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MATH3401

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Submission Topic Assignment 1

Due date of submission	14/03/2022		
Due time of submission	11:50 AM	Name of tutor (if appropriate)	Marcus
Group	05	Session attended (if appropriate)	Thursday, 8am (day / time)
Actual submission date	13/03/2022		

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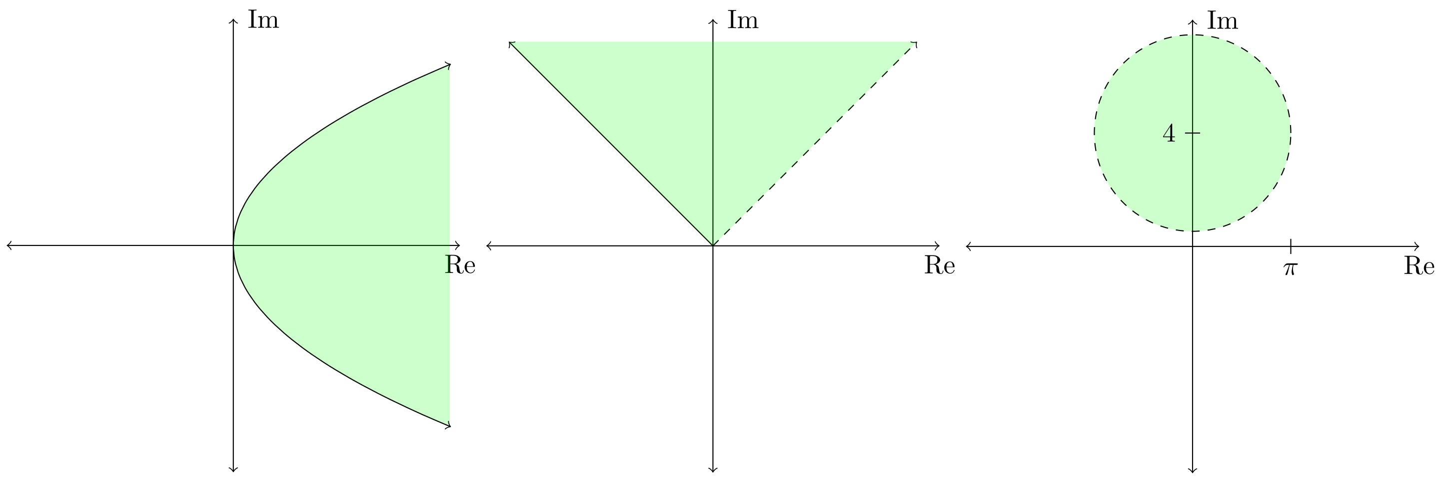
MATH3401 Assignment 1

Ryan White s4499039

14th of March 2022

Question 1:

In all of the below, the shaded green region represents the applicable region of z according to the defined domain and range in the complex plane. A solid boundary indicates inclusivity, whereas a dashed boundary represents the domain approaching, but not including said values.



(a) $\{z : \operatorname{Re}(z) \geq (\operatorname{Im}(z))^2\}$

This region represents a horizontally rotated parabola, where the applicable region of z is enclosed within the bounds since $\operatorname{Re}(z)$ is *greater* than $\operatorname{Im}(z)^2$

(b) $\pi/4 < \operatorname{Arg}(z) \leq 3\pi/4$

The allowed region is between the angle of $\pi/4$ and $3\pi/4$ for all $\operatorname{Re}(z)$ and positive $\operatorname{Im}(z)$.

(c) $|z - 4i| < \pi$

The allowed region is a circle centered at $\operatorname{Im}(z) = 4$, where the radius extends to (but does not include) π about the center.

Figure 1: Different Regions of Defined z

Question 2:

a. $z^* = z$ implies an infinite number of solutions in which $\operatorname{Im}(z) = 0 \Rightarrow z \in \mathbb{R}$. Since $\operatorname{Im}(z^*) = -\operatorname{Im}(z)$, this is the only option, as having some non-zero imaginary component of z would contradict the definition of the complex conjugate.

b. $z^* + z = 0$ implies an infinite number of solutions where $\operatorname{Re}(z) = 0 \Rightarrow z = iy \quad \forall y \in \mathbb{R}$. Similarly to part a, $\operatorname{Re}(z^*) = \operatorname{Re}(z)$, and so adding the same two non-zero real components of z would result in the sum of z and its conjugate being something other than zero.

c. Given that some complex number z is defined by $z = \operatorname{Re}(z) + i\operatorname{Im}(z)$, the equation $z^* = 9/z$ can be rearranged:

$$
\begin{aligned}
z^* &= 9/z \\
\Rightarrow \operatorname{Re}(z) - i\operatorname{Im}(z) &= \frac{9}{\operatorname{Re}(z) + i\operatorname{Im}(z)} \\
(\operatorname{Re}(z) - i\operatorname{Im}(z))(\operatorname{Re}(z) + i\operatorname{Im}(z)) &= 9 \\
\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2 &= 9 \\
\Rightarrow \operatorname{Im}(z) &= \pm\sqrt{9 - \operatorname{Re}(z)^2} \qquad \forall z : -3 < \operatorname{Re}(z) < 3
\end{aligned}
$$

There are also two solutions for $z = \pm 3$, $z \in \mathbb{R}$ for which the proof is trivial. Therefore, for $-3 \leq x \leq 3$, the solutions for z are:

$$z = x \pm i\sqrt{9 - x^2}$$

Question 3:

a.

$$\begin{aligned}
\frac{i}{1-i} + \frac{1-i}{i} &= \frac{i}{1-i} \times \frac{1+i}{1+i} + \frac{1-i}{i} \\
&= \frac{i+i^2}{1-i^2} + \frac{1-i}{i} \\
&= \frac{i-1}{1+1} + \frac{1-i}{i} \times \frac{i}{i} \\
&= \frac{i-1}{2} + \frac{i+1}{-1} \\
&= -\frac{3}{2} - \frac{1}{2}i
\end{aligned}$$

Which is of the form $z = x + iy$ with $x = -3/2$ and $y = -1/2$.

b. First take $z^3 = 8i$. Also note that a complex number $z = x + iy$ can be written in the form $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$. In this particular case, $z^3 = 8 \times i = 8(\cos(\pi/2) + i\sin(\pi/2))$ since $\cos(\pi/2) = 0$ and $i\sin(\pi/2) = i$. Going further, we have:

$$\begin{aligned}
z^3 &= 8i \\
&= 8(\cos(\pi/2) + i\sin(\pi/2)) \\
&= 8(\cos(\pi/2 + 2n\pi) + i\sin(\pi/2 + 2n\pi)) \quad \forall n \in \mathbb{N} \\
\Rightarrow z &= 8^{1/3}\left(\cos\left[\frac{\pi}{2}(1+4n)\right] + i\sin\left[\frac{\pi}{2}(1+4n)\right]\right)^{1/3} \\
&= 2\left(\cos\left[\frac{\pi}{6}(1+4n)\right] + i\sin\left[\frac{\pi}{6}(1+4n)\right]\right)
\end{aligned}$$

which has *unique* solutions for $n = 0, 1, 2$. These are, denoted by z_n and shown with their corresponding x and y values for the form $z_n = x_n + iy_n$,

$$\begin{aligned}
z_0 &= 2\cos\left[\frac{\pi}{6}\right] + 2i\sin\left[\frac{\pi}{6}\right] & x_0 &= 2\cos\left[\frac{\pi}{6}\right] & y_0 &= 2\sin\left[\frac{\pi}{6}\right] \\
z_1 &= 2\cos\left[\frac{5\pi}{6}\right] + 2i\sin\left[\frac{5\pi}{6}\right] & x_1 &= 2\cos\left[\frac{5\pi}{6}\right] & y_1 &= 2\sin\left[\frac{5\pi}{6}\right] \\
z_2 &= 2\cos\left[\frac{9\pi}{6}\right] + 2i\sin\left[\frac{9\pi}{6}\right] & x_2 &= 2\cos\left[\frac{9\pi}{6}\right] & y_2 &= 2\sin\left[\frac{9\pi}{6}\right]
\end{aligned}$$

c. Fortunately, $z = ([1+i]/\sqrt{2})^{1337}$ is already normalized with modulus $r = 1$. Plotted on the complex plane, it's clear that z makes an angle of $\theta = \pi/4$ with respect to the positive $\mathrm{Re}(z)$ axis, and so z may be represented by

$$\begin{aligned}
z &= \left(\frac{1+i}{\sqrt{2}}\right)^{1337} \\
&= \left(\cos\left[\frac{\pi}{4}\right] + i\sin\left[\frac{\pi}{4}\right]\right)^{1337} \\
&= \cos\left[\frac{1337\pi}{4}\right] + i\sin\left[\frac{1337\pi}{4}\right] \\
&= \cos\left[\frac{\pi}{4}\right] + i\sin\left[\frac{\pi}{4}\right]
\end{aligned}$$

(The last line since $1336\pi/4$ is divisible by 2π, and so the remainder is just $\pi/4$). And so there is one unique solution for z, as given above.

Question 4:

a. $\mathbb{Q}$ and $\mathbb{R}$ are fields, with $\mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$ (since $\sqrt{2} \in \mathbb{R}$ and $\mathbb{Q} \subset \mathbb{R}$), and so the associativity and commutativity field axioms are inherited into $\mathbb{Q}(\sqrt{2})$.
Take $a_n \in \mathbb{Q}(\sqrt{2})$, where $a_n = p_n + q_n\sqrt{2}\ \forall n \in \mathbb{N}$. The first field axiom that requires proving is the (F2) one.:

- Take $a_0 = p_0 + q_0\sqrt{2}$, $p_0 = q_0 = 0 \in \mathbb{Q}$.

$$\begin{aligned} a_1 + a_0 &= p_1 + q_1\sqrt{2} + p_0 + q_0\sqrt{2} \\ &= p_1 + q_1\sqrt{2} + 0 \\ &= a_1 \end{aligned}$$

  And so (F2i) holds!
- Take $b = p_b + q_b\sqrt{2}$ such that $p_b = -p_1$, $q_b = -q_1$. Then,

$$\begin{aligned} a_1 + b &= p_1 + q_1\sqrt{2} + p_b + q_b\sqrt{2} \\ &= p_1 - p_1 + q_1\sqrt{2} - q_1\sqrt{2} \\ &= 0 \end{aligned}$$

  And so (F2ii) holds, and there exists and additive identity and inverse.
-

$$\begin{aligned} a_1 \cdot 1 &= (p_1 + q_1\sqrt{2}) \cdot 1 \\ &= p_1 \cdot 1 + 1 \cdot q_1\sqrt{2} \\ &= p_1 + q_1\sqrt{2} = a_1 \end{aligned}$$

  Therefore, (F5i) and (F7) hold.
- Take

$$a_i = \frac{p_1 - q_1\sqrt{2}}{p_1^2 - 2q_1^2}$$

  for some a_1. Then,

$$\begin{aligned} a_1 \cdot a_i &= (p_1 + q_1\sqrt{2})\left(\frac{p_1 - q_1\sqrt{2}}{p_1^2 - 2q_1^2}\right) \\ &= \frac{p_1^2 - 2q_2^2}{p_1^2 - 2q_2^2} \\ &= 1 \end{aligned}$$

  And therefore (F5ii) holds, and $\mathbb{Q}(\sqrt{2})$ has a multiplicative identity and inverse. For completeness with regards to the inverse, define:

$$\begin{aligned} \frac{p_1}{p_i^2 - 2q_1^2} &= p_i \in \mathbb{Q} \\ \frac{-q_1}{p_i^2 - 2q_1^2} &= q_i \in \mathbb{Q} \end{aligned}$$

  Then, $a_i = p_i + q_i\sqrt{2} \in \mathbb{Q}(\sqrt{2})$.
  As such, the field axioms have been proven and all that is left is closure.
- Firstly with addition,

$$\begin{aligned} a_1 + a_2 &= p_1 + q_1\sqrt{2} + p_2 + q_2\sqrt{2} \\ &= (p_1 + p_2) + (q_1 + q_2)\sqrt{2} \end{aligned}$$

  Now, take $p_a = p_1 + p_2$ and $q_a = q_1 + q_2$. Then,

$$a_1 + a_2 = p_a + q_a\sqrt{2} \in \mathbb{Q}(\sqrt{2})$$

  And so $\mathbb{Q}(\sqrt{2})$ is closed under addition.

- Finally with multiplication:

$$\begin{aligned} a_1 \cdot a_2 &= (p_1 + q_1\sqrt{2}) \cdot (p_2 + q_2\sqrt{2}) \\ &= p_1p_2 + p_1q_2\sqrt{2} + p_2q_1\sqrt{2} + 2q_1q_2 \\ &= (p_1p_2 + 2q_1q_2) + (p_1q_2 + p_2q_1)\sqrt{2} \end{aligned}$$

  Now define:

$$\begin{aligned} p_m &= p_1p_2 + 2q_1q_2 \in \mathbb{Q} \\ q_m &= p_1q_2 + p_2q_1 \in \mathbb{Q} \end{aligned}$$

  And so

$$a_1 \cdot a_2 = p_m + q_m\sqrt{2} \in \mathbb{Q}(\sqrt{2})$$

  Therefore, $\mathbb{Q}(\sqrt{2})$ is closed under multiplication.

And finally, since all of the axioms and closure have been proven, $\mathbb{Q}(\sqrt{2})$ is a field, and since it is a subset of $\mathbb{R}$, it is also a subfield of $\mathbb{R}$.

b. Take $a = p + q\sqrt{2} = \sqrt{3}$, and suppose $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$.

$$\Rightarrow a^2 = p^2 + 2pq\sqrt{2} + 2q^2 = 3$$

Note that $a^2, 3 \in \mathbb{Q}(\sqrt{2})$, and define

$$\begin{aligned} & a^2 = (p^2 + 2q^2) + (2pq)\sqrt{2} = b + c\sqrt{2} \\ \text{where,} \quad & b = p^2 + 2q^2 \text{ and } c = 2pq,\ b,\, c \in \mathbb{Q} \\ \Rightarrow & b + c\sqrt{2} = 3 \\ & c\sqrt{2} = 3 - b \\ & \sqrt{2} = \frac{3-b}{c} \qquad (*) \end{aligned}$$

Now, define

$$e = \frac{3-b}{c} \in \mathbb{Q}$$

but (*) says that $e = \sqrt{2} \notin \mathbb{Q}$. And so a contradiction is found, and $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$. QED

Question 5:

Firstly, assume that $x = -1 \in \mathcal{P}$. Therefore, $x^2 \in \mathcal{P}$. However, $x^2 = -1 \times -1 = 1 \notin \mathcal{P}$. This is because it would contradict the (i) and (iii) requirements in the trichotomy law (i.e. $x \in \mathcal{P}$ *and* $-x \in \mathcal{P}$, when only one is allowed to be satisfied at one time). Therefore, $-1 \notin \mathcal{P}$.

Next, assume that $i \in \mathcal{P}$. Therefore, $i^2 \in \mathcal{P}$. But, $i^2 = -1 \notin \mathcal{P}$ as shown in the above paragraph. Therefore, $i \notin \mathcal{P}$.

Finally, assume that $-i \in \mathcal{P}$. Therefore, $(-i)^2 \in \mathcal{P}$. But, again, $(-i)^2 = (-1)^2 \cdot i^2 = i^2 \notin \mathcal{P}$ by the above.

Of course, $i \neq 0$ and so none of the requirements of the trichotomy law hold for the subset $\mathcal{P} \subset \mathbb{C}$. Therefore, $\mathbb{C}$ is not ordered. QED

Question 6:

For the mathematician now delving into the Joukowski Transformation,
they may manipulate shapes with next-to-no complication.
From a circle to an airfoil it creates in mapping,
it even transforms the airstreams in wrapping!
By utilising the powerful and confusing complex set,
one can assist those engineers without breaking a sweat.
By first plotting perfect curvature on the complex plane,
this handy transformation gets you one step closer to an aeroplane.
And how it works is quite simply as smart as sliced bread,
each point in the first plane transforms by z plus one over z!

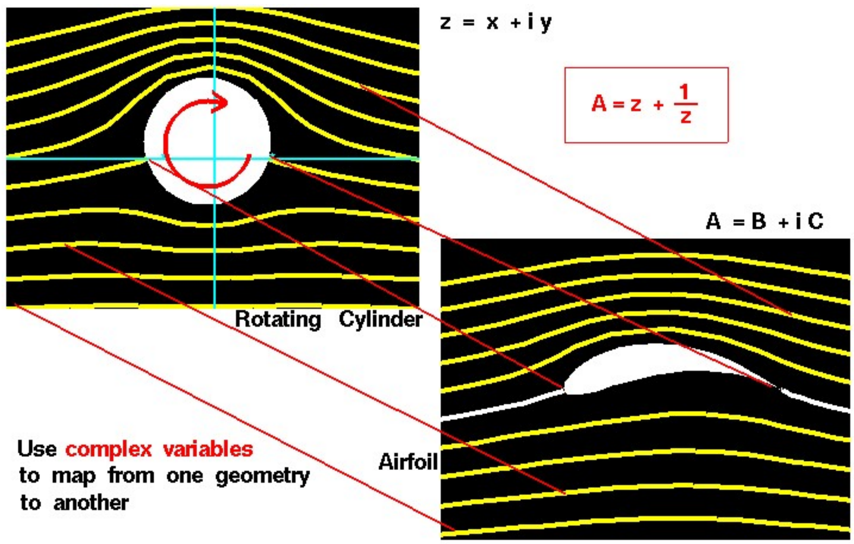


Figure 2: Joukowski Transformation from one Coordinate System to Another

The second plane then accepts the value from the first,
and with the rest I am really struggling to keep in verse.
Although the airfoil is a little bit squashed and a little bit stretched,
the wonderful world of complex transforms no longer seems so farfetched.

In all seriousness, the Joukowski Transformation maps one set of complex coordinates to another. This is most obvious with the transformation of a circle to the a wing/airfoil shape as seen in Figure 2, although *all* of the coordinates of the original system are transformed, not just the obvious circle. Figure 2 also shows the formula from which this transformation is done, in which some manipulation of $A = z + 1/z$ would need to occur in order for it to be of the form $A = B + iC$.

Source: https://www.grc.nasa.gov/www/k-12/airplane/map.html