Question 1

Part A:

a. The gradient of some variable can be described as the change of that variable with respect to some other variable. An example of this is the change in height of some land with respect to a change in latitude or longitude on the Earth's surface.

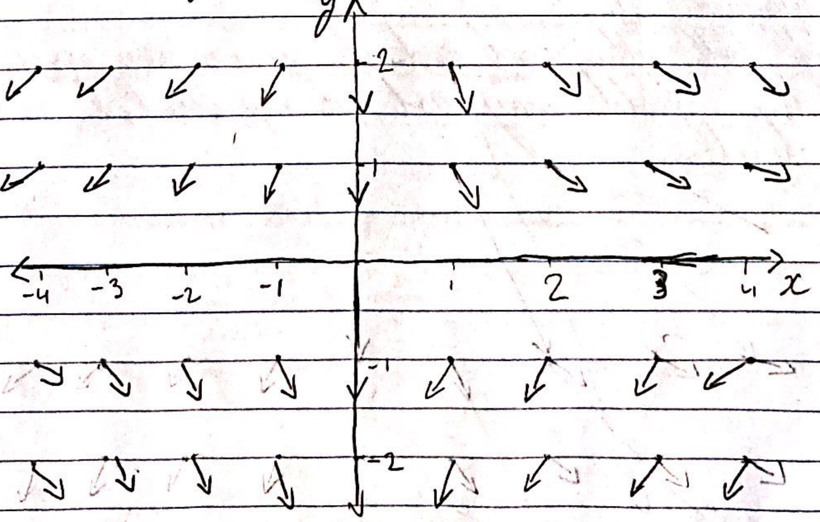
Divergence can be described as the net flux per unit volume (or area) of some variable. An example of this is some flow of water through an opening of a barrier.

Circulation is the measure of rotation of some variable field around some loop. Physically, a positive circulation corresponds to an increasing variable value around each loop, with negative circulation corresponding to a decreasing value.

b. We have that

$$f(x,y) = xy\hat{i} - (y^2+1)\hat{j}$$

So each vector at point (x,y) will have an x-component of xy and a y-component of $-(y^2+1)$.



c. The divergence of the field is calculated by

$$\nabla \cdot f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

For the field in question, we have:

$$\frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f}{\partial y} = x - 2y$$

and so

$$\nabla \cdot f = y + x - 2y$$
$$= x - y$$

From this, it can easily be seen that as x increases (with y kept constant), the divergence increases (the vector's magnitude gets larger). In the $-x$ direction, the magnitude gets larger in the $-x$ direction. For positive y (and constant x), the magnitude gets larger in the $-y$ direction as y increases. For $-y$, the magnithde gets larger in the $-y$ direction as y decreases.

All of these statements are supported by Part 6.

Part B:

d. As long as the curl of a field is zero, it may be expressed as a scalar potential function.

If the divergence of a field is zero, it may be expressed as a Vector potential function (such as with magnetism).

If both the curl and divergence are non-zero, than the field may not be represented as a potential function.

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Question 2

Part A:

a. Gauss's Law in differential form is

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

In free space, $\rho = 0$, so $\nabla \cdot \vec{E} = 0$ and

$$0 = \frac{1}{r^2}\frac{\partial (r^2 E)}{\partial r}$$

$$\Rightarrow \quad 0 = \frac{\partial (r^2 E)}{\partial r}$$

$$\Rightarrow \quad C = r^2 E$$

$$\Rightarrow \quad E = \frac{C}{r^2}$$

where C is some constant and r is the radius from the charge.

b. We have that $E = -\nabla \phi = -\frac{\partial \phi}{\partial r}$

So ϕ can be found using $E = \frac{C}{r^2}$:

$$\frac{C}{r^2} = -\frac{\partial \phi}{\partial r}$$

$$\phi = -\frac{C}{r^2}\,\partial r$$

$$= \frac{C}{r}$$

So the potential is of the form $\phi = \frac{C}{r}$ outside of the sphere.

c. Since the outer sphere is uniformly charged and it completely encloses the hollow inner sphere, there is no charge gradient inside the outer shell, and so $E = 0$ for $r < r_1$.

d. Inside the charged sphere, we use $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ to obtain:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

$$\frac{1}{r^2}\frac{\partial (r^2 E)}{\partial r} = \frac{\rho}{\varepsilon_0}$$

$$\frac{\partial (r^2 E)}{\partial r} = r^2 \frac{\rho}{\varepsilon_0}$$

$$r^2 E = \int_{r_1}^{r} r^2 \frac{\rho}{\varepsilon_0}$$

$$r^2 E = \frac{\rho}{3\varepsilon_0}(r^3 - r_1^3)$$

$$\Rightarrow E = \frac{\rho}{3\varepsilon_0}\left(r - \frac{r_1^3}{r^2}\right)$$

This equation is valid for $r_1 < r < r_2$

Part B:

a. The electric field outside of the sphere can be evaluated using the boundary conditions. At $r = r_2$, the equation derived in part d may be equated to the equation from a:

$$\frac{C}{r_2^2} = \frac{\rho}{3\varepsilon_0}\left(r_2 - \frac{r_1^3}{r_2^2}\right)$$

$$C = \frac{\rho}{3\varepsilon_0}\left(r_2^3 - r_1^3\right)$$

And therefore, outside of the sphere,

$$E = \frac{\rho}{3\varepsilon_0}\left(\frac{r_2^3 - r_1^3}{r^2}\right)$$

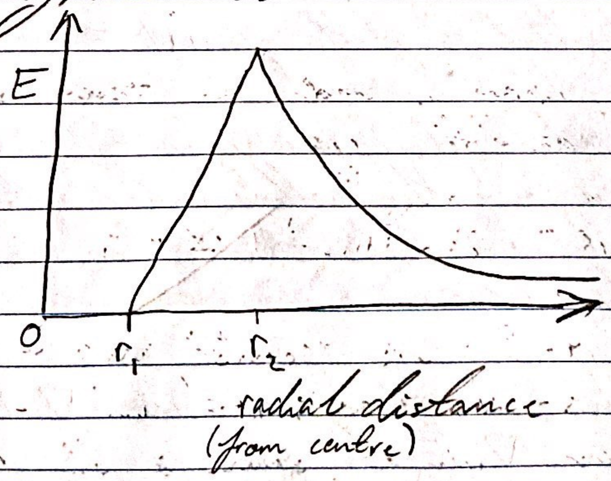
Therefore, the electric field over the range $0 \to \infty$ is

$r < r_1$: $E = 0$

$r_1 < r < r_2$: $E = \frac{\rho}{3\varepsilon_0}\left(r - \frac{r_1^3}{r^2}\right)$

$r_2 < r$: $E = \frac{\rho}{3\varepsilon_0}\left(\frac{r_2^3 - r_1^3}{r^2}\right)$

Graphically, that is



Question 3

Part A:

a. $W_{tsunami} = 20\cos(3.9\times10^{-5}x - 8.8\times10^{-3}t)$

i. Harmonic waves are of the form

$$\Psi(x,t) = A\cos(kx - \omega t + \phi_0)$$

The amplitude of the wave is then 20m.

We have that $k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k}$

$$\Rightarrow \lambda = \frac{2\pi}{(3.9\times10^{-5})}\,m = 161107.32\,m \simeq 161.1\,km$$

We also have that $V = \frac{\omega}{k}$

$$\Rightarrow V = \frac{8.8\times10^{-3}}{3.9\times10^{-5}}\,m.s^{-1} = 225.64\,m.s^{-1} = 812.3\,km.h^{-1}$$

ii. Assuming that the government could create a wave with the exact same properties as the tsunami, it would be possible to save the oil rig if the wave were created at the right moment. This is due to destructive interference for when the waves eventually meet. To avoid the interaction behaving like a standing wave, the wavelength of the artificial wave would need to be much shorter. This is, of course, neglecting the probability that the oil rig would be largely unaffected by the tsunami, as it is in deep water so the properties of the wave wouldn't change, and the tsunami would be very gradual at the rig.

b. i. $\phi_d = \frac{d}{2\pi\lambda}2\pi = \frac{800km}{2\pi\times161.1km}2\pi \cong 0.79\times2\pi$

ii. The created wave should have the exact properties as the source wave, but with 50% the amplitude and an induced phase shift of π - in order to align the peaks of the reflected wave and the troughs of the artificial wave (totally cancelling them). The artificial wave should have the equation

$$W_{artificial} = 10\cos(3.9\times10^{-5}x - 8.8\times10^{-3}t + \pi)$$

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Question 5

Part A:

a. Throughout the sections, the θ-term would not impact the flow, as the sections are radially symmetrical. In sections A and C, assuming that friction (of the walls) is negligible, the r term would not impact flow either. In these sections, though, the z term would describe the accumulated effect of gravity such that a lower z term would correspond to a greater velocity in the $-z$ direction. This effect of gravity would be present in section B also, however the gradual reduction in radius would greatly increase flow speed. The flow direction in section B would be parallel to the walls for the fluid at the boundary, also.

b. The continuity equation is defined as $\nabla \cdot (\rho \vec{v}) = -\frac{\partial \rho}{\partial t}$
As the flow is steady, there is no change in time, so the continuity equation becomes $\nabla \cdot (\rho \vec{v}) = 0$
Since the flow is steady, there must be equal parts mass leaving the output as there is entering the input.
That is $\Phi_{Output} = \Phi_{Input} \Rightarrow \rho_1 V_1 A_1 = \rho_2 V_2 A_2$

c. If flow is incompressible, input and output densities are the same, so the mass flow relation transforms
$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \Rightarrow V_1 A_1 = V_2 A_2$
Taking the input and output as circles with radius of R_1 and R_2 respectively, we have

$$V_1 \pi R_1^2 = V_2 \pi \left(\frac{R_1}{2}\right)^2$$

$$\Rightarrow V_2 = 4 V_1$$

Therefore, neglecting gravity, the flow at the output is 4 times faster than that at the input.

Part B:

d. Firstly, we have the Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + \Omega \times \vec{v} + \frac{1}{2}\nabla v^2 = -\frac{\nabla P}{\rho} - \nabla \phi + \frac{f_{visc}}{\rho}$$

As the flow is inviscid, $f_{visc} = 0$
As the flow is irrotational, $\Omega \times \vec{v} = 0$
As the flow is steady, $\frac{\partial v}{\partial t} = 0$
Therefore, we're left with

$$\frac{1}{2}\nabla v^2 = -\frac{\nabla P}{\rho} - \nabla \phi$$

$$\Rightarrow \nabla\left(\frac{1}{2}v^2 + \frac{P}{\rho} + \phi\right) = 0$$

As the tube is radially symmetric, there is no change in flow with a change in radius, so the term in the brackets is a constant. So,

$$\frac{1}{2}v^2 + \frac{P}{\rho} + \phi = \text{constant}$$

$$\Rightarrow P + \frac{1}{2}\rho v^2 + \rho\phi = \text{constant}$$

$$\Rightarrow P + \frac{1}{2}\rho v^2 + \rho g z = \text{constant}$$

Since the equation above is constant, it can be equated for two different states of the flow along the tube (1 for $z = 2$ and 2 for $z = 0$):

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g z = P_2 + \frac{1}{2}\rho v_2^2 + \rho g \times 0$$

$$\Rightarrow P_1 + \frac{1}{2}\rho v_1^2 + \rho g z = P_2 + \frac{1}{2}\rho v_2^2$$

$$\Rightarrow \frac{1}{2}\rho v_1^2 - \frac{1}{2}\rho v_2^2 + \rho g z = P_2 - P_1$$

$$\Rightarrow \Delta P = \rho\left(gz + \frac{1}{2}(v_1^2 - v_2^2)\right)$$

And so the pressure at the output (relative to the input) changes according to the equation above.