PHYS2100 Lecture Notes

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Contents

1 Part 1

1.1 Inertial Reference Frames

- A reference frame is an idealized arrangement of synchronized clocks and position markers covering all space that forms a mathematical coordinate system. Note that the origin of the reference frame, at least at t=0, is typically at the location of a free-particle or observer.
- An inertial reference frame (or inertial frame) is one such that no external forces act upon the coordinate system.
- When viewed from an inertial frame, any free particle will be seen to move in a straight line with constant velocity. As such, any free particle in the inertial frame is described by

$$\frac{d^2x}{dt^2} = 0; \qquad \frac{d^2y}{dt^2} = 0; \qquad \frac{d^2z}{dt^2} = 0;$$

- In contrast, a rotating (non-inertial) frame would have at least two spatial dimensions subject to acceleration.

1.2 The Principle of Relativity and the Invariance of the Speed of Light

- The laws of physics take the same form in all inertial frames. Special Relativity (SR), however, adds a new postulate that the speed of light, c, takes the same finite value in all inertial frames.
- Different inertial frames, K and K', will generally be associated with different times t and t'.
- Space and time are interconnected entities in SR, denoted **spacetime**, a 4-dimensional union of 3-dimensional space and 1-dimensional time.
- Universal simultaneity must be abandoned in SR, since two events that are simultaneous in one inertial frame generally won't be in another.

1.3 Events, Spacetime Diagrams and World Lines

- An event is something that happens at a location in spacetime. Note that different observers will have different coordinates for the same event, but the spacetime separation between two events is always the **same** for different reference frames.
- Useful to introduce spacetime diagrams, with a spacial dimension, x, on the x axis and ct (time multiplied by c), on the y axis. ct is shown on the y-axis as opposed to t so that both axes have units of length. At most, two spatial dimensions can be visualised by turning the spacetime diagram into a 3D diagram.
- A particle describes a curve in a spacetime diagram called a **world line**. It corresponds to the curve $x(t)$, or $X(x(t), y(t), z(t))$, describing the position x of the particle at time t, for all possible times t. Since nothing can move faster than c, the slope of $x(t)$: dx/dt must never be greater than c (or less than 1 where the y-axis is ct).

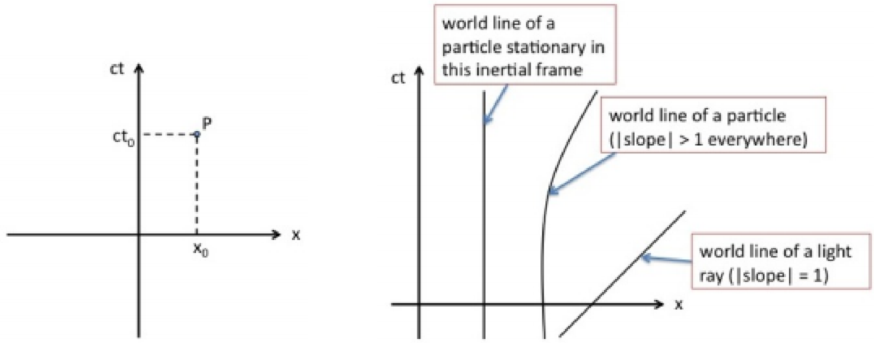

Figure 1: Spacetime diagrams. Left: An event P. Right: Examples of world lines.

- Free particles move with constant speed, so their world lines are straight lines. A special case is where a particle is stationary with respect to the inertial frame, in which case it is a vertical world line.

1.4 The Spacetime Interval and its Invariant Nature

- Define a parameter called the **spacetime interval** by

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

which effectively measures the magnitude of change in spacetime coordinates between two events.

- For two frames K and K', the spacetime intervals are equal for some two events measured in the respective frame. That is,

$$(\Delta s)^2 = (\Delta s')^2$$

So, while two observers might disagree on the spatial or time coordinates of some two events, they will always agree on the spacetime interval between them. We then say that the spacetime interval is **invariant** under change of inertial frame.

- For a photon worldline (i.e. when $v = c$), $(\Delta s)^2 = 0$. For any other worldline, the spacetime interval between two events is not equal to 0 ($(\Delta s)^2 \neq 0$).

- If two events are infinitesimally close to each other, define

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2$$

The infinitesimal spacetime interval ds is also called the **line element of flat spacetime**.

- The spacetime interval along a curve (a particle worldline) is calculated by

$$\Delta s = \int_1^2 ds$$

where the integral goes along the curve.

1.5 Types of Separation between Events and Light Cones

- It is useful to characterize pairs of events according to the sign of the square of the spacetime interval between them. The two events are said to be

$$\begin{aligned} \text{Spacelike separated if } (\Delta s)^2 &> 0 \\ \text{Null separated if } (\Delta s)^2 &= 0 \\ \text{Timelike separated if } (\Delta s)^2 &< 0 \end{aligned}$$

  Rather than remembering the sign, it is safer to think of timelike and spacelike separation in terms of whether the space part or the time part of the interval is the biggest one: if the time part is bigger, the separation is timelike; if the space part is bigger, the separation is spacelike.

- If and only if the separation between two events is spacelike is it possible to find an inertial frame in which the two events happen at the same time.

- If and only if the separation between two events is timelike is it possible to find an inertial frame in which the two events happen at the same place.

- Since the speed of a massive particle is always less than c, it follows that events on the worldline of a particle are timelike separated ($ds^2 < 0$).

- Events that are null separated can be connected by the worldline of a light ray going between the two events (light rays move on worldlines characterized by $ds^2 = 0$).

1.5.1 Light Cones

Consider an event P. The worldlines of light rays emanating from the point P (on a spacetime diagram) in both directions (on the positive ct axis) will form a wedge containing all the events that are time-like related to P in the past and future. These are all of the possible events that can be affected by the event P, in the future of P. Similarly, worldlines drawn that converge to P from the past will form a wedge that contains all the events that can affect P, and are in the past of P. Adding another spatial dimension will transform the wedges into cones (and in 3 spatial dimensions, hypercones) and so these two regions are referred to as the future and past light cones respectively. A point, P_2, that is neither in the future nor past of P is spacelike separated from it, and can change from being in the past of P to the future of P depending on the reference frame from which it is viewed. P_2 can neither affect or be affected by P, however.

1.6 Time Dilation, The Twin Paradox and Longest Distances

Consider a particle moving along its worldline. Two neighbouring points on this worldline are timelike separated, and so $ds^2 < 0$. Define

$$d\tau^2 \equiv -\frac{1}{c^2} ds^2$$

The quantity $d\tau$ is called the **proper time**, and is the time difference that would be measured by a clock carried by the particle as the particle moves between the points; i.e. proper time is the time measured by the particle in motion relative to a stationary observer. The stationary observer experiences more time than the particle by the relationship called **time dilation**:

$$d\tau = dt\sqrt{1 - v^2/c^2}$$

The time recorded by a clock following the particle as it moves along its worldline from event A to event B is calculated by

$$\tau_{AB} = \int_A^B d\tau = \int_{t_A}^{t_B} dt\sqrt{1 - v(t)^2/c^2}$$

Note that the proper time, τ_{AB}, does not just depend on the two event coordinates but also on the shape of the worldline connecting them. i.e. the worldline isn't restricted to be uniform motion, but can be an arbitrary worldline, including one subject to acceleration.
If the particle is moving at uniform motion, however, the proper time $\Delta t'$ (the time for a particle in K') is then

$$\Delta t' = \Delta t\sqrt{1 - v^2/c^2}$$

The "Twin Paradox" is in fact no paradox at all, as one of the reference frames for the two people cannot be inertial.

- For a clock within an inertial reference frame, the factor $\sqrt{1 - v^2/c^2}$ is always 1, since from the clocks perspective, $v = 0$.
- The straight line path is the *longest* distance (longest proper time) between two timelike separated points in flat four-dimensional spacetime.

1.7 Lorentz Boosts and Addition of Velocities

Two inertial frames can differ from one another by displacements, rotations (in space), or uniform motions (or any combination of these). Inertial frames that differ by uniform motion are called **Lorentz boosts** and are the generalizations to SR of the Galilean transformations in Newtonian Mechanics.

The Lorentz boost relating (ct, x) in one frame to (ct', x') in another (ignoring other spatial dimensions by $y' = y$ and $z' = z$) is given by

$$\begin{aligned} ct' &= ct\cosh\theta - x\sinh\theta \\ x' &= -ct\sinh\theta + x\cosh\theta \end{aligned}$$

Written in terms of the relative velocity (K' moving with velocity V relative to K),

$$\begin{aligned} ct' &= \gamma(ct - Vx/c) \\ x' &= \gamma(x - Vt) \\ \text{where } \gamma &= \frac{1}{\sqrt{1 - V^2/c^2}} \end{aligned}$$

The **proper length** of an object is its length in the frame in which it is at rest. Suppose an object is at rest in the frame K, and has a proper length Δx. Its length in the frame K' is obtained by finding the distance between events at each end of the object at the same time in that reference frame:

$$\Delta x' = \gamma(\Delta x - V\Delta t) = \Delta x(\gamma)^{-1}$$

In other words, the length of an object will be a factor of $\sqrt{1 - v^2/c^2} < 1$ of its proper length when measured from an inertial frame in motion.
Using the Lorentz boost, the relationship between the x-velocity of a particle as measured in two inertial frames (where $v_y = v_z = 0$) is

$$v_{x'} = \frac{dx'}{dt'} = \frac{\gamma(dx - V\,dt)}{\gamma(dt - V\,dx/c^2)} = \frac{v_x - V}{1 - Vv_x/c^2}$$

Two special cases are when $V/c \ll 1$ which gives $v_{x'} \approx v_x - V$ (which is the non-relativistic law of addition of velocities, and when $v_x = c$, which gives $v_{x'} = c$ as well, as required by the invariance of the speed of light.

2 Part 2

2.1 Units

From now on, units of time will no longer be seconds but rather light-meters - the distance light travels in one second. In these units, $c = 1$ and wherever ct was written previously, c is equivalent. Similarly, the ratio v/c becomes just v, where v is a fraction of the speed of light. In these units, 1 second $= c$ light metres.

2.2 Review of 3-Vectors

The geometrical concept of a vector is useful in SR because any relations between vectors are reference frame invariant, i.e. these relations hold after any allowed transformation of coordinates. For example, in 3D, $\vec{a} + \vec{b} = \vec{c}$ holds after any rotation or translation in space.

A given spatial reference frame (coordinate system) is defined by its base vectors $\vec{e_1}$, $\vec{e_2}$ and $\vec{e_3}$ (for example, for cartesian coordinates we could identify $1 \to x, 2 \to y, 3 \to z$). In this reference frame, each 3D vector is identified by its 3 components: $a^i = (a_1, a_2, a_3)$. This means that $\vec{a} = a_1\vec{e_1} + a_2\vec{e_2} + a_3\vec{e_3}$. In a different reference frame defined by base vectors $\vec{e_1}'$, $\vec{e_2}'$ and $\vec{e_3}'$, we have $\vec{a} = a_1'\vec{e_1}' + a_2'\vec{e_2}' + a_3'\vec{e_3}'$ and hence in this frame $a'^i = (a_1', a_2', a_3')$. In general, the components transform according to

$$a_m' = \sum_{n=1}^{3} p_{mn} a_n$$

where the 3×3 matrix p_{mn} is the same for all vectors, but depends on the choice of the two coordinate systems involved in the transformations. As usual, addition and multiplication of two vectors in the same coordinate space reads as

$$\vec{c} = \vec{a} + \vec{b} \to c^i = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$
$$\vec{c} = \alpha\vec{a} \to c^i = (\alpha a_1, \alpha a_2, \alpha a_3)$$

The orthonormality of the basis vectors is defined via their scalar (or dot) product:

$$\vec{e_i} \cdot \vec{e_j} = \delta_{ij}$$

which is true for all frames of reference. The scalar dot product between two arbitrary vectors is thus

$$\vec{a} \cdot \vec{b} = \sum_{i,j} (a_i\vec{e_i})\dot{(}b_j\vec{e_j}) = \sum_{i,j} a_i b_j (\vec{e_i}\dot{\vec{e_j}}) = \sum_i a_i b_i$$

where the length of a vector is its self-scalar product:

$$\vec{a} \cdot \vec{a} \equiv |\vec{a}|^2$$

and hence for some vector in 3D space, $|\vec{x}|^2 = x^2 + y^2 + z^2$. This vector length is invariant across changes of reference frame.

2.3 4-Vectors in SR

Spacetime 4-vectors will be written in boldface. In a given inertial frame, introduce basis 4-vectors $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ pointing along the t, x, y and z. An arbitrary 4-vector $\mathbf{a}$ can be expanded in terms of these basis vectors as

$$\mathbf{a} = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$$

where the numbers a_i are the components of $\mathbf{a}$ in this inertial frame. These components will often be listed as

$$a^\alpha = (a_0, \vec{a})$$

where $\vec{a}$ represents the spatial 3-vector with usual coordinate axes.
The **scalar product** of two 4-vectors **a** and **b** is defined in the usual way

$$\mathbf{a} \cdot \mathbf{b} = \sum_{\alpha,\,\beta} \eta_{\alpha\beta} a_\alpha b_\beta$$

where $\eta_{\alpha\beta}$ is called the **metric** of the basis vectors, $\eta_{\alpha\beta} \equiv \mathbf{e}_\alpha \mathbf{e}_\beta$. This is called the Minkowski metric, or the metric of flat space and has $-\eta_{00} = \eta_{11} = \eta_{22} = \eta_{33}$ and $\eta_{\alpha\beta} = 0$ for $\alpha \neq \beta$, which is often written in matrix form as

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

where the unlisted matrix elements are all 0. It follows that

$$\mathbf{a} \cdot \mathbf{b} = -a_0 b_0 + \vec{a} \cdot \vec{b}$$

Generalising the classification of spacetime separations, an arbitrary 4-vector **a** is classified

$$\begin{cases} \mathbf{a} \cdot \mathbf{a} > 0 & \Rightarrow \mathbf{a} \text{ is spacelike} \\ \mathbf{a} \cdot \mathbf{a} < 0 & \Rightarrow \mathbf{a} \text{ is timelike} \\ \mathbf{a} \cdot \mathbf{a} = 0 & \Rightarrow \mathbf{a} \text{ is null} \end{cases}$$

2.4 Changing Frames

So far only one inertial frame, K, has been considered whereas a different inertial frame, K', will have different components of a 4-vector, **a** (since the basis vectors are different in that frame). Under a transformation from K to K', the components of a 4-vector are defined to transform in the same way as the components of displacement vectors $d\mathbf{x}$. The transformation law for the latter is

$$dx'_\alpha = \sum_{\beta=0}^{4} \Lambda_{\alpha\beta} dx_\beta$$

where, for a boost to a new inertial frame with relative velocity $\vec{v}$ we have

$$\Lambda_{\alpha\beta} = \begin{pmatrix} \gamma & -\gamma v_x & -\gamma v_y & -\gamma v_z \\ -\gamma v_x & 1 + (\gamma - 1)\frac{{v_x}^2}{v^2} & (\gamma - 1)\frac{v_x v_y}{v^2} & (\gamma - 1)\frac{v_x v_z}{v^2} \\ -\gamma v_y & (\gamma - 1)\frac{v_x v_y}{v^2} & 1 + (\gamma - 1)\frac{{v_y}^2}{v^2} & (\gamma - 1)\frac{v_y v_z}{v^2} \\ -\gamma v_z & (\gamma - 1)\frac{v_x v_z}{v^2} & (\gamma - 1)\frac{v_y v_z}{v^2} & 1 + (\gamma - 1)\frac{{v_z}^2}{v^2} \end{pmatrix}$$

Here, $\vec{v}$ has components $(v_x,\, v_y,\, v_z)$ and $v^2 \equiv |\vec{v}|^2$. For the special case of the boost being in the x direction, the transformation reduces to the Lorenz transformations.

2.5 Kinematics and Dynamics in SR

Introduce various 4-vectors relevant to the motion of particles in SR; 4-velocity, 4-acceleration, 4-momentum, and 4-force.

3-Velocity: Particles move along worldlines in spacetime. All points along the worldlines must lie in the future light-cone of earlier points, and so intervals along the worldlines are timelike. In Newtonian mechanics, the worldline would be written as $\vec{x}(t)$. 3-velocity is invariant under translations and rotations, but is variate under boosts. Note that $\vec{x}(t)$ can't be an arbitrary function of time, and must

be constrained that $d\vec{x}/dt < 1$ so that the particle is slower than c.
In order to avoid problems, the motion of a particle along its worldline can be specified by giving its coordinates x_α in a particular inertial frame as functions $x_\alpha(\sigma)$ of some parameter σ which varies along the worldline. Ideally this parameter would be invariant across inertial frames, and often it is convenient to choose $\sigma = \tau$, the proper time, along the worldline.

4-Velocity: The 4-velocity $\mathbf{u}$ can be defined in terms of its components in a particular inertial frame K,

$$u_\alpha = \frac{dx_\alpha}{d\tau}$$

It follows from this that $\mathbf{u}$ is tangent to the particle's worldline at each point. The particle's 4-velocity can be expressed in terms of the components of it's 3-velocity, $\vec{v}$, in the same frame:

$$u^\alpha = (\gamma, \, \gamma\vec{v})$$

$\mathbf{u}$ is a timelike unit vector (it's timelike because $\mathbf{u}\cdot\mathbf{u} = -1 < 0$, and a unit vector because $|\mathbf{u}\cdot\mathbf{u}| = 1$). The property $\mathbf{u}\cdot\mathbf{u} = -1$ is called the **normalization** of the 4-velocity.
If two particles have 4-velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ respectively, then the scalar product between their 4-velocities, $\mathbf{u}_1 \cdot \mathbf{u}_2$ can be related to the magnitude of the relative 3-velocity between the particles, $|v|$, via

$$|v| = \sqrt{1 - \frac{1}{(\mathbf{u}_1 \cdot \mathbf{u}_2)^2}}$$

4-Acceleration: The 4-acceleration $\mathbf{a}$ is the derivative of the 4-velocity with respect to the proper time:

$$\mathbf{a} = \frac{d\mathbf{u}}{d\tau}$$

The 4-acceleration is orthogonal to the 4-velocity at every point in the worldline in the sense that $\mathbf{a}\cdot\mathbf{u} = 0$, which follows from differentiating both sides of $\mathbf{u}\cdot\mathbf{u} = -1$ with respect to τ and using the definition above.

4-Momentum: The 4-momentum $\mathbf{p}$ is defined as

$$\mathbf{p} = m\mathbf{u}$$

where m is the particle's mass. It follows that $p^\alpha = (m\gamma, \, m\gamma\vec{v})$. The nonrelativistic ($|\vec{v}|\|\|1$) limit of these expressions is obtained by expanding the square root in γ, which gives

$$p^0 \approx m + \frac{1}{2}m\vec{v}^2$$
$$\vec{p} \approx m\vec{v}$$

where $p^0 = E = mc^2 + \frac{1}{2}m\vec{v}^2$ is the energy (rest energy + the kinetic energy). For this, $\mathbf{p}$ is alternatively known as the **energy-momentum** 4-vector. It can otherwise be written $\mathbf{p} = (p^0, \vec{p})$. Manipulating the forms of $\mathbf{p}$ gives $\mathbf{p}\cdot\mathbf{p} = -E^2 + \vec{p}^2$, and also $\mathbf{p}\cdot\mathbf{p} = m^2\mathbf{u}\cdot\mathbf{u} = -m^2$. Equating these gives $E = \sqrt{m^2c^4 + \vec{p}^2}$.

4-Force and Equation of Motion: The SR equation of motion is

$$\boldsymbol{f} = m\mathbf{a}$$

where $\boldsymbol{f}$ is the 4-force. Equivalent forms are $\boldsymbol{f} = d\mathbf{p}/d\tau$ and $\boldsymbol{f} = md\mathbf{u}/d\tau$. Although this is a 4-vector equation, and thus represents 4 separate equations, only 3 of them are independent. Note that due to the constraint of acceleration, $\boldsymbol{f}\cdot\mathbf{u} = 0$. Next, define the 3-force $\vec{F}$ by

$$\frac{d\vec{p}}{dt} \equiv \vec{F}$$

Using some equation manipulation, you can arrive at a 4-force $\boldsymbol{f}$ in terms of the 3-force $\vec{F}$:

$$f^{\alpha} = (\gamma \vec{F} \cdot \vec{v}, \gamma \vec{F})$$

where $f^0 = \gamma \vec{F} \cdot \vec{v}$. The time component of $\boldsymbol{f}$ can also be written as

$$f^0 = \frac{dp^0}{d\tau} = \frac{dE}{d\tau} = \gamma \frac{dE}{dt}$$

This gives

$$\frac{dE}{dt} = \vec{F} \cdot \vec{v}$$

3 Part 3

3.1 Conservation of 4-Momentum

Conservation of 4-momentum is similar to conservation of 3-momentum in non-relativistic mechanics. If the 4-momenta of all of the components of the system under study, $\boldsymbol{p}_i$ are summed before an interaction to give $\boldsymbol{p}_b = \sum_i \boldsymbol{p}_i$, and summed afterwards to give $\boldsymbol{p}_a = \sum_i \boldsymbol{p}'_i$ (where $\boldsymbol{p}'_i$ are the 4-momenta of the individual components after the interaction), then $\boldsymbol{p}_a = \boldsymbol{p}_b$. As long as no forces act on the system, this is true for any inertial reference frames.

3.2 Photons (Null Geodesics)

Massless particles move at the speed of light, with null worldlines ($ds^2 = 0$). Although the following discussion is valid for any massless particle, in practice only photons are considered.

Worldline Paramtrizations: Since the proper time doesn't change along null worldlines, it cannot be used to parametrize the motion along such worldlines. Use a parameter λ, so $x^{\alpha} = x^{\alpha}(\lambda)$. With this parameter, the tangent vector to a point on the worldline is $\boldsymbol{u}$, with components defined as

$$u_{\alpha} = \frac{dx_{\alpha}}{d\lambda}$$

Consider the null worldline, $x = t + \text{const}$, representing a photon moving in the positive x direction, written as $x_{\alpha} = u_{\alpha}\lambda + \text{const}$, where λ is the parameter, and $u\alpha = (1, 1, 0, 0)$. Since $\boldsymbol{u}$ is the null vector, the equation of motion for the worldline is

$$\frac{d\boldsymbol{u}}{d\lambda} = 0$$

which takes the same form as the equation of motion for a free massive particle ($d\boldsymbol{u}/d\tau = 0$). Parameters for which the equation of motion of massless particles take the same from as for massive particles are called *affine* parameters. Affine parameters are not unique, so if λ is affine, then so is $\lambda' = k\lambda$.

Energy, Momentum, Frequency, Wavevector: In any inertial frame,

$$E = \hbar\omega \qquad \vec{p} = \hbar\vec{k} \qquad \omega = |\vec{k}|$$

where E, $\vec{p}$ and $\vec{k}$ are the photon's energy, 3-momentum, and wave 3-vector, respectively, in the inertial frame. From this, $E = \hbar\omega = \hbar|\vec{k}| = |\vec{p}|$. The photon's 4-momentum, $\boldsymbol{p}$ has components $p^{\alpha} = (E, \vec{p}) = (\hbar\omega, \hbar\vec{k}) \equiv \hbar k^{\alpha}$, where k^{α} are the components of the photon's wave 4-vector, $\boldsymbol{k}$. Thus,

$$\boldsymbol{p} = \hbar\boldsymbol{k}$$

$\boldsymbol{p}$ is a null vector (i.e. $\boldsymbol{p} \cdot \boldsymbol{p} = 0$). Since the 4-momentum of a massive particule satisfies $\boldsymbol{p} \cdot \boldsymbol{p} = -m^2$, we can conclude that photons are massless. Both $\boldsymbol{p}$ and $\boldsymbol{k}$ are tangent to the worldline of the photon.

Doppler Shift: Consider a photon source at rest in S that emits photons in all directions. Let the frequency of the emitted photon in S be ω.

Now consider the frame S' moving with respect to S with velocity $-V$ along the x-axis. Thus, the photon source is moving at $+V$ along the x' direction in S'. The frequency of the photon observed by S', ω' is solved by the Lorentz transformations

$$\omega = \gamma(\omega' - Vk^{x'})$$

Since in S' the photon's direction is at an angle α' to the x' direction, get $\cos\alpha' = k_{x'}/\omega'$. Substituting this into the equation above gives,

$$\omega' = \omega\frac{\sqrt{1-V^2}}{1-V\cos\alpha'}$$

This is relativistic Doppler shift.

Assume $V > 0$, then

$$\text{if } \alpha' = 0 \qquad \text{(photon emitted in the same direction that source is moving)}$$

$$\Rightarrow \omega' = \omega\sqrt{\frac{1+V}{1-V}} > \omega \qquad \text{(blueshift)}$$

$$\text{if } \alpha' = \pi \qquad \text{(photon emitted in the opposite direction that source is moving)}$$

$$\Rightarrow \omega' = \omega\sqrt{\frac{1-V}{1+V}} < \omega \qquad \text{(redshift)}$$

Photons emitted transverse to the direction of motion of the source ($\alpha' = \pi/2$) are red-shifted due to time dilation (transverse Doppler shift).

Relativistic Beaming: In the previous section, the direction in which light from the source arrived at the observer's reference frame was defined as $\cos\alpha' = k_{x'}/\omega'$. However we could also define the direction in which light left the source, in the source frame of reference as $\cos\alpha = k_x/\omega$. In general, $\alpha' \neq \alpha$. The change in direction of the wave vector leads to an effect called relativistic beaming, where a uniformly radiating body is brighter when it is moving towards the observer than when it is when moving away. Using the Lorentz transformations,

$$\cos\alpha' = \frac{\cos\alpha + V}{1 + V\cos\alpha}$$

Half of the photons go into the forward moving hemisphere of the source frame ($|\alpha| < \pi/2$), but the observer sees the photons in a smaller cone $|\alpha'| < \alpha'_{1/2}$ where $\cos\alpha'_{1/2} = V$. For V close to 1, this opening angle will be small.