Assignment 5

Monday, 9 October 2023 7:45 AM

Question 1:

We have a bar that occupies the domain $0 \leq x < \infty$. The bar is exposed to external forcing such that the system is described by

$$\begin{cases} R_0 \partial_{tt} U = \partial_x T = E \partial_{xx} U \\ \lim_{x \to \infty} U(x,t) = 0 \\ T(x=0,t) = b\sin(\omega t) \\ T + \tau_0 \partial_t T = E\left(\partial_x U + \tau_1 \partial_{xt} U\right) \end{cases}$$

Start by expressing stress in complex coordinates:

$$T(x=0,t) = be^{i\omega t}$$

We use the ansatz for quasistationary solutions:

$$U(x,t) = \bar{U}(x)e^{i\omega t}$$

$$T(x,t) = \bar{T}(x)e^{i\omega t}$$

and so we can substitute these into our system's constitutive relation:

$$\bar{T}(x)e^{i\omega t} + \tau_0 \cdot i\omega \bar{T}(x)e^{i\omega t} = E\left(\partial_x \bar{U}(x)e^{i\omega t} + \tau_1 i\omega \partial_x \bar{U}(x)e^{i\omega t}\right)$$

$$\Rightarrow \quad \bar{T}(x)\left(1 + i\omega\tau_0\right) = E\,\partial_x \bar{U}(x)\left(1 + i\omega\tau_1\right)$$

$$\Rightarrow \quad \bar{T}(x) = E\frac{(1+i\omega\tau_1)}{(1+i\omega\tau_0)}\partial_x \bar{U}(x)$$

$$= E^* \partial_x \bar{U}(x)$$

where $E^* = E' + iE'' \geq 0$ from the computation in the lecture.

Now, looking at the momentum equation:

$$R_0 \partial_{tt} U = E^* \partial_{xx} U$$

$$\Rightarrow -R_0\omega^2 \bar{U}(x)e^{i\omega t} = E^* \partial_{xx}\bar{U}(x)e^{i\omega t}$$

$$\Rightarrow -\frac{R_0\omega^2}{E^*}\bar{U}(x) = \partial_{xx}\bar{U}(x)$$

Setting $k^2 = \frac{R_0\omega^2}{E^*}$, we then have displacement solutions of the form

$$\bar{U}(x) = \alpha e^{ikx} + \beta e^{-ikx}$$

as in the lecture, to satisfy the far field condition $\lim_{x\to\infty} U(x,t) = 0$, we need to take $\alpha = 0$ and so

$$\bar{U}(x) = \beta e^{-ikx}$$

and so
$$\bar{U}(x) = \beta e^{-ikx}$$
$$\Rightarrow \quad \partial_x \bar{U}(x) = -ik\beta e^{-ikx} = \bar{T}(x)/E^*$$

Now we have that, at the left tip $x=0$,

$$T(0,t) = be^{i\omega t} = \bar{T}(0)e^{i\omega t}$$
$$= -ik\beta E^* e^{i\omega t}$$
$$\Rightarrow \quad b = -ik\beta E^* \quad \Rightarrow \beta = \frac{b}{-ikE^*} = \frac{ib}{kE^*}$$

and so $\quad U(x,t) = \bar{U}(x)e^{i\omega t}$
$$= \frac{ib}{kE^*} e^{i(\omega t - kx)}$$

and $\quad T(x,t) = \bar{T}(x)e^{i\omega t}$
$$= be^{i(\omega t - kx)}$$

from the lecture, we know that $E^* k = re^{i\delta}$ has phase
$$\delta = \frac{1}{2}\arctan\left(\frac{\omega(\tau_1 - \tau_0)}{1+\omega^2\tau_1\tau_0}\right)$$
and $i = e^{i\pi/2}$, hence
$$\frac{ib}{E^* k} = \frac{be^{i\pi/2}}{re^{i\delta}} = \frac{b}{r}e^{i(\pi/2 - \delta)}$$
has phase $\pi/2 - \delta$.

and so, at the left tip
$$U(0,t) = \frac{b}{r}e^{i(\omega t + (\pi/2 - \delta))}$$
$$T(0,t) = be^{i\omega t}$$
and so there is a phase difference of $\pi/2 - \delta$ between the displacement and the stress at the left tip.

This is the negative of the phase difference found in the lecture for imposed displacement.

Question 2:

We consider a rotation around the z-axis $\chi(\vec{X},t) = Q(t)\vec{X}$

We know that $U(\vec{X},t) = \chi(\vec{X},t) - \vec{X}$, and so

$$U(\vec{X},t) = (Q(t) - 1)\vec{X}$$

then,
$$V(\vec{X},t) = \frac{\partial}{\partial t}U(\vec{X},t) = \frac{\partial Q(t)}{\partial t}\vec{X}$$

where
$$\frac{\partial Q(t)}{\partial t} = \begin{pmatrix} -\omega\sin(\omega t) & -\omega\cos(\omega t) & 0 \\ \omega\cos(\omega t) & -\omega\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, in spatial coordinates,

$$u(\vec{x},t) = U(x(\vec{x},t),t)$$

where $x(\vec{x},t) = Q(t)^{-1}\vec{x}$

$$\Rightarrow \quad u(\vec{x},t) = (Q(t)-1)\,Q(t)^{-1}\,\vec{x}$$
$$= (1 - Q(t)^{-1})\,\vec{x}$$

and so, $v(\vec{x},t) = V(\chi(x,t),t)$
$$= \frac{\partial Q(t)}{\partial t}\,Q(t)^{-1}\,\vec{x}$$

as a check, compute $v(\vec{x},t) = D_t\vec{u}$:

$$v(\vec{x},t) = D_t\vec{u} = \frac{\partial u}{\partial t}\left(1 - \frac{\partial u}{\partial x}\right)^{-1}$$
$$= \frac{\partial}{\partial t}\left(-Q(t)^{-1}\vec{x}\right)\left(1 - \left(1 - Q(t)^{-1}\right)\right)^{-1}$$
$$= \frac{\partial Q(t)}{\partial t}\cdot Q(t)^{-2}\,\vec{x}\cdot\left(Q(t)^{-1}\right)^{-1}$$
$$= \frac{\partial Q(t)}{\partial t}\,Q(t)^{-1}\,\vec{x}$$

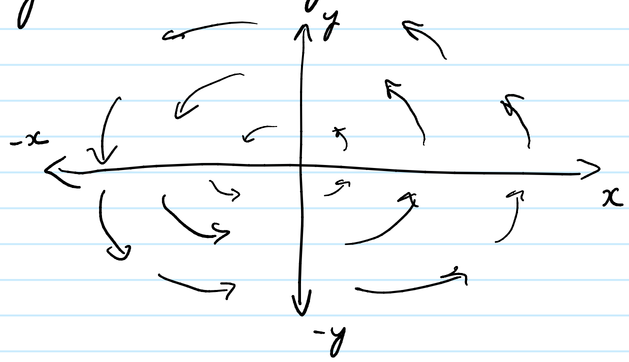
as with the first method.

To sketch a vector field in the x-y plane for v, first multiply the matrices

$$v(\vec{x},t) = \begin{pmatrix} -\omega\sin(\omega t) & -\omega\cos(\omega t) & 0 \\ \omega\cos(\omega t) & -\omega\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \omega\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This gives the vector field



Where the magnitude of the rotation is proportional to ω.