PHYS2100/2101/7200: Dynamics and Relativity

Lagrangian and Hamiltonian Dynamics lecture notes

Semester 2, 2021

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1 Single particle systems

1.1 Vector calculus

1.1.1 Curves in space

If components of a point (x, y, z) are functions of a variable t (time) then the point $(x(t), y(t), z(t))$ traces out a curve in 3-space. The coordinate equations

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

are called the **parametric equations** of the curve. We put a natural orientation on a curve

$$\gamma : (x, y, z) = (x(t), y(t), z(t)),$$

and say that point $(x(t_1), y(t_1), z(t_1))$ *precedes* point $(x(t_2), y(t_2), z(t_2))$ if $t_1 < t_2$. We put arrow heads on a curve γ to mark the positive direction.

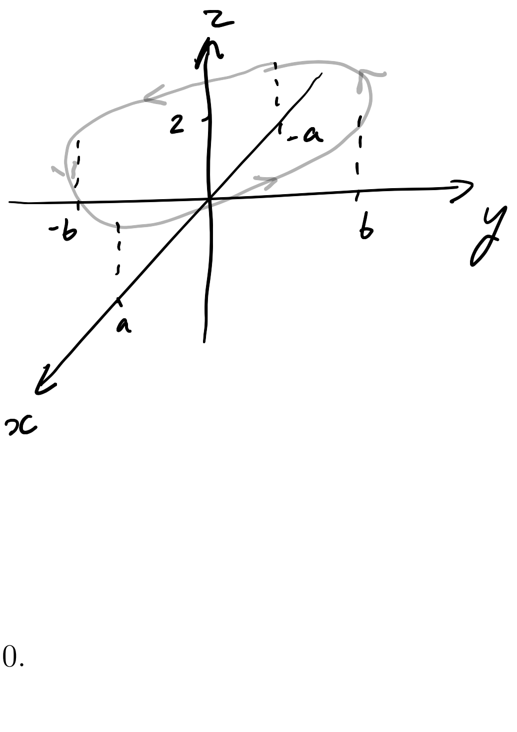
Exercise 1. *Parametrise the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the plane $z = 2$*, and draw a diagram marking the positive direction.*

Solution 1.

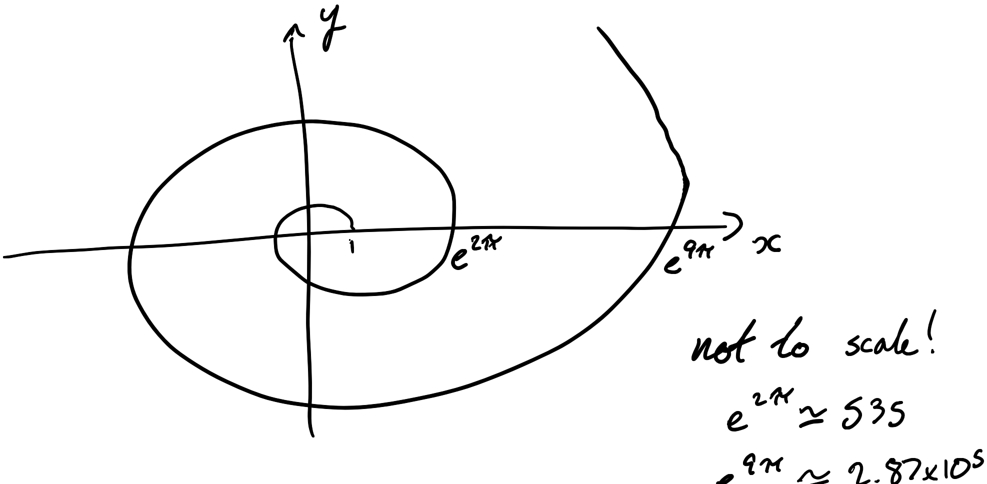
Set $\left.\begin{array}{l} x = a\cos t \\ y = b\sin t \\ z = 2 \end{array}\right\} \; 0 \le x \le 2\pi$



Exercise 2. *Roughly sketch the curve parametrised by*

$$(x, y) = (e^t \cos t, e^t \sin t), \;\; t \ge 0.$$

Solution 2.



1.1.2 Vector functions

A vector

$$\mathbf{f}(t) = f_1(t)\hat{\mathbf{i}} + f_2(t)\hat{\mathbf{j}} + f_3(t)\hat{\mathbf{k}}$$

whose components are functions of one or more variables (in this case time t) is called a **vector function**. The basic concepts of the calculus of such functions e.g. limits, differentiation etc. can be introduced in a natural way. For example

$$\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{a}$$

means that

$$\lim_{t \to t_0} f_i(t) = a_i, \quad i = 1, 2, 3.$$

Similarly, we differentiate vector functions component-wise:

$$\frac{d\mathbf{f}}{dt} = \frac{df_1}{dt}\hat{\mathbf{i}} + \frac{df_2}{dt}\hat{\mathbf{j}} + \frac{df_3}{dt}\hat{\mathbf{k}} = \dot{f}_1\hat{\mathbf{i}} + \dot{f}_2\hat{\mathbf{j}} + \dot{f}_3\hat{\mathbf{k}}.$$

Exercise 3. *If* $\mathbf{f}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + e^t\,\hat{\mathbf{k}}$, *calculate* $\lim_{t \to 0} \mathbf{f}(t)$ *and* $\dot{\mathbf{f}}(t)$. $= \dfrac{d\tilde{f}}{dt}$

Solution 3.

$\lim_{t \to 0} \tilde{f}(t) = \tilde{i} + \tilde{k}$, $\dot{\tilde{f}}(t) = -\sin t\,\tilde{i} + \cos t\,\tilde{j} + e^t\,\tilde{k}$

Also note that the usual product rules of differentiation hold for vector functions.

i.e.

$$\frac{d}{dt}\{\mathbf{g}(t) \cdot \mathbf{f}(t)\} = \dot{\mathbf{g}}(t) \cdot \mathbf{f}(t) + \mathbf{g}(t) \cdot \dot{\mathbf{f}}(t)$$

$$\frac{d}{dt}\{\mathbf{g}(t) \times \mathbf{f}(t)\} = \dot{\mathbf{g}}(t) \times \mathbf{f}(t) + \mathbf{g}(t) \times \dot{\mathbf{f}}(t).$$

Note: Unless otherwise stated, we assume throughout the course that all functions are differentiable.

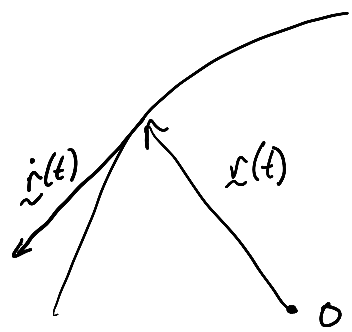
1.1.3 Position, velocity, acceleration

A particle moving in 3-space traces out a curve $(x(t), y(t), z(t))$, called the **path** of the particle, as t varies. The corresponding vector function

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

is called the **position vector** of the particle. The distance of the particle from the origin is therefore given by

$$r(t) = |\mathbf{r}(t)| = \sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)} = \sqrt{x^2 + y^2 + z^2}.$$



Thus

$$\hat{\mathbf{r}}(t) = \frac{\mathbf{r}(t)}{r(t)}$$

determines the **unit vector** in the direction of the particle.

The vector

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{x}(t)\hat{\mathbf{i}} + \dot{y}(t)\hat{\mathbf{j}} + \dot{z}(t)\hat{\mathbf{k}}$$

is called the **velocity vector** of the particle; that is,

$$\mathbf{v}(t) = \lim_{\delta t \to 0} \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t} = \dot{\mathbf{r}}(t).$$

Thus at any instant the velocity vector is tangent to the path of the particle and points in the direction of motion. It follows that

$$\hat{\mathbf{v}}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

determines a **unit tangent vector** to the curve at the point $(x(t), y(t), z(t))$. We call

$$v(t) = |\mathbf{v}(t)| = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2},$$

the **speed** of the particle and

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\mathbf{v} \cdot \mathbf{v})$$

the **kinetic energy** of the particle, where m is the particle's mass.

The vector

$$\mathbf{p} = m\mathbf{v}$$

is called the (linear) **momentum** of the particle. The **acceleration** vector of the particle is given by

$$\mathbf{a} = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t) = \ddot{x}\hat{\mathbf{i}} + \ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}.$$

According to Newton's second law, if $\mathbf{F}(t)$ is the force exerted on the particle then

$$m\ddot{\mathbf{r}} = \mathbf{F}(t)$$

which is called the **equation of motion**.

Note: In terms of momentum, we more accurately have

$$\dot{\mathbf{p}} = \mathbf{F}(t).$$

This is important only when mass m of the particle depends on time. In particular, if there is no force exerted on the particle, that is $\mathbf{F} = \mathbf{0}$, then

$$\dot{\mathbf{p}} = \mathbf{0}.$$

This is known as **conservation of linear momentum**.

Exercise 4. *A particle of mass m moves in 3-space with position vector*

$$\mathbf{r}(t) = a\cos\omega t\hat{\mathbf{i}} + a\sin\omega t\hat{\mathbf{j}} + ct\hat{\mathbf{k}}.$$

Describe the path of the particle, and find the velocity, speed and kinetic energy of the particle. Also find the unit tangent vector to curve at time $t = 0$.

Solution 4. The particle moves along a helix

The velocity

$$\underset{\sim}{v} = \dot{\underset{\sim}{r}} = -a\omega\sin\omega t\,\tilde{i} + a\omega\cos\omega t\,\tilde{j} + c\tilde{k}$$

and speed is

$$v = |\underset{\sim}{v}| = \|\underset{\sim}{v}\|$$
$$= \sqrt{a^2\omega^2 + c^2}$$

The kinetic energy is

$$T = \frac{1}{2} m \underset{\sim}{v} \cdot \underset{\sim}{v}$$

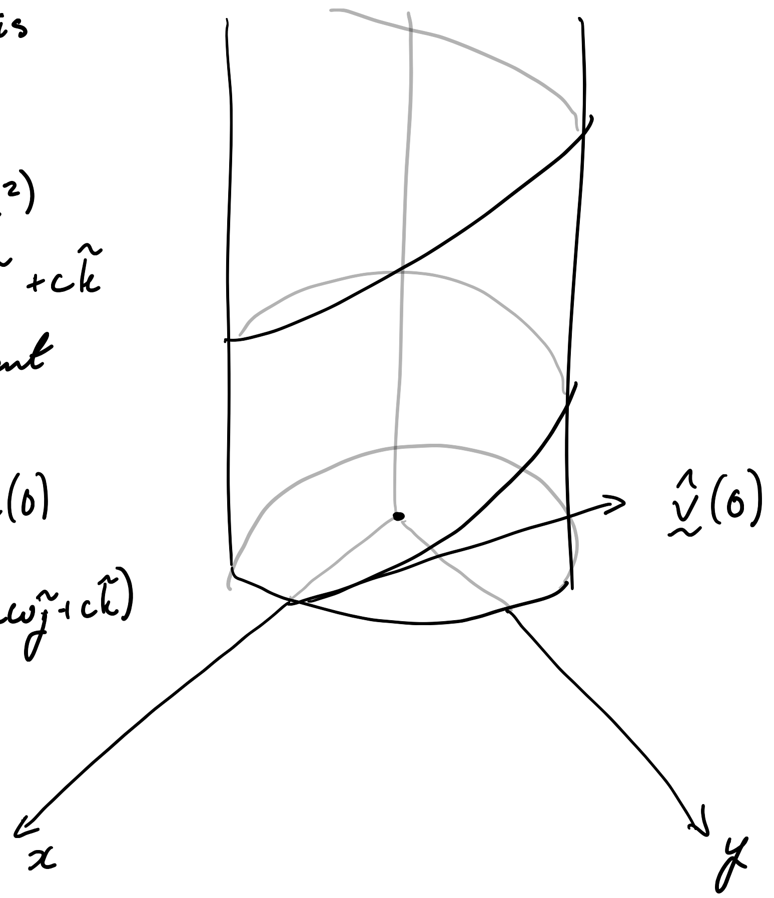
$$= \frac{1}{2} m \left(a^2 \omega^2 + c^2\right)$$

Also $\underset{\sim}{v}(0) = a\omega \tilde{j} + c\tilde{k}$

so the unit tangent vector is

$$\hat{\underset{\sim}{v}}(0) = \frac{1}{v(0)} \underset{\sim}{v}(0)$$

$$= \frac{1}{\sqrt{a^2\omega^2 + c^2}} \left(a\omega \tilde{j} + c\tilde{k}\right)$$



1.1.4 Rotational dynamics

When a system is rotating, we can describe the motion using moment-of-inertia, angular velocity, angular momentum and torque instead of mass, velocity, momentum and force.

The **moment of inertia** I of a rotating object is given by

$$I = \int \mathbf{r}.\mathbf{r}\, dm = \int r^2 dm$$

where $\mathbf{r}$ is the position of the infinitesimal point mass dm relative to the axis of rotation. The **angular velocity** $\boldsymbol{\omega}$ is related to the velocity $\mathbf{v}$ by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

If the motion of a particle is purely rotational, the magnitude of the angular velocity reduces to

$$\omega = v/r.$$

The **torque** τ is the"equivalent" of force for rotational systems, and the torque on a particle experiencing a force $\mathbf{F}$ is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

The expressions for torque and kinetic energy are parallel to those for translational mechanics, with I replacing m and $\boldsymbol{\omega}$ replacing $\mathbf{v}$. That is,

$$\boldsymbol{\tau} = I\dot{\boldsymbol{\omega}}$$

and

$$T = \frac{1}{2} I\omega^2.$$

The above expression for the torque only holds when the moment of inertia is constant. If it is changing, we instead use

$$\boldsymbol{\tau}(t) = \frac{d\boldsymbol{L}}{dt}$$

where $\boldsymbol{L}$ is the angular momentum, given by

$$\boldsymbol{L} = I\boldsymbol{\omega}.$$

1.1.5 Arclength

Let $\gamma : (x, y, z) = (x(t), y(t), z(t))$ be a curve in 3-space and let $\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$ be the corresponding position vector. Then recall that the **arclength** of the curve γ between $t = a$ and $t = b$ is given by

$$L = \int_a^b |d\mathbf{r}| = \int_a^b |\dot{\mathbf{r}}| dt = \int_a^b v(t) dt.$$

This determines the distance travelled by a particle along path γ between times $t = a$ and $t = b$.

$$\cosh t = \tfrac{1}{2}(e^{t}+e^{-t}) \qquad \sinh t = \tfrac{1}{2}(e^{t}-e^{-t})$$

Note $\cosh^2 t - \sinh^2 t = 1$

Exercise 5. *Find the arclength of the curve* $(x, y, z) = (\cosh t, \sinh t, t)$ *between* $t = 0$ *and* $t = 1$.

Solution 5. $\underset{\sim}{r}(t) = \cosh t\, \tilde{i} + \sinh t\, \tilde{j} + t\, \tilde{k}$

$$\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t) = \sinh t\, \tilde{i} + \cosh t\, \tilde{j} + \tilde{k}$$

$$v(t) = \|\underset{\sim}{v}(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t$$

$$L = \int_0^1 v(t)\, dt = \sqrt{2}\int_0^1 \cosh t\, dt = \sqrt{2}\sinh(t)\Big|_0^1 = \frac{1}{\sqrt{2}}(e - e^{-1})$$

1.2 Conservation of energy

1.2.1 Work and line integrals

Recall from classical physics that the **work** W done by a constant force $\mathbf{F}$ in moving a particle along a *straight line* from point A to point B is

$$\begin{aligned} W &= \text{(component of force in direction of motion) x (distance)} \\ &= \mathbf{F} \cdot (\mathbf{r}_B - \mathbf{r}_A). \end{aligned}$$

In general $\mathbf{F}$ is a function of t and the path of the particle is no longer a straight line but is given by a *curve*

$$\gamma : (x, y, z) = (x(t), y(t), z(t)), \quad t \in [a, b].$$

Then the work done by F over path γ is equal to the **line integral** of $\mathbf{F}(t)$ over γ, which is given by

$$\begin{aligned} W &= \int_\gamma \mathbf{F}(t) \cdot d\mathbf{r} \\ &= \int_a^b \mathbf{F}(t) \cdot \dot{\mathbf{r}}(t)\, dt. \end{aligned} \qquad \dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt}$$

Exercise 6. *Calculate the work done by a force* $\mathbf{F}(t) = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + t^2\hat{\mathbf{k}}$ *moving a particle along the helix* $\gamma : (x, y, z) = (\cos t, \sin t, t)$*,* $0 \le t \le \pi/2$*.*

$$\underset{\sim}{r}(t) = \cos t\, \underset{\sim}{i} + \sin t\, \underset{\sim}{j} + t\underset{\sim}{k}$$

Solution 6.

$$\dot{\underset{\sim}{r}}(t) = -\sin t\, \underset{\sim}{i} + \cos t\, \underset{\sim}{j} + \underset{\sim}{k}$$

$$\int_\gamma \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^{\pi/2} (\underset{\sim}{i} + 2\underset{\sim}{j} + t^2\underset{\sim}{k}) \cdot (-\sin t\, \underset{\sim}{i} + \cos t\, \underset{\sim}{j} + \underset{\sim}{k})\, dt$$

$$= \int_0^{\pi/2} (-\sin t + 2\cos t + t^2)\, dt = \left[\cos t + 2\sin t + \tfrac{1}{3}t^3\right]_0^{\pi/2}$$

$$= 1 + \frac{\pi^3}{24}$$

Lemma: Suppose a particle of mass m experiences a force $\mathbf{F}$. Then the work done by $\mathbf{F}$ in moving the particle over a path γ is equal to the increase in kinetic energy of the particle.

Proof: The work done by the force $\mathbf{F}$ in moving particle along its path γ for $a < t < b$ is given by the line integral

$$W = \int_\gamma \mathbf{F} \cdot \dot{\mathbf{r}}\, dt = \int_a^b \mathbf{F}(t) \cdot \mathbf{v}(t)\, dt.$$

$$\frac{d}{dt}(\dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}}) = \ddot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}} + \dot{\underset{\sim}{r}} \cdot \ddot{\underset{\sim}{r}} = 2\dot{\underset{\sim}{r}} \cdot \ddot{\underset{\sim}{r}}$$

But $\mathbf{F} = m\ddot{\mathbf{r}}$, so

$$\mathbf{F} \cdot \mathbf{v} = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2}m\frac{d}{dt}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = \dot{T},$$

where T is the kinetic energy. Therefore,

$$W = \int_a^b \dot{T}\, dt = T(b) - T(a).$$

i.e. The work done = increase in kinetic energy

1.2.2 Conservative systems

We say that motion is **conservative** if there exists a function $V = V(x, y, z)$, called the **potential energy** such that

$$\mathbf{F} \cdot \mathbf{v} = -\frac{dV}{dt}.$$

This potential energy V is uniquely determined up to a constant. Now we have from above

$$\mathbf{F} \cdot \mathbf{v} = \frac{dT}{dt},$$

so

$$\frac{d}{dt}(T+V)=0$$

or

$$T+V=E \text{ (const)}.$$

We call E the **energy** of the particle.

Exercise 7. *One-dimensional motion under gravity. Consider a particle of mass m at height x above the Earth's surface.*

(i) Assuming the particle is close to the Earth's surface, show that the motion is conservative, and that the total energy is given by $E=\frac{1}{2}mv^2+mgx$.

(ii) If the particle is launched vertically, solve the equation of motion, showing that with the assumption $F=-mg$ the particle will always fall back to Earth.

Solution 7. Newton's Law of Gravity

$F = -G\frac{Mm}{r^2}$

M - mass of Earth
r - distance
G - grav. const.

(1) Let "a" denote Earth radius, such that $r=a+x$ $\Rightarrow$ $F = \frac{-GMm}{(a+x)^2} \simeq -G\frac{Mm}{a^2}$ provided $x \ll a$

$= -mg$

where $g \simeq 9.8\,m/s^2$

$FV = -mg\dot{x} = -\frac{dV}{dt}$ where $V = mgx$

This is a conservative system with total energy $E = \frac{1}{2}m\dot{x}^2 + mgx$

(2) The equation of motion is

$$m\ddot{x} = -mg \quad \text{i.e.} \quad \ddot{x} = -g$$

$$v = \dot{x} = v_0 - gt \qquad (*)$$

$$\Rightarrow x = x_0 + v_0 t - \tfrac{1}{2} g t^2$$

From (*), a vertically launched particle will have zero speed when $t = \frac{v_0}{g}$, after which time the speed is negative and the particle will return to Earth.

1.2.3 Launching of artificial satellites

In certain cases of physical interest the approximation for gravitational force $F = -mg$ breaks down and we need to use the full Newtonian expression. Consider next launching a rocket ship from the Earth's surface. In this case we have

$$F = -\frac{mga^2}{(a+x)^2}$$

where a is the Earth's radius and x is the height of the rocket above the Earth.

Hence

$$Fv = -\frac{mga^2}{(a+x)^2}\dot{x} = -\frac{dV}{dt},$$

where

$$V = -\frac{mga^2}{(a+x)}.$$

Therefore the system is still conservative but now with potential energy V, and the energy is

$$E = \frac{1}{2}mv^2 - \frac{mga^2}{(a+x)},$$

which is constant in time.

Exercise 8. *Show that this reduces to the Galilean approximation (up to a constant) when $x << a$.*

Solution 8. Using the Taylor series approximation $\frac{1}{1+x} \simeq 1 - x$ for $|x| << 1$

we have $V = \frac{-mga^2}{a+x} = -\frac{mga}{1+\frac{x}{a}} \simeq -mga\left(1 - \frac{x}{a}\right)$

$= -mga + mgx$

This agrees with previous exercise up to the constant of integration for the potential

If $E < 0$, the rocket is trapped in Earth's gravitational field. Indeed $v = 0$ when

$$E = -\frac{mga^2}{(x+a)}.$$

Hence when

$$x = -\frac{a}{E}(mga + E)$$

the rocket ship falls back to Earth.

The rocket ship will escape Earth's gravitational field when $E = 0$, corresponding to

$$\frac{1}{2}mv^2 = \frac{mga^2}{(x+a)}$$

or

$$v = \sqrt{\frac{2ga^2}{(a+x)}}. \tag{1}$$

In this case v never vanishes and the rocket ship continues to rise indefinitely. Equation (1) gives the velocity needed to escape Earth's gravitational field at a height x above Earth's surface.

At the Earth's surface ($x = 0$), this velocity is

$$v_e = \sqrt{2ga} = \sqrt{\frac{2GM}{a}} \tag{2}$$

which is called the **escape velocity**. This last expression in fact gives the escape velocity from any spherical body of radius a and mass M.

1.2.4 Black holes

When $v_e = c$ (speed of light) the body is called a **black hole**. Squaring equation (2) yields the black hole equation

$$c^2 = \frac{2GM}{R}$$

or

$$R = \frac{2GM}{c^2},$$

which is known as the Schwarzchild radius. Here R gives the radius to which a spherical body of mass M must be shrunk in order to become a black hole. This agrees with the expression from general relativity.

Note: Laplace predicted the existence of such objects as far back as the 18th century.

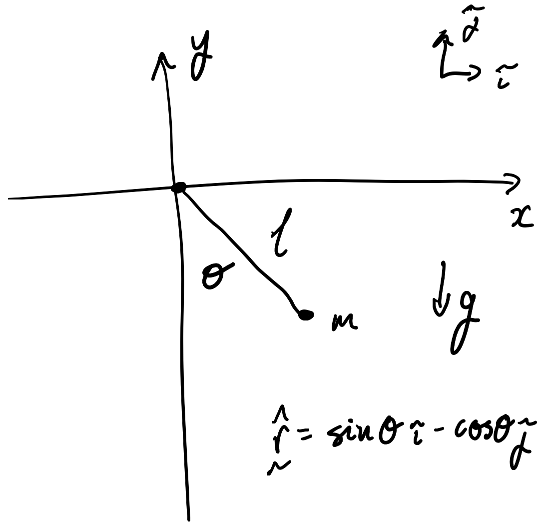
Exercise 9. *A simple pendulum.*

(i) Show that a simple pendulum is conservative and find the potential energy.

(ii) Calculate the work done by tension.

(iii) Show that if the angle is small, the pendulum will approximate simple harmonic motion.

Solution 9. $\hat{\mathbf{r}}$ is a unit direction vector



$$\mathbf{r} = l\,\hat{\mathbf{r}} = l(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})$$

T - tension

g - gravity

The equation of motion

$$m\ddot{\mathbf{r}} = \mathbf{F} = -mg\,\mathbf{j} - T\,\hat{\mathbf{r}}$$

Define $\hat{\boldsymbol{\theta}} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$

$$= \frac{d\hat{\mathbf{r}}}{d\theta}$$

And note $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = 0$ and $\dfrac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\hat{\mathbf{r}}$

$$\mathbf{v} = \dot{\mathbf{r}} = l\,\frac{d\mathbf{r}}{d\theta} \cdot \frac{d\theta}{dt} = l\dot{\theta}\,\hat{\boldsymbol{\theta}}$$ Next,

(i) $$\mathbf{F} \cdot \mathbf{v} = -(mg\,\mathbf{j} + T\,\hat{\mathbf{r}}) \cdot l\dot{\theta}\,\hat{\boldsymbol{\theta}}$$

$$= -mgl\dot{\theta}\,\mathbf{j} \cdot \hat{\boldsymbol{\theta}} - mglT\dot{\theta}\,\cancel{\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}}}$$

$$= -mgl\dot{\theta}\sin\theta$$

$$\underset{\sim}{F}\cdot\underset{\sim}{v} = -mgl\dot{\theta}\sin\theta$$

$$= -mgl\,\frac{d}{dt}(-\cos\theta)$$

$$= -mg\dot{y} \qquad\qquad y = -l\cos\theta$$

$$= -\frac{dV}{dt} \qquad\qquad \text{where } V = mgy$$

The system is conservative with total energy

$$E = \tfrac{1}{2}mv^2 + mgy = \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

(ii) The work done by tension is zero because the tension always acts perpendicularly to the motion

(iii) $\ddot{\underset{\sim}{r}} = \dot{\underset{\sim}{v}} = \frac{d}{dt}\left(l\dot{\theta}\,\hat{\underset{\sim}{\theta}}\right)$

$$\Rightarrow \dot{\underset{\sim}{v}} = l\ddot{\theta}\hat{\underset{\sim}{\theta}} + l\dot{\theta}\frac{d\hat{\underset{\sim}{\theta}}}{dt}$$

$$= l\ddot{\theta}\hat{\underset{\sim}{\theta}} + l\dot{\theta}\frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt}$$

$$= l\ddot{\theta}\hat{\underset{\sim}{\theta}} - l(\dot{\theta})^2\hat{\underset{\sim}{r}}$$

$$\Rightarrow m\ddot{\underset{\sim}{r}} = m\left(l\ddot{\theta}\hat{\underset{\sim}{\theta}} - l(\dot{\theta})^2\hat{\underset{\sim}{r}}\right) = -mg\hat{j} - T\hat{\underset{\sim}{r}} \qquad (*)$$

Take the dot product of $(*)$ with $\hat{\underset{\sim}{\theta}}$: (and simplify)

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

If θ is small, $\sin\theta \approx \theta$

$$\Rightarrow \ddot{\theta} \approx -\frac{g}{l}\theta \qquad \text{with general solution } \theta = A\cos\omega t + B\sin\omega t$$

where $\omega = \sqrt{\frac{g}{l}}$

1.3 Conservative forces

1.3.1 Gradient function

If we have a function $g : \mathbb{R}^3 \mapsto \mathbb{R}$, we call the vector function

$$\nabla g = \frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\hat{\mathbf{k}}$$

the **gradient** of g. We also call

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$$

the **del operator**.

Exercise 10. *Find* ∇g *where* $g(x, y, z) = x^2 + yz^2$.

Solution 10.

$$\nabla g = 2x\tilde{i} + z^2\tilde{j} + 2yz\tilde{k}$$

Note: ∇g is normal to the surface $g(x, y, z) = c$ at every point.

Proof: Let

$$\gamma : (x, y, z) = (x(t), y(t), z(t))$$

be any curve on the surface g. Since $g(x, y, z) = c$ we have $\dfrac{dg}{dt} = 0$. Then

$$\dot{\mathbf{r}} \cdot \nabla g = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt} = \frac{dg}{dt} = 0.$$

Therefore since $\dot{\mathbf{r}}$ is tangent to the surface, ∇g is orthogonal to the surface at the given point.

The plane orthogonal to ∇g at a point $P(x, y, z)$ on the surface $g(x, y, z) = c$ is called the **tangent plane** to the surface at P.

A force $\mathbf{F}$ is called **conservative** if there exists a function $g(x, y, z)$ such that

$$\mathbf{F} = \boldsymbol{\nabla} g.$$

We usually say more specifically, $\mathbf{F} = -\boldsymbol{\nabla} V$, where $V(x, y, z)$ is called the **potential function**, as before.

Note: Suppose a particle of mass m moves under the influence of a conservative force $\mathbf{F} = -\boldsymbol{\nabla} V$. Then the motion is conservative with potential energy V.

Proof:

$\dot{\underset{\sim}{r}} = (\dot{x}, \dot{y}, \dot{z})$

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{v} &= -\boldsymbol{\nabla} V \cdot \mathbf{v} \\
&= -\boldsymbol{\nabla} V \cdot \dot{\mathbf{r}} \\
&= -\frac{\partial V}{\partial x}\dot{x} - \frac{\partial V}{\partial y}\dot{y} - \frac{\partial V}{\partial z}\dot{z} \\
&= -\frac{\partial V}{\partial x}\frac{dx}{dt} - \frac{\partial V}{\partial y}\frac{dy}{dt} - \frac{\partial V}{\partial z}\frac{dz}{dt} \\
&= -\frac{dV}{dt}
\end{aligned}$$

so motion is conservative with potential energy V.

We can generalise this idea by using the Jacobi Matrix. The **Jacobi Matrix** of a vector function $\mathbf{f} = f_1(x_1, x_2, \ldots x_n)\hat{e}_1 + f_2(x_1, x_2, \ldots x_n)\hat{e}_2 + ... + f_n(x_1, x_2, \ldots x_n)\hat{e}_n$ is the $n \times n$ matrix with elements:

$$[J_{\mathbf{f}}]_{ij} = \frac{\partial f_i}{\partial x_j} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

In this course we are usually working in 3 dimensions, in which case $n = 3$ and x_1, x_2 and x_3 correspond to x, y and z.

Lemma: A force $\mathbf{F}$ is conservative if and only if $J_{\mathbf{F}}$ is symmetric.
Proof: Suppose $\mathbf{F} = -\boldsymbol{\nabla} V$ is conservative.

$$\begin{aligned}
[J_{\mathbf{F}}]_{ij} = \frac{\partial F_i}{\partial x_j} &= -\frac{\partial^2 V}{\partial x_j \partial x_i} \\
&= -\frac{\partial^2 V}{\partial x_i \partial x_j} \\
&= \frac{\partial F_j}{\partial x_i} = [J_{\mathbf{F}}]_{ji}
\end{aligned}$$

so $J_{\mathbf{F}}$ is symmetric. Conversely it can be shown that if $J_{\mathbf{F}}$ is symmetric then $\mathbf{F}$ is conservative - in two dimensions this is due to Green's theorem, in three dimensions this is due to Stokes' theorem (see MATH2000).

Exercise 11. *Show whether or not the following forces are conservative, and if so then deduce their potential functions.*

1. $\mathbf{F} = x^2y\,\hat{\mathbf{i}} + (x^2 - y^2)\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$
2. $\mathbf{F} = (x + y)\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$.

Solution 11.

1. $J_F = \begin{pmatrix} 2xy & x^2 & 0 \\ 2x & -2y & 0 \\ z & 0 & x \end{pmatrix} \rightarrow$ this is not symmetric, $\therefore$ F is not conservative

2. $J_F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow$ This is symmetric

$\Rightarrow$ F is conservative.

i.e. $F = -\nabla V$

So $F_1 \tilde{i} + F_2 \tilde{j} = -\dfrac{\partial V}{\partial x}\tilde{i} - \dfrac{\partial V}{\partial y}\tilde{j}$

Then, $x + y = -\dfrac{\partial V}{\partial x} \Rightarrow -V = \frac{1}{2}x^2 + xy + h(y)$ (1)

$x = -\dfrac{\partial V}{\partial y} \Rightarrow -V = xy + g(x)$ (2)

From (1) and (2), $V = -\frac{1}{2}x^2 - xy + c \qquad c \in \mathbb{R}$

1.3.2 Central forces

A force $\mathbf{F}(x, y, z)$ is called **central** if it has the form:

$$\mathbf{F} = F(r)\hat{\mathbf{r}} = \frac{F(r)}{r}\mathbf{r}$$

where $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ and $r = \sqrt{x^2 + y^2 + z^2}$. $(*)$

Central forces act either towards or away from the origin (depending on their sign) with a magnitude dependent only on the distance the object is away from the origin.

$$r = (x^2+y^2+z^2)^{1/2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x = \frac{x}{r}$$

Note:

$$\begin{aligned}
\boldsymbol{\nabla} r &= \frac{\partial r}{\partial x}\hat{\mathbf{i}} + \frac{\partial r}{\partial y}\hat{\mathbf{j}} + \frac{\partial r}{\partial z}\hat{\mathbf{k}} \\
&= \frac{x}{r}\hat{\mathbf{i}} + \frac{y}{r}\hat{\mathbf{j}} + \frac{z}{r}\hat{\mathbf{k}} \\
&= \frac{1}{r}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \\
&= \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}.
\end{aligned}$$

Exercise 12. *Let $F(r) = -V'(r)$ for some function V. Show the central force $\mathbf{F}(r) = F(r)\hat{\mathbf{r}}$ is conservative with potential function $V(r)$.*

Solution 12.

$$\begin{aligned}
\underset{\sim}{F}(r) &= -V'(r)\hat{\underline{r}} \\
&= -V'(r)\underline{\nabla} r \\
&= -\left(V'(r)\frac{\partial r}{\partial x}\hat{i} + V'(r)\frac{\partial r}{\partial y}\hat{j} + V'(r)\frac{\partial r}{\partial z}\hat{k}\right) \\
&= -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) \\
&= -\underline{\nabla} V
\end{aligned}$$

Therefore all central forces are conservative. (We assume throughout that the function $\mathbf{F}(r)$ is always integrable.) Hence a particle moving in a central force $\mathbf{F}(r) = -V'(r)\hat{\mathbf{r}}$ has energy

$$E = \frac{1}{2}mv^2 + V(r)$$

which remains constant in time.

1.3.3 Conservation of angular momentum

If we have a particle with mass m, velocity $\mathbf{v}$ and position $\mathbf{r}$, then that particle has an **angular momentum** given by:

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v}).$$

This can also be written as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where $\mathbf{p}$ is the linear momentum of the particle.

Exercise 13. *If a force* $\mathbf{F}$ *is acting on the particle, what is the rate of change in* $\mathbf{L}$*?*

Solution 13.

$$\underline{L} = m(\underline{r} \times \underline{v})$$

$$\dot{\underline{L}} = m(\underline{v} \times \underline{v}) + m(\underline{r} \times \dot{\underline{v}})$$

$$= m(\underline{r} \times \underline{a}) = \underline{r} \times m\underline{a} = \underline{r} \times \underline{F}$$

which is torque

For a *central force* $\mathbf{F} = \dfrac{F(r)}{r}\mathbf{r}$, so

$$\begin{aligned} \dot{\mathbf{L}} &= \mathbf{r} \times \mathbf{F} \\ &= F(r)(\mathbf{r} \times \mathbf{r}) \\ &= \mathbf{0}. \end{aligned}$$

Hence $\mathbf{L}$ is constant in time, so we have **conservation of angular momentum**.

Notes:

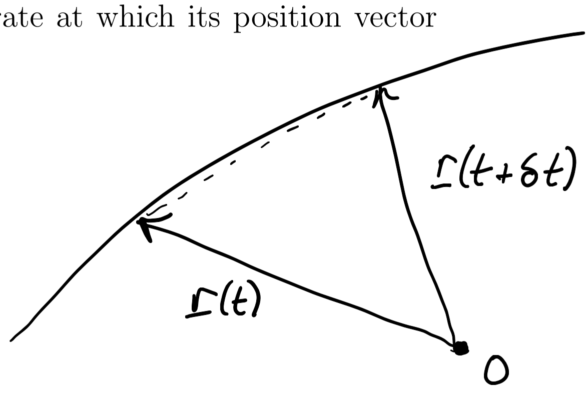
$$\begin{aligned} \mathbf{r} \cdot \mathbf{L} &= m\mathbf{r} \cdot (\mathbf{r} \times \mathbf{v}) = 0 \\ \mathbf{v} \cdot \mathbf{L} &= m\mathbf{v} \cdot (\mathbf{r} \times \mathbf{v}) = 0 \end{aligned}$$

That is, $\mathbf{r}$ and $\mathbf{v}$ are both orthogonal to $\mathbf{L}$. Thus a central force restricts movement to a plane through the origin, which is perpendicular to the angular momentum $\mathbf{L}$.

1.3.4 Kepler's second law

The angular momentum of a particle is related to the rate at which its position vector sweeps out an area: For dt small,

$$\delta A \approx \frac{1}{2}\Big|\mathbf{r}(t) \times \mathbf{r}(t + \delta t)\Big|$$



$$
\begin{aligned}
&= \frac{1}{2}\Big|\mathbf{r}(t) \times [\mathbf{r}(t+\delta t) - \mathbf{r}(t)]\Big| \\
&= \frac{1}{2}\Big|\mathbf{r}(t) \times \frac{(\mathbf{r}(t+\delta t) - \mathbf{r}(t))}{\delta t}\Big|\ \delta t \\
&\approx \frac{1}{2}\Big|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)\Big|\ \delta t,
\end{aligned}
$$

which becomes exact as $\delta t \to 0$. Therefore,

$$\frac{dA}{dt} = \frac{1}{2}|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)| = \frac{|\mathbf{L}|}{2m}.$$

So for a central field of force, where $\mathbf{L}$ is constant in time, the rate at which the position vector sweeps out an area is also constant in time. This result is known as **Kepler's second law** - empirically discovered for planetary motion by Kepler.

Note: This law, together with the planar motion of the planets, led Newton to deduce that gravity was a central force. Explicitly for a planet of mass m orbiting about the sun (assumed at origin), the force on the planet is given by

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

where M is the mass of the sun and r is the distance of the planet from the origin. We shall see later that such a law of force predicts elliptic orbits for the planets. Thus gravity is a central force with potential

$$V(r) = -\frac{K}{r}, \quad K = GMm.$$

1.3.5 Central forces and polar coordinates

We assume that the angular momentum vector $\mathbf{L}$ points in the z-direction so the particle moves in the $x - y$ plane. We introduce polar coordinates r, θ such that

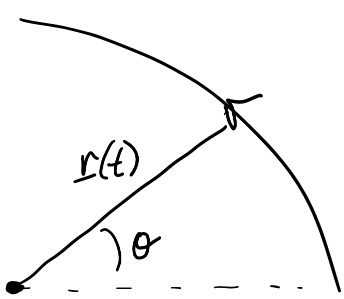
$$x = r\cos\theta, \quad y = r\sin\theta$$

and $r = \sqrt{x^2 + y^2}$ is the distance of the particle from the origin. Hence

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} = r\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$ is the unit vector in the direction of the particle. Defining

$$\hat{\boldsymbol{\theta}} = \frac{d\hat{\mathbf{r}}}{d\theta} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$$



we have

$$\frac{d\hat{\theta}}{dt} = \frac{d\hat{r}}{d\theta} \cdot \frac{d\theta}{dt}$$

$$
\begin{aligned}
\mathbf{v} &= \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}. \\
\mathbf{a} &= \dot{\mathbf{v}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}.
\end{aligned}
$$

$$\frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\underline{\hat{i}} - \sin\theta\,\hat{j}$$

$$= -\underline{\hat{r}}$$

Notes:

1. $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{r}} \cdot \hat{\mathbf{k}} = 0$.
2. $\dfrac{d\hat{\mathbf{r}}}{d\theta} = \hat{\boldsymbol{\theta}} \quad \Rightarrow \quad \dfrac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}}$.
3. $\dfrac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\hat{\mathbf{r}} \quad \Rightarrow \quad \dfrac{d\hat{\boldsymbol{\theta}}}{dt} = \dfrac{d\hat{\boldsymbol{\theta}}}{d\theta}\dfrac{d\theta}{dt} = -\dot{\theta}\hat{\mathbf{r}}$.
4. $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\mathbf{k}} = \hat{\mathbf{r}}$.

Since we are dealing with a central force

$$\mathbf{F} = -V'(r)\hat{\mathbf{r}}$$

the equation of motion is given by

$$-V'(r)\hat{\mathbf{r}} = m(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}. = m\underline{\hat{a}}$$

Equating the radial components gives us

$$\ddot{r} - r\dot{\theta}^2 = -\frac{V'(r)}{m} \qquad (*)$$

for the radial part.

Exercise 14. *Show, in these coordinates, that the angular momentum is conserved.*

Solution 14.

$\underline{L} = m\underline{r} \times \underline{V}$

where

$\underline{r} \times \underline{v} = \underline{r} \times (\dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}})$

$= r\dot{\theta}\,\underline{r} \times \hat{\underline{\theta}}$

$= r^2\dot{\theta}\hat{\underline{k}}$

$\Rightarrow |\underline{L}| = mr^2\dot{\theta}$

$\dfrac{d|\underline{L}|}{dt} = m\dfrac{d}{dt}(r^2\dot{\theta})$

$= m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$

$= 0$

Verifying that angular momentum is conserved.

The motion of the particle is thus governed by the radial equation

$$\ddot{r} = \frac{L^2}{m^2r^3} - \frac{V'(r)}{m}. \tag{3}$$

~ obtained from $(*)$ using $\dot{\theta} = \dfrac{|\underline{L}|}{mr^2}$

1.4 Inverse square law and planetary motion

According to Newton's law of gravitation a planet of mass m orbiting a star of mass M (at the origin) experiences the central force

$$\begin{aligned} \mathbf{F} &= -\frac{GMm}{r^2}\hat{\mathbf{r}} \\ &= -\frac{K}{r^2}\hat{\mathbf{r}}, \qquad\qquad K = GMm \\ &= -\boldsymbol{\nabla}V(r), \end{aligned}$$

where $V(r) = -\dfrac{K}{r}$ is the potential energy. Thus the energy of the planet is

$$E = \frac{1}{2}mv^2 - \frac{K}{r}$$

which remains constant in time.

Now introduce the Runge vector

$$\mathbf{R} = \hat{\mathbf{r}} + \frac{1}{K}(\mathbf{L} \times \mathbf{v}) = \hat{\mathbf{r}} + \frac{L}{K}(\hat{\mathbf{k}} \times \mathbf{v}).$$

Exercise 15. *Show that $\dot{\mathbf{R}} = 0$, so $\mathbf{R}$ remains constant in time.*

Solution 15.

$$\frac{d\underline{R}}{dt} = \frac{d\hat{\underline{r}}}{dt} + \frac{L}{K}\left(\hat{\underline{k}} \times \frac{d\underline{v}}{dt}\right)$$

Equation of motion $\quad m\dfrac{d\underline{v}}{dt} = \underline{F} = -\dfrac{K}{r^2}\hat{\underline{r}}$

which leads to $\quad \dot{\underline{R}} = \dfrac{d\theta}{dt}\dfrac{d\hat{\underline{r}}}{d\theta} - \dfrac{L}{mr^2}(\hat{\underline{k}} \times \hat{\underline{r}})$

$$= \dot{\theta}\hat{\underline{\theta}} - \frac{L}{mr^2}\hat{\underline{\theta}} \qquad\qquad (L = mr^2\dot{\theta})$$

$$= \dot{\theta}\hat{\underline{\theta}} - \dot{\theta}\hat{\underline{\theta}} = \underline{0}$$

$L = mr^2\theta \qquad \dot{\theta} = \frac{L}{mr^2} \qquad \underline{R} = \hat{\underline{r}} + \frac{L}{K}(\underline{k} \times \underline{v})$

Exercise 16. *Show*

$$|\mathbf{R}| = \sqrt{1 + \frac{2L^2E}{mK^2}}$$

Solution 16.

$$\underline{R} \cdot \underline{R} = \hat{\underline{r}} \cdot \hat{\underline{r}} + \frac{L}{K}\left(\hat{\underline{r}} \cdot (\underline{k} \times \underline{v}) + (\underline{k} \times \underline{v}) \cdot \hat{\underline{r}}\right) + \frac{L^2}{K^2}(\underline{k} \times \underline{v})(\underline{k} \times \underline{v})$$

$$= 1 + \frac{2L}{K} \hat{\underline{r}} \cdot (\underline{k} \times \underline{v}) + \frac{L^2}{K^2} V^2$$

Recall $\underline{k} \times \underline{v} = \underline{k} \times (\dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}) = \dot{r}\hat{\underline{\theta}} - r\dot{\theta}\hat{\underline{\theta}}$, so,

$$\underline{R} \cdot \underline{R} = 1 - \frac{2L}{K} r\theta + \frac{L^2}{K^2} V^2$$

$$= 1 - \frac{2L^2}{Kmr} + \frac{L^2}{K^2} V^2$$

$$= 1 + \frac{2L^2}{mK^2}\left(-\frac{K}{r} + \frac{1}{2}mV^2\right)$$

$$\underline{R} \cdot \underline{R} = 1 + \frac{2L^2E}{mK^2}$$

$$\Rightarrow |\underline{R}| = \sqrt{1 + \frac{2L^2E}{mK^2}}$$

We call this expression the *eccentricity* of the orbit e. That is,

$$e = |\mathbf{R}| = \sqrt{1 + \frac{2L^2E}{mK^2}} \geq 0.$$

Below we assume that $L \neq 0$ (the case $L = 0$ is trivial).

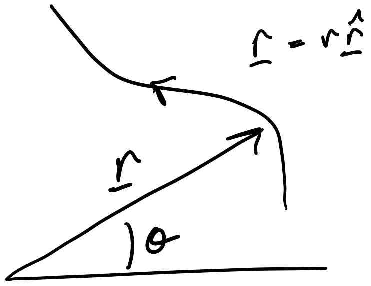
1.4.1 Planetary orbits I

To understand the motion of a planet further, we assume $\mathbf{R}$ points along the x-axis.

Then using $\hat{\mathbf{k}} \times \mathbf{v} = \dot{r}\hat{\boldsymbol{\theta}} - r\dot{\theta}\hat{\mathbf{r}}$, we may show that $r(1 - e\cos\theta) = \dfrac{L^2}{Km}$:

$$
\begin{aligned}
er\cos\theta &= |\mathbf{R}|.|\mathbf{r}|\cos\theta \\
&= \mathbf{R}\cdot\mathbf{r} \\
&= (\hat{\mathbf{r}} + \frac{L}{K}(\hat{\mathbf{k}} \times \mathbf{v}))\cdot\mathbf{r} \\
&= [\hat{\mathbf{r}} + \frac{L}{K}(\dot{r}\hat{\boldsymbol{\theta}} - r\dot{\theta}\hat{\mathbf{r}})]\cdot\mathbf{r} \\
&= r - \frac{L}{K}r^2\dot{\theta} \\
&= r - \frac{L^2}{Km}.
\end{aligned}
$$

$x = r\cos\theta$
$y = r\sin\theta$



Hence
$$r = \frac{L^2}{Km}\frac{1}{(1 - e\cos\theta)},$$
which is the equation of a conic section with eccentricity e and sun at one focus. Alternatively,
$$\frac{(x-ea)^2}{a^2} + \frac{y^2}{(1-e^2)a^2} = 1, \qquad a = \frac{L^2}{Km(1-e^2)}.$$
We have the following possibilities:

1. $0 \le e < 1$ - ellipse ($E < 0$) - the case $e = 0$ corresponds to a circle (special case of an ellipse),
2. $e = 1$ - parabola ($E = 0$),
3. $e > 1$ - hyperbola ($E > 0$).

Notes:

1. The above also applies to asteroids, comets, etc, orbiting the sun, as well as to satellites (natural or artificial) orbiting the planets.
2. Planets are trapped in sun's gravitational field and so have the energy $E < 0$ corresponding to elliptical orbits. For comets it is possible for $E \ge 0$ corresponding to parabolic or hyperbolic orbits - these are non-periodic comets.

1.4.2 Kepler's first and third laws

See tutorial Problem Sheet 1.1

1.4.3 Planetary orbits II

Exercise 17. *Consider a planet of mass m orbiting the sun - which is taken to be at the origin in the $x-y$ plane. This is a conservative (central force) system with potential $V(r) = -\frac{K}{r}$, $\quad K = GMm$, where G is the gravitational constant and M is the mass of the sun. Using equation (3), determine the equation of the orbit $r(\theta)$.*

Solution 17. Recall $\underline{\ddot{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\underline{\theta}}$

$\underline{F} = F(r)\hat{\underline{r}}$, so equating components in $\underline{F} = m\underline{\ddot{r}}$ yields

$$\frac{1}{m}F(r) = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad (*)$$

Recall $(*)$ results from the conservation of $L = mr^2\dot{\theta}$

Eliminating $\dot{\theta}$ gives $\ddot{r} - \frac{L^2}{m^2r^3} = \frac{F(r)}{m} \quad (**)$

Next, we change this to an ODE in θ by (for an arbitrary function f)

$$\frac{df}{dt} = \frac{d\theta}{dt}\frac{df}{d\theta} = \frac{L}{mr^2}\frac{df}{d\theta}$$

Set $u = r^{-1}$, so $\frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = -r^{-2}\frac{dr}{d\theta}$

Back to $(**)$:

$$m\frac{d}{dt}(\dot{r}) - \frac{L^2}{mr^3} = F(r)$$

$$m\frac{d}{dt}\left(\frac{L}{mr^2}\frac{dr}{d\theta}\right) - \frac{L^2}{mr^3} = F(r)$$

$$\Rightarrow \quad m\frac{d\theta}{dt}\frac{d}{d\theta}\left(\frac{L}{mr^2}\frac{dr}{d\theta}\right) - \frac{L^2}{mr^3} = F(r)$$

$$\frac{L^2}{mr^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{L^2}{mr^3} = F(r)$$

replace r with u $\Rightarrow \quad -\frac{L^2u^2}{m}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right) - \frac{L^2u^3}{m} = F(u^{-1})$

This reduces to $\dfrac{d^2u}{d\theta^2} + u = -\dfrac{m}{L^2u^2} F(u^{-1})$

For this example, $F(r) = -\dfrac{K}{r^2} \Rightarrow F(u^{-1}) = -Ku^2$, giving

$$\frac{d^2u}{d\theta^2} + u = \frac{Km}{L^2}$$

The general solution of this is $u = A\cos\theta + B\sin\theta + \dfrac{Km}{L^2}$

Assuming u has a maximum at $\theta = \pi$ gives $B = 0$, and

$$\frac{1}{r} = u = A\cos\theta + \frac{Km}{L^2} = \frac{Km}{L^2}(1 - e\cos\theta)$$

or rather

$$r = \frac{L^2}{Km(1 - e\cos\theta)}$$

Define $\cos\theta_0 = \dfrac{1}{e}$

$$\Rightarrow \quad e = \sqrt{1 + \frac{2L^2E}{mK^2}} \quad \rightarrow \quad \begin{array}{l} E > 0 \Rightarrow e > 1 \\ E = 0 \Rightarrow e = 1 \\ E < 0 \Rightarrow e < 1 \end{array}$$

2 Many particle systems

2.1 Constraints and generalised coordinates

If we have a system of n particles in a 3-dimensional space, $3n$ coordinates or variables are needed to specify their positions. If there are m algebraic constraint equations relating the coordinates, in principle we can eliminate m variables leaving a system depending on $3n - m$ **generalised coordinates** $q_i \, (1 \leq i \leq 3n - m)$. Such a system is said to have $3n - m$ degrees of freedom.

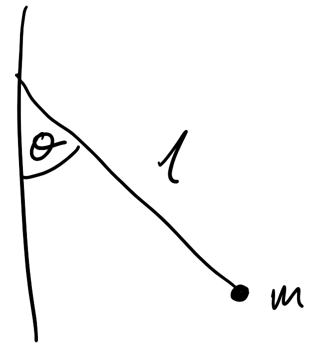
Exercise 18. *Determine the number of degrees of freedom for a simple pendulum.*

Solution 18.

$$\underline{r} = x\underline{i} + y\underline{j} \quad \text{or}$$

$$= l\sin\theta\,\underline{i} - l\cos\theta\,\underline{j}$$

Here, there is one degree of freedom, which is θ



Exercise 19. *Determine the number of degrees of freedom for a double pendulum.*

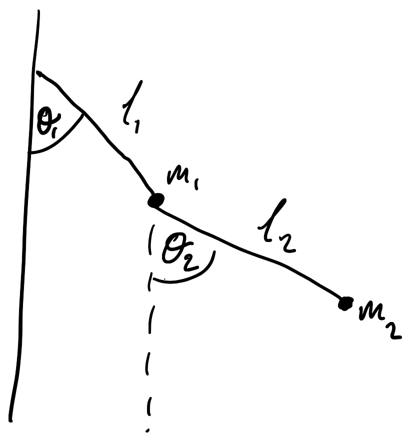
Solution 19.

Here we have $\underline{r}_1 = x_1\underline{i} + y_1\underline{j}$

$$\underline{r}_2 = x_2\underline{i} + y_2\underline{j}$$

But $x_1^2 + y_1^2 = l_1^2$ (distance of m_1 from origin)

$\Rightarrow (x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$ (distance between two masses)



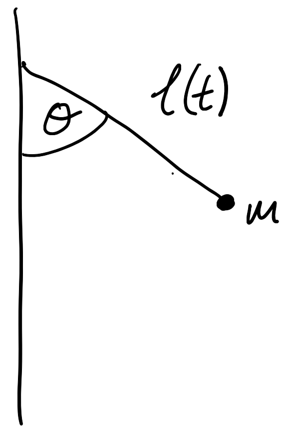
Here there are two degrees of freedom.

Exercise 20. *Determine the number of degrees of freedom for a pendulum with time-dependent length*

Solution 20.

$$\underline{r} = x\underline{i} + y\underline{j}$$
$$= l(t)\sin\theta\,\underline{i} - l(t)\cos\theta\,\underline{j}$$

Here there is only one degree of freedom (θ)



Notes: For a conservative system with m degrees of freedom and generalised coordinates $q_i \ \ (1 \le i \le m)$ (including the single and double pendulums), the kinetic energy and potential energy can be written as

$$T = T(q_i, \dot{q}_i, t), \quad V = V(q_i, t)$$

respectively. That is, V is not velocity dependent (the Lorentz force will not be considered here).

2.1.1 Generalised forces

Consider a system of n particles with m degrees of freedom and generalised coordinates $q_i \ \ (1 \le i \le m)$. The rate at which work is done is

$$\frac{dW}{dt} = \frac{dT}{dt} = \sum_{j=1}^{n} \mathbf{F}_j \cdot \mathbf{v}_j = \sum_{j=1}^{n} \mathbf{F}_j \cdot \dot{\mathbf{r}}_j$$

where $\mathbf{F}_j$ is the vector force exerted on the jth particle, and $\mathbf{r}_j, \mathbf{v}_j = \dot{\mathbf{r}}_j$, are the position and velocity vectors of the ith particle. Suppose $\mathbf{r}_j = \mathbf{r}_j(q_i, t)$. Then

$$\dot{\mathbf{r}}_j = \sum_{i=1}^{m} \left(\frac{\partial \mathbf{r}_j}{\partial q_i} \dot{q}_i \right) + \frac{\partial \mathbf{r}_j}{\partial t}.$$

Hence the rate at which work is done is now given by:

$$\begin{aligned}
\frac{dW}{dt} = \frac{dT}{dt} &= \sum_{j=1}^{n} \mathbf{F}_j \cdot \dot{\mathbf{r}}_j \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m} \mathbf{F}_j \cdot \left(\frac{\partial \mathbf{r}_j}{\partial q_i}\dot{q}_i\right) + \sum_{j=1}^{n} \mathbf{F}_j \cdot \frac{\partial \mathbf{r}_j}{\partial t} \\
&= \sum_{i=1}^{m} Q_i \dot{q}_i + \sum_{j=1}^{n} \mathbf{F}_j \cdot \frac{\partial \mathbf{r}_j}{\partial t}
\end{aligned}$$

where

$$Q_i = \sum_{j=1}^{n} \mathbf{F}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_i}, \tag{4}$$

which we call a **generalised force**.

Exercise 21. *Determine the generalised forces for a simple pendulum*

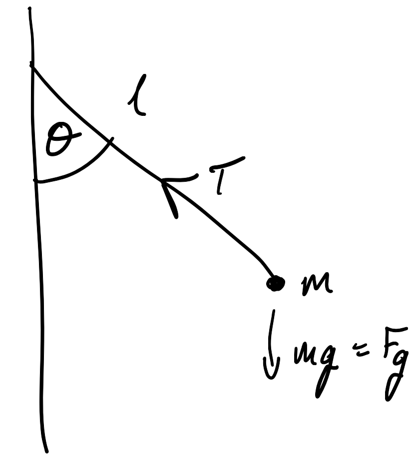
Solution 21.

$$\underline{r} = l\sin\theta\,\underline{i} - l\cos\theta\,\underline{j}$$

$$\frac{d\underline{r}}{d\theta} = l\cos\theta\,\underline{i} + l\sin\theta\,\underline{j}$$

$$\underline{F} = -mg\,\underline{j} - T\,\hat{\underline{r}}$$

$$Q = \underline{F} \cdot \frac{d\underline{r}}{d\theta} = -mgl\sin\theta$$



2.1.2 Conservative systems

A system is called **conservative** if there exists a function $V = V(q_i, t)$ called the **potential energy** such that:

$$Q_i = -\frac{\partial V}{\partial q_i}$$

and

$$\sum_{j=1}^{n} \mathbf{F}_j \cdot \frac{\partial \mathbf{r}_j}{\partial t} = -\frac{\partial V}{\partial t}.$$

Then

$$\begin{aligned}\frac{dT}{dt} &= -\sum_{i=1}^{m}\frac{\partial V}{\partial q_i}\dot{q}_i - \frac{\partial V}{\partial t}\\ &= -\frac{dV}{dt}\end{aligned}$$

So $T + V = E$, where T is our usual kinetic energy ($T = \sum_{j=1}^{n}\frac{1}{2}m_j v_j^2$).

Note that we usually say that the potential $V = V(q_i)$ has no explicit dependence on time and so $\frac{\partial V}{\partial t} = 0$. Throughout, unless otherwise stated, we make this assumption. Also, in many applications (eg. time-independent constraints)

$$\mathbf{r}_j = \mathbf{r}_j(q_1, q_2, ..., q_m) \quad \Rightarrow \quad \frac{\partial \mathbf{r}_j}{\partial t} = 0.$$

2.1.3 Equilibrium in conservative systems

For a conservative system, the generalised forces are

$$Q_i = -\frac{\partial V}{\partial q_i}.$$

If all the generalised forces in a system are zero, the system is said to be in **equilibrium**.

If we displace a conservative system from equilibrium by an arbitrarily small amount, say $(\delta q_1, \ldots, \delta q_m)$, then the change of work is zero.
Proof:

$$\begin{aligned}\delta W = \delta T &= -\delta V\\ &= -\sum_{i=1}^{m}\frac{\partial V}{\partial q_i}\delta q_i\\ &= -\sum_{i=1}^{m} 0 \cdot \delta q_i \quad \text{(because our generalised forces are zero)}\\ &= 0.\end{aligned}$$

That is, there is no work done in a virtual displacement from equilibrium.

The equilibrium is said to be **stable** if we have a local minimum at equilibrium. It is a minimum if the matrix

$$-\frac{\partial Q_j}{\partial q_i} = \frac{\partial^2 V}{\partial q_i \partial q_j} = \frac{\partial^2 V}{\partial q_j \partial q_i} = -\frac{\partial Q_i}{\partial q_j}$$

is positive definite for all j. That is all the eigenvalues are positive.

Example - Mass on a spring: Here, $V = \frac{1}{2}kx^2$, $m\ddot{x} = -kx$, $Q \equiv F = -kx$, therefore $x = 0$ is an equilibrium position. So

$$-\frac{\partial Q}{\partial x} = -\frac{dF}{dx} = k > 0 \Rightarrow \text{stable equilibrium.}$$

Example - Simple pendulum. We have

$$\begin{aligned} Q &= -mgl\sin\theta \\ &= -\frac{dV}{d\theta}, \quad V = -mgl\cos\theta, \end{aligned}$$

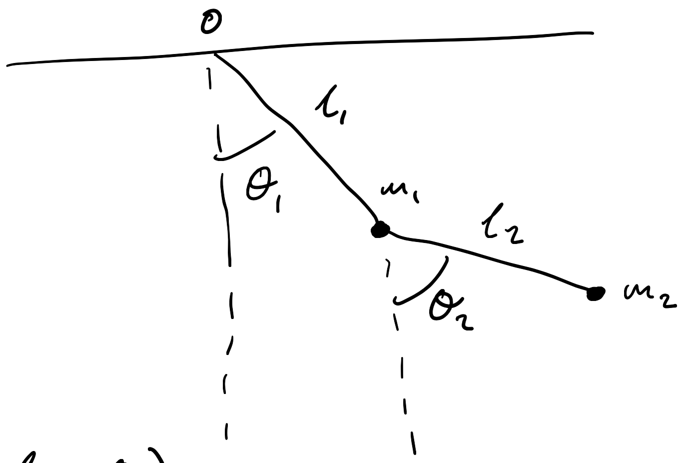
so the force is conservative with potential energy V. Therefore $Q = 0$ at $\theta = 0, \pi$ giving the equilibrium positions. To check stability,

$$\frac{d^2V}{d\theta^2} = mgl\cos\theta = \begin{cases} > 0, & \theta = 0 \text{ stable,} \\ < 0, & \theta = \pi \text{ unstable.} \end{cases}$$

Exercise 22. *Determine the stable equilibrium points for the double pendulum.*

Solution 22.

$$x_1 = l_1 \sin\theta_1$$
$$y_1 = -l_1\cos\theta_1$$
$$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2$$
$$y_2 = -l_1\cos\theta_1 - l_2\cos\theta_2$$



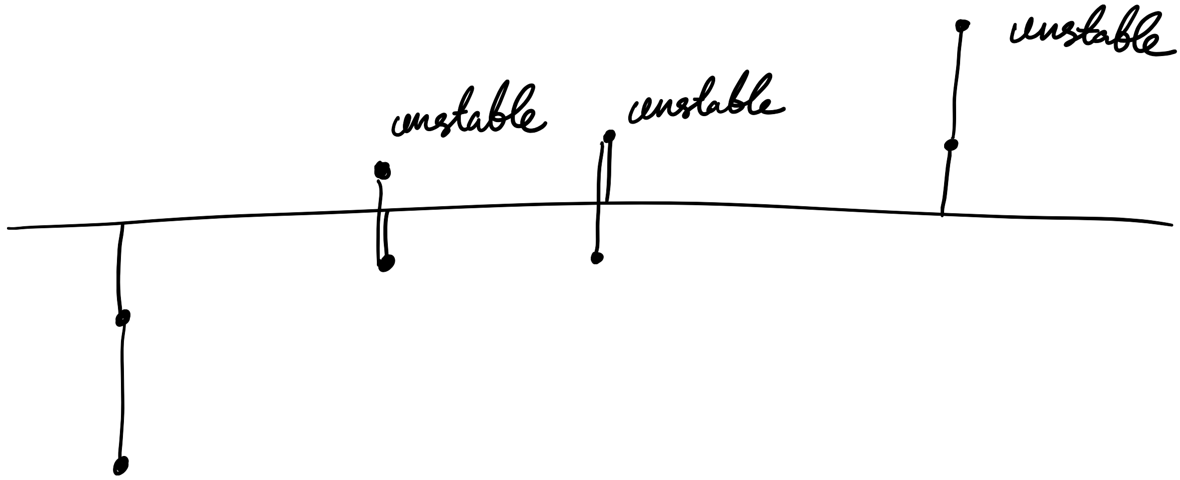
$$\begin{aligned} V &= m_1 g y_1 + m_2 g y_2 \\ &= -m_1 g l \cos\theta_1 + m_2 g(-l_1\cos\theta_1 - l_2\cos\theta_2) \\ &= -(m_1+m_2)g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \end{aligned}$$

Next,
$$-Q_1 = \frac{\partial V}{\partial \theta_1} = (m_1+m_2)g l_1 \sin\theta_1$$
$$-Q_2 = \frac{\partial V}{\partial \theta_2} = m_2 g l_2 \cos\theta_2$$

The system is in equilibrium when

$$(\theta_1, \theta_2) = (0,0), \quad (\theta_1, \theta_2) = (0, \pi), \quad (\theta_1, \theta_2) = (\pi, 0),$$
$$(\theta_1, \theta_2) = (\pi, \pi)$$

Intuitively,



Mathematically

$$\frac{\partial^2 V}{\partial \theta_1^2} = (m_1 + m_2) g l_1 \cos\theta_1 \qquad \frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} = \frac{\partial^2 V}{\partial \theta_2 \partial \theta_1} = 0$$

$$\frac{\partial^2 V}{\partial \theta_2^2} = m_2 g l_2 \cos\theta_2$$

Then,

$$\left(\frac{\partial^2 V}{\partial \theta_i \partial \theta_j} \right) = \begin{pmatrix} (m_1 + m_2) g l_1 \cos\theta_1 & 0 \\ 0 & m_2 g l_2 \cos\theta_2 \end{pmatrix}$$

This matrix has positive eigenvalues when $\theta_1 = \theta_2 = 0$, which is the only stable equilibrium solution.

$$T = \frac{1}{2}\sum_{j=1}^{n} m_j\left(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2\right)$$

2.2 Lagrangian mechanics

Recall the definition of a generalised force Eq. (4)

$$Q_i = \sum_{j=1}^{n} \mathbf{F}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_i} = \sum_{j=1}^{n} m_j \ddot{\mathbf{r}}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_i}.$$

In dynamics, the kinetic energy T of a system is a function of $q_i, \dot{q}_i$ and possibly t: $T = T(q_i, \dot{q}_i, t)$. Here we show that the generalised force is also related to the kinetic energy of the system.

Exercise 23. *Show that the generalised force is related to the kinetic energy of a system with generalised coordinates q_i, $1 \le i \le m$ by:*

$$Q_i = \frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_i}\right\} - \frac{\partial T}{\partial q_i}.$$

Solution 23. Suppose that the "k^{th}" particle has position vector $\underline{r}_k(q_1, q_2, \cdots, q_m, t)$ so that

$\dot{\underline{r}}_k(q_1, \cdots, q_m, \dot{q}_1, \cdots, \dot{q}_m, t)$. Next,

$$\dot{\underline{r}}_k = \sum_{j=1}^{m} \frac{\partial \underline{r}_k}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial \underline{r}_k}{\partial t} = \sum_{j=1}^{m} \frac{\partial \underline{r}_k}{\partial q_j}\dot{q}_j + \frac{\partial \underline{r}_k}{\partial t}$$

so
$$\frac{\partial \dot{\underline{r}}_k}{\partial \dot{q}_j} = \frac{\partial \underline{r}_k}{\partial q_j} \qquad (\nabla)$$

$$\frac{\partial}{\partial q_i}\left(\frac{d\underline{r}_k}{dt}\right) = \frac{\partial \dot{\underline{r}}_k}{\partial q_i} = \sum_{j=1}^{m} \frac{\partial^2 \underline{r}_k}{\partial q_j \partial q_i}\dot{q}_j + \frac{\partial^2 \underline{r}_k}{\partial q_i \partial t} \qquad (*)$$

Now consider

$$\frac{d}{dt}\left(\frac{\partial \underline{r}_k}{\partial q_i}\right) = \sum_{j=1}^{m} \frac{\partial^2 \underline{r}_k}{\partial q_i \partial q_j}\dot{q}_j + \frac{\partial^2 \underline{r}_k}{\partial t \partial q_i} \qquad (\Delta)$$

and compare $(*)$ and (Δ), which leads to

$$\frac{\partial \dot{\underline{r}}_k}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \underline{r}_k}{\partial q_i}\right) \qquad (\star)$$

Now,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\sum_{j=1}^{n} m_j \dot{\underline{r}}_j \cdot \dot{\underline{r}}_j\right) - \frac{1}{2}\frac{\partial}{\partial q_i}\left(\sum_{j=1}^{n} m_j \dot{\underline{r}}_j \cdot \dot{\underline{r}}_j\right)$$

$$= \frac{d}{dt}\left(\sum_{j=1}^{n} m_j \dot{\underline{r}}_j \cdot \frac{\partial \dot{\underline{r}}_j}{\partial \dot{q}_i}\right) - \sum_{j=1}^{n} m_j \dot{\underline{r}}_j \cdot \frac{\partial \dot{\underline{r}}_j}{\partial q_i}$$

$$= \frac{d}{dt}\left(\sum_{j=1}^{n} m_j \dot{\underline{r}}_j \cdot \frac{\partial \underline{r}_j}{\partial q_i}\right) - \sum_{j=1}^{n} m_j \dot{\underline{r}}_j \cdot \frac{\partial \dot{\underline{r}}_j}{\partial q_i} \qquad (\text{from } \nabla)$$

$$= \sum_{j=1}^{n}\left(m_j \ddot{\underline{r}}_j \cdot \frac{\partial \underline{r}_j}{\partial q_i} + m_j \dot{\underline{r}}_j \cdot \frac{d}{dt}\left(\frac{\partial \underline{r}_j}{\partial q_i}\right)\right) - \sum_{j=1}^{n} m_j \dot{\underline{r}}_j \cdot \frac{\partial \dot{\underline{r}}_j}{\partial q_i}$$

$$= \sum_{j=1}^{n} m_j \ddot{\underline{r}}_j \cdot \frac{\partial \underline{r}_j}{\partial q_i} + 0 \qquad (\text{from } \star)$$

$$= Q_i$$

2.2.1 Lagrange's equations

Given a conservative system with generalised coordinates $q_i, 1 \leq i \leq m$, kinetic energy $T = T(q_i, \dot{q}_i, t)$ and potential energy $V = V(q_i, t)$, we can write

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad 1 \leq i \leq m \quad \text{(Lagrange's Equations)}$$

where $L = T - V$. Here, L is known as the **Lagrangian** of the system.

Proof: We know from the above that

$$Q_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}.$$

As the system is conservative, we also know

$$Q_i = -\frac{\partial V}{\partial q_i}.$$

Moreover, note that the potential energy V is independent of $\dot{q}_i$, so

$$\frac{\partial V}{\partial \dot{q}_i} = 0.$$

Combining the first two equations we get:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i}.$$

Rearranging, we find

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial V}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} &= 0 \\
\frac{d}{dt}\left(\frac{\partial (T - V)}{\partial \dot{q}_i}\right) - \frac{\partial (T - V)}{\partial q_i} &= 0.
\end{aligned}$$

Then defining $T - V$ as the Lagrangian L, we have the desired result.

Compare with Exercise 9, page 15

Exercise 24. *Determine the equations of motion for the simple pendulum.*

Solution 24.

$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$ (see p15), $V = mgy = -mgl\cos\theta$

$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

Lagrange's equation:

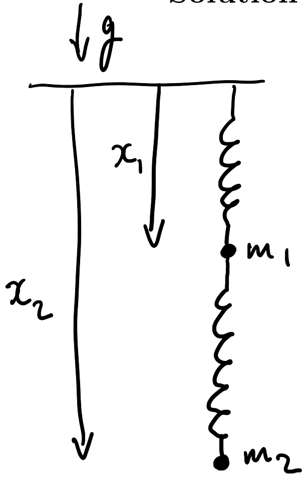
$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(ml^2\dot{\theta}\right) - (-mgl\sin\theta)$$

$$= ml^2\ddot{\theta} + mgl\sin\theta$$

$$\Rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Exercise 25. *Determine the equations of motion for a double spring system.*

Solution 25.



Let l_1 and l_2 denote the natural lengths of the springs, with constants k_1 and k_2 respectively.

$$L = T - V$$

$$= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + m_1gx_1 + m_2gx_2$$

$$- \frac{1}{2}k_1(x_1 - l_1)^2 - \frac{1}{2}k_2(x_2 - x_1 - l_2)^2$$

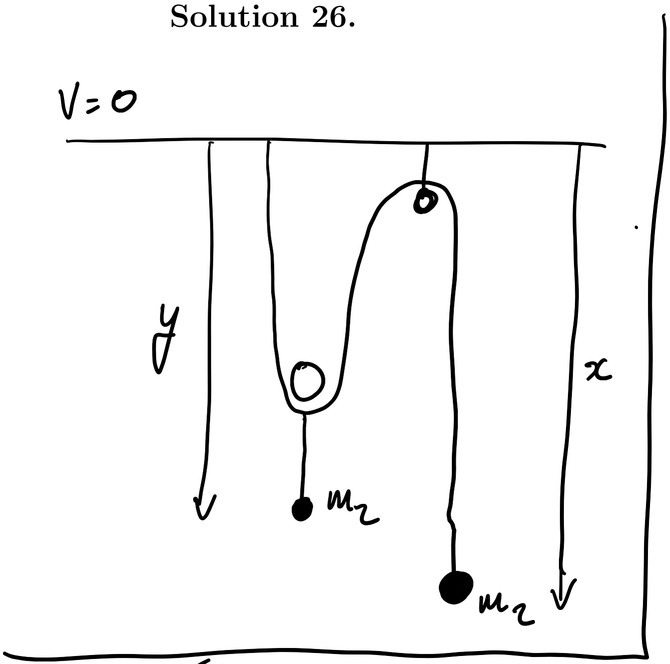
$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = m_1\ddot{x}_1 - m_1g + k_1(x_1 - l_1) - k_2(x_2 - x_1 - l_2)$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = m_2\ddot{x}_2 - m_2g + k_2(x_2 - x_1 - l_2)$$

This is a linear system $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \ddot{\underline{x}} = A\underline{x} + \underline{c}$

Exercise 26. *Determine the equations of motion for a double pulley system.*

Solution 26.



$$\mathcal{L} = T - V$$
$$= \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} m_2 \dot{x}^2 + m_1 g y + m_2 g x$$

Since the length of the rope is fixed, we have the constraint

$2y + x = l$ where l is the length of the rope.

Then, $y = \frac{1}{2}(l - x) \Rightarrow \dot{y} = -\frac{1}{2}\dot{x}$, so

$$\mathcal{L} = \frac{1}{2} m_1 \left(\frac{1}{4} \dot{x}^2\right) + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_1 g (l - x) + m_2 g x$$
$$= \frac{1}{2}\left(\frac{1}{4} m_1 + m_2\right) \dot{x}^2 + g\left(m_2 - \frac{1}{2} m_1\right) x + \frac{1}{2} m_1 g l$$

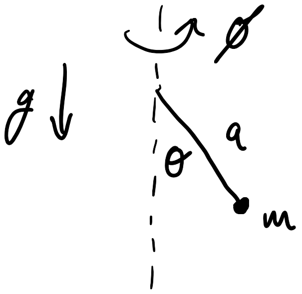
Lagranges equation is then

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \left(\frac{1}{4} m_1 + m_2\right) \ddot{x} - g\left(m_2 - \frac{1}{2} m_1\right)$$

$$\Rightarrow \quad \ddot{x} = g \frac{\left(m_2 - \frac{1}{2} m_1\right)}{\left(\frac{1}{4} m_1 + m_2\right)} = 2g \frac{(2m_2 - m_1)}{(m_1 + 4m_2)}$$

Exercise 27. *Determine the equations of motion for the spherical pendulum (equivalent to a bead on a freely rotating frictionless circular wire).*

Solution 27. Letting a denote the length of the rod, we have



$$\underline{r} = a(\cos\phi\sin\theta\,\underline{i} + \sin\phi\sin\theta\,\underline{j} - \cos\theta\,\underline{k})$$

$$\Rightarrow \dot{\underline{r}} = a\big(\dot\phi(-\sin\phi\sin\theta\,\underline{i} + \cos\phi\sin\theta\,\underline{j})$$
$$+\dot\theta(\cos\phi\cos\theta\,\underline{i} + \sin\phi\cos\theta\,\underline{j} + \sin\theta\,\underline{k})\big)$$

$\Rightarrow \dot{\underline{r}}\cdot\dot{\underline{r}} = a^2(\dot\theta^2 + \dot\phi^2\sin^2\theta)$, and then

$$T = \tfrac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}} = \tfrac{1}{2}ma^2(\dot\theta^2 + \dot\phi^2\sin^2\theta), \quad V = mga(1-\cos\theta)$$

Set $L = T - V$ and compute Lagrange's equations:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right) - \frac{\partial L}{\partial\phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right) \Rightarrow \frac{\partial L}{\partial\dot\phi} \text{ is constant}$$

i.e. $\dot\phi\sin^2\theta$ is constant (Note: this is an analogue of angular momentum)

The other equation is

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = ma^2\ddot\theta - ma^2\dot\phi^2\sin\theta\cos\theta + mga\sin\theta$$

or

$$\ddot\theta - \frac{c^2\cos\theta}{\sin^3\theta} + \frac{g\sin\theta}{a} = 0$$

where $c = \dot\phi\sin^2\theta$ (constant)

2.2.2 Generalised momenta and conservation

In the case of n free particles with cartesian coordinates (x_j, y_j, z_j), $\quad 1 \le j \le n$,

$$L = T = \sum_{j=1}^{n} \frac{1}{2} m_j (\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2).$$

We have $\dfrac{\partial L}{\partial \dot{x}_j} = m_j \dot{x}_j = x-$component of the momentum $\mathbf{p}_j$ of particle i. (Similarly for the y, z components).

In general for a system with constraints and generalised coordinates $q_i, i = 1, ..., m$, we define the **generalised momentum** associated with q_i by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, ..., m.$$

However p_i does not usually have dimensions of linear momentum.

If L depends on $\dot{q}_i$ but not on q_i then

$$\frac{\partial L}{\partial q_i} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0,$$

which means

$$\frac{dp_i}{dt} = 0 \quad \Rightarrow \quad p_i = \text{const.}$$

Several of the generalised momenta p_j may be conserved in a problem. The relevant q_i are called **cyclic** or **ignorable** coordinates.

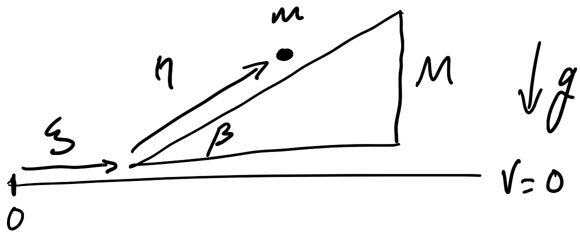
Exercise 28. *A particle of mass m slides down the smooth face of a wedge of mass M which itself is free to slide on a smooth horizontal plane. Suppose the angle of inclination of the wedge is β and at $t = 0$ the wedge is at rest and the mass is at rest a distance b along the face of the wedge. Determine the time taken for the mass to reach the bottom of the wedge.*

Solution 28.

The position vector for 'm'

$\underline{r} = (\xi + \eta\cos\beta)\underline{i} + \eta\sin\beta\,\underline{j}$

$\Rightarrow \dot{\underline{r}} = (\dot{\xi} + \dot{\eta}\cos\beta)\underline{i} + \dot{\eta}\sin\beta\,\underline{j}$



Position vector for 'M'

$\underline{R} = \xi\underline{i} \quad \dot{\underline{R}} = \dot{\xi}\underline{i}$

$$T = \tfrac{1}{2} m\, \dot{\underline{r}}\cdot\dot{\underline{r}} + \tfrac{1}{2} M \dot{\underline{R}}\cdot\dot{\underline{R}} = \tfrac{1}{2} m\left(\dot{\xi}^2 + 2\dot{\xi}\dot{\eta}\cos\beta + \dot{\eta}^2\right) + \tfrac{1}{2} M \dot{\xi}^2$$

Since frictionless, take potential due to gravity:

$V = mg\,\eta\sin\beta$ and so

$$L = T - V = \tfrac{1}{2} m\left(\dot{\xi}^2 + 2\dot{\xi}\dot{\eta}\cos\beta + \dot{\eta}^2\right) + \tfrac{1}{2} M\dot{\xi}^2 - mg\,\eta\cos\beta$$

We see that $p_\xi = \dfrac{\partial L}{\partial \dot{\xi}} = (m+M)\dot{\xi} + m\dot{\eta}\cos\beta$ is constant

Since $\dot{\eta} = \dot{\xi} = 0$ at $t=0$, we find $\dot{\xi} = -\left(\dfrac{m\cos\beta}{m+M}\right)\dot{\eta}$

The other generalised momentum is

$$p_\eta = \frac{\partial L}{\partial \dot{\eta}} = m\dot{\eta} + m\dot{\xi}\cos\beta$$
$$= \frac{m(M + m\sin^2\beta)}{m+M}\,\dot{\eta}$$

Lagrange's equation $\dfrac{dp_\eta}{dt} = \dfrac{\partial L}{\partial \eta} = -mg\sin\beta$

$$\Rightarrow \quad \ddot{\eta} = \frac{-(M+m)g\sin\beta}{M + m\sin^2\beta}$$

At $t=0$, $\eta = b$ and $\dot{\eta} = 0$ leading to

$$\eta = b - \tfrac{1}{2}\left(\frac{(M+m)g\sin\beta}{M + m\sin^2\beta}\right)t^2$$

When the mass reaches the ~~bottom~~ of the wedge, $\eta = 0$

$$\Rightarrow \quad t_f = \sqrt{\frac{2b(M + m\sin^2\beta)}{(M+m)g\sin\beta}}$$

2.2.3 Steady-state solutions and stability

Often a system will have a steady-state solution where one or more variables remain constant. Assume a system has a steady-state solution $q_i = x$. Then we can learn about the behaviour of the system by solving the equations of motion for $q_i = x + \delta x$ where δx is a *small* perturbation. If the steady state is stable, we often use this method to find the frequency of oscillation about the steady-state solution.

In order to understand the nature of the solutions we usually make a *first-order Taylor approximation.* i.e.

$$f(x + \delta x) \approx f(x) + f'(x)\delta x.$$

This is valid provided δx is sufficiently small. Note that this is equivalent to stating

$$f'(x) \approx \frac{f(x + \delta x) - f(x)}{\delta x}$$

for small δx.

Exercise 29. *For the spherical pendulum show that circular motion with a fixed angle of inclination is possible. Investigate the stability of this solution.*

Solution 29. From Ex 27,

$$L = T - V = \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\sin^2\theta\,\dot{\phi}^2) - mga(1-\cos\theta)$$

Since L is not explicitly dependent on ϕ

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ma^2\sin^2\theta\,\dot{\phi} \quad \text{is constant} \qquad (*)$$

From Lagrange's equation $0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$, we have

$$\ddot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2 - \frac{g}{a}\sin\theta \qquad (\triangle)$$

A solution for $\theta = \alpha$ (constant) is given by

$$0 = \sin\theta\cos\theta\,\dot{\phi}^2 - \frac{g}{a}\sin\theta$$

$$\Rightarrow \dot{\phi}^2 = \frac{g}{a\cos\alpha}$$

The mass moves uniformly on a circular path with angular velocity $\omega = \dot{\phi} = \sqrt{\dfrac{g}{a\cos\alpha}}$

imposing the restriction $0 \leq \alpha < \frac{\pi}{2}$

<u>Stability</u>: Consider a small perturbation $\theta = \alpha + \delta$

From $(*)$, $\quad ma^2 \sin^2(\alpha+\delta)\dot{\phi} = ma^2(\sin^2\alpha)\,\omega$

yielding, $\quad \dot{\phi} = \dfrac{\omega \sin^2\alpha}{\sin^2(\alpha+\delta)}$

From (Δ), $\quad \ddot{\delta} = \dfrac{\omega^2 \sin^4\alpha}{\sin^3(\alpha+\delta)}\cos(\alpha+\delta) - \frac{g}{a}\sin(\alpha+\delta)$

For small δ,
$$\sin(\alpha+\delta) \simeq \sin\alpha + \delta\cos\alpha$$
$$\cos(\alpha+\delta) \simeq \cos\alpha - \delta\sin\alpha$$
$$\frac{1}{\sin^3(\alpha+\delta)} \simeq \frac{\sin\alpha - 3\delta\cos\alpha}{\sin^4\alpha}$$

This leads to $\quad \delta \simeq -\omega^2(1+3\cos^2\alpha)\,\delta \qquad (\square)$

The perturbed motion is harmonic with frequency
$$p = \frac{\omega}{2\pi}\sqrt{1+3\cos^2\alpha}$$

i.e. the general solution of $(\square)$ is

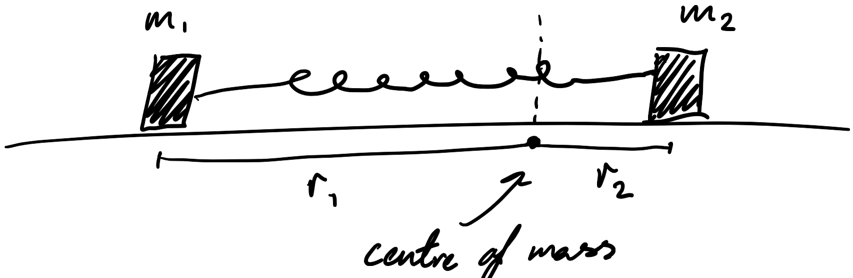
$$\delta = A\cos(pt) + B\sin(pt)$$

Recall $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$

Exercise 30. *Two masses m_1 and m_2 are connected by a (massless) spring of stiffness k. The system is set rotating about the centre of mass with an angular velocity ω and then released. If the masses are slightly disturbed along the line joining them, show that the frequency of oscillation is*

$$\frac{1}{2\pi}\sqrt{\frac{3\omega^2 m_1 m_2 + k(m_1+m_2)}{m_1 m_2}}.$$

Solution 30.



Since the system is rotating about the centre of mass, we have $m_1 r_1 = m_2 r_2$. Set $r_2 = \frac{m_1}{m_2} r_1$ Then, KE

$$T = \frac{1}{2} m_1 (\dot{r}_1^2 + r_1^2 \dot{\theta}_1^2) + \frac{1}{2} m_2 (\dot{r}_2^2 + r_2^2 \dot{\theta}_2^2), \text{ with } \dot{\theta}_1 = \dot{\theta}_2$$

$$= \frac{1}{2}\left(1 + \frac{m_1}{m_2}\right)(\dot{r}_1^2 + r_1^2 \dot{\theta}_1^2) m_1$$

Potential $V = \frac{1}{2} k (r_1 + r_2 - l)^2 = \frac{k}{2}\left(r_1\left(1 + \frac{m_1}{m_2}\right) - l\right)^2$

Lagrangian $L = T - V = \frac{1}{2}\left(1 + \frac{m_1}{m_2}\right)(\dot{r}_1^2 + r_1^2 \dot{\theta}_1^2) m_1 - \frac{k}{2}\left(r_1\left(1 + \frac{m_1}{m_2}\right) - l\right)^2$

Since L does not depend on θ_1,

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = m_1\left(1 + \frac{m_1}{m_2}\right) r_1^2 \dot{\theta}_1 \text{ is constant}$$

Set $r_1^2 \dot{\theta}_1 = c$ where c is a constant

Lagrange's equation for r_1:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_1}\right) - \frac{\partial L}{\partial r_1}$$

$$\Rightarrow \quad \ddot{r}_1 = \dot{\theta}_1^2 r_1 - \frac{k}{m_1}\left(r_1\left(1+\frac{m_1}{m_2}\right) - l\right) \qquad (\Delta)$$

Setting $\ddot{r}_1 = 0$ we obtain the steady state solution:

$$r_1 = \frac{kl}{k\left(1+\frac{m_1}{m_2}\right) - m_1\dot{\theta}_1^2} \qquad \text{where } \dot{\theta}_1 = \frac{c}{r_1^2}$$

Consider a steady state where $r_1 = d$, $\dot{\theta}_1 = \omega$ and perturbation $r_1 = d + \delta$, $\dot{\theta}_1 = \frac{d^2\omega}{r_1^2}$

Using $\frac{1}{r_1^3} = \frac{1}{(d+\delta)^3} \simeq \frac{1}{d^3} - \frac{3\delta}{d^4}$, we obtain from (Δ):

$$\ddot{\delta} \simeq -\left(\frac{3\omega^2 m_1 m_2 + k(m_1+m_2)}{m_1 m_2}\right)\delta$$

This corresponds to oscillations about the steady state with frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{3\omega^2 m_1 m_2 + k(m_1+m_2)}{m_1 m_2}}$$

2.2.4 Lagrange's equation in non-conservative systems

We derived Lagrange's equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

for a conservative system, where $L = T - V$. However the method extends to system with non-conservative forces. Let $\mathbf{F}_k^{nc}$ be a non-conservative force (e.g. a driving force) acting on a particle with associated generalised forces Q_i^{nc} defined in the usual way (see Eq. (4)). Then the equations of motion will be the following:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i^{nc}.$$

Here $L = T - V$ where the driving force $\mathbf{F}^{nc}$ contributes to the kinetic energy but not the potential energy. If a system has non-conservative forces, in general $E = T + V$ will NOT be constant.

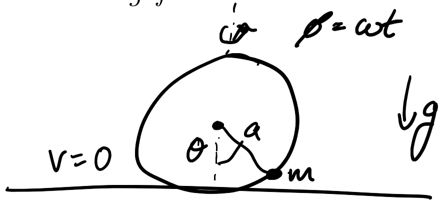
Note that if the driving force satisfies $\mathbf{F}_k^{nc} \cdot \dfrac{\partial \mathbf{r}_k}{\partial q_i} = 0$ for all $k = 1, ..., n$ and $i = 1, ..., m$ then $Q_i^{nc} = 0$ for all i, and hence

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

as before. In this case the driving force still has an effect, as it contributes to the Lagrangian.

Exercise 31. *Determine the equations of motion for a bead on a rotating frictionless wire, where the wire is rotating with constant angular velocity ω.*

Solution 31.



The position vector is

$$\underline{r} = a\sin\theta\left(\cos(\omega t)\underline{i} + \sin(\omega t)\underline{j}\right) - a\cos\theta\, \underline{k}$$

$$\frac{d\underline{r}}{d\theta} = a\cos\theta\left(\cos(\omega t)\underline{i} + \sin(\omega t)\underline{j}\right) + a\sin\theta\, \underline{k}$$

$$Q = \underline{F} \cdot \frac{d\underline{r}}{d\theta} = \left[-mg\underline{k} + \underline{F}^{nc}\right] \cdot \left[a\cos\theta\left(\cos(\omega t)\underline{i} + \sin(\omega t)\underline{j}\right) + a\sin\theta\, \underline{k}\right]$$

$$= -mga\sin\theta$$

The potential is $\quad V = mga(1-\cos\theta)$

The kinetic energy is (compare with Ex 27)

$$T = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m\left((a\dot{\theta})^2 + \omega^2 a^2 \sin^2\theta\right)$$

$$\Rightarrow L = T - V = \tfrac{1}{2} m\left((a\dot{\theta})^2 + \omega^2 a^2 \sin^2\theta\right) - mga(1-\cos\theta)$$

The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

leading to $\quad \ddot{\theta} - \sin\theta\left(\omega^2\cos\theta - \frac{g}{a}\right) = 0 \qquad (*)$

Exercise: check $E = T + V$ is not conserved by calculating $\frac{dE}{dt}$ and showing that $\frac{dE}{dt} \neq 0$ using $(*)$

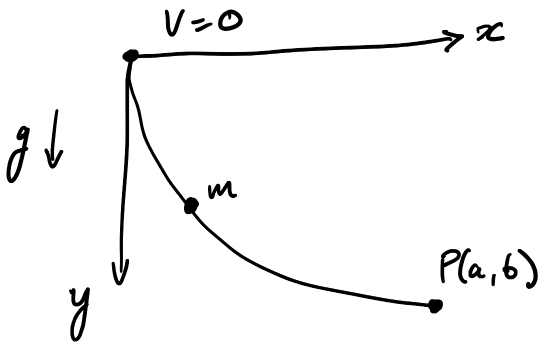
2.3 Calculus of variations

2.3.1 The Brachistochrone problem

Jacob Bernoulli solved this long-standing problem in 1696 by what turned out to be an application of the *calculus of variations.*

Exercise 32. *A particle starting at rest moves along a frictionless wire under its own weight from the origin to point $P(a, b)$. What is the path $y = y(x)$ for which the time taken is least?*

Solution 32.



This is conservative

$$E = T + V$$

$$= \frac{1}{2}mv^2 - mgy \qquad (*)$$

Initially $y = V = 0$, so $E = 0$ $\quad (**)$

We then, from (*) and (**), obtain

$$v^2 = 2gy$$

From

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2\left(\frac{dx}{dt}\right)^2}$$

$$= \frac{dx}{dt}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Then,

$$\frac{dx}{dt}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{2gy}$$

$$\Rightarrow \quad \frac{dt}{dx} = \sqrt{\frac{1 + (y')^2}{2gy}}$$

and so,

$$t = \int_0^a \sqrt{\frac{1 + (y')^2}{2gy}}\, dx$$

where $y = f(x)$

2.3.2 Calculus of variations: general approach

Given a function $F(y, y', x)$ where $y = y(x)$ is unspecified, then we consider

$$I(y) = \int_{x_1}^{x_2} F(y, y', x)dx,$$

called the **functional**. If $y(x)$ is subject to some boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$, show that $I(y)$ has a maximum or minimum value only if F satisfies the **Euler-Lagrange Equation**:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

Proof

Suppose y satisfies the given boundary conditions and consider small variations in y:

$$y \to y + \delta y = y + \epsilon\eta(x)$$

where ϵ is a small parameter independent of x and $\eta(x)$ is an arbitrary function such that $\eta(x_1) = \eta(x_2) = 0$. (This is necessary so the boundary conditions are satisfied.) Then $\delta y = \epsilon\eta(x)$ is the variation in y.

We wish to find $y(x)$ such that $I(y+\epsilon\eta(x))$ takes a maximum or minimum value at $\epsilon = 0$, for any choice of $\eta(x)$. A necessary condition for this is that

$$\frac{d}{d\epsilon} I(y + \epsilon\eta(x))\bigg|_{\epsilon=0} = 0$$

for all $\eta(x)$ satisfying $\eta(x_1) = \eta(x_2) = 0$. Then at $\epsilon = 0$ we must have

$$\begin{aligned} 0 &= \frac{d}{d\epsilon} I(y + \epsilon\eta(x))\bigg|_{\epsilon=0} \\ &= \int_{x_1}^{x_2} \frac{d}{d\epsilon} F(y + \epsilon\eta(x), y' + \epsilon\eta'(x), x)\bigg|_{\epsilon=0} dx. \end{aligned}$$

Consider the substitutions

$$\begin{aligned} u = y + \epsilon\eta(x) \qquad &\Rightarrow \qquad \frac{du}{d\epsilon} = \eta(x) \\ v = y' + \epsilon\eta'(x) \qquad &\Rightarrow \qquad \frac{dv}{d\epsilon} = \eta'(x). \end{aligned}$$

But by the chain rule

$$\begin{aligned} \frac{\partial F}{\partial u} &= \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} = \frac{\partial F}{\partial y}, \\ \frac{\partial F}{\partial v} &= \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial v} = \frac{\partial F}{\partial y'}. \end{aligned}$$

Hence

$$\begin{aligned}\left.\frac{d}{d\epsilon}F(u,v,x)\right|_{\epsilon=0} &= \left.\left(\frac{\partial F}{\partial u}\frac{du}{d\epsilon}+\frac{\partial F}{\partial v}\frac{dv}{d\epsilon}\right)\right|_{\epsilon=0}\\ &= \frac{\partial F(y,y',x)}{\partial y}\eta(x)+\frac{\partial F(y,y',x)}{\partial y'}\eta'(x)\end{aligned}$$

Substituting this into our integral, we find

$$0 = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\eta+\frac{\partial F}{\partial y'}\eta'\right)dx.$$

Now by the product rule

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\eta\right)=\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta+\frac{\partial F}{\partial y'}\eta',$$

so

$$\begin{aligned}0 &= \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\eta-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta\right)dx+\int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\eta\right)dx\\ &= \int_{x_1}^{x_2}\eta\left(\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)dx+\underbrace{\left[\eta\frac{\partial F}{\partial y'}\right]_{x_1}^{x_2}}_{0}.\end{aligned}$$

The last term equals zero because of the boundary conditions for η.

Hence we have:

$$\int_{x_1}^{x_2}\eta\left(\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)dx=0\quad\forall\eta(x).$$

As this holds for all $\eta(x)$, we must have

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0,$$

which is the Euler-Lagrange equation.

We can extend the result for systems with $F(y_1,\ldots,y_n,y_1',\ldots,y_n',x)$, where $y_j(x)$, $(1\leq j\leq n)$ are n independent functions of x whose values at x_1 and x_2 are specified and we have the functional:

$$I(y_1,\ldots,y_n)=\int_{x_1}^{x_2}F(y_1,\ldots,y_n,y_1',\ldots,y_n',x)dx.$$

The functional will have a minimum or maximum only if:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right)-\frac{\partial F}{\partial y_i}=0\qquad\forall i=1\ldots n$$

Exercise 33. *Consider the special case where F depends only on y,y', and is independent of x, that is, $F=F(y,y')$. Show the Euler-Lagrange equations give*

$$\frac{d}{dx}\left[F-y'\frac{\partial F}{\partial y'}\right]=0.$$

Solution 33.

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right)$$

$$= \frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)$$

$$= \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}y'' - \left(y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)$$

$$= y'\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)$$

$$= 0 \qquad \text{since} \quad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

2.3.3 The Brachistochrone problem solution

Exercise 34. *What is the path $y = y(x)$ for which the time taken is least?*

Solution 34.

Consider $t = \int_0^a F(y, y')\,dx$ where $F(y,y') = \sqrt{\dfrac{1+(y')^2}{2gy}}$

Since F is not explicitly dependent on x, from ex 33

$$F - y'\frac{\partial F}{\partial y'} = K \quad \text{where } K \text{ is a constant}$$

i.e. $K = \sqrt{\dfrac{1+(y')^2}{2gy}} - \sqrt{\dfrac{(y')^2}{2gy(1+y')^2}} = \dfrac{1}{\sqrt{2gy(1+(y')^2)}}$

which is equivalent to $y(1+(y')^2) = 2c$ where c is a constant

Next, $(y')^2 = \dfrac{2c}{y} - 1 \Rightarrow \dfrac{dy}{dx} = \sqrt{\dfrac{2c}{y} - 1}$.

To solve this, set $y = 2c\sin^2\theta$, so $dy = 4c\sin\theta\cos\theta\,d\theta$, and

$$dx = \sqrt{\frac{2c\sin^2\theta}{2c\cos^2\theta}} \cdot 4c\sin\theta\cos\theta\,d\theta$$

with solution $x = 2c(\theta - \frac{1}{2}\sin 2\theta)$ such that $x(0) = 0$

This gives the parametric solution

$$x = 2c(\theta - \tfrac{1}{2}\sin 2\theta)$$
$$y = c(1 - \cos 2\theta)$$

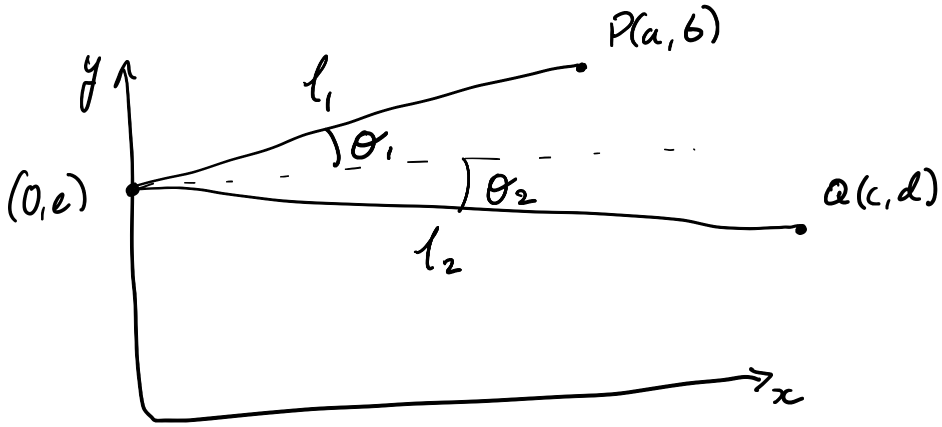
Set $2\theta = \phi \Rightarrow$
$$x = c(\phi - \sin\phi)$$
$$y = c(1 - \cos\phi)$$

The point at the origin is given by $\phi = 0$, while c and domain $[0, \phi_f]$ is a solution of $a = c(\phi_f - \sin\phi_f)$, $b = c(1 - \cos\phi_f)$

2.3.4 Reflection of light

Exercise 35. *Consider a beam of light moving from point A to point B, via a reflection off a mirror. Find the path such that the time taken is a minimum.*

Solution 35.



The total distance travelled is

$$L = l_1 + l_2$$
$$= \sqrt{a^2+(b-e)^2} + \sqrt{(e-d)^2+c^2}$$

$$\frac{dL}{de} = \frac{-(b-e)}{\sqrt{a^2+(b-e)^2}} + \frac{e-d}{\sqrt{(e-d)^2+c^2}}$$

$$= -\frac{(b-e)}{l_1} + \frac{e-d}{l_2}$$

$$= -\sin\theta_1 + \sin\theta_2$$

L is minimised when $\theta_1 = \theta_2$

2.3.5 Hamilton's principle of least action

When we first encounter dynamics, we often consider energy to be a fundamental concept. However, it turns out to be a *more* fundamental concept: action. The action of a system is related to how the system follows a particular trajectory.

Consider a conservative system with generalised coordinates $q_1, ..., q_m$ and Lagrangian $L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - V(q_i, t)$. According to Hamilton's principle, the observed motion of the system from time t_1 to t_2 is given by that trajectory which minimises the **action**:

$$A = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt.$$

Since the generalised coordinates q_i are independent, the Euler-Lagrange equations state that a necessary condition for a minimum is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, ..., n.$$

Reading example: Projectile motion.
Consider a particle launched from the origin with a particular velocity and angle. From this angle and velocity, we can work out our horizontal and vertical initial velocities. Now the question is, what path does it travel? The Lagrangian is given by:

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy.$$

Now the Principle of Least Action states that the system will evolve according to the trajectory that minimises the action functional A. Therefore the Euler-Lagrange equations must hold for both generalised coordinates. So

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \\ \frac{d}{dt}(m\dot{x}) - 0 &= 0 \\ \ddot{x} &= 0. \end{aligned}$$

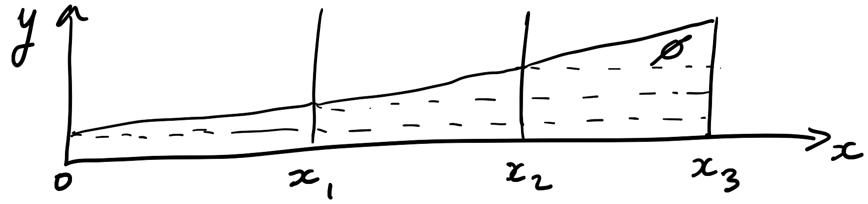
This is what we expect as there are no horizontal forces. Now let's look at the Euler-Lagrange equations for y:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= 0 \\ \frac{d}{dt}(m\dot{y}) + mg &= 0 \\ \ddot{y} &= -g. \end{aligned}$$

This is also consistent with our system (and experience). So what about our trajectory? It's easy to see (by integrating twice), that we get:

$$x = v_x t, \qquad y = y_0 + v_y t - \frac{1}{2}gt^2.$$

These two parametric equations will trace out a parabola, in accord with our experience of basic mechanics.



2.3.6 Snell's law

In elementary physics, you probably met **Snell's law**. It essentially tells us how much light bends when going from one refractive medium to another.

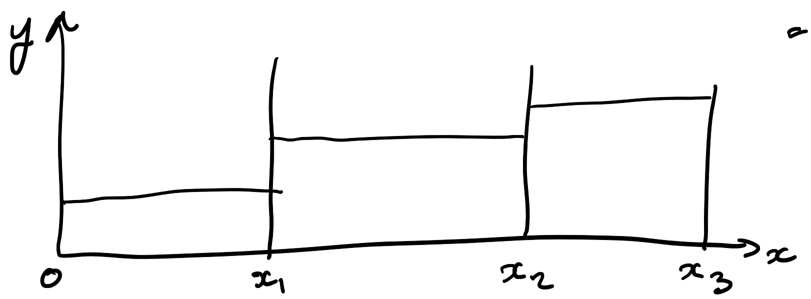
Exercise 36. *Consider a beam of light moving through a medium from the origin to a point (a, b). Find the path such that the time taken is a minimum.*

Solution 36.

Let $c(x)$ denote the speed of light

$$c = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{dx}{dt}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

e.g.



Start with $\frac{dt}{dx} = \frac{1}{c(x)}\sqrt{1+(y')^2} \equiv F(y', x)$, so that $t = \int F(y', x)\, dx$. For light to follow the path given the smallest time, the Euler-Lagrange equation

$$0 = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right), \quad \text{so}$$

$\frac{\partial F}{\partial y'} = K$, K is a constant such that $K = \frac{y'}{c(x)\sqrt{1+(y')^2}}$

Substitute $y' = \tan\phi(x)$ yielding $\frac{\sin\phi(x)}{c(x)} = K$

That
$$\frac{\sin\phi_i}{c_i} = \frac{\sin\phi_j}{c_j} \quad \text{or} \quad \frac{\sin\phi_i}{\sin\phi_j} = \frac{c_i}{c_j}$$

2.4 Hamiltonian mechanics

Consider a conservative system of n particles with generalised coordinates $q_1, \ldots, q_m$ and corresponding generalised momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad i = 1 \ldots m, \quad L = L(q_i, \dot{q}_i, t).$$

Then from Lagrange's equations

$$\begin{aligned} dL &= \sum_{i=1}^{m} \frac{\partial L}{\partial q_i} dq_i + \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt \\ &= \sum_{i=1}^{m} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) dq_i + \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt \\ &= \sum_{i=1}^{m} \dot{p}_i dq_i + \sum_{i=1}^{m} p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt. \end{aligned}$$

Let us set

$$H = \sum_{i=1}^{m} p_i \dot{q}_i - L,$$

called the **Hamiltonian** of the system. Then

$$\begin{aligned} dH &= \sum_{i=1}^{m} p_i d\dot{q}_i + \sum_{i=1}^{m} \dot{q}_i dp_i - \sum_{i=1}^{m} \dot{p}_i dq_i - \sum_{i=1}^{m} p_i d\dot{q}_i - \frac{\partial L}{\partial t} dt \\ &= -\sum_{i=1}^{m} \dot{p}_i dq_i + \sum_{i=1}^{m} \dot{q}_i dp_i - \frac{\partial L}{\partial t} dt. \end{aligned}$$

Thus changes in H depend only on changes in $p_i,\ q_i,\ t$ which suggests that H is a function of the generalised coordinates and their corresponding generalised momenta, and possibly t:

$$H = H(p_i, q_i, t).$$

$dH = \sum_j \frac{\partial H}{\partial q_i} dq_j + \sum_j \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t}$

The equation for dH then implies

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \frac{dH}{dt} = -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}.$$

These are known as **Hamilton's equations**, the last one arising from

$$\frac{dH}{dt} = -\sum_{i=1}^{m} \dot{p}_i \frac{dq_i}{dt} + \sum_{i=1}^{m} \dot{q}_i \frac{dp_i}{dt} - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}.$$

Note: The $2m$ first order Hamilton equations above *replace* the m second order Lagrange equations!

2.4.1 Physical significance of the Hamiltonian

In the case L and thus H has no explicit dependence on t, we have from Hamilton's equations

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0$$

so H is constant.

Exercise 37. *If the position functions* $\mathbf{r}_j$ *are functions of the generalised coordinates* q_i *only, i.e.* $\mathbf{r}_j = \mathbf{r}_j(q_i)$*, and* $V = V(q_i, t)$*, show that the Hamiltonian* H *is the total energy* $T + V$ *of the system.*

Note: It is possible to have $\mathbf{r}_j = \mathbf{r}_j(q_i, t)$ but $H = H(p_i, q_i)$. In that case H is still conserved, but does not denote the total energy.

Solution 37.

$$\dot{\underline{r}}_j = \sum_{i=1}^{m} \frac{\partial \underline{r}_j}{\partial q_i} \dot{q}_i$$

Assuming $V = V(q_i, t)$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_{k=1}^{n} m_k \dot{\underline{r}}_k \cdot \frac{\partial \dot{\underline{r}}_k}{\partial \dot{q}_i} = \sum_{k=1}^{n} m_k \dot{\underline{r}}_k \cdot \frac{\partial \underline{r}_k}{\partial q_i}$$

$$T = \frac{1}{2} \sum_{k=1}^{n} m_k \dot{\underline{r}}_k \cdot \dot{\underline{r}}_k = \frac{1}{2} \sum_{i,k} m_k \dot{\underline{r}}_k \cdot \frac{\partial \underline{r}_k}{\partial q_i} \dot{q}_i = \frac{1}{2} \sum_i p_i \dot{q}_i$$

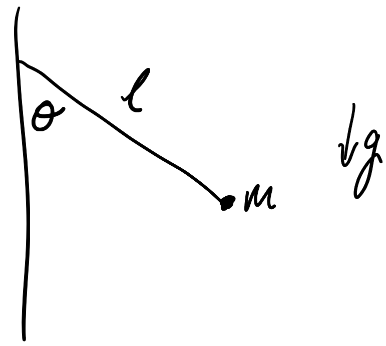
$$\Rightarrow \sum_i p_i \dot{q}_i = 2T$$

$$\Rightarrow H = T + V$$

$$H = \sum_i p_i \dot{q}_i - L = 2T - (T - V) = T + V$$

From previous calculations,

$$T = \frac{1}{2} m l^2 \dot{\theta}^2 \qquad V = -mgl\cos\theta$$



$$\Rightarrow L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

Now, $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$, or $\dot{\theta} = \frac{p_\theta}{m l^2}$

Exercise 38. *Determine the Hamiltonian for the simple pendulum.*

Solution 38.

Then,

$$\begin{aligned} H &= p_\theta \dot{\theta} - L \\ &= m l^2 \dot{\theta}^2 - \left(\frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta\right) \\ &= \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta = T + V \end{aligned}$$

as expected. Alternatively,

$$H = \frac{p_\theta^2}{2 m l^2} - mgl\cos\theta$$

The equations of motion are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m l^2} \quad (1) \quad \& \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta \quad (2)$$

From (1), we have

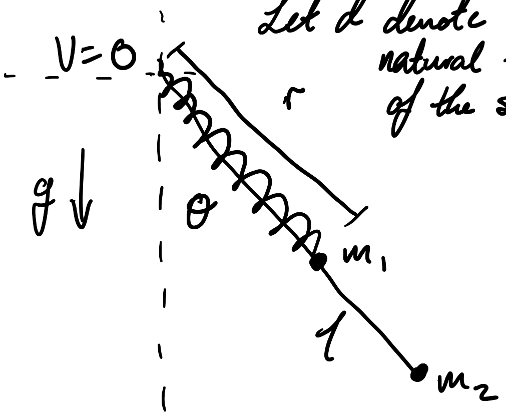
$$\ddot{\theta} = \frac{\dot{p}_\theta}{m l^2}$$

and then, using (2),

$$\ddot{\theta} = \frac{1}{m l^2}(-mgl\sin\theta) = -\frac{g}{l}\sin\theta$$

Exercise 39. *Determine the Hamiltonian for a system of two masses, one on the end of a light spring and the other on the end of a light rod. The rod runs through the spring and the mass on the end of the spring moves along the rod without friction.*

Solution 39.



Let d denote the natural length of the spring

$$T = \frac{1}{2} m_1 (r^2 \dot{\theta}^2 + \dot{r}^2) + \frac{1}{2} m_2 l^2 \dot{\theta}^2$$

$$V = -m_1 g r \cos\theta - m_2 g l \cos\theta + \frac{1}{2} k (r-d)^2$$

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m_1 (r^2 \dot{\theta}^2 + \dot{r}^2) + \frac{1}{2} m_2 l^2 \dot{\theta}^2 + m_1 g r \cos\theta \\ &\quad + m_2 g l \cos\theta - \frac{1}{2} k (r-d)^2 \end{aligned}$$

The conjugate momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = (m_1 r^2 + m_2 l^2) \dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{m_1 r^2 + m_2 l^2}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m_1 \dot{r} \quad \Rightarrow \quad \dot{r} = \frac{p_r}{m_1}$$

The Hamiltonian is defined as

$$\begin{aligned} H &= p_\theta \dot{\theta} + p_r \dot{r} - L \\ &= \frac{p_\theta^2}{2(m_1 r^2 + m_2 l^2)} + \frac{p_r^2}{2 m_1} + \frac{1}{2} k (r-d)^2 - g \cos\theta (m_1 r + m_2 l) \\ &= T + V \end{aligned}$$

2.4.2 Hamilton's equations for non-conservative systems

Recall that if a system is not conservative, the Lagrangian is defined as $L = T - V$ where V is the potential for the conservative forces, but Lagrange's equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i^{nc}$$

where Q_i^{nc} is the generalised force in the coordinate q_i associated with the non-conservative forces. In our examples, this has always been zero.

Similarly, the Hamiltonian for a non-conservative system is still defined by

$$H = \sum_{i=1}^{m} p_i \dot{q}_i - L$$

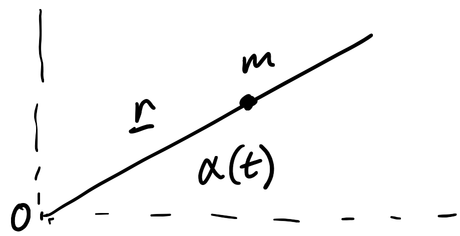
but Hamilton's equations are altered in the following way:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \qquad \frac{\partial H}{\partial q_i} = Q_i^{nc} - \dot{p}_i.$$

Hence if $Q_i^{nc} = 0$ Hamilton's equations will be unchanged.

Exercise 40. *Consider a bead on a light rod with driven rotation angle $\alpha = \alpha(t)$, on a frictionless table. Show that the Hamiltonian is not constant.*

Solution 40.



$$\underline{r} = r\cos\alpha(t)\,\underline{i} + r\sin\alpha(t)\,\underline{j}$$

$$\frac{\partial \underline{r}}{\partial r} = \cos\alpha(t)\,\underline{i} + \sin\alpha(t)\,\underline{j}$$

The generalised force is $Q = \underline{F} \cdot \frac{\partial \underline{r}}{\partial r} = 0$ (since there is no friction)

$$\dot{\underline{r}} = \dot{r}(\cos\alpha(t)\,\underline{i} + \sin\alpha(t)\,\underline{j}) + r\dot{\alpha}(-\sin\alpha(t)\,\underline{i} + \cos\alpha(t)\,\underline{j})$$

$$T = \tfrac{1}{2} m\, \dot{\underline{r}} \cdot \dot{\underline{r}} = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\alpha}^2) = L$$

Lagrange's equation $0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r}$, yielding $\ddot{r} = r\dot{\alpha}^2$

Conjugate momentum $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ and so

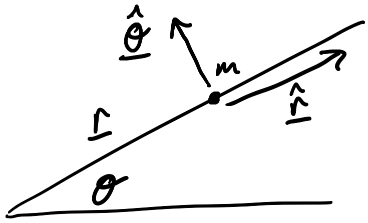
$$H = p_r \dot{r} - L = m\dot{r}^2 - \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\alpha}^2) = \tfrac{1}{2}m\dot{r}^2 - \tfrac{1}{2}mr^2\dot{\alpha}^2 \neq T$$

Using Hamilton's equations we can check:

$$\frac{dH}{dt} = (m\ddot{r} - mr\dot{\alpha}^2)\dot{r} - mr^2\dot{\alpha}\ddot{\alpha}$$
$$= -mr^2\dot{\alpha}\ddot{\alpha}$$

Exercise 41. *Consider the analogous system where the rod is not driven, but can freely rotate.*

Solution 41. Normal force has magnitude N



Here, $\underline{r} = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j}$

$$Q_\theta = \underline{F} \cdot \frac{\partial \underline{r}}{\partial \theta}$$
$$= r\underline{F} \cdot \hat{\underline{\theta}} = rN$$

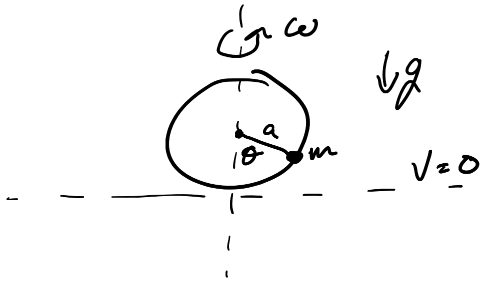
$$Q_r = \underline{F} \cdot \frac{\partial \underline{r}}{\partial r} = N\hat{\underline{\theta}} \cdot \hat{\underline{r}} = 0$$

A potential exists if $\frac{\partial Q_\theta}{\partial r} = \frac{\partial Q_r}{\partial \theta}$, but this is not the case.

For this example, the assumption that the rod has zero mass is the cause of the problems.

Exercise 42. *Show that the energy is not conserved for the problem of a bead on a wire rotating with constant angular velocity ω (Exercise 31).*

Solution 42.



From Ex 31, we have

$$L = \frac{1}{2}m(a^2\dot{\theta}^2 + \omega^2 a^2 \sin^2\theta) - mga(1-\cos\theta)$$

Then, $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}$

The Hamiltonian is

$$\begin{aligned} H = p_\theta\dot{\theta} - L &= ma^2\dot{\theta}^2 - \frac{1}{2}m(a^2\dot{\theta}^2 + \omega^2a^2\sin^2\theta) + mga(1-\cos\theta) \\ &= \frac{1}{2}ma^2(\theta^2 - \omega^2\sin^2\theta) + mga(1-\cos\theta) \\ &= \frac{p_\theta^2}{2ma^2} - \frac{1}{2}ma^2\omega^2\sin^2\theta + mga(1-\cos\theta) \end{aligned}$$

The total energy is

$$\begin{aligned} E &= T + V \\ &= \frac{p_\theta^2}{2ma^2} + \frac{1}{2}ma^2\omega^2\sin^2\theta + mga(1-\cos\theta) \\ &= H + ma^2\omega^2\sin^2\theta \end{aligned}$$

(Recall $\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$)

We see that

$$\begin{aligned} \frac{dE}{dt} &= \frac{dH}{dt} + 2ma^2\omega^2\cos\theta\sin\theta\,\dot{\theta} \\ &= 2ma^2\omega^2\cos\theta\sin\theta\,\dot{\theta} \\ &\neq 0 \quad \text{generally} \end{aligned}$$

2.4.3 Canonical transformations and Poisson brackets

Let us consider a system with one degree of freedom with generalised coordinates and momentum q, p and let $H(q,p)$ be the Hamiltonian. We recall Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{5}$$

Take a transformation to a new set of coordinates (Q,P) where $Q = Q(q,p)$ and $P = P(q,p)$. Then the Hamiltonian may be expressed in terms of $P,\ Q$

$$H'(Q,P) = H(q(Q,P), p(Q,P)) \qquad \Longleftrightarrow \qquad H(q,p) = H'(Q(q,p), P(q,p)).$$

Our aim here is to determine conditions on such a transformation so that Hamilton's equations hold for H' with Q and P,

$$\dot{Q} = \frac{\partial H'}{\partial P} \qquad \dot{P} = -\frac{\partial H'}{\partial Q}.$$

Such a transformation is called a **canonical transformation**. Now using Hamilton's equations (5) we have

$$\begin{aligned}
\dot{Q} &= \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} \\
&= \frac{\partial Q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H}{\partial q} \\
&= \{Q, H\}
\end{aligned}$$

where $\{\ \ ,\ \ \}$ is called the Poisson bracket.

Given functions $F(q,p),\ G(q,p)$ their **Poisson bracket** is defined by

$$\{F, G\} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q}.$$

Note that $\{F,G\} = -\{G,F\}$ (antisymmetry of the Poisson bracket) so that $\{F,F\} = 0$.

Now $H(q,p) = H'(Q(q,p), P(q,p))$ means that we have

$$\frac{\partial H}{\partial p} = \frac{\partial H'}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H'}{\partial P}\frac{\partial P}{\partial p}$$

and similarly for $\dfrac{\partial H}{\partial q}$.

Exercise 43. *Show that*

$$\dot{Q} = \{Q, P\}\frac{\partial H'}{\partial P}.$$

Solution 43.

$$\dot{Q} = \frac{\partial Q}{\partial q}\left(\frac{\partial H'}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H'}{\partial p}\frac{\partial P}{\partial p}\right) - \frac{\partial Q}{\partial p}\left(\frac{\partial H'}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H'}{\partial p}\frac{\partial P}{\partial q}\right)$$

$$= \frac{\partial H'}{\partial p}\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) = \frac{\partial H'}{\partial p}\{Q, P\}$$

Similarly,

$$\dot{P} = -\{Q, P\}\frac{\partial H'}{\partial Q}.$$

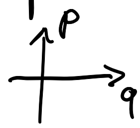
Thus Q, P obey Hamilton's equations if and only if

$$\{Q, P\} = 1.$$

This is the condition such that the variable change transformation $(q, p) \to (Q, P)$ is canonical.

Note: The Jacobian of this transformation is given by

Phase space:



$$J \equiv \frac{\partial(Q, P)}{\partial(q, p)} = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \{Q, P\}.$$

So a canonical transformation is one in which the Jacobian is $J = 1$. Thus, a canonical transformation preserves areas in phase space.

Example: Suppose that for one degree of freedom we have the transformation

$$Q = q\cos\alpha + p\sin\alpha, \qquad P = -q\sin\alpha + p\cos\alpha$$

Then

$$\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \cos^2\alpha + \sin^2\alpha = 1, \quad \{Q, Q\} = \{P, P\} = 0$$

so the transformation is canonical.

Properties of the Poisson bracket.

1. Antisymmetry:
$$\{u, v\} = -\{v, u\}.$$
2. Linearity:
$$\{au + bv, w\} = a\{u, w\} + b\{v, w\}, \qquad a,\, b \in \mathbb{R}.$$
3. Derivation:
$$\{uv, w\} = \{u, w\}v + u\{v, w\}.$$
4. Jacobi's identity: (functions with continuous 2nd derivatives)
$$\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0.$$

   That is, the sum of cyclic permutation of the double Poisson bracket of three functions is zero.

We may consider the generalised equation of motion for an arbitrary function u in the Poisson bracket formulation to be

$$\frac{du}{dt} = \{u, H\} + \frac{\partial u}{\partial t}.$$

(Canonically invariant and Hamilton's equations are a special case.)

Now if u is a constant of the motion then from the above we have

$$\{H, u\} = \frac{\partial u}{\partial t}.$$

In this way we may identify the constants of the motion. And conversely, the Poisson bracket of H with any constant of the motion must be equal to the explicit time derivative of the constant function.

If u, v are two constants of the motion and not explicit functions of t, then setting $w = H$ and using linearity, the Jacobi identity gives us that

$$\{H, \{u, v\}\} = 0.$$

(Poisson's theorem: The Poisson bracket of any two constants of the motion is also a constant of the motion.)

Suppose we have the equation of motion for u given by $\frac{du}{dt} = \{u, H\}$, with the solution

$$u(t) = u_0 + t\{u, H\}_0 + \frac{t^2}{2}\{\{u, H\}, H\}_0 + \frac{t^3}{3!}\{\{\{u, H\}, H\}, H\}_0 + ...$$

Where the subscript 0 refers to time $t = 0$.

Exercise 44. *Use this approach to obtain the solution for one-dimensional motion with constant gravitational acceleration g.*

Solution 44.
The Hamiltonian is $H = \frac{p^2}{2m} + mgy$

$$\{y, H\} = \frac{\partial y}{\partial y}\frac{\partial H}{\partial p} - \frac{\partial y}{\partial p}\frac{\partial H}{\partial y} = \frac{p}{m}$$

$$\{\{y, H\}, H\} = \frac{1}{m}\{p, H\} = \frac{1}{m}\left(\frac{\partial p}{\partial y}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial y}\right)$$
$$= -g$$

$$\{\{\{y, H\}, H\}, H\} = -\{g, H\} = 0$$

This gives the solution $y = y_0 + \frac{p_0}{m}t - \frac{gt^2}{2}$

where $\frac{p_0}{m} = v_0$

2.4.4 Generalisation to many degrees of freedom

Above it was shown that for a system with one degree of freedom that the transformation $(q,p) \to (Q,P), \;\; Q = Q(q,p), \; P = P(q,p)$ is canonical provided $\{Q,P\} = 1$ where $\{,\}$ denotes the Poisson bracket. Also by antisymmetry, we have $\{Q,Q\} = \{P,P\} = 0$. Now consider a system with m degrees of freedom with generalised coordinates and momenta $q_i, \; p_i, \; (1 \le i \le m)$. Let $H = H(q_i,p_i)$ be the Hamiltonian of the system and consider a transformation to new coordinates

$$Q_i = Q_i(q_j,p_j), \quad P_i = P_i(q_j,p_j)$$

in terms of which the Hamiltonian may be expressed

$$H(q_i(Q_j,P_j),p_i(Q_j,P_j)) = H'(Q_j,P_j) \qquad \Longleftrightarrow \qquad H(q_i,p_i) = H'(Q_j(q_i,p_i),P_j(q_i,p_i)).$$

Then the coordinate transformation Q_i, P_i is canonical if they obey Hamilton's equations for H' and provided that they satisfy

$$\{Q_i,P_j\} = \delta_{ij}, \quad \{Q_i,Q_j\} = \{P_i,P_j\} = 0,$$

where the Poisson bracket of two functions $F(q_j,p_j), G(q_j,p_j)$ is now defined by

$$\{F,G\} = \sum_{i=1}^{m} \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \right) = -\{G,F\}.$$

Above, δ_{ij} is the Kronecker delta function defined as

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Proceeding as before, using Hamilton's equations,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

we have

$$\begin{aligned}
\dot{Q}_i &= \sum_{j=1}^{m} \left(\frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j \right) \\
&= \sum_{j=1}^{m} \left(\frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j} \right) \\
&= \sum_{j=1}^{m} \frac{\partial Q_i}{\partial q_j} \sum_{k=1}^{m} \left(\frac{\partial H'}{\partial Q_k}\frac{\partial Q_k}{\partial p_j} + \frac{\partial H'}{\partial P_k}\frac{\partial P_k}{\partial p_j} \right) - \sum_{j=1}^{m} \frac{\partial Q_i}{\partial p_j} \sum_{k=1}^{m} \left(\frac{\partial H'}{\partial Q_k}\frac{\partial Q_k}{\partial q_j} + \frac{\partial H'}{\partial P_k}\frac{\partial P_k}{\partial q_j} \right) \\
&= \sum_{k=1}^{m} \frac{\partial H'}{\partial Q_k} \sum_{j=1}^{m} \left(\frac{\partial Q_i}{\partial q_j}\frac{\partial Q_k}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial Q_k}{\partial q_j} \right) + \sum_{k=1}^{m} \frac{\partial H'}{\partial P_k} \sum_{j=1}^{m} \left(\frac{\partial Q_i}{\partial q_j}\frac{\partial P_k}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial P_k}{\partial q_j} \right) \\
&= \sum_{k=1}^{m} \frac{\partial H'}{\partial Q_k}\{Q_i,Q_k\} + \sum_{k=1}^{m} \frac{\partial H'}{\partial P_k}\{Q_i,P_k\} \\
&= \frac{\partial H'}{\partial P_i}.
\end{aligned}$$

Similarly it can be shown that

$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i}.$$

As before, we call a transformation to new coordinates Q_i, P_i satisfying Hamilton's equations for $H'(Q_j, P_j) = H(q_i(Q_j, P_j), p_i(Q_j, P_j))$, a canonical transformation.
Note: It can be shown that such a transformation in phase space is volume preserving (c.f. area preserving property of canonical transformations in two-dimensional phase space).

As an example, consider the problem of two masses on springs. Let x_1 be the displacement of first mass from equilibrium and similarly for x_2. Then

$$\begin{aligned}
T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2, \quad V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3x_2^2, \\
L &= T - V, \\
p_1 &= \frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1, \quad p_2 = \frac{\partial L}{\partial \dot{x}_2} = m_2\dot{x}_2.
\end{aligned}$$

Now consider the transformation

$$\begin{aligned}
Q_1 = x_1\cos\alpha - x_2\sin\alpha, &\quad P_1 = p_1\cos\alpha - p_2\sin\alpha, \\
Q_2 = x_1\sin\alpha + x_2\cos\alpha, &\quad P_2 = p_1\sin\alpha + p_2\cos\alpha.
\end{aligned}$$

Exercise 45. *Show that this transformation is canonical.*

Solution 45.

$$\{Q_1, P_1\} = \sum_{j=1}^{2}\left(\frac{\partial Q_1}{\partial x_j}\frac{\partial P_1}{\partial p_j} - \frac{\partial Q_1}{\partial p_j}\frac{\partial P_1}{\partial x_j}\right) = \frac{\partial Q_1}{\partial x_1}\frac{\partial P_1}{\partial p_1} + \frac{\partial Q_1}{\partial x_2}\frac{\partial P_1}{\partial p_2}$$

$$= \cos^2\alpha + \sin^2\alpha = 1$$

Similarly,

$$\{Q_1, P_2\} = \frac{\partial Q_1}{\partial x_1}\frac{\partial P_2}{\partial p_1} + \frac{\partial Q_1}{\partial x_2}\frac{\partial P_2}{\partial p_2} = \cos\alpha\sin\alpha - \cos\alpha\sin\alpha = 0$$

$$\{Q_2, P_1\} = \frac{\partial Q_2}{\partial x_1}\frac{\partial P_1}{\partial p_1} + \frac{\partial Q_2}{\partial x_2}\frac{\partial P_1}{\partial p_2} = \sin\alpha\cos\alpha - \cos\alpha\sin\alpha = 0$$

$$\{Q_2, P_2\} = \frac{\partial Q_2}{\partial x_1}\frac{\partial P_2}{\partial p_1} + \frac{\partial Q_2}{\partial x_2}\frac{\partial P_2}{\partial p_2} = \cos^2\alpha + \sin^2\alpha = 1$$

Check also $\{Q_1, Q_2\} = \{P_1, P_2\} = 0$

2.4.5 Time evolution

Consider a function $F(q_i, p_i)$ of generalised coordinates and momenta. Then from Hamilton's equations we have

$$\begin{aligned} \dot{F} &= \sum_{i=1}^{m} \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i \right) \\ &= \sum_{i=1}^{m} \left(\frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = \{F, H\}. \end{aligned}$$

Thus the time evolution of a dynamical function $F(q_i, p_i)$ is governed by the Poisson bracket with H. In particular we have

$$\dot{q}_i = \{q_i, H\}, \quad \dot{p}_i = \{p_i, H\},$$

so Hamilton's equations may be expressed in terms of the Poisson bracket as

$$\{q_i, H\} = \frac{\partial H}{\partial p_i}, \quad \{p_i, H\} = -\frac{\partial H}{\partial q_i}.$$

Define an operator D_H by

$$D_H F(q_i, p_i) = \{F, H\}.$$

Thus D_H is a *first order differential operator* given by

$$D_H = \sum_{i=1}^{m} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right).$$

Then we have

$$\dot{F} = D_H F, \quad \ddot{F} = D_H \dot{F} = D_H^2 F, ..., \frac{d^n F}{dt^r} = D_H^n F.$$

Using a Taylor series expansion, we have

$$\begin{aligned} F(t) &= \sum_{r=0}^{\infty} \frac{F^{(r)}(0)}{r!} t^r \\ &= \sum_{r=0}^{\infty} \frac{t^r D_H^r}{r!} F(0) = e^{tD_H} F(0) \end{aligned}$$

where $F(t) = F(q_i(t), p_i(t))$. In this way, given the value of a function at $t = 0$ we may determine its value at any subsequent time t, at least in principle.

Exercise 46. *Obtain the general solution for the position of a mass on a spring.*

Solution 46.

$$L = T - V = \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2 , \qquad p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

and $$H = T + V = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} k x^2 = \frac{p^2}{2m} + \frac{1}{2} k x^2$$

$$D_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p} = \frac{p}{m}\frac{\partial}{\partial x} - kx\frac{\partial}{\partial p}$$

$$D_H x = \frac{p}{m} \qquad D_H p = -kx$$

$$D_H^2 x = D_H(D_H x) = \frac{1}{m} D_H p = -\frac{k}{m} x$$

$$D_H^2 p = D_H(D_H p) = -k D_H x = -\frac{k}{m} p$$

In general $$D_H^{2r} x = (-1)^r \omega^{2r} x \qquad \omega = \sqrt{\frac{k}{m}}$$

$$D_H^{2r+1} x = (-1)^r \omega^{2r+1} \frac{1}{m\omega} p$$

$$x(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} D_H^m x(0)$$

$$= \sum_{r=0}^{\infty} \frac{t^{2r}}{(2r)!} D_H^{2r} x(0) + \sum_{r=0}^{\infty} \frac{t^{2r+1}}{(2r+1)!} D_H^{2r+1} x(0)$$

$$= x(0) \sum_{r=0}^{\infty} \frac{(-1)^r t^{2r} \omega^{2r}}{(2r)!} + \frac{p(0)}{m\omega} \sum_{r=0}^{\infty} \frac{(-1)^r t^{2r+1} \omega^{2r+1}}{(2r+1)!}$$

Recall $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \qquad (*)$$

But $e^{ix} = \cos x + i \sin x$, so

From (*) we have

$$\cos x = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2r)!}$$

$$\sin x = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{(2r+1)!}$$

Then $x(t) = x(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin(\omega t)$