PHYS2055 Assignment 2

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Part A

Problem 2.1

Three positive charges, each of magnitude +q, are placed on a circle of radius a in the $x-y$ plane at angles from the x-axis of 30°, 150° and 270° respectively.

a. Calculate the scalar potential for every point in the $x-y$ plane for this configuration:

Firstly, the positions of each of the charges must be found. For ease in reference, the charge at 30° from the x-axis will be referred to as q_1, the charge at 150° q_2, and the final charge as q_3. The position of q_1 can be found to be at $(\frac{\cos 30}{a}, \frac{\sin 30}{a})$, q_2 at $(-\frac{\cos 30}{a}, \frac{\sin 30}{a})$, and q_3 at $(0, -a)$.

The distance from any point in the $x-y$ plane to these point charges can be given by

$$r_1 = \sqrt{\left(\frac{\cos 30}{a} - x\right)^2 + \left(\frac{\sin 30}{a} - y\right)^2}$$
$$r_2 = \sqrt{\left(-\frac{\cos 30}{a} - x\right)^2 + \left(\frac{\sin 30}{a} - y\right)^2}$$
$$r_3 = \sqrt{(-x)^2 + (-a-y)^2}$$

for charges q_1, q_2 and q_3 respectively.

The scalar potential solution for isolated charges is given by

$$\phi = \sum_j \frac{1}{4\pi\epsilon_0}\frac{q_j}{r_j} \tag{1}$$

Summing the each of the values for the defined charges into equation (1) gives:

$$\phi(x,y) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{\left(\frac{\cos 30}{a} - x\right)^2 + \left(\frac{\sin 30}{a} - y\right)^2}} + \frac{1}{\sqrt{\left(-\frac{\cos 30}{a} - x\right)^2 + \left(\frac{\sin 30}{a} - y\right)^2}} + \frac{1}{\sqrt{(-x)^2 + (-a-y)^2}}\right)$$

Simplifying this with exact values gives:

$$\phi(x,y) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{\left(\frac{\sqrt{3}}{2a} - x\right)^2 + \left(\frac{1}{2a} - y\right)^2}} + \frac{1}{\sqrt{\left(-\frac{\sqrt{3}}{2a} - x\right)^2 + \left(\frac{1}{2a} - y\right)^2}} + \frac{1}{\sqrt{x^2 + (-a-y)^2}}\right)$$

b. Use your answer from a. to determine the electric field in the $x - y$ plane. Create a vector plot of the field (hand drawn or using your own script):

The electric field is defined as the negative of the gradient of the scalar potential. That is,

$$\mathbf{E} = -\nabla\phi \tag{2}$$

The gradient of the potential is defined as

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right) \tag{3}$$

And finally,

$$\mathbf{E} = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right) \tag{4}$$

Where the x-component of $\mathbf{E}$ was found to be

$$\frac{\partial\phi}{\partial x} = \frac{q}{4\pi\epsilon_0}\left(\frac{\frac{\sqrt{3}}{2a} - x}{\sqrt[3]{\left(\frac{\sqrt{3}}{2a} - x\right)^2 + \left(\frac{1}{2a} - y\right)^2}} - \frac{\frac{\sqrt{3}}{2a} + x}{\sqrt[3]{\left(-\frac{\sqrt{3}}{2a} - x\right)^2 + \left(\frac{1}{2a} - y\right)^2}} - \frac{x}{\sqrt[3]{x^2 + (-a - y)^2}}\right)$$

and the y-component of $\mathbf{E}$ was found to be:

$$\frac{\partial\phi}{\partial y} = \frac{q}{4\pi\epsilon_0}\left(\frac{\frac{1}{2a} - y}{\sqrt[3]{\left(\frac{\sqrt{3}}{2a} - x\right)^2 + \left(\frac{1}{2a} - y\right)^2}} + \frac{\frac{1}{2a} - y}{\sqrt[3]{\left(-\frac{\sqrt{3}}{2a} - x\right)^2 + \left(\frac{1}{2a} - y\right)^2}} - \frac{a + y}{\sqrt[3]{x^2 + (-a - y)^2}}\right)$$

So, using the two partial derivatives above, the electric field may be found for any point in 2D Cartesian space. For a region slightly bigger than the radius of the circle on which the point charges are distributed, Figure 1 shows both the scalar potential (represented by the colour of the plane, units in Volts $-$ V) and the electric field (represented by the magnitude and direction of the overlaid vectors), where the axes are in units of radii.

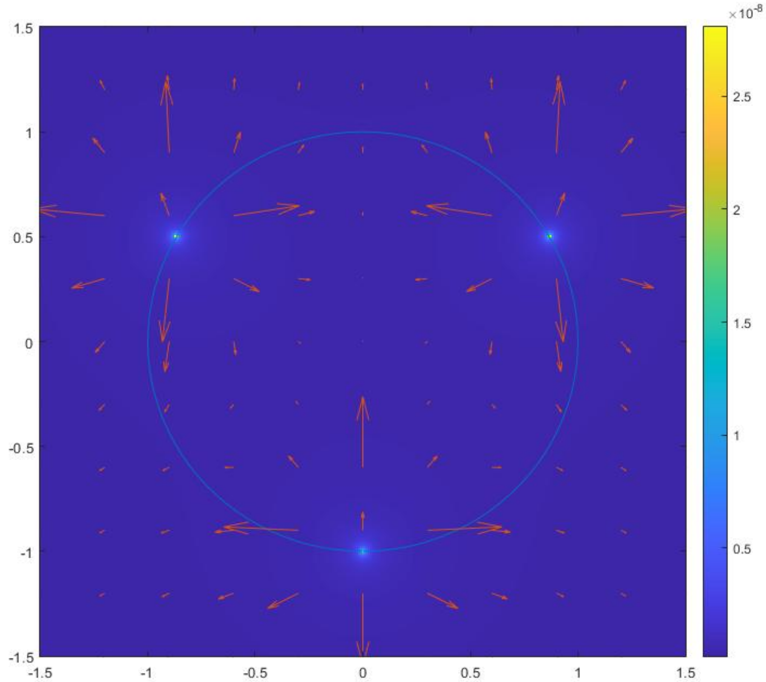

Figure 1: Electric Field Vectors and Scalar Potential for 3 Point Charges in a Circular Distribution

c. Find the magnitude of the field for distances much larger than the length a. Comment on your answer:

For x and y values far larger than a, it can be assumed that the impact of a is negligible and any terms that are influenced by a can be treated as such. So,

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{q}{4\pi\epsilon_0}\left(\frac{\frac{\sqrt{3}}{2a}-x}{\sqrt[3]{\left(\frac{\sqrt{3}}{2a}-x\right)^2+\left(\frac{1}{2a}-y\right)^2}}-\frac{\frac{\sqrt{3}}{2a}+x}{\sqrt[3]{\left(-\frac{\sqrt{3}}{2a}-x\right)^2+\left(\frac{1}{2a}-y\right)^2}}-\frac{x}{\sqrt[3]{x^2+\left(-a-y\right)^2}}\right) \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{-x}{\sqrt[3]{\left(-x\right)^2+\left(-y\right)^2}}-\frac{x}{\sqrt[3]{\left(-x\right)^2+\left(-y\right)^2}}-\frac{x}{\sqrt[3]{x^2+\left(-y\right)^2}}\right) \\
&= -\frac{3qx}{4\pi\epsilon_0}\frac{1}{\sqrt[3]{x^2+y^2}} \qquad (5)
\end{aligned}$$

and,

$$\frac{\partial \phi}{\partial y} = \frac{q}{4\pi\epsilon_0}\left(\frac{\frac{1}{2a}-y}{\sqrt[3]{\left(\frac{\sqrt{3}}{2a}-x\right)^2+\left(\frac{1}{2a}-y\right)^2}}+\frac{\frac{1}{2a}-y}{\sqrt[3]{\left(-\frac{\sqrt{3}}{2a}-x\right)^2+\left(\frac{1}{2a}-y\right)^2}}-\frac{a+y}{\sqrt[3]{x^2+(-a-y)^2}}\right)$$
$$= \frac{q}{4\pi\epsilon_0}\left(\frac{-y}{\sqrt[3]{(-x)^2+(-y)^2}}+\frac{-y}{\sqrt[3]{(-x)^2+(-y)^2}}-\frac{y}{\sqrt[3]{x^2+(-y)^2}}\right)$$
$$= -\frac{3qy}{4\pi\epsilon_0}\frac{1}{\sqrt[3]{x^2+y^2}} \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives:

$$\mathbf{E} = \left(\frac{3qx}{4\pi\epsilon_0}\frac{1}{\sqrt[3]{x^2+y^2}}, \frac{3qy}{4\pi\epsilon_0}\frac{1}{\sqrt[3]{x^2+y^2}}\right)$$

Now, for the magnitude, take a point, or a radius from the origin, r on the $x-y$ plane such that $r=x=y$. This makes the electric field (in the positive x and y direction):

$$\mathbf{E} = \left(\frac{3q}{8\sqrt{2}\pi\epsilon_0 r^2}, \frac{3q}{8\sqrt{2}\pi\epsilon_0 r^2}\right)$$

The magnitude of the field at this radius from the distribution of charges is thus:

$$|\mathbf{E}| = \sqrt{{\mathbf{E}_x}^2+{\mathbf{E}_y}^2}$$
$$= \sqrt{\left(\frac{3q}{8\sqrt{2}\pi\epsilon_0 r^2}\right)^2+\left(\frac{3q}{8\sqrt{2}\pi\epsilon_0 r^2}\right)^2}$$
$$= \frac{3q}{8\pi\epsilon_0 r^2}$$

This expression follows the inverse square law (as seen by the r^2 term in the denominator) as is expected from a field radiating from a central point. As such, this answer seems reasonable in calculating the magnitude of an electric field at some large distance, $r \gg a$.

d. Determine an expression for the electric field along the y-axis. Make a plot of the field against position. Confirm that the field you calculate is consistent with how positive test charges would move if released at any point along the y-axis:

At any point along the y axis where $x = 0$, $\mathbf{E}_x = 0$, and all x terms in $\mathbf{E}_y$ are equal to 0. Therefore,

$$\mathbf{E}_y = \frac{q}{4\pi\epsilon_0}\left(\frac{\frac{1}{2a} - y}{\sqrt[3]{\left(\frac{\sqrt{3}}{2a} - x\right)^2 + \left(\frac{1}{2a} - y\right)^2}} + \frac{\frac{1}{2a} - y}{\sqrt[3]{\left(-\frac{\sqrt{3}}{2a} - x\right)^2 + \left(\frac{1}{2a} - y\right)^2}} - \frac{a + y}{\sqrt[3]{x^2 + (-a - y)^2}}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{\frac{1}{2a} - y}{\sqrt[3]{\left(\frac{\sqrt{3}}{2a}\right)^2 + \left(\frac{1}{2a} - y\right)^2}} + \frac{\frac{1}{2a} - y}{\sqrt[3]{\left(-\frac{\sqrt{3}}{2a}\right)^2 + \left(\frac{1}{2a} - y\right)^2}} - \frac{a + y}{\sqrt[3]{(-a - y)^2}}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{\frac{1}{a} - 2y}{\sqrt[3]{\left(\frac{\sqrt{3}}{2a}\right)^2 + \left(\frac{1}{2a} - y\right)^2}} - \frac{a + y}{\sqrt[3]{(a + y)^2}}\right)$$

and,

$$|\mathbf{E}| = \mathbf{E}_y$$

Plotting this over some range of y,

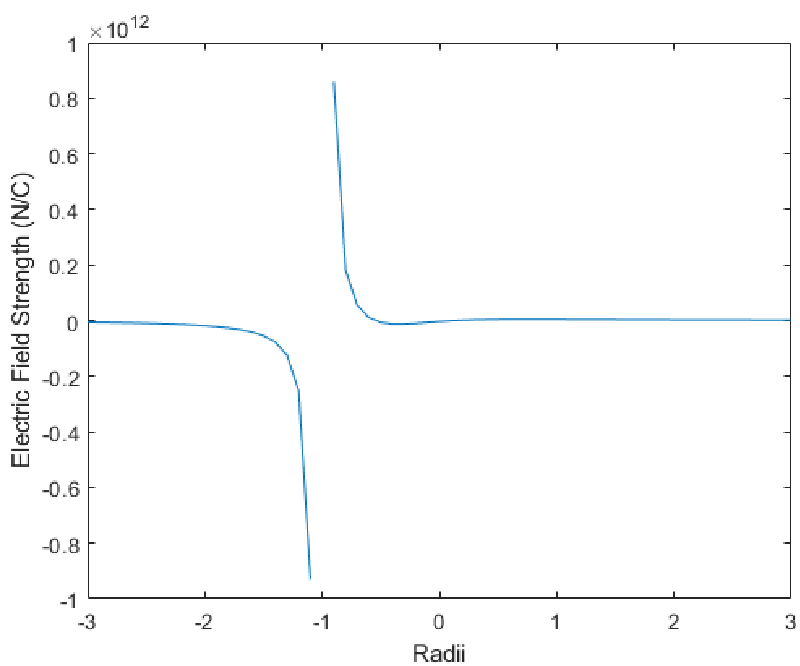

Figure 2: Electric Field Strength with respect to range in y

This shows that a positive test charge would be repelled into the negative y direction if initially released before q_3, as expected. If released closer to the centre of the circle, it would be repelled into the positive y direction, however with some initial dip in the curve due to the repulsive interaction of q_1 and q_2 which work against the field created by q_3 in this range.

Problem 2.2

Street lighting once commonly used high-pressure sodium lamps since they provided the greatest amount of human-suitable illumination for the least consumption of electricity. Their yellow colour comes chiefly from the sodium doublet: the prominent twin spectral-line feature. Consider electromagnetic radiation at the sodium doublet wavelengths propagating as a plane wave in free space with amplitude $E_0 = 5.0$ N/C.

a. Using your knowledge of electromagnetic waves, propose equations for the electric and magnetic fields of the wave of one of the lines of the doublet. Clearly explain any assumptions that you make:

   Firstly, we have Maxwell's Equations in free space:

$$\nabla \cdot \mathbf{E} = 0 \tag{7}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{8}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{9}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{10}$$

   Taking the curl of equations (8) and (10) gives:

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}) &= \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \\ &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \\ &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \end{aligned} \tag{11}$$

   and,

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{B}) &= \nabla \times \left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) \\ &= \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \\ &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} \end{aligned} \tag{12}$$

   however, we have the vector identity

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V} \tag{13}$$

   for some vector $\mathbf{V}$. Equations (11) and (12) may be shown in the form of equation (13), and simplified taking into account equations (7) and (9). This gives the two final forms:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0 \tag{14}$$

   and,

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0 \tag{15}$$

   Equations (14) and (15) are of the form of a wave propagating through a field in three dimensions, however a plane wave propagates in one dimension following an equation of the form

$$\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{16}$$

   So, equations (14) and (15) may be written in the form of equation (16) to follow the criteria of the question:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{\partial^2 \mathbf{E}}{\partial x^2} = 0 \tag{17}$$

and,

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \frac{\partial^2 \mathbf{B}}{\partial x^2} = 0 \tag{18}$$

Equations (17) and (18) describe second order harmonic waves, the general solution of which (for waves travelling at c) is of the form:

$$\psi(x,t) = A\cos\left(2\pi f\left(\frac{x}{c} - t\right) + \phi\right) \tag{19}$$

where A is the amplitude of the wave, f is the frequency of the wave, x is the x-axis displacement from the source of the wave, t is the time after initial propagation, and ϕ is the phase offset of the wave. Writing equation (19) in terms of the electromagnetic fields yields the following solutions:

$$\mathbf{E}(x,t) = A\cos\left(2\pi f\left(\frac{x}{c} - t\right) + \phi\right) \tag{20}$$

$$\mathbf{B}(x,t) = A\cos\left(2\pi f\left(\frac{x}{c} - t\right) + \phi\right) \tag{21}$$

These general solutions (for one of the lines of the doublet) are expressed under the assumption that the sodium doublet operates at a constant intensity, and that the speed of propagation is c (which may or may not be true depending on the medium in which the doublet is in).

b. Evaluate all constants in your expressions where possible:

Firstly, the phase offset may be assumed to be 0 for the simplest possible wave propagating in one dimension. As stated in the criteria, the amplitude of the electric field, E_0, is 5.0 N/C. The amplitude of the magnetic field, however, is slightly complicated. We know that this simple example of an electromagnetic wave propagating in one dimension follows Maxwell's Equations, so it will obey equations (7) through (10). Specifically, we may substitute the general solutions (20) and (21) into equation (8), taking into account the updated parameters:

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\frac{\partial}{\partial x} E_0 \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) &= -\frac{\partial}{\partial t} B_0 \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) \\
-\frac{2\pi f}{c} E_0 \sin\left(2\pi f\left(\frac{x}{c} - t\right)\right) &= -2\pi f B_0 \sin\left(2\pi f\left(\frac{x}{c} - t\right)\right) \\
B_0 &= \frac{E_0}{c}
\end{aligned} \tag{22}$$

Now the amplitudes of each of the electromagnetic waves are known, which leaves only one constant left to determine. Since the sodium doublet emits yellow light, it has a frequency of about 540 THz (5.4×10^{14} Hz) [1]. With all of these constants evaluated, the general solutions may be rewritten as

$$\mathbf{E}(x,t) = 5\cos\left(10.8\pi \times 10^{14}\left(\frac{x}{c} - t\right)\right) \tag{23}$$

$$\mathbf{B}(x,t) = \frac{5}{c}\cos\left(10.8\pi \times 10^{14}\left(\frac{x}{c} - t\right)\right) \tag{24}$$

c. Demonstrate how your answers to part a. satisfy Maxwell's Equations in free space:

Equations (17) and (18) were derived from Maxwell's Equations in free space, and may be used to check the validity of the general solutions (20) and (21). For the sake of accuracy, the solutions will be rewritten with updated parameters:

$$\mathbf{E}(x,t) = E_0 \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) \tag{25}$$

$$\mathbf{B}(x,t) = B_0 \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) \tag{26}$$

where $B_0 = \frac{E_0}{c}$. Equations (25) and (26) may be substituted into equations (17) and (18), where a final result of 0 is to be expected should they be valid solutions.

$$\begin{aligned}
0 &= \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{\partial^2 \mathbf{E}}{\partial x^2} \\
&= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E_0 \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) - \frac{\partial^2}{\partial x^2} E_0 \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) \\
&= -\frac{1}{c^2} 4\pi^2 f^2 E_0 \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) + \frac{4\pi^2 f^2 B_0}{c^2} \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) \\
&= 0
\end{aligned}$$

and,

$$\begin{aligned}
0 &= \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \frac{\partial^2 \mathbf{B}}{\partial x^2} \\
&= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} B_0 \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) - \frac{\partial^2}{\partial x^2} B_0 \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) \\
&= -\frac{1}{c^2} 4\pi^2 f^2 B_0 \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) + \frac{4\pi^2 f^2 B_0}{c^2} \cos\left(2\pi f\left(\frac{x}{c} - t\right)\right) \\
&= 0
\end{aligned}$$

This shows that the proposed general solutions obey Maxwell's Equations in free space, and the expressions are valid for the question.

d. Describe how your equations would change if the wave was spherical rather than planar:

If the wave were spherical, equations (14) and (15) would be the wave equations as opposed to equations (17) and (18). In a general form,

$$\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = 0 \tag{27}$$

Factoring in the vector identity $\nabla^2 \psi = \nabla \cdot (\nabla \psi)$ into equation (27) gives

$$\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} - \left(\frac{\partial^2 \psi_x}{\partial x^2} + \frac{\partial^2 \psi_y}{\partial y^2} + \frac{\partial^2 \psi_z}{\partial z^2}\right) = 0 \tag{28}$$

where for each $\mathbf{E}$ and $\mathbf{B}$, ψ may be replaced with the respective field. The general solutions for each field would then take into account the three dimensional axes, resulting in:

$$\psi(x,y,z,t) = \psi_0 \cos\left(k_x x + k_y y + k_z z - \omega t\right) \tag{29}$$

Part B - Advanced

Problem 2.3

(Advanced) The simulation used for the electrostatics section of the course shows the electric potential and the electric field for two-dimensional and three-dimensional situations.

a. Briefly discuss the differences in the field and potential for a single point charge in two-dimensions compared to the same situation in three-dimensions:

   Looking at the simulation on 5-Minute Physics, the potential appears to be much greater for a 3D simulation as opposed to a 2D simulation. On a related note, it appears to be much steeper, with a greater reduction of potential with respect to distance for the 3D case as opposed to the 2D case. Similarly, the electric field seems to follow this relationship. The electric field in the 3D case seems to 'fall off' much sooner with respect to distance than the 2D case.

b. The electric potential for a single point charge in two dimensions is

$$\phi = \frac{-\lambda}{2\pi\epsilon_0}\log r$$

   where r is the distance from the charge and λ is a constant representing the strength of the charge. Calculate the potential for a "dipole" in two dimensions − a positive and a negative charge of equal magnitude separated by a distance $2a$. (Note that this is a problem in electrostatics so the positions of the charges are fixed):

   Suppose the potential is being measured from some point at a distance r_+ from the positive charge (which lies at (-a, 0)), and a distance r_- from the negative charge (which lies at (a, 0)) of the dipole. In this case, the point is a distance r from the centre of the dipole (at (0,0)). The potential of the dipole, ϕ_D, can simply be represented as the superposition of the potentials of each of its charges (with one of them being the negative of the other due to the opposite charge). That is,

$$\begin{aligned} \phi_D &= \phi_+ + \phi_- \\ &= -\frac{\lambda}{2\pi\epsilon_0}\log(r_+) + \frac{\lambda}{2\pi\epsilon_0}\log(r_-) \\ &= \frac{\lambda}{2\pi\epsilon_0}\log\left(\frac{r_-}{r_+}\right) \end{aligned} \tag{30}$$

   Now, suppose the point of measurement lies at some angle θ with respect to the centre of the dipole. With the use of the cosine rule,

$$\begin{aligned} \cos\theta &= \frac{r^2 + a^2 - {r_-}^2}{2ar} \\ \Rightarrow r_- &= \sqrt{r^2 + a^2 - 2ar\cos\theta} \end{aligned} \tag{31}$$

   Similarly,

$$\begin{aligned} \cos(180 - \theta) &= \frac{r^2 + a^2 - {r_+}^2}{2ar} \\ \Rightarrow r_+ &= \sqrt{r^2 + a^2 - 2ar\cos(180 - \theta)} \end{aligned} \tag{32}$$

   Equation (30) may be rewritten with the more general conditions of the measurement point shown by equations (31) and (32):

$$\phi_D = \frac{\lambda}{2\pi\epsilon_0}\log\left(\frac{\sqrt{r^2 + a^2 - 2ar\cos\theta}}{\sqrt{r^2 + a^2 - 2ar\cos(180 - \theta)}}\right) \tag{33}$$

c. Calculate an expression for the electric field along a line perpendicular to the axis of the dipole and crossing the midpoint (so if your charges are at $x = \pm a$, then find the field along the y-axis).
Hint: Put some thought into what you expect before blunt force application of the relevant equations. There are also easier and harder forms in which the potential can be written before calculating the field:

Since the field (and consequently the potential) is only being taken through the y-axis while $x = 0$, θ can be set to equal $90°$, meaning that $\cos\theta = 0$. This simplifies equation (33) to

$$\begin{aligned}\phi_D &= \frac{\lambda}{2\pi\epsilon_0}\log\left(\frac{\sqrt{r^2+a^2}}{\sqrt{r^2+a^2}}\right)\\ &= 0\end{aligned}$$

And so the potential at any point along the y-axis is 0. Intuitively, one would say that the electric field along this axis would also be 0, however this is not the case. Firstly, along the y-axis, $r_+ = \sqrt{x^2+y^2}$, where x and y are the coordinates of the point of measurement taking the point charge to be at the origin (the same is true for r_-). Also of note is the fact that the electric field obeys the principle of superposition, so the electric field at any point along the y-axis can be found via the summation of the fields of the two charges. With this in mind,

$$\begin{aligned}\mathbf{E}_+ &= -\nabla\phi_+\\ &= \left(\frac{\partial\phi}{\partial x}\right)\mathbf{i} + \left(\frac{\partial\phi}{\partial y}\right)\mathbf{j}\\ &= \left(\frac{\partial}{\partial x}\frac{\lambda}{2\pi\epsilon_0}\log r_+\right)\mathbf{i} + \left(\frac{\partial}{\partial y}\frac{\lambda}{2\pi\epsilon_0}\log r_+\right)\mathbf{j}\\ &= \frac{\lambda}{2\pi\epsilon_0}\left(\left(\frac{\partial}{\partial x}\log\left(\sqrt{x^2+y^2}\right)\right)\mathbf{i} + \left(\frac{\partial}{\partial y}\log\left(\sqrt{x^2+y^2}\right)\right)\mathbf{j}\right)\\ &= \frac{\lambda}{2\pi\epsilon_0}\left(\left(\frac{x}{x^2+y^2}\right)\mathbf{i} + \left(\frac{y}{x^2+y^2}\right)\mathbf{j}\right) \qquad (34)\end{aligned}$$

Similarly, one can arrive at

$$\mathbf{E}_- = -\frac{\lambda}{2\pi\epsilon_0}\left(\left(\frac{x}{x^2+y^2}\right)\mathbf{i} + \left(\frac{y}{x^2+y^2}\right)\mathbf{j}\right) \qquad (35)$$

Now examine some arbitrary point along the y-axis. In the reference frame of the positive charge, this point is at $x = a$, and from the reference frame of the negative charge, the point is at $x = -a$. The y value of the point is the same for both reference frames. Substituting these values into equations (34) and (35) and adding them gives the total electric field along the y axis.

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_+ + \mathbf{E}_-\\ &= \frac{\lambda}{2\pi\epsilon_0}\left(\left(\frac{a}{a^2+y^2}\right)\mathbf{i} + \left(\frac{y}{a^2+y^2}\right)\mathbf{j}\right) - \frac{\lambda}{2\pi\epsilon_0}\left(\left(\frac{-a}{(-a)^2+y^2}\right)\mathbf{i} + \left(\frac{y}{(-a)^2+y^2}\right)\mathbf{j}\right)\\ &= \left(\frac{a\lambda}{\pi\epsilon_0\left(a^2+y^2\right)}, 0\right) \qquad (36)\end{aligned}$$

As can be seen, the electric field has no y component, and is purely acting in the x direction.

d. Plot this expression and make comparisons with a few measurements from the simulation:

A plot of the electric field strength (in the positive x-direction) with respect to position on the y-axis can be seen in Figure 3.

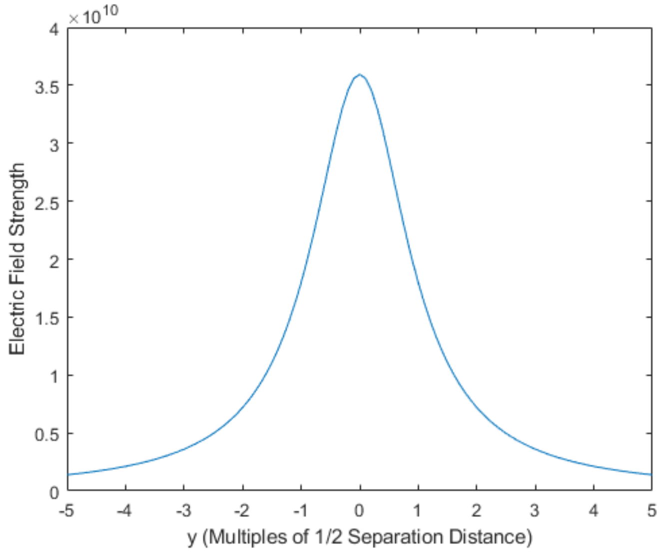

Figure 3: X-Component Electric Field Strength with Respect to Y-Axis

This plot seems largely consistent with the simulation on 5-Minute Physics. For a dipole with one +q and one -q charge, the test field on the simulation at $x = y = 0$ was measured to be approx. 68N/C. At a y-multiple of 1 times 1/2 the separation distance, the test field was measured to be approx. 33N/C in the x-direction, corresponding to a drop of about 49%. Figure 3 shows a drop of field strength of approx. 50% at the same relative y distance from the centre of the dipole. Accounting for error in manual measurement in the simulation, these two field strength drops appear to be consistent, and the model appears to be valid.

Appendices

Appendix 1 - Figure 1 Matlab Script

```
clear all;
charge = 1;
[x, y] = deal(-1.5:0.01:1.5);
[X, Y] = meshgrid(x, y);
radius = 1;
const = 1 / (4 * pi * 8.8542e-12);
r1 = sqrt(((sqrt(3) / (2 * radius)) - X).^2 + ((1 / (2 * radius)) - Y).^2);
r2 = sqrt((-(sqrt(3) / (2 * radius)) - X).^2 + ((1 / (2 * radius)) - Y).^2);
r3 = sqrt(X.^2 + (- radius - Y).^2);
pot = (charge / const) * ((1 ./ r1) + (1 ./ r2) + (1 ./ r3));
imagesc(x, y, pot); axis xy;
view(2);
colorbar
hold on

% draw the circle of radius a
circ_bounds = 0:pi/50:2*pi;
circx = radius * cos(circ_bounds);
circy = radius * sin(circ_bounds);
plot(circx, circy);
hold on

% plot electric field vectors
x = -1.5:0.3:1.5;
y = -1.5:0.3:1.5;
[X, Y] = meshgrid(x, y);
r1 = sqrt(((sqrt(3) / (2 * radius)) - X).^2 + ((1 / (2 * radius)) - Y).^2);
r2 = sqrt((-(sqrt(3) / (2 * radius)) - X).^2 + ((1 / (2 * radius)) - Y).^2);
r3 = sqrt(X.^2 + (- radius - Y).^2);
pot = (charge / const) * ((1 ./ r1) + (1 ./ r2) + (1 ./ r3));
[elecX, elecY] = gradient(pot);
elecX = -elecX;
elecY = -elecY;
elecX(isinf(elecX)) = nan;
elecY(isinf(elecY)) = nan;
quiver(X, Y, elecX, elecY);
hold off
```

Appendix 2 - Figure 2 Matlab Script

```
clear all;
y = [-3:0.1:3];
radius = 1;
const = 1 / (4 * pi * 8.8542e-12);
charge = 1;
a1 = ((1 / radius) - 2.*y) ./ (((3 / (4 * radius^2))^2 + ((1 / (2 * radius)) + y).^2).^(3/2));
a2 = (radius + y) ./ (((radius + y).^2).^(3/2));
Ey = -1 * const * charge * (a1 - a2);
plot(y, Ey);
xlabel("Radii");
ylabel("Electric Field Strength (N/C)");
```

Appendix 3 - Figure 3 Matlab Script

```
clear all;
y = [-5:0.1:5];
dist = 1;
charge = 1;
const = 1 / (pi * 8.8542e-12);
field = const * (dist * charge) ./ (dist^2 + y.^2);
plot(y, field);
xlabel("y (Multiples of 1/2 Separation Distance)")
ylabel("Electric Field Strength")
```

References

[1] Bohren, C, 2006. *Fundamentals of Atmospheric Radiation: An Introduction with 400 Problems* Wiley-VCH. p. 214. Accessed 18/04/2020