MATH2100 Assignment 2

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Tutorial 07

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Question 1

8 Marks
Consider the ODE

$$y''(t) - 6y'(t) - 3y(t) = 0, \tag{1}$$

where $y'(t)$, $y''(t)$ denote $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ respectively.

a. Find the roots of the characteristic equation corresponding to (1) and use these to write the general solution $y(t)$ to the ODE (1). (2 marks)

The roots may be found with the quadratic formula

$$\lambda_{1,\,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substituting in yields

$$\begin{aligned}\lambda_{1,\,2} &= \frac{6 \pm \sqrt{6^2 - 4 \times -3}}{2} \\ &= 3 \pm 2\sqrt{3}\end{aligned}$$

With these, the ODE's general solution is

$$y(t) = c_1 e^{(3+2\sqrt{3})t} + c_2 e^{(3-2\sqrt{3})t}$$

b. Couple the ODE (1) so that it is expressed in the form

$$\mathbf{Y}' = A\mathbf{Y} + \mathbf{b} \tag{2}$$

where $\mathbf{Y} = \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$, A is a 2×2 matrix, and $\mathbf{b}$ is a vector. (2 marks)

Firstly, equation (1) is rearranged to give

$$y''(t) = 6y'(t) + 3y(t)$$

The matrix A is then

$$A = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix}$$

and the vector $\mathbf{b}$ is the 0 vector. Expressed in the form of (2), the ODE is then

$$\mathbf{Y}' = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} \mathbf{Y}$$

c. Find the eigenvalues and eigenvectors of A and use these to plot the phase portrait for the system (2). (2 marks)

The eigenvalue equation can only be satisfied for the non-trivial $\mathbf{x}$ if

$$\det(A - \lambda I) = 0$$

Substituting in the value of A found in part b. gives

$$\begin{aligned}
0 &= \det\left(\begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \det\left(\begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) \\
&= \det\begin{pmatrix} -\lambda & 1 \\ 6 & 3-\lambda \end{pmatrix} \\
&= (-\lambda)(3-\lambda) - 1 \times 6 \\
0 &= \lambda^2 - 3\lambda - 6
\end{aligned}$$

With eigenvalues then found as

$$\begin{aligned}
\lambda_{1,\,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{3 \pm \sqrt{3^2 - 4 \times 1 \times -6}}{2} \\
&= \frac{3 \pm \sqrt{33}}{2}
\end{aligned}$$

The eigenvectors corresponding to these eigenvalues are then found by solving

$$(A - \lambda_i I)\mathbf{x}^{(i)} = 0 \,\text{for}\, \mathbf{x}^{(i)} = \begin{pmatrix} u \\ v \end{pmatrix}$$

Using the easier method of flipping the first row of $A - \lambda I$, then the sign of one of the terms gives the eigenvector for $\lambda_1 = \frac{3+\sqrt{33}}{2}$:

$$\begin{pmatrix} \frac{-3-\sqrt{33}}{2} & 1 \\ 6 & \frac{3-\sqrt{33}}{2} \end{pmatrix}\mathbf{x}^{(1)} = 0 \Rightarrow \begin{pmatrix} \frac{-3-\sqrt{33}}{2} \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ \frac{-3-\sqrt{33}}{2} \end{pmatrix} \to \begin{pmatrix} 1 \\ \frac{3+\sqrt{33}}{2} \end{pmatrix} = \mathbf{x}^{(1)}$$

And for $\lambda_2 = \frac{3-\sqrt{33}}{2}$:

$$\begin{pmatrix} \frac{-3+\sqrt{33}}{2} & 1 \\ 6 & \frac{3+\sqrt{33}}{2} \end{pmatrix}\mathbf{x}^{(2)} = 0 \Rightarrow \begin{pmatrix} \frac{-3+\sqrt{33}}{2} \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ \frac{-3+\sqrt{33}}{2} \end{pmatrix} \to \begin{pmatrix} 1 \\ \frac{3-\sqrt{33}}{2} \end{pmatrix} = \mathbf{x}^{(2)}$$

And so the general solution to the matrix equation is

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ \frac{3+\sqrt{33}}{2} \end{pmatrix} e^{\frac{3+\sqrt{33}}{2}t} + c_2 \begin{pmatrix} 1 \\ \frac{3-\sqrt{33}}{2} \end{pmatrix} e^{\frac{3-\sqrt{33}}{2}t}$$

The corresponding phase portrait is a saddle, and is shown as

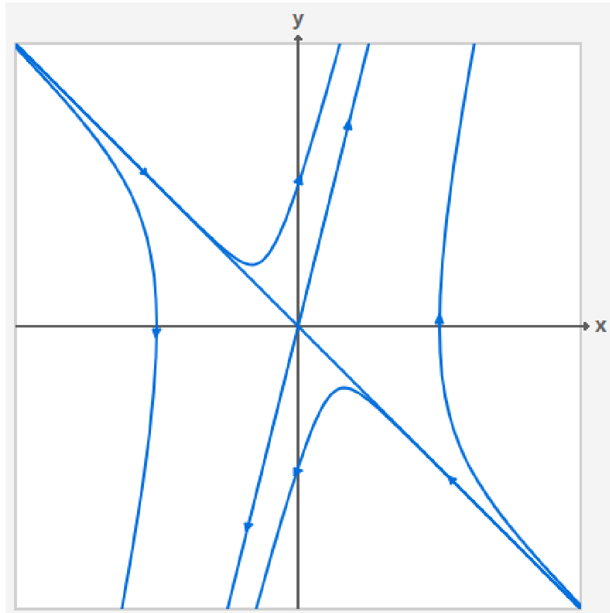

Figure 1: Saddle Phase Portrait Corresponding to Matrix A

d. Does the system (2) obey the superposition principle? Explain. (2 marks)

The system (2) is linear, as it is represented by only constant (simple) multiples of $y(t)$. Due to matrix $\mathbf{b}$ being the 0 vector, this system is also homogeneous. The superposition principle applies to only linear, homogeneous systems, and since system (2) fits this criteria, system (2) obeys the superposition principle.

Question 2

4 marks
Consider the ODE

$$-4y''(t) + 3y'(t) + 5ty(t) = \cos(t) \tag{3}$$

a. Show that this ODE (3) is linear but inhomogeneous. (2 marks)

The ODE (3) is non-linear in terms of the independent variable (see the term $5ty(t)$), however this does not affect eligibility for linearity. In terms of the unknown and it's derivatives, there are only simple (constant) multiples and so the ODE is linear.
In terms of homogeneity, first rearrange the ODE to

$$-4y''(t) + 3y'(t) + 5ty(t) - \cos(t) = 0$$

The term $-\cos(t)$ does not multiply y or it's derivatives, and so the criteria for homogeneity isn't met, and therefore the ODE isn't homogeneous.

b. Couple the ODE so that it is expressed in the form of (2). (2 marks)

Firstly, the ODE was rearranged to

$$y''(t) = \frac{3}{4}y'(t) + \frac{5}{4}ty(t) - \frac{1}{4}\cos(t)$$

The matrix A is then

$$A = \begin{pmatrix} 0 & 1 \\ \frac{3}{4} & \frac{5}{4}t \end{pmatrix}$$

and the matrix $\mathbf{b}$ as

$$\mathbf{b} = \begin{pmatrix} 0 \\ \frac{1}{4}\cos(t) \end{pmatrix}$$

The ODE can then be expressed in the form of (2) to give

$$\mathbf{Y}' = \begin{pmatrix} 0 & 1 \\ \frac{3}{4} & \frac{5}{4}t \end{pmatrix} \mathbf{Y} + \begin{pmatrix} 0 \\ \frac{1}{4}\cos(t) \end{pmatrix}$$

Question 3

10 marks

Let

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}; \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}; \qquad \mathbf{Y} = \begin{pmatrix} y \\ z \end{pmatrix}$$

and let

$$\mathbf{Y}' = A\mathbf{Y} + \mathbf{b} \tag{4}$$

a. Find a change of variables

$$p = y - a,$$
$$q = z - b$$

for some real a, b to be determined so that (4) gives the homogeneous linear system

$$\mathbf{X}' = B\mathbf{X}, \qquad \mathbf{X} = \begin{pmatrix} p \\ q \end{pmatrix} \tag{5}$$

where B is a 2×2 matrix. (2 marks)

Firstly, we have that

$$\mathbf{Y}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

The change of variables may be substituted in to yield

$$\begin{aligned} \mathbf{X}' &= \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} p+a \\ q+b \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} p + a - 2q - 2b \\ 3p + 3a - 4q - 4b + 5 \end{pmatrix} \\ &= \begin{pmatrix} p - 2q \\ 3p - 4q \end{pmatrix} + \begin{pmatrix} a - 2b \\ 3a - 4b + 5 \end{pmatrix} \end{aligned}$$

Letting each term of the matrix on the right equal zero gives

$$a = 2b \Rightarrow 6b - 4b = -5 \to 2b = -5 \to b = \frac{-5}{2} \Rightarrow a = -5$$

Meaning that the new variables have the values

$$p = y + 5,$$
$$q = z + \frac{5}{2}$$

and the homogeneous linear system is

$$\mathbf{X}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

b. Find the eigenvalues and eigenvectors of the matrix B. (2 marks)

The eigenvalues may be found with the formula $\det(B - \lambda I) = 0$:

$$\begin{aligned}
0 &= \det\left(\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \det \begin{pmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{pmatrix} \\
&= (1-\lambda)(-4-\lambda) - -2 \times 3 \\
0 &= \lambda^2 + 4\lambda + 2
\end{aligned}$$

The eigenvalues are then

$$\begin{aligned}
\lambda_{1,\,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-4 \pm \sqrt{4^2 - 4 \times 2}}{2} \\
&= -2 \pm \sqrt{2}
\end{aligned}$$

For the eigenvalue $\lambda_1 = -2 + \sqrt{2}$:

$$\begin{aligned}
&B - \lambda_1 I = \begin{pmatrix} 3-\sqrt{2} & -2 \\ 3 & -2-\sqrt{2} \end{pmatrix} \Rightarrow \begin{pmatrix} 3-\sqrt{2} & -2 \\ 3 & -2-\sqrt{2} \end{pmatrix} \mathbf{x}^{(1)} = 0 \\
&\Rightarrow \begin{pmatrix} 3-\sqrt{2} \\ -2 \end{pmatrix} \to \begin{pmatrix} -2 \\ 3-\sqrt{2} \end{pmatrix} \to \begin{pmatrix} 2 \\ 3-\sqrt{2} \end{pmatrix} = \mathbf{x}^{(1)}
\end{aligned}$$

For the eigenvalue $\lambda_2 = -2 - \sqrt{2}$:

$$\begin{aligned}
&B - \lambda_2 I = \begin{pmatrix} 3+\sqrt{2} & -2 \\ 3 & -2+\sqrt{2} \end{pmatrix} \Rightarrow \begin{pmatrix} 3+\sqrt{2} & -2 \\ 3 & -2+\sqrt{2} \end{pmatrix} \mathbf{x}^{(2)} = 0 \\
&\Rightarrow \begin{pmatrix} 3+\sqrt{2} \\ -2 \end{pmatrix} \to \begin{pmatrix} -2 \\ 3+\sqrt{2} \end{pmatrix} \to \begin{pmatrix} 2 \\ 3+\sqrt{2} \end{pmatrix} = \mathbf{x}^{(2)}
\end{aligned}$$

c. Find and classify the critical points of (5) and plot several trajectories in phase space by hand. (2 marks)

Since the system (5) is linear, there is a critical point at the origin (0, 0). If $\det A \neq 0$, then this is the only critical point.

$$\begin{aligned}
\det \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} &= 1 \times -4 - -2 \times 3 \\
&= 6 - 4 = 2 \neq 0
\end{aligned}$$

And so (0, 0) is the only critical point. If $\det B \geq 0$ and $(traceB) \leq 0$, then the point is stable. The above equation shows that $\det B = 2 > 0$, so the first half of the criteria is satisfied. The next half is

$$traceB = (a_{11} + a_{22}) = 1 - 4 = -3 < 0$$

Since the eligibility criteria is met, the critical point at the origin is stable.

d. Plot the same phase portrait using *Mathematica* (2 marks)

Using Mathematica, the phase portrait from $-3 \to 3$ was plotted using the code

```
StreamPlot[{p - 2 q, 3 p - 4 q}, {p, -3, 3}, {q, -3, 3}]
```

yielding the plot shown in Figure 2:

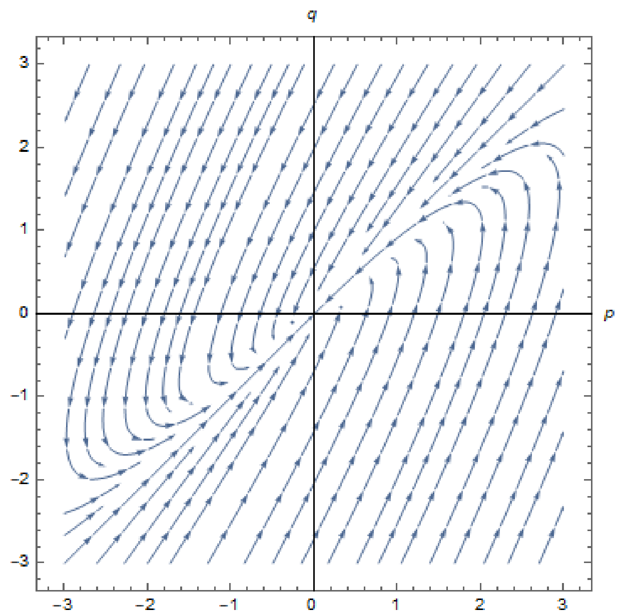

Figure 2: Phase Portrait of Equation (5)

e. Explain how the solution to (5) is related to the solution of (4). (2 marks)

The solution to (5) is the same type of phase portrait as the solution to (4), only with exact values shifted by some constant (as found in part a.).

Question 4

Consider the homogeneous polynomial

$$f(x,y) = 3x^3 + 3x^2y + 4xy^2 + 2y^3 \tag{6}$$

If $y = y(t) = \frac{dx}{dt}$ and $x = x(t)$, show that the differential equation $f(x,y) = 0$ is homogeneous but non-linear. (2 marks)

Firstly, equation (6) was rearranged to equal 0, with all values of y replaced by $\frac{dx}{dt}$:

$$2\left(\frac{dx}{dt}\right)^3 + 4x\left(\frac{dx}{dt}\right)^2 + 3x^2\left(\frac{dx}{dt}\right) + 3x^3 = 0$$

Since all of the terms of the ODE are either x or it's derivatives, the function is homogeneous. The first two terms, however, are non-linear as they are raised to powers of 3 and 2 respectively. Due to this, the ODE is non-linear.

Question 5

6 marks
Consider the following system of differential equations

$$\begin{cases} \dot{p} = 4q + p \\ \dot{q} = qp + p - q \end{cases}$$

Use *Mathematica* to

a. Find the stationary points (1 Mark)

   Using the function `Roots[-4 q∧2 - 4 q - q == 0, q]`, the values of q at the stationary points were found to be $q = -\frac{5}{4}$, 0. Since $p = -4q$ at a stationary point, the corresponding values of p are $p = 5$, 0 respectively, giving the stationary points on a (p, q) phase diagram as (5, $-\frac{5}{4}$), and (0, 0).

b. Find the Jacobian (1 Mark)

c. Analyse the linear stability of the stationary points (1 Mark)

d. Solve the system numerically for negative times starting from $p(0) = 0.0012$, $q(0) = -0.001$ and plot the solution into the phase portrait (using `Show[...]`). (2 Marks)

e. How can you classify this trajectory? (1 Mark)