MATH2100 Assignment 3

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Question 1

5 Marks
For the system,

$$y_1' = {y_1}^2 + 3y_1y_2$$
$$y_2' = 2y_1 + {y_2}^3$$

There are critical points when $(y_1', y_2') = (0, 0)$, or rather when

$$0 = {y_1}^2 + 3y_1y_2 \Rightarrow {y_1}^2 = -3y_1y_2 \Rightarrow y_1 = -3y_2$$
$$0 = 2y_1 + {y_2}^3 \Rightarrow {y_2}^3 = -2y_1 = 6y_2 \Rightarrow {y_2}^2 = 6 \Rightarrow y_2 = \pm\sqrt{6}$$

And so there are critical points at $(y_1^*, y_2^*) = (\mp 3\sqrt{6}, \pm\sqrt{6})$. The linearized system is given by

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix} \begin{pmatrix} \bar{y_1} \\ \bar{y_2} \end{pmatrix}$$

where $(\bar{y_1}, \bar{y_2}) = (y_1 - y_1^*, y_2 - y_2^*)$. The general linearized system is then

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} 2y_1 + 3y_2 & 3y_1 \\ 2 & 3{y_2}^2 \end{pmatrix} \begin{pmatrix} \bar{y_1} \\ \bar{y_2} \end{pmatrix}$$

At the critical point $(y_1^*, y_2^*) = (-3\sqrt{6}, \sqrt{6})$, the linearized system is

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} -3\sqrt{6} & -9\sqrt{6} \\ 2 & 18 \end{pmatrix} \begin{pmatrix} \bar{y_1} \\ \bar{y_2} \end{pmatrix}$$

with $\mathrm{tr}A = -3\sqrt{6} + 18 > 0$, and $\det A = -3\sqrt{6} \times 18 - -9\sqrt{6} \times 2 = -54\sqrt{6} + 18\sqrt{6} = -36\sqrt{6} < 0$. As the determinant is less than 0, this point is classified as a saddle node.
At the critical point $(y_1^*, y_2^*) = (3\sqrt{6}, -\sqrt{6})$, the linearized system is

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} 3\sqrt{6} & 9\sqrt{6} \\ 2 & 18 \end{pmatrix} \begin{pmatrix} \bar{y_1} \\ \bar{y_2} \end{pmatrix}$$

with $\mathrm{tr}A = 3\sqrt{6} + 18 > 0$, and $\det A = 3\sqrt{6} \times 18 - 9\sqrt{6} \times 2 = 54\sqrt{6} - 18\sqrt{6} = 36\sqrt{6} > 0$. Since this is not yet conclusive, square of the trace minus four times the determinant must be found. This is $(\mathrm{tr}A)^2 - 4\det A = (3\sqrt{6} + 18)^2 - 4(36\sqrt{6}) \approx 290 > 0$. Since this value is greater than 0, this critical point corresponds to an unstable focus.
The 2D phase portrait of the system can be seen in Figure 1:

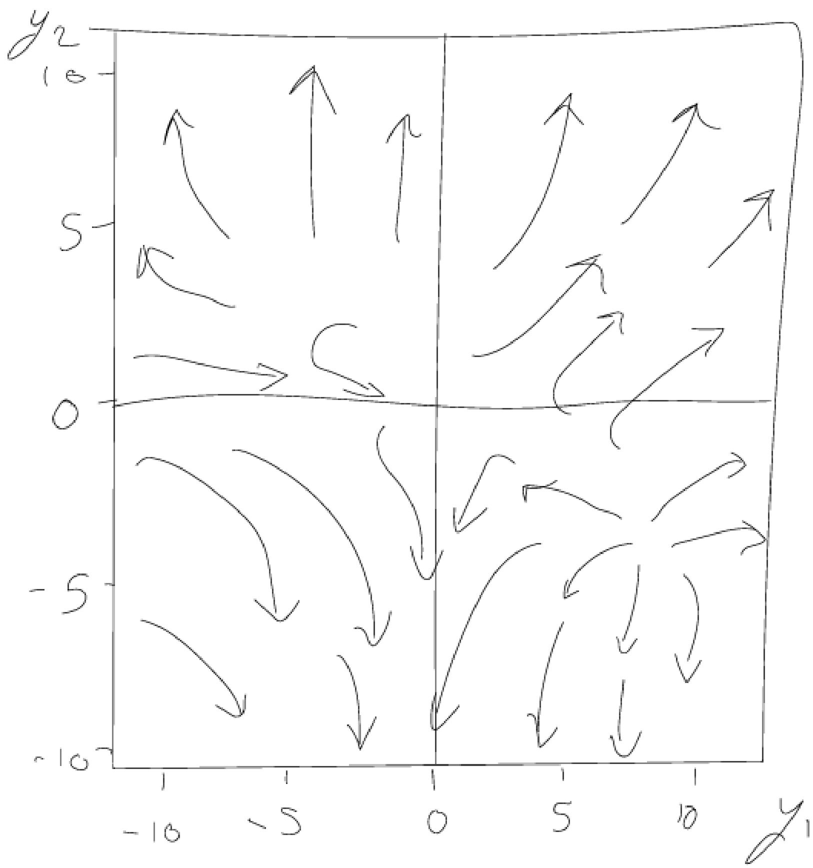
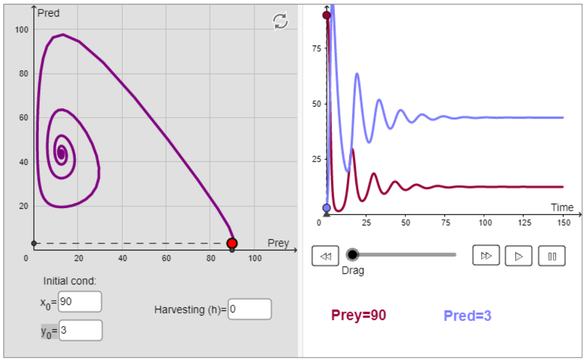


Figure 1: Hand Drawn Phase Portrait for ODE System

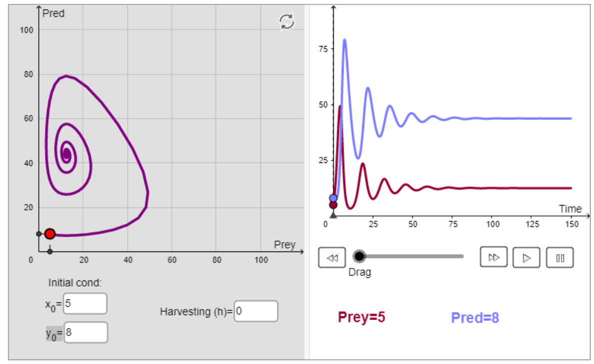
Question 2

5 Marks

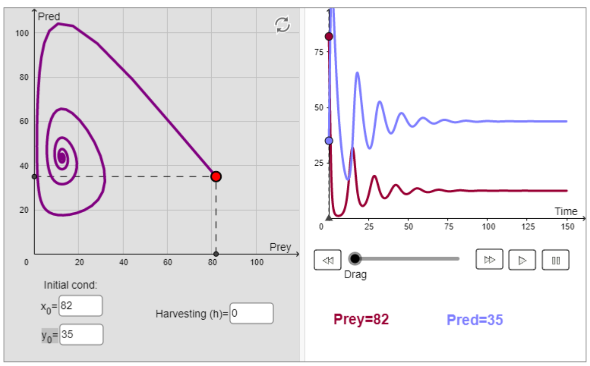
Figure 2 shows a range of variances in initial conditions for the predator-prey Lotka-Volterra Model.



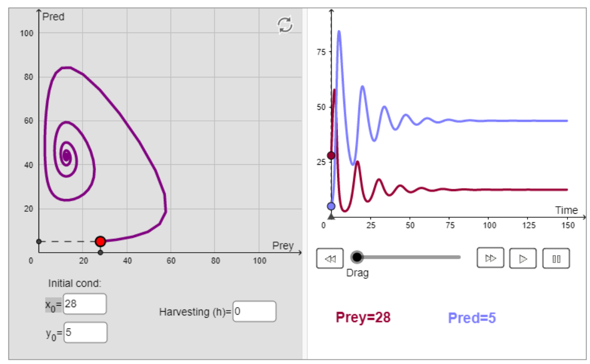
(a) Exp 1: $(x_0, y_0) = (90, 3)$



(b) Exp 2: $(x_0, y_0) = (5, 8)$



(c) Exp 3: $(x_0, y_0) = (82, 35)$



(d) Exp 4: $(x_0, y_0) = (28, 5)$

Figure 2: Variances in Initial Conditions of the Lotka-Volterra Model

As can be seen, and no matter the starting conditions, the model resembles a stable focus node, with an equilibrium position at roughly (13, 45). Although this is suitable mathematically, this model fails physically, where in Figure 2c the prey population falls below 2 (the required number to reproduce), so the prey species would (in reality) fail to extinction.

Question 3

5 Marks

For the function $f(t) = t^3e^{2t} + 4\cos(5t)$, the Laplace transform is the sum of the two respective terms of the equation

$$\mathcal{L}(f(t)) = \mathcal{L}(t^3e^{2t}) + 4\mathcal{L}(\cos(5t))$$

The Laplace transform of the first term is found by

$$\begin{aligned}
\mathcal{L}(t^3e^{2t}) &= (-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s-2}\right) \\
&= -1\frac{d}{ds}\left(\frac{2}{(s-2)^3}\right) \\
&= \frac{6}{(s-2)^4}
\end{aligned}$$

The Laplace transform of the second term is found by

$$\mathcal{L}(\cos(5t)) = \frac{s}{s^2+5^2} = \frac{s}{s^2+25}$$

And so the Laplace transform of $f(t)$ is

$$\mathcal{L}(f(t)) = \frac{6}{(s-2)^4} + \frac{4s}{s^2+25}$$

Question 4

5 Marks
Take the initial value problem

$$y' - 2y = 3e^{-t} \qquad\qquad y(0) = 4$$

The Laplace transform of the whole system is

$$\begin{aligned}
\mathcal{L}(y') - 2\mathcal{L}(y) &= 3\mathcal{L}(e^{-t}) \\
sY(s) - y(0) - 2Y(s) &= 3\frac{1}{s+1} \\
Y(s)(s-2) &= \frac{3}{s+1} + y(0) \\
Y(s) &= \frac{3}{(s+1)(s-2)} + \frac{y(0)}{s-2}
\end{aligned}$$

The final line of the above equation can be expressed in terms of partial fractions:

$$\frac{3}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$\Rightarrow 3 = A(s-2) + B(s+1)$$

If $s = 2$ $\qquad 3 = 3B \Rightarrow B = 1$

If $s = -1$ $\qquad 3 = -3A \Rightarrow A = -1$

$$\Rightarrow \frac{3}{(s+1)(s-2)} = \frac{1}{s-2} - \frac{1}{s+1}$$

And so

$$Y(s) = \frac{1}{s-2} - \frac{1}{s+1} + \frac{y(0)}{s-2}$$

Taking the inverse transform gives

$$y(t) = e^{2t} - e^{-t} + y(0)e^{2t}$$

Substituting in the initial value of $y(0)$ then gives

$$y(t) = 5e^{2t} - e^{-t}$$

Question 5

5 Marks
For the initial value problem for an RC Circuit,

$$RCy' + y = V(t), \qquad V(t) = \begin{cases} 0 & \text{if } t < 3, \\ 3 & \text{if } t \geq 3. \end{cases} \qquad R = 2, \qquad C = 1, \qquad y(0) = 4$$

the solution is found via the Laplace transform:

$$\begin{aligned}
\mathcal{L}(RCy') + \mathcal{L}(y) &= \mathcal{L}(V(t)) \\
RC(sY(s) - y(0)) + Y(s) &= 3\frac{e^{-3s}}{s} \\
RCsY(s) - RCy(0) + Y(s) &= 3\frac{e^{-3s}}{s} \\
Y(s)(RCs + 1) &= 3\frac{e^{-3s}}{s} + RCy(0) \\
Y(s) &= \frac{3e^{-3s}}{s(RCs+1)} + \frac{RCy(0)}{(RCs+1)}
\end{aligned}$$

Substituting in the given values gives

$$Y(s) = \frac{3e^{-3s}}{s(2s+1)} + \frac{4}{s+1/2}$$

The first term on the right hand side can be expressed in terms of partial fractions:

$$\begin{aligned}
\frac{3e^{-3s}}{s(2s+1)} &= \frac{A}{s} + \frac{B}{2s+1} \\
\Rightarrow 3e^{-3s} &= A(2s+1) + Bs
\end{aligned}$$

If $s = -1/2$ $\qquad 3e^{3/2} = -\frac{B}{2} \Rightarrow B = -6e^{3/2}$

If $s = 0$ $\qquad 3 = A \Rightarrow A = 3$

And so,

$$\frac{3e^{-3s}}{s(2s+1)} = \frac{3}{s} - \frac{6e^{3/2}}{2s+1}$$

And finally,

$$\begin{aligned}
Y(s) &= \frac{3}{s} - \frac{6e^{3/2}}{2s+1} + \frac{4}{s+1/2} \\
&= \frac{3}{s} + \frac{4 - 3e^{3/2}}{s+1/2} \\
&= 3\frac{1}{s} + \left(4 - 3e^{3/2}\right)\frac{1}{s+1/2}
\end{aligned}$$

Question 6

5 Marks

The system of ODEs was solved in Mathematica with the following code and results:

```
In[67]:= equas = {f'[t] == 3 f[t] + 5 g[t] - Sin[t], g'[t] == 2 f[t] - g[t] + Cos[t]};
         conds = {f[0] -> 0, g[0] -> 1};
         LaplaceTransform[equas, t, s] /. conds
```

Out[69]= $\left\{ s\,\text{LaplaceTransform}[f[t],t,s] == -\dfrac{1}{1+s^2} + 3\,\text{LaplaceTransform}[f[t],t,s] + 5\,\text{LaplaceTransform}[g[t],t,s],\ -1 + s\,\text{LaplaceTransform}[g[t],t,s] == \dfrac{s}{1+s^2} + 2\,\text{LaplaceTransform}[f[t],t,s] - \text{LaplaceTransform}[g[t],t,s] \right\}$

```
In[70]:= reduced = % /. {LaplaceTransform[f[t], t, s] → F[s], LaplaceTransform[g[t], t, s] → G[s]}
```

Out[70]= $\left\{ s\,F[s] == -\dfrac{1}{1+s^2} + 3\,F[s] + 5\,G[s],\ -1 + s\,G[s] == \dfrac{s}{1+s^2} + 2\,F[s] - G[s] \right\}$

```
In[71]:= sols = {F[s], G[s]} /. First@Solve[%, {F[s], G[s]}]
```

Out[71]= $\left\{ -\dfrac{-4-4s-5s^2}{(1+s^2)(-13-2s+s^2)},\ -\dfrac{5+2s+2s^2-s^3}{(1+s^2)(-13-2s+s^2)} \right\}$

```
In[72]:= InverseLaplaceTransform[sols, s, t]
```

Out[72]= $\left\{ \dfrac{203\,e^{(1-\sqrt{14})t} - 234\sqrt{14}\,e^{(1-\sqrt{14})t} + 203\,e^{(1+\sqrt{14})t} + 234\sqrt{14}\,e^{(1+\sqrt{14})t} - 406\,\text{Cos}[t] + 42\,\text{Sin}[t]}{1400},\ \dfrac{574\,e^{(1-\sqrt{14})t} + 53\sqrt{14}\,e^{(1-\sqrt{14})t} + 574\,e^{(1+\sqrt{14})t} - 53\sqrt{14}\,e^{(1+\sqrt{14})t} + 252\,\text{Cos}[t] + 336\,\text{Sin}[t]}{1400} \right\}$

Figure 3: Mathematica Solution for System of ODEs