# STAT2003 Assignment 3

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## 1 Question 1

a. Define $Z_1 = X_1X_2$, and $Z_2 = X_1$. By rearranging,

$$X_2 = \frac{Z_1}{X_1} = \frac{Z_1}{Z_2}$$

and the Jacobi matrix is

$$\left(\frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}}\right) = \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} X_2 & X_1 \\ 1 & 0 \end{pmatrix}$$

The determinant of this is then

$$\begin{aligned} \det \frac{\partial \boldsymbol{z}}{\partial \boldsymbol{x}} &= X_2 \cdot 0 - X_1 \dot{1} \\ &= -X_1 = -Z_2 \end{aligned}$$

and the joint distribution of $Z_1$ and $Z_2$ is

$$f_{Z_1,\ Z_2}(z_1,\ z_2) = \frac{1}{|\det \partial z/\partial x|} f_{X_1,\ X_2}(z_1,\ z_1/z_2)$$

Since $X_1$ and $X_2$ are iid,

$$\begin{aligned} f_{X_1,\ X_2}(x_1,\ x_2) &= \left(\frac{1}{x_1 \ln(2)}\right)\left(\frac{1}{x_2 \ln(2)}\right) \\ &= \frac{1}{x_1 x_2 (\ln(2))^2} \\ \Rightarrow f_{X_1,\ X_2}(z_1,\ z_1/z_2) &= \frac{1}{z_1(\ln(2))^2} \\ \Rightarrow f_{Z_1,\ Z_2}(z_1,\ z_2) &= \left(|z_2| z_1 (\ln(2))^2\right)^{-1} \end{aligned}$$

Since $z_2 \in (0.5,\ 1)$, $|z_2| = z_2$. To get the marginal distribution, $f_{Z_1}(z_1)$, we need to integrate the joint distribution with respect to $z_2$ over its domain. First, notice that the region in which the joint distribution is positive is given in Figure 1 below,

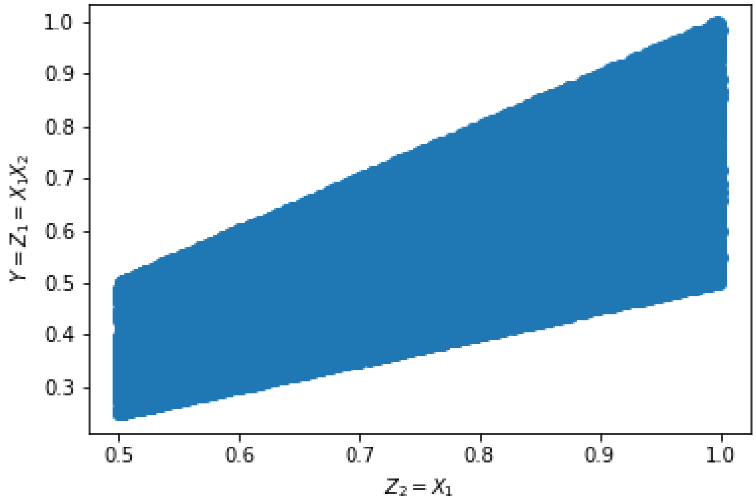


Figure 1: Domain of Joint Distribution Function $f_{Z_1,\ Z_2}(z_1,\ z_2)$

With this in mind, the domain of $Z_2$ must be split in two regions, depending on the value of $Z_1$. Firstly, notice that if $Z_1 \in (0.25,\ 0.5]$, then the lower bound of $Z_2$ is $Z_{2L} = 0.5$ and the upper bound is $Z_{2U} = 2Z_1$ (found by the line in the figure above). If $Z_1 \in (0.5,\ 1)$, then the lower bound of $Z_2$ is $Z_{2L} = Z_1$, and the upper bound is $Z_{2U} = 1$. With these in mind,

If $Z_1 \in (0.25,\ 0.5]$,

$$\begin{aligned} f_{Z_1}(z_1) &= \int_0^\infty f_{Z_1,\ Z_2}(z_1,\ z_2)\ dz_2 \\ &= \frac{1}{z_1(\ln{(2)})^2} \int_{0.5}^{2Z_1} \frac{1}{z_2}\ dz_2 \\ &= \frac{1}{z_1(\ln{(2)})^2} \left(\ln{(2Z_1)} - \ln{(0.5)}\right) \\ &= \frac{\ln{(4Z_1)}}{z_1(\ln{(2)})^2} \end{aligned}$$

If $Z_1 \in (0.5,\ 1)$,

$$\begin{aligned} f_{Z_1}(z_1) &= \int_0^\infty f_{Z_1,\ Z_2}(z_1,\ z_2)\ dz_2 \\ &= \frac{1}{z_1(\ln{(2)})^2} \int_{Z_1}^{1} \frac{1}{z_2}\ dz_2 \\ &= \frac{1}{z_1(\ln{(2)})^2} \left(\ln{(1)} - \ln{(Z_1)}\right) \\ &= -\frac{\ln{(Z_1)}}{z_1(\ln{(2)})^2} \end{aligned}$$

The marginal distribution of $Y = Z_1 = X_1X_2$ is then

$$f_Y(y) = \begin{cases} \frac{\ln(4y)}{y(\ln(2))^2} & \text{if } y \in (0.25,\ 0.5] \\ -\frac{\ln(y)}{y(\ln(2))^2} & \text{if } y \in (0.5,\ 1) \\ 0 & \text{else} \end{cases}$$

b. Let $g(y)$ be the pdf of $Z$.

If $Y \in (0.5,\ 1)$, then

$$\begin{aligned} F_Z(z) &= \mathbb{P}\left(Z \le z\right) \\ &= \mathbb{P}\left(Y \le z\right) \\ &= F_Y(z) \\ \Rightarrow g(z) &= f_Y(z) \end{aligned}$$

If $Y \in (0.25,\ 0.5]$, then

$$\begin{aligned} F_Z(z) &= \mathbb{P}\left(Z \le z\right) \\ &= \mathbb{P}\left(2Y \le z\right) \\ &= \mathbb{P}\left(Y \ge z/2\right) \\ &= 1 - F_Y(z/2) \end{aligned}$$

In the case on the right hand side, the cdf must be differentiated with respect to $z$ to obtain the pdf:

$$
\begin{aligned}
f_Z(z) &= \frac{d}{dz}(1 - F_y(z/2)) \\
&= -\frac{1}{2} f_y(z/2) \\
&= -\frac{1}{2}\frac{\ln(2z)}{\frac{z}{2}(\ln(2))^2} \\
&= -\frac{\ln(2z)}{z(\ln(2))^2}
\end{aligned}
$$

The pdf of $Z$ is then

$$
g(z) = \begin{cases} -\frac{\ln(z)}{z(\ln(2))^2} & z \in (0.5,\ 1) \\ -\frac{\ln(2z)}{z(\ln(2))^2} & z \in (0.25,\ 0.5] \\ 0 & \text{else} \end{cases}
$$

# 2 Question 2

a. With $U \sim \text{Ber}\,(1-\varrho)$, $\varrho \in [0,1)$, and $V \sim \text{Exp}(\lambda)$, $\lambda > 0$, the moment generating function of $W = UV$ is calculated as

$$
\begin{aligned}
M_W(s) &= \mathbb{E}\left(e^{sUV}\right) \\
&= \mathbb{E}\left(e^{sUV} \mid U = 0\right)(1-p) + \mathbb{E}\left(e^{sUV} \mid U = 1\right)p \\
&= \mathbb{E}\left(e^{0}\right)(1-p) + \mathbb{E}\left(e^{sV}\right)p \\
&= 1(1-p) + p \cdot M_V(s) \\
&= 1 - p + p\frac{\lambda}{\lambda - s} \qquad s < \lambda \\
&= 1 - p + \frac{p\lambda}{\lambda - s}
\end{aligned}
$$

where substitutions were made based on the probability of the Bernoulli random variables and the moment generating function of the exponential distribution.

b. First, compute the first few terms of the $X_n$ sequence

$$
\begin{aligned}
X_1 &= \varrho X_0 + W_1 \\
X_2 &= \varrho(\varrho X_0 + W_1) + W_2 \\
&= \varrho^2 X_0 + \varrho W_1 + W_2 \\
X_3 &= \varrho(\varrho^2 X_0 + \varrho W_1 + W_2) + W_3 \\
&= \varrho^3 X_0 + \varrho^2 W_1 + \varrho W_2 + W_3
\end{aligned}
$$

Naturally, the sequence $X_n$ follows the form

$$
X_n = \varrho^n X_0 + \sum_{i=1}^{n} \varrho^{n-i} W_i
$$

The moment generating function of this is then

$$
M_{X_n}(t) = \mathbb{E}\left(e^{t(\varrho^n X_0 + W_n + \varrho W_{n-1} + \dots)}\right)
$$

Since all $\{W_i\}_{i=1}^{\infty}$ are iid,

$$
\begin{aligned}
M_{X_n}(t) &= \mathbb{E}\left(e^{t\varrho^n X_0} e^{t\sum_{i=1}^{n}\varrho^{n-i}W_i}\right) \\
&= \frac{\lambda}{\lambda - t\varrho^n}\mathbb{E}\left(e^{t\sum_{i=1}^{n}\varrho^{n-i}W_i}\right)
\end{aligned}
$$

And since this mgf is of the form of that of the exponential distribution (to a constant), if $X_0 \sim \text{Exp}(\lambda)$, then so is $X_n \sim \text{Exp}(\lambda)$

c. The covariance matrix of $X_{n+k}$ and $X_n$ is

$$\begin{aligned}\operatorname{Cov}(X_{n+k}, X_n) &= \operatorname{Cov}\left(\varrho^{n+k}X_0 + \sum_{i=1}^{n+k}\varrho^{n+k-i}W_i,\ \varrho^n X_0 + \sum_{i=1}^{n}\varrho^{n-i}W_i\right)\\ &= \operatorname{Cov}\left(\sum_{i=1}^{n+k}\varrho^{n+k-i}W_i,\ \sum_{i=1}^{n}\varrho^{n-i}W_i\right) + \varrho^{n+k}\varrho^n \operatorname{Cov}(X_0, X_0)\\ &= \sum_{i=1}^{n+k}\sum_{j=1}^{n}\operatorname{Cov}\left(\varrho^{n+k-i}W_i,\ \varrho^{n-j}W_j\right) + \varrho^{2n+k}\operatorname{Var}(X_0)\end{aligned}$$

But, $\operatorname{Cov}(W_i,\ W_j) = 0$ when $i \neq j$, and so

$$\begin{aligned}\operatorname{Cov}(X_{n+k}, X_n) &= \operatorname{Cov}(W_n,\ W_n) + \varrho^{2n+k}\operatorname{Var}(X_0)\\ &= \operatorname{Var}(W_n) + \varrho^{2n+k}\operatorname{Var}(X_0)\\ &= \operatorname{Var}(W) + \varrho^{2n+k}\operatorname{Var}(X_0)\end{aligned}$$

Recall that $M_W(s) = 1 - p + \frac{p\lambda}{\lambda - s}$ and that $\operatorname{Var}(W) = M''(0) - (M'(0))^2$,

$$\begin{aligned}M'(s) &= \frac{p\lambda}{(\lambda-s)^2} \Rightarrow M'(0) = \frac{p}{\lambda}\\ M''(s) &= \frac{2p\lambda}{(\lambda-s)^3} \Rightarrow M''(0) = \frac{2p}{\lambda^2}\\ \Rightarrow \operatorname{Var}(W) &= \frac{2p}{\lambda^2} - \frac{p^2}{\lambda^2}\\ &= \frac{p}{\lambda^2}(2-p)\end{aligned}$$

Also note that $\operatorname{Var}(X_0) = 1/\lambda^2$ since it is an exponentially distributed random variable, and so

$$\begin{aligned}\operatorname{Cov}(X_{n+k}, X_n) &= \frac{p}{\lambda^2}(2-p) + \varrho^{2n+k}\frac{1}{\lambda^2}\\ &= \frac{1}{\lambda^2}\left(p(2-p) + \varrho^{2n+k}\right)\end{aligned}$$

d. Not attempted.

e. A simulated path of $X_n$ from $n = 0$ to $n = 50$ is given in Figure 2 below

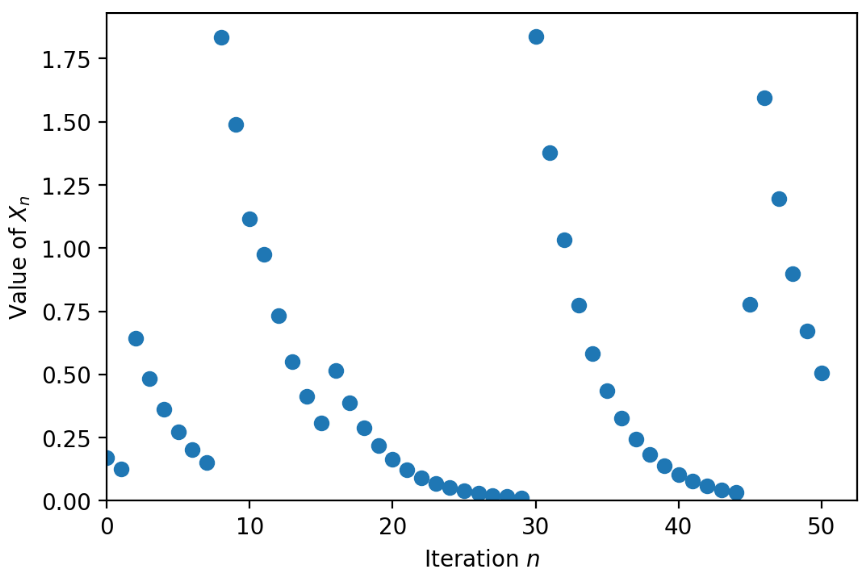


Figure 2: Value of $X_n$ from $n = 0 \to 50$

The code used to produce this is shown in the code block below

```python
# -*- coding: utf-8 -*-
"""
Created on Thu May 26 17:31:13 2022

@author: ryanw
"""

import numpy as np
import matplotlib.pyplot as plt

rho = 0.75
prob = 1 - rho
lam = 1

U = np.random.uniform(0, 1, 50)
V = np.random.exponential(lam, 50)
W = np.zeros(len(V))
X = np.zeros(len(V) + 1)
X[0] = np.random.exponential(lam)

number = np.arange(0, 50)

for num in number:
    if U[num] <= prob:
        W[num] = V[num - 1]
    else:
        W[num] = 0

for i in range(1, 51):
    X[i] = rho * X[i - 1] + W[i - 1]

plt.scatter(np.arange(0, 51), X)
plt.xlabel("Iteration $n$"); plt.ylabel("Value of $X_n$")
plt.xlim(xmin=0); plt.ylim(bottom=0)
plt.savefig("q2e.png", dpi=200, bbox_inches='tight', pad_inches = 0.01)
```

# 3 Question 3

a. Define the event that component $i$ is working as $A_i$, and the event that the system is working as $A$. Then, the probability that the system is working is

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A_1 \cap (A_2 \cup A_3)) \\ &= \mathbb{P}(A_1)\,\mathbb{P}(A_2 \cup A_3) \\ &= \mathbb{P}(A_1)\,(\mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(A_2 \cap A_3)) \\ &= \mathbb{P}(A_1)\,(\mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(A_2)\,\mathbb{P}(A_3))\end{aligned}$$

Notice that, since the time to failure of the components follows an exponential distribution,
$\mathbb{E}(t \text{ of failure}) = 1 = 1/\lambda \Rightarrow \lambda = 1$.
The cumulative distribution function of failure at some time $t$ is then

$$F(t) = 1 - \lambda e^{-\lambda t} = 1 - e^{-t}$$

And so the probability that a component is successfully running at time $t$ is

$$\begin{aligned}\mathbb{P}(A_i) &= 1 - F(t) \\ &= 1 - (1 - e^{-t}) = e^{-t}\end{aligned}$$

And so the survival function is then

$$\begin{aligned}R(t) &= e^{-t}\left(e^{-t} + e^{-t} - e^{-t}e^{-t}\right) \\ &= e^{-t}\left(2e^{-t} - e^{-2t}\right) \\ &= 2e^{-2t} - e^{-3t}\end{aligned}$$

b. The mean time to failure for the system is given by the integral of the survival function:

$$
\begin{aligned}
\mathbb{E}(t) &= \int_0^\infty R(t)\ dt \\
&= \int_0^\infty 2e^{-2t}\ dt - \int_0^\infty e^{-3t}\ dt \\
&= \left[-e^{-2t}\right]_0^\infty - \left[-\frac{e^{-3t}}{3}\right]_0^\infty \\
&= (0 - -e^0) - (-0 + e^0/3) \\
&= \frac{2}{3}
\end{aligned}
$$

And so the mean time to failure is about two thirds of a year.

c. The probability that component two is working given that the system is still working can be expressed as

$$
\mathbb{P}(A_2 \mid A) = \frac{A_2 \cap A}{\mathbb{P}(A)}
$$

Now, notice that

$$
\mathbb{P}(A_2 \mid A) = \frac{\mathbb{P}(A \mid A_2)\,\mathbb{P}(A_2)}{\mathbb{P}(A)}
$$

and notice that $\mathbb{P}(A \mid A_2) = \mathbb{P}(A_1)$ (since if component two is working, all that is needed for the system to work is that component 1 is working). Then,

$$
\begin{aligned}
\mathbb{P}(A_2 \mid A) &= \frac{\mathbb{P}(A_1)\,\mathbb{P}(A_2)}{\mathbb{P}(A)} \\
&= \frac{e^{-t}e^{-t}}{2e^{-2t} - e^{-3t}} \\
&= \frac{e^{-2t}}{e^{-2t}(2 - e^{-t})} \\
&= \frac{1}{2 - e^{-t}}
\end{aligned}
$$

Then, as $t \to \infty$, $\mathbb{P}(A_2 \mid A) \to 1/2$.

d. To determine the failure rate of the system, first the cdf of the failure must be found.

$$
\begin{aligned}
R(t) &= 1 - F(t) \\
\Rightarrow F(t) &= 1 - R(t) \\
&= 1 - 2e^{-2t} + e^{-3t}
\end{aligned}
$$

The pdf of failure given $t$ is then found by differentiating the cdf:

$$
\begin{aligned}
f(t) &= \frac{d}{dt}F(t) \\
&= 4e^{-2t} - 3e^{-3t}
\end{aligned}
$$

And finally, the failure rate can be found with these two equations by

$$
\begin{aligned}
h(t) &= \frac{f(t)}{1 - F(t)} \\
&= \frac{4e^{-2t} - 3e^{-3t}}{2e^{-t} - e^{-3t}} \\
&= \frac{4 - 3e^{-t}}{2 - e^{-t}}
\end{aligned}
$$

# 4 Question 4

a. The transition matrix for the Markov chain is given by

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & \alpha & 1-\alpha & 0 \\ 0 & 0 & \beta & 1-\beta \\ 1-\gamma & 0 & 0 & \gamma \end{pmatrix}$$

Given that $\boldsymbol{\pi} = \boldsymbol{\pi} P$ for some limiting distribution $\boldsymbol{\pi}$, the values of $(\alpha, \beta, \gamma)$ such that $\boldsymbol{\pi} = (1/10,\ 2/10,\ 3/10,\ 4/10)$ is

$$(1/10,\ 2/10,\ 3/10,\ 4/10) = (1/10,\ 2/10,\ 3/10,\ 4/10) \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & \alpha & 1-\alpha & 0 \\ 0 & 0 & \beta & 1-\beta \\ 1-\gamma & 0 & 0 & \gamma \end{pmatrix}$$

From this, a series of simultaneous equations can be produced:

$$\frac{1}{10} = \frac{1}{2} \cdot \frac{1}{10} + \frac{4}{10}(1-\gamma) \Rightarrow 1 = \frac{1}{2} + 4 - 4\gamma \Rightarrow \gamma = \frac{7}{8}$$
$$\frac{2}{10} = \frac{1}{10} \cdot \frac{1}{2} + \frac{2}{10} \cdot \alpha \Rightarrow 2 = \frac{1}{2} + 2\alpha \Rightarrow \alpha = \frac{3}{4}$$
$$\frac{3}{10} = \frac{2}{10}(1-\alpha) + \frac{3}{10}\beta \Rightarrow 3 = 2 - 2\alpha + 3\beta \Rightarrow \beta = \frac{3-2+2\alpha}{3} = \frac{5/2}{3} = \frac{5}{6}$$

And so $(\alpha, \beta, \gamma) = (3/4,\ 5/6,\ 7/8)$

b. With $(\alpha, \beta, \gamma) = (1/3,\ 1/4,\ 1/5)$, a typical random walk using this Markov chain is shown in Figure 3 below:

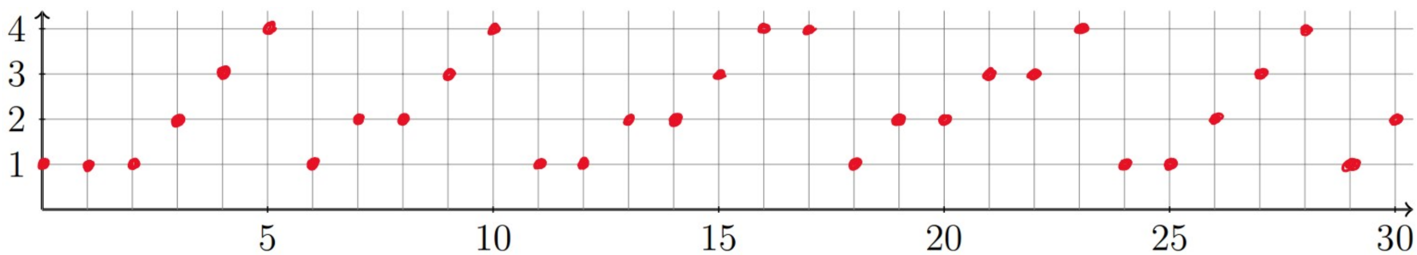


Figure 3: Example Random Walk in the Markov Chain

# 5 Question 5

a. The Chernoff bound says that

$$\mathbb{P}(h(X) \geq a) = \mathbb{P}\left(e^{th(X)} \geq e^{ta}\right) \leq \frac{\mathbb{E}\left(e^{th(x)}\right)}{e^{ta}}$$

Given that the random variables have a Geometric distribution, $\mathbb{E}(X_i) = 1/p$, and so the sum of the variables, $S_n$ will have expectation $\mathbb{E}(S_n) = n/p = \mu$, where $\mu$ is the mean of the sum of the geometric random variables. Putting these into the Chernoff bound gives

$$\begin{aligned} \mathbb{P}(S_n \geq a\mu) &\leq e^{-ta\mu}\mathbb{E}\left(e^{tS_n}\right) \\ &= e^{-ta\mu} \prod_{i=1}^{n} \mathbb{E}\left(e^{tX_i}\right) \end{aligned}$$

Now, note that the pgf for $X_i \sim \text{Geom}(p)$ is

$$\begin{aligned} G(z) &= \frac{zp}{1-z(1-p)} \\ \Rightarrow G\left(e^t\right) &= \frac{e^t p}{1-e^t(1-p)} \\ \Rightarrow \mathbb{P}(S_n \geq a\mu) &\leq e^{-ta\mu}\left(\frac{e^t p}{1-e^t(1-p)}\right)^n \end{aligned}$$

Now, substituting $\mu = n/p$ back in gives

$$\begin{aligned}
\mathbb{P}\left(S_n \geq \frac{an}{p}\right) &\leq e^{-tan/p} \cdot \frac{e^{tn}p^n}{(1-(1-p)e^t)^n} \\
\Rightarrow \mathbb{P}\left(S_n \geq \frac{an}{p}\right) &\leq \frac{p^n e^{nt(1-a/p)}}{(1-(1-p)e^t)^n} =: \mathcal{H}(t,\, a)
\end{aligned}$$

b. First, notice that

$$\frac{p^n e^{nt(1-a/p)}}{(1-(1-p)e^t)^n} = \left(\frac{pe^{t(1-a/p)}}{1-(1-p)e^t}\right)^n = \mathcal{H}(t,a)$$

If $(\mathcal{H}(t,a))^{1/n}$ is minimised, then so is $\mathcal{H}(t,a)$ (since $n \geq 1$). Now we want to minimise

$$g(t,a) := \frac{pe^{t(1-a/p)}}{1-(1-p)e^t}$$

And now, note that if $\ln(()\, g)$ is minimised, then so is $g$. Finally, we want to minimise

$$\begin{aligned}
h(t,a) &:= \ln\left(\frac{pe^{t(1-a/p)}}{1-(1-p)e^t}\right) \\
&= \ln\left(pe^{t(1-a/p)}\right) - \ln\left(1-(1-p)e^t\right) \\
&= \ln(p) \cdot t\left(1-\frac{a}{p}\right) - \ln\left(1-(1-p)e^t\right)
\end{aligned}$$

To minimise this, set $h'(t,a) = 0$.

$$\begin{aligned}
0 &= \frac{d}{dt}h(t,a) \\
&= \ln(p)\left(1-\frac{a}{p}\right) - \frac{\partial \ln(u)}{\partial u}\frac{\partial u}{\partial t}
\end{aligned}$$

Where $u = 1-(1-p)e^t \Rightarrow \partial u/\partial t = (1-p)e^t$. Then, $\partial \ln(u)/\partial u = 1/u$.

$$\begin{aligned}
\Rightarrow \ln(p)\left(1-\frac{a}{p}\right) &= \frac{(1-p)e^t}{1-(1-p)e^t} \\
&= \frac{(1-p)e^t}{1-(1-p)e^t} \cdot \frac{e^{-t}}{e^{-t}} \\
&= \frac{1-p}{e^{-t}-1+p} \\
\Rightarrow e^{-t} - 1 + p &= \frac{1-p}{\ln(p)(1-a/p)} \\
e^{-t} &= \frac{1-p}{\ln(p)(1-a/p)} - p + 1 \\
\Rightarrow t &= -\ln\left(\frac{1-p}{\ln(p)(1-a/p)} - p + 1\right)
\end{aligned}$$