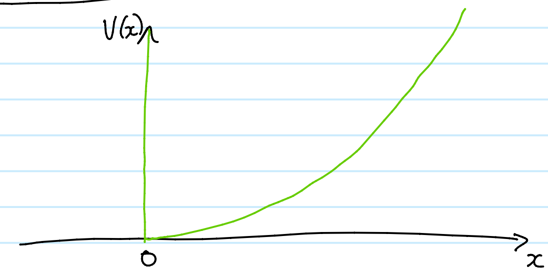
T05

Monday, 6 September 2021 4:23 PM

Problem 5.3:



The particle clearly can't have $x<0$, so the Schrodinger equation for $x \geq 0$ reads:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi$$

This is exactly Eq (2.45) in the textbook, with

$$\psi_0(x) = Ae^{-\frac{m\omega}{2\hbar}x^2}$$

Since the quadratic potential well is symmetric, the normalisation of the above equation can be multiplied by two (to account for the halved domain as a result of integration bounds changing from $(-\infty,\infty) \to [0,\infty)$).
Therefore, the normalisation is:

$$1 = 2|A|^2 \int_0^\infty e^{-m\omega x^2/2\hbar}\,dx = 2\cdot\frac{1}{2}|A|^2\sqrt{\frac{\pi\hbar}{m\omega}}$$

- Which is the same as the quadratic case
Hence, the energy and eigenfunctions will be that of the quadratic case, with

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega \quad \text{and} \quad \psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2} \quad (2.86)$$

where $H_n(\xi)$ are the Hermite polynomials.
However, there is a condition that the wavefunctions, ψ_n, must be 0 at $x=0$: $\psi(0)=0$.
Since $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$, $\xi(0)=0$. So then is the case that $H_n(\xi)=0$ at $x=0$, since no other term in eq 2.86 can be zero.
By inspection of Table 2.1 in the text, only the odd Hermite polynomials can be zero, satisfying the boundary condition. Thus, only $m=2n-1$ $(n\in\mathbb{N})$ multiples of the quadratic potential solutions are valid:

$$E_m = \left(m+\frac{1}{2}\right)\hbar\omega \qquad m=2n-1,\quad n\in\mathbb{N}$$

Problem 5.4:

$$\Psi(x,0) = A\left(\sqrt{2}\,\psi_0(x) + \psi_1(x)\right) = A\sqrt{2}\,\psi_0(x) + A\psi_1(x)$$

a. The above wave function is of the form of Eq 2.16 in the text, with $c_0 = A\sqrt{2}$ and $c_1 = A$

Take A positive, then by Eq 2.20:

$$\sum^1 |c_n|^2 = 1$$

$c_1 = A$

Take A positive, then by Eq 2.20:

$$\sum_{n=0}^{1} |c_n|^2 = 1$$

$$\Rightarrow \quad 2A^2 + A^2 = 1 \quad \Rightarrow A = \sqrt{1/3}$$

The wavefunction rewritten is then

$$\Psi(x,0) = \sqrt{\tfrac{2}{3}}\,\psi_0(x) + \sqrt{1/3}\,\psi_1(x)$$

Then, by equation 2.21, the expectation value of the total energy is

$$\langle H \rangle = \sum_{n=0}^{1} |c_n|^2 E_n$$

where $E_n = (n+\frac{1}{2})\hbar\omega$

$$\Rightarrow \langle H \rangle = |c_0|^2 E_0 + |c_1|^2 E_1$$

$$= \frac{2}{3} \times \frac{1}{2}\hbar\omega + \frac{1}{3} \times \frac{3}{2}\hbar\omega$$

$$= \frac{1}{3}\hbar\omega + \frac{1}{2}\hbar\omega = \frac{5}{6}\hbar\omega$$

But if the energy were measured, only one of E_0 or E_1 would be returned. By Eq 2.19 $|c_n|^2$ is the probability that a measurement of the energy will return E_n

Therefore,

State	Probability	Energy
0	$\lvert c_0\rvert^2 = 2/3$	$1/2\,\hbar\omega$
1	$\lvert c_1\rvert^2 = 1/3$	$3/2\,\hbar\omega$

b. According to Eq 2.17, an initial wavefunction of the form

$$\Psi(x,0) = \sum_{n=0}^{\infty} c_n \psi_n(x)$$

evolves with time as

$$\Psi(x,t) = \sum_{n=0}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

In terms of $c_0, c_1, \psi_0(x), \psi_1(x)$, $E_0 = 1/2\,\hbar\omega$ and $E_1 = 3/2\,\hbar\omega$;

$$\Psi(x,t) = \sqrt{\tfrac{2}{3}}\,\psi_0(x)e^{-i\omega t/2} + \sqrt{\tfrac{1}{3}}\,\psi_1(x)e^{-3i\omega t/2}$$

The probability density is given by $|\Psi(x,t)|^2$:

$$|\Psi(x,t)|^2 = \Psi^*\Psi$$

$$= \left(\sqrt{\tfrac{2}{3}}\,\psi_0(x)e^{-i\omega t/2} + \sqrt{\tfrac{1}{3}}\,\psi_1(x)e^{-3i\omega t/2}\right)\left(\sqrt{2/3}\,\psi_0(x)e^{i\omega t/2} + \sqrt{1/3}\,\psi_1(x)e^{3i\omega t/2}\right)$$

$$= \frac{2}{3}\psi_0^2 + \frac{\sqrt{2}}{3}\psi_0\psi_1 e^{i\omega t} + \frac{\sqrt{2}}{3}\psi_0\psi_1 e^{-i\omega t} + \frac{1}{3}\psi_1^2$$

$$= \frac{2}{3}\psi_0^2 + \frac{1}{3}\psi_1^2 + \frac{\sqrt{2}}{3}\psi_0\psi_1\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$= \frac{2}{3}\psi_0^2 + \frac{1}{3}\psi_1^2 + \frac{2\sqrt{2}}{3}\psi_0\psi_1\cos(\omega t)$$

Writing the wave functions as the stationary states for the harmonic oscillator:

$$\psi_0(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{1} \cdot e^{-\xi^2/2}$$

$$\Rightarrow \psi_0(\xi)^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\xi^2}$$

$$\Rightarrow \psi_0(\xi)^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\xi^2}$$

$$\psi_1(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2}}\, 2\xi e^{-\xi^2/2}$$

$$\Rightarrow \psi_1(\xi)^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} 2\xi^2 e^{-\xi^2}$$

$$\Rightarrow \psi_0\psi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{1}{\sqrt{2}}\, 2\xi e^{-\xi^2}$$

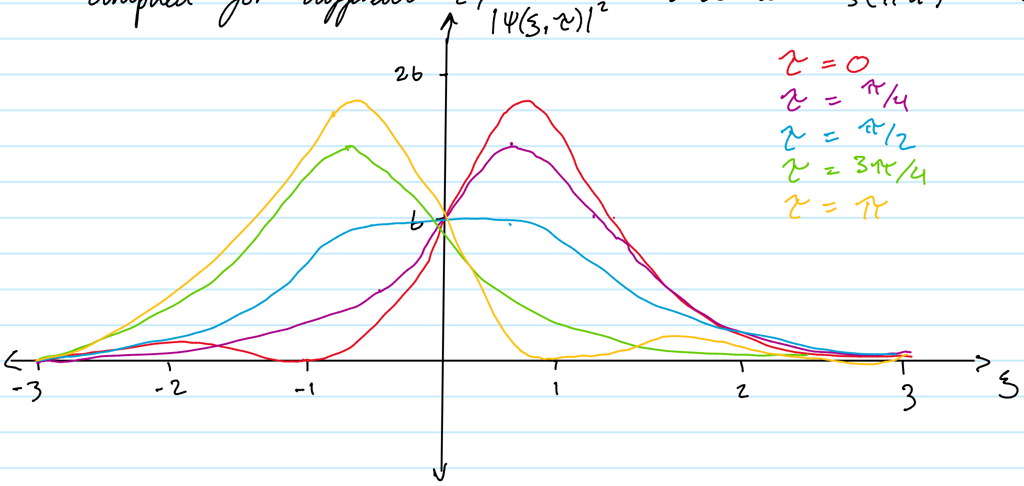
Therefore the probability density in terms of ξ is

$$|\Psi(\xi,t)|^2 = \frac{2}{3}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\xi^2}\left(1 + \xi^2 + \frac{\sqrt{2}}{\sqrt{2}}\, 2\xi\cos(\omega t)\right)$$

In terms of $\tau = \omega t$ and ξ,

$$|\Psi(\xi,\tau)|^2 = \frac{2}{3}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\xi^2}\left(1 + \xi^2 + 2\xi\cos(\tau)\right)$$

Graphed for different τ, and let $\frac{2}{3}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} = b$



c. The expectation value of x is

$$\langle x\rangle = \int_{-\infty}^{\infty} x\,|\Psi(x,t)|^2\,dx$$

The probability density $|\Psi(\xi,\tau)|^2$ can be expressed as $|\Psi(x,t)|^2$ from the transformation of variables:

$$\tau = \omega t \qquad \text{and} \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}\,x$$

$$\Rightarrow |\Psi(x,t)|^2 = \frac{2}{3}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-m\omega x^2/\hbar}\left(1 + \frac{m\omega}{\hbar}x + 2x\sqrt{\frac{m\omega}{\hbar}}\cos(\omega t)\right)$$

$$\Rightarrow \langle x\rangle = \frac{2}{3}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\int_{-\infty}^{\infty} x e^{-m\omega x^2/\hbar} + \frac{m\omega}{\hbar}\int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar} + 2\sqrt{\frac{m\omega}{\hbar}}\cos(\omega t)\int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar}\right)$$

From the list of exponential integrals from wikipedia,

$$\int_{-\infty}^{\infty} x e^{-a(x-b)^2}\,dx = b\sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}} \qquad \text{where } a = \frac{m\omega}{\hbar}$$

Using these in the expectation value of x above,

$$\langle x\rangle = \frac{2}{3}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(0 + \left(\frac{m\omega}{\hbar} + 2\sqrt{\frac{m\omega}{\hbar}}\cos(\omega t)\right)\frac{1}{2}\sqrt{\frac{\hbar^3\pi}{m^3\omega^3}}\right)$$

$$\langle x \rangle = \frac{\;}{3}\left(\frac{\;}{\pi\hbar}\right)\left(0 + \left(\frac{\;}{\hbar} + 2\sqrt{\frac{\;}{\hbar}}\cos(\omega t)\right)\frac{\;}{2}\sqrt{\frac{\;}{m^3\omega^3}}\right)$$

$$= \frac{1}{2}\cdot\frac{2}{3}\left(\frac{\hbar^2}{m^2\omega^2}\right)^{1/2}\left(\frac{m\omega}{\hbar} + 2\sqrt{\frac{m\omega}{\hbar}}\cos(\omega t)\right)$$

$$= \frac{1}{3}\frac{\hbar}{m\omega}\frac{m\omega}{\hbar} + \frac{2}{3}\frac{\hbar}{m\omega}\sqrt{\frac{m\omega}{\hbar}}\cos(\omega t)$$

Given that $\hbar$, m, and ω are positive,

$$\langle x \rangle = \frac{1}{3} + \frac{2}{3}\sqrt{\frac{\hbar}{m\omega}}\cos(\omega t)$$

Now to do the algebraic method:

$$\Psi(x,0) = \hat{a}_+\psi_0(x) + \psi_0(x)$$

$$\Rightarrow \Psi(x,t) = \hat{a}_+\psi_0(x)e^{-iE_1t/\hbar} + \psi_0(x)e^{-iE_0t/\hbar}$$

$$= \hat{a}_+\psi_0(x)e^{-3i\omega t/2} + \psi_0(x)e^{-i\omega t/2}$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega x)\psi_0(x)e^{-3i\omega t/2} + \psi_0(x)e^{-i\omega t/2}$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{\partial}{\partial x} + m\omega x\right)\psi_0(x)e^{-3i\omega t/2} + \psi_0(x)e^{-i\omega t/2}$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar e^{-3i\omega t/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega}{2\hbar}x^2}\frac{\partial}{\partial x} + m\omega\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}x e^{-\frac{m\omega}{2\hbar}x^2}e^{-3i\omega t/2}\right)$$
$$+ \left(\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega}{2\hbar}x^2}e^{-i\omega t/2}\right)$$

Let $\alpha = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$,

$$\Rightarrow \Psi(x,t) = \frac{e^{-3i\omega t/2}}{\sqrt{2\hbar m\omega}}\left(-\hbar\alpha\frac{m\omega}{\hbar}x e^{-\frac{m\omega}{2\hbar}x^2} + \alpha m\omega x e^{-\frac{m\omega}{2\hbar}x^2}\right)$$
$$+ \alpha e^{-\frac{m\omega}{2\hbar}x^2}e^{-i\omega t/2}$$

$$= \alpha e^{-\frac{m\omega}{2\hbar}x^2}e^{-i\omega t/2}$$

$$\Rightarrow |\Psi(x,t)|^2 = \Psi^*\Psi$$

$$= \alpha^2 e^{-\frac{m\omega}{\hbar}x^2}$$

$$\Rightarrow \langle x \rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx$$

$$= \alpha^2\int_{-\infty}^{\infty} x e^{-\frac{m\omega}{\hbar}x^2}\,dx$$

$$= 0$$

This is clearly wrong, but I don't have the time to work out where I went wrong.

d. I'm not going to pretend that I know what to do here. I would say that the results would be the same as in part a), as the system is always a superposition of the first two eigenfunctions?