THE UNIVERSITY OF QUEENSLAND
SCHOOL OF MATHEMATICS AND PHYSICS
PHYS2041 – Quantum Physics

Tutorial 2 Solutions

Problem 2.1

(a) The complex conjugate of $z = x + iy$ is $z^* = x - iy$, so

$$\begin{aligned} |z|^2 = z^* z &= (x - iy)(x + iy) && (1) \\ &= x^2 - ixy + ixy + y^2 && (2) \\ &= x^2 + y^2. && (3) \end{aligned}$$

In fact, the modulus $|z|$ of any complex number is real (the imaginary part is 0); it represents the length of z in the complex plane.

(b) To simplify z, multiply numerator and denominator by the complex conjugate of the denominator:

$$z = \frac{1}{2-i} = \frac{2+i}{(2-i)(2+i)} = \frac{2+i}{4+1} = \frac{2}{5} + i\frac{1}{5}. \tag{4}$$

(c) Use Euler's formula, $e^{i\varphi} = \cos\varphi + i \sin\varphi$, assuming x is real.

$$\begin{aligned} z = \left(e^{-ix}\right)^2 &= e^{-2ix} && (5) \\ &= \cos(-2x) + i\sin(-2x) && (6) \\ &= \cos(2x) - i\sin(2x). && (7) \end{aligned}$$

(d) These numbers are in polar form, $z = re^{i\varphi}$ where r is the radius on the complex plane and φ is the angle. See Figure 1.

(e) Converting to polar form,

$$\begin{aligned} r &= |z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} = \sqrt{1^2 + 1^2} = \sqrt{2} && (8) \\ \varphi &= \arctan\left(\frac{\mathrm{Re}(z)}{\mathrm{Im}(z)}\right) = \arctan(1) = \frac{\pi}{4}. && (9) \end{aligned}$$

So we have

$$z = 1 + i = \sqrt{2}e^{i\pi/4}. \tag{10}$$

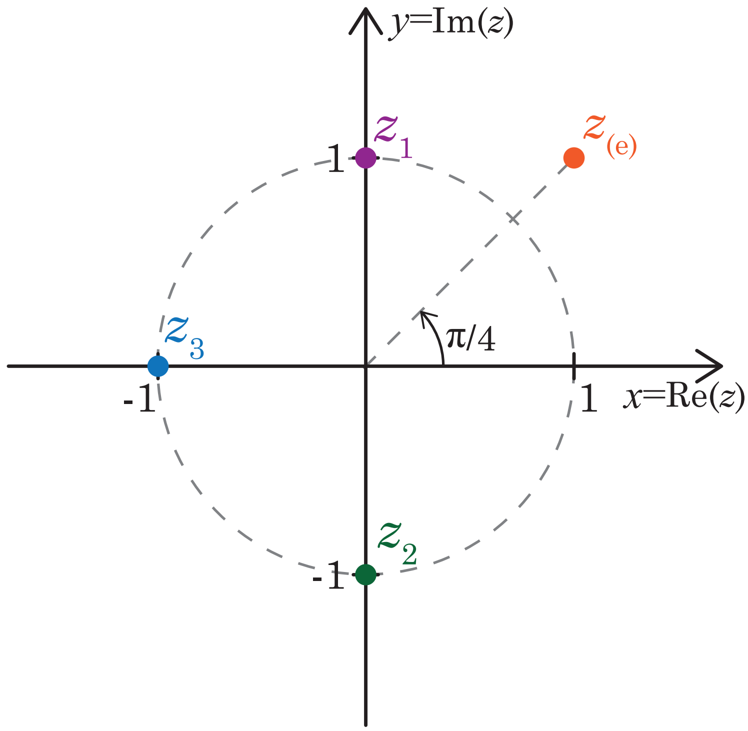

Figure 1: Plot of z_1, z_2 and z_3 from (d), and $z_{(\mathrm{e})}$ from part (e).

This is shown as $z_{(\mathrm{e})}$ in Figure 1.

(f) First use $e^{i\varphi} = \cos\varphi + i\sin\varphi$,

$$f(r) = e^{i\mathbf{k}\cdot\mathbf{r}} = \cos(\mathbf{k}\cdot\mathbf{r}) + i\sin(\mathbf{k}\cdot\mathbf{r}), \tag{11}$$

so we have

$$\begin{aligned} \mathrm{Re}(f(r)) &= \cos(\mathbf{k}\cdot\mathbf{r}) && (12) \\ &= \cos(|\mathbf{k}||\mathbf{r}|\cos(\theta)) && (13) \\ \mathrm{Im}(f(r)) &= \sin(\mathbf{k}\cdot\mathbf{r}) && (14) \\ &= \sin(|\mathbf{k}||\mathbf{r}|\cos(\theta)), && (15) \end{aligned}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{r}$. Since $e^{i\mathbf{k}\cdot\mathbf{r}}$ is in polar form already, we can see that the magnitude is 1 and the phase is $\mathbf{k}\cdot\mathbf{r} = |\mathbf{k}||\mathbf{r}|\cos(\theta)$.

Problem 2.2 **[FOR ASSIGNMENT 1; max 10 points]** We are given that $\Psi(x, t)$ solves the time dependent Schrödinger equation (TDSE) for an arbitrary potential $V(x)$, i.e.

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{-\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi. \tag{16}$$

If we change the potential by adding a **constant** energy V_0, then Ψ is no longer a solution. Let's call the new solution Ψ_0. Our job is to show that Ψ and Ψ_0 are related by

$$\boxed{\Psi_0(x,t) = \Psi(x,t)e^{-iV_0t/\hbar}}. \tag{17}$$

The first step to showing this is writing down the TDSE for with the new potential, which we have been told is solved by Ψ_0:

$$i\hbar\frac{\partial\Psi_0}{\partial t} = \frac{-\hbar}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + V(x)\Psi_0 + V_0\Psi_0. \tag{18}$$

Now let's simply substitute our ansatz into this equation. Using the product rule, the LHS becomes:

$$i\hbar\frac{\partial\Psi_0}{\partial t} = i\hbar\frac{\partial}{\partial t}\left(\Psi e^{-iV_0t/\hbar}\right) = i\hbar e^{-iV_0t/\hbar}\frac{\partial\Psi}{\partial t} + V_0\Psi e^{-iV_0t/\hbar}. \tag{19}$$

Because the constant potential V_0 is independent of x, the RHS is simply

$$\frac{-\hbar}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + V(x)\Psi_0 + V_0\Psi_0 = \frac{-\hbar}{2m}e^{-iV_0t/\hbar}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi e^{-iV_0t/\hbar} + V_0\Psi e^{-iV_0t/\hbar}. \tag{20}$$

Equating the LHS and the RHS, and cancelling where possible,

$$i\hbar\cancel{e^{-iV_0t/\hbar}}\frac{\partial\Psi}{\partial t} + V_0\Psi\cancel{e^{-iV_0t/\hbar}} = \frac{-\hbar}{2m}\cancel{e^{-iV_0t/\hbar}}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi\cancel{e^{-iV_0t/\hbar}} + V_0\Psi\cancel{e^{-iV_0t/\hbar}}, \tag{21}$$

$$i\hbar\frac{\partial\Psi}{\partial t} + \cancel{V_0\Psi} = \frac{-\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi + \cancel{V_0\Psi}, \tag{22}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{-\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi, \tag{23}$$

we are left with the TDSE for the original potential, which we already know is solved by Ψ! What this means is that we've shown that if Ψ solves the TDSE for $V(x)$, then $\Psi_0 = \Psi\exp(-iV_0t/\hbar)$ is the solution for $V(x) + V_0$, as required. The complex exponential we pick up is called a **global phase factor**, since it simply multiplies the entire wavefunction (this is compared to a relative phase).

In classical mechanics we know that a constant energy offset doesn't change anything, but is the same true in quantum mechanics? Although we just showed that the wavefunction *does* change by a global phase, remember in quantum mechanics the wavefunction is not something we can directly measure. All we can measure are probabilities and expectation values. The expectation value of an arbitrary operator $\hat{A}$ is

$$\langle\hat{A}(t)\rangle = \int_{-\infty}^{\infty}\Psi^*(x,t)\,\hat{A}(t)\,\Psi(x,t)\,dx. \tag{24}$$

Because Ψ^* is the *complex conjugate* of Ψ, calculating $\langle \hat{A}(t) \rangle$ with respect to Ψ_0 wouldn't change anything since the global phase will simply cancel.

Although the *wavefunction* changes if a constant energy is added, expectation values do not!

<u>Problem 2.3</u> [FOR ASSIGNMENT 1; max 10 points]

(a) Normalising the wavefunction means that the integral of the probability density $|\Psi(x,t)|^2$ over entire 1D space must be equal to 1:

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1. \tag{25}$$

Notice that

$$|\Psi(x,t)|^2 = |C|^2 e^{-2b|x|}. \tag{26}$$

Note also, that this is an even function. Therefore, the integral

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 2\int_{0}^{\infty} |\Psi(x,t)|^2 \, dx. \tag{27}$$

Because we are now only integration over $x > 0$, where $|x| = x$ and therefore $e^{-2b|x|} = e^{-2bx}$, we can do the integral

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 2|C|^2 \int_{0}^{\infty} e^{-2bx} \, dx \tag{28}$$

$$= 2|C|^2 \frac{-1}{2b}(0-1) \tag{29}$$

$$= \frac{|C|^2}{b}. \tag{30}$$

This equals 1, therefore, rearranging and taking the square root we find that,

$$|C| = \sqrt{b}, \tag{31}$$

which means we have normalised the wavefunction, i.e., we found the absolute value of $|C|$ such that the wavefunction $\Psi(x,t) = \sqrt{b}e^{-i\omega t}e^{-b|x|}$ is now *a priory* normalised (in the above sense that the integral of the probability density must be equal to 1). The phase of the constant C (if it was a complex constant) has no consequences on the value of the probability density $|\Psi(x,t)|^2$ and hence on the values of quantum mechanical observables, so that one can assume that the phase is zero, without loss of generality, and replace $|C|$ by a real and positive constant, which is why this property of C was assumed and stated right from the start. So the final answer here is:

$$C = \sqrt{b}. \tag{32}$$

In fact, a wavefunction $\Psi(x,t) = \sqrt{b}e^{-i\omega t}e^{-b|x|}e^{i\theta}$, where θ is an arbitrary real-valued constant, is equally normalised! So, is $\Psi(x,t) = -\sqrt{b}e^{-i\omega t}e^{-b|x|}$, for example (i.e., θ was chosen to be $\theta = \pi$ here).

(b) Find the expectation value of x and x^2. The expectation value of x is,

$$\langle x \rangle = \int_{-\infty}^{\infty} x \left|\Psi(x,t)\right|^2 \, dx. \tag{33}$$

Remember that $\left|\Psi(x,t)\right|^2$ is an even function, and $f(x) = x$ is odd. This means that the integrand $\left(x \left|\Psi(x,t)\right|^2\right)$ is an odd function and we can therefore immediately conclude that,

$$\langle x \rangle = \int_{-\infty}^{\infty} x \left|\Psi(x,t)\right|^2 \, dx = 0. \tag{34}$$

The expectation value of x^2 is,

$$\begin{aligned} \left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} x^2 \left|\Psi(x,t)\right|^2 \, dx && (35) \\ &= b \int_{-\infty}^{\infty} x^2 e^{-2b|x|} \, dx && (36) \\ &= 2b \int_{0}^{\infty} x^2 e^{-2bx} \, dx. && (37) \end{aligned}$$

In the last line we used the fact that the integrand is even. You can look up some tables to get the solution to this or you can use integration by parts. It turns out to be,

$$\left\langle x^2 \right\rangle = 2b \frac{1}{4b^3} = \frac{1}{2b^2}. \tag{38}$$

(c) The standard deviation is,

$$\begin{aligned} \sigma_x &= \sqrt{\langle x^2 \rangle - \langle x \rangle^2} && (39) \\ &= \sqrt{\frac{1}{2b^2} - 0} && (40) \\ &= \frac{1}{\sqrt{2}b}. && (41) \end{aligned}$$

(d) The probability that the particle would be found outside of the range $\langle x \rangle - \sigma_x < x < \langle x \rangle + \sigma_x$ is given by,

$$P(\text{outside}) = 1 - P(\text{inside}). \tag{42}$$

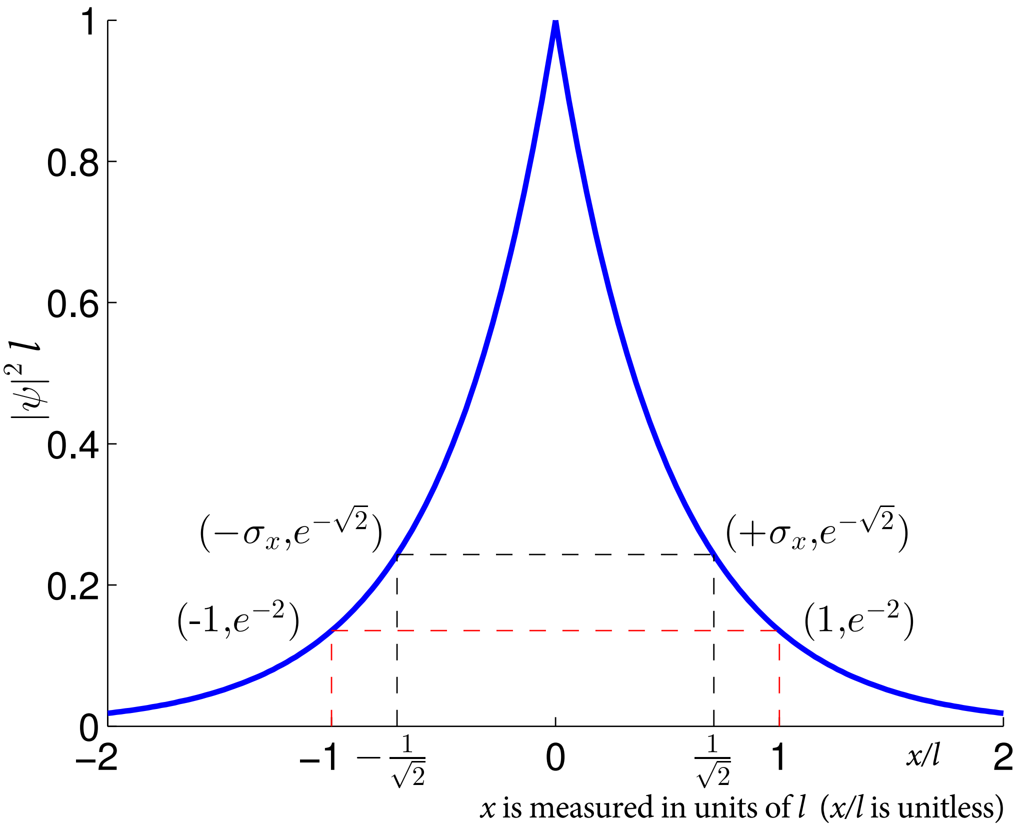

Figure 2: Plot of $|\Psi(x,t)|^2$ with standard deviation marked, as well as $\pm 1/b$. Note that $|\Psi(x,t)|$ is actually independent of time.

The probability of being inside the range is,

$$P(\text{inside}) = \int_{\langle x\rangle - \sigma_x}^{\langle x\rangle + \sigma_x} |\Psi(x,t)|^2 \; dx \tag{43}$$

$$= \int_{-\frac{1}{\sqrt{2}b}}^{\frac{1}{\sqrt{2}b}} b e^{-2b|x|} \; dx \tag{44}$$

$$= 2\int_{0}^{\frac{1}{\sqrt{2}b}} b e^{-2bx} \; dx \tag{45}$$

$$= -\left(e^{-\sqrt{2}} - e^{0}\right) \tag{46}$$

$$\approx 0.76. \tag{47}$$

We have use the fact that the integrand is even in line 3. Therefore,

$$P(\text{outside}) \approx 0.24. \tag{48}$$

Problem 2.4*

(a) To normalise the wave function, we must find C so that

$$1 = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx.$$

Substituting in our wave function and then simplifying,

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} Ce^{i\omega t}e^{-(x/l)^2}Ce^{-i\omega t}e^{-(x/l)^2}dx && (49)\\ &= C^2\int_{-\infty}^{\infty} e^{-2(x/l)^2}dx. && (50)\end{aligned}$$

We can look up the identity for the integral ($\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\pi/a}$). Here, $a = 2/l^2$. This simplifies the integral to

$$1 = C^2\sqrt{\frac{\pi l^2}{2}}.$$

Solving for C,

$$C = \left(\frac{2}{\pi l^2}\right)^{1/4}.$$

Altogether, this gives the normalised wave function, $\Psi(x,t) = \left(\frac{2}{\pi l^2}\right)^{1/4} e^{-i\omega t}e^{-(x/l)^2}$.

(b) Firstly, let's see what substituting in $t = \pi/2\omega$ does to our wave function.

$$\begin{aligned} \Psi(x,\pi/2\omega) &= \left(\frac{2}{\pi l^2}\right)^{1/4} e^{-i\omega\pi/2\omega}e^{-(x/l)^2}\\ &= \left(\frac{2}{\pi l^2}\right)^{1/4} (-i)e^{-(x/l)^2}. \end{aligned} \quad (51)$$

This means that at this time, the wave function is imaginary (no real part, $\mathrm{Re}(\Psi(x,t)) = 0$) and $\mathrm{Im}(\Psi(x,t)) = -\left(\frac{2}{\pi l^2}\right)^{1/4} e^{-(x/l)^2}$. This is not a problem (we *would* need to worry if $|\Psi(x,t)|^2$ was complex though)!

Let's also calculate $|\Psi(x,t)|^2$.

$$\begin{aligned} |\Psi(x,t)|^2 &= \Psi^\star(x,t)\Psi(x,t)\\ &= \left(\frac{2}{\pi l^2}\right)^{1/4} e^{i\omega t}e^{-(x/l)^2}\left(\frac{2}{\pi l^2}\right)^{1/4} e^{-i\omega t}e^{-(x/l)^2}\\ &= \left(\frac{2}{\pi l^2}\right)^{1/2} e^{-2(x/l)^2}. \end{aligned} \quad (52)$$

We didn't need to sub in $t = \pi/2\omega$ here since the time term cancels.

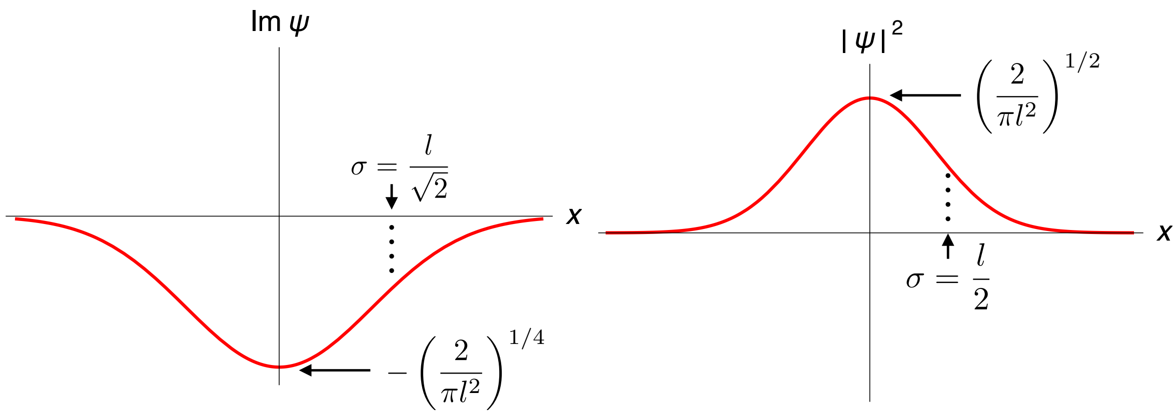

Figure 3: Plots of $\mathrm{Im}(\Psi(x,t))$ and $|\Psi(x,t)|^2$ at $t = \pi/2\omega$. The maximum values on each curve can be found by subbing in $x = 0$, and the standard deviation σ (spread of the curve) can be found by comparing the terms in the exponents with the general form of a normalised Gaussian distribution.

(c)

$$\begin{aligned}\langle x \rangle &= \int_{-\infty}^{\infty} \Psi^\star(x,t) x \Psi(x,t) dx \\ &= \left(\frac{2}{\pi l^2}\right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega t} e^{-(x/l)^2} x e^{-i\omega t} e^{-(x/l)^2} dx \\ &= \left(\frac{2}{\pi l^2}\right)^{1/2} \int_{-\infty}^{\infty} x e^{-2(x/l)^2} dx \\ &= 0.\end{aligned} \tag{53}$$

$\langle x \rangle = 0$ because the above integral is odd.

$$\begin{aligned}\langle x^2 \rangle &= \int_{-\infty}^{\infty} \Psi^\star(x,t) x^2 \Psi(x,t) dx \\ &= \left(\frac{2}{\pi l^2}\right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega t} e^{-(x/l)^2} x^2 e^{-i\omega t} e^{-(x/l)^2} dx \\ &= \left(\frac{2}{\pi l^2}\right)^{1/2} \int_{-\infty}^{\infty} x^2 e^{-2(x/l)^2} dx.\end{aligned} \tag{54}$$

We can use the identity, $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{2a^{3/2}}$, where $a = 2/l^2$. Altogether this gives,

$$\begin{aligned}\langle x^2 \rangle &= \left(\frac{2}{\pi l^2}\right)^{1/2} \frac{\sqrt{\pi}}{2(2/l^2)^{3/2}} \\ &= \frac{l^2}{4}.\end{aligned} \tag{55}$$

So we have $\langle x \rangle = 0$ and $\langle x^2 \rangle = \frac{l^2}{4}$. Now for momentum, where I'll use $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

$$\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} \Psi^\star(x,t)\hat{p}\Psi(x,t)dx \\
&= \left(\frac{2}{\pi l^2}\right)^{1/2} \int_{-\infty}^{\infty} e^{i\omega t} e^{-(x/l)^2} \left(-i\hbar\frac{\partial}{\partial x}\right) e^{-i\omega t} e^{-(x/l)^2} dx \\
&= \left(\frac{2}{\pi l^2}\right)^{1/2} \int_{-\infty}^{\infty} e^{-(x/l)^2} \left(-i\hbar\frac{\partial}{\partial x}\right) e^{-(x/l)^2} dx \\
&= \left(\frac{2}{\pi l^2}\right)^{1/2} (-i\hbar) \int_{-\infty}^{\infty} e^{-(x/l)^2} \frac{-2x}{l^2} e^{-(x/l)^2} dx \\
&= \left(\frac{2}{\pi l^2}\right)^{1/2} (-i\hbar) \left(\frac{-2}{l^2}\right) \int_{-\infty}^{\infty} x e^{-2(x/l)^2} dx \\
&= 0.
\end{aligned} \tag{56}$$

$\langle p \rangle = 0$ because the above integral is odd.

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \Psi^\star(x,t)\hat{p}^2\Psi(x,t)dx \\
&= \int_{-\infty}^{\infty} \Psi^\star(x,t) \left(-i\hbar\frac{\partial}{\partial x}\right)^2 \Psi(x,t)dx.
\end{aligned} \tag{57}$$

For $\langle p^2 \rangle$ things can easily get messy, so it'll be useful to tidy up the derivatives a little before we start:

$$\frac{\partial \Psi}{\partial x} = \frac{-2x}{l^2}\Psi$$

and

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{2\Psi}{l^2}\left(\frac{2x^2}{l^2} - 1\right).$$

Substituting this back in,

$$\langle p^2 \rangle = -\hbar^2 \left(\frac{4}{l^4}\int_{-\infty}^{\infty} \Psi^\star x^2 \Psi dx - \frac{2}{l^2}\int_{-\infty}^{\infty} \Psi^\star \Psi dx\right). \tag{58}$$

Here we know that $\int_{-\infty}^{\infty} \Psi^\star \Psi dx = 1$ and $\int_{-\infty}^{\infty} \Psi^\star x^2 \Psi dx = l^2/4$ from before. Simplifying,

$$\begin{aligned}
\langle p^2 \rangle &= -\hbar^2 \left(\frac{4}{l^4}\frac{l^2}{4} - \frac{2}{l^2}\right). \\
&= -\hbar^2 \left(\frac{1}{l^2} - \frac{2}{l^2}\right) = \frac{\hbar^2}{l^2}.
\end{aligned} \tag{59}$$

(d)

$$\begin{aligned}
\sigma_x &= \sqrt{\langle x^2 \rangle + \langle x \rangle^2} \\
&= \sqrt{l^2/4} = l/2,
\end{aligned} \tag{60}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle + \langle p \rangle^2} \\ = \frac{\hbar}{l}. \tag{61}$$

See if they satisfy Heisenberg's uncertainty principle: $\sigma_x \sigma_p \geq \hbar/2$.

$$\sigma_x \sigma_p = \frac{\hbar}{2}. \tag{62}$$

Heisenberg's uncertainty principle is (just) satisfied. States such as this which satisfy the uncertainty principle with an equality are called *minimum uncertainty* states.

(e) We can simply substitute the wavefunction into the TDSE:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi(x,t) &= \left(-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,t) \\ i\hbar\frac{\partial}{\partial t}\left(\frac{2}{\pi l^2}\right)^{1/4} e^{-i\omega t}e^{-(x/l)^2} &= \left(-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\left(\frac{2}{\pi l^2}\right)^{1/4} e^{-i\omega t}e^{-(x/l)^2} \\ i\hbar(-i\omega)e^{-i\omega t}e^{-(x/l)^2} &= \left(-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)e^{-i\omega t}e^{-(x/l)^2} \\ \hbar\omega e^{-(x/l)^2} &= -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}e^{-(x/l)^2} + V(x)e^{-(x/l)^2} \\ \hbar\omega &= -\frac{\hbar}{2m}\left(\frac{4x^2}{l^4} - \frac{2}{l^2}\right) + V(x) \\ V(x) &= \hbar\omega + \frac{\hbar}{ml^2}\left(\frac{2x^2}{l^2} - 1\right). \end{aligned} \tag{63}$$

The x^2 term in the above expression means that the potential is that of a simple harmonic oscillator (SHO), up to an unimportant constant energy offset V_0,

$$V(x) = \frac{1}{2}kx^2 + V_0, \tag{64}$$

with $V_0 = \hbar\omega - \hbar/ml^2$ and 'spring' constant $k = 4\hbar^2/ml^4$.

You'll see more examples of these later in the course because the SHO potential is super important in quantum systems. You can approximate many potentials as quadratics and we know how to solve the Schrödinger equation for harmonic potentials. A harmonic potential makes sense here because the wave function is a Gaussian, which is actually the ground state of the SHO.