MATH2001 Assignment 2

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Question 1

The matrix

$$A = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}$$

has eigenvalues 4, 2, and -2. Determine an orthogonal matrix P and a diagonal matrix D such that $PDP^{-1} = A$. Show all working.

First, the eigenvectors for each eigenvalue must be found.
<u>$\lambda = 4$:</u>

$$(A - \lambda I)\vec{v_1} = \vec{0}$$

$$\begin{pmatrix} 3-4 & 0 & 1 & 0 \\ 0 & 1-4 & 0 & 3 \\ 1 & 0 & 3-4 & 0 \\ 0 & 3 & 0 & 1-4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & 3 \\ 1 & 0 & -1 & 0 \\ 0 & 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} R_1 \to -a + c = 0 \Rightarrow c = a \\ R_2 \to -3b + 3d = 0 \Rightarrow b = d \end{matrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 2:}$

$$(A - \lambda I)\vec{v_3} = \vec{0}$$

$$\begin{pmatrix} 3-2 & 0 & 1 & 0 \\ 0 & 1-2 & 0 & 3 \\ 1 & 0 & 3-2 & 0 \\ 0 & 3 & 0 & 1-2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 3 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} R_1 \to a + c = 0 \Rightarrow c = -a \\ R_2 \to b = 3d \\ R_3 \to 3b = d \\ \Rightarrow b = 0 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \vec{v_3}$$

$\underline{\lambda = -2:}$

$$(A - \lambda I)\vec{v_4} = \vec{0}$$

$$\begin{pmatrix} 3+2 & 0 & 1 & 0 \\ 0 & 1+2 & 0 & 3 \\ 1 & 0 & 3+2 & 0 \\ 0 & 3 & 0 & 1+2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & 1 & 0 \\ 0 & 3 & 0 & 3 \\ 1 & 0 & 5 & 0 \\ 0 & 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} R_1 \to 5a + c = 0 \\ R_3 \to c + 5a = 0 \Rightarrow a = c = 0 \\ R_{2,4} \to d = -b \end{aligned}$$

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \vec{v_4}$$

Now, the eigenvectors must be normalized. First, the norm of $\vec{v_1}$ was found:

$$\begin{aligned} \|\vec{v_1}\| &= \sqrt{1^2 + 0^2 + 1^2 + 0^2} \\ &= \sqrt{2} \end{aligned}$$

Since all of the eigenvectors are comprised of entries of 2 of either 1 or -1, the norm will be the same for all of them. Hence, the normalised eigenvectors are

$$\hat{\vec{v_1}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \qquad \hat{\vec{v_2}} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \hat{\vec{v_3}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \qquad \hat{\vec{v_4}} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

The orthogonal matrix P is then,

$$\begin{aligned} P &= (\hat{\vec{v_1}}|\hat{\vec{v_2}}|\hat{\vec{v_3}}|\hat{\vec{v_4}}) \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \end{aligned}$$

With corresponding diagonal matrix D,

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

Now to check that $PDP^{-1} = A$:

$$\begin{aligned}
PDP^{-1} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 2\sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & 2\sqrt{2} & 0 & -\sqrt{2} \\ 2\sqrt{2} & 0 & -\sqrt{2} & 0 \\ 0 & 2\sqrt{2} & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \\
&= A
\end{aligned}$$

Question 2

Consider the quadratic form
$$Q(x, y, z) = 5x^2 + 2y^2 + 4z^2 + 4xy$$

a. Determine an orthogonal change of variables that removes any cross terms in $Q(x, y, z)$.

$$Q(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Firstly, the eigenvalues must be found:

$$\begin{aligned}
\det(A-\lambda I) &= 0 \\
0 &= \det\begin{pmatrix} 5-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix} \\
&= (5-\lambda)(2-\lambda)(4-\lambda) - 4(4-\lambda) \\
&= (10-7\lambda+\lambda^2)(4-\lambda) - 16 + 4\lambda \\
&= 40 - 10\lambda - 28\lambda + 7\lambda^2 + 4\lambda^2 - \lambda^3 - 16 + 4\lambda \\
&= 24 - 34\lambda + 11\lambda^2 - \lambda^3 \\
0 &= -(\lambda-6)(\lambda-4)(\lambda-1) \\
&\Rightarrow \lambda_1 = 6,\ \lambda_2 = 4,\ \lambda_3 = 1
\end{aligned}$$

Now to find the corresponding eigenvectors:
$\underline{\lambda_1 = 6:}$

$$\begin{aligned}
(A-\lambda I)\vec{v_1} &= \vec{0} \\
\begin{pmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{aligned} R_{1,2} &\to a = 2b \\ R_3 &\to c = 0 \end{aligned} \\
\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= b\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\
\Rightarrow \vec{v_1} &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}$$

$\underline{\lambda_2 = 4:}$

$$\begin{aligned}
(A-\lambda I)\vec{v_2} &= \vec{0} \\
\begin{pmatrix} 1 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{aligned} R_1 &\to a = -2b \\ R_2 &\to a = b \\ &\Rightarrow a = b = 0 \end{aligned} \\
\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
\Rightarrow \vec{v_2} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}$$

$\underline{\lambda_3 = 1:}$

$$\begin{aligned}
(A - \lambda I)\vec{v_3} &= \vec{0} \\
\begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} R_1 \to b = -2a \\ R_3 \to c = 0 \end{matrix} \\
\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= a \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \\
\Rightarrow \vec{v_3} &= \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}
\end{aligned}$$

Now the eigenvectors must be normalised to form an orthogonal P:

$$\begin{aligned}
\hat{\vec{v_1}} &= \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} \\
\hat{\vec{v_2}} &= \frac{\vec{v_2}}{\|\vec{v_2}\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
\hat{\vec{v_3}} &= \frac{\vec{v_3}}{\|\vec{v_3}\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}
\end{aligned}$$

Then, orthogonal matrix P and corresponding diagonal matrix D were constructed:

$$\begin{aligned}
P &= (\hat{\vec{v_1}} | \hat{\vec{v_3}} | \hat{\vec{v_2}}) \\
&= \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
D &= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}
\end{aligned}$$

(where λ_2 and λ_3 [and their corresponding eigenvectors] were swapped to construct a symmetric matrix). Now, $A = PDP^T$ since P is orthogonal, hence

$$\begin{aligned}
Q(x, y, z) &= \vec{x}^T A \vec{x} \\
&= \vec{x}^T PDP^T \vec{x} \\
&= \vec{u}^T D \vec{u}
\end{aligned}$$

where

$$\begin{aligned}\vec{u} &= P^T\vec{x}\\ &= \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0\\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix}\\ \begin{pmatrix} u\\ v\\ w\end{pmatrix} &= \begin{pmatrix} \frac{2x+y}{\sqrt{5}}\\ \frac{x-2y}{\sqrt{5}}\\ z\end{pmatrix}\end{aligned}$$

b. Express $Q(x, y, z)$ in terms of the new variables.

$$\begin{aligned}Q(u, v, w) &= \vec{u}^T D\vec{u}\\ &= \begin{pmatrix} u & v & w\end{pmatrix}\begin{pmatrix} 6 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{pmatrix}\begin{pmatrix} u\\ v\\ w\end{pmatrix}\\ &= 6u^2 + v^2 + 4w^2\end{aligned}$$

Question 3

Let

$$B = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$$

For all positive integers n, determine an explicit expression for the matrix B^n. That is, give an explicit formula for each matrix entry of B^n in terms of n. Show all working.

First, find the eigenvalues of B:

$$\begin{aligned}\det(B - \lambda I) &= 0\\ \lambda^2 + 1 &= 0\\ \Rightarrow \lambda &= \pm i\end{aligned}$$

Then, the eigenvectors are found:
<u>$\lambda = i$:</u>

$$\begin{aligned}(B - \lambda I)\vec{v_1} &= \vec{0}\\ \begin{pmatrix} -i & 1\\ -1 & -i\end{pmatrix}\begin{pmatrix} a\\ b\end{pmatrix} &= \begin{pmatrix} 0\\ 0\end{pmatrix} \quad \begin{matrix} R_1 \to ia = b \Rightarrow a = -ib\\ \Rightarrow 0 = -a - ib = R_2\end{matrix}\\ \Rightarrow \begin{pmatrix} a\\ b\end{pmatrix} &= b\begin{pmatrix} -i\\ 1\end{pmatrix} \Rightarrow \vec{v_1} = \begin{pmatrix} -i\\ 1\end{pmatrix}\end{aligned}$$

<u>$\lambda = -i$:</u>

$$\begin{aligned}(B - \lambda I)\vec{v_2} &= \vec{0}\\ \begin{pmatrix} i & 1\\ -1 & i\end{pmatrix}\begin{pmatrix} a\\ b\end{pmatrix} &= \begin{pmatrix} 0\\ 0\end{pmatrix} \quad \begin{matrix} R_1 \to b = -ia\\ \Rightarrow a = bi = R_2\end{matrix}\\ \Rightarrow \begin{pmatrix} a\\ b\end{pmatrix} &= b\begin{pmatrix} i\\ 1\end{pmatrix} \Rightarrow \vec{v_2} = \begin{pmatrix} i\\ 1\end{pmatrix}\end{aligned}$$

Now to normalise the eigenvectors with the complex inner product for the norm: And so,

$$\begin{aligned} P &= (\vec{v_1}|\vec{v_2}) \\ &= \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \\ D &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \end{aligned}$$

With

$$\begin{aligned} P^{-1} &= \frac{1}{\det P} \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2i} & -\frac{i}{-2i} \\ -\frac{1}{-2i} & -\frac{i}{-2i} \end{pmatrix} \\ &= \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix} \end{aligned}$$

Since the matrix B is normal,

$$\begin{aligned} B^n &= PD^nP^{-1} \\ &= \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^n \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} (-i)(i^n) & i(-i)^n \\ i^n & (-i)^n \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{i^n+(-i)^n}{2} & \frac{i(-i)^n-i(i)^n}{2} \\ \frac{i(i)^n-i(-i)^n}{2} & \frac{i^n+(-i)^n}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{i^n}{2}(1+(-1)^n) & \frac{i^{n+1}}{2}((-1)^n-1) \\ \frac{i^{n+1}}{2}(1-(-1)^n) & \frac{i^n}{2}(1+(-1)^n) \end{pmatrix} \end{aligned}$$

Question 4

Let D be the region in the $x-y$ plane defined only for $x \geq 0$ and bounded by the y-axis, the line $y = 4$ and the curve $y = x^2$. Evaluate the double integral

$$\iint_D xy^2 dA$$

Show all working.

Firstly, the point of intersection of $y = 4$ and $y = x^2$ is

$$\begin{aligned} y = 4 &= x^2 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Since we're only considering the positive x-axis, the only point of intersection occurs at $x = 2$, and so the region D is

$$D = \left\{(x, y) \,|\, 0 \leq x \leq 2,\, x^2 \leq y \leq 4\right\}$$

Thus the double integral becomes

$$\begin{aligned} \iint_D xy^2 dA &= \int_0^2 \left(\int_{x^2}^4 xy^2 dy \right) dx \\ &= \int_0^2 \left[\frac{1}{3} xy^3 \right]_{y=x^2}^{y=4} dx \\ &= \int_0^2 \frac{64}{3} x - \frac{1}{3} x^7 dx \\ &= \left[\frac{64}{6} x^2 - \frac{1}{24} x^8 \right]_{x=0}^{x=2} \\ &= 32 \end{aligned}$$