MATH3403 Assignment 5

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Question 1

We have

$$\begin{cases} u_t - ku_{xx} = \delta(t-1) & x > 0,\, t > 0 \\ u(0,t) = 0 \\ u(x,0) = \delta(x-2) \end{cases}$$

First, reflect the initial condition about the origin, and take $u(x,0) = f_0(x)$, $u_t - ku_{xx} = g_0(x,\,t)$, where

$$f_0(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}$$

$$g_0(x,t) = \begin{cases} g(x,t) & x \geq 0 \\ -g(-x,t) & x < 0 \end{cases}$$

To satisfy this, the solution must be 0 at the boundary, but $u(0,t) = 0$ and so this criteria is satisfied. Now, take $v(x,t) = u(x,t)\big|_{x\geq 0}$, which gives the system

$$\begin{cases} v_t - kv_{xx} = g_0(x,t) \\ \quad v(0,t) = 0 \\ \quad v(x,0) = f_0(x,t) \end{cases}$$

Since $v(x,t)$ is odd, by result in Q2 Tutorial 8,

$$\begin{aligned} v(x,t) &= \int_0^t \int_0^\infty \left(\Phi(x-y,t-s) - \Phi(x+y,t-s)\right) g(x,s)\,dy\,ds + \int_0^\infty \left(\Phi(x-y,t) - \Phi(x+y,t)\right) f(y)\,dy \\ &= \int_0^t \left(\int_0^\infty \frac{1}{\sqrt{4\pi k(t-s)}} e^{\frac{-(x-y)^2}{4k(t-s)}} \delta(s-1) - \frac{1}{\sqrt{4\pi k(t-s)}} e^{\frac{-(x+y)^2}{4k(t-s)}} \delta(s-1)\,dy \right) ds \\ &+ \int_0^\infty \Phi(x-y,t)\delta(y-2) - \Phi(x+y,t)\delta(y-2)\,dy \end{aligned}$$

Since $\int_{\mathbb{R}} \Phi(x-y,t)\delta(y+a)\,dy = \Phi(x+a,t)$,

$$v(x,t) = \int_0^t \left(\int_0^\infty \frac{1}{\sqrt{4\pi k(t-s)}} e^{\frac{-(x-y)^2}{4k(t-s)}} \delta(s-1) - \frac{1}{\sqrt{4\pi k(t-s)}} e^{\frac{-(x+y)^2}{4k(t-s)}} \delta(s-1)\,dy \right) ds + \Phi(x-2,t) + \Phi(x+2,t)$$

Now, introduce change of variables:

$$-A^2 = -\frac{(x-y)^2}{4k(t-s)} \qquad\qquad -B^2 = -\frac{(x+y)^2}{4k(t-s)}$$

$$\Rightarrow A = \frac{x-y}{\sqrt{4k(t-s)}} \qquad \Rightarrow B = \frac{x+y}{\sqrt{4k(t-s)}}$$

$$\frac{dA}{dy} = -\frac{1}{\sqrt{4k(t-s)}} \qquad \frac{dB}{dy} = \frac{1}{\sqrt{4k(t-s)}}$$

$$A(y=0) = \frac{x}{\sqrt{4k(t-s)}} \qquad B(y=0) = \frac{x}{\sqrt{4k(t-s)}}$$

$$A(y\to\infty) = -\infty \qquad B(y\to\infty) = \infty$$

Under this change of variables,

$$\begin{aligned}
&\int_0^\infty \frac{1}{\sqrt{4\pi k(t-s)}} e^{\frac{-(x-y)^2}{4k(t-s)}}\delta(s-1) - \frac{1}{\sqrt{4\pi k(t-s)}} e^{\frac{-(x+y)^2}{4k(t-s)}}\delta(s-1)\,dy \\
&= \int_{-\frac{x}{\sqrt{4k(t-s)}}}^{-\infty} \frac{-\sqrt{4k(t-s)}}{\sqrt{4\pi k(t-s)}} e^{-A^2}\,dA - \int_{\frac{x}{\sqrt{4k(t-s)}}}^{\infty} \frac{\sqrt{4k(t-s)}}{\sqrt{4\pi k(t-s)}} e^{-B^2}\,dB \\
&= \frac{1}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{4k(t-s)}}}^{\infty} e^{-A^2}dA - \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4k(t-s)}}}^{\infty} e^{-B^2}dB \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-A^2}dA + \frac{1}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{4k(t-s)}}}^{0} e^{-A^2}dA - \left(\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-B^2}dB - \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4k(t-s)}}} e^{-B^2}dB \right) \\
&= \frac{1}{2}\operatorname{erf}(\infty) + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4k(t-s)}}} e^{-A^2}dA - \frac{1}{2}\operatorname{erf}(\infty) + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{4k(t-s)}}\right) \\
&= \operatorname{erf}\left(\frac{x}{\sqrt{4k(t-s)}}\right)
\end{aligned}$$

$$\Rightarrow v(x,t) = \int_0^t \operatorname{erf}\left(\frac{x}{\sqrt{4k(t-s)}}\right)\delta(s-1)\,ds + \Phi(x-2,t) + \Phi(x+2,t)$$

Since $f(y)\delta(y-a) = f(a)\delta(y-a)$,

$$\begin{aligned}
v(x,t) &= \int_0^t \operatorname{erf}\left(\frac{x}{\sqrt{4k(t-1)}}\right)\delta(s-1)\,ds + \Phi(x-2,t) + \Phi(x+2,t) \\
&= \operatorname{erf}\left(\frac{x}{\sqrt{4k(t-1)}}\right)\Theta(t-1) + \Phi(x-2,t) + \Phi(x+2,t)
\end{aligned}$$

where $\Theta(t-a) = \int_0^t \delta(s-a)\,ds$. Since $v(x,t) = u(x,t)\big|_{x\geq 0}$, the solution $u(x,t)$ is then

$$u(x,t) = \operatorname{erf}\left(\frac{x}{\sqrt{4k(t-1)}}\right)\Theta(t-1) + \Phi(x-2,t) + \Phi(x+2,t)$$

Question 2

Suppose the heat equation, represented by the system

$$\begin{cases} u_t - ku_{xx} = 0 \\ \quad u(x,0) = f_0(x) \\ \quad u(0,t) = 0 \end{cases}$$

where

$$f_0(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}$$

has solution $u(x,t)$ with odd initial condition $f_0(x)$. The solution is then

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \Phi(x-y,t) f_0(y)\, dy \\ &= \int_0^{\infty} \Phi(x-y,t) f(y)\, dy - \int_{-\infty}^{0} \Phi(x-y,t) f(-y)\, dy \\ &= \int_0^{\infty} \Phi(x-y,t) f(y)\, dy - \int_0^{\infty} \Phi(x+y,t) f(y)\, dy \\ &= \int_0^{\infty} \left(\Phi(x-y,t) - \Phi(x+y,t)\right) f(y)\, dy \end{aligned}$$

If the solution is odd, then $u(x,t) = -u(-x,t)$. Computing the latter gives

$$\begin{aligned} -u(-x,t) &= -\int_{-\infty}^{\infty} \Phi(-x+y,t) f_0(y)\, dy \\ &= -\int_0^{\infty} \Phi(-x+y,t) f(y)\, dy + \int_{-\infty}^{0} \Phi(-x+y,t) f(-y)\, dy \\ &= \int_0^{\infty} \Phi(x-y,t) f(y)\, dy + \int_0^{\infty} \Phi(-x-y,t) f(y)\, dy \\ &= \int_0^{\infty} \left(\Phi(x-y,t) - \Phi(x+y,t)\right) f(y)\, dy \\ &= u(x,t) \end{aligned}$$

And so, by definition of odd functions, a solution to the heat equation is odd with odd initial conditions.

Question 3

a. Define $v(x,t) = u(x,t)e^{-at}$, for $a < 0$ and $t > 0$ on the domain $x \in (0,1)$. So,

$$\begin{aligned} v_t &= u_t(x,t)e^{-at} - au(x,t)e^{-at} \\ v_{xx} &= u_{xx}(x,t)e^{-at} \end{aligned}$$

Then,

$$\begin{aligned} v_t - v_{xx} &= u_t(x,t)e^{-at} - au(x,t)e^{-at} - u_{xx}(x,t)e^{-at} \\ &= e^{-at}(u_t - au - u_{xx}) \\ &= e^{-at} \cdot 0 = 0 \end{aligned}$$

And since $u(x,t)$ solves the heat equation, then so does $v(x,t)$. By the maximum (minimum principle), the maximum of v is on the boundary of the domain. Computing the initial/boundary conditions for v gives

$$\begin{aligned} v(x,0) &= u(x,0)e^{-a\cdot 0} \\ &= \sin(\pi x) \\ v(0,t) &= u(0,t)e^{-at} \\ &= 0 \\ v(1,t) &= u(1,t)e^{-at} \\ &= 0 \end{aligned}$$

But $u(x,t) = v(x,t)e^{at} = v(x,t)e^{-bt}$ for some $b > 0$, implying that $u(x,t)$ is decreasing exponentially for all $t > 0$. Therefore, the maximum will occur along the $t = 0$ boundary (or rather, as $t \to 0$). At $t = 0$, $u(x,0) = \sin(\pi x)$, implying that along the boundary $0 \leq u(x,0) \leq 1$ for $x \in [0,1]$. However, the domain is not inclusive of 0 and 1, so $0 < u(x,t) < 1$ at $t = 0$, and since the value of $u(x,t)$ was established to be exponentially decreasing for all $t > 0$ (asymptotically approaching 0 as $t \to \infty$),

$$0 < u(x,t) < 1 \qquad \forall t > 0,\, x \in (0,1)$$

b. We have the system

$$\begin{cases} u_t - ku_{xx} - au = 0 \qquad x \in (0,1)\, t > 0 \\ u(x,0) = \sin(\pi x) \\ u(0,t) = 0 = u(1,t) \end{cases}$$

where $u(x,t)$ solves the heat equation. Take a function $v(x,t) = u(1-x,t)e^{-at}$ with $a < 0$.

$$\begin{aligned} \Rightarrow v_t(x,t) &= u_t(1-x,t)e^{-at} - au(1-x,t)e^{-at} \\ v_x(x,t) &= -u_x(1-x,t)e^{-at} \\ v_{xx}(x,t) &= u_{xx}(1-x,t)e^{-at} \end{aligned}$$

And so

$$v_t(x,t) - kv_{xx}(x,t) = u_t(1-x,t)e^{-at} - au(1-x,t)e^{-at} - ku_{xx}(1-x,t)e^{-at}$$

with initial condition

$$\begin{aligned} v(x,0) &= e^{-a\cdot 0}u(1-x,0) \\ &= \sin(\pi(1-x)) \\ &= \sin(\pi - \pi x) \\ &= \sin(\pi x) = u(x,0) \end{aligned}$$

where $\sin(\pi x) = \sin(\pi - \pi x)$ by phase shift properties of the sine function. The boundary conditions for $v(x,t)$ are then

$$v(0,t) = e^{-at}u(1-0,t) = e^{-at}u(1,t) = 0 = e^{-at}\cdot 0 = e^{-at}u(0,t) = e^{-at}u(1-1,t) = v(1,t)$$

Therefore $v(x,t)$ solves the heat equation and by uniqueness of solutions to the heat equation,

$$v(x,t) = u(1-x,t) = u(x,t)$$

Question 4

Suppose functions $u(x,t) \leq v(x,t)$ are solutions to the heat equation on $x \in [0, L]$, and for $t = 0$. Now, take $w = u - v \leq 0$. By linearity of the heat equation solutions, w is also a solution at $t = 0$:

$$\begin{aligned} w_t - w_{xx} &= u_t - v_t - u_{xx} + v_{xx} \\ &= u_t - u_{xx} - (v_t - v_{xx}) \\ &= 0 - 0 = 0 \end{aligned}$$

However, $w(x,t) \leq 0$ for $t = 0$ and so the maximum (minimum) principle states that the minimum is on the boundary. Since a solution to the heat equation will tend towards unity of the boundary conditions as $t \to \infty$,

$$\lim_{t\to\infty} w(x,t) \to 0^-$$

and so $w(x,t) \leq 0$ for all $t \geq 0$,

$$\begin{aligned} \Rightarrow u(x,t) - v(x,t) &\leq 0 \\ u(x,t) &\leq v(x,t) \qquad \forall t \geq 0 \end{aligned}$$

QED