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MATH3401
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Submission Topic Assignment 3

Due date of submission	9/05/2022		
Due time of submission	11:50 AM	Name of tutor (if appropriate)	
Group	05	Session attended (if appropriate)	day time
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MATH3401 Assignment 3

Ryan White s4499039

9th of May, 2022

Question 1

a. Let

$$f(z) = \frac{7z^3}{z^3 - 13z}$$

If $\lim_{z\to\infty} f(z) = 7$, then $\lim_{z\to 0} f(1/z) = 7$.

$$\begin{aligned}
\lim_{z\to 0} f\left(\frac{1}{z}\right) &= \lim_{z\to 0} \frac{7\left(\frac{1}{z}\right)^3}{\left(\frac{1}{z}\right)^3 - 13\left(\frac{1}{z}\right)} \\
&= \lim_{z\to 0} \frac{7}{z^3\left(\frac{1}{z^3} - \frac{13}{z}\right)} \\
&= \lim_{z\to 0} \frac{7}{1 - 13z^2} \\
&= \frac{7}{1 - 13\cdot 0} = 7
\end{aligned}$$

Therefore, $\lim_{z\to\infty} f(z) = 7$.

b. Let

$$f(z) = \frac{7z^3}{z^2 + 13z}$$

If $\lim_{z\to\infty} f(z) = \infty$, then $\lim_{z\to 0} 1/f(1/z) = 0$.

$$\begin{aligned}
\lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} &= \lim_{z\to 0} \frac{\left(\frac{1}{z}\right)^2 + 13\left(\frac{1}{z}\right)}{\left(\frac{1}{z}\right)^3} \\
&= \lim_{z\to 0} z^3\left(\frac{1}{z^2} + \frac{13}{z}\right) \\
&= \lim_{z\to 0} z + 13z^2 \\
&= 0
\end{aligned}$$

Therefore, $\lim_{z\to\infty} f(z) = \infty$.

c. Let

$$f(z) = \frac{(az+b)^2}{(cz+d)^2} = \frac{a^2z^2 + 2abz + b^2}{c^2z^2 + 2cdz + d^2}$$

If $\lim_{z\to\infty} f(z) = a^2/c^2$, then $\lim_{z\to 0} f(1/z) = a^2/c^2$.

$$\begin{aligned}
\lim_{z\to 0} f\left(\frac{1}{z}\right) &= \lim_{z\to 0} \frac{a^2\left(\frac{1}{z}\right)^2 + 2ab\left(\frac{1}{z}\right) + b^2}{c^2\left(\frac{1}{z}\right)^2 + 2cd\left(\frac{1}{z}\right) + d^2} \\
&= \lim_{z\to 0} \frac{a^2\left(\frac{1}{z}\right)^2 + 2ab\left(\frac{1}{z}\right) + b^2}{c^2\left(\frac{1}{z}\right)^2 + 2cd\left(\frac{1}{z}\right) + d^2} \cdot \frac{z^2}{z^2} \\
&= \lim_{z\to 0} \frac{a^2 + 2abz + b^2z^2}{c^2 + 2cdz + d^2z^2} = \frac{a^2}{c^2}
\end{aligned}$$

Therefore, provided $c \neq 0$, $\lim_{z\to\infty} f(z) = a^2/c^2$.

Question 2

a. If $f(z) = 2xy + i(x^2 - y^2)$ is analytic, the Cauchy-Riemann equations must hold. Since $f(z)$ is of the form of a bivariate polynomial, it is defined on all $\mathbb{C}$ (since polynomials are defined everywhere).
$f(z)$ is of the form $f(x, y) = u(x, y) + iv(x, y)$, with

$$u(x, y) = 2xy \qquad v(x, y) = x^2 - y^2$$
$$\Rightarrow u_x = 2y \qquad \Rightarrow v_x = 2x$$
$$u_y = 2x \qquad v_y = -2y$$

For C/R to be satisfied, $u_x = v_y$, but $2y \neq -2y$ anywhere except $y = 0$. Since $f(z)$ is not $\mathbb{C}$-differentiable in the neighbourhood of $y = 0$ (and hence nowhere else), it is nowhere analytic.

b. We have

$$\begin{aligned} f(z) &= \sin(\overline{z}) \\ &= \sin(x + i(-y)) \\ &= \sin(x)\cos(-(iy)) + \cos(x)\sin(-(iy)) \\ &= \sin(x)\cosh(y) - i\cos(x)\sinh(y) \end{aligned}$$

The function $\sin(\overline{z})$ is defined on all $\mathbb{C}$ by the result in lectures, and $f(z)$ is of the form $f(x, y) = u(x, y)+iv(x, y)$ with

$$u(x, y) = \sin(x)\cosh(y) \qquad v(x, y) = -\cos(x)\sinh(y)$$
$$\Rightarrow u_x = \cos(x)\cosh(y) \qquad \Rightarrow v_x = \sin(x)\sinh(y)$$
$$u_y = \sin(x)\sinh(y) \qquad v_y = -\cos(x)\cosh(y)$$

Clearly, $u_x \neq v_y$ and $-v_x \neq u_y$ in general. Only at points where $u_x = v_y = 0$ and $u_y = -v_x = 0$ will $f(z)$ be complex differentiable.
Recall that C/R $\Leftrightarrow df/d\overline{z} = 0$.

$$\Rightarrow \frac{df}{d\overline{z}} = \cos(\overline{z})$$

which is only 0 at $\overline{z} = n\pi + \pi/2$, where $n \in \mathbb{Z}$.
Therefore, $f(z) = \sin(\overline{z})$ is differentiable at some points in $\mathbb{C}$, but not on their neighbourhoods and so $f(z)$ is analytic nowhere.

Question 3

The function

$$f(z) = x^4 + i(1 - y)^4$$

is of the form $f(x, y) = u(x, y) + iv(x, y)$, with

$$u(x, y) = x^4 \qquad v(x, y) = (1 - y)^4$$
$$\Rightarrow u_x = 4x^3 \qquad \Rightarrow v_x = 0$$
$$u_y = 0 \qquad v_y = -4(1 - y)^3$$

As before, $f(z)$ is defined on all $z \in \mathbb{C}$ by polynomial existence.
Clearly, $-v_x = u_y$ everywhere. u_x equals v_y only when

$$\begin{aligned} x = -(1 - y) \Rightarrow x^3 &= -(1 - y)^3 \\ 4x^3 &= -4(1 - y)^3 \\ \Rightarrow u_x &= v_y \end{aligned}$$

And so $f(z)$ is only differentiable along the line $x = y - 1$, or equivalently $y = x + 1$. However, $f(z)$ is not differentiable in any neighbourhood of $y = x + 1$ and so is nowhere analytic.

Question 4

a. Since x and y are real numbers, $f(z) = \sqrt{|xy|}$ is a real-valued function (due to the absolute value being taken). As such, $f(z) = \sqrt{|xy|} + i \cdot 0$. Therefore, $u(x,\, y) = \sqrt{|xy|}$ and $v(x,\, y) = 0 \Rightarrow v_x = v_y = 0$.
u_x and u_y can be calculated using the definition of the derivative. Firstly, consider approaching the origin along the x-axis (since the derivative is independent of approach path)

$$\begin{aligned} u_x(0,0) &= \lim_{(\Delta x, 0) \to (0,0)} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \\ &= \lim_{(\Delta x, 0) \to (0,0)} \frac{\sqrt{|0 \cdot 0 + \Delta x \cdot 0|} - \sqrt{|0 \cdot 0|}}{\Delta x} \\ &= \lim_{(\Delta x, 0) \to (0,0)} \frac{0}{\Delta x} = 0 \end{aligned}$$

Now, approaching the origin along the y-axis:

$$\begin{aligned} u_y(0,0) &= \lim_{(0, \Delta y) \to (0,0)} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} \\ &= \lim_{(0, \Delta y) \to (0,0)} \frac{\sqrt{|0 \cdot 0 + 0 \cdot \Delta y|} - \sqrt{|0 \cdot 0|}}{\Delta y} \\ &= \lim_{(0, \Delta y) \to (0,0)} \frac{0}{\Delta y} = 0 \end{aligned}$$

Therefore, $u_x = 0 = v_y$ and $-v_x = 0 = u_y$ at the origin, and so C/R is satisfied at the origin.

b. Since f differentiability is independent of approach path, choose the path $y = x$ to approach the origin.

$$\Rightarrow f(z) = \sqrt{|x \cdot x|} = \sqrt{|x^2|}$$

Since x is real-valued, $x^2 \geq 0$ and so

$$f(z) = \sqrt{x^2} = |x|$$

but $|x|$ is not differentiable at the origin and so neither is $f(z)$.

c. The definitions from lectures state that C/R is necessary for differentiability but not sufficient. As such, if C/R doesn't hold, $f(z)$ is definitely not differentiable but if C/R does hold, $f(z)$ is ***not*** necessarily differentiable and other tests are needed to properly determine the differentiability of f.

Question 5

a. Boisgerault defines a rectifiable path as $C : [0,1] \to \mathbb{C}$ if the curve C is piecewise continuous differentiable, with finite arc length. This holds if there are consecutive continuously differentiable paths over a partition over the domain such that the joining of all of the piecewise paths results in the path C.

Source: Boisgerault, S, 2017. *Complex Analysis and Applications*. Accessed at:
`https://direns.mines-paristech.fr/Sites/Complex-analysis/Complex%20Analysis%20and%20Applications%20(a4).pdf`

b. An example of a non-rectifiable curve in $\mathbb{C}$ is the curve $f(z = x + iy) = x \sin(1/x)$. As x approaches 0, the curve oscillates more frequently and the path length increases towards infinity.