MATH2001 Assignment 3

Ryan White
s4499039

21st of May 2021

Question 1

Considering the lamina on the domain $D = \{(x,y) |\, x^2+y^2 \leq 1,\, x \leq 0,\, y \leq 0\}$, described by the mass density function

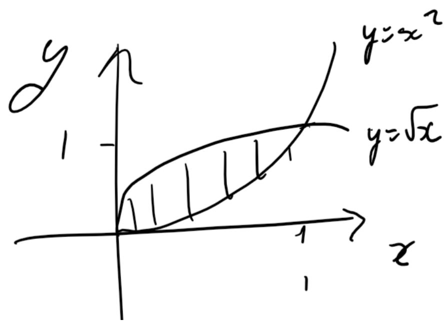
$$\rho(x,y) = kxy\sqrt{1+x^2+y^2}$$

the mass of the lamina may be determined by taking the double integral of the mass density function over the domain, i.e. from $y=-1$ to $y=0$ and from $x=0$ to $x^2+y^2=1 \Rightarrow x = -\sqrt{1-y^2}$ (since taking the x and y axis each less than or equal to 0). That is,

$$\begin{aligned}
m &= \int_{-1}^{0}\int_{-\sqrt{1-y^2}}^{0} kxy\sqrt{1+x^2+y^2}\; dx\; dy \\
\Rightarrow m &= \int_{-1}^{0}\int_{-\sqrt{1-y^2}}^{0} \frac{ky}{2}\sqrt{u}\; du\; dy \qquad \text{Where } u = 1+x^2+y^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x} \\
&= \int_{-1}^{0}\left[\frac{ky}{3}u^{3/2}\right]_{-\sqrt{1-y^2}}^{0} dy \\
&= \frac{k}{3}\int_{-1}^{0} y\left[(1+x^2+y^2)^{3/2}\right]_{x=-\sqrt{1-y^2}}^{x=0} dy \\
&= \frac{k}{3}\int_{-1}^{0} y\left((1+y^2)^{3/2} - (1+y^2+1-y^2)^{3/2}\right) dy \\
&= \frac{k}{3}\int_{-1}^{0} y\left((1+y^2)^{3/2} - 2\sqrt{2}\right) dy \\
&= \frac{k}{3}\int_{-1}^{0} \frac{1}{2}\left(u^{3/2} - 2\sqrt{2}\right) du \qquad \text{Where } u = y^2+1 \Rightarrow \frac{du}{dy} = 2y \Rightarrow dy = \frac{du}{2y} \\
&= \frac{k}{6}\left(\int_{-1}^{0} u^{3/2}\, du - \int_{-1}^{0} 2\sqrt{2}\, du\right) \\
&= \frac{k}{6}\left(\left[\frac{2}{5}(y^2+1)\right]_{y=-1}^{y=0} - \left[2\sqrt{2}(y^2+1)\right]_{y=-1}^{y=0}\right) \\
&= \frac{k}{6}\left(\frac{2}{5}(1-\sqrt{32}) - (2\sqrt{2}-4\sqrt{2})\right) \\
&= \frac{k}{6}\left(\frac{2}{5} - \frac{8\sqrt{2}}{5} + \frac{10\sqrt{2}}{5}\right) \\
&= \frac{k}{15}\left(1+\sqrt{2}\right)
\end{aligned}$$

Question 2

Take the region $E = \{(x, y, z) |\, 0 \leq x \leq 1,\, x^2 \leq y \leq \sqrt{x},\, 0 \leq z \leq x + y\}$, the x-y plane of the region being shown below

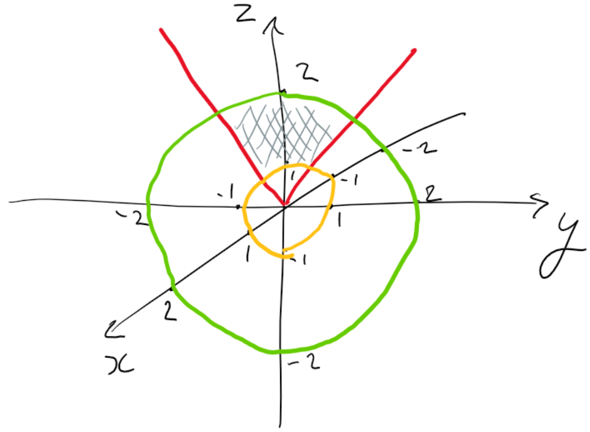


On this plane, the region is clearly enclosed by $y = x^2$ as a lower bound for $0 \leq x \leq 1$, and $x = y^2 \Rightarrow y = \sqrt{x}$ as an upper bound (positive square root since only taking positive x-y plane. The bounds of x (0 and 1 for the lower and upper respectively) were determined by equating $x = y^2$ and $y = x^2$, of which only 0 and 1 are solutions for x. The volume of the region enclosed in E is then,

$$\begin{aligned}
\iiint_E xy\, dV &= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy\, dz\, dy\, dx \\
&= \int_0^1 \int_{x^2}^{\sqrt{x}} [zxy]_{z=0}^{z=x+y}\ dy\, dx \\
&= \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y + xy^2\ dy\, dx \\
&= \int_0^1 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^{\sqrt{x}}\ dx \\
&= \int_0^1 \frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{x^6}{2} - \frac{x^7}{3}\ dx \\
&= \left[\frac{x^4}{8} + \frac{2x^{7/2}}{21} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1 \\
&= \frac{1}{8} + \frac{2}{21} - \frac{1}{14} - \frac{1}{24} \\
&= \frac{3}{28}
\end{aligned}$$

Question 3

Take the region T (shaded in grey in the diagram below) enclosed below by a sphere of radius $r^2 = 1$, above by a sphere of radius $r^2 = 4 \Rightarrow r = 2$, and by the cone $z = \sqrt{x^2 + y^2}$



First, the coordinate system was converted from cartesian to spherical by the transformations:

$$\begin{aligned} x &= r \sin\phi \cos\theta \\ y &= r \sin\phi \sin\theta \\ z &= r \cos\phi \\ dV = dx\,dy\,dz &= r^2 \sin\phi \,d\phi\, d\theta\, dr \end{aligned}$$

By these transformations, the cone then has the formula $r\cos\phi = \sqrt{r^2 \sin^2\phi} = r\sin\phi \Rightarrow \tan\phi = 1$ and so $\phi = \pi/4$. Thus, the region T is bounded by $0 \leq \phi \leq \pi/4$. Since T is between two spheres of radius 1 and 2, and encompasses a full 2π rotation about the z axis, the region is defined by

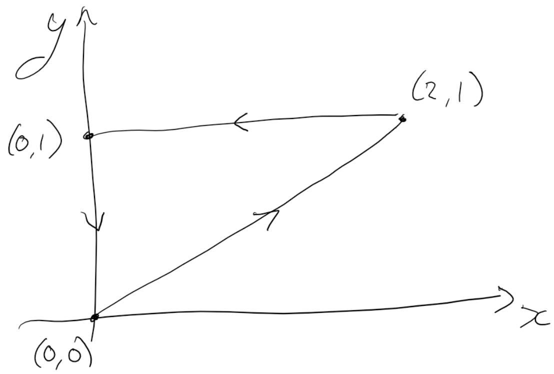
$$T = \{(r, \theta, \phi) |\, 1 \leq r \leq 2,\, 0 \leq \theta \leq 2\pi,\, 0 \leq \phi \leq \pi/4\}$$

The volume of the region was then calculated:

$$\begin{aligned} \iiint_T (x^2 + y^2 + z^2)\; dV &= \int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \left(r^2 \sin^2\phi \cos^2\theta + r^2 \sin^2\phi \sin^2\theta + r^2\cos^2\phi\right) r^2 \sin\phi \;d\phi\, d\theta\, dr \\ &= \int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \left(r^2 \sin^2\phi + r^2\cos^2\phi\right) r^2 \sin\phi \;d\phi\, d\theta\, dr \\ &= \int_1^2 \int_0^{2\pi} \int_0^{\pi/4} r^4 \sin\phi \;d\phi\, d\theta\, dr \\ &= \left(\int_1^2 r^4\, dr\right) \left(\int_0^{2\pi} 1\, d\theta\right) \left(\int_0^{\pi/4} \sin\phi\, d\phi\right) \\ &= \left(\frac{32}{5} - \frac{1}{5}\right)(2\pi)\left(-\cos\left(\frac{\pi}{4}\right) + 1\right) \\ &= \frac{31\pi}{5}\left(2 - \frac{2}{\sqrt{2}}\right) = \frac{31\pi}{5}\left(2 - \sqrt{2}\right) \qquad \text{Since } \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{aligned}$$

Question 4

Take the path C through a potential field on the x-y axis shown by the diagram below:



The bounds of the region above clearly go from $x = 0 \to 2$, with the y-bounds of the region being bounded above by 1 and below by the straight line of gradient 1/2. That is, $x/2 \leq y \leq 1$.
By Green's Theorem, the path integral can be expressed as a double integral of the region enclosed by the path. That is,

$$\oint_C F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Substituting in the potential equation gives

$$\begin{aligned}
\oint_C (x^2+y^2)\,dx + (x^2-y^2)\,dy &= \iint_D \left(\frac{\partial(x^2-y^2)}{\partial x} - \frac{\partial(x^2+y^2)}{\partial y}\right) dy\,dx \\
&= \int_0^2 \int_{x/2}^1 2x - 2y \;dy\,dx \\
&= \int_0^2 \left[2xy - y^2\right]_{x/2}^1 \,dx \\
&= \int_0^2 2x - 1 - x^2 + \frac{x^2}{4}\,dx \\
&= \int_0^2 2x - 1 - \frac{3x^2}{4}\,dx \\
&= \left[x^2 - x - \frac{x^3}{4}\right]_0^2 \\
&= 4 - 2 - 2 \\
&= 0
\end{aligned}$$

So, even though the field is not conservative, the path integral yields a value of 0.