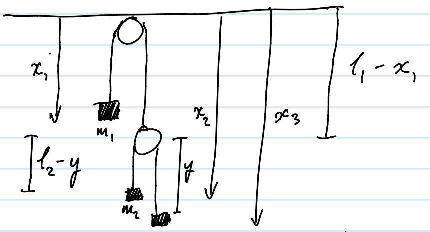
Dynamics Assignment 3

Question 1:



Define new variable, y, as the distance of m_3 from its pulley.
Define length of rope on top pulley as l_1 and rope on bottom pulley as l_2.
These dont impact final result, and it's clear to see that $\dot{y} = \dot{x}_3$

i.

$$T = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 + \tfrac{1}{2} m_3 \dot{x}_3^2$$

$$\dot{x}_2 = -\dot{x}_1 - \dot{y} \qquad \dot{x}_3 = \dot{y} - \dot{x}_1$$

$$\Rightarrow T = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 (\dot{x}_1 + \dot{y})^2 + \tfrac{1}{2} m_3 (\dot{y} - \dot{x}_1)$$

$$V = -g m_1 x_1 - g m_2 (l_1 - x_1 + l_2 - y) - g m_3 (l_1 - x_1 + y)$$

$$\Rightarrow L = T - V$$
$$= \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 (\dot{x}_1 + \dot{y})^2 + \tfrac{1}{2} m_3 (\dot{y} - \dot{x}_1) + g m_1 x_1 + g m_2 (l_1 - x_1 + l_2 - y) + g m_3 (l_1 - x_1 + y)$$

for x_1,
$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1}$$
$$= \frac{d}{dt}(m_1 \dot{x}_1 + m_2 \dot{x}_1 + m_2 \dot{y} + m_3 \dot{x}_1 - m_3 \dot{y}) - g m_1 + g m_2 + g m_3$$
$$0 = \ddot{x}_1 (m_1 + m_2 + m_3) + \ddot{y}(m_2 - m_3) - g(m_1 - m_2 - m_3)$$

for y,
$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y}$$
$$= \frac{d}{dt}\left(m_2 \dot{y} + m_2 \dot{x}_1 + m_3 \dot{y} - m_3 \dot{x}_1\right) + g m_2 - g m_3$$
$$0 = \ddot{y}(m_2 + m_3) + \ddot{x}_1 (m_2 - m_3) + g(m_2 - m_3)$$

ii. Rearranging the second EOM and substituting into the first allows $\ddot{x}_1$ to be found.

$$\Rightarrow \ddot{y} = -\ddot{x}_1 \frac{(m_2 - m_3)}{(m_2 + m_3)} - g \frac{(m_2 - m_3)}{(m_2 + m_3)}$$

$$\Rightarrow 0 = \ddot{x}_1 (m_1 + m_2 + m_3) - \ddot{x}_1 \frac{(m_2 - m_3)^2}{(m_2 + m_3)} - g \frac{(m_2 - m_3)^2}{(m_2 + m_3)} - g(m_1 - m_2 - m_3)$$

$$\Rightarrow \ddot{x}_1 \left(\frac{(m_2 - m_3)^2}{(m_2 + m_3)} - (m_1 + m_2 + m_3) \right) = -g \left(\frac{m_2^2 - 2 m_2 m_3 + m_3^2 + m_1 (m_2 + m_3) - m_2^2 - m_2 m_3 - m_3^2 - m_2 m_3}{m_2 + m_3} \right)$$

$$\ddot{x}_1 \left(\frac{m_2^2 - 2 m_2 m_3 + m_3^2 - m_1 (m_2 + m_3) - m_2 (m_2 + m_3) - m_3 (m_2 + m_3)}{m_2 + m_3} \right) = -g \left(\frac{m_1 (m_2 + m_3) - 4 m_2 m_3}{m_2 + m_3} \right)$$

$$\Rightarrow -\ddot{x}_1 \left(\frac{m_1 (m_2 + m_3) + 4 m_2 m_3}{m_2 + m_3} \right) = -g \left(\frac{m_1 (m_2 + m_3) - 4 m_2 m_3}{m_2 + m_3} \right)$$

$$\Rightarrow \ddot{x}_1 (m_1 (m_2 + m_3) + 4 m_2 m_3) = g(m_1 (m_2 + m_3) - 4 m_2 m_3)$$

$$\Rightarrow \ddot{x}_1 = \left(\frac{m_1 (m_2 + m_3) - 4 m_2 m_3}{m_1 (m_2 + m_3) + 4 m_2 m_3} \right) g$$

Question 2:

i. Define l as the equilibrium length of the spring.

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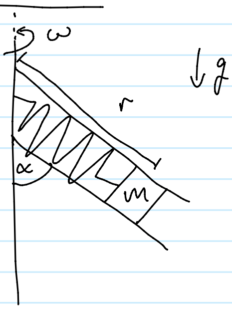


i. Define l as the equilibrium length of the spring.

$$\underline{r} = r\left(\cos(\omega t)\sin\alpha\,\underline{i} + \sin(\omega t)\sin\alpha\,\underline{j} - \cos\alpha\,\underline{k}\right)$$

where $r = l + x \Rightarrow \frac{dr}{dt} = \dot{r} = \dot{x} = \frac{dx}{dt}$

(x being displacement from l)

$$\dot{\underline{r}} = \left(\dot{r}\cos(\omega t)\sin\alpha - r\omega\sin(\omega t)\sin\alpha\right)\underline{i}$$
$$+\left(\dot{r}\sin(\omega t)\sin\alpha + r\omega\cos(\omega t)\sin\alpha\right)\underline{j}$$
$$-\dot{r}\cos\alpha\,\underline{k}$$

$$\dot{\underline{r}}\cdot\dot{\underline{r}} = \dot{r}^2\cos^2(\omega t)\sin^2\alpha - 2\omega r\dot{r}\cos(\omega t)\sin(\omega t)\sin^2\alpha$$
$$+ r^2\omega^2\sin^2(\omega t)\sin^2\alpha + \dot{r}^2\sin^2(\omega t)\sin^2\alpha$$
$$+ 2\omega r\dot{r}\cos(\omega t)\sin(\omega t)\sin^2\alpha + r^2\omega^2\cos^2(\omega t)\sin^2\alpha$$
$$+ \dot{r}^2\cos^2\alpha$$
$$= \dot{r}^2 + r^2\omega^2\sin^2\alpha$$

$$\Rightarrow T = \tfrac{1}{2}m\,\dot{\underline{r}}\cdot\dot{\underline{r}} = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\omega^2\sin^2\alpha$$

$$V = -mgr\cos\alpha + \tfrac{1}{2}kr^2 + \tfrac{1}{2}kl^2 - krl$$

$$\Rightarrow L = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\omega^2\sin^2\alpha + mgr\cos\alpha - \tfrac{1}{2}kr^2 - kl^2 + krl$$

$$\Rightarrow 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r}$$
$$= \frac{\partial}{\partial t}(m\dot{r}) - mr\omega^2\sin^2\alpha - mg\cos\alpha + k(r-l)$$
$$0 = m\ddot{r} - mr\omega^2\sin^2\alpha - mg\cos\alpha + k(r-l)$$

ii. If mass at a fixed r, r_0, then $\ddot{r} = 0$ and

$$0 = -mr_0\omega^2\sin^2\alpha - mg\cos\alpha + k(r-l)$$
$$r_0\omega^2 = -\frac{g\cos\alpha}{\sin^2\alpha} + \frac{kr}{m\sin^2\alpha} - \frac{kl}{m\sin^2\alpha}$$
$$r_0\left(\omega^2 - \frac{k}{m\sin^2\alpha}\right) = -\frac{g\cos\alpha}{\sin^2\alpha} - \frac{kl}{m\sin^2\alpha}$$

Since r_0 must be positive, the left side must be negative, and so

$$\omega^2 < -\frac{k}{m\sin^2\alpha}$$

Then, r_0 is

$$r_0 = -\frac{g\cos\alpha}{\omega^2\sin^2\alpha - \frac{k}{m}} - \frac{kl}{m\omega^2\sin^2\alpha - k}$$
$$r_0 = \frac{g\cos\alpha}{\frac{k}{m} - \omega^2\sin^2\alpha} + \frac{kl}{k - m\omega^2\sin^2\alpha}$$

iii. $m\ddot{r} = mr\omega^2\sin^2\alpha + mg\cos\alpha - k(r-l)$

Consider perturbation $r = r_0 + \delta$

iii. $m\ddot{r} = mr\omega^2\sin^2\alpha + mg\cos\alpha - k(r-l)$

Consider perturbation $r = r_0 + \delta$

$\Rightarrow m\ddot{r} = m(r+\delta)\omega^2\sin^2\alpha + mg\cos\alpha - k(r+\delta-l)$

$= mr_0\omega^2\sin^2\alpha + m\delta\omega^2\sin^2\alpha + mg\cos\alpha - k(r-l) - k\delta$

Since $m\ddot{r} = mr_0\omega^2\sin^2\alpha + mg\cos\alpha - k(r-l)$ corresponds to the steady state of $m\ddot{r} = 0$, the $m\ddot{r}$ left over is from the perturbation, with

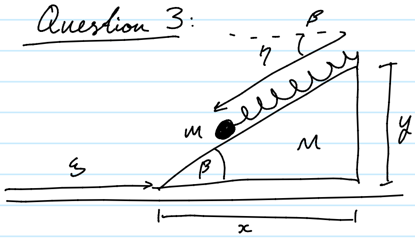
$$m\ddot{r} = m\delta\omega^2\sin^2\alpha - k\delta$$

$$= -\delta(k - m\omega^2\sin^2\alpha)$$

This is of the form of the harmonic oscillator, with frequency

$$f = \sqrt{\frac{k}{m} - \omega^2\sin^2\alpha}$$

<u>Question 3:</u>



$\eta = l + x$

$\Rightarrow x = \eta - l$

i. $T = \sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}m(\dot{\xi} - \dot{\eta}\cos\beta - \dot{\eta}\sin\beta)^2$

$V = -mg\eta\sin\beta + \frac{1}{2}k(\eta - l)^2$

$\Rightarrow L = \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}m(\dot{\xi} - \dot{\eta}\cos\beta - \dot{\eta}\sin\beta)^2 + mg\eta\sin\beta - \frac{1}{2}k\eta^2 - \frac{1}{2}kl^2 + k\eta l$

$= \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}m(\dot{\xi}^2 - 2\dot{\xi}\dot{\eta}(\cos\beta + \sin\beta) + \dot{\eta}^2(\cos\beta + \sin\beta)^2) + mg\eta\sin\beta - \frac{1}{2}k\eta^2 - \frac{1}{2}kl^2 + k\eta l$

Since only acting on $\hat{\underline{\xi}}$, $2\dot{\xi}\dot{\eta}(\cos\beta + \sin\beta) = 2\dot{\xi}\dot{\eta}\cos\beta$

and $\dot{\eta}^2(\cos\beta + \sin\beta)^2 = \dot{\eta}^2$

$\Rightarrow L = \frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}m(\dot{\xi}^2 - 2\dot{\xi}\dot{\eta}\cos\beta + \dot{\eta}^2) + mg\eta\sin\beta - \frac{1}{2}k\eta^2 - \frac{1}{2}kl^2 + k\eta l$

$\Rightarrow 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) - \frac{\partial L}{\partial \xi}$

$= \frac{d}{dt}(M\dot{\xi} + m\dot{\xi} - m\dot{\eta}\cos\beta)$

$0 = M\ddot{\xi} + m\ddot{\xi} - m\ddot{\eta}\cos\beta$

and,

$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}}\right) - \frac{\partial L}{\partial \eta}$

$= \frac{d}{dt}(-m\dot{\xi}\cos\beta + m\dot{\eta}) - mg\sin\beta + k\eta - kl$

$0 = m\ddot{\eta} - m\ddot{\xi}\cos\beta - mg\sin\beta + k(\eta - l)$

ii. Rearranging the first equation of motion gives

$\ddot{\xi}(M+m) = m\ddot{\eta}\cos\beta$

$\Rightarrow \ddot{\xi} = \frac{m\ddot{\eta}\cos\beta}{M+m}$

Substituting this into the second equation of motion,

$\Rightarrow 0 = m\ddot{\eta} - \frac{m^2\ddot{\eta}\cos^2\beta}{M+m} - mg\sin\beta + k(\eta - l)$

$\ddot{\eta}\left(m\left(\frac{M+m-m\cos^2\beta}{M+m}\right)\right) = mg\sin\beta - k(\eta - l)$

$$\Rightarrow \quad 0 = m\ddot{\eta} - \frac{m^2\ddot{\eta}\cos^2\beta}{M+m} - mg\sin\beta + k(\eta - l)$$

$$\ddot{\eta}\left(m\left(\frac{M+m-m\cos^2\beta}{M+m}\right)\right) = mg\sin\beta - k(\eta-l)$$

$$\ddot{\eta} = \frac{(M+m)\left(g\sin\beta - \frac{k}{m}(\eta-l)\right)}{M+m-m\cos^2\beta}$$

$$= \frac{(M+m)\left(g\sin\beta - \frac{k}{m}(\eta-l)\right)}{M+m\sin^2\beta}$$

$$\ddot{\eta} = \frac{(M+m)g\sin\beta}{M+m\sin^2\beta} - (\eta-l)\frac{k(M+m)}{m(M+m\sin^2\beta)}$$

Which is of the form of a harmonic oscillator ODE which is

$$\ddot{x} = a - \omega^2 x$$

where a is some constant, $x = \eta - l$ and ω is the frequency. This has general solution $x = A\cos(\omega t + \varphi)$

Clearly,
$$\omega^2 = \frac{k(M+m)}{m(M+m\sin^2\beta)}$$

and so
$$\omega = \sqrt{\frac{k(M+m)}{m(M+m\sin^2\beta)}}$$

(positive square root since ω is strictly positive).