Assignment 6

Tuesday, 24 October 2023   10:13 AM

Question 1:

a. To formulate the governing equations, we assume laminar flow in the x-direction,

$$\vec{v} = \begin{pmatrix} u(t,y) \\ 0 \\ 0 \end{pmatrix}; \quad \partial_x p = 0$$

with homogeneous flow across z, $\partial_z p = 0$, and hence no flow in the y-direction.

With no-slip boundary conditions, we have that the velocities at the upper and lower plates at time $t \geq 0$ are

$$\vec{v}(t, x, y=h, z) = u_0$$

and

$$\vec{v}(t, x, y=0, z) = 0$$

respectively.

We assume incompressible flow, that is

$$\partial_t \rho = 0 \ ; \quad \partial_x \rho = \partial_y \rho = \partial_z \rho = 0$$

Using the NS and continuity equations,

$$\rho D_t \vec{v} = -\nabla p + \mu \Delta \vec{v} + \rho \vec{f}$$

$$\nabla \cdot \vec{v} = 0$$

$$\Rightarrow \quad \rho(\partial_t v + (\vec{v} \cdot \nabla)\vec{v}) = -\nabla p + \mu \Delta \vec{v} + \rho \vec{f}$$

Assuming there are no body forces, $\vec{f} = 0$, we can simplify the equations to

$$\rho(\partial_t u(t,y) + (\partial_x u(t,y))u) = -\partial_x p + \mu\left(\frac{\partial^2 u(t,y)}{\partial y^2}\right) \qquad (1)$$

$$\rho(\partial_t(0) + (\partial_y(0)) \cdot 0) = -\partial_y p + \mu \cdot 0$$

$$\rho(\partial_t(0) + (\partial_z(0)) \cdot 0) = -\partial_z p + \mu \cdot 0$$

$$\partial_x u + 0 + 0 = 0 \qquad (2)$$

Using our laminar flow assumption, $\partial_x p = 0$, and inserting (2) into (1), we get,

$$(*) \quad \begin{cases} \rho\, \partial_t u(t,y) = \mu\left(\frac{\partial^2 u(t,y)}{\partial y^2}\right) \\ 0 = -\partial_y p \\ 0 = -\partial_z p \\ \partial_x u(t,y) = 0 \end{cases}$$

$$\Big(\quad \partial_x u(t,y) = 0$$

as our governing equations.

b. We had that the steady state solution is

$$u_\infty = \frac{u_0 y}{h} = u_\infty(y)$$

Now define $\bar{u}(t,y) = u_\infty - u(t,y)$

$$= \frac{u_0 y}{h} - u(t,y)$$

so that $u(t,y) = \frac{u_0 y}{h} - \bar{u}(t,y)$

Now, we still have that

$$\bar{u}(t, y=0) = u_\infty(0) - u(t, y=0)$$
$$= 0$$

but now we also have

$$\bar{u}(t, y=h) = u_\infty(h) - u(t, y=h)$$
$$= u_0 - u_0$$
$$= 0$$

i.e, we now have two zero boundary conditions.

also, note that $\partial_t \bar{u}(t,y) = \partial_t u_\infty(y) - \partial_t u(t,y)$

$$= -\partial_t u(t,y)$$

and $\frac{\partial^2 \bar{u}(t,y)}{\partial y^2} = \frac{\partial^2 u_\infty(y)}{\partial y^2} - \frac{\partial^2 u(t,y)}{\partial y^2}$

$$= -\frac{\partial^2 u(t,y)}{\partial y^2}$$

and finally, $\partial_x \bar{u}(t,y) = \partial_x u_\infty - \partial_x \bar{u}(t,y)$

$$= -\partial_x \bar{u}(t,y)$$

Hence our system (*) becomes

$$\begin{cases} \partial_t \bar{u}(t,y) = \frac{\mu}{\rho}\left(\frac{\partial^2 \bar{u}(t,y)}{\partial y^2}\right) \\ \partial_x \bar{u}(t,y) = 0 \end{cases}$$

subject to the boundary and initial conditions

$$\bar{u}(t, y=0) = \bar{u}(t, y=h) = 0 \qquad \forall t$$

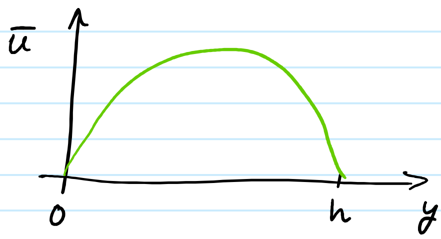
$$\bar{u}(t=0, y) = u_\infty(y) - u(t=0, y)$$
$$= u_\infty(y)$$
$$> 0 \qquad \forall y \in (0, h)$$

$$= u_\infty(y) > 0 \quad \forall y \in (0,h)$$

c. From the information in the assignment, we have that

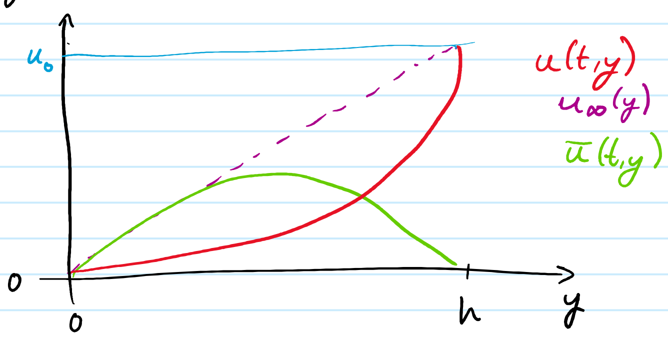
$$\bar{u}(t,y) \approx C\sin\left(\frac{n\pi}{h}y\right) \quad \text{for } t >> 0.$$

That is



And so, $u(t,y) \approx \frac{u_0 y}{h} - C\sin\left(\frac{n\pi}{h}y\right)$

has the form,



for some snapshot at sufficiently large time (but not so large that we are in a steady state!)