PHYS2100 Assignment 4

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Question 1

a. The arclength of a point traversed by a point along a circle is given by

$$|\dot{\mathbf{r}}|dt = \left|\frac{d\mathbf{r}}{d\theta}\right| \frac{d\theta}{dt} dt = \left|\frac{d\mathbf{r}}{d\theta}\right| d\theta$$

Take the parameterisation

$$\begin{aligned}
\mathbf{r} &= (-a\sin\theta)\mathbf{i} + (-a\cos\theta)\mathbf{j} \\
\Rightarrow \frac{d\mathbf{r}}{d\theta} &= -a\cos\theta\mathbf{i} + a\sin\theta\mathbf{j} \\
\Rightarrow |\dot{\mathbf{r}}|dt &= \left|\frac{d\mathbf{r}}{d\theta}\right| d\theta \\
&= \int_0^{\pi/2} \sqrt{(-a\cos\theta)^2 + (a\sin\theta)^2} d\theta \\
&= \int_0^{\pi/2} \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} d\theta \\
&= \int_0^{\pi/2} a d\theta \\
&= [a\theta]_0^{\pi/2} = \frac{a\pi}{2}
\end{aligned}$$

Therefore the point P had an arclength of $a\pi/2$ from $\theta = 0 \to \pi/2$.

b. The velocity of P is given by

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\frac{d\hat{\mathbf{r}}}{d\theta}$$

where $r = a \Rightarrow \dot{r} = 0$.

$$\dot{\theta} = \omega \qquad \text{and} \qquad \begin{aligned}
\frac{d\hat{\mathbf{r}}}{d\theta} &= \frac{1}{r}\frac{d\mathbf{r}}{d\theta} \\
&= -\frac{a}{a}\cos\theta\mathbf{i} + \frac{a}{a}\sin\theta\mathbf{j} \\
&= -\cos\theta\mathbf{i} + \sin\theta\mathbf{j}
\end{aligned}$$

Thus,

$$\mathbf{v} = a\omega(-\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

Acceleration is given by

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

Given that $\dot{r} = 0 \Rightarrow \ddot{r} = 0$. Since $\dot{\theta} = \omega$ is constant, $\ddot{\theta} = \dot{\omega} = 0$.

$$\Rightarrow \mathbf{a} = -a\omega^2 \hat{\mathbf{r}} + 0\hat{\theta}$$

Since $\mathbf{r} = r\hat{\mathbf{r}} \Rightarrow \hat{\mathbf{r}} = \mathbf{r} \cdot 1/r$

$$\begin{aligned}
\mathbf{a} &= -a\omega^2(-\sin\theta\mathbf{i} - \cos\theta\mathbf{j}) \\
&= a\omega^2(\sin\theta\mathbf{i} + \cos\theta\mathbf{j})
\end{aligned}$$

Question 2

a. Angular momentum is given by $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$. Note that $\mathbf{r} = r\hat{\mathbf{r}} = r\mathbf{e}_r \Rightarrow \hat{\mathbf{r}} = \mathbf{e}_r$. The velocity is then

$$\begin{aligned}
\mathbf{v} = \frac{d\mathbf{r}}{dt} &= \frac{dr\mathbf{e}_r}{dt} \\
&= \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} \\
&= \dot{r}\mathbf{e}_r + r\left(\frac{d}{dt}\cos\phi\sin\theta\mathbf{i} + \frac{d}{dt}\sin\phi\sin\theta\mathbf{j} + \frac{d}{dt}\cos\theta\mathbf{k}\right) \\
&= \dot{r}\mathbf{e}_r + r\left(\frac{d\theta}{dt}\frac{d(\cos\phi\sin\theta)}{dt}\mathbf{i} + \frac{d\phi}{dt}\frac{d(\cos\phi\sin\theta}{d\phi}\mathbf{i} + \frac{d\theta}{dt}\frac{d(\sin\phi\sin\theta)}{d\theta}\mathbf{j}\right. \\
&\qquad\qquad \left. + \frac{d\phi}{dt}\frac{d(\sin\phi\sin\theta)}{d\phi}\mathbf{j} + \frac{d\theta}{dt}\frac{d\cos\theta}{d\theta}\mathbf{k} + \frac{d\phi}{dt}\frac{d\cos\theta}{d\phi}\mathbf{k}\right) \\
&= \dot{r}\mathbf{e}_r + r\left(\dot{\theta}\cos\phi\cos\theta\mathbf{i} - \dot{\phi}\sin\phi\sin\theta\mathbf{i} + \dot{\theta}\sin\phi\cos\theta\mathbf{j} + \dot{\phi}\cos\phi\sin\theta\mathbf{j} - \dot{\theta}\sin\theta\mathbf{k}\right) \\
&= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\dot{\phi}\sin\theta\mathbf{e}_\phi
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{L} &= m\left(\mathbf{r} \times \left(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\dot{\phi}\sin\theta\mathbf{e}_\phi\right)\right) \\
&= mr\left((\mathbf{e}_r \times \dot{r}\mathbf{e}_r) + (\mathbf{e}_r \times r\dot{\theta}\mathbf{e}_\theta) + (\mathbf{e}_r \times r\dot{\phi}\sin\theta\mathbf{e}_\phi)\right) \\
&= mr\left(\dot{r}(\mathbf{e}_r \times \mathbf{e}_r) + r\dot{\theta}(\mathbf{e}_r \times \mathbf{e}_\theta) + r\dot{\phi}\sin\theta(\mathbf{e}_r \times \mathbf{e}_\phi)\right) \\
&= mr\left(r\dot{\theta}\mathbf{e}_\phi - r\dot{\phi}\sin\theta(\mathbf{e}_\phi \times \mathbf{e}_r)\right) \\
&= m\left(r^2\dot{\theta}\mathbf{e}_\phi - r^2\dot{\phi}\sin\theta\mathbf{e}_\theta\right)
\end{aligned}$$

QED

b. The kinetic energy is expressed by

$$\begin{aligned}
T &= \frac{1}{2}m\mathbf{v} \cdot \mathbf{v} \\
&= \frac{1}{2}m\left(v_1v_1 + v_2v_2 + v_3v_3\right) \\
&= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right)
\end{aligned}$$

QED

c. If angular momentum is conserved,

$$
\begin{aligned}
\frac{d\mathbf{L}}{dt} &= 0 \\
\Rightarrow 0 &= \frac{d}{dt}\left(m\left(r^2\dot{\theta}\mathbf{e}_\phi - r^2\dot{\phi}\sin\theta\mathbf{e}_\theta\right)\right) \\
\Rightarrow \frac{d}{dt}\left(r^2\dot{\theta}\mathbf{e}_\phi\right) &= \frac{d}{dt}r^2\dot{\phi}\sin\theta\mathbf{e}_\theta \\
&= r^2\left(\ddot{\phi}\sin\theta\mathbf{e}_\theta + \dot{\phi}\frac{d\sin\theta}{dt}\mathbf{e}_\theta + \dot{\phi}\sin\theta\frac{d\mathbf{e}_\theta}{dt}\right) \\
&= r^2\left(\ddot{\phi}\sin\theta\mathbf{e}_\theta + \dot{\phi}\dot{\theta}\cos\theta\mathbf{e}_\theta + \dot{\phi}\sin\theta\left(\frac{d}{dt}\cos\phi\cos\theta\mathbf{i} + \frac{d}{dt}\sin\phi\cos\theta\mathbf{j} - \frac{d}{dt}\sin\theta\mathbf{k}\right)\right) \\
&= r^2\left(\ddot{\phi}\sin\theta\mathbf{e}_\theta + \dot{\phi}\dot{\theta}\cos\theta\mathbf{e}_\theta + \dot{\phi}\sin\theta\left(-\dot{\phi}\sin\phi\cos\theta\mathbf{i} - \dot{\theta}\cos\phi\sin\theta\mathbf{i} + \dot{\phi}\cos\phi\cos\theta\mathbf{j}\right.\right. \\
&\qquad \left.\left. -\dot{\phi}\sin\phi\sin\theta\mathbf{j} - \dot{\theta}\cos\theta\mathbf{k} - 0\mathbf{k}\right)\right) \\
&= r^2\left(\ddot{\phi}\sin\theta\mathbf{e}_\theta + \dot{\phi}\dot{\theta}\cos\theta\mathbf{e}_\theta + \dot{\phi}\sin\theta\left(-\dot{\theta}\mathbf{e}_r + \dot{\phi}\cos\theta\mathbf{e}_\phi\right)\right) \\
&= r^2\left(\ddot{\phi}\sin\theta\mathbf{e}_\theta + \dot{\phi}\dot{\theta}\cos\theta\mathbf{e}_\theta - \dot{\phi}\dot{\theta}\sin\theta\mathbf{e}_r + \dot{\phi}^2\sin\theta\cos\theta\mathbf{e}_\phi\right)
\end{aligned}
$$

Looking only at the $\mathbf{e}_\phi$ coordinate space,

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = \dot{\phi}^2\sin\theta\cos\theta$$

QED

Also, if angular momentum is conserved, $\mathbf{L}$ is a constant. Take $\mathbf{L} = D$,

$$\Rightarrow D = m\left(r^2\dot{\theta}\mathbf{e}_\phi - r^2\dot{\phi}\sin\theta\mathbf{e}_\theta\right)$$

Looking at only $\mathbf{e}_\theta$ (and take d as the *constant* component of $\mathbf{L}$ in that coordinate space),

$$d = -mr^2\dot{\phi}\sin\theta\left(\cos\phi\cos\theta\mathbf{i} + \sin\phi\cos\theta\mathbf{j} - \sin\theta\mathbf{k}\right)$$

Now, looking only at $\mathbf{k}$, and absorbing m into the constant to give a new constant c,

$$
\begin{aligned}
c &= -r^2\dot{\phi}\sin\theta(-\sin\theta) \\
&= r^2\dot{\phi}\sin^2\theta
\end{aligned}
$$

QED

Question 3

a. Take the parameterisation

$$\mathbf{r} = r\sin\theta\mathbf{i} - r\cos\theta\mathbf{j}$$

and let r be the current radial displacement of the mass from the origin, and l be the equilibrium radial distance of the spring from the origin. Then,

$$\begin{aligned}\frac{d\mathbf{r}}{d\theta} &= r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} \\ \frac{d\mathbf{r}}{dr} &= \sin\theta\mathbf{i} - \cos\theta\mathbf{j} = \hat{\mathbf{r}}\end{aligned}$$

The force is then,

$$\mathbf{F} = -mg\mathbf{j} - k\left(\frac{r-l}{r}\right)\hat{\mathbf{r}} = -k(r-l)\sin\theta\mathbf{i} - mg\mathbf{j} + k(r-l)\cos\theta\mathbf{j}$$

The factor out the front of $\hat{\mathbf{r}}$ is to make it so that the force due to the spring acts as a proportion of radial displacement from the equilibrium length of the spring. This results in the generalised forces of

$$\begin{aligned}Q_r &= \mathbf{F}\cdot\frac{d\mathbf{r}}{dr} = -k(r-l)\sin^2\theta + mg\cos\theta - k(r-l)\cos^2\theta \\ &= mg\cos\theta - k(r-l) \\ Q_\theta &= \mathbf{F}\cdot\frac{d\mathbf{r}}{d\theta} = -kr(r-l)\cos\theta\sin\theta - mgr\sin\theta + kr(r-l)\cos\theta\sin\theta \\ &= -mgr\sin\theta\end{aligned}$$

b. Since generalised forces are related to the respective derivative of the potential function,

$$\begin{aligned}Q_\theta &= -\frac{\partial V}{\partial\theta} \\ \Rightarrow V_\theta &= -\int Q_\theta d\theta \\ &= -mgr\cos\theta \\ Q_r &= -\frac{\partial V}{\partial r} \\ \Rightarrow V_r &= -\int Q_r dr \\ &= mgr\cos\theta + \frac{1}{2}kr^2 - krl\end{aligned}$$

The potential is the sum of it's components, so

$$\begin{aligned}V &= V_\theta + V_r \\ &= \frac{1}{2}kr^2 - krl\end{aligned}$$