Assignment 3

Tuesday, 29 August 2023 9:05 PM

Question 1 (5.10):

Using Greenshield's law, with $v_M = 1$, and $\rho_M = 6$. Hence

$$c(\rho) = v_M\left(1 - \frac{2\rho}{\rho_M}\right)$$
$$= 1 - \frac{\rho}{3}$$

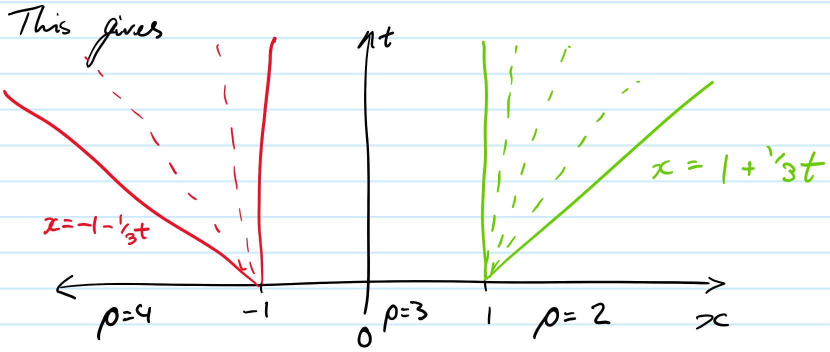
We assume $\rho(x,0) = f(x)$ is a piecewise constant.

a. We want to choose an $f(x)$ such that the solution $\rho(x,t)$ will have two fans and no shocks.
Of course, if the fans overlap, we'll have a shockwave, and so we can't have the fans overlapping.
To have two fans, we'll need a gradual increase in speed from one side to the other, and therefore a gradual decrease in density. Choose

$$\rho(x,0) = f(x) = \begin{cases} 4 & \text{if } x < -1 \\ 3 & \text{if } -1 \le x \le 1 \\ 2 & \text{if } x > 1 \end{cases}$$

$$\Rightarrow c(\rho) = \begin{cases} -1/3 & \text{if } x < -1 \\ 0 & \text{if } -1 \le x \le 1 \\ 1/3 & \text{if } x > 1 \end{cases}$$

This gives



Which has two expansion fans as needed, and solution

$$\rho(x,t) = \begin{cases} 4 & \text{if } x < -\frac{1}{3}t \\ 3 - 3x/t & \text{if } -\frac{1}{3}t \le x < -1 \\ 3 & \text{if } -1 \le x \le 1 \\ 3 - 3x/t & \text{if } 1 < x \le \frac{1}{3}t \\ 2 & \text{if } x > 1 \end{cases}$$

b. Now we want to start out with two shock waves, and no expansion fans. We want an increasingly negative $c(\rho)$.
Consider

$$\rho(x,0) = f(x) = \begin{cases} 3 & \text{if } x < -1 \\ 4 & \text{if } -1 \le x \le 0 \\ 5 & \text{if } 0 < x \end{cases}$$

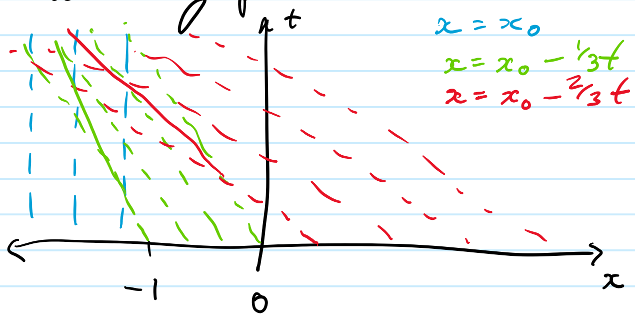
This has corresponding wavespeed

$$c(\rho) = \begin{cases} 0 & \text{if } x < -1 \\ -1/3 & \text{if } -1 \le x \le 0 \end{cases}$$

This has corresponding wavespeed

$$c(\rho) = \begin{cases} 0 & \text{if } x < -1 \\ -1/3 & \text{if } -1 \le x \le 0 \\ -2/3 & \text{if } 0 < x \end{cases}$$

With the resulting plot



We can see that at high enough t, one of the shocks will eventually disappear.

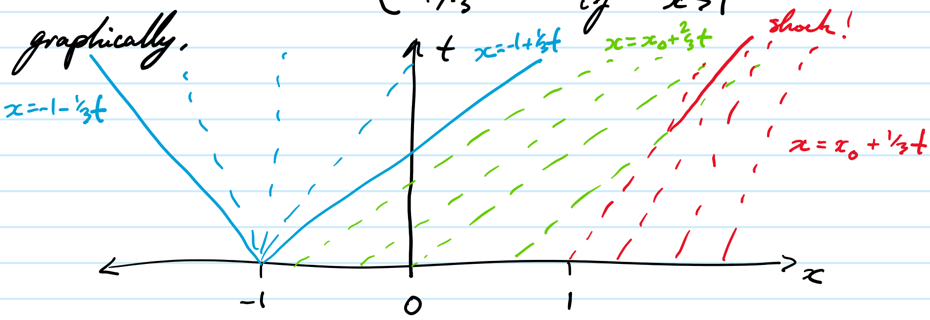
c. Now we want an initial condition that gives a shockwave on the right and expansion fan on the left.
So, we want low velocity → highest velocity → high velocity.
Choose

$$\rho(x,0) = f(x) = \begin{cases} 4 & \text{if } x < -1 \\ 1 & \text{if } -1 \le x \le 1 \\ 2 & \text{if } x > 1 \end{cases}$$

This gives speeds of

$$c(\rho) = \begin{cases} -1/3 & \text{if } x < -1 \\ 2/3 & \text{if } -1 \le x \le 1 \\ 1/3 & \text{if } x > 1 \end{cases}$$

graphically,



d. The example in Fig. 5.30 is analogous to the above case (albeit mirrored). It is not possible to have an expansion fan and shock wave and not have them intersect at some time. The condition for their non-intersection would be that the shock would need a lower slope (in magnitude) with the same sign as the slope of the fan edge. However, if this were the case, we'd get another fan and the shock would disappear.
Therefore it is not possible for them to <u>never</u> intersect.

Question 2 (5.11):

Here we use Greenshield's law again, with $v_M = 1$ and $\rho_M = 10$, and a piecewise constant initial condition $\rho(x,0) = f(x)$.
Hence $c(\rho) = 1 - \frac{\rho}{\ldots}$

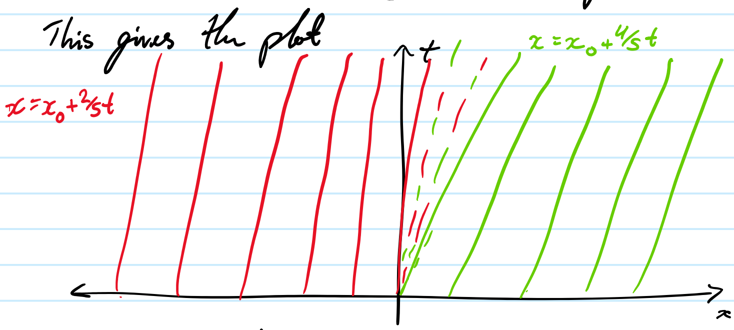
and $\rho_m = 10$, and a piecewise constant initial condition $\rho(x,0) = f(x)$.
Hence $c(\rho) = 1 - \frac{\rho}{5}$

a. Assume $3 \le f(x) \le 4$. It is impossible to obtain a value of $\rho(x,t) = 5$ or $\rho(x,t) = 2$ at any (x,t), since there would be a step function density gradient between 3 and 4 in the event of a shock, and a smooth gradient between 3 and 4 in the event of an expansion fan. Both of these cases would be monotonically increasing/decreasing.

b. We suppose that $f(x)$ only takes on the values $f(x) = 1$ and $f(x) = 3$. For $\rho(x,t)$ to take on values $1 \le \rho(x,t) \le 3$, we'd need an expansion fan. i.e. low velocity → high velocity
Choose

$$\rho(x,0) = f(x) = \begin{cases} 3 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

This gives the plot

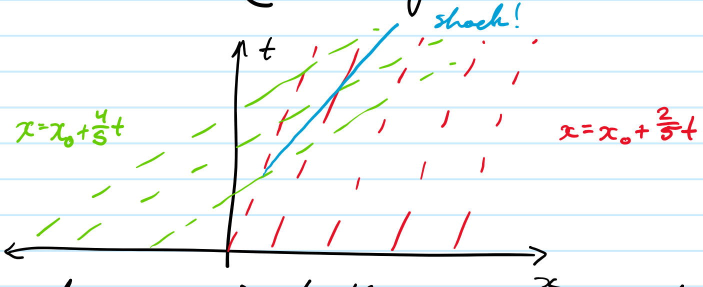


The intermediate region between the red and green characteristics has $1 \le \rho(x,t) \le 3$.

Conversely, if we wanted only values of $\rho(x,t) = 1$ or $\rho(x,t) = 3$, we'd need a shock. We can flip the values in the above case to get

$$\rho(x,0) = f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 3 & \text{if } x \geq 0 \end{cases}$$

with



And so values of (x,t) to the left of the shock will have $\rho(x,t) = 1$ and values to the right will have $\rho(x,t) = 3$.

c. Here, we want an $f(x)$ so that $\rho(x,t) \in \{1, 2, 3\}$.
We'll want a double shock system, so choose

$$\rho(x,0) = f(x) = \begin{cases} 1 & \text{if } x < -1 \\ 2 & \text{if } -1 \le x \le 1 \\ 3 & \text{if } x > 1 \end{cases}$$

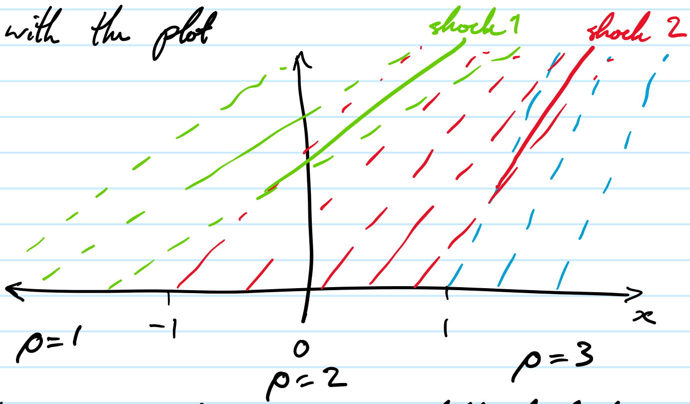
So

$$\rho(x,0) = f(x) = \begin{cases} 2 & \text{if } -1 \le x \le 1 \\ 3 & \text{if } x > 1 \end{cases}$$

So,

$$c(\rho) = \begin{cases} 4/5 & \text{if } x < -1 \\ 3/5 & \text{if } -1 \le x \le 1 \\ 2/5 & \text{if } x > 1 \end{cases}$$

with the plot



Hence $\rho(x,t) = 1$ to the left of shock 1.
$\rho(x,t) = 2$ between shocks 1 and 2
$\rho(x,t) = 3$ to the right of shock 2

Question 3 (6.2):

At $t=0$, a bar occupies $0 \le X \le 1$, with $\mathcal{X}(X,t) = X + Xt^2$

a. At $t=2$, we'd expect the left end of the bar to remain at $x=0$ since there is a 0 trajectory for $X=0$.

i.e. $x_L = \mathcal{X}(0,2) = 0$

for the right end, which is at $X=1$ at $t=0$, at $t=2$, we have

$$x_R = \mathcal{X}(1,2) = 1 + 1\cdot 2^2 = 5$$

$\Rightarrow$ the spatial extent at $t=2$ is $0 \le x \le 5$

b. We have the displacement in material coordinates as

$$U(X,t) = \mathcal{X}(X,t) - X$$
$$= Xt^2$$

The velocity is then found by

$$V(X,t) = \frac{\partial U(X,t)}{\partial t}$$
$$= 2Xt$$

The limits on X are imposed by the initial condition $0 \le X \le 1$

c. The velocity in spatial coordinates is given by

$$v(x,t) = V\Big|_{X = \frac{x}{1+t^2}}$$ (since $\mathcal{X}(X,t) = X + Xt^2 \Rightarrow X = \frac{x}{1+t^2}$)

$$= \frac{2xt}{1+t^2}$$

The limits imposed on x are imposed by $x = \chi(X,t)$,
so
$$\chi(0,t) \le x \le \chi(1,t)$$
$$\Downarrow$$
$$0 \le x \le 1+t^2$$

d. If the temperature of the bar is defined as
$$\theta(x,t) = xt^3$$
the rate of change of the temperature following the material section is given by the material derivative:
$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + v\frac{\partial\theta}{\partial x}$$
$$= 3xt^2 + \left(\frac{2xt}{1+t^2}\right)t^3$$
$$= 3xt^2 + \frac{2xt^4}{1+t^2}$$
$$= 3xt^2\left(\frac{1+3t^2}{1+t^2}\right)$$

Question 4 (6.8):

a. One reason why we can't have
$$\chi(X,t) = \frac{1}{2}X\cos(t)$$
is because we require that $\chi(X,0) = X$. In this case, we have
$$\chi(X,0) = \frac{1}{2}X\cdot 1 \neq X$$
and so the requirement isn't satisfied.

b. According to the condition (6.20) in the textbook,
$$\frac{\partial u}{\partial x} < 1$$
we can't have some $u(0,t) = -1$ and $u(1,t) = 1$ for some t. For this to be true, we'd have $\partial u/\partial x = 2$ which is obviously greater than 1, hence violating (effectively) the impenetrability of matter assumption.

c. Analogously to part b., we also can't have $U(0,t) = 1$ and $U(\frac{1}{2},t) = 0$ for some t, since this would imply
$$\frac{\partial U}{\partial X} = -2$$
but we have the impenetrability of matter assumption that relies on the condition (6.19).
$$\frac{\partial U}{\partial X} > -1$$

d. We want to prove that if $X_1 < X_2$, then
$$U(X_1,t) < X_2 - X_1 + U(X_2,t)$$
First, we know that
$$U(X_1,t) = \chi(X_1,t) - X_1$$
$$\Rightarrow \chi(X_1,t) = U(X_1,t) + X_1$$
$$\Rightarrow \chi(X_2,t) = U(X_2,t) + X_2$$

$$U(X_1,t) = \chi(X_1,t) - X_1$$
$$\Rightarrow \chi(X_1,t) = U(X_1,t) + X_1$$
$$\Rightarrow \chi(X_2,t) = U(X_2,t) + X_2$$

By the impenetrability of matter assumption, we have that if $X_1 < X_2$, then $\chi(X_1,t) < \chi(X_2,t)$
In other words,

$$\chi(X_1,t) < \chi(X_2,t)$$
$$\Rightarrow U(X_1,t) + X_1 < U(X_2,t) + X_2$$
$$\Rightarrow U(X_1,t) < X_2 - X_1 + U(X_2,t)$$

QED.

e. Consider a hypothetical displacement function of the form $U = \alpha \sin(X)$, with $\alpha > 1$.

By the definition of displacement functions, we need $U(X,0) = 0$
However U is not a function of t, and $\alpha > 1$, and so $U \neq 0$ for $X \neq n\pi$, $n \in \mathbb{Z}$, which is one argument against $U = \alpha \sin(X)$ being a valid function.
Another argument considers the requirement that

$$\frac{\partial U}{\partial X} > -1$$
$$\Rightarrow \frac{\partial}{\partial X}(\alpha \sin(X)) > -1$$
$$\Rightarrow \alpha \cos(X) > -1$$

However, since $\alpha > 1$, the above inequality is not always true $\forall X$ (notably near values of $X \sim (2n+1)\pi$, $n \in \mathbb{Z}$).
With these two arguments together, we see that the conditions for a valid displacement function do not hold for practically all values of X, and so this function is not possible.