PHYS2055 Assignment 3

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Part A

Problem 3.1

Imagine you are standing on a rock, fishing in the ocean. You observe two travelling water waves coming towards you. Consider these two waves as harmonic plane waves originating from two sources, S_1 and S_2, in phase with each other (see Figure 1). They are defined by their speed v, wavelength λ, amplitude W_0. The distances between you and the two sources are d_1 and d_2, with a angle θ between the two as shown in Figure 1.

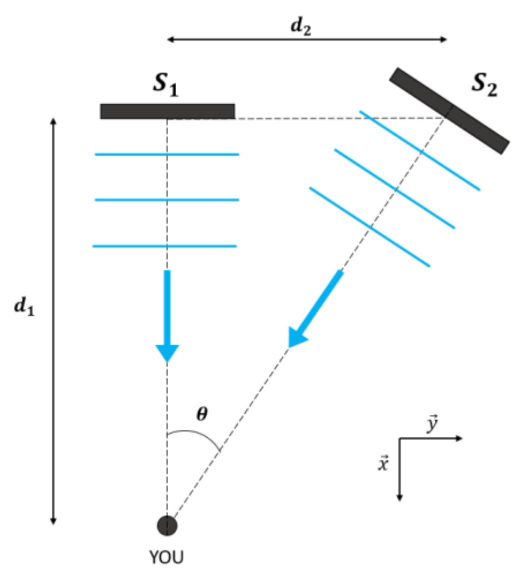

Figure 1: Total Impulse over Time

a. Write an expression for the travelling waves $\vec{W}_1$ and $\vec{W}_2$ in terms of x, y, t, v, λ, θ and W_0.

   Any two dimensional harmonic plane wave may be written in the form

$$\psi(x, y, t) = A\cos\left(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi\right) \tag{1}$$

   where $\mathbf{k} = (k_x, k_y)$ and $\mathbf{r} = (x, y)$. Considering that the plane waves begin with peaks at their sources, ϕ can be taken to be 0. As defined in the question, the amplitude of each wave is W_0. Also of note is that $\omega = 2\pi f$, where $f = \frac{v}{\lambda}$, so $\omega = \frac{2\pi v}{\lambda}$. So, equation (1) may be rewritten as

$$\psi(x, y, t) = W_0 \cos\left((k_x, k_y) \cdot (x, y) - \frac{2\pi v}{\lambda} t\right) \tag{2}$$

The wave number may be represented as $k = \frac{2\pi}{\lambda}$, and the x and y components of the wave vector $\mathbf{k}$ are found to be

$$k_x = \frac{2\pi}{\lambda} \cos\theta \tag{3}$$

$$k_y = \frac{2\pi}{\lambda} \sin\theta \tag{4}$$

Substituting equations (3) and (4) into equation (2), and solving the dot product gives

$$\psi(x, y, t) = W_0 \cos\left(x\frac{2\pi}{\lambda} \cos\theta + y\frac{2\pi}{\lambda} \sin\theta - \frac{2\pi v}{\lambda} t\right)$$

which simplifies down to

$$\psi(x, y, t) = W_0 \cos\left(\frac{2\pi}{\lambda} (x \cos\theta + y \sin\theta - vt)\right) \tag{5}$$

Equation (5) is suitable for $\vec{W}_2$, however $\vec{W}_1$ may be simplified further as it is travelling in the x direction only. Thus, $\vec{W}_1(x, t) = W_0 \cos\left(\frac{2\pi}{\lambda} (x - vt)\right)$

b. Find the expression of the phase difference at your location, $\Delta\phi$, of $\vec{W}_2$ relative to $\vec{W}_1$, in terms of the parameters given.

The phase difference between the two waves may be given by the difference in full cycles that the waves experience along their path to the point of observation. This may be found by the path difference, multiplied by the wave number (since the two waves are identical, no other variables need be considered). That is,

$$\Delta\phi = \frac{2\pi \Delta d}{\lambda}$$

which may be represented in terms of parameters given, where the distance from S_2 to the point of observation is $\sqrt{{d_1}^2 + {d_2}^2}$, as

$$\Delta\phi = \frac{2\pi\left(\sqrt{{d_1}^2 + {d_2}^2} - d_1\right)}{\lambda} \tag{6}$$

c. Calculate $\Delta\phi$ as a multiple of 2π and conclude on how the waves interfere. A drawing can help. Parameter values are wavelength $\lambda = 2.00 \times 10^1$m, amplitude $W_0 = 5.00 \times 10^{-1}$m, distances $d_1 = 1.30 \times 10^3$m and $d_2 = 9.75 \times 10^2$m.

Substituting the values given into equation (6) gives

$$\begin{aligned}\Delta\phi &= \frac{2\pi\left(\sqrt{1300^2 + 975^2} - 1300\right)}{20} \\ &= 16.25 \times 2\pi \\ &= \frac{\pi}{2}\end{aligned}$$

Since constructive interference occurs when the phase difference is an even multiple of 2π, and destructive interference occurs when the phase difference is an odd multiple of 2π, this interference is neither constructive nor destructive.

d. The rock you are standing on is just at sea level. Calculate the wave amplitude at your location and conclude on your action: stay fishing on the rock, retreat or find a better spot. Justify your answer. *Hint:* $\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$

The amplitude of the wave at the rock may be found by the superposition of the two propagating waves,

$$W = \vec{W}_1 + \vec{W}_2 = W_0 \cos\left(\frac{2\pi}{\lambda}(x - vt)\right) + W_0 \cos\left(\frac{2\pi}{\lambda}(x\cos\theta + y\sin\theta - vt)\right)$$

Using the trigonometric identity given, this expands to

$$W_0\left(2\cos\frac{\frac{2\pi}{\lambda}(x-vt) + \frac{2\pi}{\lambda}(x\cos\theta + y\sin\theta - vt)}{2}\cos\frac{\frac{2\pi}{\lambda}(x-vt) - \frac{2\pi}{\lambda}(x\cos\theta + y\sin\theta - vt)}{2}\right) \quad (7)$$

Since the distance between the observation point and S_2 is larger than the distance between the point and S_1, $\vec{W}_2$ will arrive at the point later than $\vec{W}_1$ assuming that their speeds are equal. Also assuming that $\vec{W}_1$ persists at the point during the time difference between the waves, the term vt may be shown in the form $v\frac{\sqrt{{d_1}^2+{d_2}^2}}{v} = \sqrt{{d_1}^2+{d_2}^2}$. Substituting this, and all other variables into equation (7) yields

$$\begin{aligned} W &= W_0\left(2\cos\frac{\frac{2\pi}{\lambda}\left(d_1 + d_1\cos\theta + d_2\sin\theta - 2\sqrt{{d_1}^2+{d_2}^2}\right)}{2}\cos\frac{\frac{2\pi}{\lambda}(d_1 - d_1\cos\theta - d_2\sin\theta)}{2}\right) \\ &= W_0\left(2\cos\frac{\pi\left(d_1 + d_1\cos\theta + d_2\sin\theta - 2\sqrt{{d_1}^2+{d_2}^2}\right)}{\lambda}\cos\frac{\pi(d_1 - d_1\cos\theta - d_2\sin\theta)}{\lambda}\right) \quad (8) \end{aligned}$$

where θ in equation (8) may be represented by $\theta = \arctan\left(\frac{d_2}{d_1}\right)$. Substituting this, and all values given in part c., into equation (8) gives

$$\begin{aligned} W &= \frac{1}{2}\left(2\cos\frac{-325\pi}{20}\cos\frac{-325\pi}{20}\right) \\ &= \cos^2\left(\frac{-65\pi}{4}\right) \\ &= 5\times10^{-1}\text{m} \end{aligned}$$

This means that the superposition of both waves, at least at the time at which $\vec{w}_2$ arrives the the observation point, totals to a wave height of 50cm. This is not the highest at which the waves may get, however, assuming that each of the two waves persist. Given that the waves will swell even higher than this 50cm, the fisherman should ideally find a better spot at which the waves destructively interfere. Given that the wavelength is 20m, the fisherman should not have to walk very far to find such a point.

Problem 3.2

a. Consider a travelling wave given by the function $\psi(y,t)$. At time $t=0$ the shape of the wave is:

$$\psi(y,0) = ae^{-b(y-c)^2}$$

where a, b and c are positive constants.

i. Write the expression of the corresponding travelling wave, having a speed v in the positive y direction.

For a wave with a velocity v in the positive y direction, the wave function will have inside of it the function $y - vt$. Since the travelling wave given in the criteria was described at $t = 0$, the term vt would be 0. So, one can add the $-vt$ term into the equation directly after the y variable to give the equation as

$$\psi(y,t) = ae^{-b(y-vt-c)^2}$$

The equation may be broken up to simplify constants. It may be seen that

$$\psi(y,t) = Ae^{-(y-vt-c)^2}$$

where $A = ae^b$.

ii. Sketch by hand the profile of this wave at $t = 0$s and $t = 1$s if $v = 2$m/s. Assume $c = 2$m.

At $t = 0$, the wave equation will resemble

$$\psi(y,t) = Ae^{-(y-2)^2}$$

and at $t = 1$ and $v = 2$, the wave equation will resemble

$$\psi(y,t) = Ae^{-(y-4)^2}$$

These two points in time are shown graphically on Figure 2:

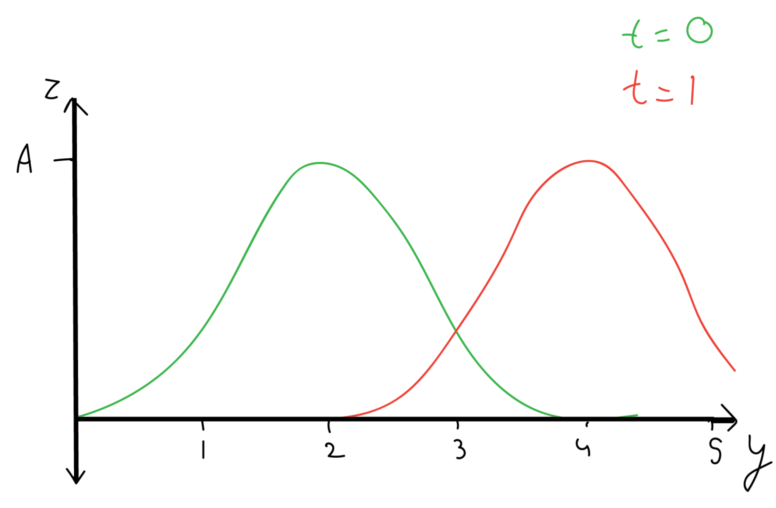

Figure 2: Comparison of Travelling Wave at Different Points in Time

b. Determine if the following functions describe a travelling wave:

i. $\psi(x,t) = (ax^2 + b)(ct^2 + d)$, where a, b, c and d are positive constants:

Since the function is not able to be split into a function of the form $g(x \pm vt)$, where v is some constant velocity, ψ is not a travelling wave.

ii. $\psi(y,t) = Ay^2 - Bt^2 + C$, where A, B and C are positive constants:

Factoring this wave equation yields

$$\psi(y,t) = \left(\sqrt{A}y + \sqrt{B}t\right)\left(\sqrt{A}y - \sqrt{B}t\right) + C$$

This is not of the form of a travelling wave, as a travelling wave needs to be of the form of either one, or the superposition of non-equally opposite functions of the form $g(y \pm vt)$. This is a multiplication of the two, and does not result in a travelling wave.

Part B - Advanced

Problem 3.3

In a galaxy far far away, the Empire fleet has built a massively strong linearly polarized laser beam to destroy one of the Republic's battle ships. Luckily, the Republic ship has the technology to create a shield to protect it, using either a polariser, half wave plate or quarter wave plate.

a. Not knowing the direction of polarization of the Empire's laser beam, answer the following questions:

   i. What can you say about its polarisation on the Poincare sphere? Justify your answer.

      Since the laser beam is quoted as being linearly polarised, the beam has no elliptical/circular polarisation, so it is solely horizontally/vertically polarised or diagonally polarised. Linearly polarised light only oscillates in one plane, and the plane of oscillation will determine which linear polarisation it is. Worth noting is that linearly polarised light (propagating in the z direction) may have electric field components in both the x and y direction, although they must be in phase.

   ii. Which shield is the best choice for Republic ship? Justify your answer.

      Since the direction of polarisation is not known, completely reducing the intensity of the laser beam isn't feasible (or rather, likely). Instead, it's be worth reducing the amplitude of the electric field of the beam to some more manageable magnitude. To do this, a polariser placed at any arbitrary angle would be the best option. Since the direction of polarisation is not known, there's no way to deduce which angle the polariser should be oriented at, although it is likely that there will be some reduction in intensity according to Malus' Law.

b. When the Empire discovers the Republic defence strategy, it places a quarter wave plate at the output of their laser beam.

   i. What now is its polarization state on the Poincare sphere? Justify your answer.

      Since a quarter wave plate has been placed on the output of the beam, there is now an additional phase difference of $\frac{\pi}{2}$ between the x and y components (in no particular order) of the beam's electric field. This means that the polarisation state has been shifted towards either right-circular or left-circular (depending on whether the induced phase difference was $\pm\frac{\pi}{2}$, and the initial state of polarisation).

   ii. Which shield is the best choice for Republic ship? Would a combination of two of those three shields help? Justify your answer.

      The best choice for the Republic ship's shield remains the polariser, as the wave plates only change the polarisation state of the incident beam with no loss of intensity. A combination of a quarter wave plate and a linear polariser could help even moreso, specifically if the exact state of polarisation of the incident beam was known. If the quarter wave plate were to be placed before the polariser, with the polariser at an angle of 45° to the incident light beam, the intensity would be greatly reduced.