

PHYS3020 Lab Report 1

Boltzmann Factor

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Abstract

Herein we derived an experimentally-derived value for Boltzmann's constant using various statistical methods on a set of obtained data: $k_B = (1.38 \pm 0.03) \times 10^{-23} \text{ J.K}^{-1}$. This value was found to be on the order of only $\sim 0.3\%$ out of agreement with the officially defined value, and was found by an average of values obtained for 5 different temperature regimes. Each of those values were found from a weighted linear regression on a plot of observed current vs externally applied voltage for a doped transistor within a thermal bath. In the process, the band gap value of the transistor silicon was estimated to be $\phi = 0.991 \pm 0.004 \text{ eV}$, and some inferences were made as to the saturation voltage of this particular setup for which higher voltages no longer yielded correspondingly higher observed currents. Of particular note, it was found that, at least in the temperature regimes observed, the saturation voltage had a linear dependence on temperature. Some points of error were identified and recommendations made for any future investigations using a similar methodology.

1 Introduction

The Boltzmann constant is a fundamental physical constant which is used in practically all fields of physics, from astrophysics to quantum mechanics and electronics. Its significance lies in relating the kinetic properties of particles to their thermal energies [3]. Due to its widespread use and impact on all manner of calculations, knowing the value of the constant to high precision is essential to minimise uncertainty in all other experimental physics. As it stands, the value for Boltzmann's constant is now fixed at $k_B = 1.380649 \times 10^{-23} \text{ J/K}$ [2], with no associated uncertainty. This experiment aims to derive this value, within uncertainty, using well-tested approximations for electron energy distributions in the context of an electrical circuit with a doped component.

the transistor) is a direct consequence of the associated electron energies of the base region. Since we'd expect that no current would flow when no external voltage is applied, the diffusion current must be equal to the generation current when V_{BE} (the voltage difference between base and emitter) is 0 and so the diffusion current is expressed as

$$I_{\text{diff}} = \exp\left(\frac{Vq_e}{k_B T}\right) \int_0^\infty j_0 D(\varepsilon) \exp\left(-\frac{\varepsilon + \phi}{k_B T}\right) d\varepsilon \quad (2)$$

where the two exponential terms are modelled by Boltzmann distributions, and the integral term is equal to I_{gen} .

The Boltzmann distribution of electron energies serves as a useful approximation to the Fermi-Dirac distribution,

$$\mathcal{P}(\varepsilon) \propto \frac{1}{1 + \exp\left(\frac{E - \mu}{k_B T}\right)} \quad \text{Fermi-Dirac} \quad (3)$$

in both the low temperature or high energy limits where the exponential term in the denominator dominates over the 1, i.e. $(E - \mu)/k_B T \gg 1$ [3]. Neglecting the 1 and approximating this reduces the Fermi-Dirac formula to the Boltzmann distribution,

$$\mathcal{P}(\varepsilon) \propto \exp\left(\frac{\mu - E}{k_B T}\right) \quad \text{Boltzmann} \quad (4)$$

The total current detected (collected) by an ammeter in a circuit with a voltage source and doped transistor is equal to the current due to diffusion (about the base region) minus the generation current due to thermal excitation;

$$I_c = I_{\text{diff}} - I_{\text{gen}} \quad (1)$$

As such, the generation current I_{gen} is proportional to energy distribution and thermal qualities of the conduction band as given in equation (1) in the laboratory notes.

The magnitude of the diffusion current (due to a potential difference between the base and emitter regions of

for which all of the calculations in the report will be done with. Since temperatures close to room temperature (\sim

293K) are in this low-temperature regime (compared to the interiors of stars, for example), this approximation is suitable.

Since the diffusion current can safely be modelled by a Boltzmann distribution in the low-temperature regime, equation (2) is approximately true. Because of this, we can express the collector current I_c (from equation (1) as

$$I_c = j_0 \left(\exp \left(\frac{V_{q_e}}{k_B T} \right) - 1 \right) \int_0^\infty D(\varepsilon) \exp \left(-\frac{\varepsilon + \phi}{k_B T} \right) d\varepsilon \quad (5)$$

In the low (read: roughly room temperature) temperature regime, we can ignore the -1 term in the above equation and be subject to some minor error. We can do this in the same conditions as that postulated in the Boltzmann distribution justification, i.e. when $V_{q_e}/k_B T \gg 1$. At room temperature (293K), we would expect about a 10-15% error from this alone, which increases to $\sim 22\%$ error at 373K. Hence, for sufficiently low temperatures we can work with the drawbacks.

We have that $D(\varepsilon) \propto \varepsilon^{1/2}$ in our three dimensional case of current flow. For the sake of derivation, set $D(\varepsilon) = a\varepsilon^{1/2} = x$ for some constant of proportionality a . With the well defined property of exponential integrals,

$$\int_0^\infty x^n e^{-bx} dx = \frac{n!}{b^{n+1}} \quad \text{for } b > 0 \quad (6)$$

we can set $n = 1/2 \Rightarrow 2n + 1 = 3/2$ and $b = (k_B T)^{-1}$, and so

$$\int_0^\infty D(\varepsilon) \exp \left(-\frac{\varepsilon}{k_B T} \right) = a k_B^{3/2} T^{3/2} (1/2)! \propto T^{3/2} \quad (7)$$

This shows that the diffusion current is directly proportional to $T^{3/2}$ up to a constant. As such, the collector current can be expressed as

$$I_c = \sigma T^{3/2} \exp \left(\frac{V_{BE} q_e}{k_B T} \right) \exp \left(-\frac{\phi}{k_B T} \right) \quad (8)$$

where σ is composed of all of the previous constants (multiplied together). Taking the natural logarithm of this gives

$$\ln(I_c) = \ln(\sigma) + \frac{V_{BE} q_e}{k_B T} - \frac{\phi}{k_B T} + \ln(T^{3/2}) \quad (9)$$

which is a suitable expression to be able to determine Boltzmann's constant. In order to do this, measuring currents at multiple Band-Emitter voltage differences (V_{BE}) should present a linear trend, since the equation

$$\ln(I_c) = \frac{q_e}{k_B T} V_{BE} + \ln(I_0) \quad (10)$$

is of the form of $y = mx + c$, where the gradient m and y -intercept (in this case where the voltage is equal to 0) are given by

$$m_{\ln(I_c)V} = \frac{q_e}{k_B T} \quad (11)$$

$$c = \ln(I_0) = \ln(\sigma) - \frac{\phi}{k_B T} + \ln(T^{3/2}) \quad (12)$$

At sufficiently high voltage (dependent on temperature), we expect that the collector current will have some maximum defined as the saturation current I_s . As per the laboratory notes, this current is expected to be reached when

$$I_s = \alpha T^{3/2} \exp \left(\frac{\varepsilon - \phi}{k_B T} \right) \quad (13)$$

where α is some proportionality constant. Setting $I_s = I_c$ allows us to derive a formula to determine the voltage at which the current should become saturated for a given temperature:

$$\alpha T^{3/2} \exp \left(\frac{\varepsilon - \phi}{k_B T} \right) = \sigma T^{3/2} \exp \left(\frac{V_{BE} q_e}{k_B T} \right) \exp \left(-\frac{\phi}{k_B T} \right) \quad (14)$$

$$\alpha \exp \left(\frac{\varepsilon}{k_B T} \right) = \sigma \exp \left(\frac{V_{BE} q_e}{k_B T} \right) \quad (15)$$

Taking the natural logarithm of both sides yields

$$\ln a + \frac{\varepsilon}{k_B T} = \ln \sigma + \frac{V_{BE} q_e}{k_B T} \quad (16)$$

$$\Rightarrow V_{BE} = \frac{k_B c}{q_e} T + \frac{\varepsilon}{q_e} \quad (17)$$

where $c = \ln a - \ln \sigma$. Therefore, we'd expect that the applied voltage for which the current becomes saturated has a linear dependence on temperature.

3 Experiment

To measure the collector current across different regimes, the transistor was immersed in different temperature environments. The basic setup for the experiment is shown in Figure 1, which shows the circuit diagram of the apparatus.

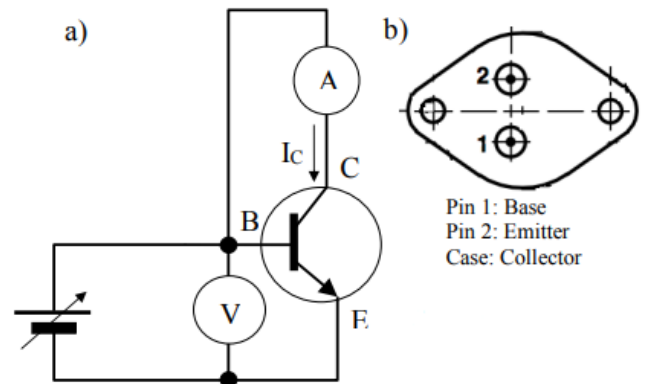


Figure. 1 a) The circuit diagram for the experimental setup, where the circled region is represented by Figure b). b) Diagram of the transistor connections.

Source: [1]

3.1 Method

To begin with, we measured the collector currents in a room temperature environment. The transistor was connected to the ammeter, voltmeter and power supply as in Figure 1, and the transistor was situated within a glass beaker (without being in contact with the glass walls). Using a digital thermometer with 0.01K precision, the ambient temperature was measured before any data was collected. We checked this temperature again periodically throughout the experiment (in roughly 3 minute intervals) to make sure that the temperature remained consistent within uncertainty.

Once the temperature was recorded, we began taking measurements. Beginning from 0 applied volts on the power supply, we raised the voltage by 0.05V increments slowly until a current was detected on the ammeter (with 0.01 μ A precision). The last voltage to not read a current was recorded (using the voltmeter with 0.001V precision) with the associated 0A observation from the ammeter. After this, we incremented the power supply by 0.05V and recorded the corresponding current. We did this repeatedly until the recorded current approached 10⁶ μ A, as the power supply used had a limit of 1A and we did not want to risk overloading the system.

An analogous process to the above was performed for a range of other temperatures. After the room temperature observations were made, we filled a different 500mL beaker up to approximately the 150mL level with dry ice (that had been kept in an insulated container prior). Once this was done, we poured isopropyl alcohol into the beaker until it reached approximately the 300mL level (where the transistor could feasibly be submerged in the cool alcohol bath). The transistor was then submerged in the alcohol bath and left for about 60 seconds to stabilise in temperature. While we were waiting, we put the digital thermometer prod into the bath to monitor the temperature. After the \sim 60 seconds had elapsed and the temperature appeared reasonably stable, we began measurements in the same way as in the second paragraph of the method (previous paragraph).

While two of the three group members were taking measurements with the cold alcohol bath, the third group member filled a different 500mL beaker up to approximately the 300mL level with rice bran oil, and placed it into the slot on a heating mantle. An external thermometer (compatible with the heating mantle) was plugged into the back of the apparatus, and the thermometer probe stuck into the oil while being held above the heating mantle with a clamp attached to a retort stand. Initially the heating mantle was set to heat up to 60 degrees Celsius. Conveniently, the mantle reached this temperature just as the group finished taking measurements with the cold bath, and so the

transistor was then submerged into the oil bath and the process in paragraph 2 was repeated again. The temperature was more stable for the oil bath than the cold alcohol bath, as the oil was constantly heated by a small amount to remain a constant temperature.

After measurements were taken at 60 degrees Celsius, the mantle was set to heat to 90 degrees Celsius. When it reached this, and stopped fluctuating in temperature, the measurements were taken again. Finally, this process was repeated for a 101 degree Celsius data run.

4 Results

Almost all of the data collected is given in Table 2 in the Appendix. Another run of data was collected at approximately 241K (as an initial value), but was omitted as the temperature varied significantly over the course of data collection which would have affected the accuracy of subsequent analysis.

A log plot of all included current readings against the associated voltage is shown in Figure 2.

In order to maximise the accuracy of the obtained linear regression parameters, a successive-maximisation algorithm was implemented in Python using the `numpy` package. Starting from the lowest voltage value, the algorithm would evaluate the R^2 value of a trendline connecting that and the next data point. It was programmed to iterate over all data points for a given temperature and to stop when one R^2 of the trendline was lower than the previous value. This method was implemented to minimise the error of including data points approximating the saturation current (where the linear trend in Figure 2 appears to break into a curve at high voltage/current), while also maximising the accuracy of the overall trendline parameters. The overall trendline parameters are given in Table 1, where the fits were consistently characterised by $R^2 > 0.99$.

Temperature (K)	Gradient (V ⁻¹)	Intercept (ln μ A)	Boltzmann $\times 10^{-25}$ (J.K ⁻¹)
221 \pm 3	49.3 \pm 1.4	-24.6 \pm 0.1	147 \pm 5
293 \pm 1.5	39.2 \pm 1.4	-12.2 \pm 0.1	139 \pm 5
333 \pm 1.5	35.0 \pm 1.4	-6.71 \pm 0.16	137 \pm 6
363 \pm 2	33.2 \pm 1.4	-3.84 \pm 0.20	133 \pm 6
374 \pm 2	31.5 \pm 1.4	-2.45 \pm 0.25	136 \pm 6

Table 1 Linear regression results and associated calculated Boltzmann constant. The value of the Boltzmann constant was calculated according to the right side formula in equation (18), where the uncertainties were propagated from the temperature and gradient uncertainties according to propagation relations.

To determine the value of Boltzmann's constant from

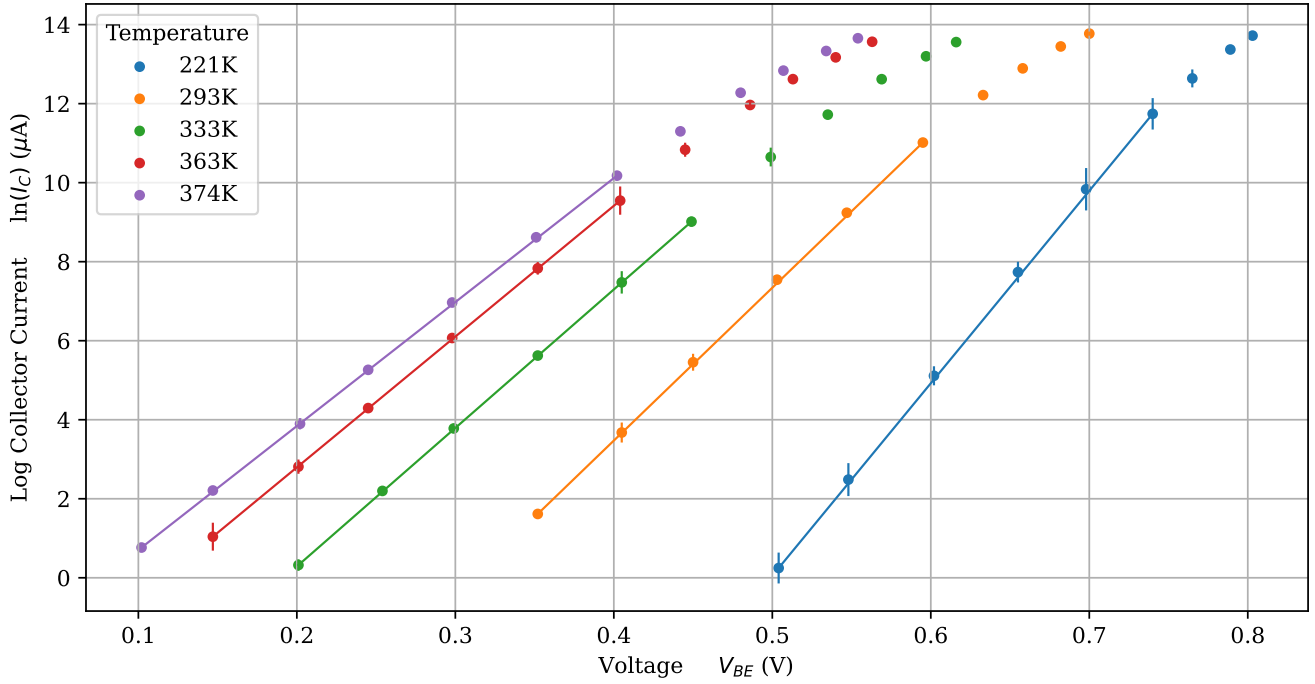


Figure. 2 Log collector current as a function of voltage. Error bars for the log current are present on all points, but are too small to be resolved for most. As the legend shows, colour is separated based on the temperature of the current-voltage readings. The trendlines were formulated according to the description in the Results section, where the included data points were chosen to maximise accuracy of the trend parameters.

this data alone, we note that gradient and associated temperature of each trendline can be used according to equation (11). We have that the gradient has units V^{-1} , or equivalently $C.J^{-1}$. With the relation

$$m_{\ln(I_c)V} = \frac{q_e}{k_B T} \Rightarrow k_B = \frac{q_e}{m_{\ln(I_c)V} T} \quad (18)$$

we obtain the desired units of Boltzmann's constant, namely $J.K^{-1}$. Averaging all of the values for Boltzmann's constant in Table 1 yields a final value of

$$k_B = (1.38 \pm 0.03) \times 10^{-23} J.K^{-1} \quad (19)$$

which is in excellent agreement with the *defined* value of $k_B = 1.380649 \times 10^{-23} J.K^{-1}$ [2] (in fact, when not truncating the digits to the appropriate significant figures, the error comes out to be only 0.24%).

As a next objective for this analysis, the band gap in the silicon of the transistor could be estimated. Using the relation given in equation (12), plotting the y -intercepts of the previous trendlines (minus the log 3 halves power of temperature) against the inverse of temperature should yield a new trendline with a gradient of $-\phi/k_B$:

$$\ln(I_0) - \ln(T^{3/2}) = \ln \sigma - \frac{\phi}{k_B} \left(\frac{1}{T} \right) \quad (20)$$

Noting that $\ln(I_0) - \ln(T^{3/2}) = \ln(I_0 T^{-3/2})$, the data was plotted is shown in Figure 3.

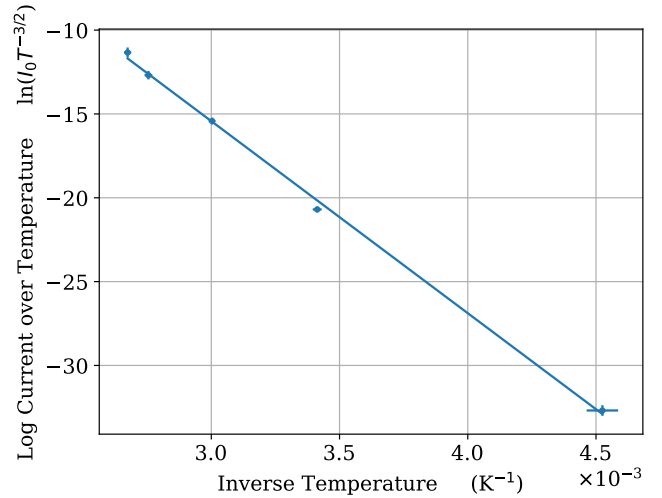


Figure. 3 The plot of $\log I_0 T^{-3/2}$ vs inverse temperature yields a gradient that can be used to calculate the band gap energy. Error bars associated with both y -values and x -values are plotted, but are usually too small to be resolved against the scale of the figure. The trendline was determined via a weighted least-squares linear regression.

The gradient of this trendline is then

$$-\frac{\phi}{k_B} \simeq (-1.15 \pm 0.01) \times 10^4 \text{ K} \quad (21)$$

Using the value of Boltzmann's constant experimentally obtained previously, we obtain

$$\phi = 0.991 \pm 0.004 \text{ eV} \quad (22)$$

where the value in joules was converted to electron volts. This is reasonably out of agreement with the quoted silicon transistor band gap of $\sim 1.11 \text{ eV}$ as quoted in the literature [1], but is close considering it as a secondary objective of the report.

It's reasonably clear to see in Figure 2 that at high voltages and currents, the trends are no longer linear. As explained in the Theory section, this is roughly where the current becomes saturated and the relationship between saturation voltage and temperature was claimed to be linear. The saturation voltages were taken to be the x -value of the first data point that significantly deviated from the linear trend in Figure 2. The plot of these points against their associated temperature is shown in Figure 4.

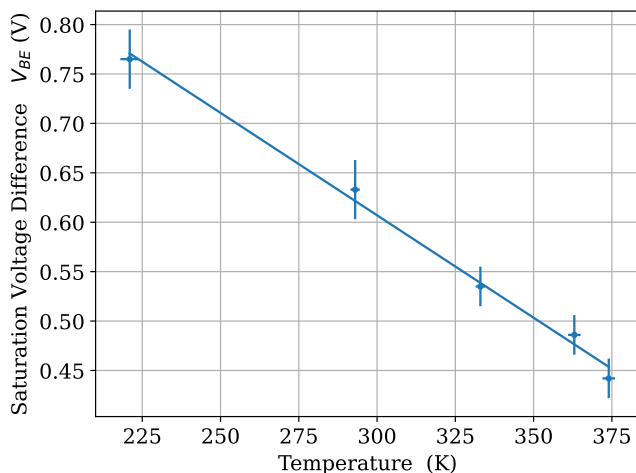


Figure. 4 A plot of the estimated saturation voltage against their associated temperature can be used to test the linear relationship postulated in equation (17). Error bars are present for both x and y values, where the uncertainty in saturation voltage arose from doubts about where exactly the collector current became saturated. As before, the trendline was formulated via a weighted least-squares linear regression.

We claim that a linear trend is clearly in agreement with the data, and so equation (17) is valid at least in the scope of this paper.

5 Discussion

Significant shortfalls were present in the experiment, particularly when testing with cold temperatures. A large source of uncertainty in the measurements was as a direct result of the isopropyl alcohol bath slowly heating up to room temperature in the absence of constant cooling. This systematically affected the accuracy of measurements taken, and for future analyses we recommend performing these measurements with the bath in an insulated (or constantly cooled) environment. Similarly in the hot oil baths, the heating mantle would occasionally heat the oil past the set temperature which impacted the results in a similar way. We recommend performing high temperature measurements with a larger thermal sink, or in a more homogeneously heated environment (such as in an oven, for example).

On the note of error source, the uncertainty in temperature was taken to be range for which the observed temperature values fluctuated across the period of data collection, plus 1 to be conservative. The uncertainty in current was taken in much the same way, with the uncertainty being half of the difference between the maximum and minimum observed values across reading fluctuations.

No fluctuation was observed in the independent voltmeter connected to the circuit, and so uncertainty was taken as 4 times the magnitude of the smallest significant figure to be conservative (i.e. $\Delta V_{BE} = \pm 0.004 \text{ V}$).

The experimentally derived value of Boltzmann's constant aligning perfectly with the value in the literature is an excellent result. Although the methodology of this experiment is just one way of finding Boltzmann's constant, we're excited that a value from an undergraduate lab report could be substituted into any manner of applicable equations to yield an accurate result.

6 Conclusions

Using a doped transistor submerged in a thermal bath, we experimentally derived a value for Boltzmann's constant ($k_B = (1.38 \pm 0.03) \times 10^{-23} \text{ J.K}^{-1}$) that aligns perfectly, within uncertainty, to the defined value of $k_B = 1.380649 \times 10^{-23} \text{ J.K}^{-1}$ [2]. This was calculated using the results of a series of weighted linear regressions, using an original successive-maximisation algorithm to determine the most accurate fit parameters. In the process, the silicon band gap was estimated at $\phi = 0.991 \pm 0.004 \text{ eV}$ which was not in agreement with the literature [1].

On top of this, a linear trend between saturation current and temperature was postulated, which was supported from the obtained data within uncertainty. Finally, We suggest potential improvements for future experiments.

References

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7 Appendices

7.1 Collected Data

Table 2 Collected data across the methodology described in section 3. The “Set V_{BE} ” column refers to the manually set voltage on the power supply, while the “Real V_{BE} ” column refers to the voltage observed on a much more accurate, independent voltmeter.

Temperature (K)	ΔT (K)	Real V_{BE} (V)	Set V_{BE} (V)	I_c (μA)	ΔI_c (μA)
221	3	0.45	0.45	0	1
221	3	0.504	0.5	1.28	0.05
221	3	0.548	0.55	12	0.5
221	3	0.602	0.6	166	4
221	3	0.655	0.65	2290	60
221	3	0.698	0.7	18650	1000
221	3	0.74	0.75	125800	5000
221	3	0.765	0.8	308300	7000
221	3	0.789	0.85	640800	7000
221	3	0.803	0.9	910300	9000
293	1.5	0.302	0.3	0	1
293	1.5	0.352	0.35	5.03	0.03
293	1.5	0.405	0.4	39.5	1
293	1.5	0.45	0.45	234	5
293	1.5	0.503	0.5	1890	15
293	1.5	0.547	0.55	10290	80
293	1.5	0.595	0.6	60800	400
293	1.5	0.633	0.65	202000	800
293	1.5	0.658	0.7	397500	2000
293	1.5	0.682	0.75	692000	4000
293	1.5	0.7	0.8	958000	5000
333	1.5	0.147	0.15	0	1
333	1.5	0.201	0.2	1.38	0.02
333	1.5	0.254	0.25	9.01	0.05
333	1.5	0.299	0.3	43.8	0.3
333	1.5	0.352	0.35	277	2
333	1.5	0.405	0.4	1766	50
333	1.5	0.449	0.45	8220	100
333	1.5	0.499	0.5	42150	1000
333	1.5	0.535	0.55	123200	800
333	1.5	0.569	0.6	302300	1000
333	1.5	0.597	0.65	539600	1000
333	1.5	0.616	0.7	772900	1000
363	2	0.102	0.1	0	1
363	2	0.147	0.15	2.83	0.1
363	2	0.201	0.2	16.655	0.3
363	2	0.245	0.25	73.3	0.8
363	2	0.298	0.3	432	5
363	2	0.352	0.35	2520	40
363	2	0.404	0.4	13990	500
363	2	0.445	0.45	50700	900
363	2	0.486	0.5	157300	1500
363	2	0.513	0.55	302600	2000
363	2	0.54	0.6	525300	2500
363	2	0.563	0.65	779300	3000
374	2	0.102	0.05	0	1
374	2	0.102	0.1	2.15	0.01
374	2	0.147	0.15	9.11	0.05
374	2	0.202	0.2	49.3	0.7
374	2	0.245	0.25	193	2
374	2	0.298	0.3	1061	5
374	2	0.351	0.35	5530	70
374	2	0.402	0.4	26300	150
374	2	0.442	0.45	80752	800
374	2	0.48	0.5	214500	1000
374	2	0.507	0.55	375500	1500
374	2	0.534	0.6	617000	2000
374	2	0.554	0.65	852000	4000