MATH2400 Assignment 3

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Question 1

Show the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for any positive integer } n, \\ 0 & \text{otherwise,} \end{cases}$$

has an infinite number of discontinuities.

This may be proven based on two statements: firstly that any x where $f(x) = 1$ is discontinuous, and that there are an infinite number of points where $f(x) = 1$ in the domain $[0, 1]$.
Starting with the first statement, a function is continuous at a if $\lim_{x\to a} f(x) = f(a)$. For the function in question, as x approaches $a = \frac{1}{n}$ for any positive integer n (but still does not equal it), $x \neq \frac{1}{n}$ and $f(x) = 0$. Therefore, $\lim_{x\to a} f(x) = 0 \neq 1 = f(\frac{1}{n})$, and so the function is not continuous at any point where $f(x) = 1$.
As for the final statement, the function is discontinuous if x is the inverse of a positive integer. As all of the inverses of the positive integers are less than or equal to one (the positive integers are strictly increasing, and so the inverse is strictly decreasing, so the smallest positive integer will be the largest inverse positive integer, meaning that the largest inverse positive integer occurs when $n = 1 \Rightarrow x = 1/1 = 1$) and greater than zero (as the inverses of the positive integers approach 0 as $n \to \infty$ but never actually equal it), the whole set of the inverses of the positive integers fits within the domain [0,1], and this set is infinite. As all of these points lie on the domain, there will be an infinite amount of points where $f(x) = 1$ as x goes from $0 \to 1$.
As the two prerequisites have been satisfied, it has been shown that the function f has an infinite number of discontinuities. QED

Question 2

Show that the function f from Problem (1) is Riemann integrable. What is $\int_0^1 f(x)\,dx$?

Recall that a function is Riemann integrable if, $\forall\,\epsilon > 0$, there exists a partition of the the function's domain such that $U(P,f) - L(P,f) < \epsilon$. Also note that f is bounded above by 1 (when $x = \frac{1}{n}$ for some positive integer n), and bounded below by 0 (for any other case) by construction of the function.
Firstly, let $\epsilon > 0$, and $f : [0,1] \to \mathbb{R}$. Consider a partition $P = \{c, 1\}$ of [0, 1], where $0 < c < 1$. The lower and upper darboux sums of this partition are

$$\begin{aligned} L(P,f) &= \sum_{i=1}^{n} m_i \Delta x_i \\ &= 0 \end{aligned}$$

$$\begin{aligned} U(P,f) &= \sum_{i=1}^{n} M_i \Delta x_i \\ &= (1-c) \end{aligned}$$

For the function to be Riemann integrable, take $(1-c) < \epsilon \Rightarrow (1-\epsilon) < c$, and therefore $U(P,f)-L(P,f) < \epsilon$. Since the function's points where $f(x) = 1$ are infinitely thin (i.e. $\Delta x_i = 0$), the area under the 'curve' of the function is 0, and $\int_0^1 f(x)\,dx = 0$. Alternatively, as the function was shown to be Riemann integrable, $\int_0^1 f(x)\,dx = \underline{\int_0^1} f(x)\,dx = \sup\{L(P,f) \mid [0,1]\} = \sum_{i=1}^{n} m_i \Delta x_i = 0$.

Question 3

Find the derivatives of the functions for $x \in [0, \pi/4]$.

$$F(x) = \int_0^x \frac{1}{\cos t}\,dt, \qquad G(x) = \int_0^{x^2} 3^s\,ds$$

Using theorem 5.3.3 from Lebl, $F(x)$ may be represented as $F(x) = \int_0^x f(t)dt$, where $f(t) = \frac{1}{\cos t}$. The theorem states that if $f(t)$ is continuous over $[a,b]$, then $F(x)$ is differentiable over the domain with $F'(x) = f(x)$. Observe that the trigonometric functions are continuous over their entire domain, so $\cos t$ is continuous over $[0, \pi/4]$, and some function $h(x) := c^d$, $d \in \mathbb{R}$, is continuous (by corollary in course lecture). The function $f(t)$ may be shown as $f(t) = (\cos t)^{-1}$, where $-1 \in \mathbb{R}$, which is of the form of $h(x)$ where $c = \cos t$ and so $f(t)$ is continuous. Since $f(t)$ is continuous over it's entire domain,

$$F'(x) = f(x) = \frac{1}{\cos x}$$

The same method may be used for $G(x)$. If $g(s) := 3^s$ is continuous over the domain $[0, \pi/4]$, then $G'(x) = g(x)$. By Travis Scrimshaw (on Piazza), 3^s may be assumed to be continuous, and that x^2 is continuously differentiable. Thus,

$$G'(x) = g(x) = 3^{x^2}$$

due to $G(x)$ going from $0 \to x^2$.

Question 4

Let f be a continuous function. Prove that

$$\int_0^x \left[\int_0^u f(t)\,dt\right] du = \int_0^x f(u)(x-u)\,du$$

Hint: Apply the fundamental theorem of calculus to $F(u) = u\int_0^u f(t)\,dt$.

Firstly, consider some g continuous function at c. Let $\epsilon > 0$, and take $\delta > 0$ such that $|x-c| < \delta \Rightarrow |g(x) - g(c)| < \epsilon$. Thus, $g(c) - \epsilon < g(x) < g(c) + \epsilon \Rightarrow (g(c) - \epsilon)(x-c) \leq \int_c^x g \leq (g(c)+\epsilon)(x-c)$. As $\epsilon \to 0$, $g(c)(x-c) = \int_c^x g = G(c)$.

Returning to the original equation, each side may be multiplied by u, yielding

$$\int_0^x \left[u\int_0^u f(t)\,dt\right] du = \int_0^x uf(u)(x-u)\,du$$

By the hint, take $F(u) = u\int_0^u f(t)\,dt$, and thus

$$\int_0^x F(u)du = \int_0^x uf(u)(x-u)\,du$$

It may also be calculated that $F'(u) = uf(u)$ as per the fundamental theorem of calculus. So,

$$\int_0^x F(u)du = \int_0^x F'(u)(x-u)\,du$$

Let $F'(u) = g(u)$, and $F(u) = G(u)$. Then as per the first derivation,

$$\int_0^x F(u)du = \int_0^x F(u)\,du$$

and the two sides equal each other. QED

Question 5

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a,b]$. Suppose $\int_a^b f = 0$. Show that $f(x) = 0$ for all $x \in [a,b]$.

Suppose that, in the domain $[a,b]$ $a \neq b$, there exists some point $c \in \mathbb{R}$ such that $a < c < b$ and $f(c) \neq 0$. Then, $f(c) > 0$ and $\int_a^b f = \int_a^c f + \int_c^b f = 0$. Since f is Riemann integrable, f is a bounded function and $f(a)$, $f(b)$, $f(c) \in \mathbb{R}$. Take the interval $[a,c]$. Since $f(c) > 0$, as the function goes from any $a \geq 0$, $a \in \mathbb{R} \to c$, there will be some finite area under the 'curve', as the curve must rise (or fall if $f(a) > f(c)$) to meet c. Thus, $\int_a^c f > 0$. The same may be shown for the interval $[c,b]$, where $c \to b > 0$, $b \in \mathbb{R}$, leading to $\int_c^b f > 0$. As both of the integrals over the integrals are greater than 0, it follows that $\int_a^c f + \int_c^b f > 0 \Rightarrow \int_a^b f > 0$, and so a contradiction is found. This proof was supposing that there exists some point between a and b such that the function is not 0 at this point. As this was contradicted, it has been proven that $f(x) = 0$ for all $x \in [a,b]$. QED