MATH3403 Assignment 6

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Question 1

a. Since we can assume radial symmetry, the solution to the steady-state ($u_t = 0$) heat equation on the annulus is Laplace's 2D solution (solved in lecture 10):

$$(ru_r)_r = 0 \implies u(r) = c_1 \ln(r) + c_2$$

with boundary conditions

$$\begin{cases} u(r) = b & r = 1 \\ \frac{\partial u}{\partial r} = -a < 0 & a \in \mathbb{R},\ r = 2 \end{cases}$$

Evaluating the boundary conditions gives

$$b = u(1) = c_1 \ln(1) + c_2 = c_2 \Rightarrow b = c_2$$

and so

$$u(r) = c_1 \ln(r) + b$$

with

$$\frac{du}{dr} = \frac{c_1}{r}$$

At $r = 2$, $du/dr = -a$ and so $c_1 = -2a$ which gives a formula for the temperature at any radius:

$$u(r) = -2a \ln(r) + b$$

b. By the maximum (minimum) principle, the maximum temperature lies on the boundary $r = 1$ with $u(r) = b$. For all $r > 1$, $u(r) < b$ and so the minimum is on the boundary $r = 2$ with $u_{\min} = -2a \ln(2) + b$.

c. If $b = 115^{\text{o}}$, the value of a such that $u_{\min} = 40^{\text{o}}$ is

$$40^{\text{o}} = -2a \ln(2) + 115^{\text{o}}$$
$$a = \frac{75^{\text{o}}}{2\ln(2)} \simeq 54.1^{\text{o}}$$

Question 2

Consider

$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial n} = g & \text{on } \partial U \end{cases}$$

and assume that there are two solutions, u_1 and u_2, such that $u_1 - u_2 = w$ where w solves the Neumann problem

$$\begin{cases} -\Delta w = 0 & \text{in } U \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial U \end{cases}$$

Now, Green's first identity says that

$$\iint_{\partial U} w \frac{\partial w}{\partial n} \, dS = \iiint_U \nabla w \cdot \nabla w \, dx + \iiint_U w \Delta w \, dx \tag{1}$$

but $\partial w / \partial n = 0 = \Delta w$ (by Neumann conditions), and so equation (1) becomes

$$\iiint_U \nabla w^2 \, dx = 0$$

and since ∇w^2 is negative nowhere, then $\iiint \nabla w^2 = 0 \implies \nabla w = 0$ in all U, which implies that w is constant in all U and so

$$u_1 - u_2 = \text{constant}$$

and the solution u is unique to a constant. QED

Question 3

Suppose we have the BVP on domain U:

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

and that a unique u solves the BVP with a Green's function $G(x, x_0)$ such that

$$\begin{cases} \Delta G = \delta_{x_0} & \text{in } U \\ G = 0 & \text{on } \partial U \end{cases}$$

Now, suppose that there are two Green's functions G_1 and G_2 that satisfy this, and define

$$G_1 - G_2 = G$$

Then,

$$\begin{aligned} \Delta G &= \Delta G_1 - \Delta G_2 \qquad \text{in } U \\ &= \delta_{x_0} - \delta_{x_0} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} G &= G_1 - G_2 \qquad \text{on } \partial U \\ &= 0 - 0 = 0 \end{aligned}$$

And so since the solution u on U is unique, $G_1 - G_2 = 0 \Longrightarrow G_1 = G_2$ and there is only one unique Green's Function for a unique solution on a given domain.

Question 4

Take the upper half sphere

$$B^* = \left\{x^2 + y^2 + x^2 < a \mid z > 0\right\}$$

Then, by method of reflection about the z-axis, Green's Function on this domain is:

$$G(x, y, z; x_0, y_0, z_0) = G_B(x, y, z; x_0, y_0, z_0) - G_B(x, y, -z; x_0, y_0, z_0) \tag{2}$$

where $G_B = \Phi_L(|\vec{x} - \vec{x_0}|) - \Phi_L(\frac{1}{a}|\vec{x_0}||\vec{x} - \vec{x_0}^*|)$, and

$$\vec{x_0}^* = \frac{a^2\vec{x_0}}{|\vec{x_0}|^2} \qquad \vec{x_0}^* = (x_0^*, y_0^*, z_0^*)$$

(as shown in lectures 17 and 18). Also note that

$$\Phi_L = \frac{1}{4\pi}\left(\frac{-1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}\right)$$

With all of these, then equation (2) becomes

$$\begin{aligned} G(x, y, z; x_0, y_0, z_0) = \frac{1}{4\pi}&\left(\frac{-1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} - \frac{-1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (-z-z_0)^2}}\right) \\ &- \frac{1}{4\pi}\left(\frac{-1}{\left|\frac{|\vec{x_0}|\vec{x}}{a} - \frac{a\vec{x_0}}{|\vec{x_0}|}\right|} - \frac{-1}{\left|\frac{|\vec{x_0}|\vec{x_2}}{a} - \frac{a\vec{x_0}}{|\vec{x_0}|}\right|}\right) \end{aligned}$$

where $x_2 = (x, y, -z)$. Then,

$$\begin{aligned} G(x, y, z; x_0, y_0, z_0) = \frac{1}{4\pi}&\left(\frac{-1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} - \frac{-1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (-z-z_0)^2}}\right. \\ &- \frac{a}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}\left(\frac{-1}{\sqrt{(x-dx_0)^2 + (y-dy_0)^2 + (z-dz_0)^2}}\right. \\ &\left.\left. - \frac{-1}{\sqrt{(x-dx_0)^2 + (y-dy_0)^2 + (-z-dz_0)^2}}\right)\right) \end{aligned}$$

where d is a constant defined by

$$d = \frac{a^2}{x_0^2 + y_0^2 + z_0^2}$$

and so $G(x, y, z; x_0, y_0, z_0)$ is Green's function for the upper half sphere of radius a.