MATH1071 Course Notes

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Release 1.0
10th March, 2022

Contents

Preface

The content of these notes are my own personal study notes for MATH1071, taken during the 2020 Semester 1 offering and lectured by Associate Professor Artem Pulemotov. Converting the course notes to a digital format was done independently of the course coordinator and commenced in November 2020. Progress was accelerated during the 2021 offering of the course, where work-in-progress revisions were sent to students for feedback. After a hiatus from studying for Semester 2 exams, these notes were finalised at the start of Semester 1 2022.

These notes intend to cover most lecture material in written format. I have also added summaries and alternate explanations throughout the document to give students a different perspective. These notes are not a replacement for lectures, as those sessions offer immediate feedback to questions and go into greater detail than described here. Watching lectures in full, taking your own notes then comparing them with these would be an ideal strategy to tackle the course (plus tutorials, of course).

I would like to thank Ciarán Komarakul-Greene, Ben Kruger, Arwen Nugteren and Max Orchard for their significant contributions to proofreading the notes and rephrasing technical sections to improve clarity. Their assistance helped fill in the minor details to produce a complete cohesive document, which I really appreciate. I would also like to thank the lecturer for his engaging and entertaining teaching, which made a rough entry into university enjoyable.

ducky
March, 2022

Chapter 1

Set Theory and Fields

1.1 Sets and Functions

We usually think of a set as a list of elements or a collection of things, but those definitions are just synonyms of the word set. Although they can be defined in terms of axioms, in this course we appeal to our intuition.

Definition 1.1.1: Set

A *set* is an unordered list of elements.

The set containing the elements a, b, c and d is written as $\{a, b, c, d\}$. For some set A, $a \in A$ means that a is an element of set A. If all elements in set A are within set B, we say that A is an (improper) *subset* of B, written as $A \subseteq B$. A and B not being equal changes the improper subset to a *proper subset*, $A \subset B$, like $\leq$ and $<$ for numbers. The empty set $\emptyset$ contains no elements, and is considered to be a subset of every other set.

It is also possible to define sets without listing all of their elements. The set of all a which satisfy property $P(a)$ is written as $\{a \mid P(a)\}$. For example, the set of all numbers greater than 3 is written as $\{x \mid x > 3\}$. This notation can be used to define some operations on sets:

- The *union* of A and B is defined as $A \cup B = \{c \mid c \in A \text{ or } c \in B\}$ and gives a set with all elements in both A and B.
- The *intersection* of A and B is defined as $A \cap B = \{c \mid c \in A \text{ and } c \in B\}$ and gives a set with only the elements common to both A and B.
- The *set difference* of B from A is defined as $A \setminus B = \{c \mid c \in A \text{ and } c \notin B\}$ and gives all elements of A that are not also in B.
- The *Cartesian product* of A and B is defined as $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ and gives all possible pairs of elements from both sets.

Note that there is no notion of duplicate elements in sets – an element is either in the set or it is not. Taking the union of sets with common elements makes those elements appear "once" in the result.

1.2 Sets of Numbers

Here are common sets of numbers. For this course, we will take the following definition for the natural numbers for granted.

Definition 1.2.1: Natural Numbers

The set of *natural numbers* $\mathbb{N}$ is the set of "counting numbers", i.e. $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$.

Definition 1.2.2: Integers

The set of *integers* $\mathbb{Z}$ is defined as $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n \mid n \in \mathbb{N}\}$.

Definition 1.2.3: Rational Numbers

The set of *rational numbers* $\mathbb{Q}$ is defined as

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z}, q \in \mathbb{N} \right\}.$$

We identify p_1/q_1 and p_2/q_2 to be the same if $p_1 = kp_2$ or $q_1 = kq_2$ for some $k \in \mathbb{Z}, k \neq 0$. This is so that $1/2$ and $2/4$ are identified as the same quantity as the numbers can be cancelled.

Definition 1.2.4: Real Numbers

The construction of the set of *real numbers* $\mathbb{R}$ will not be covered in this course. A simplified definition which isn't completely true but will suffice is the set of "finite and infinite decimals".

Definition 1.2.5: Complex Numbers

The set of *complex numbers* $\mathbb{C}$ is defined as $\mathbb{C} = \{a+ib \mid a, b \in \mathbb{R}\}$ where the *imaginary unit* i is such that $i^2 = -1$.

1.3 Common Functions

The notation $f : A \to B$ defines f as a function or mapping from the set A to the set B.

Definition 1.3.1: Absolute Value

The *absolute value* function $|\cdot| : \mathbb{R} \to [0, \infty)$ is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Definition 1.3.2: Floor Function

The *floor* function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is defined by the property that $\lfloor x \rfloor$ is the greatest integer such that $\lfloor x \rfloor \leq x$.

Definition 1.3.3: Positive Square Root

The positive square root $\sqrt{\cdot} : [0, \infty) \to [0, \infty)$ has the property that $\sqrt{x}$ is the unique non-negative number such that $(\sqrt{x})^2 = x$.

1.4 Fields

For the rest of the notes, the symbols $\forall$ and $\exists$ within mathematical statements stand for *for all* and *there exists* respectively. $\forall\, a\, \exists\, b$: $a + 2b = 0$ is read as "for all a, there exists b such that $a + 2b = 0$".

Definition 1.4.1: Field

A *field* is a set $\mathbb{F}$ equipped with addition $+ : \mathbb{F}\times\mathbb{F} \to \mathbb{F}$ and multiplication $\cdot : \mathbb{F}\times\mathbb{F} \to \mathbb{F}$, such that for all $a, b, c \in \mathbb{F}$, the following *field axioms* hold:

Name	$+$	$\cdot$
Associativity	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
Identity	$\exists\, 0 \in \mathbb{F}$: $a + 0 = a$	$\exists\, 1 \in \mathbb{F}$: $a \cdot 1 = a$
Inverse	$\exists\, {-a} \in \mathbb{F}$: $a + (-a) = 0$	$\forall\, a \neq 0\, \exists\, a^{-1} \in \mathbb{F}$: $a \cdot a^{-1} = 1$
Distributivity	$a \cdot (b + c) = a \cdot b + a \cdot c$	

These operations and axioms are taken for granted when manipulating expressions in algebra. However, it is helpful to look at them from a more general point of view as in the following examples.

Example 1.4.1

$\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ are fields when equipped with the standard addition and multiplication operations. Note that fields have *additional structure* compared to sets, so we need to specify what operations are defined.

Example 1.4.2

The set $\mathbb{F} = \{0, 1\}$ is a field if we define operations according to the following *Cayley tables*:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Field axioms can be partially verified from Cayley tables using shortcuts:

- symmetry along the main diagonal of a table indicates commutativity,
- identity can be checked by looking across the 0 or 1 column,
- inverse holds if each row/column has the identity element.

Distributivity and associativity need to be verified manually for each triple of elements. As with other "shortcuts", the above results should *not* be used in assessment unless justification (such as a proof) is provided.

Setting $0 = 1$ in $\mathbb{F}$ still satisfies the axioms but isn't really useful as a field.

Example 1.4.3
$\mathbb{N}$ and $\mathbb{Z}$ are *not* fields with standard operations. This is because $\mathbb{N}$ does not contain 0, the additive identity, and $\mathbb{Z}$ does not contain multiplicative inverses (except for 1).

Example 1.4.4
Note that as long as we can define operations on $\mathbb{F}$ which satisfy the axioms, $\mathbb{F}$ will be a field. For instance, consider $\mathbb{F} = \{\square, \triangle\}$. We can define addition and multiplication:

$+$	$\square$	$\triangle$
$\square$	$\square$	$\triangle$
$\triangle$	$\triangle$	$\square$

$\cdot$	$\square$	$\triangle$
$\square$	$\square$	$\square$
$\triangle$	$\square$	$\triangle$

Comparing this to the tables defined in Example 1.4.2, we can gather that $\square$ is the 0 element and $\triangle$ is the 1 element. $\mathbb{F}$ is therefore a field.

Example 1.4.5
What if we change the definition of multiplication in the above example to the following?

$\cdot$	$\square$	$\triangle$
$\square$	$\triangle$	$\square$
$\triangle$	$\square$	$\square$

Now, $\square$ is not the multiplicative identity because $\square \cdot \square = \triangle \neq \square$, and $\triangle$ isn't the multiplicative identity either because $\triangle \cdot \triangle = \square \neq \triangle$. Thus, the new $\mathbb{F}$ is not a field.

Theorem 1.4.2
For all $a \in \mathbb{F}$, $a \cdot 0 = 0 \cdot a = 0$.

Proof. We use the field axioms.

$$\begin{aligned}
a \cdot 0 &= a \cdot (0 + 0) && \text{(id.)}\\
a \cdot 0 &= a \cdot 0 + a \cdot 0 && \text{(distr.)}\\
a \cdot 0 + -(a \cdot 0) &= (a \cdot 0 + a \cdot 0) + -(a \cdot 0) && \text{(equality)}\\
0 &= (a \cdot 0 + a \cdot 0) + -(a \cdot 0) && \text{(inv.)}\\
0 &= (a \cdot 0) + (a \cdot 0 + -(a \cdot 0)) && \text{(assoc.)}\\
0 &= (a \cdot 0) + 0 && \text{(inv.)}\\
0 &= a \cdot 0. && \text{(id.)}
\end{aligned}$$

Thus, $0 = a \cdot 0$, and by commutativity, $0 \cdot a = 0$. ∎

Theorem 1.4.3
The additive identity is unique. In other words, for all $a \in \mathbb{F}$, if there exists some $\tilde{0}$ such that $\tilde{0} + a = a + \tilde{0} = a$, then $0 = \tilde{0}$.

Proof. By commutativity of addition,

$$0 + \tilde{0} = \tilde{0} + 0 = \tilde{0}.$$

By definition of $\tilde{0}$ in the theorem statement, with $a = 0$,

$$\tilde{0} + 0 = 0 + \tilde{0} = 0.$$

Thus, $0 = \tilde{0}$. □

Theorem 1.4.4

If $a \cdot b = 0$ for some $a, b \in \mathbb{F}$, then $a = 0$ or $b = 0$ or both.

Proof. If $a = 0$ the result follows so assume $a \neq 0$. Then, there exists some $a^{-1} \in \mathbb{F}$. We then use the field axioms.

$$\begin{aligned} a \cdot b &= 0 \\ a^{-1} \cdot (a \cdot b) &= a^{-1} \cdot 0 && \text{(equality)} \\ (a^{-1} \cdot a) \cdot b &= a^{-1} \cdot 0 && \text{(assoc.)} \\ 1 \cdot b &= a^{-1} \cdot 0 && \text{(inv.)} \\ b &= a^{-1} \cdot 0 && \text{(id.)} \\ b &= 0. && \text{(Theorem 1.4.2)} \end{aligned}$$

Thus, $b = 0$. □

Chapter 2

Sequences and Limits

2.1 Upper and Lower Bounds

Definition 2.1.1: Upper and Lower Bounds

Let $\Omega \subset \mathbb{R}$ be a set. The number $b \in \mathbb{R}$ is

- an *upper bound* of Ω if $b \geq x$ for all $x \in \Omega$
- a *lower bound* of Ω if $b \leq x$ for all $x \in \Omega$.

Definition 2.1.2: Supremum and Infimum

b is a least upper bound or *supremum* of Ω, written $b = \sup \Omega$, if

1. b is an upper bound of Ω, and
2. $b \leq C$ for any upper bound C of Ω.

The greatest lower bound or *infimum* of Ω, written $\inf \Omega$, is defined analogously.

Suprema and infima are useful since they formalise the maximum element of a set without requiring that they are elements of the set. For example,

- $\sup[0,1] = \sup(0,1) = 1$.
- $\max[0,1] = 1$ but $\max(0,1)$ does not exist.
- $\inf(0,1) = 0$.
- $\sup((0,1) \cup \{10\}) = 10$.
- $\sup \mathbb{N}$ does not exist, and $\inf \mathbb{N} = 1$.

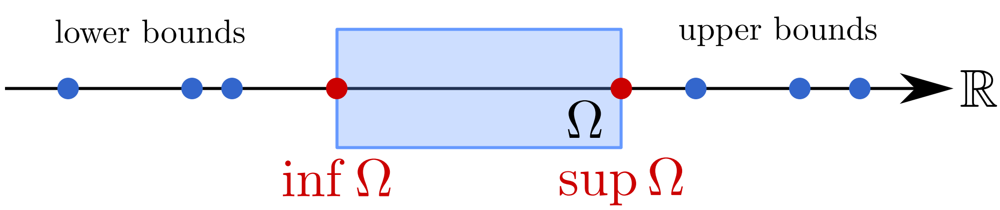


Figure 2.1: Supremum and infimum of a set.

Every subset of $\mathbb{R}$ that has an upper bound also has a supremum in $\mathbb{R}$. This is called the *least upper bound property* of $\mathbb{R}$. The field $\mathbb{Q}$ does not have this property – for instance, the set $\mathbb{Q} \cap [0, \pi)$ has no supremum in $\mathbb{Q}$ since π is irrational.

Proposition 2.1.3

The supremum and infimum of a set are unique.

Proof. Suppose S and $\tilde{S}$ are suprema of Ω. Then, both are upper bounds of Ω. Since a supremum is less than or equal to all upper bounds, $S \leq \tilde{S}$ and $\tilde{S} \leq S$, thus $S = \tilde{S}$.

The proof is analogous for $\inf \Omega$. $\square$

Instead of defining suprema and infima in terms of other upper or lower bounds, we can also define them in terms of their proximity to elements of the set. The following theorem formalises the idea that $\sup \Omega$ is as close as possible to the elements of Ω.

Theorem 2.1.4: Epsilon Definition of Supremum

Assume Ω is not empty. Then, $S = \sup \Omega$ if and only if

1. *S is an upper bound of Ω (for all $x \in \Omega$, $S \geq x$), and*
2. *for all $\varepsilon > 0$, there exists some $x \in \Omega$ such that $S - \varepsilon < x$.*

Here, ε acts as the width of a window that some $x \in \Omega$ has to fit into in order for S to be the supremum. This is described by $S - \varepsilon < x \leq S$ where $x \leq S$ comes from the first requirement. We can also write the inequality as $S - x < \varepsilon$, indicating that as ε shrinks, we will always find x close enough to S if S is the supremum. This form will be used in more definitions throughout the course, starting with limits.

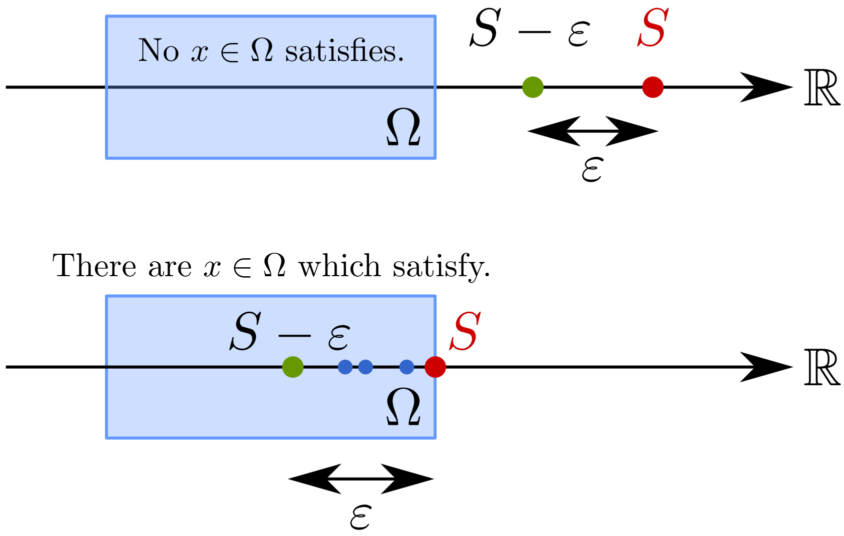


Figure 2.2: Intuition behind Theorem 2.1.4. Blue points are values of x such that $S - \varepsilon < x$.

Proof. We first prove in the forward direction, showing that the supremum of a set satisfies both requirements. Assume $S = \sup \Omega$. By definition, $\sup \Omega$ is an upper bound of Ω, so requirement 1 is satisfied. We argue by contradiction to prove requirement 2 is satisfied. Assume requirement 2 is false. Then, there exists $\varepsilon > 0$ such that no $x \in \Omega$ satisfies $S - \varepsilon < x$. In other words, for all $x \in (S - \varepsilon, S]$, x cannot be in Ω, so $S - \varepsilon$ is an upper bound of Ω. However, because $S - \varepsilon < s$, this is impossible since S is the lowest upper bound. This is a contradiction which proves requirement 2 is satisfied.

We now prove in the reverse direction, trying to show that if S fulfils both requirements then S is the supremum of Ω.
Assume both requirements hold. By the first requirement, S is an upper bound of Ω. To show that S is the supremum, we want to show that S is less than or equal to all other upper bounds of Ω. Take another upper bound $\tilde{S}$, and we will argue by contradiction to prove $S \leq \tilde{S}$.

Assume $S > \tilde{S}$ and choose $\varepsilon = \frac{S-\tilde{S}}{2} > 0$. We claim there is no $x \in \Omega$ such that $x > S - \varepsilon$. If $x > S - \varepsilon$, then

$$x > S - \frac{S-\tilde{S}}{2} = \frac{2S - S + \tilde{S}}{2} = \frac{S+\tilde{S}}{2} > \frac{\tilde{S}+\tilde{S}}{2} = \tilde{S}.$$

Thus, if $x > S - \varepsilon$, then $x > \tilde{S}$, and $x \notin \Omega$ because $\tilde{S}$ is an upper bound of Ω. However, requirement 2 fails because we have found an ε such that x is no longer in Ω under the conditions. This contradiction proves that $S \leq \tilde{S}$, so $S = \sup \Omega$. □

2.2 Limits and their Properties

Definition 2.2.1: Sequence

A *sequence* of elements from a set X is a map from $\mathbb{N}$ to X, assigning every index (natural number) to something in X.
For this course, sequences are represented as $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ etc. where $a_1, a_2, \cdots$ are its elements. Note the distinction between $(a_n)_{n=1}^{\infty}$ and a_n, where the former indicates the sequence and the latter indicates its n^{th} element.

Examples of sequences include $1, 2, 3, 4, \cdots$ and $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$, but they don't have to have a pattern – an infinite list of random numbers would be a seequence. Taking $a_n = \frac{1}{n}$ as an example, as n gets large a_n gets smaller and will approach 0. Sequences like $a_n = n$ or $a_n = (-1)^n$ (giving $-1, 1, -1, 1, \cdots$) don't approach any number as n gets large. This idea of the value of a_n for very large n is made precise by the definition of the limit.

Definition 2.2.2: Limit of a Sequence

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. The *limit* of $(a_n)_{n=1}^{\infty}$ equals $a \in \mathbb{R}$, written $\lim_{n\to\infty} a_n = a$, if for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - a| < \varepsilon$.

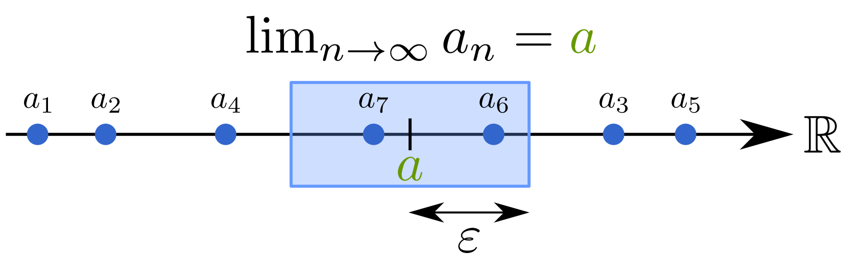


Figure 2.3: Illustration of the definition of the limit. The blue dots are a sequence a_n converging to a.

Depicted in Figure 2.3, for a sequence with $\lim_{n\to\infty} a_n = a$, we can take a neighbourhood around a with radius ε – the definition guarantees there is some cutoff point N where all elements of index greater than N will lie inside that neighbourhood. $|a_n - a| < \varepsilon$ means a_n is within ε of a, so we can make ε smaller and smaller so a_n can fit inside with large enough N.

Definition 2.2.3: Convergence of a Sequence

The sequence $(a_n)_{n=1}^{\infty}$ *converges* to a if $\lim_{n\to\infty} a_n = a$. If no a satisfying the definition of the limit exists, we say $(a_n)_{n=1}^{\infty}$ *diverges* or that the limit does not exist.

Definition 2.2.4: Epsilon Neighbourhood of a Sequence

For some $\varepsilon > 0$, the set $\{x \in \mathbb{R} \mid |x - a| < \varepsilon\}$ is called the *epsilon neighbourhood* of a.

Note that $|x - a| < \varepsilon$ if and only if $-\varepsilon < x - a < \varepsilon$, if and only if $a - \varepsilon < x < a + \varepsilon$. Writing the condition in this fashion may be useful for certain proofs.

Example 2.2.1

We want to prove the following limit:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Recalling the definition of the limit, we want to show that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon.$$

Fix a positive value of ε. We can simplify the absolute value:

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}$$

We want $\frac{1}{n-1} < \varepsilon$, equivalently $n + 1 > 1/\varepsilon$, so we find an expression for N such that this holds if $n \geq N$. We cannot set $N = \frac{1}{\varepsilon}$ since N is natural, so we take $N = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$, adding one to avoid N being zero. Now, if $n \geq N$, then:

$$n + 1 \geq N + 1 = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1 + 1 = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 2 > \frac{1}{\varepsilon}.$$

So, $n + 1 > \frac{1}{\varepsilon}$ which is what we wanted to show. Thus, if $N = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$ and $n \geq N$, then $\left|\frac{n}{n+1} - 1\right| < \varepsilon$, hence $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Example 2.2.2

We want to prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Fix $\varepsilon > 0$. We want to find $N \in \mathbb{N}$ such that $\left|\frac{1}{n} - 0\right| < \varepsilon$. Equivalently, $\frac{1}{n} < \varepsilon$, which holds if $n \geq \left\lfloor \frac{1}{n} \right\rfloor + 1$, using the same floor function technique from the previous example. Thus, if $N = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$ and $n \geq N$, then $\left| \frac{1}{n} - 0 \right| < \varepsilon$ and $\lim_{n \to \infty} \frac{1}{n} = 0$.

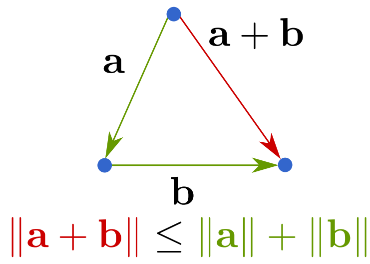


Figure 2.4: Visual representation of the Triangle Inequality.

Theorem 2.2.5: Triangle Inequality

For all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

Proof. An efficient approach is to note that

$$|a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2 \leq |a|^2 + |b|^2 + 2|a||b| = (|a| + |b|)^2 ,$$

so taking the square root of both sides[1] gives $|a + b| \leq |a| + |b|$. □

Alternate Proof. We can also prove by splitting the problem into cases.

Case 1: If a and b are both positive, then $a + b \geq 0$ so $|a + b| = a + b = |a| + |b|$.

Case 2: Similarly, if both are negative, then $a + b \leq 0$ so

$$|a + b| = -(a + b) = -a + -b = |a| + |b| .$$

From here, without loss of generality, assume $a \geq 0$ and $b \leq 0$. If this is not the case, swap a and b. We must ensure that the following cases are exhaustive (cover all options) to ensure it is valid to swap the values.

Case 3: If $|a| \geq |b|$, then $a + b \geq 0$. Since $|x| \leq x$ for all x, we have $a + b \leq |a| + |b|$. However, as $a + b$ is positive, $a + b = |a + b|$, so we have $|a + b| \leq |a| + |b|$.

Case 4: To make the last case exhaustive, we calculate the negation of case 3, $|a| \geq |b|$, which is $|a| < |b|$. Assuming this, since $a \geq 0$ and $b \leq 0$, then $a + b \leq 0$. So,

$$\begin{aligned} |a + b| &= -(a + b) \\ &= -a + -b \\ &= -|a| + |b| \\ &< -|a| + |b| + 2|a| \\ &= |a| + |b| . \end{aligned}$$

All cases are covered which completes the proof. □

[1] Strictly speaking, we can square root both sides because $\sqrt{x}$ is monotone increasing (Definition 3.1.1) and both $|a + b|^2$ and $(|a| + |b|)^2$ are positive. These requirements allow the inequality to be preserved after applying the square root.

Theorem 2.2.6: Second Triangle Inequality

For all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$.

Proof. This was an assignment question. □

Theorem 2.2.7: Limit is unique

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$, then $a = b$.

Proof. Fix $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, there exists $N \in \mathbb{N}$ such that if $n \geq N_1$, then $|a_n - a| < \frac{\varepsilon}{2}$. Also, since $\lim_{n\to\infty} a_n = b$, then there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $|a_n - b| < \frac{\varepsilon}{2}$. Now, if $n \geq \max\{N_1, N_2\}$, then using the triangle inequality,

$$\begin{aligned} |a - b| &= |a - a_n + a_n - b| \\ &\leq |a - a_n| + |a_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

This holds for all $\varepsilon > 0$. Thus, $|a - b| = 0$, so $a = b$. □

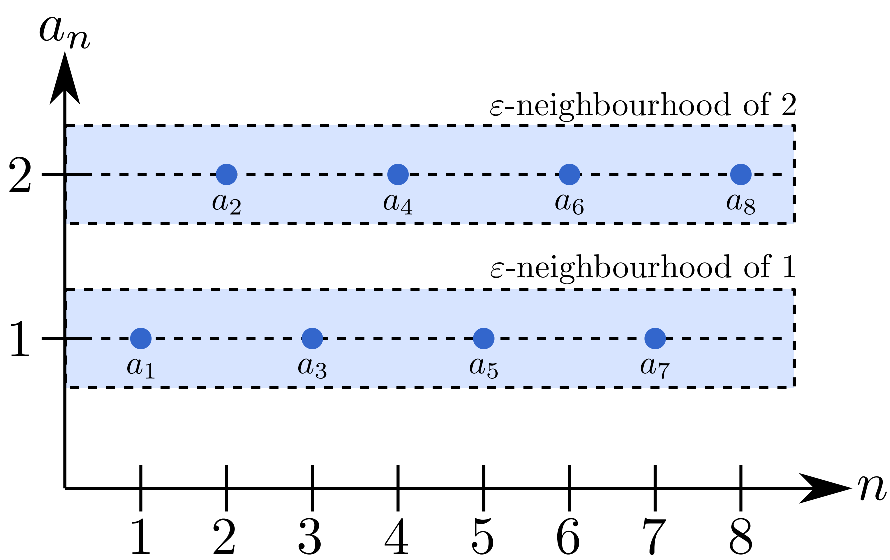


Figure 2.5: Intuition for Theorem 2.2.7. For the above sequence $(a_n)_{n=1}^{\infty}$, there is no limit because we can shrink ε enough that the neighbourhood around 2 does not contain any points of value 1. If a limit does exist, it has to be unique.

Theorem 2.2.8: Squeeze Theorem

Suppose sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are such that:

1. $a_n \leq b_n \leq c_n$ *for all* $n \in \mathbb{N}$
2. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

Then, $\lim_{n\to\infty} b_n = L$.

Proof. Observe that:

$$|b_n - L| = |b_n - a_n + a_n - L|$$

$$\begin{aligned}
&\le |b_n - a_n| + |a_n - L| \\
&= b_n - a_n + |a_n - L| \\
&\le c_n - a_n + |a_n - L| \\
&= c_n - L + L - a_n + |a_n - L| \\
&= |c_n - L + L - a_n| + |a_n - L| \\
&\le |c_n - L| + |L - a_n| + |a_n - L| .
\end{aligned}$$

Fix $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, by definition there exists $N_1 \in \mathbb{N}$ such that if $n \ge N_1$, then $|a_n - L| = |L - a_n| < \frac{\varepsilon}{3}$. Similarly, since $\lim_{n\to\infty} c_n = L$, by definition there exists $N_2 \in \mathbb{N}$ such that if $n \ge N_2$, then $|c_n - L| = |L - c_n| < \frac{\varepsilon}{3}$.

Let $N = \max\{N_1, N_2\}$. If $n \ge N$, then:

$$\begin{aligned}
|b_n - L| &\le |c_n - L| + |L - a_n| + |a_n - L| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Thus, $\lim_{n\to\infty} b_n = L$. □

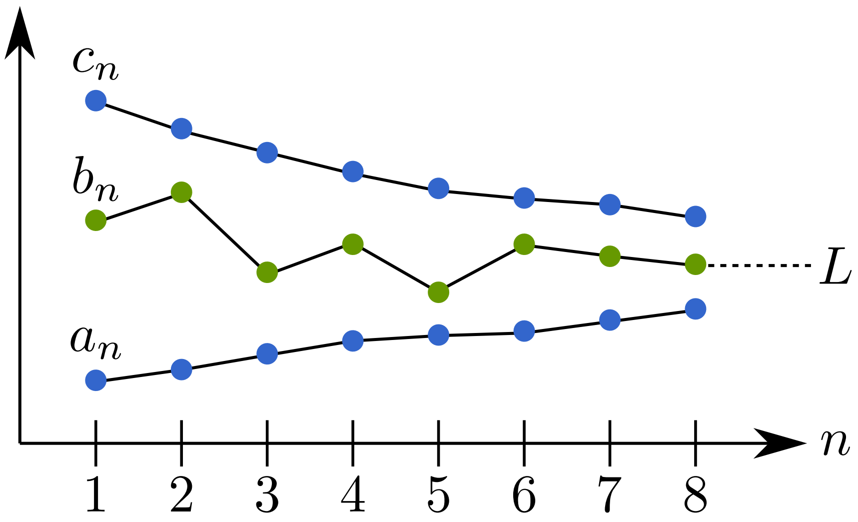


Figure 2.6: An illustration of Theorem 2.2.8.

Example 2.2.3

We will find the following limit using the Squeeze Theorem:

$$\lim_{n\to\infty} \frac{|\sin n^3|}{n^2}.$$

For all $n \in \mathbb{N}$, set $a_n = 0$ and $c_n = \frac{1}{n^2}$. Since $-1 \le \sin n \le 1$, apply $|\cdot|$ and divide by n^2 on all parts of the inequality to get

$$0 \le \frac{|\sin n^3|}{n^2} \le \frac{1}{n^2}.$$

Note $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = 0$. Next, we want to find $\lim_{n\to\infty} \frac{1}{n^2}$. We know $0 \le \frac{1}{n^2} \le \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$ so by the Squeeze Theorem, $\lim_{n\to\infty} \frac{1}{n^2} = 0$. Using the

Squeeze Theorem again on the original limit, since a_n and c_n both go to 0,

$$\lim_{n\to\infty} \frac{|\sin n^3|}{n^2} = 0.$$

Example 2.2.4

We will find $\lim_{n\to\infty} \frac{n-1}{n^4+2}$. Observe that

$$0 \le \frac{n-1}{n^4+2} \le \frac{n}{n^4+2} \le \frac{n}{n^4} = \frac{1}{n^3} < \frac{1}{n}.$$

Thus, $0 \le \frac{n-1}{n^4+2} \le \frac{1}{n}$. We note that $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$, so by applying the Squeeze Theorem we obtain

$$\lim_{n\to\infty} \frac{n-1}{n^4+2} = 0.$$

Example 2.2.5

Try finding $\lim_{n\to\infty} \frac{1}{n!}$ using the Squeeze Theorem with $0 \le \frac{1}{n!} < \frac{1}{n}$.

Example 2.2.6

To find the limit $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$, observe that

$$\begin{aligned}\sqrt{n+1} - \sqrt{n} &= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ &\le \frac{1}{\sqrt{n}}.\end{aligned}$$

Thus, $0 \le \sqrt{n+1} - \sqrt{n} \le \frac{1}{\sqrt{n}}$. By the Squeeze Theorem,

$$\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = 0.$$

Theorem 2.2.9: Convergence implies Boundedness

Convergent sequences are bounded. More precisely, if $(a_n)_{n=1}^{\infty}$ converges, then there exists $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \le M$.

Proof. Assume $\lim_{n\to\infty} a_n = L$. By the definition of the limit, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \ge N$, then $|a_n - L| < \varepsilon$. This holds, for instance, if $\varepsilon = 1$. So, there exists $N_1 \in \mathbb{N}$ such that if $n \ge N_1$, then $|a_n - L| < 1$. By the second triangle inequality (Theorem 2.2.6),

$$1 > |a_n - L| \ge ||a_n| - |L|| \ge |a_n| - |L|$$

since $|x| \geq x$ for all x. Thus, $|a_n| - |L| < 1$ so $|a_n| < 1 + |L|$. This is a bound for the "tail" of the sequence. For the rest of the elements, we simply use a maximum to bound the "head", thus

$$|a_n| \leq M = \max\{|a_1|, |a_2|, \ldots, |a_{N_1-1}|, |L| + 1\}$$

for all $n \in \mathbb{N}$. □

The following result allows us to evaluate limits without having to resort to an ε-N proof or the squeeze theorem every time.

Theorem 2.2.10: Sequence Limit Properties

Assume $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Fix* $\lambda \in \mathbb{R}$. *Then,*

1. $\lim_{n\to\infty}(a_n + b_n) = a + b$
2. $\lim_{n\to\infty} \lambda a_n = \lambda a$
3. $\lim_{n\to\infty} a_n b_n = ab$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, *provided that* $b \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$.

Remark. As a consequence of this theorem,

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty}(a_n + (-1)b_n) = \lim_{n\to\infty} a_n + (-1)\lim_{n\to\infty} b_n = a - b.$$

Proof. **Part 1** and **Part 2** are left as exercises.

Part 3: Observe that for $n \in \mathbb{N}$,

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\
&\leq |a_n(b_n - b)| + |b(a_n - a)| \\
&= |a_n| |b_n - b| + |b| |a_n - a|.
\end{aligned}$$

We know that $|a_n| < M$ for some $M > 0$ by Theorem 2.2.9. Thus,

$$|a_n b_n - ab| \leq M |b_n - b| + |b| |a_n - a|.$$

Fix $\varepsilon > 0$. There exists some $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|b_n - b| < \frac{\varepsilon}{2M}$. Also, there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|a_n - a| < \frac{\varepsilon}{2(|b|+1)}$, adding 1 to ensure a non-zero denominator. If $n \geq N = \max\{N_1, N_2\}$, then

$$\begin{aligned}
|a_n b_n - ab| &\leq M |b_n - b| + |b| |a_n - a| \\
&< M \cdot \frac{\varepsilon}{2M} + |b| \cdot \frac{\varepsilon}{2(|b|+1)} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Part 4: we will first prove that $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$. Fix $\varepsilon > 0$. Note

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b \cdot b_n}\right| = \frac{|b - b_n|}{|b| |b_n|}.$$

We need to show that $|b_n|$ does not become "small". If it is small, then the magnitude of the fraction will increase as $|b_n|$ get smaller, so we cannot bound it by ε. We will use the fact that $\lim_{n\to\infty} b_n = b \neq 0$, and we will show that since $b \neq 0$, $|b_n| > \frac{|b|}{2}$ for large enough n.

We use the definition of the limit for $(b_n)_{n=1}^{\infty}$ with $\varepsilon = \frac{|b|}{2}$. This tells us that there exists some $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $|b_n - b| < \frac{|b|}{2}$. By the second triangle inequality,

$$|b| - |b_n| \leq |b_n - b| < \frac{|b|}{2}.$$

So, $|b_n| > |b| - \frac{|b|}{2} = \frac{|b|}{2} > 0$. Thus, if $n \geq N_1$, then

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b|\,|b_n|} \leq \frac{b_n - b}{|b| \cdot \frac{|b|}{2}} = 2 \cdot \frac{1}{|b|^2} \cdot |b_n - b|\,.$$

Also, since $(b_n)_{n=1}^{\infty}$ converges, there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|b_n - b| < \frac{\varepsilon |b|^2}{2}$. If $n \geq \max\{N_1, N_2\}$, then

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| \leq \frac{2}{|b|^2} \cdot \frac{\varepsilon\,|b|^2}{2} = \varepsilon.$$

Thus, $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$. Using the result of **Part 3**, we obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n} = \frac{a}{b}.$$ □

Example 2.2.7

To find the limit $\lim_{n\to\infty} \frac{3n^2+2n}{5n^2+3}$, we can use Theorem 2.2.10 instead of a full ε-N proof as follows.

$$\begin{aligned}
\lim_{n\to\infty} \frac{3n^2+2n}{5n^2+3} &= \lim_{n\to\infty} \frac{3+\frac{2}{n}}{5+\frac{3}{n^2}} \\
&= \frac{\lim_{n\to\infty} 3+\frac{2}{n}}{\lim_{n\to\infty} 5+\frac{3}{n^2}} \\
&= \frac{\lim_{n\to\infty} 3 + 2\lim_{n\to\infty} \frac{1}{n}}{\lim_{n\to\infty} 5 + 3\lim_{n\to\infty} \frac{1}{n^2}} \\
&= \frac{3+2\cdot 0}{5+3\cdot 0} \\
&= \frac{3}{5}.
\end{aligned}$$

Proposition 2.2.11

If $b_k \neq 0$, then:

$$\lim_{n\to\infty} \frac{a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0}{b_k n^k + b_{k-1} n^{k-1} + \ldots + b_1 n^1 + b_0} = \frac{a_k}{b_k}.$$

Proof. Divide both the numerator and denominator by n^k and apply Theorem 2.2.10. □

Theorem 2.2.12

Here are some important limits whose proofs will not be given in this course.

1. *If* $\lambda \in (-1, 1)$, $\lim_{n\to\infty} \lambda^n = 0$.
2. *If* $c > 0$, $\lim_{n\to\infty} c^{\frac{1}{n}} = \lim_{n\to\infty} \sqrt[n]{c} = 1$.
3. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Chapter 3

More Sequences

3.1 Monotonicity

Definition 3.1.1: Monotone Sequence

The sequence $(a_n)_{n=1}^{\infty}$ is *monotone increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$. It is *monotone decreasing* if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$. $(a_n)_{n=1}^{\infty}$ is *monotone* if it is monotone increasing or monotone decreasing. Sometimes the terms non-increasing, non-decreasing, strictly increasing and strictly decreasing are used, akin to the difference between $\geq$ or $>$.

Theorem 3.1.2: Monotone Sequence Converges iff Bounded

A monotone sequence converges if and only if it is bounded.

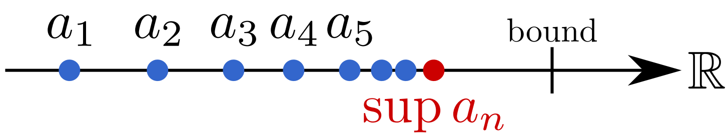


Figure 3.1: Illustration of Theorem 3.1.2.

Proof. The forward direction is already proven in Theorem 2.2.9. For the reverse direction, without loss of generality, assume $(a_n)_{n=1}^{\infty}$ is monotone increasing. If it is not, apply the theorem to $-a_n$. Define $\alpha = \sup\{a_1, a_2, \dots\} \in \mathbb{R}$, which exists as the sequence is bounded[1]. Let us show that $\alpha = \lim_{n\to\infty} a_n$.

Choose $\varepsilon > 0$. By Theorem 2.1.4, there exists $N \in \mathbb{N}$ such that $a_N \in (\alpha - \varepsilon, \alpha]$. By monotonicity, if $n \geq N$, then $a_n \geq a_N > \alpha - \varepsilon$. However, $a_n \leq \alpha$. Thus, for $n \geq N$,

$$\alpha - \varepsilon < a_n \leq \alpha < \alpha + \varepsilon,$$

so $|a_n - \alpha| < \varepsilon$. This is precisely the definition of the limit, so $\lim_{n\to\infty} a_n = \alpha$ and $(a_n)_{n=1}^{\infty}$ is convergent. $\square$

Lemma 3.1.1 (Bernoulli Inequality)**.** *If $x \geq -1$ and $n \in \mathbb{N} \cup \{0\}$, and $x = -1$ and $n = 0$ do not occur simultaneously, then $(1+x)^n \geq 1 + nx$.*

[1]This is due to $\mathbb{R}$ having the *least upper bound property* introduced in Section 2.1.

Proof. The equation in the lemma holds if $x = -1$. Assume $x \neq -1$. We prove by induction. For the base case, if $n = 0$, then $(1+x)^0 = 1 \geq 1$ as required. As the induction hypothesis, assume the inequality holds for $n = k$, i.e.

$$(1+x)^k \geq 1 + kx.$$

We now prove the inequality for $n = k+1$ as follows.

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)(1+x)^k \\
&\geq (1+x)(1+kx) \\
&= 1 + kx^2 + x + kx \\
&= 1 + (x + kx) + kx^2 \\
&= 1 + x(k+1) + kx^2 \\
&\geq 1 + x(k+1).
\end{aligned}$$

□

Theorem 3.1.3: Euler's Number and Limits

Consider the sequences $a_n = \left(1 + \frac{1}{n}\right)^n$ *and* $b_n = \left(1 + \frac{1}{n}\right)^{n+1}$*, for* $n \in \mathbb{N}$*. Then,* $(a_n)_{n=1}^{\infty}$ *is increasing,* $(b_n)_{n=1}^{\infty}$ *is decreasing, both converge, and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Proof. We will first prove the monotonicity of the sequences. Observe that

$$\begin{aligned}
a_n &= \left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n, \\
b_{n-1} &= \left(1 + \frac{1}{n-1}\right)^n = \left(\frac{n-1+1}{n-1}\right)^n = \left(\frac{n}{n-1}\right)^n.
\end{aligned}$$

If $n > 1$, then

$$\frac{a_n}{b_{n-1}} = \left(\frac{(n+1)(n-1)}{n^2}\right)^n = \left(\frac{n^2-1}{n^2}\right)^n \left(1 - \frac{1}{n^2}\right)^n = \left(1 + \left(\frac{-1}{n^2}\right)\right)^n.$$

Apply Lemma 3.1.1 with $x = \frac{-1}{n^2}$, which gives

$$\frac{a_n}{b_{n-1}} = \left(1 + \left(\frac{-1}{n^2}\right)\right)^n \geq 1 + n \cdot \frac{-1}{n^2} = 1 - \frac{1}{n} = \frac{n-1}{n}.$$

Similarly,

$$\frac{b_{n-1}}{a_n} = \left(\frac{n^2}{n^2-1}\right)^n = \left(\frac{n^2-1+1}{n^2-1}\right)^n = \left(1 + \frac{1}{n^2-1}\right) > \left(1 + \frac{1}{n^2}\right)^n.$$

Use Lemma 3.1.1 again with $x = \frac{1}{n^2}$ which gives

$$\frac{b_{n-1}}{a_n} > \left(1 + \frac{1}{n^2}\right)^n \geq 1 + n \cdot \frac{1}{n^2} = 1 + \frac{1}{n}.$$

To summarise, $\frac{a_n}{b_{n-1}} \geq \frac{n-1}{n}$ and $\frac{b_{n-1}}{a_n} > 1 + \frac{1}{n}$. This implies that

$$a_n \geq \frac{n-1}{n} \cdot b_{n-1}$$

$$\begin{aligned}
&= \left(\frac{n}{n-1}\right)^{-1}\left(\frac{n}{n-1}\right)^{n} \\
&= \left(\frac{n-1+1}{n-1}\right)^{n-1} \\
&= \left(1+\frac{1}{n-1}\right)^{n-1} \\
&= a_{n-1},
\end{aligned}$$

so $(a_n)_{n=1}^{\infty}$ is increasing. Also,

$$b_{n-1} > \left(1+\frac{1}{n}\right) a_n = \left(1+\frac{1}{n}\right)^{n+1} = b_n,$$

so $(b_n)_{n=1}^{\infty}$ is decreasing. Furthermore, $a_n < b_n$ by their definitions, so for all $n \geq 2$,

$$2 = a_1 \leq a_n < b_n \leq b_1 = 4.$$

Thus, both (monotone) sequences are bounded, so they converge. Lastly,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty} a_n\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} a_n,$$

showing both sequences converge to the same limit. □

Remark. The common limit of the above sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ is called $e \approx 2.71828\ldots$. Somewhat related, the Russian writer Leo Tolstoy was born in 1828.

Definition 3.1.4: Sequence Limits to Infinity

The sequence $(a_n)_{n=1}^{\infty}$ *goes to infinity*, written $\lim_{n\to\infty} a_n = \infty$, if for every $M > 0$, there exists $N \in \mathbb{N}$ such that $a_n > M$ when $n \geq N$. Similarly, we define $\lim_{n\to\infty} a_n = -\infty$ with $M < 0$, $a_n < M$.

Example 3.1.1

Examples of sequences going to infinity include $\lim_{n\to\infty} n^3 = \infty$ and $\lim_{n\to\infty} n - n^2 = \infty$.

3.2 Subsequences and Upper/Lower Limits

Definition 3.2.1: Upper and Lower Limits

Given $(a_n)_{n=1}^{\infty}$, define

$$\begin{aligned}
x_n &= \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} \\
y_n &= \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}.
\end{aligned}$$

$\lim_{n\to\infty} x_n$ is called the *upper limit* of $(a_n)_{n=1}^{\infty}$, written $\limsup_{n\to\infty} a_n$. Similarly, $\lim_{n\to\infty} y_n$ is called the *lower limit* of $(a_n)_{n=1}^{\infty}$, written $\liminf_{n\to\infty} a_n$. Note that $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ may be ∞ or $-\infty$ respectively.

This definition gives a notion of limiting value for sequences that don't converge, as shown in the following examples.

Example 3.2.1

Given $a_n = \{1, -1, 0, 1, -1, 0, \cdots\}$, $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$. Indeed, $x_n = 1$ and $y_n = -1$ for all $n \in \mathbb{N}$. However, $\lim_{n\to\infty} a_n$ doesn't exist.

Example 3.2.2

Let $a_n = 1 + \frac{1}{n} = \{2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \ldots\}$. Then, $x_n = \{2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, \ldots\}$, so $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n = 1$. Likewise, $y_n = \{1, 1, 1, \ldots\}$ so $\liminf_{n\to\infty} a_n = 1$. Also, $\lim_{n\to\infty} a_n = 1$ as well. Coincedence? I think not.

From the definition of upper and lower limits, all *bounded* sequences have lim sup and lim inf that aren't ∞ or $-\infty$. Since all convergent sequences are bounded (Theorem 2.2.9), convergent sequences also have such lim sup and lim inf. In fact, we have the following result which I could not find in my notes, but must at least have been mentioned in passing.

Theorem 3.2.2

A bounded sequence $(a_n)_{n=1}^{\infty}$ *converges if and only if* $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$.

Definition 3.2.3: Subsequence

Let $(a_n)_{n=1}^{\infty}$ be a sequence and $\{n_1, n_2, n_3, \ldots\}$ be a strictly increasing sequence of natural numbers. Then, $(a_{n_k})_{k=1}^{\infty}$ is a *subsequence* of $(a_n)_{n=1}^{\infty}$.

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad \cdots$$

Figure 3.2: For this sequence we pick $n_1 = 3$, $n_2 = 4$, $n_3 = 6$ etc. The subsequence is $\{a_3, a_4, a_6, a_8, \ldots\}$.

Example 3.2.3

Consider the sequence $a_n = \{1, -1, 1, -1, \ldots\}$. Two subsequences are

$$\{a_1, a_3, a_5, a_7, \ldots\} = \{1, 1, 1, 1, \ldots\}$$
$$\{a_2, a_4, a_6, a_8, \ldots\} = \{-1, -1, -1, -1, \ldots\}.$$

Proposition 3.2.4

The sequence $a_n = \{1, -1, 1, -1, \ldots\}$ *diverges.*

Proof. Assume $\lim_{n\to\infty} a_n = \alpha$. α can either be equal to 1, -1 or not equal to either. For this proof, assume $\alpha \neq \pm 1$ since the other cases are treated similarly. We know that for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - \alpha| < \varepsilon$.

Since we prove by contradiction, we want to show that there exists some $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, $n \geq N$ implies $|a_n - \alpha| \geq \varepsilon$. Take $\varepsilon = \min\{\frac{|\alpha-1|}{2}, \frac{|\alpha-(-1)|}{2}\}$. If $a_n = 1$, then

$$|\alpha - a_n| = |\alpha - 1| > \frac{|\alpha - 1|}{2} \geq \varepsilon.$$

If $a_n = -1$, then

$$|a_n - \alpha| = |-1 - \alpha| = |\alpha - -1| > \frac{|\alpha - -1|}{2} \geq \varepsilon.$$

Thus, $|a_n - \alpha| > \varepsilon$ for all $n \in \mathbb{N}$. This contradicts the assumption that $|a_n - \alpha| < \varepsilon$ for all $n \in \mathbb{N}$, so $(a_n)_{n=1}^{\infty}$ diverges. □

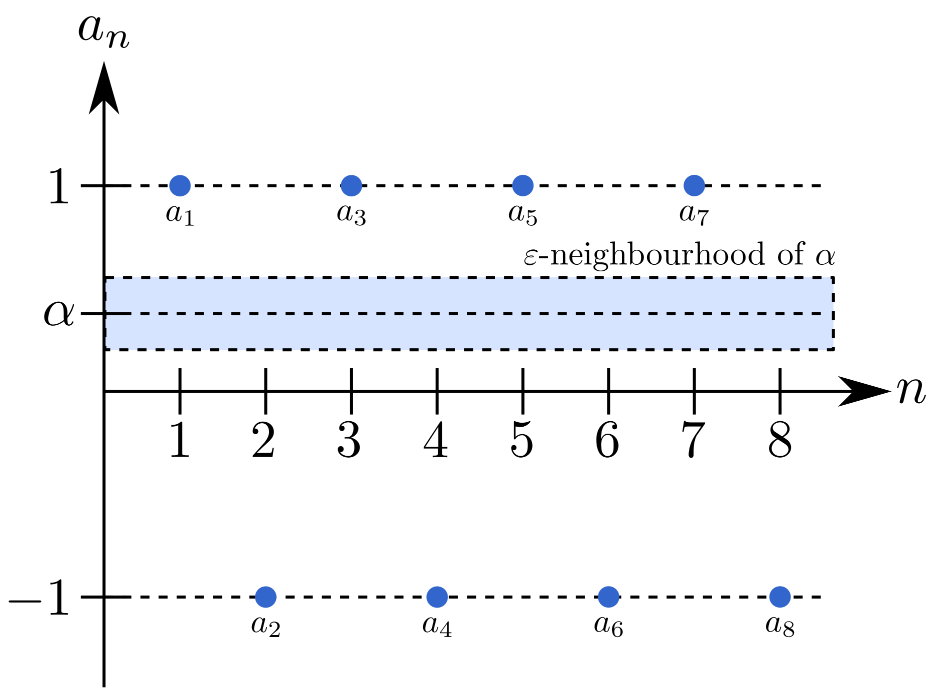


Figure 3.3: Illustration of the proof to Proposition 3.2.4. Note that we assume $\alpha \neq \pm 1$, so there are no sequence terms within the ε-neighbourhood of α.

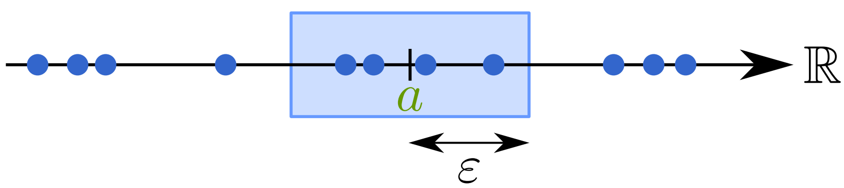


Figure 3.4: Illustration of cluster points. There are infinitely many points of the sequence within the indicated ε-neighbourhood. Note that one sequence may have multiple cluster points.

Definition 3.2.5: Cluster Point

The number $a \in \mathbb{R}$ is a *cluster point* of $(a_n)_{n=1}^{\infty}$ if for all $\varepsilon > 0$, there exist infinitely many $n \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$.

Remark. $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are the greatest and smallest cluster points of $(a_n)_{n=1}^{\infty}$ respectively.

Theorem 3.2.6

Let $(a_n)_{n=1}^{\infty}$ be a sequence, and take $a \in \mathbb{R}$. Then,

1. *a is a cluster point if and only if for all $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $n \geq N$ such that $|a_n - a| < \varepsilon$.*
2. *a is a cluster point if and only if $(a_n)_{n=1}^{\infty}$ has a subsequence $(a_{n_k})_{k=1}^{\infty}$ where $\lim_{k\to\infty} a_{n_k} = a$.*
3. *$\lim_{n\to\infty} a_n = a$ if and only if every subsequence of $(a_n)_{n=1}^{\infty}$ converges to a.*

Proof. **Part 1:** For the forward direction, let a be a cluster point. Let $\varepsilon > 0$ and $N \in \mathbb{N}$. By the definition of a cluster point, there are infinitely many $n \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$. At least some of these must be greater than N since there are infinitely many of them.

For the reverse direction, fix $\varepsilon > 0$. We want to show there are infinitely many $n \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$. Applying the condition with $N = 1$, we find that there is some n_1 such that $|a_{n_1} - a| < \varepsilon$. Apply the condition again with $N = n_1 + 1$. There exists $n_2 \in \mathbb{N}$ such that $n_2 \geq N = n_1 + 1 > n_1$, where $|a_{n_2} - a| < \varepsilon$. Continue with this process to obtain a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $|a_{n_k} - a| < \varepsilon$ for all $k \in \mathbb{N}$. Thus, a is a cluster point.

Part 2: For the forward direction, assume a is a cluster point. Let us construct a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $\lim_{k\to\infty} a_{k_k} = a$. Apply **Part 1** of the theorem setting $\varepsilon = 1, N = 1$, which tells us that there exists $n_1 \in \mathbb{N}$ such that $|a_{n_1} - a| < 1$. Apply **Part 1** again with $\varepsilon = \frac{1}{2}, N = n_1 + 1$, so there exists $n_2 \in \mathbb{N}, n_2 \geq n_1 + 1$ such that $|a_{n_2} - a| < \frac{1}{2}$. Apply **Part 1** again with $\varepsilon = \frac{1}{3}, N = n_2 + 1$. This can be continued forever to obtain a subsequence $(a_{n_k})_{k=1}^{\infty}$. Showing that $\lim_{k\to\infty} a_{n_k} = a$ is left as an exercise.

For the reverse direction, assume there exists $(a_{n_k})_{k=1}^{\infty}$ such that $\lim_{k\to\infty} a_{n_k} = a$. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $k \geq N$, then $|a_{n_k} - a| < \varepsilon$. Since there are infinitely many k satisfying this inequality, a is a cluster point.

Part 3: This was given as an assignment question so the proof is not provided here. □

3.3 Cauchy Sequences

Definition 3.3.1: Cauchy Sequences

$(a_n)_{n=1}^{\infty}$ is a *Cauchy sequence* if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n, k \geq N$ then $|a_n - a_k| < \varepsilon$.

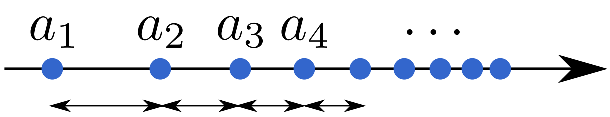


Figure 3.5: Illustration of a Cauchy sequence. The distance between successive elements eventually gets less than ε.

Within this course, there are not much uses for Cauchy sequences other than as a different viewpoint to understand convergence, as we will see soon. Instead of elements of a sequence getting arbitrarily close to a *fixed number* ($|a_n - a| < \varepsilon$), we just require the elements get arbitrarily close to *each other* ($|a_n - a_k| < \varepsilon$).

Theorem 3.3.2: Convergence implies Cauchy

If $(a_n)_{n=1}^{\infty}$ converges, then it is Cauchy.

Proof. Assume $\lim_{n\to\infty} a_n = a$. Fix $\varepsilon > 0$. We know that there exists $N \in \mathbb{N}$ such that if $n, k \geq N$ then $|a_n - a| < \frac{\varepsilon}{2}$ and $|a_k - a| < \frac{\varepsilon}{2}$. Since $n, k \geq N$,

$$\begin{aligned} |a_n - a_k| &= |a_n - a + a - a_k| \\ &\leq |a_n - a| + |a - a_k| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$ □

Definition 3.3.3: Peak Point

We call $k \in \mathbb{N}$ a *peak point* of $(a_n)_{n=1}^{\infty}$ if $a_k > a_n$ for all $n > k$.

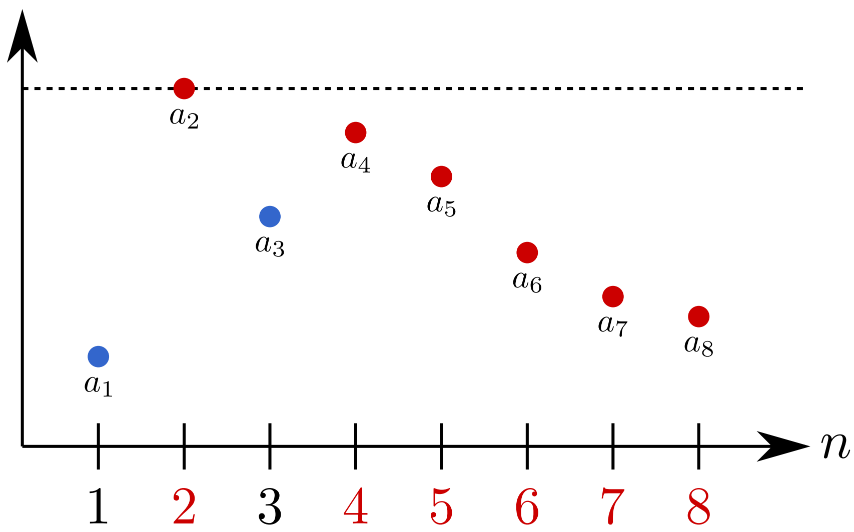


Figure 3.6: Illustration of the peak points of a sequence.

Theorem 3.3.4: Every Sequence has a Monotone Subsequence

Every sequence of real numbers has a monotone subsequence.

Proof. Considering a sequence $(a_n)_{n=1}^{\infty}$, we will split the proof into two cases.

Case 1: There are infinitely many peak points, $n_1, n_2, n_3, \ldots$, where $n_1 < n_2 < n_3 < \ldots$. By definition, $a_{n_1} > a_{n_2} > a_{n_3} > \ldots$, thus $(a_{n_k})_{k=1}^{\infty}$ is monotone.

Case 2: There are finitely many peak points, namely $k_1, k_2, k_3, \ldots, k_m$ for some $m \in \mathbb{N}$. Set $n_1 = \max\{k_1, \ldots, k_m\} + 1$. Since $n_1 \notin \{k_1, \ldots, k_m\}$, n_1 is not a peak point. Therefore, there exists some $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$. However, n_2 is also not a peak point, so there exists $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$. Continue in this way to obtain a monotone subsequence $(a_{n_k})_{k=1}^{\infty}$. □

Theorem 3.3.5: Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

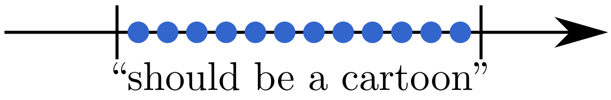


Figure 3.7: Illustration of Theorem 3.3.5.

Proof. Every sequence has a monotone subsequence by Theorem 3.3.4. Every subsequence of a bounded sequence must also be bounded. So, we have a bounded monotone subsequence which must converge by Theorem 3.1.2. Thus, our sequence has a convergent subsequence. □

Lemma 3.3.1. *All Cauchy sequences are bounded.*

Proof. Apply the definition of a Cauchy sequence with $\varepsilon = 1$. This tells us that there exists some $N \in \mathbb{N}$ such that if $n, k \geq N$, then $|a_n - a_k| < 1$. This holds for $k = N$. So, if $n \geq N$, then $|a_n - a_N| < 1$, and by the second triangle inequality, $|a_n| - |a_N| < 1$, so $|a_n| < |a_N| + 1$. Thus, for all $n \in \mathbb{N}$, $|a_n| \leq \max\{|a_1|, \ldots, |a_{N-1}|, |a_N| + 1\}$.

Note that this proof is similar to the proof for Theorem 2.2.9. □

Theorem 3.3.6: Cauchy implies Convergence

If $(a_n)_{n=1}^{\infty}$ is Cauchy, then it converges.

Proof. By the Bolzano-Weierstrass Theorem, $(a_n)_{n=1}^{\infty}$ has a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$. Assume $\lim_{k\to\infty} a_{n_k} = a$. We claim that $\lim_{n\to\infty} a_n = a$.

Fix $\varepsilon > 0$. Since $\lim_{k\to\infty} a_{n_k} = a$, then there exists some $K \in \mathbb{N}$ such that if $k \geq K$, then $|a_{n_k} - a| < \frac{\varepsilon}{2}$. Since $(a_n)_{n=1}^{\infty}$ is Cauchy, there exists $N \in \mathbb{N}$ such that if $n, l \geq N$, then $|a_n - a_l| < \frac{\varepsilon}{2}$. Choose n and k such that $n \geq N$, $k \geq K$ and $n_k \geq N$. Then,

$$\begin{aligned}
|a_n - a| &= |a_n - a_{n_k} + a_{n_k} - a| \\
&\leq |a_n - a_{n_k}| + |a_{n_k} - a| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Thus, if $n \geq N$, then $|a_n - a| < \varepsilon$ which means $\lim_{n\to\infty} a_n = a$. □

We've thus shown that for sequences in $\mathbb{R}$, notions of a convergent sequence and a Cauchy sequence are *equivalent*. Sometimes it is nicer to work with Cauchy sequences for less "nice" spaces than $\mathbb{R}$, but that isn't necessary for this course.

Chapter 4

Functions and Continuity

4.1 Limits of Functions

Definition 4.1.1: Limit Point

Given $X \subset \mathbb{R}$, we call $a \in \mathbb{R}$ a *limit point* of X if for all $\varepsilon > 0$, there exists some $x \in X$ such that $x \in (a - \varepsilon, a + \varepsilon) \setminus \{a\}$.

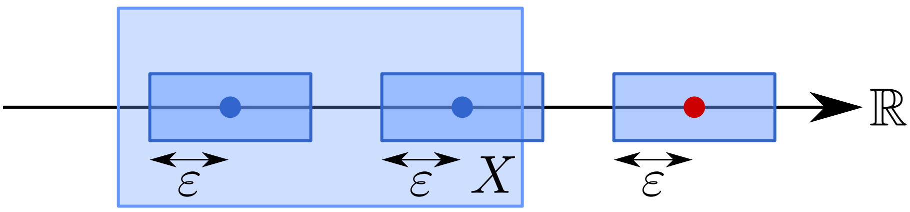


Figure 4.1: An illustration of some limit points of a set shown in blue. The red point is not a limit point because there is no overlap with its ε-neighbourhood and the set X.

Example 4.1.1

Here are some examples of limit points.

- The set of the limit points of $X = (0, 1]$ is $[0, 1]$.
- $\mathbb{N}$ has no limit points.
- The only limit point of $X = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is 0.

Definition 4.1.2: Limit of a Function

Let $f : X \to \mathbb{R}$ be a function. Assume a is a limit point of X. The number $l \in \mathbb{R}$ is called the *limit* of f as x goes to a, written $\lim_{x \to a} f(x)$, if for all $\varepsilon > 0$, there exists some $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - l| < \varepsilon$.

Note that the distinction of a being a limit point of X and not just a member of X only really matters when X doesn't have many 'gaps', like $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ from the previous example. When f's domain is $\mathbb{R}$ or an interval of $\mathbb{R}$, any point within X (and its endpoints if applicable) will be a limit point.

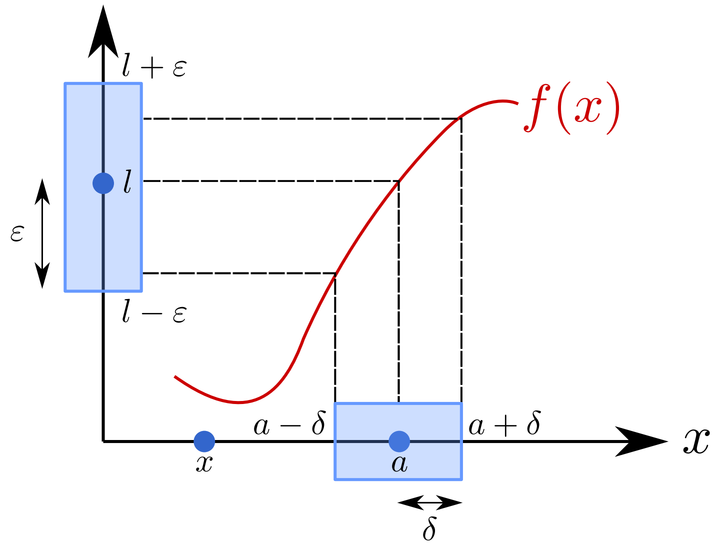


Figure 4.2: Illustration of the definition of the limit. Any point chosen within the δ-neighbourhood of a will lie within the ε-neighbourhood of l.

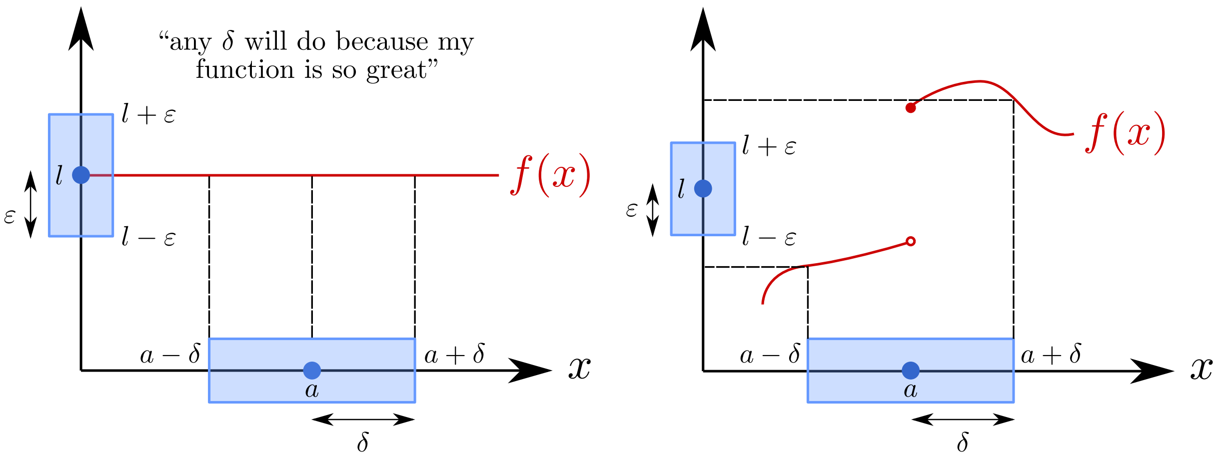


Figure 4.3: Examples of limits of different kinds of functions.

The graph on the left of Figure 4.3 shows a constant function. All points within The δ-neighbourhood will fall within the ε-neighbourhood no matter the δ. This shows that the limit at any point a will always be l.

The function in the graph on the right has a jump at $x = a$. The value of l shown will not be the limit at $x = a$, since no matter how small δ is, $a - \delta$ and $a + \delta$ will never land within the ε-neighbourhood. Since this won't work for any value of l, the limit *doesn't exist* at $x = a$.

For the graph in Figure 4.4, l is the limit of f. Note that in the definition we require $0 < |x - a|$, as we are interested in the behaviour of the function around a, not necessarily at a.

We can prove limits of functions using the ε-δ definition, as shown in the following examples.

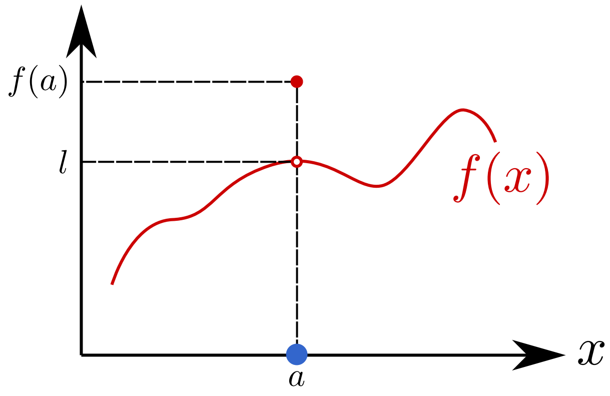
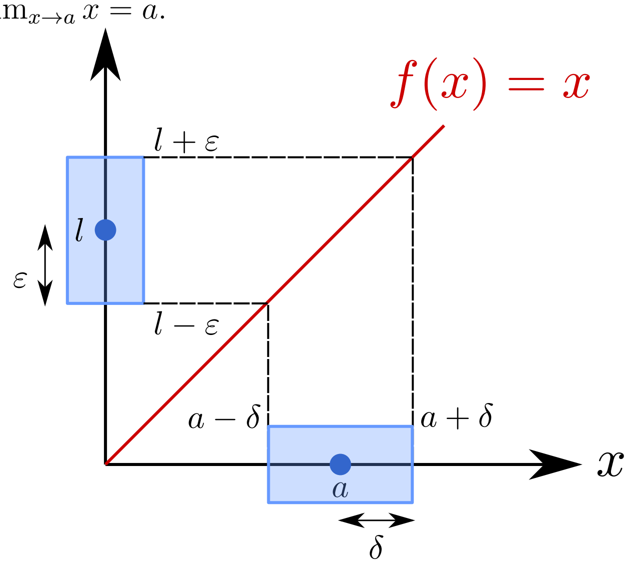


Figure 4.4: Example of the limit of a function which "jumps up" at a point.

Example 4.1.2
We wish to prove $\lim_{x\to a} x = a$.



Take $\varepsilon > 0$ and choose $\delta = \varepsilon$. If $0 < |x - a| < \delta$, then

$$|f(x) - a| = |x - a| < \delta = \varepsilon.$$

Thus, $|f(x) - a| < \varepsilon$, and $\lim_{x\to a} f(x) = a$.

Example 4.1.3
To prove that $\lim_{x\to 2} x^2 + x = 6$, take $\varepsilon > 0$. By the definition, we need to find $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|x^2 + x - 6| < \varepsilon$. Observe that

$$\left|x^2 + x - 6\right| = |x - 2|\,|x + 3|\,.$$

If $|x - 2| < 1$ then $|x| - 2 < 1$ and $|x| < 3$ by the second triangle inequality. Therefore,

$$|x + 3| \leq |x| + 3 < 3 + 3 = 6.$$

Thus, if $|x - 2| < 1$ then $|x + 3| < 6$ so

$$\left|x^2 + x - 6\right| = |x - 2|\,|x + 3| < 6\,|x - 2|\,.$$

If, in addition, $|x-2| < \frac{\varepsilon}{6}$, then $|x^2+6-6| < 6 \cdot \frac{\varepsilon}{6} = \varepsilon$. Thus, we set $\delta = \min\{1, \frac{\varepsilon}{6}\}$. If $0 < |x-2| < \delta$, then

$$\left|x^2+x-6\right| = |x-2|\,|x+3| < 6\,|x-2| < \varepsilon.$$

Thus, $\lim_{x\to 2} x^2 + x = 6$.

Note that δ may depend on both ε and a, as shown in the following diagram.

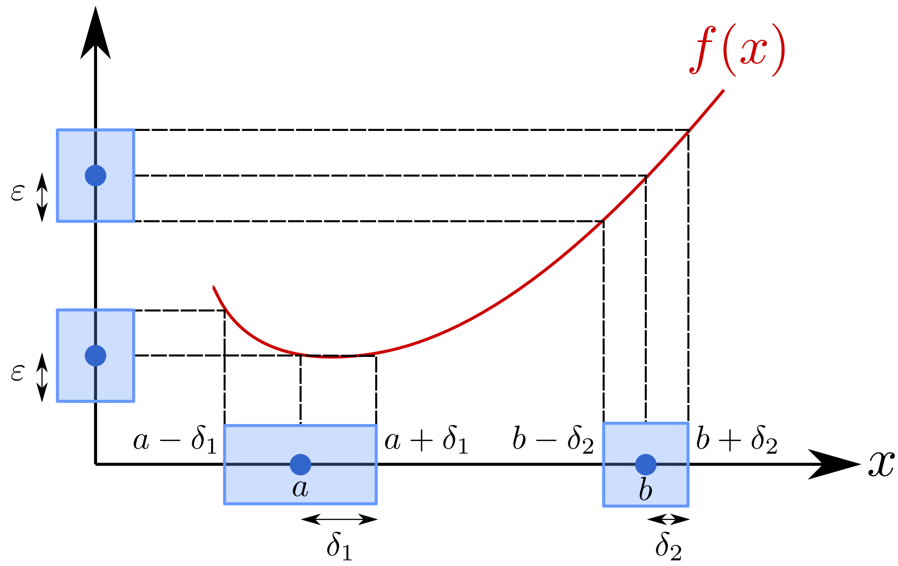


Figure 4.5: Illustration of the possible dependence of δ on x in the limit definition.

Theorem 4.1.3: Function Limit Properties

Assume $\lim_{x\to a} f(x) = l$ *and* $\lim_{x\to a} g(x) = m$. *Then,*

1. $\lim_{x\to a} (f(x) + g(x)) = l + m$.
2. $\lim_{x\to a} (f(x)g(x)) = lm$.
3. $\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{l}{m}$, *provided that* $m \neq 0$.

Proof. **Part 1:** This is an exercise. Try using the triangle inequality.

Part 2: First fix $\varepsilon > 0$. We need to find $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x)g(x) - lm| < \varepsilon$. Observe that

$$\begin{aligned} |f(x)g(x) - lm| &= |f(x)g(x) - f(x)m + f(x)m - lm| \\ &= |f(x)[g(x) - m] + m[f(x) - l]| \\ &\le |f(x)|\,|g(x) - m| + |m|\,|f(x) - l|\,. \end{aligned}$$

The only problematic term here is $|f(x)|$ as all others are either constant or will be less than ε for close enough x. We want to show that $|f(x)|$ does not rush to ∞. Since $\lim_{x\to a} f(x) = l$, there exists $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$ then $|f(x) - l| < 1$. By the second triangle inequality, $|f(x)| - |l| < 1$ so $|f(x)| < 1 + |l|$. Thus, given that $0 < |x-a| < \delta_1$,

$$|f(x)g(x) - ml| \le (1 + |l|)\,|g(x) - m| + |m|\,|f(x) - l|\,.$$

We can also reason that there exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$ then $|g(x)-m| < \frac{\varepsilon}{2(1+|l|)}$. Similarly, we know there exists $\delta_3 > 0$ such that if $0 < |x-a| < \delta_3$ then $|f(x)-l| < \frac{\varepsilon}{2(1+|m|)}$. Set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. If $0 < |x-a| < \delta$, then

$$\begin{aligned}
|f(x)g(x) - lm| &\leq (1+|l|)\,|g(x)-m| + |m|\,|f(x)-l| \\
&< (1+|l|) \cdot \frac{\varepsilon}{2(1+|l|)} + |m| \cdot \frac{\varepsilon}{2(1+|m|)} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Part 3: This is left as an exercise. As a hint, first show $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{\lim_{x\to a} f(x)}$, then use the result of **Part 2**. $\square$

Definition 4.1.4: Function Limits to Infinity

$\lim_{x\to\infty} f(x) = l$ if for all $\varepsilon > 0$, there exists some $M > 0$ such that if $x > M$ then $|f(x) - l| < \varepsilon$. Similarly, we define $\lim_{x\to-\infty} f(x) = l$ using $M < 0$ and $x < M$. Compare this definition to Definition 2.2.2 describing sequence limits.

Definition 4.1.5: Function Limits at Infinity

$\lim_{x\to a} f(x) = \infty$ if for all $M > 0$, there exists some $\delta > 0$ such that if $0 < |x-a| < \delta$ then $f(x) > M$. We can also define $\lim_{x\to a} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, etc. Compare this definition to Definition 3.1.4 describing sequences going to infinity.

Example 4.1.4

We will prove $\lim_{x\to\infty} \frac{x^2+x}{x^2+4x+7} = 1$ using the ε-δ definition. Take $\varepsilon > 0$. We need to find $M > 0$ such that if $x > M$, then

$$\left| \frac{x^2+x}{x^2+4x+7} - 1 \right| < \varepsilon.$$

We simplify as follows:

$$\left| \frac{x^2+x}{x^2+4x+7} - 1 \right| = \left| \frac{x^2+x-x^2-4x-7}{x^2+4x+7} \right| = \left| \frac{-3x-7}{x^2+4x+7} \right| = \left| \frac{3x+7}{x^2+4x+7} \right|.$$

If $x > 0$ then $\left|\frac{3x+7}{x^2+4x+7}\right| = \frac{3x+7}{x^2+4x+7} \leq \frac{3x+7}{x^2}$. We want to get rid of this 7, which we can do if $x > 3$ giving $3x = 9 > 7$. So, assuming $x > 3$,

$$\frac{3x+7}{x^2} < \frac{3x+3x}{x^2} = \frac{6x}{x^2} = \frac{6}{x}.$$

Also, if $x > \frac{6}{\varepsilon}$, then $\left|\frac{x^2+x}{x^2+4x+7} - 1\right| < \frac{6}{x} < \varepsilon$. Thus, let $M = \max\{0, 3, \frac{6}{\varepsilon}\} = \max\{3, \frac{6}{\varepsilon}\}$. If $x > M$, then the previous simiplifications are valid so $\left|\frac{x^2+x}{x^2+4x+7} - 1\right| < \varepsilon$ and the limit holds.

Example 4.1.5
We can also evaluate the limit in Example 4.1.4 using Theorem 4.1.3. Observe that

$$\lim_{x\to\infty} \frac{x^2+x}{x^2+4x+7} = \lim_{x\to\infty} \frac{x^2+x}{x^2+4x+7} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{1+\frac{4}{x}+\frac{7}{x^2}} = 1.$$

We can use this technique to prove the following proposition.

Proposition 4.1.6

If $b_n \neq 0$, then

$$\lim_{x\to\infty} \frac{a_n x^n + \cdots + a_0}{b_n x^n + \cdots + b_0} = \frac{a_n}{b_n}.$$

Proof. The proof is very similar to the proof of Proposition 2.2.11. □

Definition 4.1.7: Right and Left Limits

L is the *right limit* or *upper limit* of f at a, written $\lim_{x\to a^+} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$. Similarly, L is the *left limit* or *lower limit* of f at a, written $\lim_{x\to a^-} f(x) = L$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < a - x < \delta$, then $|f(x) - L| < \varepsilon$.

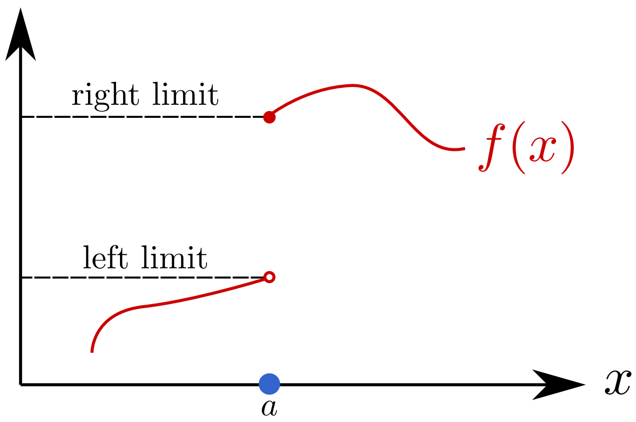


Figure 4.6: Illustration of one-sided limits of a function.

Proposition 4.1.8

If f has right and left limits at a, and they are both equal to L, then $\lim_{x\to a} f(x) = L$.

Proof. This is an exercise. □

Theorem 4.1.9: Squeeze Theorem for Functions

Given functions $f, g, h : X \to \mathbb{R}$, if $f(x) \leq h(x) \leq g(x)$ for all $x \in X$, and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$, *then* $\lim_{x\to a} h(x) = L$. *Note that we allow $a = \pm\infty$.*

Proof. The proof is very similar to the proof of the Squeeze Theorem for sequences – see Theorem 2.2.8. □

Example 4.1.6

We can evaluate the limit $\lim_{x\to 0} x^2 \sin\frac{1}{x}$ using the Squeeze Theorem. Note $-1 \leq \sin\frac{1}{x} \leq 1$, so $-x^2 \leq x^2\sin\frac{1}{x} \leq x^2$, and $\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0$. Thus, by the Squeeze Theorem, $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

Example 4.1.7

Similar to Example 4.1.6, let's evaluate the limit $\lim_{x\to 0} x\sin\frac{1}{x}$, again using the Squeeze Theorem. We can use the fact that $-1 \leq \sin\frac{1}{x} \leq 1$ as before, but we must now consider positive and negative cases when multiplying all parts of the inequality by x.

If x is positive, then $-x \leq \sin\frac{1}{x} \leq x$, and $\lim_{x\to 0} x = \lim_{x\to 0} -x = 0$, so

$$\lim_{x\to 0^+} x\sin\frac{1}{x} = 0.$$

If x is negative, then $x \leq x\sin\frac{1}{x} \leq -x$, and $\lim_{x\to 0} x = \lim_{x\to 0} -x = 0$, so

$$\lim_{x\to 0^-} x\sin\frac{1}{x} = 0.$$

Since both one-sided limits are equal, then $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

Theorem 4.1.10

The following are equivalent:

1. $\lim_{x\to a} f(x) = L$
2. $\lim_{n\to\infty} f(x_n) = L$ *for all* $(x_n)_{n=1}^{\infty}$ *such that* $\lim_{n\to\infty} x_n = a$.

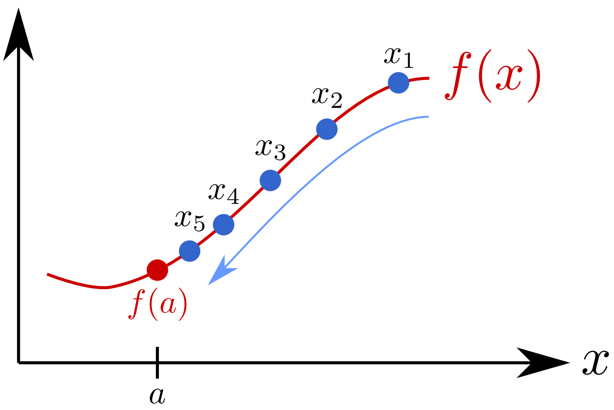


Figure 4.7: Illustration of Theorem 4.1.10.

Proof. The proof that statement 1 implies statement 2 was an assignment question so it will not be provided here.

To show that statement 2 implies statement 1, we prove by contradiction. Assume statement 2 holds and suppose statement 1 does not. Then, $\lim_{x\to a} f(x)$ does not exist, or does not equal L. Therefore, by negating the definition of the limit, there exists an $\varepsilon_0 > 0$ such that for all $\delta > 0$, there exists some $x \in (a-\delta, a+\delta)\setminus\{a\}$ such that $|f(x) - L| \geq \varepsilon_0$.

Take $\delta = 1$. Then, there exists $x_1 \in (a-1, a+1) \setminus \{a\}$ such that $|f(x_1) - L| > \varepsilon_0$.

Take $\delta = \frac{1}{2}$. Then, there exists $x_2 \in (a - \frac{1}{2}, a + \frac{1}{2}) \setminus \{a\}$ such that $|f(x_2) - L| > \varepsilon_0$.

Continue setting $\delta = \frac{1}{3}, \frac{1}{4}, \ldots$ to obtain a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = a$ and $x_n \in (a - \frac{1}{n}, a + \frac{1}{n}) \setminus \{a\}$. In addition, by negating the definition of the limit, $|f(x_n) - L| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. However, we assumed that statement 2 was true: we assumed that if we have a sequence $(x_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = a$, then we have $\lim_{n\to\infty} f(x_n) = L$. This is a contradiction, so statement 2 implies statement 1. $\square$

Here are some important limits which can be used in this course without proof.

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Intuitively, very close to 0, the graph of $\sin x$ looks quite similar to x, so their ratio will be 1.
2. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.
3. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$. To show this, make use of the trigonometric identity $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$. Take $\alpha = \frac{x}{2}$. Then,

$$\begin{aligned}\cos x &= \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\\ &= \cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) - 2\sin^2\left(\frac{x}{2}\right)\\ &= 1 - 2\sin^2\left(\frac{x}{2}\right).\end{aligned}$$

So, $1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$. Thus,

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x^2} = \lim_{x\to 0} 2\cdot\frac{\sin^2\left(\frac{x}{2}\right)}{4\cdot\left(\frac{x}{2}\right)^2} = \frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right).$$

Set $t = \frac{x}{2}$. Note that $t \to 0$ if and only if $x \to 0$ – this is crucial to verify for changing variables in limits. So,

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}\lim_{t\to 0}\left(\frac{\sin t}{2}\right)^2 = \frac{1}{2}\cdot 1^2 = \frac{1}{2},$$

using the first limit in this list.
4. $\lim_{x\to 0}\frac{e^x - 1}{x} = 1$.
5. $\lim_{x\to 0}\frac{a^x-1}{x} = \ln a$, for $a > 0$, $a \neq 1$. To show this, note

$$\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{x\to 0}\frac{e^{\ln(a^x)-1}}{x} = \lim_{x\to 0}\frac{e^{x\ln a}-1}{x} = \lim_{x\to 0}\frac{\ln a\cdot e^{x\ln a}-1}{x\ln a}.$$

Let $t = x\ln a$, noting that $t \to 0$ if and only if $x \to 0$. Now,

$$\lim_{x\to 0}\frac{a^x-1}{x} = \ln a\lim_{t\to 0}\frac{e^t-1}{1} = \ln a,$$

using the third limit in this list.

4.2 Continuous Functions

Definition 4.2.1: Continuity

If we have $X \subset \mathbb{R}$, $f : X \to \mathbb{R}$ and $a \in X$, we say f is *continuous* at a if

$$\lim_{x \to a} f(x) = f(a).$$

Alternatively, f is continuous at a if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.
We say $f : X \to \mathbb{R}$ is continuous on X if f is continuous at every $x \in X$.

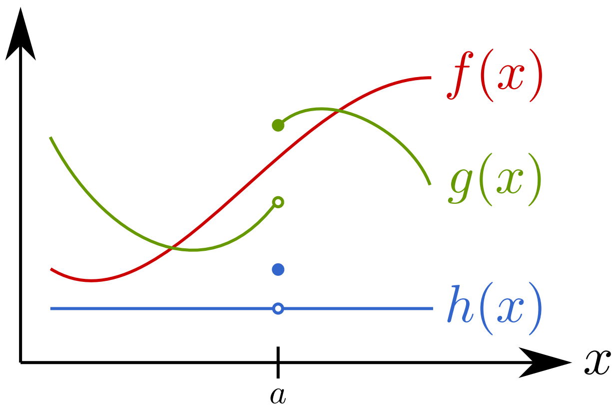


Figure 4.8: Examples of continuous and discontinuous functions. On the interval drawn, $f(x)$ is continuous, and $g(x)$ and $h(x)$ are discontinuous at a.

Intuitively, you can draw a continuous function without lifting your pen. For example, $f(x) = x$ and $f(x) = a$, $a \in \mathbb{R}$ are continuous on $\mathbb{R}$.

Remark. In Figure 4.8, $g(x)$ is said to have a *jump discontinuity* at a because the function jumps up. $h(x)$ is said to have a *removable discontinuity* at a, because the left and right limits of $h(x)$ are the same. The discontinuity can be "removed" by redefining the function to have the value of $\lim_{x \to a} h(x)$ at a.

Theorem 4.2.2

Assume $f, g : (a, b) \to \mathbb{R}$ are continuous at some $x_0 \in (a, b)$. Then,

1. *The function $f + g$ where $(f + g)(x) = f(x) + g(x)$ is continuous at x_0.*
2. *The function fg is continuous at x_0.*
3. *The function $\frac{f}{g}$ is continuous at x_0 if $g(x_0) \neq 0$.*

Proof. This theorem follows immediately from the corresponding theorem involving limits, Theorem 4.1.3. □

Remark. As a result of Theorem 4.2.2, note that every polynomial function is continuous on $\mathbb{R}$. Also, all rational functions, which are functions of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, are continuous for all x such that $Q(x) \neq 0$.

Example 4.2.1

$\frac{x^4+11x+1}{3x-2}$ is continuous on $\mathbb{R} \setminus \{\frac{2}{3}\}$ as this is where the denominator is non-zero.

Example 4.2.2

$f(x) = \frac{1}{x}$ is continuous on $(0, \infty)$. We can prove this using the ε-δ definition.

Fix $x_0 \in (0, \infty)$, and let $\varepsilon > 0$. We will show that there exists some $\delta > 0$ such that if $0 < |x - x_0| < \delta$, then $\left|\frac{1}{x} - \frac{1}{x_0}\right| < \varepsilon$. Observe that

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{xx_0}\right| = \frac{|x - x_0|}{|x|\,|x_0|}.$$

Assume $\delta < \frac{|x_0|}{2}$. Then, $|x - x_0| < \frac{x_0}{2}$, so $\frac{-x_0}{2} < x - x_0 < \frac{x_0}{2}$ and

$$\frac{x_0}{2} < x < \frac{3x_0}{2}.$$

So,

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{|x|\,|x_0|} = \frac{|x - x_0|}{xx_0} < \frac{2\,|x - x_0|}{x_0x_0} = \frac{2\,|x - x_0|}{x_0^2}.$$

Thus,

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \frac{2\,|x - x_0|}{x_0^2} < \frac{2\delta}{x_0^2}.$$

From here, let $\delta = \min\left\{\frac{x_0}{4}, \frac{\varepsilon x_0^2}{2}\right\}$. If $|x - x_0| < \delta$, then

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \frac{2\delta}{x_0^2} \leq \frac{2\varepsilon x_0^2}{x_0^2 \cdot 2} = \varepsilon,$$

so $f(x)$ is continuous on $(0, \infty)$.

4.3 Uniform Continuity

Definition 4.3.1: Uniform Continuity

Suppose I is an interval on $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be some function. We say that f is *uniformly continuous* on I if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in I$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

In this definition, δ is only allowed to depend only on ε and not some point a. x and y are simply two different values in the interval; they don't correspond to x or y axes. Additionally, uniform continuity is a *global* property (on an interval) in contrast to continuity which is a *local* property (at a point).

Example 4.3.1

$f(x) = x$ is uniformly continuous on $\mathbb{R}$. Try proving this using the ε-δ definition, setting $\delta = \varepsilon$ as in Example 4.1.2.

Example 4.3.2

$f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, \infty)$, even though it is continuous on that interval. However, on $[1, \infty)$, it *is* uniformly continuous.

Example 4.3.3

$f(x) = x^2$ on $(0, \infty)$ is not uniformly continuous. We can prove this using the ε-δ definition. Assume on the contrary that f *is* uniformly continuous on $(0, \infty)$. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in (0, \infty)$, if $|x - y| < \delta$, then $|x^2 - y^2| < \varepsilon$.

Take $\varepsilon = 1$. Then, we have some δ. Let $x \in (0, \infty)$, and $y = x + \frac{\delta}{2}$, so

$$|x - y| = \left|x - \left(x + \frac{\delta}{2}\right)\right| = \left|\frac{\delta}{2}\right| < \delta,$$

which satisfies the first inequality in the ε-δ definition. Now, we have

$$\begin{aligned}
1 &> \left|x^2 - y^2\right| \\
&= \left|x^2 - \left(x + \frac{\delta}{2}\right)^2\right| \\
&= \left|x^2 - x^2 - x\delta - \left(\frac{\delta}{2}\right)^2\right| \\
&= \left|-x\delta - \frac{\delta^2}{4}\right| \\
&= \left|x\delta + \frac{\delta_2}{4}\right| \\
&= x\delta + \frac{\delta^2}{4} \\
&> x\delta.
\end{aligned}$$

The last two lines use the fact that $x, \delta \geq 0$, and that $\frac{\delta^2}{4} > 0$. To conclude, $1 > x\delta$ for all $x \in (0, \infty)$. If we take some $x > \frac{1}{\delta}$, then we have $x\delta > 1$. This is a contradiction. Therefore, $f(x) = x^2$ is not uniformly continuous on $(0, \infty)$.

Proposition 4.3.2

If $f : I \to \mathbb{R}$ is uniformly continuous on I, then f is continuous on I.

Proof. Essentially, for some $x_0 \in I$, choose $y = x_0$ in the definition for uniform continuity. The ε-δ definition for continuity follows. □

Theorem 4.3.3: Continuity on closed interval implies Uniform Continuity

Suppose f is continuous on a closed and bounded interval $[a, b]$. Then, f is uniformly continuous on $[a, b]$.

Remark. We want our interval to be *closed* and *bounded*, so $f(x)$ can't go to infinity. As shown in Example 4.3.2, $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, \infty)$.

Proof. We prove by contradiction. Assume that f is not uniformly continuous on $[a, b]$. We negate the definition of uniform continuity by swapping "for all" and "there exists", and flip the implication, so negating $A \implies B$ gives $A \implies \neg B$. So, *there exists* some $\varepsilon_0 > 0$ such that *for all* $\delta > 0$, *there exists* some $x, y \in [a, b]$ where $|x - y| < \delta$ but

$$|f(x) - f(y)| \geq \varepsilon_0.$$

Take $\delta = 1$. We know there exists some $x_1, y_1 \in [a, b]$ such that $|x_1 - y_1| < 1$ but $|f(x_1) - f(y_1)| \geq \varepsilon_0$. Then, take $\delta = \frac{1}{2}$. There exists some $x_2, y_2 \in [a, b]$ such that $|x_2 - y_2| < \frac{1}{2}$, but $|f(x_2) - f(y_2)| \geq \varepsilon_0$. Continue in this fashion. For some $\delta = \frac{1}{n}$, there exists some $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < \frac{1}{n}$, but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

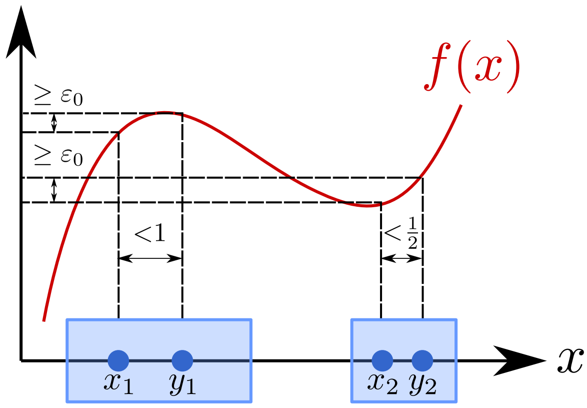


Figure 4.9: Two terms of the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ whose difference gets closer together but the difference between $f(x_n)$ and $f(y_n)$ is still larger than ε_0.

From here, we've constructed two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ with elements inside $[a, b]$. Here, we use the Bolzano-Weierstrass Theorem: since $[a, b]$ is closed and bounded, $(x_n)_{n=1}^{\infty}$ must have some subsequence $(x_{n_k})_{k=1}^{\infty}$ which converges to some $x_0 \in [a, b]$, i.e.

$$\lim_{k \to \infty} x_{n_k} = x_0.$$

Similarly, we have some subsequence $(y_{n_k})_{k=1}^{\infty}$ which converges to some $y_0 \in [a, b]$. We claim that $(x_{n_k})_{k=1}^{\infty}$ and $(y_{n_k})_{k=1}^{\infty}$ converge to the same limit x_0, so that $y_0 = x_0$.

To prove the claim, we want to show that for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that if $k \geq N$, then $|y_{n_k} - x_0| < \varepsilon$. Observe that

$$\begin{aligned}
|y_{n_k} - x_0| &= |y_{n_k} - x_{n_k} + x_{n_k} - x_0| \\
&\leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x_0| \\
&< \frac{1}{n_k} + |x_{n_k} - x_0|.
\end{aligned}$$

The last line follows from the definition of x_n and y_n, since they are required to be closer than $\frac{1}{n}$.

Since $(x_{n_k})_{k=1}^{\infty}$ converges to x_0 for large enough k, we have $|x_{n_k} - x_0| < \frac{\varepsilon}{2}$. If we take k large enough such that $\frac{1}{n_k} < \frac{\varepsilon}{2}$, then

$$|y_{n_k} - x_0| < \frac{1}{n_k} + |x_{n_k} - x_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Although some small details are left out, this concludes the proof of the claim.

Now, let us prove that the claim above leads to a contradiction. Since f is continuous and $\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k} = x_0$, we have that $\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k}) = f(x_0)$ by Theorem 4.1.10. Thus, for some $N \in \mathbb{N}$, if $k \geq N$,

$$|f(x_{n_k}) - f(x_0)| < \frac{\varepsilon_0}{4} \text{ and } |f(y_{n_k}) - f(x_0)| < \frac{\varepsilon_0}{4}.$$

From here, observe that

$$\begin{aligned} |f(x_{n_k}) - f(y_{n_k})| &= |f(x_{n_k}) - f(x_0) + f(x_0) - f(y_{n_k})| \\ &\leq |f(x_{n_k}) - f(x_0)| + |f(y_{n_k}) - f(x_0)| \\ &< \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} \\ &= \frac{\varepsilon_0}{2}. \end{aligned}$$

Notice that this is a contradiction, because by negating the uniform continuity definition,

$$|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0,$$

but we've shown $|f(x_{n_k}) - f(y_{n_k})| < \frac{\varepsilon_0}{2}$. Thus, f is uniformly continuous on $[a, b]$. □

4.4 IVT and EVT

The following theorems will be stated without proof.

Theorem 4.4.1: Intermediate Value Theorem

Assume $f : [a, b] \to \mathbb{R}$ is continuous. Fix λ such that $f(a) < \lambda < f(b)$, or $f(b) < \lambda < f(a)$. Then, there exists some $c \in (a, b)$ such that $f(c) = \lambda$.

Example 4.4.1

We claim that there is always at least 1 solution to the cubic equation $x^3+ax^2+bx+c = 0$. Modify this argument to be more rigorous as an exercise; the idea is that making x large and positive results in $f(x) > 0$, and making x large and negative results in $f(x) < 0$. Then, apply the Intermediate Value Theorem at $\lambda = 0$.

Definition 4.4.2: Maximum and Minimum of a Function

Let $f : X \to \mathbb{R}$ be a function. f attains a *maximum* (or *minimum*) at $a \in X$ if $f(a) \geq f(x)$ (or $f(a) \leq f(x)$ for minimum) for all $x \in X$. $f(a)$ is called the extremal value of f. Maxima and minima are collectively called *extrema*.

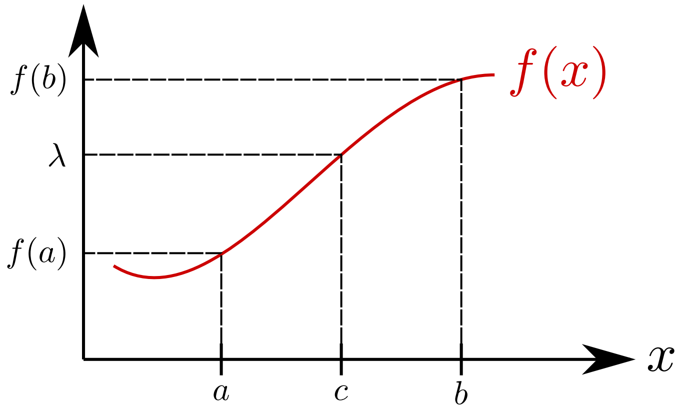


Figure 4.10: Illustration of Theorem 4.4.1.

Example 4.4.2

The function $f : (-1, 1) \to \mathbb{R}$, $f(x) = x$ has no maximum or minimum because the domain is an open interval.

Example 4.4.3

$f : [1, \infty) \to \mathbb{R}$, $f(x) = x^2$ has no maximum, but has a minimum at $x = 1$.

Theorem 4.4.3: Extreme Value Theorem

Assume $f : [a, b] \to \mathbb{R}$ is continuous. Then, f attains its maximum and minimum value on $[a, b]$.

Chapter 5

Differential Calculus

5.1 Derivatives and their Properties

Definition 5.1.1: Derivative

Suppose $x_0 \in (a, b)$ for some $a, b \in \mathbb{R}$. Let $f : (a, b) \to \mathbb{R}$ be a function. The *derivative* of f at x_0, denoted $\frac{\mathrm{d}}{\mathrm{d}x} f(x_0)$ or $f'(x_0)$, is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Both of the limits calculate the same quantity, but one might be more convenient to use in some situations. If the limit exists we say that f is *differentiable* at x_0.

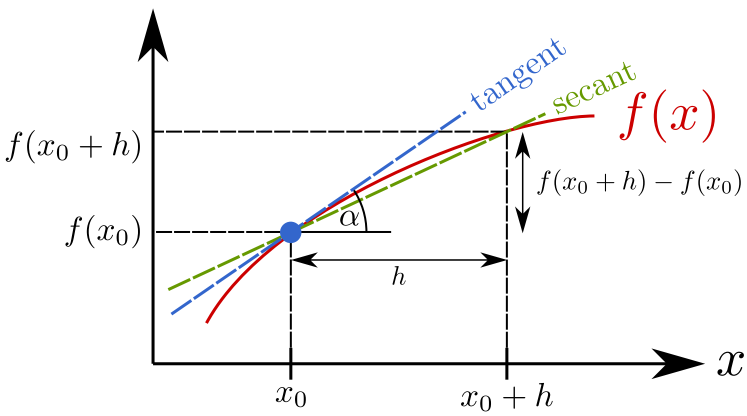


Figure 5.1: Illustration of the limit definition of the derivative. $f'(x_0)$ is the gradient of the green secant line going through the points at x_0 at $x_0 + h$, as h goes to 0. Note that $f'(x_0) = \tan \alpha$, where α is the angle between the tangent line and the horizontal.

Remark. The equation of the tangent line at a point is given by $y = f'(x_0) \cdot (x - x_0) + f(x_0)$.

Proposition 5.1.2: Differentiability implies Continuity

If f is differentiable at x_0, then it is continuous at x_0.

Proof. We want to show that $\lim_{x\to x_0} f(x) = f(x_0)$. So,

$$\begin{aligned}
\lim_{x\to x_0} f(x) &= \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0)+f(x_0)\right)\\
&= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\cdot\lim_{x\to x_0}(x-x_0)+f(x_0)\\
&= f'(x_0)\cdot 0 + f(x_0)\\
&= f(x_0).
\end{aligned}$$

□

Example 5.1.1

We will find the derivative of $f(x) = x^3$ using the definition, often referred to as calculating the derivative by first principles.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}\\
&= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}\\
&= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}\\
&= \lim_{h\to 0} 3x^2+3xh+h^2\\
&= 3x^2.
\end{aligned}$$

So, $f'(x) = 3x^2$.

Example 5.1.2

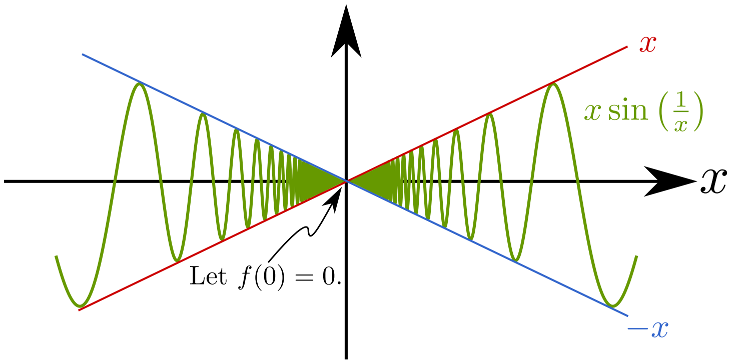
For $f(x) = |x|$, f is not differentiable at $x = 0$ because the slope of the function is not continuous at $x = 0$. The gradient is 1 for $x > 0$ and -1 for $x < 0$). Note that f is continuous and uniformly continuous but not differentiable.

Example 5.1.3

Let

$$f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0\\ 0, & x = 0.\end{cases}$$

We define $f(0) = \lim_{x\to 0} f(x) = 0$ to make f continuous, since there is a removable discontinuity at $x = 0$.



To determine if f is differentiable at $x_0 = 0$, observe that

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} \sin\frac{1}{h}.$$

We prove by contradiction, firstly assuming this limit does exist with value α. Then, for some sequence $(h_n)_{n=1}^{\infty}$ which converges to 0, $\lim_{n\to\infty} \frac{1}{h_n} = \alpha$ as well.

Consider defining the terms of $(h_n)_{n=1}^{\infty}$ by $\frac{1}{h_n} = \frac{\pi}{2} + n\pi$, so $h_n = \frac{1}{\frac{\pi}{2}+n\pi}$. Note that $\lim_{n\to\infty} h_n = 0$, but

$$\sin\frac{1}{h_n} = \sin\left(\frac{\pi}{2} + n\pi\right) = (-1)^n,$$

and $\lim_{n\to\infty}(-1)^n$ does not exist. Hence, $\lim_{h\to 0}\sin\left(\frac{1}{h}\right)$ does not exist and f is not differentiable at $x = 0$.

Theorem 5.1.3: Derivative Rules

Suppose $f, g : (a,b) \to \mathbb{R}$ are differentiable at $x \in (a,b)$. Then, $f+g$, fg and $\frac{f}{g}$ are differentiable at x (the last is true if $g(x) \neq 0$). Moreover,

1. $(f+g)'(x) = f'(x) + g'(x)$.
2. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.
3. $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$.

Proof. **Part 1:** This is left as an exercise.

Part 2: We have

$$\begin{aligned}(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}\\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}\\ &= \lim_{h\to 0}\left[f(x+h)\cdot\frac{g(x+h)-g(x)}{h} + g(x)\cdot\frac{f(x+h)-f(x)}{h}\right]\\ &= \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= f(x)g'(x) + g(x)f'(x).\end{aligned}$$

Note that $\lim_{h\to 0} f(x+h) = f(x)$ since f is continuous at x, implied by differentiability.

Part 3: This is left as an exercise. □

Corollary 5.1.1. *If $c \in \mathbb{R}$ and f is defined as in Theorem 5.1.3, then cf is differentiable and $(cf)'(x) = cf'(x)$.*

Proof. By the product rule,

$$(cf)'(x) = (c)'f(x) + cf'(x) = 0 \cdot f(x) + cf'(x) = cf'(x).$$

Note that $(c)' = \lim_{h\to 0} \frac{c-c}{h} = 0$. □

Corollary 5.1.2. *Polynomials and rational functions are differentiable where they are defined.*

Theorem 5.1.4: Chain Rule

Suppose I, J are open intervals in $\mathbb{R}$. Consider $f : I \to J$ and $g : J \to R$. Assume f is differentiable at some $c \in I$, and g is differentiable at $f(c)$. Also, assume f and g are continuous. Define the composition of the functions g and f by

$$g \circ f : I \to \mathbb{R}, \quad (g \circ f)(x) = g(f(x)).$$

Then, $g \circ f$ is differentiable at c, and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

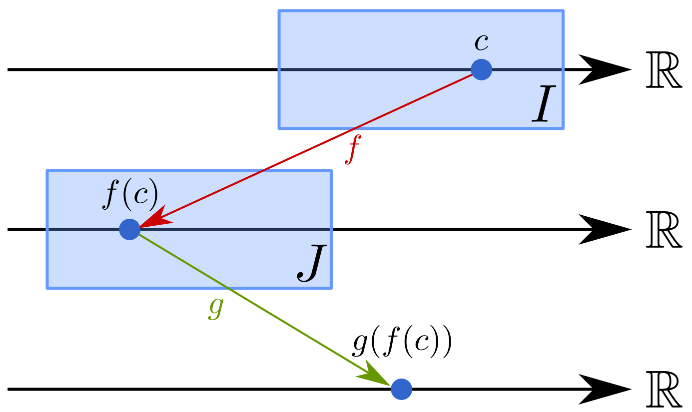


Figure 5.2: An illustration of the composition of functions used in the Chain Rule.

For some intuition on why this theorem is true, observe that

$$\begin{aligned}(g \circ f)'(c) &= \lim_{h\to 0} \frac{g(f(c+h)) - g(f(c))}{h} \\ &= \lim_{h\to 0} \frac{g(f(c+h)) - g(f(c))}{f(c+h) - f(c)} \cdot \frac{f(c+h) - f(c)}{h}.\end{aligned}$$

Let $t = f(c+h)$ and $t_0 = f(c)$. Then,

$$\begin{aligned}(g \circ f)'(c) &\approx \lim_{t\to t_0} \frac{g(t) - g(t_0)}{t - t_0} \cdot f'(c) \\ &= g'(t_0)f'(c) \\ &= g'(f(c))f'(c).\end{aligned}$$

However, if f is a constant function, then $t - t_0 = 0$, leading to problems with the denominator. The following proof aims to get around this by carefully defining piecewise functions which will keep the denominator non-zero. To understand the flow of this proof, it is helpful to fully rewrite the lines of the proof using the definitions of the piecewise functions used.

Proof. Define $d = f(c)$ and functions $\alpha : I \to \mathbb{R}$, $\beta : J \to \mathbb{R}$ by:

$$\alpha(x) = \begin{cases} \frac{f(x)-d}{x-c} - f'(c), & x \neq c \\ 0, & x = c \end{cases}$$

$$\beta(y) = \begin{cases} \frac{g(y)-g(d)}{y-d} - g'(d), & y \neq d \\ 0, & y = d. \end{cases}$$

This is a safety definition against the case where the denominator equals 0. Note that α and β are continuous. Setting $y = f(x)$ for $x \neq c$, we see that

$$\begin{aligned} (g \circ f)(x) - (g \circ f)(c) &= g(f(x)) - g(f(c)) \\ &= g(y) - g(d) \\ &= (y - d)(g'(d) + \beta(y)) \\ &= (f(x) - f(c))(g'(d) + \beta(y)) \\ &= (x - c)(f'(c) + \alpha(x))(g'(d) + \beta(y)). \end{aligned}$$

Therefore,

$$\frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = (f'(c) + \alpha(x))(g'(d) + \beta(y)).$$

Taking the limit as $x \to c$ on both sides, we obtain

$$\begin{aligned} (g \circ f)'(c) &= \lim_{x \to c} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} \\ &= \lim_{x \to c} [(f'(c) + \alpha(x))(g'(d) + \beta(f(x))] \\ &= (f'(c) + \alpha(c))(g'(f(c)) + \beta(f(c))) \qquad (*) \\ &= f'(c) \cdot g'(f(c)), \end{aligned}$$

since $\alpha(c) = 0$ and $\beta(f(c)) = 0$. □

Note that the line marked $(*)$ would be flawed if not for the special $x = c$ cases for α and β to ensure that α and β stay continuous even when $x = c$.

Example 5.1.4

These formulas for derivatives can be used without proof.

- $\frac{\mathrm{d}}{\mathrm{d}x} c = 0$.
- $\frac{\mathrm{d}}{\mathrm{d}x} x = 1$.
- $\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$.
- $\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x$.
- $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$.
- $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$ for $x > 0$.

Example 5.1.5

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x} = \frac{(\sin x)'(\cos x) - (\cos x)'(\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}$$

when $\cos x \neq 0$.

Example 5.1.6

To find $\frac{\mathrm{d}}{\mathrm{d}x}a^x$, note that $\frac{\mathrm{d}}{\mathrm{d}x}a^x = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\ln a}\right)^x = \frac{\mathrm{d}}{\mathrm{d}x}e^{x\ln a}$. Use the Chain Rule with $f(x) = x\ln a$, $g(y) = e^y$. So,

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = \frac{\mathrm{d}}{\mathrm{d}x}(g \circ f)(x) = g'(f(x))f'(x) = e^{x\ln a}\ln a = a^x \ln a$$

for $a > 0$.

Example 5.1.7

We want to find $\frac{\mathrm{d}}{\mathrm{d}x}x^\alpha$. Instead, consider $\ln x^\alpha$. We have $\frac{\mathrm{d}}{\mathrm{d}x}\ln x^\alpha = \frac{\mathrm{d}}{\mathrm{d}x}\alpha\ln x = \frac{\alpha}{x}$. Also, $\frac{\mathrm{d}}{\mathrm{d}x}\ln x^\alpha = \frac{1}{x^\alpha}\cdot(x^\alpha)'$ by the chain rule. Thus, $\frac{\alpha}{x} = \frac{(x^\alpha)'}{x^\alpha}$, so

$$(x^\alpha)' = \alpha x^{\alpha-1}$$

for positive x.

Example 5.1.8

To find $\frac{\mathrm{d}}{\mathrm{d}x}x^x$, $x > 0$, we do a similar trick as in the above example. $\frac{\mathrm{d}}{\mathrm{d}x}\ln x^x = \frac{\mathrm{d}}{\mathrm{d}x}x\ln x = \ln x + 1$. By the chain rule, $\frac{\mathrm{d}}{\mathrm{d}x}\ln x^x = \frac{1}{x^x}\cdot(x^x)'$. Thus, $\frac{(x^x)'}{x^x} = lnx + 1$, so

$$(x^x)' = x^x(\ln x + 1).$$

Definition 5.1.5: Injective Function

$f : X \to \mathbb{R}$ is *injective* if $f(a) = f(b)$ implies $a = b$ for all $a, b \in X$.

Definition 5.1.6: Inverse Function

Let X be some set and f be a function, and define the *image* of X to be the set

$$f(X) = \{f(x) \mid x \in X\}.$$

If f is injective, there exists some $f^{-1} : f(X) \to X$ such that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$ for all $x \in X$, $y \in f(X)$. We call f^{-1} the *inverse function* of f.

Remark. Note that in Definition 5.1.6 we define the inverse of $f : X \to f(X)$ to be a function $f^{-1} : f(X) \to X$. This may not be true if we define $f : X \to Y$ for some larger set $Y \supseteq f(X)$ – if the image of X doesn't "fill" *all* of Y (i.e. $f(X) \neq Y$), then we say f is not *surjective* and f is not invertible. This distinction between $f : X \to f(X)$ and $f : X \to Y$ becomes relevant in MATH1061 and higher-level math courses, but does not matter too much for this course.

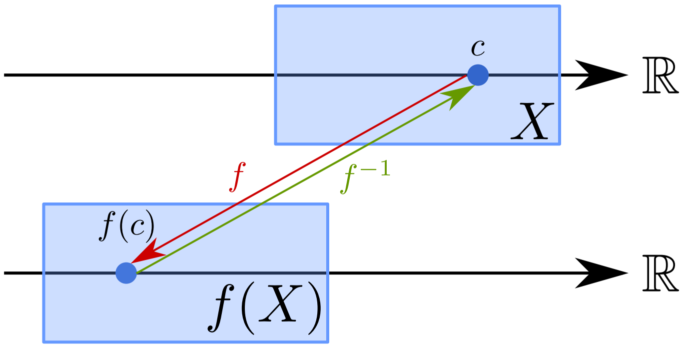


Figure 5.3: Illustration of a function $f : X \to \mathbb{R}$ and its inverse. $f(X)$ is defined as above in Definition 5.1.6.

Example 5.1.9

$f(x) = x^2$ is not injective on $\mathbb{R}$, but it *is* injective on $[0, \infty)$. Its inverse defined on $[0, \infty)$ is $g(x) = \sqrt{x}$.

The proofs for the next two theorems will be left to MATH2401.

Theorem 5.1.7: Continuous and Injective implies Monotone

Assume f is continuous and injective on an interval I. Then, f is strictly monotone on I. Moreover, f^{-1} is continuous.

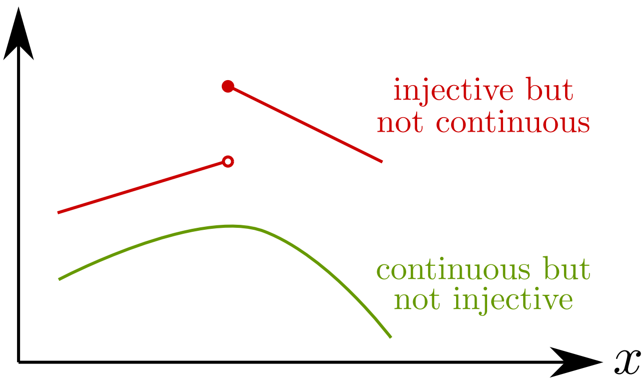


Figure 5.4: Illustration on what can happen to the monotonicity of functions if either condition on f is removed in Theorem 5.1.7.

Theorem 5.1.8: Inverse Function Theorem

Suppose f is continuous and injective on I. Assume f is differentiable at $f^{-1}(c)$ where $c \in f(I)$. Also, assume that $f'(f^{-1}(c)) \neq 0$. Then,

1. *f^{-1} is differentiable at c.*
2. *$\left(f^{-1}\right)'(c) = \frac{1}{f'(f^{-1}(c))}$, or equivalently, $f'(c) = \frac{1}{(f^{-1})'(f(c))}$; the second form is used more commonly for computation.*

Proof. In this course, we'll only go over a non-rigorous argument to **Part 2**. Denote

$y = f^{-1}(x)$ and $y_0 = f^{-1}(c)$. Then,

$$\begin{aligned}
\left(f^{-1}\right)'(c) &= \lim_{x\to c} \frac{f^{-1}(x) - f^{-1}(c)}{x - c} \\
&= \lim_{x\to c} \frac{y - y_0}{f(y) - f(y_0)} \\
&= \lim_{y\to y_0} \frac{y - y_0}{f(y) - f(y_0)} \qquad (*) \\
&= \frac{1}{\lim_{y\to y_0} \frac{f(y)-f(y_0)}{y-y_0}} \\
&= \frac{1}{f'(y_0)} \\
&= \frac{1}{f'(f^{-1}(c))}.
\end{aligned}$$

The line indicated by $(*)$ above holds because f^{-1} is continuous by Theorem 5.1.7, allowing us to change limit variables. □

Example 5.1.10

Define $f : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], f(x) = \sin^{-1}(x)$. To find $\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x)$, note that $f^{-1}(x) = \sin x$, so by the Inverse Function Theorem,

$$\begin{aligned}
f'(x) &= \frac{1}{\left(f^{-1}\right)'(f(x))} \\
&= \frac{1}{\cos(\sin^{-1} x)} \\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1}(x))}} \\
&= \frac{1}{\sqrt{1 - x^2}}
\end{aligned}$$

for $x \in (-1, 1)$.

Here, we discuss higher-order derivatives.

Example 5.1.11

Note that if $f : X \to \mathbb{R}$ is differentiable on X, then f' is also a function from $X \to \mathbb{R}$. f' is *not necessarily* differentiable on X. For instance, let

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0. \end{cases}$$

We have $f'(x) = 2x$ for $x \in (0, \infty)$, and $f'(x) = -2x$ for $x \in (-\infty, 0)$. At $x = 0$, $f'(x) = 0$ – proving this using the limit definition of the derivative is left as an exercise. Thus, $f'(x) = 2\,|x|$ which is not differentiable at 0, and $f''(0)$ does not exist. $f''(x)$ is not differentiable on $\mathbb{R}$.

Note that for notation, we can write $f(x), f'(x), f''(x), f'''(x), f^{(4)}(x), f^{(5)}(x), \cdots$ to indicate higher-order derivatives. Alternatively, $f^{(n)}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)$.

Theorem 5.1.9: L'Hôpital's Rule

Let $f, g \in (a,b) \to \mathbb{R}$ be differentiable on $(a,b) \setminus \{c\}$ for some $c \in (a,b)$. Assume $g(x) \neq 0$ on $(a,b) \setminus \{c\}$. Also, assume that either

1. $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$, *or*
2. $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty$.

If $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists, and the limits are equal.

Remark. To use L'Hôpital's Rule, c may be ∞, and this can be used for one-sided limits. Make sure that all conditions are verified before applying the theorem.

Example 5.1.12

To calculate $\lim_{x\to 1} \frac{\ln x}{x-1}$, note that both $\ln x$ and $x-1$ are differentiable for $x > 0$, and that $\lim_{x\to 1} \ln x = \lim_{x\to 1} x - 1 = 0$. We can then calculate the limit

$$\lim_{x\to 1} \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)}{\frac{\mathrm{d}}{\mathrm{d}x}(x-1)} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = 1.$$

Thus, by L'Hôpital's Rule, $\lim_{x\to 1} \frac{\ln x}{x-1} = 1$.

Example 5.1.13

To find $\lim_{x\to\infty} \frac{x^2}{e^x}$, note that $\lim_{x\to\infty} x^2 = \lim_{x\to\infty} e^x = \infty$ and both functions are differentiable on $\mathbb{R}$. We calculate

$$\lim_{x\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}x}(x^2)}{\frac{\mathrm{d}}{\mathrm{d}x}(e^x)} = \lim_{x\to\infty} \frac{2x}{e^x}.$$

We need to apply L'Hôpital's Rule again, finding

$$\lim_{x\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}x}(2x)}{\frac{\mathrm{d}}{\mathrm{d}x}(e^x)} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

Thus, $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$.

Example 5.1.14

For the limit $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$, note that

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = -\lim_{x\to 0^+} \frac{-\ln x}{\frac{1}{x}}.$$

Now, in this form, $\lim_{x\to 0^+} -\ln x = \lim_{x\to 0^+} \frac{1}{x} = \infty$, so we calculate

$$-\lim_{x\to 0^+} \frac{\frac{\mathrm{d}}{\mathrm{d}x}(-\ln x)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)} = -\lim_{x\to 0^+} \frac{-\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} x = 0.$$

Thus, $\lim_{x\to 0^+} x\ln x = 0$. Note that we needed to manipulate the expression to get it in a fractional form for L'Hôpital's Rule to be applicable, and the limit was negated to ensure the numerator goes to $+\infty$, not $-\infty$.

Example 5.1.15

The calculation

$$\lim_{x\to 0}\frac{x+1}{2x+3} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$$

is incorrect, because L'Hôpital's Rule requires *both* the numerator and denominator to go to 0 or ∞. The correct answer $\frac{1}{3}$ is obtained by substituting $x = 0$, as rational functions are continuous where defined.

5.2 Local and Global Extrema

Definition 5.2.1: Local Extrema

A function $f : X \to \mathbb{R}$ has a *local maximum* at $c \in X$ if there exists an open interval $U \subset \mathbb{R}$ such that $c \in U$ and $f(c) \geq f(x)$ for all $x \in U \cap X$. The *local minumum* is defined similarly. We say c is a *local extremum* if it is either a local maximum, minumum, or both.

Strict local extremum are when we replace $\geq$ or $\leq$ with $>$ or $<$.

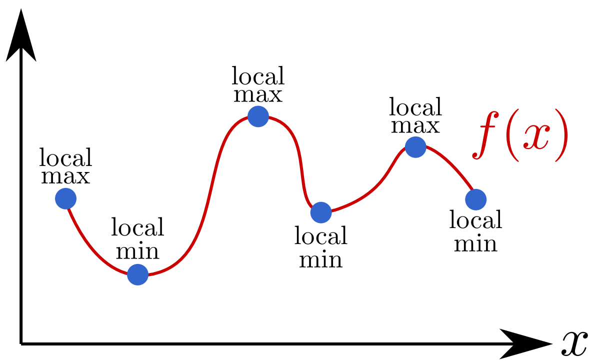


Figure 5.5: Illustration of local extrema of a function. Note that the endpoints are also classified as local extrema because Definition 5.2.1 only requires an *open* U.

Lemma 5.2.1. *Assume $g : (\alpha, \beta) \to \mathbb{R}$ is such that $\lim_{x\to\alpha^+} g(x)$ and $\lim_{x\to\beta^-} g(x)$ exist. If $g(x) \geq 0$ on (α,β), then $\lim_{x\to\alpha^+} g(x) \geq 0$ and $\lim_{x\to\beta^-} g(x) \geq 0$.*

Proof. This is left as an exercise. □

Theorem 5.2.2

Assume $f : [a, b] \to \mathbb{R}$ is a function and $c \in (a, b)$ is a local extremum. If $f'(c)$ exists, then $f'(c) = 0$.

Remark. This is not an if and only if statement; consider horizontal inflection points like x^3 at $x = 0$ as a counterexample. Additionally, f might not be differentiable at a local extremum. For example, consider $f(x) = |x|$ on $\mathbb{R}$. At $x = 0$, we have a local minimum, but $f'(0)$ does not exist.

Proof. Without loss of generality, assume c is a local maximum. If c is a local minumum, apply the theorem to $-f$. By the above lemma, since $f(x) - f(c) \leq 0$ near c, then

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0.$$

By the same reasoning,

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0.$$

Now,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

Thus, $f'(c) \leq 0$ and $f'(c) \geq 0$, so $f'(c) = 0$. □

Example 5.2.1

Consider $f(x) = x^3 - 3x^2 + 1$ on $\left[-\frac{1}{2}, 4\right]$. To find local extrema, we have $f'(x) = 3x^2 - 6x = 3x(x-2)$. Thus, the *candidates* for local extrema are $x = 0, x = 2$ and *also* $x = -\frac{1}{2}, x = 4$. It is essential to check the endpoints of the domain as the derivative isn't defined there.

From here, let's find the global extrema of f on $\left[-\frac{1}{2}, 4\right]$. We know that global extrema are also local extrema, so we pick the maximum and minimum of the evaluated values at each identified local extrema.

$$f(\frac{-1}{2}) = \frac{1}{8}, \quad f(0) = 1, \quad f(4) = 17, \quad f(2) = -3.$$

Thus, the global maximum of f is 17 at $x = 4$, and the global minimum of f is -3 at $x = 2$.

Theorem 5.2.3: Rolle's Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

Proof. If $f(x) = f(a)$ for all $x \in (a, b)$, then the theorem immediately follows. Assume f is not constant.

Without loss of generality, assume there exists some $x_0 \in (a, b)$ such that $f(x_0) > f(a)$. If no such x_0 exists, then there exists $x_1 \in (a, b)$ such that $f(x_1) < f(a)$. A similar argument works, or you could apply the result to $-f$.

By the Extreme Value Theorem, f has a global maximum on $[a, b]$. Call it c. Since $f(c) \geq f(x_0) > f(a) = f(b)$, we know $c \neq a$ and $c \neq b$. Thus, $c \in (a, b)$ and $f'(c) = 0$ by Theorem 5.2.2. □

Theorem 5.2.4: Mean Value Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then, there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

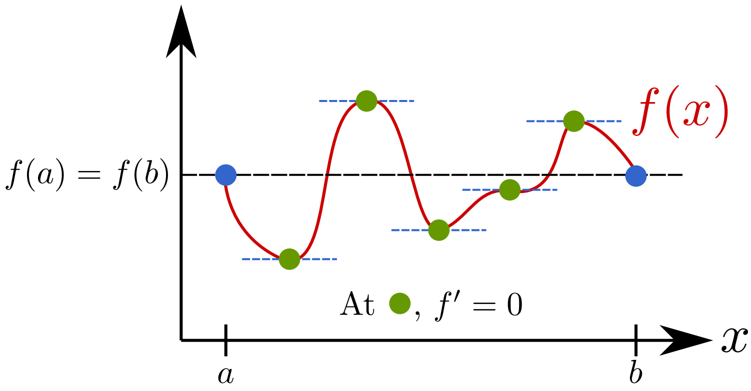


Figure 5.6: Illustration of Theorem 5.2.3.

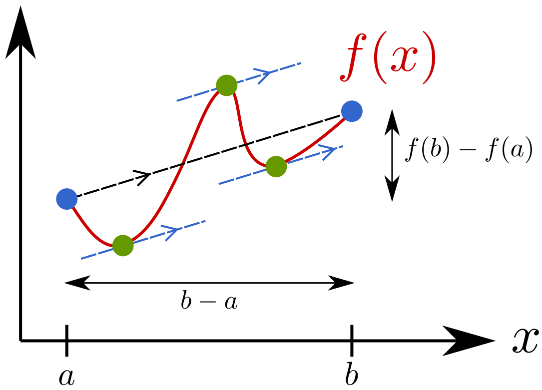


Figure 5.7: Illustration of the Section 5.2. Note that the tangent lines to the green points and the line from the endpoints of the function are parallel with gradient $\frac{f(b)-f(a)}{b-a}$; the theorem asserts the existence of these green points.

Proof. Consider

$$\varphi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Observe that

$$\varphi(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0, \text{ and}$$
$$\varphi(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - f(a) - f(b) + f(a) = 0.$$

Applying Rolle's Theorem to $\varphi(x)$ as $\varphi(a) = \varphi(b) = 0$, we know that there exists some $c \in (a, b)$ such that $\varphi'(c) = 0$. Observe that

$$\varphi'(c) = f'(c) - 0 - \frac{f(b) - f(a)}{b - a} \cdot (x - a)' = f'(c) - \frac{f(b) - f(a)}{b - a},$$

therefore $f'(c) = \frac{f(b)-f(a)}{b-a}$. □

We can use the MVT as an aid to prove interesting results that would be difficult to prove otherwise. Assume $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) for the following examples.

Example 5.2.2

If $f'(x) = 0$ on (a, b), then f is constant on $[a, b]$.

Indeed, take $x \in (a, b]$, and apply MVT on $[a, x]$. We conclude that $f(x) - f(a) = f'(c)(x - a)$ for some $c \in [a, x]$. Therefore, $f(x) - f(a) = 0$, and $f(x) = f(a)$ for all $a \in [a, b]$.

Example 5.2.3

The following are all true:

1. If $f' \geq 0$ then f is non-decreasing.
2. If $f' \leq 0$ then f is non-increasing.
3. If $f' > 0$ then f is strictly increasing.
4. If $f' < 0$ then f is strictly decreasing.

Part 1: Take $x, y \in [a, b]$. Assume $x < y$. We want to prove $f(x) \leq f(y)$. Apply MVT on $[x, y]$. We find $f(y) - f(x) = f'(c)(y - x)$ for some $c \in (x, y)$. By assumption, $f'(c) \geq 0$. Therefore, $f(y) - f(x) \geq 0$, and $f(y) \geq f(x)$. Note that $y - x \geq 0$ since $y > x$.

Part 2: Apply **Part 1** to $-f$.

Parts 3 and 4 are analogous.

Note that the converse to statements 3 and 4 are false. Consider $f(x) = x^3$ with a horizontal inflection point at $x = 0$.

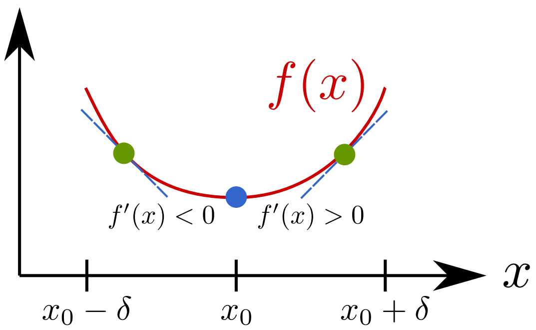
Lemma 5.2.2. *Consider $g : (a, b) \to \mathbb{R}$, and a point $c \in (a, b)$. If $\lim_{x \to c} g(x) > 0$, then there exists some $\delta > 0$ such that $g(x) > 0$ for all $x \in (c - \delta, c + \delta) \setminus \{c\}$.*

Proof. This was an assignment question so the proof will not be provided here. □

Theorem 5.2.5: Second Derivative Test

Assume f is twice differentiable on (a, b). Let $x_0 \in (a, b)$. Then,

1. *If $f'(x_0) = 0$ and $f''(x) > 0$, then x_0 is a local minimum.*
2. *If $f'(x_0) = 0$ and $f''(x) < 0$, then x_0 is a local maximum.*



Proof. **Part 1:** By definition,

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x)}{x - x_0}.$$

This is positive by assumption. Lemma 5.2.2 tells us that $\frac{f'(x)}{x-x_0} > 0$ on $(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$. This means $f'(x) > 0$ on $(x_0, x_0 + \delta)$ and $f'(x) < 0$ on $(x_0 - \delta, x_0)$, because the numerator and denominator must have the same sign to make the fraction positive.

Thus, f is increasing on $(x_0, x_0 + \delta)$ and decreasing on $(x_0 - \delta, x_0)$, so x_0 must be a local minimum.

Part 2: Apply **Part 1** to $-f$. □

Here are some examples of single-variable optimisation.

Example 5.2.4

Consider $f(x) = \frac{x^5}{5} - x + 1$ on $[-2, 2]$. Note that $f'(x) = x^4 - 1$, and $f'(x) = 0$ if and only if $x = \pm 1$. Thus, candidates for local extrema are $x = \pm 1$, $x = \pm 2$. We calculate $f''(x) = 4x^3$, so $f''(1) = 4, f''(-1) = -4$. This means that $x = 1$ is a local minimum and $x = -1$ is a local maximum. Also,

$$f(1) = \frac{1}{5}, \quad f(-1) = \frac{9}{5}, \quad f(2) = \frac{27}{5}, \quad f(-2) = \frac{-17}{5}.$$

Thus, $x = 2$ is a global maximum and $x = -2$ is a global minimum. You can check this by using a graphing calculator.

Example 5.2.5

What is the largest possible area of a rectangle with perimeter 4?

Let x and y be the lengths for two perpendicular sides of a rectangle. We have $2x + 2y = 4$, so $y = 2 - x$. The area is $A(x) = x(2 - x) = -x^2 + 2x$. We have $a'(x) = -2x + 2 = 0$ if and only if $x = 1$. Also, $A''(x) = -2 < 0$, so $x = 1$ is a local maximum. At the endpoints of $[0, 2]$, $A = 0$. Thus, $x = 1$ is the global maximum, and $A(1) = 1$ is the largest possible area.

Chapter 6

Riemann Integration

For this entire chapter, we will assume that the functions we consider are *bounded*. See Section 7.3 for a discussion on how to integrate unbounded functions.

6.1 The Riemann Integral

Definition 6.1.1: Partition

A *partition* of $[a, b]$ is a set of points $\{x_0, x_1, x_2, \cdots, x_n\}$ such that $x_0 = a < x_1 < x_2 < \cdots < x_n = b$. Note that there is no requirement on the spacing, only that they are strictly increasing, start at a, and end at b.

Definition 6.1.2: Lower and Upper Sums

The *lower sum* of f with respect to the partition $P = \{x_0, x_1, \cdots, x_n\}$ is

$$\mathrm{L}(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x).$$

Similarly, the *upper sum* of f is

$$\mathrm{U}(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}), \quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).$$

These can be thought as approximations to the *signed area* of a function.

Remark. If $|f| \leq M$ for some $M \in \mathbb{R}$, then

$$\begin{aligned}
\mathrm{L}(f, P) &\geq \sum_{i=1}^{n} (-M)(x_i - x_{i-1}) \\
&= -M \sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= -M\left(\cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + \cancel{x_3} - \cancel{x_2} + \cdots + x_n - \cancel{x_{n-1}}\right) \\
&= -M(x_n - x_0) \\
&= -M(b - a).
\end{aligned}$$

Similarly, $\mathrm{U}(f,P) \leq M(b-a)$. To summarise,

$$-M(b-a) \leq \mathrm{L}(f,P) \leq \mathrm{U}(f,P) \leq M(b-a).$$

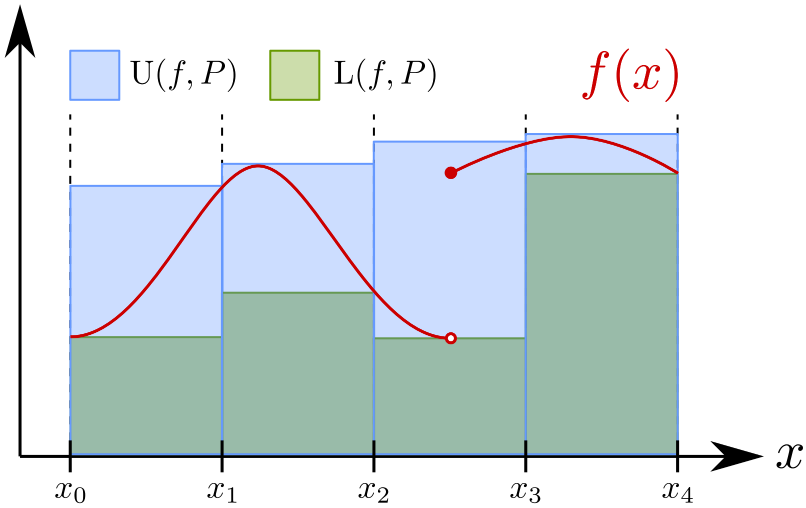
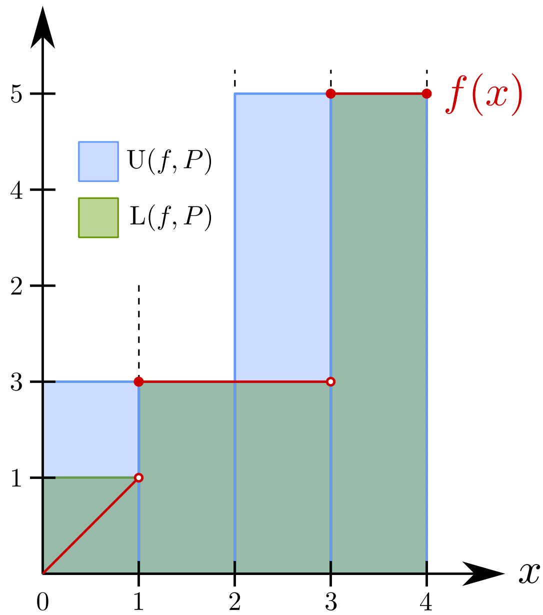


Figure 6.1: Illustration of lower and upper sums. The lower sum is the total green area and the upper sum is the total blue area. For this function, as the space between points in the partition decreases, the lower and upper sums should get closer to the actual area under the curve.

Example 6.1.1

Here is an example on how to evaluate upper and lower sums. Let

$$f(x) = \begin{cases} x, & x \in [0,1) \\ 2, & x \in [1,3) \\ 5, & x \in [3,4]. \end{cases}$$



Let partition $P_1 = \{0, 2, 4\}$. We can then calculate:

$$\mathrm{L}(f, P_1) = \sum_{i=1}^{2} \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) = 0 \cdot 2 + 2 \cdot 2 = 4,$$
$$\mathrm{U}(f, P_1) = \sum_{i=1}^{2} \sup_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) = 2 \cdot 2 + 5 \cdot 2 = 14.$$

Additionally, let partition $P_2 = \{0, 1, 2, 3, 4\}$. Then,

$$\mathrm{L}(f, P_2) = 0 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 5 \cdot 1 = 9,$$
$$\mathrm{U}(f, P_2) = 2 \cdot 1 + 2 \cdot 1 + 5 \cdot 1 + 5 \cdot 1 = 14.$$

Note that the lower sum increased when we added more elements to the partition. This is a consequence of Lemma 6.1.1 below.

Definition 6.1.3: Refinement of a Partition

A partition P' is called a *refinement* of P if $P \subset P'$.

Lemma 6.1.1. *If there exists a refinement P' of partition P (on the same interval), then* $\mathrm{L}(f, P) \leq \mathrm{L}(f, P')$ *and* $\mathrm{U}(f, P) \geq \mathrm{U}(f, P')$.

Proof. We will first consider what happens when P' differs from P just once. Let $P = \{x_0, x_1, \cdots, x_n\}$, and assume

$$P' = \{x_0, x_1, \cdots, x_{j-1}, \gamma, x_j, \cdots, x_n\}.$$

Then,

$$\begin{aligned}
\mathrm{L}(f, P) &= \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) \\
&= \sum_{i=1, i \neq j}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) + \inf_{x \in [x_{j-1}, x_j]} f(x).
\end{aligned}$$

This just splits the sum into one term for $i = j$ and another sum for the other elements. Observe that the quantity

$$\inf_{x \in [x_{j-1}, x_j]} f(x) = \inf\{f(x) \mid x \in [x_{j-1}, x_j]\}$$

is *less than or equal to* $\inf\{f(x) \mid x \in [x_{j-1}, \gamma]\}$ and $\inf\{f(x) \mid x \in [\gamma, x_j]\}$. This is because as you shrink the set, the infimum will either stay the same or increase. Therefore,

$$\begin{aligned}
\mathrm{L}(f, P) \leq &\sum_{i=1, i \neq j}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) + \inf_{x \in [x_{j-1}, \gamma]} f(x)(\gamma - x_{j-1}) \\
&+ \inf_{x \in [\gamma, x_j]} f(x)(x_j - \gamma) \\
= &\mathrm{L}(f, P').
\end{aligned}$$

Similarly, $\mathrm{U}(f, P) \geq \mathrm{U}(f, P')$. We use a similar inequality for suprema.

If P' differs from P by m times, simply repeat the above argument m times. From Definition 6.1.1, both P and P' must be finite sets, so this is sufficient to prove the theorem. □

Definition 6.1.4: Lower and Upper Integrals

The quantity

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\{\mathrm{L}(f,P) \mid P \text{ is a partition of } [a,b]\}$$

is called the *lower integral* of f over $[a,b]$. Similarly, the quantity

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\{\mathrm{U}(f,P) \mid P \text{ is a partition of } [a,b]\}$$

is called the *upper integral* of f over $[a,b]$.

Lemma 6.1.2. *If P_1 and P_2 are partitions of $[a,b]$, then $\mathrm{L}(f,P_1) \le \mathrm{U}(f,P_2)$.*

Proof. Define $P = P_1 \cup P_2$. Now, P_1 and P_2 are refinements of P. So,

$$\mathrm{L}(f,P_1) \le \mathrm{L}(f,P) \le \mathrm{U}(f,P) \le \mathrm{U}(f,P_2). \qquad \square$$

Definition 6.1.5: Integral

If $\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x$, then f is *integrable*, and

$$\int_a^b f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x$$

is the *integral* of f on $[a,b]$. It can be thought of as the *signed area* of f.

6.2 Integrability

Under what conditions can functions actually be integrable? In other words, when is the equality in Definition 6.1.5 true? It turns out that not a lot is required.

Theorem 6.2.1

The function $f : [a,b] \to \mathbb{R}$ is integrable if and only if for all $\varepsilon > 0$, there exists some partition P such that $\mathrm{U}(f,P) - \mathrm{L}(f,P) < \varepsilon$.

Note that because this is an "if and only if" statement, this theorem is another way to check the integrability of a function instead of comparing upper and lower integrals. Some proofs may require *both* approaches.

Proof. For the forward direction, assume f is integrable. Fix $\varepsilon > 0$. There exists some partition P_1 such that

$$\int_a^b f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x < \mathrm{L}(f,P_1) + \frac{\varepsilon}{2}.$$

As a consequence,

$$\mathrm{L}(f,P_1) > \underline{\int_a^b} f(x)\,\mathrm{d}x - \frac{\varepsilon}{2}.$$

Also, there exists some partition P_2 such that

$$\int_a^b f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x > \mathrm{U}(f, P_2) - \frac{\varepsilon}{2}.$$

This means

$$\mathrm{U}(f, P_2) < \overline{\int_a^b} f(x)\,\mathrm{d}x + \frac{\varepsilon}{2}.$$

Define $P = P_1 \cup P_2$. Then,

$$\begin{aligned}\mathrm{U}(f, P) - \mathrm{L}(f, P) &\leq \mathrm{U}(f, P_2) - \mathrm{L}(f, P_1)\\ &< \int_a^b f(x)\,\mathrm{d}x + \frac{\varepsilon}{2} - \left(\int_a^b f(x)\,\mathrm{d}x - \frac{\varepsilon}{2}\right)\\ &= \varepsilon.\end{aligned}$$

For the reverse direction, fix $\varepsilon > 0$. We assume that there exists some partition P such that $\mathrm{U}(f, P) - \mathrm{L}(f, P) < \varepsilon$. Then,

$$\overline{\int_a^b} f(x)\,\mathrm{d}x - \underline{\int_a^b} f(x)\,\mathrm{d}x \leq \mathrm{U}(f, P) - \mathrm{L}(f, P) < \varepsilon.$$

This is because upper integral is less than or equal to the upper sum, and the lower integral is greater than or equal to the lower sum. Subtracting the inequality makes the result bigger.

Since $\overline{\int_a^b} f(x)\,\mathrm{d}x \geq \underline{\int_a^b} f(x)\,\mathrm{d}x$, then $\overline{\int_a^b} f(x)\,\mathrm{d}x - \underline{\int_a^b} f(x)\,\mathrm{d}x = 0$ because it is less than all ε greater than 0. Thus, f is integrable. $\square$

Example 6.2.1

Let $f(x) = 1$ where $x \in [0, 1]$. We want to determine if f is integrable, and will use the approach involving upper and lower integrals. Take a partition $P = \{x_0, x_1, \cdots, x_n\}$. Then,

$$\begin{aligned}\mathrm{L}(f, P) &= \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1})\\ &= \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1})\\ &= \cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + \cdots + x_n - \cancel{x_{n-1}}\\ &= x_n - x_0\\ &= 1.\end{aligned}$$

Similarly, $\mathrm{U}(f, P) = 1$ for all partitions P. Because $\mathrm{L}(f, P) = \mathrm{U}(f, P) = 1$, then the upper and lower integrals must be equal, so f is integrable and $\int_0^1 f(x)\,\mathrm{d}x = 1$.

Example 6.2.2
Define
$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$
We claim that f is not integrable on $[0, 1]$. Note that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are said to be *dense*, which means that if you take $x, y \in \mathbb{Q}$, you can find $c \in (x, y) \cap \mathbb{Q}$ as well. The same goes for $\mathbb{R} \setminus \mathbb{Q}$.

Take any partition $P = \{x_0, \cdots, x_n\}$. Then,
$$\mathrm{L}(f, P) = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) = 0,$$
so $\underline{\int_0^1} f(x)\,\mathrm{d}x = 0$. Also,
$$\mathrm{U}(f, P) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 = 1.$$
So, $\overline{\int_0^1} f(x)\,\mathrm{d}x = 1$. But $\overline{\int_0^1} f(x)\,\mathrm{d}x \neq \underline{\int_0^1} f(x)\,\mathrm{d}x$, so f is not integrable. As a side note, f is discontinuous *everywhere*.

Theorem 6.2.2

If $f : [a, b] \to \mathbb{R}$ is continuous at all but finitely many points, then f is integrable.

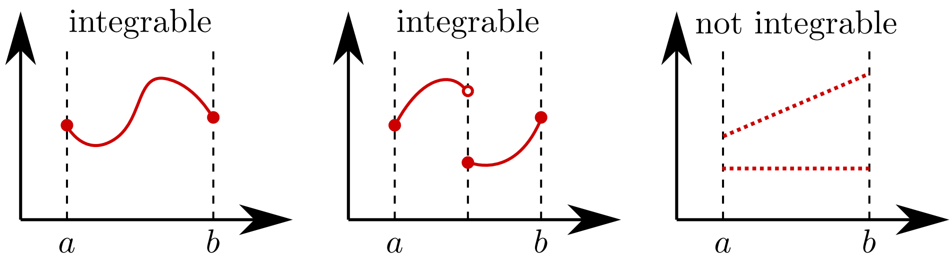


Figure 6.2: An illustration of Theorem 6.2.2. The third function has an infinite amount of discontinuities.

Proof. **Part 1:** We first prove the result if f has no discontinuities within its domain.

Assume f is continuous on $[a, b]$. Then, f is uniformly continuous on $[a, b]$ by Theorem 4.3.3. Fix $\varepsilon > 0$. We will find a partition P such that $\mathrm{U}(f, P) - \mathrm{L}(f, P) < \varepsilon$, which will imply integrability.

From the uniform continuity of f, there exists some $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{|b-a|}$. It will be clear why we choose $\frac{\varepsilon}{|b-a|}$ later.

Choose P such that $|x_i - x_{i-1}| < \delta$ for all $i = 1, \cdots, n$, and refer to its elements by $P = \{x_0, x_1, \cdots, x_n\}$. Then,
$$\mathrm{U}(f, P) - \mathrm{L}(f, P) = \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}).$$

This just comes from the definitions of upper and lower sums. Since f is continuous, by the Extreme Value Theorem, there exists some $x_i' \in [x_{i-1}, x_i]$ such that

$$f(x_i') = \sup_{x \in [x_{i-1}, x_i]} f(x),$$

and there exists some $x_i'' \in [x_{i-1}, x_i]$ such that

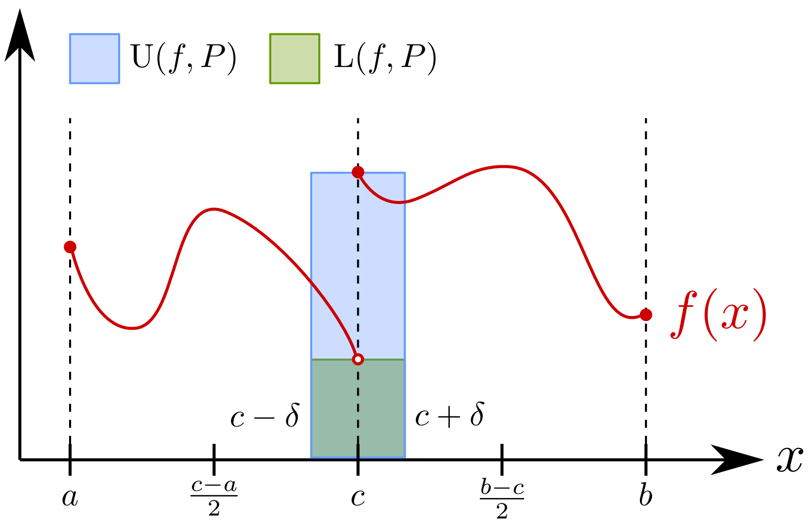
$$f(x_i'') = \inf_{x \in [x_{i-1}, x_i]} f(x).$$

Now,

$$\begin{aligned} \mathrm{U}(f,P) - \mathrm{L}(f,P) &= \sum_{i=1}^{n} \left(f(x_i') - f(x_i'')\right)(x_i - x_{i-1}) \\ &< \sum_{i=1}^{n} \frac{\varepsilon}{|b-a|}(x_i - x_{i-1}) \qquad (*) \\ &= \frac{\varepsilon}{|b-a|}(x_n - x_0) \\ &= \frac{\varepsilon}{|b-a|}|b-a| \\ &= \varepsilon. \end{aligned}$$

Thus, f is integrable. Note that for the line marked with $(*)$, since x_i' and x_i'' are in $[x_{i-1}, x_i]$, they must be closer than δ by how we've constructed P. This means that $|f(x_i') - f(x_i'')| < \frac{\varepsilon}{|b-a|}$.

Part 2: We now deal with the case where f has one discontinuity $c \in (a,b)$. The cases where $c = a$ or $c = b$ are treated similarly. Fix $\varepsilon > 0$. We will find a partition P such that $\mathrm{U}(f,P) - \mathrm{L}(f,P) < \varepsilon$.



Since f is bounded, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Define $\delta = \min\{\frac{\varepsilon}{8M}, \frac{b-c}{2}, \frac{c-a}{2}\}$, where the last two fractions are included because we don't want the δ-neighbourhood of c to be outside of $[a,b]$. By our choice of δ, $[c-\delta, c+\delta] \subset [a,b]$.

By **Part 1**, f is integrable on $[a, c-\delta]$ and $[c+\delta, b]$, since f is continuous on those intervals. Thus, there exists partitions P_1 of $[a, c-\delta]$ and P_2 of $[c+\delta, b]$ such that

$$\mathrm{U}(f, P_1) - \mathrm{L}(f, P_1) < \frac{\varepsilon}{4} \quad \text{and}\, \mathrm{U}(f, P_2) - \mathrm{L}(f, P_2) < \frac{\varepsilon}{4}.$$

Define $P = P_1 \cup P_2$, which is also a partition of $[a, b]$. Then,

$$\begin{aligned}
\mathrm{U}(f,P) - \mathrm{L}(f,P) &= (\mathrm{U}(f,P_1) - \mathrm{L}(f,P_1)) + (\mathrm{U}(f,P_2) - \mathrm{L}(f,P_2)) && (1)\\
&\quad + \left(\sup_{x\in[c-\delta,c+\delta]} f(x) - \inf_{x\in[c-\delta,c+\delta]} f(x)\right)(c+\delta-(c-\delta)) \\
&< \left(\frac{\varepsilon}{4}\right) + \left(\frac{\varepsilon}{4}\right) + \left(\sup_{x\in[c-\delta,c+\delta]} f(x) - \inf_{x\in[c-\delta,c+\delta]} f(x)\right)(2\delta) && (2)\\
&< \frac{\varepsilon}{2} + 2M \cdot 2\delta && (3)\\
&= \frac{\varepsilon}{2} + 4M \cdot \delta \\
&\le \frac{\varepsilon}{2} + 4M \cdot \frac{\varepsilon}{8M} && (4)\\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

(1) involves splitting $\mathrm{U}(f,P) - \mathrm{L}(f,P)$ into the upper and lower sums for the regions left and right of c, and adding the area of the the rectangle around the discontinuity.

(2) uses our bounds on the left and right sections since we know f is integrable there.

(3) holds because $f(x)$ is bounded by M, so it is at most $\sup f(x) = \inf f(x) = M$.

(4) comes from our definition of δ at the beginning of **Part 2**.

Part 3: If f has more than one discontinuity (but still a finite amount), just apply **Part 2** enough times. □

Theorem 6.2.3

If $f : [a, b] \to \mathbb{R}$ is monotone, it is integrable.

Proof. Assume f is monotone increasing. The proof is similar if f is monotone decreasing. Fix $\varepsilon > 0$, so we will find $P = \{x_0, \cdots, x_n\}$ such that $\mathrm{U}(f,P) - \mathrm{L}(f,P) < \varepsilon$.

Let P be the partition that splits $[a, b]$ into n equal parts. Namely, each element x_k has the formula $x_k = a + k \cdot \frac{b-a}{n}$. Then,

$$\begin{aligned}
\mathrm{U}(f,P) - \mathrm{L}(f,P) &= \sum_{i=1}^{n} \left(\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) && \text{(monotone)}\\
&= \left(\cancel{f(x_1)} - f(x_0) + \cancel{f(x_2)} - \cancel{f(x_1)} + \cdots + f(x_n) - \cancel{f(x_{n-1})}\right)\frac{b-a}{n} \\
&= \frac{b-a}{n}(f(x_n) - f(x_0))
\end{aligned}$$

$$= \frac{b-a}{n}(f(b) - f(a)).$$

Choose n to be greater than $\frac{(b-a)(f(b)-f(a))}{\varepsilon}$. Then, this will give $\mathrm{U}(f,P) - \mathrm{L}(f,P) < \varepsilon$, thus f is integrable. $\square$

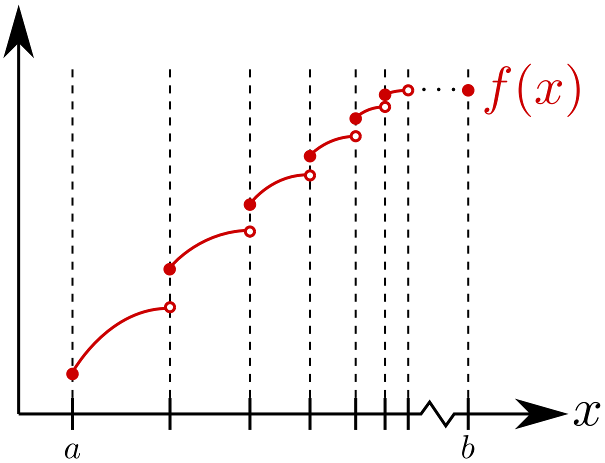


Figure 6.3: An example of a function which has infinite discontinuities but is monotone and bounded. Intuitively, if you make δ near b, then on $[a,\delta]$ there will be finitely many discontinuities. So, f is integrable on $[a,\delta]$, and since the discontinuities are concentrated near b, they will be insignificant since the width gets smaller, making the whole function integrable.

Note that a monotone function on $[a,b]$ is bounded.

6.3 Integral Properties

Lemma 6.3.1. *If $a \le x \le b$ and $a \le y \le b$ for $a,b,x,y \in \mathbb{R}$, then $|x-y| \le b-a$.*

Proof. From $a \le x \le b$, we have $a - y \le x - y \le b - y$. Using $-b \le -y$ and $-y \le -a$,

$$a - b \le a - y \le x - y \le b - y \le b - a.$$

So, $a - b = -(b-a) \le x - y \le b - a$, giving $|x-y| \le b-a$ as required. $\square$

Theorem 6.3.1

1. *If f is integrable on $[a,b]$, then so is λf for all $\lambda \in \mathbb{R}$, and*

$$\int_a^b \lambda f(x)\,\mathrm{d}x = \lambda \int_a^b f(x)\,\mathrm{d}x.$$

2. *If f,g are integrable on $[a,b]$, then so is $f+g$, and*

$$\int_a^b (f+g)(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x.$$

3. *If f,g are integrable and $f \le g$, then*

$$\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x.$$

4. *If f is integrable on $[a,b]$ and $[b,c]$, then f is integrable on $[a,c]$. Furthermore,*

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x.$$

5. *If f is integrable on $[a,b]$, then so is the function $|f|$, and*

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

Proof. **Part 1:** U and L both get multiplied by λ. Try as an exercise.

Part 2: Let $P = \{x_0, \cdots, x_n\}$ be a partition of $[a,b]$. For $i = 1, \cdots, n$, we claim that

$$\inf_{x\in[x_{i-1},x_i]} f(x) + \inf_{x\in[x_{i-1},x_i]} g(x) \leq \inf_{x\in[x_{i-1},x_i]} (f+g)(x).$$

Indeed, comparing the following two sets,

$$\{f(x') + g(x'') \mid x', x'' \in [x_{i-1}, x_i]\} \supseteq \{f(x) + g(x) \mid x \in [x_{i-1}, x_i]\}.$$

Since the second set is smaller, its infimum will be at least the infimum of the bigger set since there are less "options" for the infimum. This implies that

$$\begin{aligned}
\mathrm{L}(f,P) + \mathrm{L}(g,P) &= \sum_{i=1}^n \left(\inf_{x\in[x_{i-1},x_i]} f(x) + \inf_{x\in[x_{i-1},x_i]} g(x)\right)(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^n \left(\inf_{x\in[x_{i-1},x_i]} (f+g)(x)\right)(x_i - x_{i-1}) \\
&= \mathrm{L}(f+g,P).
\end{aligned}$$

Similarly,

$$\sup_{x\in[x_{i-1},x_i]} (f+g)(x) \leq \sup_{x\in[x_{i-1},x_i]} f(x) + \sup_{x\in[x_{i-1},x_i]} g(x),$$

and $\mathrm{U}(f,P) + \mathrm{U}(g,P) \geq \mathrm{U}(f+g,P)$. This means

$$\mathrm{L}(f,P) + \mathrm{L}(g,P) \leq \underbrace{\mathrm{L}(f+g,P) \leq \mathrm{U}(f+g,P)}_{\mathrm{L}\,\leq\,\mathrm{U}\text{ for same partition}} \leq \mathrm{U}(f,P) + \mathrm{U}(g,P). \tag{$*$}$$

Fix $\varepsilon > 0$. There exists partitions P' and P'' such that $\mathrm{U}(f,P') - \mathrm{L}(f,P') < \frac{\varepsilon}{2}$, and $\mathrm{U}(g,P'') - \mathrm{L}(g,P'') < \frac{\varepsilon}{2}$. If $P = P' \cup P''$, then adding the two equations gives

$$\mathrm{U}(f,P) + \mathrm{U}(g,P) - (\mathrm{L}(f,P) + \mathrm{L}(g,P)) < \varepsilon.$$

Using $(*)$, we can change this inequality to involve $f+g$. Since $\mathrm{U}(f+g,P) \leq \mathrm{U}(f,P) + \mathrm{U}(g,P)$ (and similar for L), we have

$$\mathrm{U}(f+g,P) - \mathrm{L}(f+g,P) < \varepsilon,$$

and $f+g$ is integrable.

Alternatively, we could have used Lemma 6.3.1, noticing that $\mathrm{L}(f+g,P)$ and $\mathrm{U}(f+g,P)$ are both within the same bounds.

To find an explicit formula for the integral of $f + g$, we can write

$$\mathrm{L}(f, P) + \mathrm{L}(g, P) \le \mathrm{L}(f + g, P) \le \int_a^b (f + g)(x)\,\mathrm{d}x \le \mathrm{U}(f + g, P) \le \mathrm{U}(f, P) + \mathrm{U}(g, P).$$

Also, by the definition of the integral,

$$\mathrm{L}(f, P) + \mathrm{L}(g, P) \le \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x \le \mathrm{U}(f, P) + \mathrm{U}(g, P).$$

We use Lemma 6.3.1 since the two previous inequality chains have the same left and right bounds. Therefore,

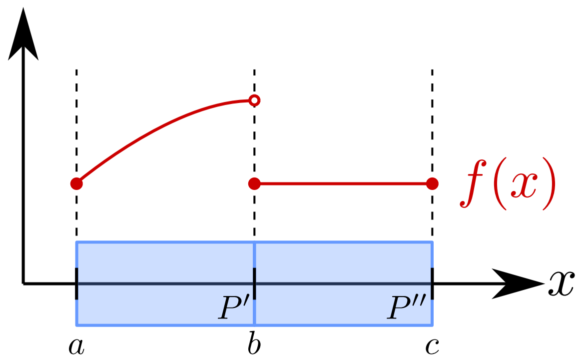
$$\begin{aligned}&\left|\int_a^b (f + g)(x)\,\mathrm{d}x - \left(\int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x\right)\right| \\ &\le \mathrm{U}(f, P) + \mathrm{U}(g, P) - (\mathrm{L}(f, P) + \mathrm{L}(g, P)) \\ &< \varepsilon.\end{aligned}$$

Since the absolute value is less than ε for all $\varepsilon > 0$ it must be equal to 0. Thus,

$$\int_a^b (f + g)(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x.$$

Part 3: Use the fact that $\mathrm{U} \ge \mathrm{L}$ for different partitions. Try as an exercise.

Part 4: We use the diagram below for intuition.



Fix $\varepsilon > 0$. Since f is integrable on $[a, b]$, there exists some partition P' of $[a, b]$ such that $\mathrm{U}(f, P') - \mathrm{L}(f, P') < \frac{\varepsilon}{2}$. Also, there exists some partition P'' of $[b, c]$ such that $\mathrm{U}(f, P'') - \mathrm{L}(f, P'') < \frac{\varepsilon}{2}$. Define $P = P' \cup P''$. Then, P is a partition of $[a, c]$, and

$$\mathrm{U}(f, P) - \mathrm{L}(f, P) = \mathrm{U}(f, P') + \mathrm{U}(f, P'') - (\mathrm{L}(f, P') + \mathrm{L}(f, P'')) < \varepsilon.$$

by adding the inequalities together. Thus, f is integrable on $[a, c]$. Next,

$$\mathrm{L}(f, P') + \mathrm{L}(f, P'') \le \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x \le \mathrm{U}(f, P') + \mathrm{U}(f, P'').$$

Also,

$$\mathrm{L}(f, P') + \mathrm{L}(f, P'') = \mathrm{L}(f, P) \le \int_a^c f(x)\,\mathrm{d}x \le \mathrm{U}(f, P) = \mathrm{U}(f, P') + \mathrm{U}(f, P'').$$

Using Lemma 6.3.1 as in **Part 2**, we have

$$\begin{aligned}
&\left|\int_a^c f(x)\,\mathrm{d}x - \left(\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x\right)\right| \\
&\le \mathrm{U}(f,P') + \mathrm{U}(f,P'') - (\mathrm{L}(f,P') + \mathrm{L}(f,P'')) \\
&< \varepsilon.
\end{aligned}$$

This holds for all $\varepsilon > 0$, so

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x.$$

Part 5: Try as an exercise using the triangle inequality. $\square$

6.4 The Fundamental Theorem of Calculus

Theorem 6.4.1: Mean Value Theorem for Integrals

If f is continuous on $[a,b]$, then there exists some $c \in [a,b]$ such that

$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a).$$

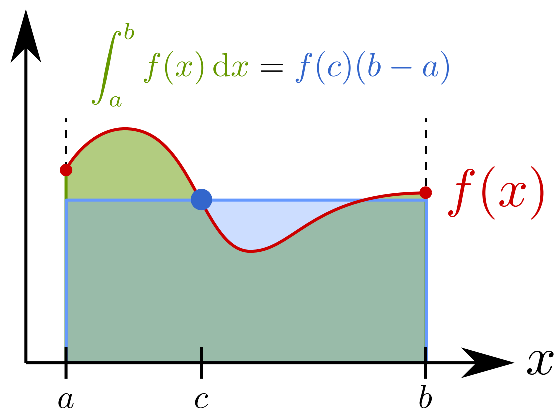


Figure 6.4: Illustration of Theorem 6.4.1.

Remark. You can interpret $f(c) = \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}$ as the average value of $f(x)$ from a to b.

Proof. If f is constant on $[a,b]$, then the result just comes from the formula for the area of a rectangle where c is the value of f. Assume f is not constant.

Denote $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$. Since f is continuous, by the Extreme Value Theorem, there exists some $x_m, x_M \in [a,b]$ such that $m = f(x_m)$ and $M = f(x_M)$. Without loss of generality, assume $x_m < x_M$. If this is not the case, apply the result to $-f$. x_m and x_M will not be equal as we assumed f is not constant. Observe that

$$m(b-a) \le \int_a^b f(x)\,\mathrm{d}x \le M(b-a).$$

So, $m \leq \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a} \leq M$.

Apply the Intermediate Value Theorem on $[x_m, x_M]$. Since $f(x_m) = m$ and $f(x_M) = M$, then there exists some $c \in [x_m, x_M]$ such that

$$f(c) = \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}.$$

Thus, $\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a)$ for some point $c \in [a,b]$. □

Theorem 6.4.2

If f is integrable on $[a,b]$, then the function

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is continuous on $[a,b]$.

This theorem shows that integrating a function 'upgrades' it: integrating an *integrable* function makes it *continuous*.

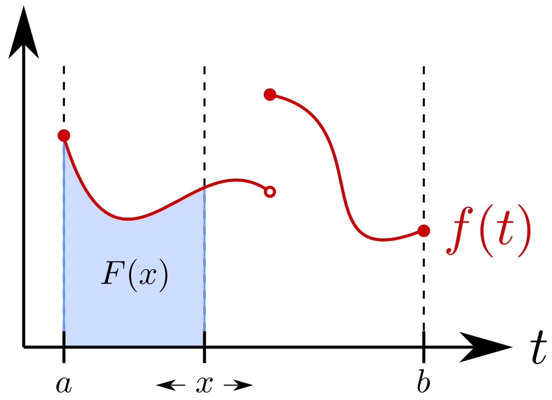


Figure 6.5: Illustration of Theorem 6.4.2. $F(x)$ is the area under the curve from a to x.

Proof. Let $c \in [a,b]$. We will show that F is continuous at c. Let M be such that $|f(x)| \leq M$ for all $x \in [a,b]$. We want to show that $\lim_{h\to 0} F(c+h) = F(c)$, or alternatively,

$$\lim_{h\to 0} (F(c+h) - F(c)) = 0.$$

We begin by showing $\lim_{h\to 0^+} (F(c+h) - F(c)) = 0$. Indeed, if $h > 0$, then

$$F(c+h) - F(c) = \int_a^{c+h} f(x)\,\mathrm{d}x - \int_a^c f(x)\,\mathrm{d}x = \int_c^{c+h} f(x)\,\mathrm{d}x.$$

Since $-M \leq f(x) \leq M$ on $[a,b]$, then

$$\int_c^{c+h} f(x)\,\mathrm{d}x \leq M(c+h-c) = Mh.$$

Also, $-Mh \leq \int_c^{c+h} f(x)\,\mathrm{d}x$. Therefore,

$$|F(c+h) - F(c)| = \left|\int_c^{c+h} f(x)\,\mathrm{d}x\right| \leq Mh.$$

From here, we can show that $\lim_{h\to 0^+} |F(c+h) - F(c)| = 0$. Using the definition of the upper limit found in Definition 4.1.7, pick $\delta = \frac{\varepsilon}{M}$. This is left as an exercise.

Similarly, $\lim_{h\to 0^-} |F(c+h) - F(c)| = 0$, so $F(x)$ is continuous for all $c \in [a, b]$ – that is, over the whole interval. $\square$

Theorem 6.4.3: Fundamental Theorem of Calculus

Assume $f : [a, b] \to \mathbb{R}$ is continuous. Define $F(x) = \int_a^x f(t)\,\mathrm{d}t$. Then, F is differentiable on (a, b), $F'(x) = f(x)$, and

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

This formula is sometimes called the Newton-Leibniz formula.

The following diagram illustrates how integration ‘upgrades’ the class of a function.

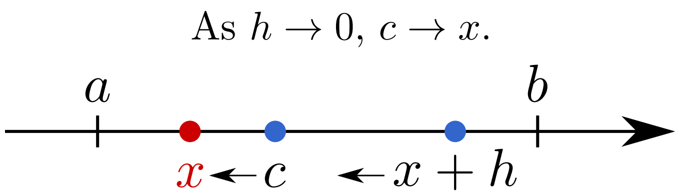
$$\text{Integrable} \xrightarrow[\text{(Theorem 6.4.2)}]{\text{integrate}} \text{Continuous} \xrightarrow[\text{(FTC)}]{\text{integrate}} \text{Differentiable}$$

Proof. We will show first that F is differentiable on (a, b) and $F'(x) = f(x)$. Let us compute $F'(x)$ for some $x \in (a, b)$. We want to find $\lim_{h\to 0} \frac{F(x+h)-f(x)}{h}$, and we will begin by finding the upper limit.

Given $h \in (0, b - x)$, we see that $F(x + h) - F(x) = \int_x^{x+h} f(t)\,\mathrm{d}t$. By Theorem 6.4.1,

$$\int_x^{x+h} f(t)\,\mathrm{d}t = f(c)(x + h - x) = f(c)h,$$

where $c \in [x, x + h]$.



Since f is continuous on a closed, bounded interval, it is uniformly continuous on $[a, b]$ (Theorem 4.3.3). Therefore, given some $\varepsilon > 0$, there exists some $\delta > 0$ such that if $|x - y| < \delta$ for all $x, y \in [a, b]$, then $|f(x) - f(y)| < \varepsilon$. If $h < \delta$, then $x \le c \le x+h < x+\delta$ so $|c - x| < \delta$. This is illustrated by the above image. Thus,

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| = \left|\frac{f(c)h}{h} - f(x)\right| = |f(c) - f(x)| < \varepsilon.$$

So, $\lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h} = f(x)$, as we've reworked the definition of uniform continuity into the definition of the limit. We can also show the lower limit is equal to $f(x)$ by a similar argument. Thus, $F'(x) = f(x)$.

The Newton-Leibniz formula follows from the definition of F. $\square$

Definition 6.4.4: Antiderivatives

The function F is an *antiderivative* of f if $F'(x) = f(x)$.

Remark. If F is defined as in the FTC, and G is some antiderivative of f, then $G(x) = F(x) + c$ for some $c \in \mathbb{R}$, and $\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a)$. This is because since $F'(x) = G'(x) = f(x)$, we find that

$$(F - G)'(x) = (f - f)(x) = 0,$$

so $F(x) - G(x) = c$ for some $c \in \mathbb{R}$. Also,

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = G(b) - c - (G(a) - c) = G(b) - G(a).$$

Additionally, if f is discontinuous, then $F(x) = \int_a^x f(t)\,\mathrm{d}t$ may not be differentiable. For example, take

$$f(x) = \operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0. \end{cases}$$

Note that $\int_{-1}^x f(t)\,\mathrm{d}t = |x| - 1$ on $[-1, 1]$, which is continuous but not differentiable.

Example 6.4.1

For $F(x) = \int_a^x \sin^3(t)\,\mathrm{d}t$, we can calculate $F'(x)$ by using the FTC. This gives $F'(x) = \sin^3 x$.

Example 6.4.2

We want to find the derivative of

$$F(x) = \int_a^{e^{x^2}} \sin^3 t\,\mathrm{d}t.$$

By FTC and the chain rule,

$$F'(x) = \sin^3(e^{x^2}) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2}\right) = 2xe^{x^2}\sin^3\left(e^{x^2}\right).$$

To get this result, we just consider $F(x) = G(H(x))$ where $G(x) = e^{x^2}$ and $H(x) = \int_a^x \sin^3 t\,\mathrm{d}t$.

Chapter 7

Integration Techniques and Applications

7.1 Integration Techniques

Definition 7.1.1: Indefinite Integrals

The *indefinite integral* of f is the family of functions $\int f(x)\,\mathrm{d}x = F(x) + c$, where F is an antiderivative of f and $c \in \mathbb{R}$.

Remark. If f is continuous, then we can take $F(x) = \int_a^x f(t)\,\mathrm{d}t$ using the FTC. However, we want an explicit formula for F. This isn't always possible to find, as in the case of functions such as $f(x) = e^{x^2}$.

Proposition 7.1.2

We have the following integral formulae for some $c \in \mathbb{R}$:

- $\int a\,\mathrm{d}x = ax + c$
- $\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$ *for* $n \neq -1$
- $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c$
- $\int \sin x\,\mathrm{d}x = -\cos x + c$
- $\int \cos x\,\mathrm{d}x = \sin x + c$
- $\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1} x + c$
- $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1} x + c$
- $\int e^x\,\mathrm{d}x = e^x + c.$

Proof. These can be verified by differentiating, using formulas in Example 5.1.4. □

Integration by Parts

Theorem 7.1.3: Integration by Parts

If $u, v : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then

$$\int uv' \,\mathrm{d}x = uv - \int u'v \,\mathrm{d}x \quad \textit{and} \quad \int_a^b uv' \,\mathrm{d}x = uv\Big|_a^b - \int_a^b u'v \,\mathrm{d}x,$$

where $uv|_a^b = u(b)v(b) - u(a)v(a)$.

Remark. Informally, we can write $u' \,\mathrm{d}x = \frac{\mathrm{d}u}{\cancel{\mathrm{d}x}} \cancel{\mathrm{d}x} = \mathrm{d}u$. The IBP formula then becomes $\int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u$.

Proof. We know $(uv)' = u'v + v'u$. Integrate both sides: $\int (uv)' \,\mathrm{d}x = \int u'v \,\mathrm{d}x + \int v'u \,\mathrm{d}x$. So, $uv = \int u'v \,\mathrm{d}x + \int uv' \,\mathrm{d}x$, and rearranging gives the result. The formula for the definite integral follows from the FTC. □

For some trickier IBP questions, LIATE *may* help choose the right u:

- **L**: Logarithmic
- **I**: Inverse trigonometric
- **A**: Arithmetic (just polynomials and the like)
- **T**: Trigonometric
- **E**: Exponential

Aim to identify u as a function in the first category that matches from the list. For example, with $\int xe^x \,\mathrm{d}x$, Arithmetic comes before Exponential, so you would pick $u = x$ and $\mathrm{d}v = e^x \,\mathrm{d}x$. The video at `https://www.youtube.com/watch?v=-reFBJ4R9iA` explains LIATE in the form of song.

Example 7.1.1

We want to evaluate $\int xe^x \,\mathrm{d}x$. Let $u = x$ and $\mathrm{d}v = e^x \,\mathrm{d}x$, so $\mathrm{d}u = 1 \,\mathrm{d}x$ and $v = \int e^x \,\mathrm{d}x = e^x$, ignoring the constant as we only need one constant term at the end of evaluation. So,

$$\begin{aligned} \int xe^x \,\mathrm{d}x &= \int u \,\mathrm{d}v \\ &= uv - \int v \,\mathrm{d}u \\ &= xe^x - \int e^x \cdot 1 \,\mathrm{d}x \\ &= xe^x - e^x + c, \quad c \in \mathbb{R}. \end{aligned}$$

Example 7.1.2
To evaluate $\int \ln x \,\mathrm{d}x$, let $u = \ln x$ and $\mathrm{d}v = 1\,\mathrm{d}x$ so $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and $v = \int 1\,\mathrm{d}x = x$. Then,

$$\begin{aligned}\int \ln x \,\mathrm{d}x &= \int u \,\mathrm{d}v \\ &= uv - \int v \,\mathrm{d}u \\ &= x \ln x - \int x \cdot \frac{1}{x} \,\mathrm{d}x \\ &= x \ln x - x + c, \quad c \in \mathbb{R}.\end{aligned}$$

Example 7.1.3
Let $I = \int e^x \sin x \,\mathrm{d}x$. Using LIATE, apply IBP with $u = \sin x$ and $\mathrm{d}v = e^x\,\mathrm{d}x$ so $\mathrm{d}u = \cos x\,\mathrm{d}x$ and $v = e^x$. So,

$$\begin{aligned}I &= \int u \,\mathrm{d}v \\ &= uv - \int v \,\mathrm{d}u \\ &= e^x \sin x - \int e^x \cos x \,\mathrm{d}x.\end{aligned}$$

From here, apply IBP again with $u = \cos x$ and $\mathrm{d}v = e^x\,\mathrm{d}x$ so $\mathrm{d}u = -\sin x$ and $v = e^x$, giving

$$\begin{aligned}I &= e^x \sin x - \int u \,\mathrm{d}v \\ &= e^x \sin x - \left(uv - \int v \,\mathrm{d}u\right) \\ &= e^x \sin x - \left(e^x \cos x - \int -e^x \sin x \,\mathrm{d}x\right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \,\mathrm{d}x \\ &= e^x \sin x - e^x \cos x - I + c \\ 2I &= e^x \sin x - e^x \cos x + c \\ I &= \frac{e^x}{2}(\sin x - \cos x) + c, c \in \mathbb{R}.\end{aligned}$$

Usually you will need to apply IBP multiple times if you have e^x multiplied by another function as the integrand.

Example 7.1.4
Let $I = \int \sin^n x \,\mathrm{d}x$ for $n \geq 2$. Observe that

$$I = \int \sin^n x \,\mathrm{d}x = \int \sin^{n-1} x \cdot \sin x \,\mathrm{d}x,$$

so we can choose $u = \sin^{n-1} x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$ so $\mathrm{d}u = (n-1)\sin^{n-2} x \cdot \cos x\,\mathrm{d}x$ by the Chain Rule and $v = \int \sin x\,\mathrm{d}x = -\cos x$. Then,

$$\begin{aligned}
I &= \int u \, \mathrm{d}v \\
&= uv - \int v \, \mathrm{d}u \\
&= -\sin^{n-1} x \cos x - \int -(n-1) \sin^{n-2} x \cos x \cdot \cos x \, \mathrm{d}x \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, \mathrm{d}x \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \left(1 - \sin^2 x\right) \, \mathrm{d}x \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, \mathrm{d}x - (n-1) \int \sin^n x \, \mathrm{d}x \\
&= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, \mathrm{d}x - (n-1) I \\
nI &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, \mathrm{d}x \\
I &= \frac{-1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, \mathrm{d}x.
\end{aligned}$$

This is called a *reduction formula* for $\int \sin^n x \, \mathrm{d}x$. There is a similar formula for $\int \cos^n x \, \mathrm{d}x$ – try it as an exercise.

Note that if you want to evaluate an integral consisting of a product of powers of sine and cosine, the general strategy is to use the identity $\sin^2 x + \cos^2 x = 1$ (rearranged for $\sin^2$ or $\cos^2$) and simplify until you have a product of only $\sin^1 x$ (or $\cos^1 x$) and an expression only using the other trig ratio. Then, use a u-substitution aiming to eliminate that $\sin x$ or $\cos x$. Integration by parts is often not needed.

Integration by Substitution

Theorem 7.1.4: Integration by Substitution

If f, g are continuous on $[a, b]$ and g is differentiable on (a, b) with continuous g', then

$$\int_{g(a)}^{g(b)} f(u) \, \mathrm{d}u = \int_a^b f(g(x)) g'(x) \, \mathrm{d}x.$$

Remark. Informally, $\int f(u(x)) \, \mathrm{d}u = \int f(u(x)) \cdot \frac{\mathrm{d}u}{\cancel{\mathrm{d}x}} \cdot \cancel{\mathrm{d}x} = \int f(u(x)) u'(x) \, \mathrm{d}x$.

Proof. Suppose F is an antiderivative of f. The left side of the result equals $F(g(b)) - F(g(a))$ by the FTC. Next, $(F \circ g)' = (F' \circ g) \cdot g' = (f \circ g)\, g'$ by the Chain Rule. This means that $F \circ g$ is an antiderivative of $(f \circ g)\, g'$. So, by the FTC,

$$\int_a^b (f \circ g)(x) g'(x) \, \mathrm{d}x = (F \circ g)(b) - (F \circ g)(a) = F(g(b)) - F(g(a)) = \text{LHS}. \qquad \square$$

Note that we can also use this theorem to find indefinite integrals. Make sure you change the bounds when using this with definite integrals!

Example 7.1.5

To find $I = \int_a^b \tan x \, \mathrm{d}x = \int_a^b \frac{\sin x}{\cos x} \, \mathrm{d}x$ for $a, b \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, let $g(x) = \cos x$ and $f(u) = \frac{1}{u}$, so

$$\begin{aligned} I &= \int_a^b f(g(x))g'(x) \, \mathrm{d}x \\ &= \int_{g(a)}^{g(b)} f(u) \, \mathrm{d}u \\ &= \int_{\cos a}^{\cos b} \frac{-1}{u} \, \mathrm{d}u \\ &= -\ln|u| \Big|_{\cos a}^{\cos b} \\ &= -\ln\cos|b| + \ln\cos|a| \, . \end{aligned}$$

Note that in practice, working is usually shown in the following manner instead of repeating the formula. Let $u = \cos x$ so $\mathrm{d}u = -\sin x \, \mathrm{d}x$ and thus $\mathrm{d}x = \frac{-1}{\sin x} \, \mathrm{d}u$. Then,

$$\begin{aligned} I &= \int_{\cos a}^{\cos b} \frac{\sin x}{u} \cdot \frac{-1}{\sin x} \, \mathrm{d}u \\ &= \int_{\cos a}^{\cos b} \frac{-1}{u} \, \mathrm{d}u \\ &= -\ln\cos b + \ln\cos a. \end{aligned}$$

Example 7.1.6

To find $I = \int \sin^3 x \cos x \, \mathrm{d}x$, let $u = \sin x$ so $\mathrm{d}u = \cos x \, \mathrm{d}x$. So,

$$\int \sin^3 x \cos x \, \mathrm{d}x = \int u^3 \, \mathrm{d}u = \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c$$

for some $c \in \mathbb{R}$.

Example 7.1.7

To find $I = \int_0^1 \sqrt{1 - x^2} \, \mathrm{d}x$, let $x = \sin u$ so $u = \sin^{-1} x$ and $\mathrm{d}x = \cos u \, \mathrm{d}u$. This gives

$$\begin{aligned} \int_0^1 \sqrt{1 - x^2} \, \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 u} \cos u \, \mathrm{d}u \\ &= \int_0^{\frac{\pi}{2}} \cos u \cdot \cos u \, \mathrm{d}u \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 2u) \, \mathrm{d}u \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{\cos 2u}{2} \, \mathrm{d}u \\ &= \left(\frac{u}{2} + \frac{\sin 2u}{4}\right) \Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4}. \end{aligned}$$

Example 7.1.8

To find $I = \int \frac{2}{(x^2+1)^2}\,\mathrm{d}x$, let $x = \tan u$ so $u = \tan^{-1} x$ and $\mathrm{d}x = \frac{1}{\cos^2 u}\,\mathrm{d}u$. This gives

$$\begin{aligned}
\int \frac{2}{\left(x^2+1\right)^2}\,\mathrm{d}x &= \int \frac{2}{\left(\tan^2 u + 1\right)^2} \cdot \frac{1}{\cos^2 u}\,\mathrm{d}u \\
&= \int \frac{2}{\left(\frac{1}{\cos^2 u}\right)^2} \cdot \frac{1}{\cos^2 u}\,\mathrm{d}u \\
&= \int \frac{2\cos^4 u}{\cos^2 u}\,\mathrm{d}u \\
&= \int 2\cos^2 u\,\mathrm{d}u \\
&= 2\int \frac{1}{2}\left(1+\cos 2u\right)\,\mathrm{d}u \\
&= u + \frac{1}{2}\sin 2u + c \\
&= \sin^{-1} x + \frac{1}{2}\sin\left(2\tan^{-1} x\right) + c, c \in \mathbb{R}.
\end{aligned}$$

Note that this expression (specifically $\sin\left(2\tan^{-1} x\right)$) can be simplified using the process described in Example 7.1.11.

As a general tip, if your integral contains something along the lines of $\sqrt{1-x^2}$, try something like $x = \sin u$ and use the identity $\sin^2 x + \cos^2 x = 1$. If your integral has something like x^2+1, try $x = \tan u$ and use the identity $\sec^2 x = \tan^2 x + 1$.

Integration of Rational Functions

Definition 7.1.5: Rational Functions

A *rational function* is a ratio of two polynomials, $\frac{p(x)}{q(x)}$, where $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$.

For this subsecton, we assume that $a_n = 1$. If not, just factor out a_n. Also, assume $m > n$. If not, rewrite $\frac{p(x)}{q(x)}$ as a polynomial added to a rational function such that $m > n$. For example,

$$\frac{u^2}{u-1} = \frac{u^2-1+1}{u-1} = \frac{(u-1)(u+1)+1}{u-1} = u+1+\frac{1}{u-1}.$$

We state two technical algebraic lemmas without proof.

Lemma 7.1.1. *Every polynomial can be factored as a product of functions of the form $ax+b$ and of the form ax^2+bx+c with $b^2-4ac<0$.*

As an example, $x^4 - 16 = \left(x^2\right)^2 - 4^2 = (x^2-4)(x^2+4) = (x+2)(x-2)(x^2+4)$. We are not diving into $\mathbb{C}$.

Lemma 7.1.2. *If $p(x)$ and $q(x)$ are such that*

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

$$\begin{aligned} q(x) = &(x-\alpha_1)^{r_1}(x-\alpha_2)^{r_2}\cdot\ldots\cdot(x-\alpha_k)^{r_k} \\ &\cdot\left(x^2+\beta_1 x+\gamma_1\right)^{s_1}\cdot\ldots\cdot\left(x^2+\beta_2 x+\gamma_l\right)^{s_l}, \end{aligned}$$

then

$$\begin{aligned} \underbrace{\frac{p(x)}{q(x)}}_{\text{hard-to-int. fraction}} &= \underbrace{\left[\frac{w_1}{x-\alpha_1}+\frac{w_2}{(x-\alpha_1)^2}+\cdots+\frac{w_{r_1}}{(x-\alpha_1)^{r_1}}\right]}_{\text{easy-to-integrate fractions}} \\ &+\cdots\text{ (similar terms for each linear factor)} \\ &+\underbrace{\left[\frac{y_1 x+z_1}{x^2+\beta_1 x+\gamma_1}+\frac{y_2 x+z_2}{(x^2+\beta_1 x+\gamma_1)^2}+\cdots+\frac{y_{s_1}x+z_{s_1}}{(x^2+\beta_1 x+\gamma_1)^{s_1}}\right]}_{\text{possibly try u-substitution such as } x=\tan u} \\ &+\cdots\text{ (similar terms for each quadratic factor).} \end{aligned}$$

This is called partial fraction decomposition*.*

Essentially, the lemma just says that we can decompose some rational function (given the assumptions stated earlier) into a sum of fractions that look like $\frac{a}{(x-b)^n}$ and $\frac{ax+b}{(x^2+cx+d)^n}$. There are techniques to integrate these kinds of fractions shown in the following examples.

Example 7.1.9

Let's perform partial fraction decomposition on $\frac{x+2}{x^2+x}$. By Lemma 7.1.2, we can express this as $\frac{x+2}{x^2+x}=\frac{x+2}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}$. How do we find A and B? Note that we can turn this into

$$\frac{A}{x}+\frac{B}{x+1}=\frac{A(x+1)+Bx}{x(x+1)}=\frac{(A+B)x+A}{x(x+1)}.$$

So, we can equate the numerator of this fraction with the original fraction from the question, giving us

$$(A+B)x+A=x+2,$$

so we have $A=2$ and $B=-1$. Then,

$$\frac{x+2}{x^2+x}=\frac{2}{x}-\frac{1}{x+1},$$

and

$$\int\frac{x+2}{x^2+x}\,\mathrm{d}x=\int\frac{2}{x}-\frac{1}{x+1}\,\mathrm{d}x=2\ln|x|-\ln|x+1|+c$$

for $c\in\mathbb{R}$.

Example 7.1.10

To find the integral $\int\frac{x^3+x+2}{x^4+2x^2+1}\,\mathrm{d}x$, observe that

$$\begin{aligned} \frac{x^3+x+2}{x^4+2x^2+1} &= \frac{x^3+x+2}{(x^2+1)^2} \\ &= \frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2} \end{aligned}$$

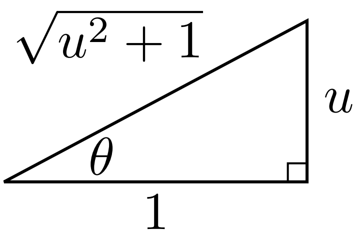
$$= \frac{(Ax+B)(x^2+1)+Cx+D}{(x^2+1)^2}$$
$$= \frac{Ax^3+Bx^2+(A+C)x+B+D}{(x^2+1)^2}.$$

So, we have
$$x^3+x+2 = Ax^3+Bx^2+(A+C)x+B+D,$$
giving $A=1$, $B=0$, $C=0$ and $D=2$. Then,
$$\int \frac{x^3+x+2}{x^4+2x^2+1} = \underbrace{\int \frac{1}{2}\frac{x}{x^2+1}\,\mathrm{d}x}_{\text{u-sub}} + 2\underbrace{\int \frac{1}{(x^2+1)^2}\,\mathrm{d}x}_{\text{see Example 7.1.8}},$$
$$= \frac{1}{2}\ln\left|x^2+1\right| + \tan^{-1}x + \frac{1}{2}\sin\left(2\tan^{-1}x\right) + c$$
for $c \in \mathbb{R}$.

For linear terms, use some substitution on the denominator like $u = x - \alpha$. For quadratic terms, use a substution like $x = \tan u$. Here's an example that relies heavily on trigonometric identities to simplify the integral.

Example 7.1.11

Consider the integral $\int \frac{1}{1+\sin x}\,\mathrm{d}x$. Let's use the substitution[a] $u = \tan\frac{x}{2}$, so $x = 2\tan^{-1}u$ and $\mathrm{d}x = \frac{2}{1+u^2}\,\mathrm{d}u$. Using this substitution, what will $\sin x$ equal to?
$$\sin x = \sin\left(2\tan^{-1}u\right) = 2\sin\left(\tan^{-1}u\right)\cos\left(\tan^{-1}u\right),$$
using the identity $\sin 2\theta = 2\sin\theta\cos\theta$. Both $\sin(\tan^{-1}u)$ and $\cos(\tan^{-1}u)$ can be calculuated using a trick involving right triangles. Let's start by setting $\theta = \tan^{-1}u$ so $u = \tan\theta$, and we want to find $\sin(\tan^{-1}u) = \sin\theta$. Using trigonometric ratios, $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{u}{1}$, so we draw the following triangle:



By Pythagoras' Theorem, the hypotenuse is $\sqrt{u^2+1}$. Thus,
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{u}{\sqrt{u^2+1}},$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{u^2+1}}.$$

So,
$$\sin x = 2\sin\left(\tan^{-1}u\right)\cos\left(\tan^{-1}u\right) = 2\frac{u}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{u^2+1}} = \frac{2u}{u^2+1}.$$

From here, substitute in for $\sin x$ and $\mathrm{d}x$.

$$\begin{aligned}\int \frac{1}{1+\sin x}\,\mathrm{d}x &= \int \frac{1}{1+\frac{2u}{u^2+1}} \cdot \frac{2}{1+u^2}\,\mathrm{d}u \\ &= 2\int \frac{1}{(1+u^2)+2u}\,\mathrm{d}u \\ &= 2\int \frac{1}{(u+1)^2}\,\mathrm{d}u.\end{aligned}$$

Let $t = u+1$ so $\mathrm{d}u = \mathrm{d}t$. Then,

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x = 2\int \frac{1}{t^2}\,\mathrm{d}t = \frac{-2}{t} + c = \frac{-2}{u+1} + c = \frac{-2}{\tan\frac{x}{2}+1} + c, c \in \mathbb{R}.$$

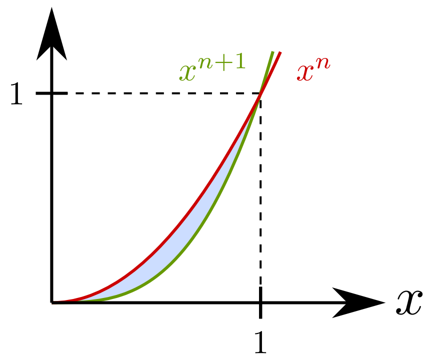
[a]This is a very sneaky substitution known as the Weierstrass substitution.

7.2 Areas between Curves

To find areas between curves within some region, first calculate their intersection points and determine the intervals which you'll need to integrate. Then, subtract the function on the bottom from the function on top and integrate over each interval. Here are some examples.

Example 7.2.1

Find the area of the region bounded by the curves $y = x^n$ and $y = x^{n+1}$ for $n \in \mathbb{N}\setminus\{1\}$, $x \geq 0$.



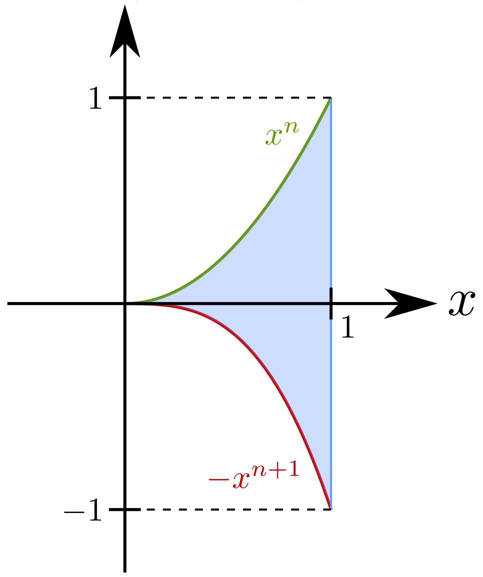
For intersection points, $x^n = x^{n+1} \implies x^n(1-x) = 0 \implies x = 0, x = 1$. So, the area is

$$\begin{aligned}\int_0^1 x^n - x^{n+1}\,\mathrm{d}x &= \left.\frac{x^{n+1}}{n+1}\right|_0^1 - \left.\frac{x^{n+2}}{n+2}\right|_0^1 \\ &= \frac{1}{n+1} - \frac{1}{n+2}.\end{aligned}$$

Note that we do $x^n - x^{n+1}$ because $x^n \geq x^{n+1}$ for $x \in [0,1]$.

Example 7.2.2

Find the area between the curves $y = x^n$ and $y = -x^{n+1}$ for $x \in [0, 1]$.



The area is just

$$\int_0^1 x^n \,\mathrm{d}x - \int_0^1 \left(-x^{n+1}\right) \,\mathrm{d}x = \left.\frac{x^{n+1}}{n+1}\right|_0^1 + \left.\frac{x^{n+2}}{n+2}\right|_0^1$$
$$= \frac{1}{n+1} + \frac{1}{n+2}.$$

7.3 Improper Integrals

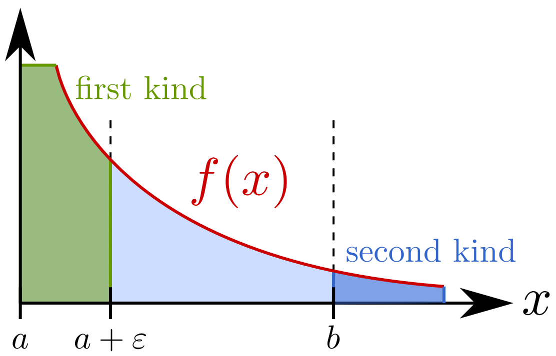


Figure 7.1: Illustration of improper integrals of the first and second kind.

Definition 7.3.1: Improper Integrals of First and Second Kind

Consider a function $f : [a, b] \to \mathbb{R}$ which is *not necessarily bounded.* Assume f is integrable on $[a + \varepsilon, b]$ for $\varepsilon > 0$ (technically, $\varepsilon \in (0, b - a)$). Then,

$$\lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} f(x) \,\mathrm{d}x$$

is called the *improper integral of the first kind* of f on $[a, b]$. It is usually denoted with the same notation as $\int_a^b f(x)\,\mathrm{d}x$.
Now, assume $f : [a, \infty) \to \mathbb{R}$ is integrable on $[a, b]$ for all $b > a$. Then,

$$\lim_{b\to\infty} \int_a^b f(x)\,\mathrm{d}x$$

is called the *improper integral of the second kind* of f on $[a, \infty)$. It is denoted $\int_a^\infty f(x)\,\mathrm{d}x$.
Similarly, we define $\int_{-\infty}^b f(x)\,\mathrm{d}x$.
Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable on $[a, b]$ for $a, b \in \mathbb{R}$. We can define

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^0 f(x)\,\mathrm{d}x + \int_0^\infty f(x)\,\mathrm{d}x,$$

provided both improper integrals on the RHS exist.

Remark. In general,

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x \neq \lim_{a\to\infty} \int_{-a}^a f(x)\,\mathrm{d}x.$$

For example, $\int_{-\infty}^\infty x\,\mathrm{d}x$ does not exist, since

$$\lim_{a\to\infty} \int_0^a x\,\mathrm{d}x = \lim_{a\to\infty} \frac{x^2}{2}\bigg|_0^a = \lim_{a\to\infty} \frac{a^2}{2} = \infty.$$

However,

$$\lim_{a\to\infty} \int_{-a}^a x\,\mathrm{d}x = \lim_{a\to\infty} \frac{x^2}{2}\bigg|_{-a}^a = \lim_{a\to\infty} 0 = 0,$$

since the areas to the left and right of $x = 0$ cancel out.

Here's an example of evaluating improper integrals.

Example 7.3.1

Consider $\int_0^1 x^p\,\mathrm{d}x$ for $p \in \mathbb{R}$. We will split this into five cases.

Case 1: If $p > 0$, we can just use the power rule.

Case 2: If $p = 0$, the integrand just degenerates to 1.

Case 3: If $p \in (-1, 0)$, we have problems at $x = 0$ since x^{-1} is undefined at 0. This is an improper integral of the first kind, so we compute

$$\begin{aligned}\int_0^1 x^p\,\mathrm{d}x &= \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 x^p\,\mathrm{d}x \\ &= \frac{1}{p+1} \lim_{\varepsilon\to 0^+} x^{p+1}\bigg|_\varepsilon^1 \\ &= \frac{1}{p+1} \lim_{\varepsilon\to 0^+} \left(1 - \varepsilon^{p+1}\right) \\ &= \frac{1}{p+1},\end{aligned}$$

since the power $p + 1$ is now positive.

Case 4: If $p < -1$, the integral diverges by a similar argument as in **Part 3**. The power $p - 1$ is negative so $\lim_{\varepsilon\to 0^+} \left(1 - \varepsilon^{p+1}\right) = -\infty$.

Case 5: If $p = -1$, then

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x} \, \mathrm{d}x = \lim_{\varepsilon \to 0^+} \ln x \Big|_\varepsilon^1 = \lim_{\varepsilon \to 0^+} \left(- \ln \varepsilon \right).$$

This limit does not exist so the integral diverges.

Chapter 8

Series

8.1 Series, Partial Sums and Convergence

Definition 8.1.1: Series

A *series* is a formal sum $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ where $(a_n)_{n=1}^{\infty}$ is a sequence. The index n doesn't need to begin at 1 – an index shift changes the formula of a_n to have the first element start at a chosen n.

Definition 8.1.2: Partial Sum

The number $S_k = \sum_{n=1}^{k} a_n$ is called the k^{th} *partial sum* of the series $\sum_{n=1}^{\infty} a_n$.

Definition 8.1.3: Convergent/Divergent Series

If the sequence $(S_k)_{k=1}^{\infty}$ converges to some $S \in \mathbb{R}$, we say that $\sum_{n=1}^{\infty} a_n$ *converges* and its sum is S. If $(S_k)_{k=1}^{\infty}$ diverges, then $\sum_{n=1}^{\infty} a_n$ *diverges.*

Look up Achilles and the Turtle - this is just an example to show that an infinite number of small things can add to something finite.

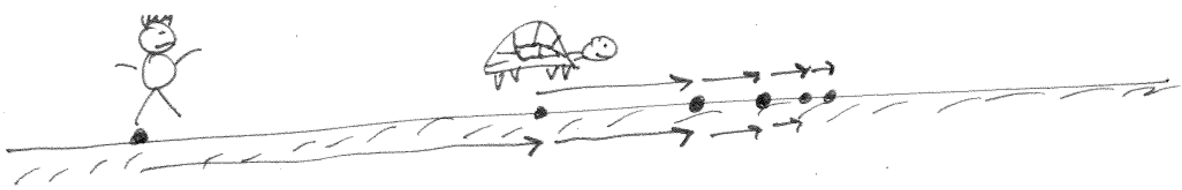


Figure 8.1: Illustration of Achilles and the Turtle by Dr. Pulemotov, 2019.

8.2 Series Convergence Tests

This section specifically deals with determining whether a given series converges. Note that it is often much harder to calculate what the series *converges to* – we will only be able to consistently do that for a very specific type of series in this course.

Theorem 8.2.1

If $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Denote $S_k = \sum_{n=1}^{k} a_n$. Then, $a_n = S_n - S_{n-1}$. Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = 0,$$

since both individual limits exist. □

Remark. The converse is **false**. We most often use this theorem in its contrapositive form – if $\lim_{n\to\infty} a_n \neq 0$, then the series *must diverge*.

Theorem 8.2.2: Cauchy Criterion

The series $\sum_{n=1}^{\infty} a_n$ *converges if and only if for all* $\varepsilon > 0$, *there exists some* $N \in \mathbb{N}$ *such that if* $q > p \geq N$, *then*

$$\left| \sum_{k=p+1}^{q} a_k \right| < \varepsilon.$$

Proof. $\sum_{n=1}^{\infty} a_n$ converges if and only if $(S_n)_{n=1}^{\infty}$ converges. This sequence converges if and only if it is Cauchy. This means that for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that if $p, q \geq N$, then $|S_q - S_p| < \varepsilon$. Now,

$$|S_q - S_p| = \left| \sum_{n=1}^{q} a_n - \sum_{n=1}^{p} a_n \right| = \left| \sum_{n=p+1}^{q} a_n \right| < \varepsilon.$$ □

Here's an example using the Cauchy Criterion.

Example 8.2.1

$\sum_{n=1}^{\infty} \frac{1}{n}$, called the Harmonic Series, diverges. To show this, assume it converges and apply the Cauchy Criterion. Set $\varepsilon = \frac{1}{3}$. Then, there exists some $N \in \mathbb{N}$ such that if $q > p \geq N$, then $\left|\sum_{n=p+1}^{q} \frac{1}{n}\right| < \frac{1}{3}$. In particular, this must hold for $p = N$, $q = 2N$. Then,

$$\begin{aligned} \left| \sum_{n=N+1}^{2N} \frac{1}{n} \right| &= \underbrace{\frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{2N}}_{N \text{ terms}} \\ &> \frac{1}{2N} + \frac{1}{2N} + \cdots + \frac{1}{2N} \\ &= \frac{N}{2N} = \frac{1}{2} > \frac{1}{3}, \end{aligned}$$

which is a contradiction. Thus, the Harmonic Series diverges.

Theorem 8.2.3

Assume $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, with $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$. Then,

1. $\sum_{n=1}^{\infty} (a_n + b_n) = a + b$.
2. $\sum_{n=1}^{\infty} \lambda a_n = \lambda a$ *for* $\lambda \in \mathbb{R}$.

Proof. **Part 1:** This is left as an exercise.

Part 2: Since the series $\sum_{k=1}^{n} \lambda a_n$ adds up a finite amount of terms, we can factor λ out of the sum, so

$$\sum_{n=1}^{\infty} \lambda a_n = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda a_k = \lambda \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lambda \lim_{n\to\infty} S_n = \lambda a. \qquad \square$$

Here are a number of tests you can use to check convergence.

Theorem 8.2.4: Comparison Test

If $0 \le a_n \le b_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Consider the sequence of partial sums $(S_n)_{n=1}^{\infty}$ where $S_n = \sum_{k=1}^{n} a_k$. Since $a_n \ge 0$, $(S_n)_{n=1}^{\infty}$ is non-decreasing. Also, $S_n \le \sum_{k=1}^{\infty} b_k$ for all $n \in \mathbb{N}$. Thus, both $(S_n)_{n=1}^{\infty}$ and $\sum_{n=1}^{\infty} a_n$ converge by Theorem 3.1.2. $\square$

Here are a few examples.

Example 8.2.2

We want to show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. We don't actually need the comparison test here. Observe that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Thus,

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_2 &= \frac{1}{2} + \frac{1}{2\cdot 3}\frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3} \\ S_3 &= \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{1}{2} + \frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} = 1 - \frac{1}{4}. \end{aligned}$$

By induction, we can show that $S_n = 1 - \frac{1}{n+1}$, so $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$.

Example 8.2.3

Using the contrapositive of the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \in (0,1)$, because $\frac{1}{n^p} > \frac{1}{n}$ for such p.

Example 8.2.4

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. This is *not* because $\frac{1}{n^2} > \frac{1}{n(n+1)}$ – this inequality goes the opposite direction from what we want.

Observe that $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$. Now, $0 \le \frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$. Since we have already shown $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges in Example 8.2.2, so does $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ by comparison. Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Example 8.2.5

Try showing as an exercise that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p \in (2, \infty)$ by comparison.

Theorem 8.2.5: Integral Test

Assume f is a non-negative, non-increasing, continuous function on $[1, \infty)$. Then, $\int_1^{\infty} f(x)\,\mathrm{d}x$ and $\sum_{n=1}^{\infty} f(n)$ converge or diverge together *(they probably won't have the same value, though).*

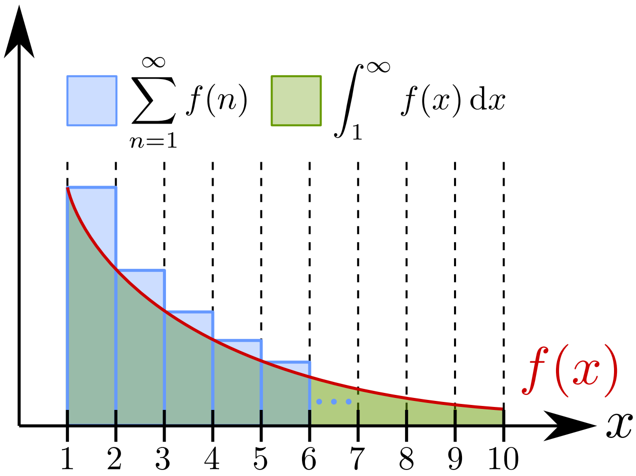


Figure 8.2: Illustration of the integral test, which asserts that the integral (green area) and infinite sum (blue area) must both converge or both diverge. You can think of the infinite sum as the upper sum of $f(x)$ over an infinitely long partition with points spaced distance 1 apart.

Proof. Consider the interval $[1, n+1]$. The set $P = \{1, 2, 3, 4, \cdots, n+1\}$ is a partition of $[1, n+1]$. Since f is non-increasing, $\inf_{x \in [x_{i-1}, x_i]} f(x) = f(x_i)$, and $\sup_{x \in [x_{i-1}, x_i]} f(x) = f(x_{i-1})$. So,

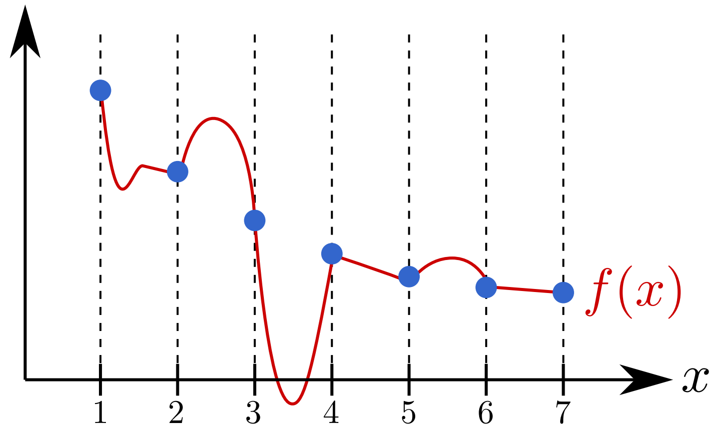
$$\mathrm{U}(f, P) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_{i-1})(1) = \sum_{i=1}^{n} f(i),$$

because of how we defined P. Also, $\mathrm{L}(f, P) = \sum_{i=1}^{n} f(i+1)$.

We know that $0 \leq \mathrm{L}(f, P) \leq \int_1^{n+1} f(x)\,\mathrm{d}x \leq \mathrm{U}(f, P)$. If $\int_1^{\infty} f(x)\,\mathrm{d}x$ converges, then $\mathrm{L}(f, P) = \sum_{i=1}^{n} f(i+1) = \sum_{i=2}^{n+1} f(i)$ converges as $n \to \infty$ by comparison. Thus, $\sum_{i=1}^{\infty} f(i)$ converges.

If $\int_1^{\infty} f(x)\,\mathrm{d}x$ diverges, then $\mathrm{U}(f, P) = \sum_{i=1}^{n} f(i)$ diverges as $n \to \infty$, so $\sum_{i=1}^{\infty} f(i)$ diverges. $\square$

Remark. The assumptions made on f are essential. The below image shows that if f is not non-increasing, then the area is not a valid approximation to whether the series converges or not.



Example 8.2.6

Does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge or $p \in (1,2)$? Take $f(x) = \frac{1}{x^p}$. Continuity of f follows from Theorem 4.2.2. f is non-negative as none of the variables in $\frac{1}{n^p}$ will be negative, and f being non-increasing can be shown by verifying $f'(x) \leq 0$. So, we can apply the integral test as follows.

$$\int_1^{\infty} \frac{1}{x^p}\,\mathrm{d}x = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,\mathrm{d}x = \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\bigg|_1^b = 0 - \frac{1}{-p+1} = \frac{-1}{1-p}.$$

Since the improper integral converges, the series must also converge by the Integral Test.

Theorem 8.2.6: p-Series Test

For some $p \in \mathbb{R}$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges for $p > 1$ and diverges for $p \leq 1$.

Proof. **Case 1:** If $p < 0$, the exponent to n^{-p} is positive and hence $\lim_{n\to\infty} n^{-p} \neq 0$ so the series diverges.

Case 2: If $p = 0$, the summand becomes $\frac{1}{p^0} = 1$ and $\lim_{n\to\infty} 1 \neq 0$ so the series diverges.

Case 3: If $p = 1$, the series is the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

Case 4: If $p \in (0,1)$, see Example 8.2.3.

Case 5: If $p \in (1,2)$, see Example 8.2.6.

Case 6: If $p = 2$, see Example 8.2.4.

Case 7: If $p > 2$, see Example 8.2.5. □

Theorem 8.2.7: Limit Comparison Test

Suppose $a_n, b_n > 0$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ (where $c \neq \infty$), then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

Proof. First assume that $\sum_{n=1}^{\infty} b_n$ converges. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, there exists some $N \in \mathbb{N}$ such that if $n \geq N$, then $\frac{a_n}{b_n} < 2c$ and $a_n < 2cb_n$.

Now, the series $\sum_{n=1}^{\infty} 2cb_n = 2c\sum_{n=1}^{\infty} b_n$ converges by assumption. The series $\sum_{n=N}^{\infty} a_n$ converges by comparison, since a_n is positive giving

$$0 < \sum_{n=N}^{\infty} a_n \leq 2c\sum_{n=N}^{\infty} b_n \leq 2c\sum_{n=1}^{\infty} b_n.$$

Since they differ by only finitely many terms. $\sum_{n=1}^{\infty} a_n$ converges as well.

If $\sum_{n=1}^{\infty} a_n$ converges, then, $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{c} > 0$, and $\sum_{n=1}^{\infty} b_n$ converges by the previous argument. ☐

Example 8.2.7

$\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$ diverges. Use limit comparison with $\frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n+1}{n^2+1}} = \lim_{n\to\infty} \frac{n^2+1}{n(n+1)} = 1 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$ does as well.

Remark. When you have a series with a rational function as the summand, try using limit comparison with the highest degree term on numerator divided by highest degree term on denominator. For instance, in the above example, we compared with $\frac{n}{n^2} = \frac{1}{n}$.

Definition 8.2.8: Geometric Series

Given $r \in \mathbb{R} \setminus \{0\}$, the series $\sum_{n=0}^{\infty} r^n$ is called a *geometric series.*

Proposition 8.2.9: Convergence of Geometric Series

Consider the series $\sum_{n=0}^{\infty} r^n$. If $|r| \geq 1$, then the series diverges. If $0 < |r| < 1$, then the series converges and its sum is $\frac{1}{1-r}$.

Proof. If $|r| \geq 1$, then $\lim_{n\to\infty} r^n \neq 0$ so the series diverges.
For $0 < |r| < 1$, We compute the partial sum:

$$\begin{aligned} S_n &= 1 + r + r^2 + \cdots + r^{n-1} + r^n \\ rS_n &= r + r^2 + r^3 + \cdots + r^n + r^{n+1} \\ S_n - rS_n &= 1 - r^{n+1} \\ S_n &= \frac{1 - r^{n+1}}{1 - r}. \end{aligned}$$

This means $\sum_{n=0}^{\infty} r^n = \lim_{n\to\infty} S_n = \frac{1}{1-r}$. ☐

Theorem 8.2.10: Ratio Test

Assume $a_n > 0$ for all $n \in \mathbb{N}$. Then,

1. *If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.*
2. *If $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.*
3. *Otherwise, the test is inconclusive.*

In practice it is often sufficient to only find lim *instead of* lim sup *and* lim inf*, since* lim = lim sup = lim inf *if a sequence converges (Theorem 3.2.6), but this statement is a stronger result.*

Proof. We assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r \in \mathbb{R}$ to simplify this proof.

Part 1: Fix s such that $r < s < 1$. Because $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < s$, then there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $\frac{a_{n+1}}{a_n} < s$. Then, $\frac{a_{N+1}}{a_N} < s$ so $a_{N+1} < sa_N$. Also, $\frac{a_{N+2}}{a_{N+1}} < s$ so $a_{N+2} < sa_{N+1} < s^2a_N$. Continuing like this we obtain $a_{N+k} < s^k a_N$. Thus,

$$\sum_{n=N}^{\infty} a_n \leq \sum_{k=0}^{\infty} s^k a_N = a_N \sum_{k=0}^{\infty} s^k = a_N \frac{1}{1-s}.$$

So, $\sum_{n=N}^{\infty} a_n$ converges and thus $\sum_{n=1}^{\infty} a_n$ converges.

Part 2: Assume $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r > 1$. Then, there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $\frac{a_{n+1}}{a_n} > 1$. Then, $\frac{a_{N+1}}{a_N} > 1$, so $a_{N+1} > a_N$. Also, $\frac{a_{N+2}}{a_{N+1}} > 1$ so $a_{N+2} > a_{N+1} > a_N$ etc. Thus, $a_{N+k} > a_N$ for all $k \in \mathbb{N}$, which means that a_n cannot approach 0 as $n \to \infty$, so the series diverges. $\square$

Example 8.2.8

Consider $\sum_{n=1}^{\infty} \frac{n!}{5^n}$. With series questions involving factorials, the Ratio Test is most likely the best way to approach the problem. Observe that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{5^{n+1}}}{\frac{n!}{5^n}} = \lim_{n\to\infty} \frac{(n+1)!5^n}{5^{n+1}n!} = \lim_{n\to\infty} \frac{n+1}{5} = \infty > 1,$$

so $\sum_{n=1}^{\infty} \frac{n!}{5^n}$ diverges.

Example 8.2.9

Consider $\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)!}$. We apply the Ratio Test as follows.

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{(n+1)^2}{(2(n+1)-1)!}}{\frac{n^2}{(2n-1)!}} \\
&= \lim_{n\to\infty} \frac{(n+1)^2(2n-1)!}{n^2(2n+1)!} \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{1}{(2n)(2n+1)} \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{4n^4+2n^3} \\
&= 0 < 1.
\end{aligned}$$

So, $\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)!}$ converges.

Example 8.2.10

To illustrate that we can't deduce the convergence of a series if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ using the Ratio Test, consider the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$. This diverges, however

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1.$$

Additionally, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as shown in Example 8.2.4, but

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = 1.$$

So, both convergent and divergent series could result in a value of 1 when taking the limit required by the Ratio Test.

Theorem 8.2.11: Alternating Series Test

Assume $(a_n)_{n=1}^{\infty}$ *is a sequence such that* $a_n > 0$ *and* $a_n \geq a_{n+1}$ *for all* $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} a_n = 0$. *Then, the series* $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ *converges.*

Remark. This test is also called the Leibniz Test. Also, any $(-1)^n$ term will do – the index does not need to be $n+1$ to use the test. Apply an index shift if you're not certain.

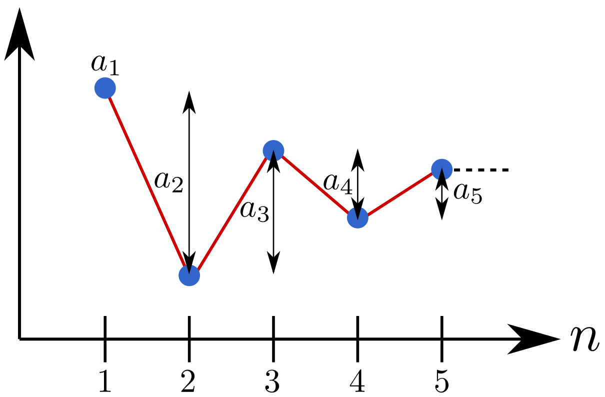


Figure 8.3: Illustration of the Alternating Series Test. The elements in a_n describe the difference between successive elements in the summand of the series.

Proof. Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$. Observe that $S_1 \geq S_3 \geq S_5 \geq \cdots$. Indeed,

$$S_{2n+3} = S_{2n+1} - a_{2n+2} + a_{2n+3} = S_{2n+1} + (a_{2n+3} - a_{2n+2}) \geq S_{2n+1},$$

since $a_{2n+3} - a_{2n+2} \geq 0$ by monotonicity. In addition,

$$S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2}) \leq S_{2n},$$

since $a_{2n+1} - a_{2n+2} \geq 0$ by monotonicity. Moreover, $S_{2n+1} = S_n + a_{2n+1} \geq S_{2n}$. In summary,

$$S_{2n} \leq S_{2n+1} \leq S_1, \quad \text{and} \quad S_{2n+1} \geq S_{2n} \geq S_2.$$

This shows that $(S_{2n})_{n=1}^{\infty}$ and $(S_{2n+1})_{n=1}^{\infty}$ are bounded. Since they are monotone and bounded, they converge by Theorem 3.1.2.

Let $\lim_{n\to\infty} S_{2n} = \alpha$, $\lim_{n\to\infty} S_{2n+1} = \beta$. We want to show $\alpha = \beta$. Now,

$$\beta - \alpha = \lim_{n\to\infty} S_{2n+1} - \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} (S_{2n+1} - S_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0$$

by assumption that $(a_n)_{n=1}^{\infty}$ goes to 0. Thus, $\beta - \alpha = 0$ so $\beta = \alpha$, and $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n+1}$, so $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges. □

Example 8.2.11

Try showing that the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges using the Alternating Series test on $a_n = \frac{1}{\ln n}$.

Definition 8.2.12: Absolute and Conditional Convergence

$\sum_{n=1}^{\infty} a_n$ *converges absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ *converges conditionally.* This distinction is necessary as alternating series need a lot less to converge than non-alternating ones.

Proposition 8.2.13

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

Proof. Use the Cauchy criterion. Fix $\varepsilon > 0$. We want to show that there exists some $N \in \mathbb{N}$ such that if $p > q \geq N$, then $\left|\sum_{n=q}^{p} a_n\right| < \varepsilon$. Since $\sum_{n=1}^{\infty} |a_n|$ converges, then there exists some $N \in \mathbb{N}$ such that $\left|\sum_{n=q}^{p} |a_n|\right| < \varepsilon$. By the triangle inequality,

$$\left|\sum_{n=q}^{p} a_n\right| \leq \sum_{n=q}^{p} |a_n| = \left|\sum_{n=q}^{p} |a_n|\right| < \varepsilon.$$

We are allowed to apply the triangle inequality to any finite number of terms. □

As an example, $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$ converges since it converges absolutely, as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

8.3 Taylor and Maclaurin Series

Consider an infinitely differentiable function $f : (b, c) \to \mathbb{R}$. Here, b may be $-\infty$ and c may be ∞. This introduces a bit of intuition on where Taylor and Maclaurin Series actually come from. I'd recommend re-reading this section once you get to the formal definition, and comparing what we had here to the definition.

Assume we can write the function f as $f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$, for some $a \in (b, c)$ and $x \in (b, c)$. This means that our function can be represented as infinitely many kx^n terms centered around $x = a$. Observe that

$$f(a) = a_0 + a_1(a-a) + a_2(a-a)^2 + \cdots = a_0.$$

Next,

$$f'(a) = a_1 + 2a_2(a-a) + 3a_3(a-a)^2 + \cdots = a_1$$

$$f''(a) = 2a_2 + 6a_3(a-a) + 12a_4(a-a)^2 + \cdots = 2a_2.$$

Continuing in this way, we find that $f^{(n)}(a) = n!a_n$, so each constant is given by

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

Definition 8.3.1: Taylor and Maclaurin Series

The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the *Taylor Series* of f at a. Taking the Taylor Series at $a = 0$ gives the *Maclaurin Series* of f, which is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.
Note that $0! = 1$. For the purposes of simplifying notation, we often define $0^0 = 1$, but note that this is *purely* just to simplify the formula (0^0 is indeterminate in most contexts).

Example 8.3.1

For $f(x) = e^x$, let us write down the Maclaurin Series. We have

$$f^{(n)}(0) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^x \bigg|_{x=0} = e^0 = 1.$$

Therefore, the series is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. We can use the Ratio Test to show this converges absolutely for all $x \in \mathbb{R}$:

$$\lim_{n\to\infty} \frac{|x|^{n+1} n!}{(n+1)! \, |x|^n} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1.$$

Example 8.3.2

For $f(x) = \sin x$, it is straightforward to show that for all $k \in \mathbb{N}$, $f^{(4k)}(x) = \sin x$, $f^{(4k+1)}(x) = \cos x$, $f^{(4k+2)}(x) = -\sin x$, etc. So, the Maclaurin Series of f is

$$\begin{aligned}
&\sin 0 + \frac{\cos 0}{1!}x - \frac{\sin 0}{2!}x^2 - \frac{\cos 0}{3!}x^3 + \frac{\sin 0}{4!}x^4 + \cdots \\
&= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.
\end{aligned}$$

By the Ratio Test, this is absolutely convergent for all $x \in \mathbb{R}$.

From this, we can determine $0 = \sin \pi = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!}$. Note that if you're ever given a sum that looks suspiciously like a Taylor Series you know, try manipulating the summand to get the expression to simplify.

Example 8.3.3

The Maclaurin Series of $\cos x$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$. This can be derived similarly to Example 8.3.2. You can memorise the differences between these formulas by recalling that cos is an even function, so the indices are even ($2n$), while sin is an odd function so the indices are odd $(2n+1)$.

Example 8.3.4

For $f(x) = \frac{1}{1-x}$, note that this is the formula for the limiting sum of a geometric series (see Proposition 8.2.9). Because of this, we can reason that the Maclaurin Series of f is $\sum_{n=0}^{\infty} x^n$. This converges if and only if $x \in (-1,1)$ as shown in that proposition.

Example 8.3.5

For $f(x) = \ln(1+x)$, it can be shown as an exercise that the Maclaurin Series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$. We find where it is convergent.

Case 1: If $x \in (-1,1)$, then the series converges absolutely by comparison, as $\left|\frac{(-1)^{n+1}}{n} x^n\right| \leq |x^n|$, and $\sum_{n=1}^{\infty} |x^n|$ converges for such x.

Case 2: If $|x| > 1$, then it diverges since $\frac{x^n}{n}$ does not go to 0.

Case 3: If $x = -1$, then

$$\frac{(-1)^{n+1}}{n} x^n = \frac{(-1)^{n+1+n}}{n} = \frac{(-1)^{2n+1}}{n} = \frac{-1}{n},$$

and $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

Case 4: If $x = 1$, then the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges conditionally by the Alternating Series test.

So, the Maclaurin Series converges for $x \in (-1,1]$.

Example 8.3.6

For the function

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

what is its Maclaurin Series? We compute the following derivatives:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{e^{\frac{-1}{h^2}}}{h} \\ &= \lim_{t \to \infty} \frac{e^{-t^2}}{\frac{1}{t}} && (\text{set } t = \tfrac{1}{h}) \\ &= \lim_{t \to \infty} \frac{t}{e^{t^2}} \\ &= \lim_{t \to \infty} \frac{1}{2e^{t^2}} \\ &= 0, \end{aligned}$$

using L'Hôpital's Rule. Similarly,

$$\begin{aligned} f''(0) &= \lim_{h\to 0} \frac{f'(h) - f'(0)}{h} \\ &= \lim_{h\to 0} \left(\frac{-2h^{-3}e^{\frac{-1}{h^2}}}{h} \right) \\ &= \lim_{h\to 0} -2h^{-4}e^{\frac{-1}{h^2}} \\ &= \lim_{t\to\infty} -2\frac{t^4}{e^{t^2}} \\ &= -2\lim_{t\to\infty} \frac{4t^3}{2te^{t^2}} \\ &= \cdots \\ &= 0, \end{aligned}$$

using the same substitution $t = \frac{1}{h}$ as before. We can show that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Thus, the Maclaurin Series of f is 0. This is an example of how calculating the Maclaurin series may not yield an accurate approximation to your function. As a side note, this function is 'weird' since it is nowhere analytic, which will be explained in MATH2401.

Example 8.3.7

For $f(x) = xe^{x^2}$, set $t = x^2$, so

$$e^{x^2} = e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

So, $f(x) = xe^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$.

You can calculate Taylor Series of more complicated functions if you're able to express it as a multiplication or composition of a function whose Taylor Series you already know, but be careful to note where the series converges.

Definition 8.3.2: Radius of Convergence

If a Taylor Series converges absolutely at some $x = a + r$ for some $r > 0$, then it converges absolutely for $x \in [a - r, a + r]$ by comparison (since for $x < a + r$, then $x^n < (a + r)^n$). The number

$$\sup\{r \geq 0 \mid \text{series conv. abs. on } [a - r, a + r]\}$$

is called the *radius of convergence* of the Taylor Series. It may be ∞.

This is covered in more detail in MATH2401.

Informal Discussion on the Euler Formula

How would you define e^z where $z \in \mathbb{C}$? If $z = \alpha + \beta i$ for some $\alpha, \beta \in \mathbb{R}$, then $e^z = e^\alpha \cdot e^{\beta i}$. We will try using a Maclaurin Series to define $e^{\beta i}$:

$$\begin{aligned}
e^{\beta i} &= \sum_{n=0}^{\infty} \frac{(i\beta)^n}{n!} \\
&= 1 + \frac{i\beta}{1} - \frac{\beta^2}{2!} - \frac{i\beta^3}{3!} + \frac{\beta^4}{4!} + \frac{i\beta^5}{5!} + \cdots \\
&= \text{(wizardry goes here)} \\
&= \left(1 - \frac{\beta^2}{2} + \frac{\beta^4}{4!} - \cdots\right) + i\left(\frac{\beta}{1!} - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \cdots\right) \\
&= \cos\beta + i\sin\beta.
\end{aligned}$$

This needs to be done *very carefully*, but it does work. The equation $e^{i\beta} = \cos\beta + i\sin\beta$ is called the Euler formula.

Chapter 9

Matrices

Notation

As a quick note, for this chapter and the next, boldface lowercase font like $\mathbf{x}$ is used for vectors and boldface uppercase font like $\mathbf{A}$ is used for matrices. Elements are written as x_i for vectors and A_{ij} for matrices. In handwriting it is common to use either no additional notation, $\vec{x}$ or a tilde underneath the letter for vectors. Matrices are commonly just handwritten as normal capital letters. When concatenating multiple vectors to form a matrix, I use square brackets to surround the elements instead of the usual parentheses. These are only guidelines – it only matters that notation is consistent and clear.

9.1 A Historical Introduction to Linear Algebra

This section will discuss ways to solve the following system of linear equations where $x_1, \cdots, x_n$ are the unknowns and the other terms are constants.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases} \tag{$*$}$$

Definition 9.1.1: Matrix

A *matrix* is essentially a rectangular table. An $m \times n$ (read m-by-n) matrix has m rows and n columns. A matrix is *square* if $m = n$.

Definition 9.1.2: Coefficient and Augmented Matrices

We can associate two matrices to the system $(*)$. The coefficient matrix corresponding to $(*)$ is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

and the *augmented matrix* corresponding to $(*)$ is

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

formed by concatenating the constants $b_1, \cdots, b_m$ to the right of the coefficient matrix.

Definition 9.1.3: Elementary Row Operations

The three *elementary row operations* on matrices are to

1. interchange two rows,
2. multiply a row by some non-zero constant;
3. add some constant multiple of a row to another row.

Definition 9.1.4: Row-Echelon Form

A matrix is in *row-echelon form* if:

1. All the rows containing zeroes are at the bottom.
2. The first non-zero element of every row is 1.
3. All the entries below the first non-zero entry of every row are zeroes, such as

$$\begin{pmatrix} a_{11} & \cdots & \cdots & \cdots \\ 0 & a_{22} & \cdots & \cdots \\ 0 & 0 & a_{33} & \cdots \end{pmatrix}.$$

Remark. Elementary row operations don't change the set of solutions to a matrix equation (i.e. the system of linear equations represented by the augmented matrix).

Definition 9.1.5: Rank of a Matrix

The number of non-zero rows in a matrix reduced to row-echelon form by elementary row operations is called the *rank* of the matrix. This also applies to augmented matrices.

Remark. A system like $(*)$ from the beginning of this subsection has at least one solution if and only after row reducing, all rows of zeros on the LHS are equal to 0 on RHS. Equivalently, the rank of the coefficient matrix is the same as the rank of the augmented matrix. If the rank also equals the number of unknowns, there is a *unique* solution to the system. Otherwise, there are infinitely many solutions.

Here are some examples of reducing matrices to row-echelon form using a process called *Gaussian Elimination.*

Example 9.1.1

Consider the system

$$\begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ x + y - z = 1. \end{cases}$$

We can form the following augmented matrix, and row reduce:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right)$$
$$\xrightarrow[R_3 \leftarrow \frac{-1}{2} R_3]{R_2 \leftarrow \frac{-1}{2} R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

This resulting matrix is now in row-echelon form. The ranks of both coefficient and augmented matrices are both 3. The system becomes

$$\begin{cases} x + y + z = 3 \\ y = 1 \\ z = 1 \end{cases}$$

so there is a unique solution $x = y = z = 1$.

Example 9.1.2

Consider the system

$$\begin{cases} x + 3y - 2x = 5 \\ -x + y - 2z = -9 \\ 2x + 4y - 2z = 12. \end{cases}$$

Then,

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ -1 & 1 & -2 & -9 \\ 2 & 4 & -2 & 12 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 4 & -4 & -4 \\ 0 & -2 & 2 & 2 \end{array}\right)$$
$$\xrightarrow[R_3 \leftarrow \frac{-1}{2} R_3]{R_2 \leftarrow \frac{1}{4} R_2} \left(\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right)$$
$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Note that the rank of the coefficient and augmented matrices are 2, but this does not equal the number of variables. Our system is

$$\begin{cases} x + 3y - 2z = 5 \\ y - z = -1. \end{cases}$$

Let $z = \alpha \in \mathbb{R}$, so our solution is

$$z = \alpha,\ y = -1 + \alpha,\ x = 8 - \alpha$$

for all $\alpha \in \mathbb{R}$. There are infinite solutions parameterised by the variable α.

Note that in some cases, it may be easier to reduce a matrix by first swapping rows to get the top left element in the left part of the augmented matrix as 1 before adding rows together.

Definition 9.1.6: Vector (Simplified)

An n-dimensional *vector* is an array of n numbers, written as a bold lowercase letter like $\mathbf{x}$, represented as

$$(x_1, x_2, \cdots, x_n) \quad \text{or} \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

The n-dimensional zero vector $\mathbf{0}$ has all of its elements set to zero. Vectors can be added together and be multiplied by a scalar, both operations being elementwise. In this course, we take vectors to be *column vectors* where the elements are stacked vertically. This can be interpreted as a $n \times 1$ matrix. The distinction between row and column vectors matters for the following operation, as explained in Section 9.2.

Definition 9.1.7: Multiplication between Matrices and Vectors

Let the matrix $\mathbf{A}$ and vector $\mathbf{x}$ be as follows:

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

We define the *multiplication* of $\mathbf{A}$ and $\mathbf{x}$ to be the m-dimensional vector

$$\mathbf{Ax} = \begin{pmatrix} A_{11}x_1 + \cdots + A_{1n}x_n \\ A_{21}x_1 + \cdots + A_{2n}x_n \\ \vdots \\ A_{m1}x_1 + \cdots + A_{mn}x_n \end{pmatrix}.$$

This can also be written in terms of each element of $\mathbf{Ax}$ as

$$(\mathbf{Ax})_i = \sum_{p=1}^{n} A_{ip}x_p$$

for $i = 1, \cdots, n$. The operation has the property that $\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{Ax} + \mathbf{Ay}$ and $\mathbf{A}(\lambda\mathbf{x}) = \lambda\mathbf{Ax}$ for all n-dimensional $\mathbf{x}, \mathbf{y}$ and $\lambda \in \mathbb{R}$. We discuss why $\mathbf{xA}$ is not defined in Section 9.2.

Remark. If we let

$$\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

we can denote the system $(*)$ as $\mathbf{Ax} = \mathbf{b}$. This is the matrix equation corresponding to the system.

Definition 9.1.8: Homogeneous and Inhomogeneous Systems

The system $\mathbf{Ax} = \mathbf{b}$ is called *homogeneous* if $\mathbf{b} = \mathbf{0}$, and *inhomogeneous* otherwise.

Definition 9.1.9: n-dimensional Real Space

We define the *n-dimensional real space* $\mathbb{R}^n$ as

$$\mathbb{R}^n = \{(x_1, \cdots, x_n) \mid x_1, \cdots, x_n \in \mathbb{R}\}.$$

Definition 9.1.10: Null-Space

The set $\{x \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{0}\}$ (the set of vectors $\mathbf{x}$ such that you get $\mathbf{0}$ when you apply $\mathbf{A}$ to $\mathbf{x}$) is called the *null-space* of $\mathbf{A}$. We denote this by $\mathrm{NS}(\mathbf{A})$.

Proposition 9.1.11

If $\mathbf{p}$ is a vector such that $\mathbf{Ap} = \mathbf{b}$, then

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{b}\} = \{\mathbf{y} + \mathbf{p} \mid \mathbf{y} \in \mathrm{NS}(\mathbf{A})\}.$$

Proof. Assume $\mathbf{y} \in \mathrm{NS}(\mathbf{A})$ and $\mathbf{Ap} = \mathbf{b}$. Note that

$$\mathbf{A}(\mathbf{y} + \mathbf{p}) = \mathbf{Ay} + \mathbf{Ap} = \mathbf{0} + \mathbf{b} = \mathbf{b}.$$

Because all vectors $\mathbf{y} + \mathbf{p}$ on the right side can be written as an element of the set on the left, then

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{b}\} \supseteq \{\mathbf{y} + \mathbf{p} \mid \mathbf{y} \in \mathrm{NS}(\mathbf{A})\}.$$

Now, assume $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ap} = \mathbf{b}$. We can write $\mathbf{x} = \mathbf{p} + (\mathbf{x} - \mathbf{p})$. The vector $\mathbf{y} = \mathbf{x} - \mathbf{p}$ is in $\mathrm{NS}(A)$ since $\mathbf{A}(\mathbf{x} - \mathbf{p}) = \mathbf{Ax} - \mathbf{Ap} = \mathbf{b} - \mathbf{b} = \mathbf{0}$. By the same logic as before,

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{b}\} \subseteq \{\mathbf{y} + \mathbf{p} \mid \mathbf{y} \in \mathrm{NS}(\mathbf{A})\}.$$

Since both sets are improper subsets of each other, they must be equal. □

9.2 Matrix Multiplication

Suppose we have some $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ with elements A_{ij} and B_{ij}. We can define $\mathbf{A} + \mathbf{B}$ and $\lambda\mathbf{A}$ (for $\lambda \in \mathbb{R}$) just by applying the operations to each element. This is associative and commutative, and also distributes under multiplication between a vector and a matrix as follows:

$$(\mathbf{A} + \mathbf{B})\mathbf{x} = \mathbf{Ax} + \mathbf{Bx}$$

How would we define multiplication between matrices? We could do it elementwise, i.e. $(\mathbf{AB})_{ij} = A_{ij}B_{ij}$. Although elementwise multiplication between matrices does have some uses, the following presents a more natural definition.

Assume $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times k$ for some $m, n, k \in \mathbb{N}$. Take some $\mathbf{x} \in \mathbb{R}^n$. Observe that the vector resulting from the multiplication of $\mathbf{Ax}$ is in $\mathbb{R}^m$, because $\mathbf{A}$ has the same amount of columns that $\mathbf{x}$ has rows. The matrix $\mathbf{A}$ can then be interpreted as a "mapping" which sends a vector from $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e. $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m$. Similarly, $\mathbf{B}$ can be interpreted as the mapping $\mathbf{B} : \mathbb{R}^k \to \mathbb{R}^n$.

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "\mathbf{B}"] \arrow[dr, "\mathbf{C}"'] & \mathbb{R}^n \arrow[d, "\mathbf{A}"] \\ & \mathbb{R}^m \end{tikzcd}$$

Figure 9.1: Illustration of matrix multiplication represented as mappings between $\mathbb{R}^k$, $\mathbb{R}^m$ and $\mathbb{R}^n$.

From Figure 9.1, the map $\mathbf{C} : \mathbb{R}^k \to \mathbb{R}^m$, given by $\mathbf{Cx} = \mathbf{A}(\mathbf{Bx})$ is a good candidate for the product of $\mathbf{A}$ and $\mathbf{B}$. $\mathbf{Cx}$ can be thought of as the *composition* of the two "transformations" $\mathbf{A}$ and $\mathbf{B}$ applied to $\mathbf{x}$, just like $h(x) = f(g(x))$ is the composition of the functions f and g applied to x.

To show $\mathbf{C}$ is given by a matrix, let us compute $\mathbf{A}(\mathbf{Bx})$ for some $\mathbf{x} \in \mathbb{R}^k$. We keep the same $m, n, k \in \mathbb{N}$ as before. By Definition 9.1.7,

$$(B\mathbf{x})_i = \sum_{j=1}^{k} B_{ij} x_j,$$

where j is just a dummy variable that we are summing over representing the column of $\mathbf{B}$ are looking at, and $(\mathbf{Bx})_i$ is the element in the i^{th} row. Now,

$$\begin{aligned} (\mathbf{A}(\mathbf{Bx}))_i &= \sum_{q=1}^{m} A_{iq} (\mathbf{Bx})_q \\ &= \sum_{q=1}^{m} A_{iq} \left(\sum_{j=1}^{k} B_{qj} x_j \right) \\ &= \sum_{q=1}^{m} \sum_{j=1}^{k} A_{iq} B_{qj} x_j \\ &= \sum_{j=1}^{k} \left(\sum_{q=1}^{m} A_{iq} B_{qj} \right) x_j \end{aligned}$$

where q is another indexing dummy variable. We are able to switch the sums because they are finite and addition of the elements is commutative. Thus, we have the following definition.

Definition 9.2.1: Matrix Multiplication

The *multiplication* $\mathbf{C} = \mathbf{AB}$ for $m \times n$ $\mathbf{A}$ and $n \times k$ $\mathbf{B}$ is given by a $m \times k$ matrix whose entry at the $(i, j)^{\text{th}}$ position is

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Theorem 9.2.2: Properties of Matrix Multiplication

For matrices of appropriate size,

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
2. $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$
3. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$

Proof. **Part 1:** Let $\mathbf{A}$ be $m \times n$, $\mathbf{B}$ be $n \times o$ and $\mathbf{C}$ be $o \times p$. We can then determine that $\mathbf{AB}$ is $m \times o$ and $\mathbf{BC}$ is $n \times p$. For the left-hand side,

$$\begin{aligned}(\mathbf{A}(\mathbf{BC}))_{ij} &= \sum_{k=1}^{m} A_{ik}(BC)_{kj} \\ &= \sum_{k=1}^{m} A_{ik}\left(\sum_{l=1}^{o} B_{kl}C_{lj}\right) \\ &= \sum_{k=1}^{m}\sum_{l=1}^{o} A_{ik}B_{kl}C_{lj}.\end{aligned}$$

For the right-hand side,

$$\begin{aligned}((\mathbf{AB})\mathbf{C})_{ij} &= \sum_{l=1}^{o}(AB)_{il}C_{lj} \\ &= \sum_{l=1}^{o}\left(\sum_{k=1}^{n} A_{ik}B_{kl}\right)C_{lj} \\ &= \sum_{k=1}^{n}\sum_{l=1}^{o} A_{ik}B_{kl}C_{lj} \\ &= \text{LHS}.\end{aligned}$$

Both sides are equal so $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

Part 2 and **Part 3** are left as exercises. □

Remark. Roses are red, stocks are lucrative; matrix muliplication is not commutative! In nearly all cases, $\mathbf{AB} \neq \mathbf{BA}$.

Here are some examples of matrix multiplication.

Example 9.2.1

If we had some

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

then

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

Example 9.2.2

Consider the matrix

$$\mathbf{Q} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

When $\mathbf{Q}$ is applied to some vector in $\mathbb{R}^2$, the vector gets rotated counter-clockwise about the origin by 90°. As an example,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

rotates by 180°, which is alternatively expressed as $\mathbf{Q}^2$. This can be interpreted as first applying a 90° rotation and then applying it again; $\mathbf{QQx} = \mathbf{Q}^2\mathbf{x}$. Similarly, the matrix for rotating by 270° counter-clockwise is

$$\mathbf{Q}^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The video at `https://www.youtube.com/watch?v=kYB8IZa5AuE` provides a good explanation of how this works.

Example 9.2.3

Referring to Definition 9.1.7, why is $\mathbf{xA}$ not defined? Since we consider $\mathbf{x} \in \mathbb{R}^n$ as a column vector, this can be interpreted as an $n \times 1$ matrix. If $\mathbf{A}$ is $m \times n$, we cannot multiply an $n \times 1$ matrix with an $m \times n$ matrix as the number of columns of $\mathbf{x}$ does not equal the number of rows in $\mathbf{A}$. However, $\mathbf{Ax}$ *is* defined which gives an $m \times 1$ matrix, or just an m-dimensional vector, as discussed earlier.

Definition 9.2.3: Identity Matrix

The $n \times n$ *identity matrix* $\mathbf{I_n}$, or just $\mathbf{I}$ for short, is the square matrix that has ones on the main diagonal and zeroes everywhere else. That is,

$$(\mathbf{I_n})_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

For example, in the 2×2 and the 3×3 case,

$$\mathbf{I_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{I_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is the identity element for matrix multiplication. For $n \times n$ $\mathbf{A}$, $\mathbf{AI_n} = \mathbf{I_nA} = \mathbf{A}$.

Definition 9.2.4: Elementary Matrix

Every elementary row operation is given by multiplication by one of the *elementary matrices.*

1. Multiplying row i of $\mathbf{A}$ by some $\lambda \in \mathbb{R}$ is the same as multiplying $\mathbf{A}$ by

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \lambda & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

   This is the identity matrix modified by adding λ in the $(i,i)^{\text{th}}$ spot.

2. Swapping rows i and j is the same as multiplying by

$$\begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix}$$

   which is the identity matrix modified by adding 1 in the $(i,j)^{\text{th}}$ and $(j,i)^{\text{th}}$ spots and adding 0 in the $(i,i)^{\text{th}}$ and $(j,j)^{\text{th}}$ spots.

3. Adding a row multiplied by λ to another row is the same as multiplying by

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & \vdots & \ddots & 0 \\ 0 & \cdots & \lambda & \cdots & 1 \end{pmatrix}$$

   This is the identity matrix modified by adding λ in the $(i,j)^{\text{th}}$ spot.

Corollary 9.2.1. *The process of Gaussian Elimination implies that every $n \times n$ matrix can be expressed as a product of elementary matrices $E_1E_2\cdots E_k$.*

9.3 Transposes and Inverses

Definition 9.3.1: Transpose of a Matrix

If $\mathbf{A}$ is $m \times n$, then the *transpose* of $\mathbf{A}$ denoted $\mathbf{A}^{\mathrm{T}}$ is an $n \times m$ matrix defined by $\left(A^{\mathrm{T}}\right)_{ij} = A_{ji}$. You can think of this as reflecting the elements along the diagonal, or where the n^{th} row is now the n^{th} column.

Theorem 9.3.2: Transpose Properties

For matrices of appropriate size,

1. $(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$
2. $\left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{A}$
3. $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

Proof. **Part 1** and **Part 2** are left as exercises.

Part 3: Assume **A** and **B** are $n \times n$. A very similar argument applies in the general case. Now, by Definition 9.2.1,

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj},$$

and we swap i and j when taking the transpose, so

$$\left((\mathbf{AB})^{\mathrm{T}}\right)_{ij} = \sum_{k=1}^{n} A_{jk}B_{ki}.$$

Also,

$$\left(\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\right)_{ij} = \sum_{k=1}^{n} B^{\mathrm{T}}_{ik}A^{\mathrm{T}}_{kj} = \sum_{k=1}^{n} B_{ki}A_{jk} = \sum_{k=1}^{n} A_{jk}B_{ki}$$

since scalar multiplication is commutative. Both sides are the same, so $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$. □

Definition 9.3.3: Matrix Inverse

If **A** is an $n \times n$ matrix, then an *inverse* of A is a matrix B such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I_n}$$

where $\mathbf{I_n}$ is the $n \times n$ identity matrix. We call A *invertible* if such a matrix exists.

Example 9.3.1

The matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

does not have an inverse, because if

$$\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an inverse of A, then

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for all values of a and b.

Proposition 9.3.4: Uniqueness of Matrix Inverse

If an inverse of **A** *exists, denoted by* $\mathbf{A}^{-1}$, *then it is unique.*

Proof. Assume $\mathbf{B}$ and $\mathbf{C}$ are inverses of $\mathbf{A}$. Then,

$$\mathbf{C} = \mathbf{I_n}\mathbf{C} = (\mathbf{BA})\mathbf{C} = \mathbf{B}(\mathbf{AC}) = \mathbf{B}\mathbf{I_n} = \mathbf{B}. \qquad \square$$

Proposition 9.3.5

If $\mathbf{AB} = \mathbf{I_n}$ *and* $\mathbf{CA} = \mathbf{I_n}$*, then* $\mathbf{B} = \mathbf{C}$*.*

Proof. This proof is left as an exercise and is similar to the one for Proposition 9.3.4. □

Note that Proposition 9.3.5 says that if we *already have* both a left and right inverse, they are the same. This is different to the existence of a left inverse implying the existence of a right inverse, which will be proved later in Theorem 9.3.7.

Example 9.3.2

$\mathbf{A}^{-1}$ can be thought of as reversing the transformation applied to a vector by the matrix $\mathbf{A}$. Referring back to Example 9.2.2 where $\mathbf{Q}$ was defined to be the matrix corresponding to a 90° counterclockwise rotation about $\mathbb{R}^2$, it is intuitive that $\mathbf{Q}^3\mathbf{Q} = \mathbf{I_2}$ because rotating by 270° counterclockwise and then by an additional 90° brings the rotation back to the initial position, represented by $\mathbf{I_2}$. By the same reasoning, $\mathbf{Q}\mathbf{Q}^3 = \mathbf{I_2}$, so we can write

$$\mathbf{Q}^{-1} = \mathbf{Q}^3.$$

Proposition 9.3.6

If $\mathbf{A}$ *is invertible, then* $\left(\mathbf{A}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}}$*.*

Proof. Note that

$$\left(\mathbf{A}^{-1}\right)^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \left(\mathbf{A}\mathbf{A}^{-1}\right)^{\mathrm{T}} = \mathbf{I_n}^{\mathrm{T}} = \mathbf{I_n},$$

and

$$\mathbf{A}^{\mathrm{T}}\left(\mathbf{A}^{-1}\right)^{\mathrm{T}} = \left(\mathbf{A}^{-1}\mathbf{A}\right)^{\mathrm{T}} = \mathbf{I_n}^{\mathrm{T}} = \mathbf{I_n},$$

so the inverse of $\mathbf{A}^{\mathrm{T}}$ is $\left(\mathbf{A}^{-1}\right)^{\mathrm{T}}$. □

If we have some $n \times n$ matrix $\mathbf{A}$, how do we find $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{I_n}$? We want

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix} = \mathbf{I_n}.$$

Let each $\mathbf{x_i}$ be a column vector representing the i^{th} column of $\mathbf{X}$, as in

$$\mathbf{x_1} = \begin{pmatrix} X_{11} \\ \vdots \\ X_{n1} \end{pmatrix}, \cdots, \mathbf{x_n} = \begin{pmatrix} X_{1n} \\ \vdots \\ X_{nn} \end{pmatrix}.$$

Now, the matrix equation $\mathbf{AX} = \mathbf{I_n}$ can be written in terms of each $\mathbf{x_i}$ as

$$\mathbf{A}\mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \mathbf{A}\mathbf{x_n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

This can be thought of as n systems of n equations, where we want $\mathbf{Ax_i}$ to equal the i^{th} row of $\mathbf{I_n}$. We can solve this using Gaussian Elimination as in the following example.

Remark. $\mathbf{A}$ is invertible if and only if it can be reduced to $\mathbf{I_n}$ using elementary row operations.

Example 9.3.3

We wish to find the inverse of

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & -3 \\ -2 & -2 & 5 \end{pmatrix}.$$

We can form the augmented matrix $[\mathbf{A} \mid \mathbf{I_3}]$ and reduce using Gaussian Elimination until the left side of the augmented matrix becomes $\mathbf{I_3}$. This is equivalent to solving the n systems of n equations $\mathbf{Ax_i} = (\mathbf{I_n})_i$ as discussed above.

$$\left(\begin{array}{ccc|ccc} 2 & -3 & 1 & 1 & 0 & 0 \\ 1 & -2 & -3 & 0 & 1 & 0 \\ -2 & -2 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow 2R_2} \left(\begin{array}{ccc|ccc} 2 & -3 & 1 & 1 & 0 & 0 \\ 2 & -4 & -6 & 0 & 2 & 0 \\ -2 & -2 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{ccc|ccc} 2 & -3 & 1 & 1 & 0 & 0 \\ 0 & -1 & -7 & -1 & 2 & 0 \\ 0 & -5 & 6 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \leftarrow R_3 - 5R_2]{R_1 \leftarrow R_1 - 3R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & 22 & 4 & -6 & 0 \\ 0 & -1 & -7 & -1 & 2 & 0 \\ 0 & 0 & 41 & 6 & -10 & 1 \end{array}\right)$$

$$\xrightarrow{\substack{R_1 \leftarrow \frac{1}{2}R_1 \\ R_2 \leftarrow -R_2 \\ R_3 \leftarrow \frac{1}{41}R_3}} \left(\begin{array}{ccc|ccc} 1 & 0 & 11 & 2 & -3 & 0 \\ 0 & 1 & 7 & 1 & -2 & 0 \\ 0 & 0 & 1 & \frac{6}{41} & \frac{-10}{41} & \frac{1}{41} \end{array}\right)$$

$$\xrightarrow[R_2 \leftarrow R_2 - 7R_3]{R_1 \leftarrow R_1 - 11R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{16}{41} & \frac{-13}{41} & \frac{-11}{41} \\ 0 & 1 & 0 & \frac{-1}{41} & \frac{-12}{41} & \frac{-7}{41} \\ 0 & 0 & 1 & \frac{6}{41} & \frac{-10}{41} & \frac{1}{41} \end{array}\right).$$

Here, the matrix on the right side of the augmented matrix is $\mathbf{A}^{-1}$. So,

$$\mathbf{A}^{-1} = \frac{1}{41}\begin{pmatrix} 16 & -13 & -11 \\ -1 & -12 & -7 \\ 6 & -10 & 1 \end{pmatrix}.$$

Theorem 9.3.7: Invertible iff Null-Space is 0

A square matrix $\mathbf{A}$ *is invertible if and only if* $\mathrm{NS}(\mathbf{A}) = \{\mathbf{0}\}$.

Proof. For the forward direction, we'll just assume $\mathbf{A}$ has a left inverse, which we know since $\mathbf{A}$ is invertible in the theorem statement. We know $\mathbf{0} \in \mathrm{NS}(\mathbf{A})$ because $\mathbf{A0} = \mathbf{0}$

always. We want to show that if $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$. Now, if $\mathbf{Ax} = \mathbf{0}$, then

$$\mathbf{x} = \left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{x} = \mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0},$$

so $\mathbf{A}$ having a left inverse means $\text{NS}(\mathbf{A}) = \{\mathbf{0}\}$.

For the reverse direction, we claim that if $\text{NS}(\mathbf{A}) = \{\mathbf{0}\}$, then $\mathbf{A}$ has a *right inverse.*

Proof (Claim). If $\text{NS}(\mathbf{A}) = \{\mathbf{0}\}$, then the system $\mathbf{Ax} = \mathbf{0}$ has a unique solution. By Proposition 9.1.11, the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution[1] for all $\mathbf{b} \in \mathbb{R}^n$. Because of this uniqueness, we can construct a right inverse $\mathbf{C}$ of $\mathbf{A}$ as follows.

Let the column vector $\mathbf{b_i}$ be the i^{th} column of the identity matrix $\mathbf{I_n}$. We can obtain vectors $\mathbf{x_i}$ by solving $\mathbf{Ax_i} = \mathbf{b_i}$ for each $i = 1, \cdots, n$. The matrix $\mathbf{C}$ can be constructed by concatenating the column vectors $\mathbf{x_i}$ together:

$$\mathbf{C} = \begin{bmatrix} \mathbf{x_1} & \mathbf{x_2} & \cdots & \mathbf{x_n} \end{bmatrix}.$$

$\mathbf{C}$ is the right inverse of $\mathbf{A}$, or $\mathbf{AC} = \mathbf{I_n}$, because of how each $\mathbf{x_i}$ was constructed. Thus, since $\text{NS}(\mathbf{A}) = \{\mathbf{0}\}$, $\mathbf{A}$ has a right inverse. □

From this, we will show that $\mathbf{A}$ has a left inverse. Note that the left inverse of $\mathbf{C}$ from the claim is $\mathbf{A}$. By the same argument as in the forward direction, this implies $\text{NS}(\mathbf{C}) = \{\mathbf{0}\}$. We use this with the above claim to say that $\mathbf{C}$ has a right inverse. By Proposition 9.3.5, since $\mathbf{C}$ has both a left and right inverse, both must be the same. So, $\mathbf{AC} = \mathbf{CA} = I_n$, hence $\mathbf{A}$ is invertible. □

Summary. Here is a summary of the major steps of the proof.

1. We prove that a matrix with a left inverse implies that its null-space is $\{\mathbf{0}\}$. Hence, if $\mathbf{A}$ is invertible, its null-space is $\{\mathbf{0}\}$ which proves the forward direction of the proof.
2. We prove that if a matrix $\mathbf{A}$ has a null-space of $\{\mathbf{0}\}$, then it has a right inverse. In doing so, we construct $\mathbf{C}$ which is a right inverse of $\mathbf{A}$.
3. Since $\mathbf{C}$ has a left inverse (which is $\mathbf{A}$), we use the argument in the first step to show $\text{NS}(\mathbf{C}) = \{\mathbf{0}\}$.
4. Proposition 9.3.5 says that if a matrix has both a left and right inverse, they are the same. From this, $\mathbf{C}$'s inverse is $\mathbf{A}$ and $\mathbf{A}$'s inverse is $\mathbf{C}$, thus $\mathbf{A}$ is invertible if its null-space is $\{\mathbf{0}\}$.

Note that this proof also shows that if a right inverse exists, then a left inverse exists.

Theorem 9.3.8: Matrix Inverse Properties

If $\mathbf{A}$ *and* $\mathbf{B}$ *are invertible, then*

1. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
2. $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$.

Proof. Both parts are left as exercises. □

Briefly, matrices *do not* form a field (see Definition 1.4.1) because matrix multiplication is not commutative and a non-zero matrix may not be invertible.

[1] I'm not certain of this – will need to check with someone.

9.4 Determinants

Definition 9.4.1: Determinant

Let $\mathbf{A}$ be an $n \times n$ matrix indexed by A_{ij}. The *determinant* of $\mathbf{A}$ is defined recursively as follows. If $n = 1$, then $\det \mathbf{A} = A_{11}$. Otherwise, we define a matrix $\mathbf{C}$ by

$$C_{ij} = (-1)^{i+j} \det \tilde{\mathbf{A}}_{\mathbf{ij}}$$

where

$$\tilde{\mathbf{A}}_{\mathbf{ij}} = \begin{pmatrix} A_{11} & \cdots & A_{1\,j-1} & A_{1\,j+1} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{i-1\,1} & \cdots & A_{i-1\,j-1} & A_{i-1\,j-1} & \cdots & A_{i-1\,n} \\ A_{i+1\,1} & \cdots & A_{i+1\,j-1} & A_{i+1\,j+1} & \cdots & A_{i+1\,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{n\,j-1} & A_{n\,j+1} & \cdots & A_{nn} \end{pmatrix}.$$

$\tilde{\mathbf{A}}_{\mathbf{ij}}$ is the matrix $\mathbf{A}$ with the i^{th} row and j^{th} column removed, so $\tilde{\mathbf{A}}_{\mathbf{ij}}$'s dimensions are $(n-1) \times (n-1)$. We call C_{ij} the $(i,j)^{\text{th}}$ *cofactor* of $\mathbf{A}$, and $\mathbf{C}$ is the *cofactor matrix* of $\mathbf{A}$. The determinant is then

$$\det \mathbf{A} = \sum_{i=1}^{n} A_{1i} C_{1i} = \sum_{i=1}^{n} (-1)^{i+1} A_{1i} \det \tilde{\mathbf{A}}_{\mathbf{1i}}.$$

Determinants of matrices are sometimes written as the matrix with vertical lines surrounding the elements instead of parentheses.

Remark. In terms of computing the determinant, the webpage `https://www.mathsisfun.com/algebra/matrix-determinant.html` provides a simple explanation. One thing of note is that for some 2×2 matrix,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Theorem 9.4.2

Suppose $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{a_1}, \cdots, \mathbf{a_n}$ *are row vectors in* $\mathbb{R}^n$. *Consider the matrices*

$$\mathbf{A} = \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{u} + \lambda \mathbf{v} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{u} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_{r-1}} \\ \mathbf{v} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{bmatrix},$$

for some $\lambda \in \mathbb{R}$, *which are formed by concatenating the row vectors vertically. Then,* $\det \mathbf{A} = \det \mathbf{B} + \lambda \det \mathbf{C}$.

Proof. We prove by induction. The result follows immediately if $n = 1$, since if $\mathbf{A} =$

$(u + \lambda v)$, $\mathbf{B} = (u)$ and $\mathbf{C} = (v)$ for $u, v, \lambda \in \mathbb{R}$, then

$$\det \mathbf{A} = u + \lambda v = \det \mathbf{B} + \lambda \det \mathbf{C}.$$

Now, assume the result holds for $(n-1) \times (n-1)$ matrices, and that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $n \times n$. We split the rest of the proof into two cases.

Case 1: $r = 1$, or in other words, the first row is being modified. Here, we will not be using the induction hypothesis. In this case, if we index $\mathbf{u}$ and $\mathbf{v}$ by u_j and v_j where j represents each column, then

$$\begin{aligned}
\det \mathbf{A} &= \sum_{j=1}^{n} (-1)^{1+j} (u_j - \lambda v_j) \det \tilde{\mathbf{A}}_{\mathbf{1j}} \\
&= \sum_{j=1}^{n} (-1)^{1+j} u_j \det \tilde{\mathbf{A}}_{\mathbf{1j}} + \lambda \sum_{j=1}^{n} (-1)^{1+j} v_j \det \tilde{\mathbf{A}}_{\mathbf{1j}}.
\end{aligned}$$

Since $r = 1$, then $\tilde{\mathbf{A}}_{\mathbf{1j}} = \tilde{\mathbf{B}}_{\mathbf{1j}} = \tilde{\mathbf{C}}_{\mathbf{1j}}$ because all other rows are the same if the first row is removed. So,

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{1+j} u_j \det \tilde{\mathbf{B}}_{\mathbf{1j}} + \lambda \sum_{j=1}^{n} (-1)^{1+j} v_j \det \tilde{\mathbf{C}}_{\mathbf{1j}} = \det \mathbf{B} + \lambda \det \mathbf{C}.$$

Case 2: $r > 1$. In this case, the first rows of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ will be the same. We know that $\tilde{\mathbf{A}}_{\mathbf{ij}}$ is $(n-1) \times (n-1)$. Since matrices $\tilde{\mathbf{A}}_{\mathbf{ij}}$, $\tilde{\mathbf{B}}_{\mathbf{ij}}$ and $\tilde{\mathbf{C}}_{\mathbf{ij}}$ will only differ by one row (or no rows if $i = r$), the induction hypothesis can be applied giving

$$\det \tilde{\mathbf{A}}_{\mathbf{ij}} = \det \tilde{\mathbf{B}}_{\mathbf{ij}} + \lambda \det \tilde{\mathbf{C}}_{\mathbf{ij}}.$$

Therefore,

$$\begin{aligned}
\det A &= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det \tilde{\mathbf{A}}_{\mathbf{1j}} \\
&= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \left(\det \tilde{\mathbf{B}}_{\mathbf{1j}} + \lambda \det \tilde{\mathbf{C}}_{\mathbf{1j}} \right) \\
&= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det \tilde{\mathbf{B}}_{\mathbf{1j}} + \lambda \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det \tilde{\mathbf{C}}_{\mathbf{ij}} \\
&= \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \det \tilde{\mathbf{B}}_{\mathbf{ij}} + \lambda \sum_{j=1}^{n} (-1)^{1+j} C_{1j} \det \tilde{\mathbf{C}}_{\mathbf{ij}} \\
&= \det \mathbf{B} + \lambda \det \mathbf{C}.
\end{aligned}$$

□

Corollary 9.4.1. *If* $\mathbf{A}$ *has a row of zeroes, then* $\det \mathbf{A} = 0$.

Proof. Assume the matrix $\mathbf{A}$ can be written as

$$A = \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_r} \\ \mathbf{0} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{bmatrix}$$

where $\mathbf{a_1}, \cdots, \mathbf{a_n}$ are row vectors. Then, we split $\mathbf{0}$ into $\mathbf{0} + \mathbf{0}$ and use the result of Theorem 9.4.2.

$$\det \mathbf{A} = \det \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_r} \\ \mathbf{0} + \mathbf{0} \\ \mathbf{a_{r+1}} \\ \vdots \\ \mathbf{a_n} \end{bmatrix} = \det \mathbf{A} + \det \mathbf{A} = 2 \det \mathbf{A} \implies \det \mathbf{A} = 0. \qquad \square$$

For the rest of this chapter, assume we deal with $n \times n$ matrices where $n \geq 2$ unless specified.

Theorem 9.4.3: Determinant along any Row

For all $i = 1, \cdots, n$,

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det \mathbf{\tilde{A}_{ij}}.$$

This allows us to take the determinant along any row as long as the signs of each A_{ij} are correct. Instead of alternating $+$ and $-$ along every column, $(-1)^{i+j}$ gives the following sign pattern, extended to any length:

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

Important: This proof given by the lecturer was not assessed when I took the course, but I am unsure about the current course offering. I have included it here for completion.

We will prove the following lemma first.

Lemma 9.4.1. *Assume* $\mathbf{B}$ *is a matrix whose i^{th} row is*

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

with a 1 *in the k^{th} spot. Then,*

$$\det \mathbf{B} = (-1)^{i+k} \det \mathbf{\tilde{B}_{ik}}.$$

Note that there is no condition on the elements of $\mathbf{B}$ that aren't in row i.

Proof (Lemma). We use induction. Proving this for the 2×2 base case is straightforward and will not be shown here. Assume the result holds for any $(n-1) \times (n-1)$ matrix. We will prove the lemma for the $n \times n$ case.

If the changed row is the first one, i.e. $i = 1$, then the result just restates the definition of the determinant. Assume $i > 1$. Then,

$$\det B = \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \det \tilde{B}_{1j}$$

$$= \sum_{j=1}^{k-1}(-1)^{1+j}B_{1j}\det\tilde{B}_{1j} + (-1)^{1+k}B_{1k}\det\tilde{B}_{1k} + \sum_{j=k+1}^{n}(-1)^{1+j}B_{1j}\det\tilde{B}_{1j}.$$

Note that $\det\tilde{B}_{1k} = 0$ by Corollary 9.4.1 because we remove the k^{th} column, so the new matrix has a row of zeroes. The second term disappears, so we will apply the induction hypothesis to the other two terms.

Consider the first term where $j < k$. For $\tilde{\mathbf{B}}_{\mathbf{1j}}$, the 1 will be in the $(i-1, k-1)^{\text{th}}$ spot. This is because we remove a row before row i shifting the first coordinate back by 1, and a column before k shifting the second coordinate back by 1. The rest of the $(i-1)^{\text{th}}$ row will be 0, which is left from the original matrix $\mathbf{B}$. We can immediately apply the induction hypothesis on $\tilde{\mathbf{B}}_{\mathbf{1j}}$, so

$$\det\tilde{\mathbf{B}}_{\mathbf{1j}} = (-1)^{(i-1)+(k-1)}B_{ij}\det\mathbf{C}_{\mathbf{ij}} = (-1)^{i+k}B_{ij}\det\mathbf{C}_{\mathbf{ij}},$$

where $\mathbf{C}_{\mathbf{ij}}$ is obtained from $\tilde{\mathbf{B}}_{\mathbf{1j}}$ by removing row $i-1$ and column $k-1$. Equivalently, $\mathbf{C}_{\mathbf{ij}}$ is obtained from $\mathbf{B}$ by removing rows 1 and i, and columns j and k. Consider all four removals to occur at once, applied to the original $\mathbf{B}$ to get $\mathbf{C}_{\mathbf{ij}}$. Note that we do not have a summation here as when traversing the $i-1^{\text{th}}$ row, all elements except the 1 will be 0 and will not contribute to the sum.

Similarly, for the third term where $j > k$, the 1 is in the $(i-1, k)^{\text{th}}$ spot, and

$$\det\tilde{\mathbf{B}}_{\mathbf{1j}} = (-1)^{(i-1)+k}B_{ij}\det\mathbf{C}_{\mathbf{ij}}.$$

Note that this is the same $\mathbf{C}_{\mathbf{ij}}$ from the first term, as the same rows and columns are being removed from the original $\mathbf{B}$. The differences in the exponents of (-1) are due to the position of k relative to the removed rows and columns. Then,

$$\begin{aligned}\det\mathbf{B} &= \sum_{j=1}^{k-1}(-1)^{1+j}(-1)^{i+k}B_{ij}\det\mathbf{C}_{\mathbf{ij}} + \sum_{j=k+1}^{n}(-1)^{1+j}(-1)^{i+k-1}B_{ij}\det\mathbf{C}_{\mathbf{ij}} \\ &= (-1)^{i+k}\left(\sum_{j=1}^{k-1}(-1)^{1+j}B_{ij}\det\mathbf{C}_{\mathbf{ij}} + \sum_{j=k+1}^{n}(-1)^{j}B_{ij}\det\mathbf{C}_{\mathbf{ij}}\right).\end{aligned}$$

However, the part inside the brackets is the same as $\det\tilde{\mathbf{B}}_{\mathbf{ik}}$, because we have removed row 1 and column k as k is not in the sum. So,

$$\det\mathbf{B} = (-1)^{i+k}\det\tilde{\mathbf{B}}_{\mathbf{ik}}.$$ □

Now, we are able to prove the theorem.

Proof (Theorem). Using Lemma 9.4.1, if $i = 1$, the theorem is just the definition of the determinant. If $i > 1$, we will be able to write $\mathbf{A}$ as a sum of coefficients multiplied by n matrices, which each look like the following.

$$\mathbf{D}_{\mathbf{k}} = \begin{pmatrix} A_{11} & \cdots & A_{1\,k-1} & A_{1k} & A_{1\,k+1} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n_1} & \cdots & A_{n\,k-1} & A_{nk} & a_{n\,k+1} & \cdots & A_{nn} \end{pmatrix}$$

When we add all $\mathbf{D_k}$ together, we can multiply each of them by the corresponding A_{ik} value to set the element in the $(i,k)^{\text{th}}$ position of the total matrix as that value. Recalling that the determininant of this new matrix is simply $A_{ik}\det \mathbf{D_k}$ by Theorem 9.4.2, we can then write

$$\det \mathbf{A} = \sum_{k=1}^{n} A_{ik}\det \mathbf{D_k}.$$

From here, we apply the lemma to reach the desired result.

$$\begin{aligned}\det \mathbf{A} &= \sum_{k=1}^{n} A_{ik}\det \mathbf{D_k}\\ &= \sum_{k=1}^{n} A_{ik}\left((-1)^{i+k}\det \tilde{\mathbf{A}}_{\mathbf{ik}}\right)\\ &= \sum_{k=1}^{n}(-1)^{i+k} A_{ik}\det \tilde{\mathbf{A}}_{\mathbf{ik}}.\end{aligned}$$ □

Corollary 9.4.2. *If two rows of* $\mathbf{A}$ *are the same, then* $\det \mathbf{A} = 0$.

Proof. We again use induction. For the 2×2 case, using the formula listed after Definition 9.4.1, given $a, b \in \mathbb{R}$,

$$\det\begin{pmatrix} a & b\\ a & b\end{pmatrix} = ab - ab = 0.$$

Assume the result holds for $(n-1)\times(n-1)$ matrices. Given some $n\times n$ $\mathbf{A}$, assume rows l and k are the same. Choose i such that $i \neq l$ and $i \neq k$. Then, using Theorem 9.4.3,

$$\det \mathbf{A} = \sum_{m=1}^{n}(-1)^{i+m} A_{im}\det \tilde{\mathbf{A}}_{\mathbf{im}}.$$

However, $\tilde{\mathbf{A}}_{\mathbf{im}}$ is $(n-1)\times(n-1)$ with two identical rows as we didn't remove the identical ones from $\mathbf{A}$, so by the induction hypothesis, $\det \tilde{\mathbf{A}}_{\mathbf{im}} = 0$ and thus $\det \mathbf{A} = 0$. □

Theorem 9.4.4: Swapping Rows Flips Sign of Determinant

If $\mathbf{A}$ *is* $n\times n$ *and* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by swapping 2 rows, then* $\det \mathbf{A} = -\det \mathbf{B}$.

Proof. Here, we use a "cheap but effective trick": suppose that for some row vectors $\mathbf{a_1}, \cdots, \mathbf{a_n} \in \mathbb{R}^n$, define $\mathbf{A}$ and $\mathbf{B}$ to be

$$\mathbf{A} = \begin{bmatrix}\mathbf{a_1}\\ \vdots\\ \mathbf{a_r}\\ \vdots\\ \mathbf{a_s}\\ \vdots\\ \mathbf{a_n}\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}\mathbf{a_1}\\ \vdots\\ \mathbf{a_s}\\ \vdots\\ \mathbf{a_r}\\ \vdots\\ \mathbf{a_n}\end{bmatrix}.$$

Then, using Corollary 9.4.2,

$$
\begin{aligned}
0 = \det \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_r} + \mathbf{a_s} \\ \vdots \\ \mathbf{a_r} + \mathbf{a_s} \\ \vdots \\ a_n \end{bmatrix} &= \det \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_r} \\ \vdots \\ \mathbf{a_r} \\ \vdots \\ \mathbf{a_n} \end{bmatrix} + \det \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_s} \\ \vdots \\ \mathbf{a_r} \\ \vdots \\ \mathbf{a_n} \end{bmatrix} + \det \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_s} \\ \vdots \\ \mathbf{a_s} \\ \vdots \\ \mathbf{a_n} \end{bmatrix} + \det \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_r} \\ \vdots \\ \mathbf{a_s} \\ \vdots \\ \mathbf{a_n} \end{bmatrix} \\
&= 0 + \det \mathbf{B} + 0 + \det \mathbf{A} \\
&= \det \mathbf{A} + \det \mathbf{B}.
\end{aligned}
$$

So, $\det \mathbf{A} = -\det \mathbf{B}$. □

Theorem 9.4.5

Assume $\mathbf{A}$ *is obtained from* $\mathbf{B}$ *by adding a multiple of one row to another row. Then,* $\det \mathbf{A} = \det \mathbf{B}$.

Proof. If for some row vectors $\mathbf{a_1}, \cdots, \mathbf{a_n} \in \mathbb{R}^n$,

$$
A = \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_k} \\ \vdots \\ \mathbf{a_l} \\ \vdots \\ \mathbf{a_n} \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_k} + \lambda \mathbf{a_l} \\ \vdots \\ \mathbf{a_l} \\ \vdots \\ \mathbf{a_n} \end{bmatrix},
$$

then by Corollary 9.4.2 since the new matrix has repeated rows,

$$
\det \mathbf{B} = \det \mathbf{A} + \lambda \det \begin{bmatrix} \mathbf{a_1} \\ \vdots \\ \mathbf{a_l} \\ \vdots \\ \mathbf{a_l} \\ \vdots \\ \mathbf{a_n} \end{bmatrix} = \det \mathbf{A}.
$$

□

Corollary 9.4.3. *For an* $n \times n$ *matrix, if* $\operatorname{rank} \mathbf{A} < n$ *then* $\det \mathbf{A} = 0$.

Proof. We reduce the matrix $\mathbf{A}$ to row-echelon form by elementary row operations. Each operation will multiply the determinant by a non-zero number. To be precise, Theorem 9.4.2 proves this for multiplying one row by a number, Theorem 9.4.4 proves this for swapping two rows and Theorem 9.4.5 proves this for adding a multiple of one row to another.

Thus, the determinant of the reduced matrix is $c \det \mathbf{A}$ where $c \neq 0$. However, since $\operatorname{rank} \mathbf{A} < n$, the reduced matrix will have a full row of zeroes so its determinant is zero by Corollary 9.4.1. This gives $c \det \mathbf{A} = 0$, and because c is non-zero, $\det \mathbf{A}$ must be 0. $\square$

Remark. If the elements of $\mathbf{A}$ under the main diagonal are all 0, we say that A is *upper triangular*. In this case,

$$\det \mathbf{A} = A_{11} A_{22} \cdots A_{nn},$$

which is the product of the diagonal elements. We can use this property to quickly find the determininant by row reducing the matrix until it is upper triangular, making sure to flip the sign of the final product if we swapped rows an odd number of times during the Gaussian Elimination process.

Theorem 9.4.6: Determinant of Product is Product of Determinant

If $\mathbf{A}$ *and* $\mathbf{B}$ *are* $n \times n$*, then* $\det \mathbf{AB} = \det \mathbf{A} \cdot \det \mathbf{B}$.

Remark. The determinant is one of the only functions on matrices which has this property.

Important: This proof is quite technical and was not covered in full when I took the course. The lecturer did present a sketch of the proof, which is what I have included here.

Proof. Here is a sketch of the proof.

1. Assume $\mathbf{A}$ is invertible. There is a separate case to handle if it is not.
2. Since $\mathbf{A}$ is invertible, we can reduce it to $\mathbf{I_n}$ by elementary row operations.
3. Every elementary row operation is given by the multiplication of an elementary matrix, introduced in Definition 9.2.4. So,

$$\mathbf{A} = \mathbf{E_1 E_2} \cdots \mathbf{E_k I_n}$$

   where $\mathbf{E_1}, \cdots, \mathbf{E_k}$ are elementary matrices.
4. Verify that $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$ when $\mathbf{A}$ is elementary.
5. Conclude with something similar to

$$\begin{aligned} \det \mathbf{AB} &= \det (\mathbf{E_1 E_2} \cdots \mathbf{E_k I_n B}) \\ &= \det \mathbf{E_1} (\mathbf{E_2} \cdots \mathbf{E_k B}) \\ &= \det \mathbf{E_1} \det (\mathbf{E_2} \cdots \mathbf{E_k B}) \\ &= \det \mathbf{E_1} \det \mathbf{E_2} \det (\mathbf{E_3} \cdots \mathbf{E_k B}) \\ &= \cdots \\ &= \det \mathbf{E_1} \cdots \det \mathbf{E_k} \det \mathbf{B} \\ &= \det (\mathbf{E_1} \cdots \mathbf{E_k}) \det \mathbf{B} \\ &= \det \mathbf{A} \det \mathbf{B}, \end{aligned}$$

   when $\mathbf{A}$ is invertible and hence can be expressed as a product of elementary matrices.

$\square$

Corollary 9.4.4. *If* $\mathbf{A}$ *is invertible, then* $\det \mathbf{A} \neq 0$ *and* $\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}$.

Proof. Since $\mathbf{AA}^{-1} = \mathbf{I_n}$, then $\det(\mathbf{AA}^{-1}) = \det \mathbf{A} \cdot \det \mathbf{A}^{-1} = 1$. Both results immediately follow. $\square$

Theorem 9.4.7: Determinant is Unaffected by Transpose

> *For square* $\mathbf{A}$*,* $\det \mathbf{A} = \det \mathbf{A}^{\mathrm{T}}$*.*

Proof. If $\mathbf{A}$ is not invertible, try proving this case as an exercise.

Assume $\mathbf{A}$ is invertible. It is straightforward to verify $\det \mathbf{E} = \det \mathbf{E}^{\mathrm{T}}$ if $\mathbf{E}$ is an elementary matrix. Two of the elementary matrices are symmetric ($\mathbf{E} = \mathbf{E}^{\mathrm{T}}$), and the other is triangular.

Since $\mathbf{A} = \mathbf{E_1} \cdots \mathbf{E_k} \mathbf{I_n}$ for elementary $\mathbf{E_i}$, then $\mathbf{A}^{\mathrm{T}} = (\mathbf{E_1} \cdots \mathbf{E_k})^{\mathrm{T}} = \mathbf{E_k}^{\mathrm{T}} \cdots \mathbf{E_1}^{\mathrm{T}}$, and

$$\begin{aligned}
\det \mathbf{A}^{\mathrm{T}} &= \det \left(\mathbf{E_k}^{\mathrm{T}} \cdots \mathbf{E_1}^{\mathrm{T}} \right) \\
&= \det \mathbf{E_k}^{\mathrm{T}} \cdots \det \mathbf{E_1}^{\mathrm{T}} \\
&= \det \mathbf{E_1}^{\mathrm{T}} \cdots \det \mathbf{E_k}^{\mathrm{T}} \\
&= \det \mathbf{E_1} \cdots \det \mathbf{E_k} \\
&= \det(\mathbf{E_1} \cdots \mathbf{E_k}) \\
&= \det \mathbf{A}.
\end{aligned}$$

□

Alternate Proof. This can also be proven directly using the definition of the determinant (Definition 9.4.1) – try as an exercise. □

Theorem 9.4.8: Invertible iff Nonzero Determinant

> $\mathbf{A}$ *is invertible if and only if* $\det \mathbf{A} \neq 0$*.*

Proof. We have proved the forward direction in Corollary 9.4.4. For the reverse direction of the proof, consider the matrix

$$\mathbf{G} = \frac{1}{\det \mathbf{A}} (\mathbf{C})^{\mathrm{T}}, \quad C_{ij} = (-1)^{i+j} \det \tilde{\mathbf{A}}_{\mathbf{ij}}.$$

The matrix $\mathbf{C}$ is the cofactor matrix from Definition 9.4.1. $\mathbf{G}$ is defined since $\det \mathbf{A} \neq 0$. We claim $\mathbf{GA} = \mathbf{AG} = \mathbf{I_n}$. For $k = 1, \cdots, n$, we find that the diagonal elements of $\mathbf{AG}$ are:

$$\begin{aligned}
(\mathbf{AG})_{kk} &= \sum_{i=1}^{n} A_{ki} G_{ik} \\
&= \sum_{i=1}^{n} A_{ki} \frac{1}{\det \mathbf{A}} (-1)^{i+k} \det \tilde{\mathbf{A}}_{\mathbf{ki}} \\
&= \frac{1}{\det \mathbf{A}} \sum_{i=1}^{n} A_{ki} (-1)^{i+k} \det \tilde{\mathbf{A}}_{\mathbf{ki}} \\
&= \frac{1}{\det \mathbf{A}} \det \mathbf{A} = 1.
\end{aligned}$$

Thus, $\mathbf{AG}$ has ones on the diagonal. If $k \neq l$, then

$$(\mathbf{AG})_{kl} = \sum_{i=1}^{n} A_{ki} G_{il} = \frac{1}{\det \mathbf{A}} \sum_{i=1}^{n} (-1)^{i+l} A_{ki} \det \tilde{\mathbf{A}}_{\mathbf{li}}.$$

The sum in the above equation looks similar to the determinant of $\mathbf{A}$ going along row l:

$$\det \mathbf{A} = \sum_{i=1}^{n} (-1)^{l+i} A_{li} \det \tilde{\mathbf{A}}_{\mathbf{li}}.$$

Compare this with the equation in Theorem 9.4.3. However, the A_{li} term has changed to A_{ki}. This means that to calculate the value of the sum, we calculate $\det \mathbf{A}$ along row l but multiply by the elements in row k instead of row l. Equivalently, the sum is the determinant of the following matrix, where row l has been changed to have row k's elements:

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kn} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kn} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix},$$

where the repeated elements are on rows k and l. Since two rows are the same, the determinant of this matrix is 0. This means that $(\mathbf{AG})_{kl} = 0$, so $\mathbf{AG}$ has 0s off the diagonal and 1s on the diagonal, thus $\mathbf{AG} = \mathbf{I_n}$.

Since right inverse implies left inverse as shown in the proof of Theorem 9.3.7, both are the same and thus $\mathbf{A}$ is invertible, where $\mathbf{G}$ is the inverse of $\mathbf{A}$. □

Remark. The expression for $\mathbf{G}$ is an explicit formula for $\mathbf{A}^{-1}$ which can be used for computation. However, calculating $\mathbf{G}$ is much more inefficient compared to Gaussian Elimination.

9.5 Eigenvalues and Eigenvectors

Definition 9.5.1: Eigenvalues and Eigenvectors

> Given some $n \times n$ matrix $\mathbf{A}$, a *non-zero* vector $\mathbf{x} \in \mathbb{C}^n$ is an *eigenvector* of $\mathbf{A}$ with the corresponding *eigenvalue* $\lambda \in \mathbb{C}$ if $\mathbf{Ax} = \lambda\mathbf{x}$.

Eigenvectors can be thought of vectors that do not change their direction when multiplied by $\mathbf{A}$. Instead, they simply scale in length, where the scaling factor is the corresponding eigenvalue. This is what the equation $\mathbf{Ax} = \lambda\mathbf{x}$ represents. The video at `https://www.youtube.com/watch?v=PFDu9oVAE-g` offers a great explanation – it will be helpful to read Section 10.2 discussing the span of a set before watching. The other videos in the series cover this content as well.

Remark. Both $\mathbf{x}$ and λ do not always have complex values – it depends on the matrix. The following process works in the general, complex case.

To find the eigenvalues and eigenvectors of a matrix, we want to solve $\mathbf{Ax} = \lambda\mathbf{x}$ for all possible values of λ and $\mathbf{x}$. If $\mathbf{Ax} = \lambda\mathbf{x}$, then $\mathbf{0} = \mathbf{Ax} = \lambda\mathbf{x} = (\mathbf{A} - \lambda\mathbf{I_n})\mathbf{x}$, so $\mathbf{x}$ is in the null-space of $\mathbf{A} - \lambda\mathbf{I_n}$. We only consider non-zero $\mathbf{x}$ as per the definition so we can solve

$$\det(\mathbf{A} - \lambda\mathbf{I_n}) = 0$$

for λ. The left-hand side of this equation is said to be the *characteristic polynomial* in λ for $\mathbf{A}$. After obtaining values for λ, solve $(\mathbf{A} - \lambda\mathbf{I_n})\mathbf{x} = \mathbf{0}$ for $\mathbf{A}$, for each value of λ. This gives each eigenvalue and its corresponding eigenvectors.

Example 9.5.1

Let

$$\mathbf{A} = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}.$$

To find the eigenvalues of $\mathbf{A}$, we first calculate

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I_2}) &= \det \begin{pmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix} \\ &= (5+\lambda)(2+\lambda) - 4 \\ &= \lambda^2 + 7\lambda + 6 \\ &= (\lambda+1)(\lambda+6) = 0, \end{aligned}$$

so $\lambda = -1$ and $\lambda = -6$ are the eigenvalues of A.

We now calculate eigenvectors. For $\lambda = -1$, $(\mathbf{A} - \lambda \mathbf{I_2})\mathbf{x} = (\mathbf{A} + \mathbf{I_2})\mathbf{x} = \mathbf{0}$, so

$$\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \left(\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

From here, if $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, let $x_1 = \alpha \in \mathbb{C} \setminus \{0\}$. Then, from the top row of the matrix, $-4\alpha + 2x_2 = 0$ so $x_2 = 2\alpha$, and the eigenvectors corresponding to $\lambda = -1$ are of the form

$$\mathbf{x} = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

Similarly, for $\lambda = -6$, $(\mathbf{A} - \lambda \mathbf{I_2})\mathbf{x} = (\mathbf{A} + 6\mathbf{I_2})\mathbf{x} = \mathbf{0}$, so

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then let $x_2 = \alpha \in \mathbb{C} \setminus \{0\}$. Then, $x_1 + 2\alpha = 0$ so $x_1 = -2\alpha$, and the eigenvectors corresponding to $\lambda = -6$ are of the form

$$\mathbf{x} = \alpha \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

Example 9.5.2

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

To find eigenvalues,

$$\begin{aligned} \det(A - \lambda I_2) &= \det \begin{pmatrix} -\lambda & 2 \\ -2 & -\lambda \end{pmatrix} \\ &= \lambda^2 + 4 = 0, \end{aligned}$$

so $\lambda = 2i$ and $\lambda = -2i$ are the eigenvalues of $\mathbf{A}$.

For $\lambda = 2i$, we solve $(\mathbf{A} - 2i\mathbf{I_2})\mathbf{x} = \mathbf{0}$, giving

$$\left(\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow iR_2} \left(\begin{array}{cc|c} -2i & 2 & 0 \\ -2i & 2 & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} -2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

For $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, let $x_1 = \alpha \in \mathbb{C} \setminus \{0\}$. Then, $-2i\alpha + 2x_2 = 0$ giving $x_2 = i\alpha$, and

$$\mathbf{x} = \alpha \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

For $\lambda = -2i$, we solve $(\mathbf{A} + 2i\mathbf{I_2})\mathbf{x} = \mathbf{0}$, giving

$$\left(\begin{array}{cc|c} 2i & 2 & 0 \\ -2 & 2i & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow -iR_2} \left(\begin{array}{cc|c} 2i & 2 & 0 \\ 2i & 2 & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} 2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

For $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, let $x_1 = \alpha \in \mathbb{C} \setminus \{0\}$. Then, $2i\alpha + 2x_2 = 0$ giving $x_2 = -i\alpha$, and

$$\mathbf{x} = \alpha \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \alpha \in \mathbb{C} \setminus \{0\}.$$

Note that the matrix $\mathbf{A}$ can be written as $\mathbf{A} = 2\mathbf{I_2}\mathbf{Q}^3$, where $\mathbf{Q}^3$ is the 270° counter-clockwise rotation matrix from Example 9.2.2. Since eigenvectors are vectors which only get scaled and not rotated when a matrix is applied, the reason why $\mathbf{A}$'s eigenvectors are not real is because *all* of the vectors in $\mathbb{R}^2$ get rotated due to them being multiplied by $\mathbf{Q}^3$. The only vectors which don't get rotated are not in $\mathbb{R}^2$ – rather, they are elements of $\mathbb{C}^2$.

Chapter 10

Linear Algebra

10.1 Vector Spaces

Definition 10.1.1: Vector Space

A *vector space* over a field $\mathbb{F}$ is a non-empty set V equipped with two operations, $+ : V \times V \to V$ and $\cdot : \mathbb{F} \times V \to V$, such that for some $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and $a, b \in \mathbb{F}$:

1. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. there exists $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. for all $\mathbf{u} \in V$, there exists $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $a(b\mathbf{v}) = (ab)\mathbf{v}$
6. there exists $1 \in \mathbb{F}$ such that $1\mathbf{u} = \mathbf{u}$
7. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
8. $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$.

Definition 10.1.2: Vector

A *vector* is an element of a vector space.

This definition of a vector space and a vector generalises the previous idea of vectors as an array of n numbers in Definition 9.1.6 from the previous chapter. Compare these vector space axioms with the field axioms in Definition 1.4.1. Showing $\mathbf{0}$ is unique is very similar to the proof for fields (Theorem 1.4.3).

Note that vector spaces do not define multiplication between vectors – this is the job of other mathematical structures which will not be discussed in this course.

Example 10.1.1

Here are some examples of vector spaces. These sets satisfy all of the above axioms.

1. $\mathbb{R}^n$ for all $n \in \mathbb{N}$ is a vector space over $\mathbb{R}$.
2. $\mathbb{C}^n$ for all $n \in \mathbb{N}$ is a vector space over $\mathbb{C}$.
3. $\{0\}$ is a vector space over any field. This is straightforward to verify.
4. The set of polynomials in x with coefficients in $\mathbb{R}$, sometimes denoted as $\mathbb{R}[x]$ or

$P(\mathbb{R})$, is a vector space over $\mathbb{R}$.

5. The set of polynomials in x with degree at most n and coefficients in $\mathbb{R}$, sometimes denoted as $\mathbb{R}_n[x]$ or $P_n(\mathbb{R})$, is a vector space over $\mathbb{R}$.
6. The set of continuous functions defined on $[a,b]$, denoted as $C[a,b]$, is a vector space over $\mathbb{R} \cap [a,b]$.
7. $M_{m\times n}(\mathbb{F})$, the set of all m-by-n matrices with entries in $\mathbb{F}$, form a vector space over $\mathbb{F}$.

Example 10.1.2

The set $\{(a_1,\cdots,a_n) \in \mathbb{R}^n \mid a_i > 0, i = 1,\cdots n\}$ is not a vector space under standard operations because there are no additive inverses. Similarly, the set $\{(1,2),(3,4)\} \subset \mathbb{R}^2$ is not a vector space over $\mathbb{R}$ under standard operations, but we can make it a vector space over $\mathbb{F} = \{0,1\}$ defining $(1,2)+(3,4) = (1,2)$, etc.

Example 10.1.3

Let V be a vector space over $\mathbb{R}$, and let $\mathbf{x} \in V$. Define new operations

$$\begin{aligned} \oplus : V \times V \to V, &\qquad \mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} + \mathbf{x} \\ \otimes : \mathbb{R} \times V \to V, &\qquad k \otimes \mathbf{v} = k\mathbf{v} + (k-1)\mathbf{x} \end{aligned}$$

for $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$. Show as an exercise that V is a vector space using the new operations $\oplus$ and $\otimes$. As a hint, the additive identity is $-\mathbf{x}$ and additive inverse of $\mathbf{v}$ is $-\mathbf{v} - 2\mathbf{x}$.

Example 10.1.4

Let $V = \{a \in \mathbb{R} \mid a > 0\}$ and define new operations

$$\begin{aligned} \oplus : V \times V \to V, &\qquad x \oplus y = xy \\ \otimes : \mathbb{R} \times V \to V, &\qquad k \otimes x = x^k \end{aligned}$$

for $x, y \in V$ and $k \in \mathbb{R}$. Show as an exercise that V is a vector space under the operations $\oplus$ and $\otimes$. As a hint, the additive identity is 1 and additive inverse of x is x^{-1}.

Definition 10.1.3: Subspace

A subset of the vector space V is a *subspace* if it is also a vector space under the same addition and scalar multiplication defined on V.

Theorem 10.1.4

Let W be a non-empty subset of the vector space V. Then, W is a subspace if and only if for all $\mathbf{u}, \mathbf{v} \in W$ and $a \in \mathbb{F}$,

1. $\mathbf{u} + \mathbf{v} \in W$
2. $a\mathbf{u} \in W$.

Proof. This is an exercise. □

Remark. This theorem allows you to verify some set W is a subspace by verifying

1. W is non-empty, usually by showing $\mathbf{0} \in W$
2. addition is closed under W
3. scalar multiplication is closed under W.

Example 10.1.5

A line or plane going through the origin *is* a subspace of $\mathbb{R}^n$. However, a line of plane that does not go through the origin *is not* a subspace of $\mathbb{R}^n$ as it does not contain an additive identity.

Example 10.1.6

The set $W = \{(a, b, c) \in \mathbb{R}^3 \mid a + b = 2c\}$ is a subspace of $\mathbb{R}^3$. To show this,

1. $\mathbf{0} = (0, 0, 0) \in W$ since $\mathbf{0} \in \mathbb{R}^3$ and $0 + 0 = 2(0)$.
2. For some $(a, b, c), (a', b', c') \in W$,
$$(a + a') + (b + b') = (a + b) + (a' + b') = 2c + 2c' = 2(c + c'),$$
so $(a, b, c) + (a', b', c') \in W$ and addition is closed under W.
3. For some $(a, b, c) \in W$ and $k \in \mathbb{R}$,
$$ka + kb = k(a + b) = k(2c) = 2(kc),$$
so $k(a, b, c) = (ka, kb, kc) \in W$ and scalar multiplication is closed under W.

Proposition 10.1.5

Let $\mathbf{A} \in M_{n\times n}(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Then, the set
$$W = \{\mathbf{v} \in \mathbb{C}^n \mid \mathbf{A}\mathbf{v} = \lambda\mathbf{v}\}$$
is a subspace of $\mathbb{C}^n$.

Proof. Setting $\lambda = 0$ shows that $\mathbf{0} \in W$ so W is non-empty. Let $\mathbf{u}, \mathbf{v} \in W$ and $k \in \mathbb{C}$. Observe that
$$\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} = \lambda\mathbf{u} + \lambda\mathbf{v} = \lambda(\mathbf{u} + \mathbf{v}),$$
so $\mathbf{u} + \mathbf{v} \in W$. Also,
$$\mathbf{A}(k\mathbf{v}) = k(\mathbf{A}\mathbf{v}) = k(\lambda\mathbf{v}) = \lambda(k\mathbf{v}),$$
so $k\mathbf{v} \in W$. Thus, W is a subspace of $\mathbb{C}^n$. □

Definition 10.1.6: Eigenspace

We call the set
$$W = \{\mathbf{v} \in \mathbb{C}^n \mid \mathbf{A}\mathbf{v} = \lambda\mathbf{v}\},$$
if $W \neq \{0\}$, the *eigenspace* of $\mathbf{A}$ corresponding to the eigenvalue λ.

Eigenspaces are further discussed in MATH2001 and MATH2301.

Corollary 10.1.1. NS(**A**) *is a subspace of* $\mathbb{C}^n$.

Proof. Use Proposition 10.1.5 with eigenvalue $\lambda = 0$. □

10.2 Span

Definition 10.2.1: Linear Combination

A *linear combination* of the vectors $\mathbf{v_1}, \cdots, \mathbf{v_n} \in V$ is a vector of the form

$$a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n}$$

where $a_1, \cdots, a_n \in \mathbb{F}$.

Definition 10.2.2: Span

Let Ω be a possibly infinite set of vectors in V. The *span* of Ω, denoted $\operatorname{span}\Omega$, is the set of all finite linear combinations of vectors in Ω.

Example 10.2.1

Let $V = \mathbb{R}_4[x]$ (defined in Example 10.1.1) and $\Omega = \{1, x, x^2\}$. Then,

$$\operatorname{span}\Omega = \{a \cdot 1 + b \cdot x + c \cdot x^2 \mid a, b, c \in \mathbb{R}\} = \mathbb{R}_2[x].$$

A good exercise is to prove that $\mathbb{R}_k[x]$ is a subspace of $\mathbb{R}_n[x]$ if $n \geq k$.

Example 10.2.2

Let

$$\Omega = \{(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\}$$

where Ω contains n vectors each containing n elements. We claim that $\operatorname{span}\Omega = \mathbb{R}^n$. To show this, note that $\operatorname{span}\Omega \supseteq \mathbb{R}^n$ since all of these elements of $\mathbb{R}^n$ will be in the span. Set each of the coefficients in the linear combination to the value of each element of some vector in $\mathbb{R}^n$.

Also, $\operatorname{span}\Omega \subseteq \mathbb{R}^n$ because vectors in Ω are in $\mathbb{R}^n$ and $\mathbb{R}^n$ is closed under linear combinations. Thus, $\operatorname{span}\Omega = \mathbb{R}^n$.

Example 10.2.3

We claim that for $\Omega = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$, $\operatorname{span}\Omega = \mathbb{R}^3$.

Showing $\operatorname{span}\Omega \subseteq \mathbb{R}^3$ is straightforward, as each of the vectors in Ω are in $\mathbb{R}^3$, and $\mathbb{R}^3$ is closed under linear combinations. To show $\operatorname{span}\Omega \supseteq \mathbb{R}^3$, we want to show that every vector in $\mathbb{R}^3$ can be written as a vector in $\operatorname{span}\Omega$. An element in $\operatorname{span}\Omega$ can be written as

$$a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}] = \begin{pmatrix} a + b + c \\ b + c \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

for $a, b, c \in \mathbb{R}^3$. We set this to an arbitrary vector $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ and solve for the coefficients, which gives

$$c = z, \quad b = y - z, \quad a = x - y.$$

Thus,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x-y)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (y-z)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

and any vector in $\mathbb{R}^3$ can be written as a linear combination of the vectors in Ω. Thus, $\operatorname{span}\Omega = \mathbb{R}^3$.

Proposition 10.2.3: Span is a Subspace

Let $\Omega \subseteq V$. Then, $\operatorname{span}\Omega$ *is a subspace of V.*

Proof. This is an exercise. $\square$

Definition 10.2.4: Linear Dependence and Independence

A set $\Omega \subseteq V$ is *linearly dependent* if there exists distinct vectors $\mathbf{v_1}, \cdots, \mathbf{v_n} \in \Omega$ such that

$$a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n} = \mathbf{0}$$

for some $a_1, \cdots, a_n \in \mathbb{F}$ not all 0. In other words, Ω is linearly dependent if one or more of the vectors in Ω can be removed such that the span stays the same.
If Ω is not linearly dependent, it is *linearly independent*.

Remark. Here are some ways to check if sets are linearly dependent or not.

1. If $\mathbf{0} \in \Omega$ then $1 \cdot \mathbf{0} = \mathbf{0}$, so Ω is automatically linearly dependent.
2. Let $\Omega = \{\mathbf{u}, \mathbf{v}\}$ for distinct $\mathbf{u}, \mathbf{v}$. Then, Ω is linearly dependent if and only if $\mathbf{u} = \alpha\mathbf{v}$ or $\mathbf{v} = \alpha\mathbf{u}$ for $\alpha \in \mathbb{F}$. This follows immediately from the definition.
3. Let $\Omega = \{\mathbf{v_1}, \cdots, \mathbf{v_n}\} \subset V$. Then, Ω is linearly *independent* if and only if the only solution to $a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n} = \mathbf{0}$ is $a_1 = \cdots = a_n = 0$. This is a useful test to check linear independence.

Proposition 10.2.5

Let $\Omega = \{\mathbf{v_1}, \cdots, \mathbf{v_n}\} \subset \mathbb{C}^n$ and form $\mathbf{A} = \begin{bmatrix} \mathbf{v_1} & \cdots & \mathbf{v_n} \end{bmatrix}$ by concatenating the vectors so each vector forms a column of $\mathbf{A}$. Then, the following are equivalent.

1. *Ω is linearly independent.*
2. $\mathrm{NS}(\mathbf{A}) = \{0\}$.
3. *$\mathbf{A}$ is invertible.*
4. $\det \mathbf{A} \neq 0$.

Proof. Instead of showing **Part 1** $\iff$ **Part 2** $\iff$ **Part 3** $\iff$ **Part 4**, which requires 6 implications to be proved, show that **Part 1** $\implies$ **Part 2** $\implies$ **Part 3** $\implies$ **Part 4** $\implies$ **Part 1**, which only requires 4 implications. The second implication has been proven as Theorem 9.3.7 and the third has been proven as Corollary 9.4.4. The first and fourth implications have been left as exercises. $\square$

10.3 Bases and Dimension

Definition 10.3.1: Basis of a Vector Space

A *basis* for a vector space V is a set $\Omega \subset V$ such that

1. Ω is linearly independent
2. $\operatorname{span} \Omega = V$.

The second requirement is sometimes stated as "Ω spans V". Bases can be thought of the smallest set where every vector in V can be uniquely expressed as a linear combination of the basis vectors – this is proven in Theorem 10.3.3. Note that bases for a vector space are not necessarily unique, but all bases for a vector space have the same number of elements. We will not prove this second fact in this course.

Example 10.3.1

Here are some examples of bases for various vector spaces.

1. $\mathbb{R}^2$ has a basis $\Omega = \{(1,0),(0,1)\}$. These vectors are indeed linearly independent and span $\mathbb{R}^2$.
2. $\mathbb{R}^n$ has a basis $\Omega = \{(1,0,\cdots,0),\cdots,(0,\cdots,0,1)\}$ which is called the "standard" or "canonical" basis of $\mathbb{R}^n$. We have proved that Ω spans $\mathbb{R}^n$ in Example 10.2.2.
3. $\Omega = \{1, x, x^2, \cdots, x^n\}$ is a basis for $\mathbb{R}_n[x]$.

Example 10.3.2

Define the vector space V by

$$V = \{(a,b,c) \in \mathbb{R}^3 \mid a+b=c\} = \{(a,b,a+b) \mid a,b \in \mathbb{R}\}.$$

One way to find a basis for vector spaces defined like this is to set the first arbitrary coefficient to 1 and everything else to 0, and add that vector to a set Ω. Repeat this for all variables. In this example, $\Omega = \{(1,0,1),(0,1,1)\}$. Verifying that Ω is a basis for V is left as an exercise.

Definition 10.3.2: Dimension of a Vector Space

If V has a finite basis, V is *finite-dimensional*, and the number of vectors in the basis is the *dimension* of V denoted as $\dim V$.

Note that $\dim \mathbb{R}^n = n$. Additionally, the vector space

$$\mathbb{R}^\infty = \{(a_1, a_2, \cdots) \mid a_i \in \mathbb{R}, i \in \mathbb{N}\}$$

is infinite-dimensional. This can be thought of as the set of all sequences with elements in $\mathbb{R}$.

Remark. Let V be a finite-dimensional vector space where $\dim V = n$, and let $\Omega = \{\mathbf{v_1}, \cdots, \mathbf{v_m}\} \subset V$.

1. If $m = n$ then Ω is a basis for V.
2. If $m > n$ then Ω is linearly dependent.
3. If $m < n$ then Ω cannot span V.

Additionally, if $W \subseteq V$ is a subspace of V, then $\dim W \leq \dim V$.

Theorem 10.3.3

Let $\Omega = \{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ be a basis for V. Then, for all $\mathbf{v} \in V$, there exists unique $a_1, \cdots, a_n \in \mathbb{F}$ such that $\mathbf{v} = a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n}$. In other words, any vector in V can be expressed as a unique linear combination of the basis vectors.

Proof. Since Ω is a basis for V, we know that $\operatorname{span}\Omega = V$ which implies that a vector in V can be written as a linear combination of the vectors in Ω with coefficients $a_1, \cdots, a_n$. To show that these coefficients are unique, introduce $b_1, \cdots, b_n$ such that

$$\mathbf{v} = a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n} = b_1\mathbf{v_1} + \cdots + b_n\mathbf{v_n}.$$

Then,

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = (a_1 - b_1)\mathbf{v_1} + \cdots + (a_n - b_n)v_n.$$

Since Ω is linearly independent, $a_1 - b_1 = \cdots = a_n - b_n = 0$, so $a_1 = b_1, \cdots, a_n = b_n$. Thus, the coefficients are unique. $\square$

Changelog

I'm keeping a changelog of typos fixed and sections added between releases.

0.1 (22nd February, 2022)

- Finalise content and presentation

0.2 (28th February, 2022)

- Fix various hyperlinks

0.3 (4th March, 2022)

- Reword Example 1.4.3

1.0 (10th March, 2022)

- First full release!!!