T01

Sunday, 8 August 2021 8:16 PM

Problem 1.1

$$f(x) = f(0)e^{-x^2/2\sigma^2}$$

a. What should $f(0)$ be so that $\int_{-\infty}^{\infty} f(x)\,dx = 1$?

$$\int_{-\infty}^{\infty} f(0)e^{-x^2/2\sigma^2}\,dx = 1$$

$$\Rightarrow 1 = 2f(0)\int_0^{\infty} e^{-x^2/2\sigma^2}\,dx$$

$$= f(0)\,\pi\sqrt{2\pi\sigma^2}$$

$$\Rightarrow f(0) = \frac{1}{\pi\sqrt{2\pi\sigma^2}}$$

b. What is the average value x, $\langle x\rangle = \int_{-\infty}^{\infty} x f(x)\,dx$

$$\langle x\rangle = \int_{-\infty}^{0} x f(x)\,dx + \int_0^{\infty} x f(x)\,dx$$

Since $x f(x)$ is odd and convergent,

$$\int_{-\infty}^{0} x f(x)\,dx = -\int_0^{\infty} x f(x)\,dx$$

$$\Rightarrow \langle x\rangle = 0$$

c. What is the average value of x^2, $\langle x^2\rangle = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$\langle x^2\rangle = 2\int_0^{\infty} \frac{1}{\pi\sqrt{2\pi\sigma^2}}\, x^2 e^{-x^2/2\sigma^2}\,dx$$

$$= \frac{1}{2\pi\sqrt{2\pi\sigma^2}}\sqrt{8\pi\sigma^6}$$

$$= \frac{\sigma^2}{\pi}$$

d. Define r.m.s of $f(x)$ as $\sqrt{\langle x^2\rangle - \langle x\rangle^2}$.

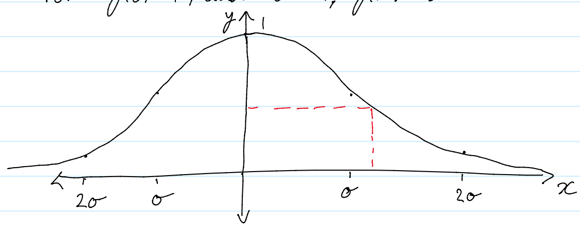
$$w_{rms} = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\sigma^2}{\pi} - 0^2}$$

$$= \frac{\sigma}{\sqrt{\pi}}$$

σ represents the standard deviation of $f(x)$

e. Plot $f(x)$.

For $f(0)=1$, and $\sigma = 1$, $f(x)$ is



f. $\frac{1}{2}f(0) = f(0)e^{-w_{1/2}^2/2\sigma^2}$

$$\Rightarrow \ln\left(\tfrac{1}{2}\right) = -\frac{w_{1/2}^2}{2\sigma^2}$$

$$w_{1/2} = \sqrt{2\sigma^2 \ln(2)}$$

$$= \sqrt{2\ln(2)}$$

$$\simeq 1.177$$

$w_{1/2}$ is 0.177 larger than σ, as expected from the plot in part e.

Problem 1.3:

a. Evaluate the infinite series:

$$Z = \sum_{n=0}^{\infty} e^{-nx} \qquad (n = 0, 1, 2 \ldots)$$

by the integral test for series convergence, if $\int_0^{\infty} a_n\,dn$ converges, then so does $\sum_{n=0}^{\infty} a_n$

let $a_n = e^{-nx}$

$$\Rightarrow \int_0^{\infty} e^{-nx}\,dn$$

$$= \frac{1}{x}$$

for $\sum_{n=0}^{\infty} e^{-nx}$ to be convergent, x must be greater than 0. Thus, $\int_0^{\infty} a_n$ is convergent for $0 < x < \infty$

$\Rightarrow \sum a_n$ is convergent for $0 < x < \infty$.

Let $x = 1$, and the series becomes

$$Z = \sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{e}\right)^n$$

which is a geometric series evaluating to

$$Z = \frac{1}{1-\frac{1}{e}} = \frac{e}{e}\,\frac{1}{1-\frac{1}{e}}$$

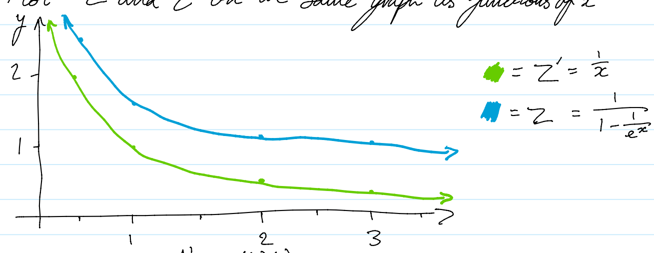
$$= \frac{e}{e-1} \simeq 1.582$$

b. As shown in part a,

$$Z' = \int_0^{\infty} e^{-nx} = \frac{1}{x}$$

for $x = 1$, as above, $Z' = 1$

c. Plot Z and Z' on the same graph as functions of x



$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}(x)^n$$

$$= \frac{f^{(0)}(0)}{0!}(x)^0 + \frac{f^{(1)}(0)}{1!}x$$

$$= e^{-0} - e^{-0}x$$

$$= 1 - x$$

putting this in place of e^{-x} in Z gives

$$Z = \frac{1}{1-1+x} = \frac{1}{x} = Z' \,!$$

$\therefore$ the first two taylor expansion terms for e^{-x} gives $Z = Z'$

T02

Friday, 6 August 2021 1:04 PM

Problem 2.2:

$$\varphi(t) = e^{-iVt/\hbar}$$

$$\varphi_0(t) = e^{-i(V+V_0)t/\hbar}$$

$$= e^{(-iVt/\hbar)+(-iV_0t/\hbar)}$$

$$= e^{-iVt/\hbar} e^{-iV_0t/\hbar}$$

$$\varphi_0(t) = \varphi e^{-iV_0t/\hbar}$$

$\Rightarrow$ calculate difference between

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x,t)|^2 dx$$

and

$$\langle x_0 \rangle = \int_{-\infty}^{\infty} x |\Psi_0(x,t)|^2 dx$$

$$\Rightarrow \langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x,t)|^2 dx$$

$$= \int_{-\infty}^{\infty} x \Psi^* \Psi \, dx$$

$$= \int_{-\infty}^{\infty} x e^{iVt/\hbar} e^{-iVt/\hbar} dx$$

$$= \int_{-\infty}^{\infty} x e^0 \, dx = \int_0^{\infty} x \, dx - \int_0^{\infty} x \, dx$$

$$\Rightarrow \langle x \rangle = 0$$

$$\langle x_0 \rangle = \int_{-\infty}^{\infty} x |\Psi_0(x,t)|^2 dx$$

$$= \int_{-\infty}^{\infty} x \Psi_0^* \Psi_0 \, dx$$

$$= \int_{-\infty}^{\infty} x e^{i(V+V_0)t/\hbar} e^{-i(V+V_0)t/\hbar} dx$$

$$= \int_{-\infty}^{\infty} x e^0 \, dx$$

$$= 0 \quad \text{(as above)}$$

$\therefore$ a change in the magnitude of the potential has no effect on the expectation value.

Problem 2.3:

Consider $\Psi(x,t) = Ce^{-i\omega t} e^{-b|x|}$

a. Normalise Ψ (find C in terms of b)

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1 \Rightarrow \int_{-\infty}^{\infty} Ce^{i\omega t} e^{-b|x|} Ce^{-i\omega t} e^{-b|x|} dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} C^2 e^0 e^{-2b|x|} dx = 1$$

$$\Rightarrow \int_0^{\infty} e^{-2bx} dx + \int_{-\infty}^{0} e^{2bx} dx = \frac{1}{C^2}$$

$$\Rightarrow 2\int_0^{\infty} e^{-2bx} dx = \frac{1}{C^2}$$

$$\Rightarrow \frac{1}{2b} = \frac{1}{2C^2} \quad \text{(as per result in 1.3b)}$$

$$\Rightarrow C = \sqrt{b}$$

b. Calculate the expectation values of x, x^2

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\Psi(x,t)|^2 dx$$

$$= \int_{-\infty}^{\infty} bxe^{-2b|x|} dx$$

$$= \int_0^{\infty} bxe^{-2bx} dx + \int_{-\infty}^{0} bxe^{2bx} dx$$

$$= \int_0^{\infty} bxe^{-2bx} dx - \int_0^{\infty} bxe^{-2bx} dx$$

$$\therefore \langle x \rangle = 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\Psi(x,t)|^2 dx$$

$$= \int_{-\infty}^{\infty} bx^2 e^{-2b|x|} dx$$

$$= \int_0^{\infty} bx^2 e^{-2bx} dx + \int_{-\infty}^{0} bx^2 e^{2bx} dx$$

$$= 2\int_0^{\infty} bx^2 e^{-2bx} dx$$

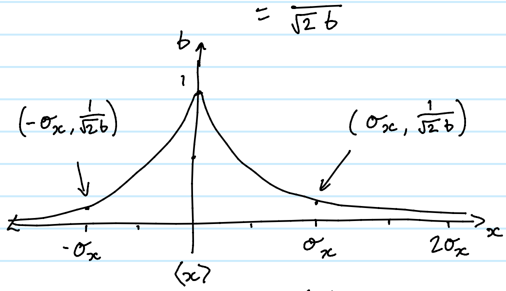
$$= 2b \frac{2!}{(2b)^3} = \frac{1}{2b^2} \quad \text{(as per wikipedia list of integrals)}$$

c. Determine σ_x of x. Graph $|\Psi|^2$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{\frac{1}{2b^2} - 0}$$

$$= \frac{1}{\sqrt{2}\, b}$$



d. What is the probability that you would find x outside of $\langle x \rangle - \sigma_x < x < \langle x \rangle + \sigma_x$?

$$P = 1 - \int_{-\sigma_x}^{\sigma_x} |\Psi(x,t)|^2 dx$$

outside of $\langle x \rangle - \sigma_x < x < \langle x \rangle + \sigma_x$:

$$P = 1 - \int_{-\sigma_x}^{\sigma_x} |\Psi(x,t)|^2 \, dx$$

$$= 1 - \int_{-1/\sqrt{2}b}^{1/\sqrt{2}b} \Psi^* \Psi \, dx$$

$$= 1 - 2\int_0^{1/\sqrt{2}b} b e^{-2bx} \, dx$$

$$= 1 - \left(-e^{-2bx} \Big|_0^{1/\sqrt{2}b} \right) = 1 - \left(1 - e^{-\sqrt{2}}\right)$$

$$= e^{-\sqrt{2}} \simeq 0.2431$$

$\therefore$ The probility of finding the particle outside two deviations from the expected value is approx. 0.2431

e. The probability that the particle would be found <u>within</u> $2\sigma_x$ of the expectation value is just

$$P(-\sigma_x \le x \le \sigma_x) = 1 - P(-\infty < x < \infty; -\sigma_x \nleq x \nleq \sigma_x)$$

$$= 1 - 0.2431$$

$$= 0.7569$$