MATH4105 Assignment 4

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Question 1

For a timelike geodesic, we have

$$l = r^2 \sin^2\theta \frac{d\phi}{d\tau}; \qquad e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$$

Hence,

$$\frac{l}{e} = \frac{r^2 \sin^2\theta}{\left(1 - \frac{2M}{r}\right)} \frac{d\phi}{d\tau}\frac{d\tau}{dt}$$

In the plane $\theta = \pi/2$, $\sin^2\theta = 1$ and so the equation simplifies down to

$$\frac{l}{e} = \frac{r^2}{\left(1 - \frac{2M}{r}\right)}\frac{d\phi}{dt}$$
$$\Rightarrow \frac{d\phi}{dt} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\frac{l}{e} = \Omega$$

Question 2

We have the constant

$$\varepsilon = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) \equiv \frac{e^2 - 1}{2}$$

In Schwarzschild coordinates, we have the effective potential as

$$V_{\text{eff}}(r) = \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right]$$

And so

$$e^2 - 1 = \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1$$
$$e^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right)$$

If the orbit is circular, we know that there is no change in radius over time and so $dr/d\tau = 0$, giving

$$e^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right)$$

as desired.

Question 3

As per the lectures, we have a circular orbit when there's no change in the effective potential with a change in radius. That is,

$$\frac{d(V_{\text{eff}}/m)}{dr} = 0 = \frac{M}{r^2} - \frac{(L/m)^2}{r^3} + \frac{3M(L/m)^2}{r^4}$$

$$\Rightarrow \left(\frac{L}{m}\right)^2 = \frac{M}{1/r - 3M/r^2} = \frac{r^2 M}{r - 3M}$$

But $L/m = l$, and so

$$l^2 = \frac{r^2 M}{r - 3M}$$

Substituting this into equation (2) in the assignment sheet, and taking l^2/e^2 gives

$$\begin{aligned}
\frac{l^2}{e^2} &= \frac{r^2 M}{r - 3M}\left(1 - \frac{2M}{r}\right)^{-1}\left(1 + \frac{l^2}{r^2}\right)^{-1} \\
&= \frac{r^2 M}{r - 3M}\left(1 - \frac{2M}{r}\right)^{-1}\left(1 + \frac{M}{r - 3M}\right)^{-1} \\
&= \frac{r^2 M}{r - 3M + M}\left(1 - \frac{2M}{r}\right)^{-1} \\
&= \frac{r^2 M}{r(1 - 2M/r)}\left(1 - \frac{2M}{r}\right)^{-1} \\
&= Mr\left(1 - \frac{2M}{r}\right)^{-2} \\
\Rightarrow \frac{l}{e} &= (Mr)^{1/2}\left(1 - \frac{2M}{r}\right)^{-1}
\end{aligned}$$

as desired.

Question 4

Substituting the result from question 3 into the l/e term in the result of question 1 gives

$$\begin{aligned}
\Omega &= \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)(Mr)^{1/2}\left(1 - \frac{2M}{r}\right)^{-1} \\
&= \left(\frac{M}{r^3}\right)^{1/2} \\
\Rightarrow \Omega^2 &= \frac{M}{r^3}
\end{aligned}$$

where we've used the results from questions 1 and 3 which hold for timelike geodesics in circular orbits.

Question 5

For a spacecraft moving in a circular orbit (of radius $r = 7M$) around a Schwarzschild black hole, the period of the orbit as viewed from an observer at infinity will be measured as one revolution with respect to their time t. That is,

$$\begin{aligned}
\Omega^2 = \frac{M}{r^3} &= \left(\frac{d\phi}{dt}\right)^2 \\
\Rightarrow \frac{d\phi}{dt} &= \frac{M^{1/2}}{r^{3/2}} \\
\int_0^P dt &= \int_0^{2\pi} \frac{r^{3/2}}{M^{1/2}}\, d\phi \\
P &= 2\pi r\left(\frac{r}{M}\right)^{1/2} \\
&= 2\pi \cdot 7M\sqrt{7} = 2\pi \cdot 7^{3/2} M
\end{aligned}$$

Question 6

For a timelike geodesic, we have that $d\tau^2 = -ds^2$, and so

$$d\tau^2 = -ds^2$$

$$
\begin{aligned}
&= \left(1 - \frac{2M}{r}\right) dt^2 - r^2 d\phi^2 \\
\Rightarrow \frac{d\tau^2}{dt^2} &= \left(1 - \frac{2M}{r}\right) - r^2 \left(\frac{d\phi}{dt}\right)^2 \\
&= \left(1 - \frac{2M}{r}\right) - r^2 \frac{M}{r^3} \\
&= 1 - \frac{3M}{r}
\end{aligned}
$$

$$
\Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{3M}{r}\right)^{-1/2}
$$

In the context of the period as measured from a clock on the spacecraft in orbit, we require the period with respect to the proper time τ, and hence need $d\phi/d\tau$ to get there:

$$
\begin{aligned}
\frac{d\phi}{d\tau} &= \frac{d\phi}{dt}\frac{dt}{d\tau} \\
&= \left(\frac{M}{r^3}\right)^{1/2} \left(1 - \frac{3M}{r}\right)^{-1/2} \\
&= \left(\frac{M}{r^3 - 3Mr^2}\right)^{1/2} \\
\int_0^{P'} d\tau &= \int_0^{2\pi} \left(\frac{r^3 - 3Mr^2}{M}\right)^{1/2} d\phi \\
P' &= 2\pi r \left(\frac{r - 3M}{M}\right)^{1/2}
\end{aligned}
$$

For $r = 7M$, we get

$$
\begin{aligned}
P' &= 2\pi \cdot 7M \left(\frac{7M - 3M}{M}\right)^{1/2} \\
&= 28\pi M
\end{aligned}
$$

Question 7

For a stationary observer, we have that $dr/d\tau = d\phi/d\tau = d\theta/d\tau = 0$. Therefore the 4-velocity of the observer is

$$
\mathbf{u}_{\text{obs}} = (dt/d\tau,\ 0,\ 0,\ 0)
$$

The normalization of the 4-velocity tells us that

$$
\begin{aligned}
-1 &= \mathbf{u}_{\text{obs}} \cdot \mathbf{u}_{\text{obs}} \\
&= g_{tt} \left(\frac{dt}{d\tau}\right)^2 \\
\Rightarrow \frac{dt}{d\tau} &= \left(1 - \frac{2M}{r}\right)^{-1/2} \\
\Rightarrow \mathbf{u}_{\text{obs}} &= \left(\left(1 - \frac{2M}{r}\right)^{-1/2},\ 0,\ 0,\ 0\right)
\end{aligned}
$$

The first particle falls radially inwards with $e = 1$, where

$$
\begin{aligned}
e &= \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \\
\Rightarrow \frac{dt}{d\tau} &= \left(1 - \frac{2M}{r}\right)^{-1}
\end{aligned}
$$

Since the particle is falling radially inwards, $d\phi/d\tau = d\theta/d\tau = 0$ and so its 4-velocity is

$$\mathbf{u}_1 = \left(\left(1 - \frac{2M}{r}\right)^{-1},\ \frac{dr}{d\tau},\ 0,\ 0\right)$$

The dot product of the observer and the first particle is then

$$\begin{aligned}\mathbf{u}_{\text{obs}} \cdot \mathbf{u}_1 &= g_{tt}\left(\frac{dt}{d\tau}\right)_{\text{obs}}\left(\frac{dt}{d\tau}\right)_1 + 0 + 0 + 0 \\ &= -\left(1 - \frac{2M}{r}\right)\left(1 - \frac{2M}{r}\right)^{-1/2}\left(1 - \frac{2M}{r}\right)^{-1}\end{aligned}$$

At $r = 6M$, this is

$$\begin{aligned}\mathbf{u}_{\text{obs}} \cdot \mathbf{u}_1 &= -\left(1 - \frac{2M}{6M}\right)^{-1/2} \\ &= -\sqrt{\frac{3}{2}}\end{aligned}$$

And so the magnitude of the relative velocity between the observer and the first particle is

$$\begin{aligned}|V|_1 &= \sqrt{1 - (\mathbf{u}_{\text{obs}} \cdot \mathbf{u}_1)^{-2}} \\ &= \sqrt{1 - \left(-\sqrt{\frac{3}{2}}\right)^{-2}} \\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

which is in natural units.
The second particle begins its radial infall with $e = 2$, and so via the same process as above we arrive at

$$\begin{aligned}\mathbf{u}_2 &= \left(2\left(1 - \frac{2M}{r}\right)^{-1},\ \frac{dr}{d\tau},\ 0,\ 0\right) \\ \Rightarrow \mathbf{u}_{\text{obs}} \cdot \mathbf{u}_2 &= g_{tt}\left(\frac{dt}{d\tau}\right)_{\text{obs}}\left(\frac{dt}{d\tau}\right)_1 + 0 + 0 + 0 \\ &= 2\mathbf{u}_{\text{obs}} \cdot \mathbf{u}_1 \\ &= -2\sqrt{\frac{3}{2}} = -\sqrt{6}\end{aligned}$$

at $r = 6M$. Therefore the relative velocity between the second particle and the observer is

$$\begin{aligned}|V|_2 &= \sqrt{1 - (\mathbf{u}_{\text{obs}} \cdot \mathbf{u}_2)^{-2}} \\ &= \sqrt{1 - (-\sqrt{6})^{-2}} \\ &= \sqrt{\frac{5}{6}}\end{aligned}$$

which is given in natural units.
The absolute difference in observed values is then

$$\Delta|V| = |V|_2 - |V|_1 = \sqrt{\frac{5}{6}} - \frac{1}{\sqrt{3}} \simeq 0.336 \text{ (natural units)}$$

And the relative difference in relative velocities with respect to the observer is

$$\frac{|V|_2}{|V|_1} = \frac{\sqrt{10}}{2} \simeq 1.58$$

That is, particle 2 is moving about 1.58 times faster with respect to the observer at $r = 6M$ than particle 1.

Question 8

i. Assume that the object has no angular motion and only moves radially. That is, $d\phi/d\tau = d\theta/d\tau = 0$. The 4-velocity is then given by

$$\mathbf{u} = (dt/d\tau,\ dr/d\tau,\ 0,\ 0)$$

By the normalization of the 4-velocity, we have that

$$-1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + 0 + 0$$

And since the particle is moving along a geodesic (radial infall), we know the $\varepsilon = 0$ giving

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = -V_{\text{eff}}(r)$$

which holds for any radius along the geodesic. Since we have radial motion, $l^2 = (r^2 \sin^2\theta[d\phi/d\tau])^2 = 0$, and the effective potential becomes

$$V_{\text{eff}}(r) = \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right) - 1\right]$$

And so the change in radius with respect to the proper time is the difference between the geodesic velocity at that point minus its initial velocity at $r = r_0$:

$$\begin{aligned}
\left(\frac{dr}{d\tau}\right)^2 &= \left(\frac{dr}{d\tau}\right)^2_{r=r} - \left(\frac{dr}{d\tau}\right)^2_{r=r_0} \\
&= \left(1 - \frac{2M}{r}\right) + 1 - \left(1 - \frac{2M}{r_0}\right) - 1 \\
&= \frac{2M}{r} - \frac{2M}{r_0} \\
\Rightarrow \frac{dr}{d\tau} &= \pm\left(\frac{2M}{r} - \frac{2M}{r_0}\right)^{1/2}
\end{aligned}$$

where we can take the positive root corresponding to radial infall.
With this, the 4-velocity normalisation becomes

$$\begin{aligned}
-1 &= g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2 \\
&= -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{2M}{r} - \frac{2M}{r_0}\right) \\
\Rightarrow \left(\frac{dt}{d\tau}\right)^2 &= \left(1 - \frac{2M}{r}\right)^{-1}\left(1 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{2M}{r} - \frac{2M}{r_0}\right)\right) \\
&= \left(1 - \frac{2M}{r}\right)^{-2}\left(1 - \frac{2M}{r_0}\right) \\
\Rightarrow \frac{dt}{d\tau} &= \pm\left(1 - \frac{2M}{r}\right)^{-1}\left(1 - \frac{2M}{r_0}\right)^{1/2}
\end{aligned}$$

Where we take the positive root for forward passage of time. Finally, we get the 4-velocity of the particle at r released from r_0 as being

$$\mathbf{u} = \left(\left(1 - \frac{2M}{r}\right)^{-1}\left(1 - \frac{2M}{r_0}\right)^{1/2},\ \left(\frac{2M}{r} - \frac{2M}{r_0}\right)^{1/2},\ 0,\ 0\right)$$

ii. With

$$dr_s = \left(1 - \frac{2M}{r_s}\right)^{-1/2} dr; \qquad dt_s = \left(1 - \frac{2M}{r_s}\right)^{1/2} dt$$

we can take the relative velocity of a falling object and an observer at radius r as being

$$V = \frac{dr_s}{dt_s} = \frac{\left(1 - \frac{2M}{r_s}\right)^{-1/2}}{\left(1 - \frac{2M}{r_s}\right)^{1/2}}\frac{dr}{dt}$$

$$
\begin{aligned}
&= \left(1-\frac{2M}{r_s}\right)^{-1}\frac{dr}{d\tau}\frac{d\tau}{dt} \\
&= \left(1-\frac{2M}{r_s}\right)^{-1}\left(\frac{2M}{r}-\frac{2M}{r_0}\right)^{1/2}\left(\frac{dt}{d\tau}\right)^{-1} \\
&= \left(1-\frac{2M}{r_s}\right)^{-1}\left(\frac{2M}{r}-\frac{2M}{r_0}\right)^{1/2}\left(1-\frac{2M}{r}\right)\left(1-\frac{2M}{r_0}\right)^{-1/2}
\end{aligned}
$$

On setting $r_s = r$, we obtain

$$
V = \left(\frac{2M}{r}-\frac{2M}{r_0}\right)^{1/2}\left(1-\frac{2M}{r_0}\right)^{-1/2}
$$

as required.

iii. Our 4-velocity in shell coordinates is in the same form as a 4-velocity in special relativity if $\mathbf{u} = (\gamma,\ \gamma v)$. That is,

$$
\mathbf{u} = (\gamma,\ \gamma v) \Leftrightarrow \left(\frac{dt_s}{d\tau},\ \frac{dt_s}{d\tau}v\right)
$$

We know that

$$
\begin{aligned}
\frac{dt_s}{d\tau} &= \left(1-\frac{2M}{r_s}\right)^{1/2}\frac{dt}{d\tau} \\
&= \left(1-\frac{2M}{r_s}\right)^{1/2}\left(1-\frac{2M}{r_s}\right)^{1/2}\left(1-\frac{2M}{r}\right)^{-1}
\end{aligned}
$$

on setting $r_s = r$, we get

$$
\frac{dt_s}{d\tau} = \left(1-\frac{2M}{r_0}\right)^{1/2}\left(1-\frac{2M}{r}\right)^{-1/2}
$$

Now for the radial component, we get

$$
\begin{aligned}
\frac{dr_s}{d\tau} &= \left(1-\frac{2M}{r_s}\right)^{-1/2}\frac{dr}{d\tau} \\
&= \left(1-\frac{2M}{r_s}\right)^{-1/2}\left(\frac{2M}{r}-\frac{2M}{r_0}\right)^{1/2}
\end{aligned}
$$

On setting $r_s = r$,

$$
\begin{aligned}
\frac{dr_s}{d\tau} &= \left(1-\frac{2M}{r_0}\right)^{1/2}\left(1-\frac{2M}{r}\right)^{-1/2}\left(\frac{2M}{r(1-2M/r_0)}-\frac{2M}{r_0(1-2M/r_0)}\right)^{1/2} \\
&= \frac{dt_s}{d\tau}\left(\frac{2M(r_0-r)}{r(r_0-2M)}\right)^{1/2} \\
&= \frac{dt_s}{d\tau}v
\end{aligned}
$$

To check that this is the correct expression for velocity, note that a special relativistic 4-velocity dictates that

$$
\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-v^2}} \\
\left(\frac{dt_s}{d\tau}\right)^2 &= \frac{1}{1-v^2} \\
1-v^2 &= \left(\frac{dt_s}{d\tau}\right)^{-2} \\
v &= \pm\sqrt{1-\left(\frac{dt_s}{d\tau}\right)^{-2}} \\
&= \sqrt{1-\left(1-\frac{2M}{r}\right)\left(1-\frac{2M}{r_0}\right)^{-1}}
\end{aligned}
$$

$$= \sqrt{\frac{r(r_0 - 2M) - rr_0 + 2Mr_0}{r(r_0 - 2M)}}$$
$$= \left(\frac{2M(r_0 - r)}{r(r_0 - 2M)}\right)^{1/2}$$

as derived earlier. Hence the 4-velocity in shell coordinates does indeed have the same form as that in special relativity, with a velocity valid for $r \leq r_0$ and $r_0 > 2M$.

iv. To obtain the 3-acceleration in these shell coordinates, we take the time derivative of the 3-velocity:

$$\begin{aligned}
a &= \frac{d^2 r_s}{dt_s^2} = \frac{d}{dt_s}\frac{dr_s}{dt_s} \\
&= \frac{dr_s}{dt_s}\frac{d}{dr_s}\frac{dr_s}{dt_s} \\
&= \left(\frac{2M(r_0 - r)}{r(r_0 - 2M)}\right)^{1/2}\left(1 - \frac{2M}{r_s}\right)^{1/2}\frac{d}{dr}\left(\frac{2M(r_0 - r)}{r(r_0 - 2M)}\right)^{1/2} \\
&= \left(1 - \frac{2M}{r_s}\right)^{1/2}\left(\frac{-M(r_0 - r)}{r^2(r_0 - 2M)} - \frac{M}{r(r_0 - 2M)}\right) \\
&= \left(1 - \frac{2M}{r_s}\right)^{1/2} \cdot \frac{M}{r}\left(\frac{-(r_0 - r)}{r(r_0 - 2M)} - \frac{1}{r_0 - 2M}\right)
\end{aligned}$$

On setting $r_s = r$, we finally obtain

$$a = \frac{M}{r}\left(1 - \frac{2M}{r}\right)^{1/2}\left(\frac{r/r_0 - 1}{r(1 - 2M/r_0)} - \frac{1}{r_0(1 - 2M/r_0)}\right)$$

v. To check we have the right expression with gives the rest acceleration, we set $r = r_0$ which yields

$$\begin{aligned}
a|_{r=r_0} &= \frac{M}{r_0}\left(1 - \frac{2M}{r_0}\right)^{1/2}\left(0 - \frac{1}{r_0(1 - 2M/r_0)}\right) \\
&= -\frac{M}{{r_0}^2}\left(1 - \frac{2M}{r_0}\right)^{-1/2}
\end{aligned}$$

which has the same magnitude as the result obtained in the lectures.
Now, setting the initial release distance of the particle to $r_0 \to \infty$ gives

$$\begin{aligned}
\lim_{r_0 \to \infty} a &= \frac{M}{r}\left(1 - \frac{2M}{r}\right)^{1/2}\left(\frac{0 - 1}{r(1 - 0)} - 0\right) \\
&= -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{1/2}
\end{aligned}$$

Now, as $r \to 2M$ (i.e. it's approaching the event horizon of the Schwarzschild black hole)

$$\begin{aligned}
\lim_{r \to 2M}\left(\lim_{r_0 \to \infty} a\right) &= -\frac{M}{4M^2}\left(1 - \frac{2M}{2M}\right)^{1/2} \\
&= 0
\end{aligned}$$

and so the magnitude of the 3-acceleration approaches 0 as $r \to 2M$. This is the case since the escape velocity of the mass M at $r = 2M$ is c, and so a particle launched at c from $r = 2M$ would approach a velocity of 0 as $r \to \infty$. In this case we have a particle beginning at infinity with no velocity which accelerates continuously on its infall so that it's travelling at effectively c by the time it reaches the event horizon and cannot accelerate any more by the principles of relativity.

Question 9

The Schwarzschild line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

The Eddington-Finkelstein change of coordinates introduces two new variables:

$$\tilde{t} \equiv t + 2M \log\left(\left|\frac{r}{2M} - 1\right|\right); \qquad v = \tilde{t} + r \qquad \Longrightarrow \qquad v = t + r + 2M \log\left(\left|\frac{r}{2M} - 1\right|\right)$$

Taking the differential of this, we get

$$\begin{aligned} dv &= dt + dr + 2M\,(r - 2M)^{-1}\,dr \\ &= dt + \left(1 + \frac{2M}{r - 2M}\right) dr \\ &= dt + \left(1 - \frac{2M}{r}\right)^{-1} dr \\ \Rightarrow dt^2 &= dv^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dv\,dr + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 \end{aligned}$$

On making this substitution, the Schwarzschild line element becomes

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right) dv^2 + 2dv\,dr - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\ &= -\left(1 - \frac{2M}{r}\right) dv^2 + 2dv\,dr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{aligned}$$

Which is exactly the Eddington-Finkelstein line element as in the course notes.