# STAT2003 Assignment 2

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## Question 1

a. The probability generating function of $Y$ is given by

$$\begin{aligned}
G_Y(z) = \mathbb{E}\left(z^Y\right) &= \sum_{n=0}^{\infty} \mathbb{E}\left(z^Y \mid N = n\right) f_N(n) \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left(z^{\sum_{i=1}^{N} X_n} \mid N = n\right) \cdot \frac{4^n}{n!}\ e^{-4} \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left(\prod_{i=1}^{N} z^{X_i} \mid N = n\right) \cdot \frac{4^n}{n!}\ e^{-4} \\
&= \sum_{n=0}^{\infty} \prod_{i=1}^{n} \mathbb{E}\left(z^{X_i}\right) \cdot \frac{4^n}{n!}\ e^{-4} \\
&= \sum_{n=0}^{\infty} \left((1 - p + zp)^{n_B}\right)^n \cdot \frac{4^n}{n!}\ e^{-4}
\end{aligned}$$

where, in the last step, the probability generating function of the binomial distribution was substituted in. In this context, we have $X_i \sim \text{Bin}(2,\ 1/2)$ and so $p = 1/2$ and $n_B = 2$.

$$\begin{aligned}
\Rightarrow G_Y(z) = \mathbb{E}\left(z^Y\right) &= \sum_{n=0}^{\infty} \frac{((1/2 + 1/2z)^2)^n \cdot 4^n}{n!}\ e^{-4} \\
&= \sum_{n=0}^{\infty} \frac{(z^2 + 2z + 1)^n}{n!}\ e^{-4} \\
&= e^{(z^2+2z+1)} \cdot e^{-4} \\
&= e^{(z^2+2z-3)}
\end{aligned}$$

where the second last line was due to the Taylor series equivalence with exponentiation.

b. The probability that $Y = k$ can be found via equation 1:

$$\mathbb{P}\left(Y = k\right) = \frac{{G_Y}^{(k)}(0)}{k!} \tag{1}$$

As such,

$$\begin{aligned}
\mathbb{P}\left(Y = 0\right) &= \frac{{G_Y}^{(0)}(0)}{0!} \\
&= \frac{e^{(0^2+2\cdot 0-3)}}{1} \\
&= e^{-3}
\end{aligned}$$

For $Y = 1$, note that ${G_Y}'(z) = dG_Y/dz = (2z + 2) \cdot \exp\left(z^2 + 2z - 3\right)$. With this in mind,

$$\begin{aligned}
\mathbb{P}\left(Y = 1\right) &= \frac{{G_Y}'(0)}{1!} \\
&= \frac{2e^{-3}}{1} = 2e^{-3}
\end{aligned}$$

c. The expectation value of $Y$ can be found from the first derivative of the probability generating function at $z = 1$:

$$\begin{aligned}\mathbb{E}\left(Y\right) &= G'(1) \\ &= (2 \cdot 1 + 2) \cdot \exp\left(1^2 + 2 \cdot 1 - 3\right) \\ &= 4e^0 = 4\end{aligned}$$

Now, to find the variance of $Y$, first the double derivative of the pgf must be computed:

$$\begin{aligned}G''(z) &= \frac{d}{dz}G'(z) = \frac{d}{dz}\left((2z+2) \cdot \exp\left(z^2 + 2z - 3\right)\right) \\ &= 4 \cdot \exp\left(z^2 + 2z - 3\right) + (2z+2)^2 \cdot \exp\left(z^2 + 2z - 3\right) \\ &= (4z^2 + 8z + 8) \cdot \exp\left(z^2 + 2z - 3\right) \\ \Rightarrow \mathrm{Var}\left(Y\right) &= G''(1) + G'(1) - (G(1))^2 \\ &= 20 + 4 - 1 = 23\end{aligned}$$

d. Define $V = V_1 + 2V_2$, where $V_i \sim \mathrm{Poi}(\lambda_i)$, $i = 1,\, 2$. The probability generating function of this is then

$$\begin{aligned}G_V(z) = \mathbb{E}\left(z^V\right) &= \mathbb{E}\left(z^{V_1 + 2V_2}\right) \\ &= \mathbb{E}\left(z^{V_1}\right)\mathbb{E}\left(z^{2V_2}\right) \\ &= \mathbb{E}\left(z^{V_1}\right)\mathbb{E}\left((z^2)^{V_2}\right) \\ &= G_{V_1}(z)G_{V_2}(z^2) \\ &= e^{\lambda_1(z-1)} \cdot e^{\lambda_2(z^2-1)} \\ &= \exp\left(\lambda_1(z-1) + \lambda_2(z^2-1)\right)\end{aligned}$$

By the uniqueness property, $V$ and $Y$ have the same distribution if their PGFs are the same. By setting them equal to each other, the values of $\lambda_1$ and $\lambda_2$ such that this holds true can be found.

$$\begin{aligned}G_V(z) &= G_Y(z) \\ \exp\left(\lambda_1(z-1) + \lambda_2(z^2-1)\right) &= \exp\left(z^2 + 2z - 3\right) \\ \lambda_1(z-1) + \lambda_2(z^2-1) &= z^2 + 2z - 3 \\ \lambda_2 z^2 + \lambda_1 z - (\lambda_1 + \lambda_2) &= z^2 + 2z - 3 \\ \Rightarrow \lambda_1 &= 2 \\ \Rightarrow \lambda_2 &= 1\end{aligned}$$

And so, if $\lambda_1 = 2$ and $\lambda_2 = 1$, $V$ and $Y$ have the same distribution.

# Question 2

a. Firstly, the cumulative distribution function of $g(x)$, $G(x)$, must be found. This is the integral from 0 to $x$ of the probability distribution function, given by

$$\begin{aligned} G(x) &= \int_0^x g(u)\ du \\ &= \int_0^x \frac{\alpha\beta u^{\alpha-1}}{(\beta+u^\alpha)^2}\ du \end{aligned}$$

Into this, make the substitution $s = \beta + u^\alpha \Rightarrow ds = \alpha u^{\alpha-1} du$:

$$\begin{aligned} G(x) &= \int_0^x \frac{\beta}{s^2} ds \\ &= \left[-\frac{\beta}{s}\right]_0^x \\ &= \left[-\frac{\beta}{\beta+u^\alpha}\right]_0^x \\ &= -\frac{\beta}{x^\alpha+\beta} - \left(-\frac{\beta}{0+\beta}\right) \\ &= 1 - \frac{\beta}{x^\alpha+\beta} \end{aligned}$$

Now, let some $G(x) = y$ such that $x = G^{-1}(y)$. The inverse function is then found by rearrange the cumulative distribution function $G(x)$:

$$\begin{aligned} y &= 1 - \frac{\beta}{x^\alpha+\beta} \\ 1-y &= \frac{\beta}{x^\alpha+\beta} \\ x^\alpha+\beta &= \frac{\beta}{1-y} \\ x^\alpha &= \beta\left(\frac{1}{1-y} - 1\right) \\ \Rightarrow G^{-1}(x) &= \left(\beta\left(\frac{1}{1-x} - 1\right)\right)^{1/\alpha} \end{aligned} \tag{2}$$

With this, the algorithm to simulate a variable from $g$ using the inverse transform method is:

1. Generate $U \sim U(0,1)$
2. Return $X \sim G^{-1}(U)$, where $G^{-1}$ is given by equation 2.

b. A Python implementation of the above inverse transform algorithm is provided in the following cell.

```python
# -*- coding: utf-8 -*-
"""
Created on Tue Apr 26 20:20:19 2022

@author: ryanw
"""

from numpy.random import uniform, seed
from numpy import arange
import matplotlib.pyplot as plt


def inverse_function(sample):
    '''This is the inverse function from question 2, part a.
    '''
    b = 3
    a = 2
    output = (b * ((1 / (1 - sample)) - 1))**(1/a)
```

```
    return output

def prob_dens_func(x):
    '''This is the probability density function as per the question description.
    '''
    b = 3
    a = 2
    output = a * b * (x**(a-1)) / (b + x**a)**2
    return output


seed(58268)          #this is a nice seed :)
values = uniform(0, 1, 10**5)       #values is an array of 10^5 uniform random variables, with min=0 and max=1

variables = inverse_function(values)    #calculates the inverse-transform variables
X = arange(0, max(variables), 0.1)      #this is the range of the pdf function line

fig, ax = plt.subplots()

ax.hist(variables, bins=500)
plt.xlim(0, 20)
ax.set_xlabel("Random Variable")
ax.set_ylabel("Number of Instances")

ax2 = ax.twinx()

ax2.plot(X, prob_dens_func(X), 'r-')
plt.ylim(0, 0.4)
ax2.set_ylabel("Probability")

fig.savefig('histogram.pdf', dpi=200, bbox_inches='tight', pad_inches = 0.01)
```

This code produces the histogram shown in figure 1, with the probability density function of $g$ overlaid in red.

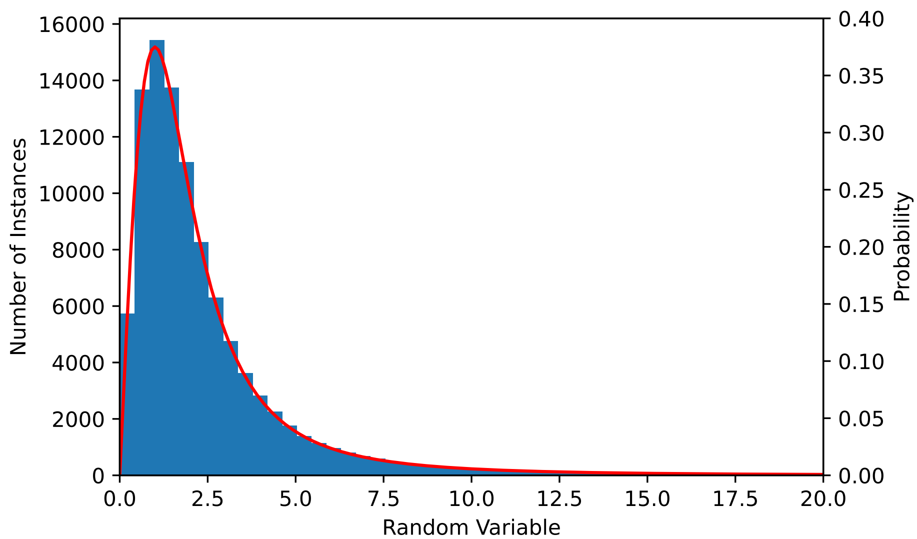


Figure 1: Histogram of Random Variables Generated from $g$

c. i. With $\alpha = 1/2$ and $\beta = 1/\sqrt{3}$, $g(x)$ becomes

$$g(x) = \frac{x^{-1/2}}{2\sqrt{3}x^{1/2}(1/\sqrt{3} + x^{1/2})^2}$$
$$= \left(2\sqrt{3}x^{3/2} + 4x + \frac{2\sqrt{3}}{3}x^{1/2}\right)^{-1}$$

Now, noting that $\Gamma(3/2) = \sqrt{\pi}/2$, $f(x)$ becomes

$$f(x) = \frac{x^{-1/2}}{\pi/2(x+1)^2}$$

$$
\begin{aligned}
&= \frac{2}{\pi x^{5/2} + 2\pi x^{3/2} + \pi x^{1/2}} \\
\Rightarrow (f/g)(x) &= \frac{2\left(2\sqrt{3}x^{3/2} + 4x + \frac{2\sqrt{3}}{3}x^{1/2}\right)}{\pi x^{5/2} + 2\pi x^{3/2} + \pi x^{1/2}} \\
&= \frac{4\sqrt{3}x + 8x^{1/2} + \frac{4\sqrt{3}}{3}}{\pi x^2 + 2\pi x + \pi}
\end{aligned}
$$

To find the derivative of $(f/g)$ and hence any stationary points, the quotient rule can be used:

$$
(h/k)' = \frac{k(x)\ h'(x) - k'(x)\ h(x)}{(k(x))^2} \tag{3}
$$

Set $h(x) = 4\sqrt{3}x + 8x^{1/2} + 4\sqrt{3}/3 \Rightarrow h'(x) = 4\sqrt{3} + 4x^{-1/2}$, and $k(x) = \pi x^2 + 2\pi x + \pi \Rightarrow k'(x) = 2\pi x + 2\pi$. Since the stationary point occurs when $(f/g)' = 0$, the numerator of equation (3) must be 0 and so we can look at strictly the numerator to make computation easier. Define a new function, $w(x)$ such that roots of $w$ occur at stationary points of $(f/g)(x)$ (i.e. when $w(x) = 0$, $(f/g)'(x) = 0$).

$$
\begin{aligned}
w(x) &= k(x)\ h'(x) - k'(x)\ h(x) \\
&= (\pi x^2 + 2\pi x + \pi)(4\sqrt{3} + 4x^{-1/2}) - (2\pi x + 2\pi)(4\sqrt{3}x + 8x^{1/2} + 4\sqrt{3}/3)
\end{aligned}
$$

Setting $x = 1/3$ then gives:

$$
\begin{aligned}
w(1/3) &= \left(\frac{\pi + 6\pi + 9\pi}{9}\right)\left(4\sqrt{3} + 4\sqrt{3}\right) - \left(\frac{2\pi + 6\pi}{3}\right)\left(\frac{4\sqrt{3} + 8\sqrt{3} + 4\sqrt{3}}{3}\right) \\
&= \left(\frac{16\pi}{9}\right)\left(8\sqrt{3}\right) - \left(\frac{24\pi}{9}\right)\left(\frac{16\sqrt{3}}{3}\right) \\
&= 0
\end{aligned}
$$

And so there is a stationary point at $x^* = 1/3$. Since this point lies between $x_l = 3/10$ and $x_u = 4/10$, substituting in the lower and upper bound $x$ values into $w$ shows whether $x^*$ is a minimum or a maximum point. Note that $w(x)$ doesn't correspond to the exact value of $(f/g)'$ (except when $w = 0$), but that the sign is correct regardless due to the strictly positive denominator of $(f/g)'$.

$$
\begin{aligned}
w(3/10) &= k(3/10)\ h'(3/10) - k'(3/10)\ h(3/10) \\
&\simeq 3.93 \\
w(4/10) &= k(4/10)\ h'(4/10) - k'(4/10)\ h(4/10) \\
&\simeq -7.59
\end{aligned}
$$

And so $x^* = 1/3$ is a maximum, since $(f/g)'(x)$ is positive for $x < 1/3$ and negative for $x > 1/3$, with $x \sim x^*$.

ii. The acceptance — rejection algorithm to generate a random variable from the distribution $f(x)$ is as follows:
   1. Generate $X \sim g$ as per part b) of this question.
   2. Generate $Y \sim U(0, C \cdot g(x))$
   3. If $Y \leq f(X)$, return $Z = X$. Otherwise, return to step 1 and generate new variables.

iii. Notice that in the algorithm above, there is an undefined constant $C$. This constant can be evaluated by substituting in $x = 1/3$ into the function $(f/g)$:

$$
\begin{aligned}
C = (f/g)(1/3) &= \frac{\frac{4\sqrt{3}}{3} + \frac{8\sqrt{3}}{3} + \frac{4\sqrt{3}}{3}}{\frac{\pi}{9} + \frac{2\pi}{3} + \pi} \\
&= \frac{3\sqrt{3}}{\pi}
\end{aligned}
$$

When simulating random variables of a distribution via the acceptance—rejection method, we would expect $N \cdot C$ iterations to produce $N$ random variables. To produce one random variable from $f$, we would expect to require $1 \cdot C \approx 1.654$ random variables from $g$.

d. A python implementation of the algorithm from part c)ii is provided in the cell below.

```python
# -*- coding: utf-8 -*-
"""
Created on Sun May  1 16:58:32 2022

@author: ryanw
"""

from numpy.random import uniform, seed
from numpy import arange, pi, zeros, sqrt
import matplotlib.pyplot as plt


def inverse_function(sample):
    '''This is the inverse function from question 2, part a.
    '''
    b = 1 / sqrt(3)
    a = 1 / 2
    output = (b * ((1 / (1 - sample)) - 1))**(1/a)
    return output

def prob_dens_funcG(x):
    '''This is the probability density function of g(x).
    '''
    b = 1 / sqrt(3)
    a = 1 / 2
    output = a * b * (x**(a - 1)) / (b + x**a)**2
    return output

def prob_dens_funcF(x):
    '''This is the probability density function of f(x).
    '''
    output = (x**(-1/2)) / ((pi / 2) * (x + 1)**2)
    return output

seed(4)          #this is a nice seed :)

n = 10**5        #how many rand. vars. we want.
z = zeros(n)
C = 3 * sqrt(3) / pi
for i in range(n):
    found = False
    while not found:
        x = inverse_function(uniform(0, 1))     #X ~ g  as per q2b
        y = uniform(0, C * prob_dens_funcG(x))  #Y ~ U(0, Cg(x))
        if (y <= prob_dens_funcF(x)):
            found = True
            z[i] = x

xmax = 2
X = arange(0, xmax+1, 0.01)

fig, ax = plt.subplots()
#now to plot the histogram of random variables and their frequencies
ax.hist(z, bins=40, range=(0, xmax))
plt.xlim(0, xmax)
ax.set_xlabel("Random Variable")
ax.set_ylabel("Number of Instances")

ax2 = ax.twinx()
#now to plot the pdf overlaid on top of the histogram
ax2.plot(X, prob_dens_funcF(X), 'r-')
plt.ylim(bottom=0)
ax2.set_ylabel("Probability")

fig.savefig('histogram2d.pdf', dpi=200, bbox_inches='tight', pad_inches = 0.01)
```

The histogram that this code produces is shown in figure 2, with the probability density function of $f$ overlaid on top.

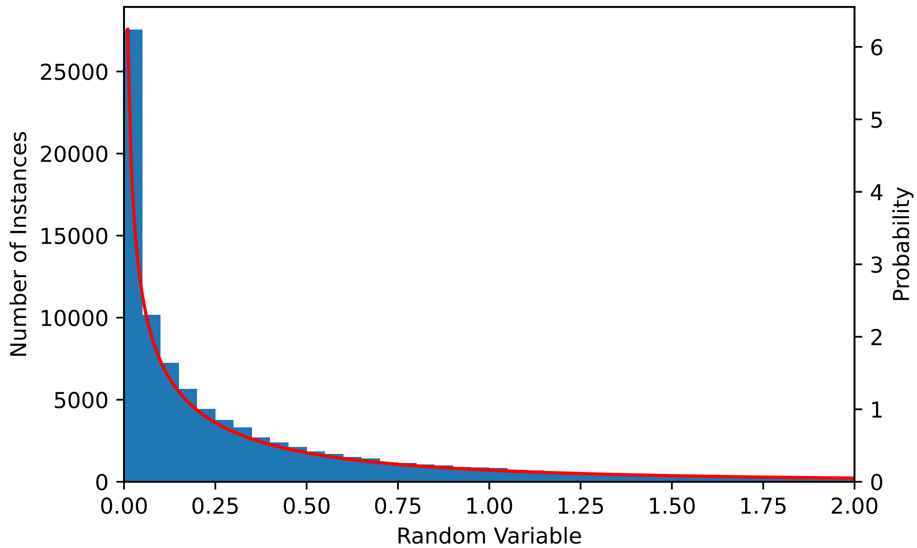


Figure 2: Histogram of Random Variables Generated from $f$

## Question 3

a. Since the random variable $X \sim U(0,\, 1)$, it has the probability density function

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Since $(Y \mid X = x) \sim U(-3x,\, 5x)$, the conditional probability density function of $Y$ is then

$$f_{Y|X}(y \mid x) = \begin{cases} \frac{1}{8x} & \text{if } -3x \le y \le 5x \\ 0 & \text{otherwise} \end{cases}$$

To evaluate the integral of the conditional pdf and the pdf of $X$ (and hence find the pdf of $Y$), first the range needs to be split into the upper and lower half space. Take an upper bound of $x = 1$ and a lower bound of $y = 5x \Rightarrow x = y/5$ if $y \in [0,\, 5]$, or a lower bound of $y = -3x \Rightarrow x = -y/3$ if $y \in (0,\, -3]$.
When $y \in [0, 5]$,

$$\begin{aligned} f_Y(y) &= \int_{y/5}^{1} f_{Y|X}(y \mid x)\; f_X(x)\; dx \\ &= \int_{y/5}^{1} \frac{1}{8x}\; dx \\ &= \frac{1}{8}\left[\ln(x)\right]_{y/5}^{1} \\ &= -\frac{1}{8}\ln(y/5) \end{aligned}$$

When $y \in (0, -3]$,

$$\begin{aligned} f_Y(y) &= \int_{-y/3}^{1} f_{Y|X}(y \mid x)\; f_X(x)\; dx \\ &= \int_{-y/3}^{1} \frac{1}{8x}\; dx \\ &= \frac{1}{8}\left[\ln(x)\right]_{-y/3}^{1} \\ &= -\frac{1}{8}\ln(-y/3) \end{aligned}$$

The probability density function of $Y$ is then

$$f_Y(y) = \begin{cases} -\frac{1}{8}\ln(y/5) & y \in [0,\, 5] \\ -\frac{1}{8}\ln(-y/3) & y \in (0,\, -3] \\ 0 & \text{otherwise} \end{cases}$$

b. The expectation of $Y$ can be found via

$$\begin{aligned}
\mathbb{E}\left(Y\right) &= \int_{-\infty}^{\infty} \mathbb{E}\left(Y \mid X = x\right) f_X(x)\; dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{Y|X}(y \mid x)\; dy\; f_X(x)\; dx \\
&= \int_0^1 \int_{-3x}^{5x} \frac{y}{8x}\; dy\; f_X(x)\; dx \\
&= \int_0^1 \left[\frac{y^2}{16x}\right]_{-3x}^{5x} f_X(x)\; dx \\
&= \int_0^1 \left(\frac{25x}{16} - \frac{9x}{16}\right) f_X(x)\; dx \\
&= \int_0^1 x \cdot 1\; dx \\
&= \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}
\end{aligned}$$

c. The correlation between $X$ and $Y$ is given by

$$\begin{aligned}
\varrho(X,\, Y) &= \frac{\operatorname{Cov}\left(X,\, Y\right)}{\sqrt{\operatorname{Var}\left(X\right)\operatorname{Var}\left(Y\right)}} \\
&= \frac{\mathbb{E}\left(XY\right) - \mathbb{E}\left(X\right)\mathbb{E}\left(Y\right)}{\sqrt{\operatorname{Var}\left(X\right)\operatorname{Var}\left(Y\right)}}
\end{aligned}$$

Since $X$ is a simple binomial distribution, it has

$$\begin{aligned}
\operatorname{Var}\left(X\right) &= \frac{1}{12}(b-a)^2 = \frac{1}{12} \\
\mathbb{E}\left(X\right) &= \frac{1}{2}(a+b) = \frac{1}{2}
\end{aligned}$$

In the previous part, it was found that $\mathbb{E}\left(Y\right) = 1/2$ also. The variance in $Y$ is then

$$\begin{aligned}
\operatorname{Var}\left(Y\right) &= \mathbb{E}\left(Y^2\right) - \mathbb{E}\left(Y\right)^2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2\, f_{Y|X}(y \mid x)\; dy\; f_X(x)\; dx - (1/2)^2 \\
&= \int_0^1 \int_{-3x}^{5x} \frac{y^2}{8x}\; dy\; f_X(x)\; dx - \frac{1}{4} \\
&= \int_0^1 \left[\frac{y^3}{24x}\right]_{-3x}^{5x} f_X(x)\; dx - \frac{1}{4} \\
&= \int_0^1 \frac{152x^2}{24} \cdot 1\; dx - \frac{1}{4} \\
&= \left[\frac{152x^3}{72}\right]_0^1 - \frac{1}{4} \\
&= \frac{19}{9} - \frac{1}{4} = \frac{67}{36}
\end{aligned}$$

The expectation of $XY$ can then be calculated,

$$\begin{aligned}
\mathbb{E}\left(XY\right) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{Y\mid X}(y \mid x) f_X(x)\; dy\; dx \\
&= \int_0^1 \int_{-3x}^{5x} \frac{y}{8}\; dy\; dx \\
&= \int_0^1 \left[\frac{y^2}{16}\right]_{-3x}^{5x} dx \\
&= \int_0^1 \left(\frac{25x^2}{16} - \frac{9x^2}{16}\right) dx \\
&= \int_0^1 x^2\; dx \\
&= \frac{1}{3}
\end{aligned}$$

With all of these values defined, the correlation can then be calculated:

$$\begin{aligned}
\varrho(X,\, Y) &= \frac{\mathbb{E}\left(XY\right) - \mathbb{E}\left(X\right)\mathbb{E}\left(Y\right)}{\sqrt{\operatorname{Var}\left(X\right)\operatorname{Var}\left(Y\right)}} \\
&= \frac{1/3 - (1/2)\cdot(1/2)}{\sqrt{(1/12)\cdot(67/36)}} \\
&\simeq 0.2116
\end{aligned}$$

# Question 4

The random variable $Y$ is defined by $Y = g(X) = -\ln(X) \Rightarrow g^{-1}(X) = \exp\left(-X\right)$ and $g'(X) = -1/X$. The probability density function of $Y$ can then be found by the transformation rule:

$$\begin{aligned}
f_Y(y) &= \frac{f_X(g^{-1}(y)}{|g'(g^{-1}(y))|} \\
&= \frac{-\ln(e^{-y})}{|-1/e^{-y}|} \\
&= ye^{-y}
\end{aligned}$$

# Question 5

a. Note that

$$\operatorname{Cov}\left(X_n,\, X_{n+k}\right) = \mathbb{E}\left(X_n\, X_{n+k}\right) - \mathbb{E}\left(X_n\right)\mathbb{E}\left(X_{n+k}\right)$$

When $k = 0$,

$$\begin{aligned}
\operatorname{Cov}\left(X_n,\, X_{n+k}\right) &= \operatorname{Cov}\left(X_n,\, X_n\right) \\
&= \operatorname{Var}\left(X_n\right) \\
&= \mathbb{E}\left(\left(Z_n + 1/3Z_{n-1}\right)^2\right) - \mathbb{E}\left(Z_n + 1/3Z_{n-1}\right)^2 \\
&= \mathbb{E}\left(Z_n^2 + 2/3Z_{n-1}Z_n + 1/9Z_{n-1}^2\right) - \mathbb{E}\left(Z_n\right)^2 - 2/3\mathbb{E}\left(Z_{n-1}Z_n\right) - 1/9\mathbb{E}\left(Z_{n-1}\right)^2 \\
&= \left(\mathbb{E}\left(Z_n^2\right) - \mathbb{E}\left(Z_n\right)^2\right) + 1/9\left(\mathbb{E}\left(Z_{n-1}^2\right) - \mathbb{E}\left(Z_{n-1}\right)^2\right) \\
&= \operatorname{Var}\left(Z_n\right) + 1/9\operatorname{Var}\left(Z_{n-1}\right) \\
&= 1 + 1/9 = 10/9
\end{aligned}$$

When $k > 1$, $X_n$ and $X_{n+k}$ are completely independent, and so

$$\begin{aligned}
\operatorname{Cov}\left(X_n,\, X_{n+k}\right) &= \mathbb{E}\left(X_n\, X_{n+k}\right) - \mathbb{E}\left(X_n\right)\mathbb{E}\left(X_{n+k}\right) \\
&= \mathbb{E}\left(X_n\right)\mathbb{E}\left(X_{n+k}\right) - \mathbb{E}\left(X_n\right)\mathbb{E}\left(X_{n+k}\right) \\
&= 0
\end{aligned}$$

And finally, when $k = 1$,

$$\begin{aligned}
\operatorname{Cov}(X_n,\, X_{n+1}) &= \mathbb{E}(X_n\, X_{n+1}) - \mathbb{E}(X_n)\,\mathbb{E}(X_{n+1}) \\
&= \mathbb{E}((Z_n + 1/3 Z_{n-1})(Z_{n+1} + 1/3 Z_n)) - \mathbb{E}(Z_n + 1/3 Z_{n-1})\,\mathbb{E}(Z_{n+1} + 1/3 Z_n) \\
&= \mathbb{E}\left(Z_n Z_{n+1} + 1/3 Z_n^2 + 1/3 Z_{n-1} Z_{n+1} + 1/9 Z_{n-1} Z_n\right) - (\mathbb{E}(Z_n) + 1/3\mathbb{E}(Z_{n-1}))\,(\mathbb{E}(Z_{n+1}) + 1/3\mathbb{E}(Z_n)) \\
&= \mathbb{E}(Z_n Z_{n+1}) + 1/3\mathbb{E}\left(Z_n^2\right) + 1/3\mathbb{E}(Z_{n-1} Z_{n+1}) + 1/9\mathbb{E}(Z_{n-1} Z_n) \\
&\quad - \mathbb{E}(Z_n)\,\mathbb{E}(Z_{n+1}) - 1/3\mathbb{E}(Z_n)^2 - 1/3\mathbb{E}(Z_{n-1})\,\mathbb{E}(Z_{n+1}) - 1/9\mathbb{E}(Z_{n-1})\,\mathbb{E}(Z_n)
\end{aligned}$$

Since all variables are independent, $\mathbb{E}(Z_i Z_j) = \mathbb{E}(Z_i)\,\mathbb{E}(Z_j)$, and so

$$\begin{aligned}
\operatorname{Cov}(X_n,\, X_{n+1}) &= 1/3\mathbb{E}\left(Z_n^2\right) - 1/3\mathbb{E}(Z_n)^2 \\
&= 1/3\left(\mathbb{E}\left(Z_n^2\right) - \mathbb{E}(Z_n)^2\right) \\
&= 1/3 \cdot \operatorname{Var}(Z_n) \\
&= 1/3
\end{aligned}$$

Therefore, the covariance (for all $n \geq 1$) is

$$\operatorname{Cov}(X_n,\, X_{n+k}) = \begin{cases} 10/9 & \text{if } k = 0 \\ 1/3 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

b. If $U$ and $V$ are uncorrelated, $\varrho = 0 \Rightarrow \operatorname{Cov}(U,\, V) = 0 \Rightarrow \mathbb{E}(UV) = \mathbb{E}(U)\,\mathbb{E}(V)$. Now, take the function $f$,

$$\begin{aligned}
f(c) &= \mathbb{E}\left((U - cV)^2\right) \\
&= \mathbb{E}\left(U^2 - 2cUV + c^2 V^2\right) \\
&= \mathbb{E}\left(U^2\right) - 2c\mathbb{E}(UV) + c^2\mathbb{E}\left(V^2\right) \\
&= \mathbb{E}\left(U^2\right) - 2c\mathbb{E}(U)\,\mathbb{E}(V) + c^2\mathbb{E}\left(V^2\right)
\end{aligned}$$

but since $U$ and $V$ have $\mu = \mathbb{E}(U) = \mathbb{E}(V) = 0$,

$$f(c) = \mathbb{E}\left(U^2\right) + c^2\mathbb{E}\left(V^2\right)$$

and since the function $f$ is strictly positive, both $\mathbb{E}\left(U^2\right)$ and $\mathbb{E}\left(V^2\right)$ are positive. Therefore, setting $c = 0$ minimises $f$, since

$$\mathbb{E}\left(U^2\right) \leq \mathbb{E}\left(U^2\right) + c^2\mathbb{E}\left(V^2\right)$$