

Assignment 5

Monday, 9 October 2023 7:45 AM

Question 1:

We have a bar that occupies the domain $0 \leq x < \infty$.
The bar is exposed to external forcing such that the system
is described by

$$\begin{cases} R_0 \partial_{ttt} U = \partial_x T = E \partial_{xx} U \\ \lim_{x \rightarrow \infty} U(x, t) = 0 \\ T(x=0, t) = b \sin(\omega t) \\ T + \gamma_0 \partial_t T = E (\partial_x U + \tau, \partial_{xt} U) \end{cases}$$

Start by expressing others in complex coordinates:

$$T(x=0, t) = b e^{i\omega t}$$

We use the ansatz for quasistationary solutions:

$$U(x, t) = \bar{U}(x) e^{i\omega t}$$

$$T(x, t) = \bar{T}(x) e^{i\omega t}$$

and so we can substitute these into our system's constitutive relation:

$$\bar{T}(x) e^{i\omega t} + \tau_0 \cdot i\omega \bar{T}(x) e^{i\omega t} = E (\partial_x \bar{U}(x) e^{i\omega t} + \tau, i\omega \partial_x \bar{U}(x) e^{i\omega t})$$

$$\Rightarrow \bar{T}(x) (1 + i\omega \tau_0) = E \partial_x \bar{U}(x) (1 + i\omega \tau_1)$$

$$\Rightarrow \bar{T}(x) = E \frac{(1 + i\omega \tau_1)}{(1 + i\omega \tau_0)} \partial_x \bar{U}(x)$$

$$= E^* \partial_x \bar{U}(x)$$

where $E^* = E' + iE'' \geq 0$ from the computation in the lecture.

Now, looking at the momentum equation:

$$R_0 \partial_{ttt} U = E^* \partial_{xx} U$$

$$\Rightarrow -R_0 \omega^2 \bar{U}(x) e^{i\omega t} = E^* \partial_{xx} \bar{U}(x) e^{i\omega t}$$

$$\Rightarrow -\frac{R_0 \omega^2}{E^*} \bar{U}(x) = \partial_{xx} \bar{U}(x)$$

Setting $K^2 = \frac{R_0 \omega^2}{E^*}$, we then have displacement solutions of the form

$$\bar{U}(x) = \alpha e^{iKx} + \beta e^{-iKx}$$

as in the lecture, to satisfy the far field condition

$\lim_{x \rightarrow \infty} U(x, t) = 0$, we need to take $\alpha = 0$
and so

$$\bar{U}(x) = \beta e^{-iKx}$$

- - - .. -iKx - - -

and so

$$\bar{U}(x) = \beta e^{-ikx}$$

$$\Rightarrow \partial_x \bar{U}(x) = -ik\beta e^{-ikx} = \bar{T}(x)/E^*$$

Now we have that, at the left tip $x=0$,

$$\begin{aligned} T(0,t) &= b e^{i\omega t} = \bar{T}(0) e^{i\omega t} \\ &= -ik\beta E^* e^{i\omega t} \\ \Rightarrow b &= -ik\beta E^* \Rightarrow \beta = \frac{b}{-ikE^*} = \frac{ib}{kE^*} \end{aligned}$$

and so $U(x,t) = \bar{U}(x) e^{i\omega t}$

$$= \frac{ib}{kE^*} e^{i(\omega t - kx)}$$

and $T(x,t) = \bar{T}(x) e^{i\omega t}$

$$= b e^{i(\omega t - kx)}$$

from the lecture, we know that $E^* K = r e^{i\delta}$

has $\delta = \frac{1}{2} \arctan \left(\frac{\omega(x_1 - x_0)}{1 + \omega^2 x_1 x_0} \right)$

phase, and $i = e^{i\pi/2}$,

hence

$$\frac{ib}{kE^* K} = \frac{be^{i\pi/2}}{re^{i\delta}} = \frac{b}{r} e^{i(\pi/2 - \delta)}$$

has phase $\pi/2 - \delta$.

and so, at the left tip

$$U(0,t) = \frac{b}{r} e^{i(\omega t + (\pi/2 - \delta))}$$

$$T(0,t) = b e^{i\omega t}$$

and so there is a phase difference of $\pi/2 - \delta$ between the displacement and the stress at the left tip.

This is the negative of the phase difference found in the lecture for imposed displacement.

Question 2:

We consider a rotation around the z-axis $X(\vec{x},t) = Q(t)\vec{x}$

We know that $U(\vec{x},t) = X(\vec{x},t) - \vec{x}$, and so

$$U(\vec{x},t) = (Q(t) - 1)\vec{x}$$

then,

$$V(\vec{x},t) = \frac{\partial}{\partial t} U(\vec{x},t) = \frac{\partial Q(t)}{\partial t} \vec{x}$$

where $\frac{\partial Q(t)}{\partial t} = \begin{pmatrix} -\omega \sin(\omega t) & -\omega \cos(\omega t) & 0 \\ \omega \cos(\omega t) & -\omega \sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then, in spatial coordinates,

$$u(\vec{x},t) = U(x(\vec{x},t), t)$$

where $x(\vec{x},t) = Q(t)^{-1} \vec{x}$

$$\Rightarrow u(\vec{x}, t) = (Q(t) - I) Q(t)^{-1} \vec{x}$$

$$= (I - Q(t)^{-1}) \vec{x}$$

and so, $v(\vec{x}, t) = V(\gamma(x, t), t)$

$$= \frac{\partial Q(t)}{\partial t} Q(t)^{-1} \vec{x}$$

as a check, compute $v(\vec{x}, t) = D_t \vec{u}$:

$$v(\vec{x}, t) = D_t \vec{u} = \frac{\partial \vec{u}}{\partial t} (I - \frac{\partial \vec{u}}{\partial x})^{-1}$$

$$= \frac{\partial}{\partial t} (-Q(t)^{-1} \vec{x}) (I - (I - Q(t)^{-1}))^{-1}$$

$$= \frac{\partial Q(t)}{\partial t} \cdot Q(t)^{-2} \vec{x} \cdot (Q(t)^{-1})^{-1}$$

$$= \frac{\partial Q(t)}{\partial t} Q(t)^{-1} \vec{x}$$

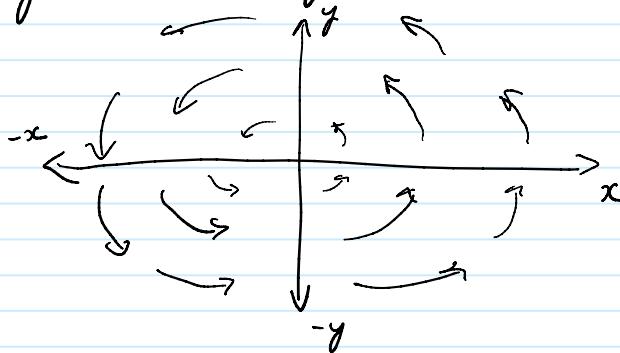
as with the first method.

To sketch a vector field in the x - y plane for v , first multiply the matrices

$$v(\vec{x}, t) = \begin{pmatrix} -\omega \sin(\omega t) & -\omega \cos(\omega t) & 0 \\ \omega \cos(\omega t) & -\omega \sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \omega \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This gives the vector field



where the magnitude of the rotation is proportional to ω .