\frac{2mB}{kT}}$$

$N + U/mB$

$$\Rightarrow \frac{NmB - U}{NmB + U} = e^{\frac{2mB}{kT}}$$

$$NmB - U = e^{\frac{2mB}{kT}}(NmB + U)$$

$$NmB\left(1 - e^{\frac{2mB}{kT}}\right) = Ue^{\frac{2mB}{kT}} + U$$

$$= \left(1 + e^{\frac{2mB}{kT}}\right)U$$

$$\Rightarrow U = NmB\left(\frac{1 - e^{\frac{2mB}{kT}}}{1 + e^{\frac{2mB}{kT}}}\right)$$

$$\langle U \rangle = -NmB\left(\frac{e^{\frac{2mB}{kT}} - 1}{e^{\frac{2mB}{kT}} + 1}\right)$$

Since $\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$,

$$\langle U \rangle = -NmB \tanh\left(\frac{mB}{kT}\right)$$

d. In the high temperature limit $kT \gg mB$, $\frac{mB}{kT} \ll 1$
The Taylor series expansion of $\tanh(x)$ is

$$\tanh(x) \approx x - \frac{x^3}{3} + \frac{2x^5}{15} - \dots$$

So, keeping the first term of the Taylor series expansion of $\langle U \rangle$ gives

$$\langle U \rangle \approx -NmB\frac{mB}{kT}$$

$$= -N\frac{(mB)^2}{kT} \quad \propto \frac{1}{T}$$

Clearly this agrees (approximately) with Curie's Law which predicts that the net fractional magnetization decreases as $1/T$ in the high temperature limit.

e. Due to the $1/T$ factor, as $T \to 0$ we would expect $\langle U \rangle \to -\infty$, i.e. the system aligns totally with the external magnetic field.