PHYS3071
Assignment one rubric

Semester 2, 2024

Question I – The SIR model

This question investigates a so-called compartment model from epidemiology to study the spread of infections in a population. A correct answer does not require any background knowledge of this subject.

Aside from the analytical parts to the question, marks should be awarded for correct, working Python code.

(a) Adding together the three equations for the SIR model gives

$$\begin{aligned}\frac{d}{dt}S(t) + \frac{d}{dt}I(t) + \frac{d}{dt}R(t) &= \frac{d}{dt}\Big(S(t) + I(t) + R(t)\Big)\\ &= -\beta I(t)S(t) + \beta I(t)S(t) - \gamma I(t) + \gamma I(t)\\ &= 0.\end{aligned}$$

Hence the sum of the compartments $S(t)+I(t)+R(t)$ is a constant, N - the total number of individuals. [**2 marks** for correct working]

(b) Without the recovered compartment, The SIR equations reduce to

$$\begin{aligned}\frac{d}{dt}S(t) &= -\beta I(t)S(t),\\ \frac{d}{dt}I(t) &= \beta I(t)S(t).\end{aligned}$$

We can solve the equation for $S(t)$ using part (a) to eliminate the dependence on $I(t)$, so that

$$\frac{d}{dt}S(t) = -\beta S(t)\big(N - S(t)\big).$$

This ordinary differential equation can then be straight-forwardly integrated to give

$$\begin{aligned}\int \frac{dt}{S(t)\big(N - S(t)\big)}\frac{dS}{dt} &= \int -\beta dt,\\ \frac{1}{N}\int\left(\frac{dS}{S} + \frac{dS}{N-S}\right) &= -\beta\int dt.\end{aligned}$$

Applying the initial conditions $S_0 = S(t=0)$ and $I_0 = I(t=0)$ we obtain

$$\ln\left(\frac{S}{N-S}\right) = -\beta N t + \ln\left(\frac{S_0}{N-S_0}\right),$$

which can be written as

$$S(t) = N\frac{\exp(-\beta N t)}{(N-S_0)/S_0 + \exp(-\beta N t)}.$$

Similar working also gives

$$I(t) = N\frac{I_0/(N-I_0)}{I_0/(N-I_0) + \exp(-\beta N t)}.$$

The time at which the two populations $S(t)$ and $I(t)$ cross each other is found by setting $S(t) = I(t)$, which is straight-forwardly found to be $t_{\text{cross}} = (1/\beta N)\ln(S_0/I_0)$.

[**2 marks** for working, **2 marks** for both compartment eqs and **1 mark** for correct crossing time]

(c) The plot showing the solutions for S(t) and I(t) is shown below in Fig. 1.

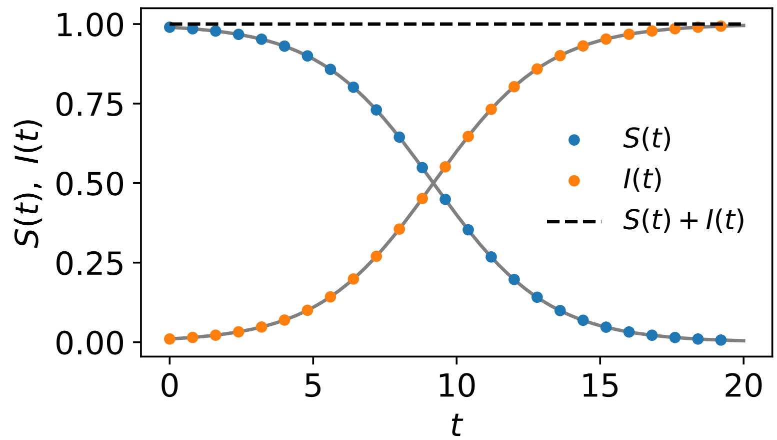

Figure 1: Grey solid lines show the exact solutions for $S(t)$ and $I(t)$, while the blue and orange points show the corresponding numerical values obtained from RK4. The black dashed line is the total population $S(t) + I(t) = N$.

The numerical solutions for $S(t)$ and $I(t)$ with $S(t = 0) = 0.99$ and $I(t = 0) = 0.01$ are shown in Fig. 1 (coloured points), with the corresponding exact solutions plotted in solid grey.

Initially at $t = 0$ almost all of the population are susceptible, with only one percent being infected. The number of infected then begins to grow, which after a while causes the two compartments to cross each other at $t = t_{\text{cross}}$. For long times ($t \gg 1$) the situation is reversed.

[**3 marks** for correct figure, **3 marks** for correct description]

(d) The plot showing the solutions for $S(t)$, $I(t)$ and $R(t)$ is shown below in Fig. 2

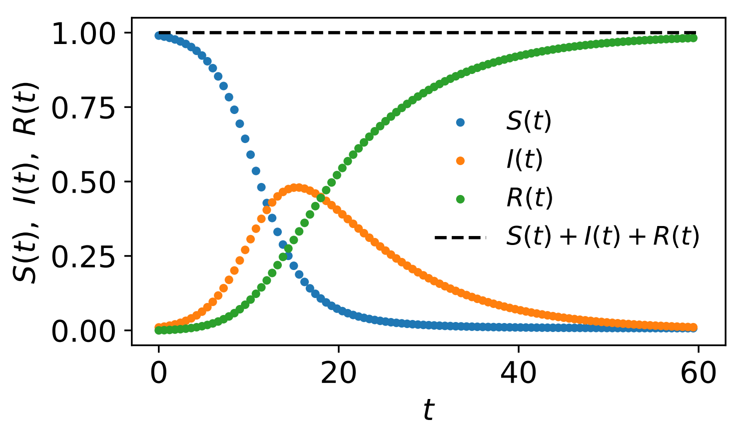

Figure 2: Coloured data shows the RK4 solutions for $S(t)$, $I(t)$ and $R(t)$. The black dashed line gives the total population.

The numerical solutions for $S(t)$, $I(t)$ and $R(t)$ with $S(t = 0) = 0.99$ and $I(t = 0) = 0.01$ are shown in Fig. 2 (coloured points), while the total population is given by the black dashed line.

For short times almost all of the population is susceptible, again only one percent is infected, and there are almost no recovered individuals.

For intermediate times, the number of infected increases at a faster rate than the number of recovered individuals. The infected compartment reaches a peak before declining.

For long times, both the infected and susceptible compartments decrease towards zero, as almost all the population has moved into the recovered compartment.

[**3 marks** for correct figure, **3 marks** for correct description]

(e) We know that

$$\frac{dI}{dt} = \beta I(t)S(t) - \gamma I(t),$$

so that if $dI/dt > 0$ we can write $\beta I(t)S(t) - \gamma I(t) > 0$ or

$$\beta/\gamma \gtrsim 1$$

which assumes that $S(t = 0) \approx N$.

[**1 mark** for correct working, **1 mark** for correct assumption]

(f) The SIR equations can be written in terms of the reproduction number R_0 using either of the scalings $t \to t/\gamma$ or $t \to t/\beta$ so that in both cases the new time variable does not carry any units. Then for the first choice of scaling we have

$$\frac{d}{dt}S(t) = -R_0 I(t)S(t), \quad \frac{d}{dt}I(t) = R_0 I(t)S(t) - I(t), \quad \frac{dR}{dt} = I(t),$$

while for the second scaling we instead obtain

$$\frac{d}{dt}S(t) = -I(t)S(t), \quad \frac{d}{dt}I(t) = I(t)S(t) - \frac{I(t)}{R_0}, \quad \frac{dR}{dt} = \frac{I(t)}{R_0}.$$

Figure 3 shows the solutions for $I(t)$ as the reproduction number R_0 is increased from below to above

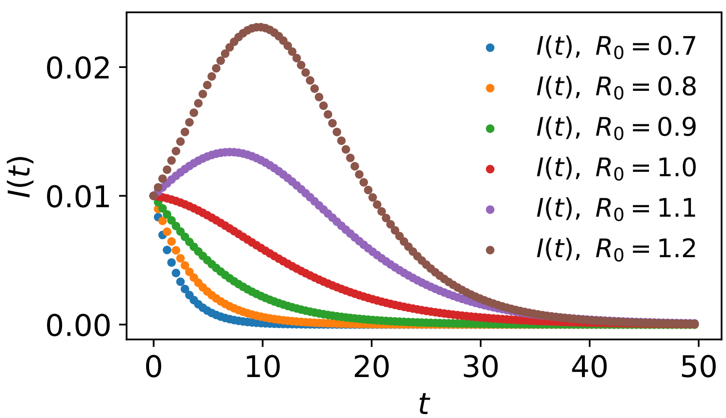

Figure 3: percentage of infected individuals I(t) for different values of the reproduction number R_0.

one. It does not depend on which one of the two scalings above the student has chosen. Approximately, the number of infected individuals decreases as a function of time for $R_0 < 1$ and increases when $R_0 > 1$.

The data for $R_0 = 1$ slowly decreases. This is because we have chosen a value of $S(t = 0)$ slightly less than one. As $S(t = 0)$ approaches one this curve starts to flatten off.

[**1 mark** for correct scaling, **2 marks** for correct figure and **1 mark** for correct description]