PHYS2100 Semester 2 2021 Practice Exam

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November 17, 2021

1.

$$\begin{aligned}
F(r) = -V'(r) &= -\frac{GMm}{x^2} \\
V'(r) &= \frac{GMm}{x^2} \\
V(r) &= -\frac{GMm}{x} \\
T = \frac{1}{2}mv^2 &= \frac{1}{2}m\dot{x}^2 \\
E = T + V &= \frac{1}{2}mv^2 - \frac{GMm}{x}
\end{aligned}$$

At infinite distance,

$$\begin{aligned}
E &= 0 - 0 = 0 \\
\therefore \frac{GMm}{x} &= \frac{1}{2}mv^2 \\
\frac{2GM}{x} &= v^2 \\
\frac{Rc^2}{x} &= v^2 \\
v &= -c\sqrt{\frac{R}{x}}
\end{aligned}$$

2. (i) Let y be the verticle length of string from pulley A to mass m_2. So then the remaining length of string on the incline would have length $l - y$. y will be the free variable in this solution. Let the vector space be centred at the tip of the inclined plane.

$$\begin{aligned}
\mathbf{r}_1 &= h - (l-y)\cos\beta\hat{\mathbf{i}} + h\tan\beta - (l-y)\sin\beta\hat{\mathbf{j}} \\
\mathbf{r}_2 &= h\hat{\mathbf{i}} + h\tan\beta - y\hat{\mathbf{j}} \\
\mathbf{v}_1 &= \dot{y}(\cos\beta\hat{\mathbf{i}} + \sin\beta\hat{\mathbf{j}}) \\
\mathbf{v}_2 &= -\dot{y}\hat{\mathbf{j}} \\
T &= \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_2\dot{y}^2 \\
V &= m_1 g(h\tan\beta + (y-l)\sin\beta) + m_2 g(h\tan\beta - y) \\
L &= T - V = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_2\dot{y}^2 - m_1 g(h\tan\beta + (y-l)\sin\beta) - m_2 g(h\tan\beta - y)
\end{aligned}$$

(ii)

$$
\begin{aligned}
0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} \\
&= m_1\ddot{y} + m_2\ddot{y} + m_1 g\sin\beta - m_2 g \\
&= \ddot{y} + \frac{m_1 g\sin\beta - m_2 g}{m_1+m_2} \\
\ddot{y} &= -\frac{m_1 g\sin\beta - m_2 g}{m_1+m_2} \\
\Rightarrow y &= -\frac{m_1 g\sin\beta - m_2 g}{2(m_1+m_2)}t^2 + c_1 t + c_2
\end{aligned}
$$

(iii) The system is in equilibrium when $\ddot{y} = \dot{y} = 0$. I.e.

$$
\begin{aligned}
0 = \ddot{y} &= -\frac{m_1 g\sin\beta - m_2 g}{m_1+m_2} \\
0 = \dot{y} &= -\frac{m_1 g\sin\beta - m_2 g}{m_1+m_2}t + c_1 = c_1 \\
\Rightarrow y_0 &= -\frac{m_1 g\sin\beta - m_2 g}{2(m_1+m_2)}t^2 + c_1 t + c_2 = c_2
\end{aligned}
$$

3. (i)

$$
\begin{aligned}
\mathbf{F}(x,y) &= -k_1 x\hat{\mathbf{i}} - k_2 y\hat{\mathbf{j}} = -\nabla V \\
&= -\frac{\partial V}{\partial x}\hat{\mathbf{i}} - \frac{\partial V}{\partial y}\hat{\mathbf{j}} \\
\Rightarrow V &= \frac{k_1}{2}x^2 + g(y) = \frac{k_2}{2}y^2 + g(x) \\
\Rightarrow V &= \frac{k_1}{2}x^2 + \frac{k_2}{2}y^2 + c \text{ let } c = 0 \text{ so that potential is 0 at origin} \\
\dot{\mathbf{r}} &= -\frac{k_1}{m}xt\hat{\mathbf{i}} - \frac{k_2}{m}yt\hat{\mathbf{j}} \\
T &= \frac{(k_1^2x^2 + k_2^2y^2)t^2}{2m} \\
\therefore L = T - V &= \frac{(k_1^2x^2 + k_2^2y^2)t^2}{2m} - \frac{k_1x^2 + k_2y^2}{2} \\
&= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{k_1x^2 + k_2y^2}{2} \\
p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\
p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y} \\
H &= p_x\dot{x} + p_y\dot{y} - L \\
&= m\dot{x}^2 + m\dot{y}^2 - \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{k_1x^2 + k_2y^2}{2} \\
&= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{k_1x^2 + k_2y^2}{2} \\
&= \frac{p_x^2 + p_y^2}{2m} + \frac{k_1x^2 + k_2y^2}{2}
\end{aligned}
$$

(ii)

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$
$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$
$$\dot{p_x} = -\frac{\partial H}{\partial x} = -k_1 x$$
$$\dot{p_y} = -\frac{\partial H}{\partial y} = -k_2 y$$

(iii)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + k_1 x = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} + k_2 y = 0$$
$$\ddot{x} = -\frac{k_1}{m}x,\ \ddot{y} = -\frac{k_2}{m}y$$
$$\Rightarrow x = A\sin\left(\sqrt{\frac{k_1}{m}}t\right) + B\cos\left(\sqrt{\frac{k_1}{m}}t\right)$$
$$1 = A\sin(0) + B\cos(0)$$
$$\Rightarrow B = 1$$
$$\dot{x} = -\sqrt{\frac{k_1}{m}}\sin\left(\sqrt{\frac{k_1}{m}}t\right) + \sqrt{\frac{k_1}{m}}A\cos\left(\sqrt{\frac{k_1}{m}}t\right)$$
$$-1 = \sqrt{\frac{k_1}{m}}A$$
$$\Rightarrow A = -\sqrt{\frac{m}{k_1}}$$
$$y = C\sin\left(\sqrt{\frac{k_2}{m}}t\right) + D\cos\left(\sqrt{\frac{k_2}{m}}t\right)$$
$$0 = C = D$$
$$\therefore x = -\sqrt{\frac{m}{k_1}}\sin\left(\sqrt{\frac{k_1}{m}}t\right) + \cos\left(\sqrt{\frac{k_1}{m}}t\right),\ y = 0$$

4.

$$\dot{q} = \frac{\partial H}{\partial p} = p$$
$$\dot{p} = -\frac{\partial H}{\partial q} = -V'(q) = \begin{cases} -2q(|q|-a) - q^2 & \text{if } q \text{ is positive} \\ -2q(|q|-a) + q^2 & \text{if } q \text{ is negative} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Consider the case when $q > 0$

$$\begin{aligned}
(\dot{q}, \dot{p}) &= (0,0) = (p, -2q(q-a) - q^2) \\
&\Rightarrow p = 0 \\
2q(q-a) &= -q^2 \\
q - a &= -\frac{q}{2} \\
q + \frac{q}{2} &= a \\
\frac{3q}{2} &= a \\
q &= \frac{2}{3}a
\end{aligned}$$

Otherwise if q is negative,

$$\begin{aligned}
\dot{p} &= 0 = -2q(-q-a) + q^2 \\
-q^2 &= 2q(q+a) \\
-q &= 2(q+a) \\
3q &= -2a \\
q &= -\frac{2}{3}a
\end{aligned}$$

Note, in either case, $a > 0$

$$\begin{aligned}
H &= 0 + \frac{4}{9}a^2(\frac{2}{3}a - a) \\
&= -\frac{4}{27}a^3
\end{aligned}$$

5. (a) Note that $\frac{df^n(x)}{dx} = \frac{df^{n-1}(x)}{dx}\frac{df(x)}{dx}$, so

$$\lim_{n\to\infty} \frac{1}{n} \ln \left| \left.\frac{df^n(x)}{dx}\right|_{x=x_0} \right|$$

$$\lim_{n\to\infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} \left.\frac{df(x)}{dx}\right|_{x=x_i} \right|$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \left.\frac{df(x)}{dx}\right|_{x=x_i} \right|$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

(b) In a fixed point, $f'(x_i) = 0$, so the exponent would be $-\infty$. Else for attractors, p-cycles and, in general, cases where x converges, $\lambda < 0$.

(c)

$$f'(x) = \begin{cases} r, & \text{if } 0 \le x \le \frac{1}{2} \\ -r, & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

$$\begin{aligned} \lambda &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln r \\ &= \lim_{n\to\infty} \ln r = \ln r \end{aligned}$$

So λ must diverge when $r > 1$.

6. (i)

$$\begin{aligned} d\tau^2 &= dt^2(1 - 0.8^2) = 0.36dt^2 \\ \tau^2 &= 0.36t^2 \\ \tau &= 0.6t \end{aligned}$$

So total time for Alice is $\tau = 0.6 \cdot 4$ years, or 2.4 years.

(ii) Due to length contraction, the journey that Alice actually takes is a lot shorter than what Bob perceives. And so Alice is covering less distance in the same amount of time, thus, isn't travelling faster than light.

7. (i)

$$\begin{aligned} d\tau &= dt_1\sqrt{1 - \overrightarrow{v_1}} = dt_2\sqrt{1 - \overrightarrow{v_2}} \\ \mathbf{u}_1 &= (\frac{dt_1}{d\tau}, \frac{d\overrightarrow{x_1}}{d\tau}) = (\frac{dt_1}{d\tau}, \frac{d\overrightarrow{x_1}}{dt_1}\frac{dt_1}{d\tau}) \\ &= (\frac{1}{\sqrt{1 - \overrightarrow{v_1}}}, \overrightarrow{v_1}\frac{1}{\sqrt{1 - \overrightarrow{v_1}}}) = (\gamma_1, \gamma_1\overrightarrow{v_1}) \end{aligned}$$

Similarly, $\mathbf{u}_2 = (\gamma_2, \gamma_2\overrightarrow{v_2})$ where $\gamma_2 = (1 - \overrightarrow{v_2})^{-\frac{1}{2}}$

(ii)

$$
\begin{aligned}
V_{rel} &= \sqrt{1 - (\mathbf{u}_1 \cdot \mathbf{u}_2)^{-2}} \\
&= \sqrt{1 - (-\gamma_1\gamma_2 + \gamma_1\gamma_2 v_1 v_2)^{-2}} \\
&= \sqrt{1 - \left(\frac{-1 + v_1 v_2}{\sqrt{(1 - v_1)(1 - v_2)}}\right)^{-2}} \\
&= \sqrt{1 - \left(\frac{(1 - v_1)(1 - v_2)}{(v_1 v_2 - 1)^2}\right)} \\
&= \sqrt{\frac{(v_1 v_2 - 1)^2 - (1 - v_1)(1 - v_2)}{(v_1 v_2 - 1)^2}} \\
&= \sqrt{\frac{1 + v_1^2 v_2^2 - 2v_1 v_2 - 1 - v_1 v_2 + v_1 + v_2}{1 + v_1^2 v_2^2 - 2v_1 v_2}} \\
&= \sqrt{\frac{v_1^2 v_2^2 - 3v_1 v_2 + v_1 + v_2}{1 + v_1^2 v_2^2 - 2v_1 v_2}}
\end{aligned}
$$

Fuck this

8.

$$
\begin{aligned}
E &= \sqrt{m^2 + p^2} \\
&= \sqrt{m^2 (1 + \left(\frac{p}{m}\right)^2)} \\
&= m\sqrt{1 + (\gamma v)^2} \\
&\approx m\left(1 + \frac{\gamma^2 v^2}{2}\right) \\
&\approx m + \frac{mv^2}{2}
\end{aligned}
$$

The $\frac{mv^2}{2}$ term contributes kinetic energy, while m contributes mass energy