# MATH3202 Assignment 2

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## Section A

The mathematical formulation for our model pertaining to the *Pacific Paradise Gas* client is listed below. We identify the relevant data to the optimisation problem, as well as the variables (and their constraints) that provide the optimal solution to the cost minimisation problem.

### Sets

| | |
|---|---|
| $N$ | Nodes |
| $S$ | Supplying Nodes, $S \subset N$ |
| $E$ | Edges (pipes that link nodes) |
| $T$ | Forecasted demand period (time) |
| $O$ | Options for upgrades (0th option is no upgrade) |
| $F$ | Forecast scenarios for demand in 10 years |

### Data

| | |
|---|---|
| $l_e$ | Length of edge $e \in E$ (km) |
| $d_{nt}$ | Forecast daily gas demand at node $n \in N$ at time $t \in T$ (MJ) |
| $c_s$ | Cost per unit gas from supplier $s \in S$ (\$/MJ) |
| $SC_s$ | Maximum supply (supplying capacity) of supplier $s \in S$ over one day (MJ) |
| $f_e$, $t_e$ | "From" and "to" nodes of each edge $e \in E$ |
| $SU_{os}$ | Supply upgrade for option $o \in O$ for supplying node $s$ in $S$ (MJ) |
| $UC_{os}$ | Upgrade cost for option $o \in O$ for supplying node $s \in S$ (\$) |
| $\mathrm{P}_{\max}$ | Maximum capacity of pipeline (MJ) |
| $\mathrm{P}_{\text{cost}}$ | Pipeline upgrade cost (\$/km) |
| $\mathrm{CMult}_t$ | Cost multiplier for $t \in T$ (equal to the list `[1, 0.7]`) |
| $\mathrm{DMult}_{tf}$ | Demand forecast multiplier for $t \in T$, $f \in F$, i.e. `[[1, 1, 1], [0.8, 1, 1.2]]` |

### Variables

| | | |
|---|---|---|
| $x_{ntf}$ | Gas supplied by supplier $n \in N$ at time $t \in T$ for scenario $f \in F$ | (continuous) |
| $y_{etf}$ | Net gas transmission along edge $e \in E$ on day $t \in T$ for scenario $f \in F$ | (continuous) |
| $W_{ostf}$ | 1 if using upgrade $o \in O$ for supplier $s \in S$ at time $t \in T$ for scenario $f \in F$, 0 otherwise | (binary) |
| $P_{etf}$ | 1 if duplicating pipeline $e \in E$ at time $t \in T$ for scenario $f \in F$, otherwise 0 | (binary) |

### Objective

$$\min \sum_{f \in F} \frac{1}{3} \left( \left[ \sum_{\substack{o \in O \\ s \in S \\ t \in T}} \mathrm{CMult}_t \cdot UC_{os} \cdot W_{ostf} \right] + \left[ \sum_{\substack{e \in E \\ t \in T}} \mathrm{CMult}_t \cdot P_{etf} \cdot l_e \cdot \mathrm{P}_{\text{cost}} \right] \right)$$

### Constraints

$$x_{stf} \leq SC_s + \sum_{\substack{o \in O \\ j \leq t \in T}} (W_{osjf} \cdot SU_{os}) \qquad \forall s \in S,\ \forall t \in T,\ \forall f \in F \quad (1)$$

$$x_{ntf} \leq 0 \quad \forall n \in N \setminus S,\ \forall t \in T,\ \forall f \in F \quad (2)$$

$$x_{ntf} + \sum_{\substack{e\in E\\ s.t.\ t_e=n}} y_{etf} = \mathrm{DMult}_{tf}\cdot d_{nt} + \sum_{\substack{e\in E\\ s.t.\ f_e=n}} y_{etf} \qquad \forall n\in N,\ \forall t\in T,\ \forall f\in F \quad (3)$$

$$\sum_{\substack{o\in O\\ t\in T}} W_{ostf} = 1 \qquad \forall s\in S,\ \forall f\in F \quad (4)$$

$$y_{etf} \leq \mathrm{P}_{\max}\cdot\left(1+\sum_{j\leq t\in T} P_{ejf}\right) \qquad \forall e\in E,\ \forall t\in T,\ \forall f\in F \quad (5)$$

$$\sum_{t\in T} P_{etf} \leq 1 \qquad \forall e\in E,\ \forall f\in F \quad (6)$$

$$\sum_{f\in F} P_{e0f} = \texttt{len}(F)\cdot P_{e00} \qquad \forall e\in E \quad (7)$$

$$\sum_{\substack{o\in O\\ f\in F}} W_{os0f} = \texttt{len}(F)\cdot\sum_{o\in O} W_{os00} \qquad \forall s\in S \quad (8)$$

$$\sum_{\substack{o\in O\\ s\in S}} W_{os1f}\cdot UC_{os} + \sum_{e\in E} P_{e1f}\cdot l_e\cdot\mathrm{P}_{\text{cost}} \leq 2\left(\sum_{\substack{o\in O\\ s\in S}} W_{os0f}\cdot UC_{os} + \sum_{e\in E} P_{e0f}\cdot l_e\cdot\mathrm{P}_{\text{cost}}\right) \qquad \forall f\in F \quad (9)$$

Constraints (1) and (2) ensure that no supplying node is supplying more than its capacity, and no non-supplying node is supplying any gas into the pipeline system. Constraint (3) ensures that the pipeline gas flow between nodes is balanced, up to the demand multiplier (given by $\mathrm{DMult}_{tf}$) for the different forecast scenarios in 10 years. Constraint (4) ensures that only one upgrade (or no upgrade) is done to the supplying node over the course of time periods. Constraint (5) ensures that the gas flow across a pipe is no more than the maximum capacity of the pipe (including any doubling upgrades done to it). Constraint (6) is analogous to constraint (4) in that a maximum of one upgrade can be done to the pipe. Constraints (7 and 8) ensure that the initial upgrades done are the same no matter the 10-year forecast scenario. Finally, constraint (9) makes it so that the undiscounted cost of the 2nd period upgrades is no more than twice the cost of the first period upgrades.

### Gurobi Implementation

The model, programmed utilising `gurobipy`, is available in the file attached with this document.

## Section B

### Responses to Communications

#### Communication 6

Communication 6 saw us ignore the majority of information from the previous 5 communications and turn our focus onto forecasting, and the required upgrades needed to meet gas demand in years time. Using an integer programming formulation, we found that the optimal upgrade cost to meet the 10-year supply was $113,909,000. This solution involved no upgrade for the supplier at node 20, upgrade 3 for the supplier at node 27, and the 1st upgrades for the suppliers at nodes 36 and 45. This result is intuitive as there were no distance constrains on gas transmission (e.g. cost per distance per unit flow) in our model, and so the model would prefer a large and cost effective upgrade on a single supplying node, with small upgrades to other nodes.

#### Communication 7

The next communication involved an analogous modification in the model to communication 6. Now, we were able to upgrade the pipelines (to double their flow capacity) which were subject to a maximum flow of 358MJ across each pipe per day in a desired time period. Since this further constrained the model, the optimal solution rose about $6.2 million (for a total cost of $120,060,626) due to the $200,000 cost per kilometer of upgraded pipeline. We also note here that the supplier upgrades in this forecast differ to that in communication 6, where all suppliers are now upgraded; we see that node 20 uses upgrade 3, node 36 uses upgrade 2, and nodes 27 and 45 use upgrade 1. Further, the pipelines that link nodes (36, 29) and (15, 33) were upgraded at costs of approx. $2.9 million and $2.1 million respectively.

**Communication 8**

The eighth communication saw us implement a time-dependent upgrade solution to the model, where nodes and pipes could be upgraded initially or after a 5-year period with an associated 30% discount. The optimal solution of the upgrade cost was significantly lower as a result of the aforementioned discount, at a total cost of approximately $87,610,940. The optimal solution sees the model upgrade node 20 with upgrade 1 to meet the 5-year demand, and no upgrade to node 27. The remainder of the upgrades take place after the first 5-year period, where nodes 36 and 45 utilise upgrades 2 and 3 respectively, and the pipeline connecting nodes (36, 29) is upgraded at a (discounted) cost of about $2 million.

**Communication 9**

Communication 9 saw us make the largest change to our model since its initial formulation, with the added functionality to account for different demand forecasts at the 10-year mark. To do this, we equally weighted the optimal solution for each possible demand outcome to yield the *expected* optimal upgrade solution of approximately $90,262,350. Although we will know the true demand in 5 years time, the expected cost is a good estimation for what we will actually need to pay to upgrade the system. In each of the three possible forecast demand scenarios, the upgrades are listed below.

| Forecast Scenario | Initial Node Upgrades | | | | 5-Year Node Upgrades | | | | Initial Pipe Upgrades | 5-Year Pipe Upgrades |
|---|---|---|---|---|---|---|---|---|---|---|
| | 20 | 27 | 36 | 45 | 20 | 27 | 36 | 45 | | |
| 1 (80% Demand) | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | N/A | N/A |
| 2 (100% Demand) | 1 | 0 | 0 | 0 | 0 | 0 | 2 | 3 | N/A | (36, 49); (45, 42) |
| 3 (120% Demand) | 1 | 0 | 0 | 0 | 0 | 3 | 2 | 3 | N/A | (27, 2); (36, 29); (27, 12); (45, 42) |

Table 1: The list of upgrades for each scenario clearly shows more and costlier upgrades for higher forecast demand. The node upgrade columns are separated by which node is being upgraded, where an upgrade of ‘0’ equates to no upgrade for that time period, ‘1’ is the first upgrade, etc. The pipeline upgrades are listed by the connected nodes for which the pipes link.

Even though this model accounts for the discounted cost of upgrade after the first 5-years, we note that this solution is still more expensive to upgrade the pipeline system than that seen in the eighth communication. This is due to the influence of the ‘worst-case’ demand scenario disproportionately impacting the expected system upgrade cost to a greater cost.

**Communication 10**

In response to the Pacific Paradise government restriction on upgrades per time period, we amended the model so that the undiscounted 5-year upgrade cost was no more than twice that of the initial upgrade cost. The resulting upgrades are listed in Table 2.

| Forecast Scenario | Initial Node Upgrades | | | | 5-Year Node Upgrades | | | | Initial Pipe Upgrades | 5-Year Pipe Upgrades |
|---|---|---|---|---|---|---|---|---|---|---|
| | 20 | 27 | 36 | 45 | 20 | 27 | 36 | 45 | | |
| 1 (80% Demand) | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 1 | N/A | N/A |
| 2 (100% Demand) | 0 | 2 | 0 | 0 | 1 | 0 | 0 | 3 | N/A | (27, 2); (45, 42) |
| 3 (120% Demand) | 0 | 3 | 0 | 0 | 0 | 0 | 2 | 3 | N/A | (27, 2); (36, 29); (27, 12); (15, 33); (45, 42) |

Table 2: Analogous to Table 1, we see similar behaviour with the constraints of communication 10 albeit with typically costlier upgrades in the first period.

Of course, this restriction just further constrains the model and raises the expected optimal solution to approximately $100,414,117.
The slack on this additional restriction is about $13.1 million, $5.8 million, and $39.4 million for each of the forecast scenarios 1 to 3 respectively. That is to say that we'd be able to make significantly more upgrades to the pipeline system in the 120% demand scenario after the initial 5-year period if we needed to, while remaining within the governments requirements.