Gravitational Waves

Summary

- The linearised Einstein vacuum equation
- Gravitational waves
- Detection of gravitational waves

These notes are mainly based on selected parts of Ch. 16 in Hartle.

1 Solution of the linearised Einstein vacuum equation

Recall that the Einstein equation in the vacuum is

$$R_{\alpha\beta} = \partial_\gamma \Gamma^\gamma_{\alpha\beta} - \partial_\beta \Gamma^\gamma_{\alpha\gamma} + \Gamma^\eta_{\alpha\beta}\Gamma^\gamma_{\eta\gamma} - \Gamma^\eta_{\alpha\gamma}\Gamma^\gamma_{\eta\beta} = 0 \tag{1}$$

We wish to find the solution to the vacuum Einstein equation when there is a small perturbation to the Minkowski metric, i.e.

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{2}$$

where $h_{\alpha\beta}$ is a "small" perturbation. The first order perturbations in the Christoffel symbols are given by

$$\delta\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}\eta^{\gamma\delta}\left(\frac{\partial h_{\delta\alpha}}{\partial x^\beta} + \frac{\partial h_{\delta\beta}}{\partial x^\alpha} - \frac{\partial h_{\alpha\beta}}{\partial x^\delta}\right) \tag{3}$$

as the zeroth order vanishes. In addition the perturbation in the Ricci curvature simplifies because the last 2 terms are 2nd order in the perturbation and so can be neglected. Thus the linearised Einstein vacuum equation becomes

$$\begin{aligned} \delta R_{\alpha\beta} &= \partial_\gamma(\delta\Gamma^\gamma_{\alpha\beta}) - \partial_\beta(\delta\Gamma^\gamma_{\alpha\gamma}) && (4) \\ &= \frac{1}{2}[-\Box h_{\alpha\beta} + \partial_\alpha V_\beta + \partial_\beta V_\alpha] = 0, && (5) \end{aligned}$$

where $\Box \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ is the D'Alembertian and $V_\alpha \equiv \partial_\gamma h^\gamma_\alpha - \frac{1}{2}\partial_\alpha h^\gamma_\gamma$ with $h^\gamma_\alpha = \eta^{\gamma\delta}h_{\delta\alpha}$.

Choosing Coordinates – Gauge

Recall that coordinates are arbitrary. By careful choice of coordinates the solution to the linearised vacuum Einstein equation can be simplified. Conversely we can note that $\delta R_{\alpha\beta} = 0$ cannot uniquely determine $h_{\alpha\beta}(x)$ without some conditions on the coordinates.

Consider transformations of the kind

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x) \tag{6}$$

where the $\xi^{\alpha}(x)$ are arbitrary functions that are of the same small size as the metric perturbations. In general recall that metrics transform via

$$g'_{\alpha\beta}(x') = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} g_{\gamma\delta}(x). \tag{7}$$

To first order in $\xi^{\alpha}(x)$ we have expressions like

$$x^{\alpha} = x'^{\alpha} - \xi^{\alpha}(x^{\beta}) = x'^{\alpha} - \xi^{\alpha}(x'^{\beta}) \tag{8}$$

and

$$\frac{\partial x^{\alpha}}{\partial x'^{\beta}} = \delta^{\alpha}_{\beta} - \frac{\partial \xi^{\alpha}}{\partial x'^{\beta}} = \delta^{\alpha}_{\beta} - \frac{\partial \xi^{\alpha}}{\partial x^{\beta}}. \tag{9}$$

Applying this to our linearised metric we find

$$\eta_{\alpha\beta} + h_{\alpha\beta}(x) \to \eta_{\alpha\beta} + h'_{\alpha\beta}(x') \tag{10}$$

where $h'_{\alpha\beta} = h_{\alpha\beta} - \partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha}$. This is often referred to as a gauge transformation because of its similarities with gauge transformations in EM.

Lorentz Gauge Conditions

We now require that the coordinate choice is such that

$$V_{\alpha} = \partial_{\gamma} h^{\gamma}_{\alpha} - \frac{1}{2}\partial_{\alpha} h^{\gamma}_{\gamma} = 0. \tag{11}$$

This is referred to as the Lorentz gauge condition. Given this then the linearised Einstein vacuum equation becomes the wave equation:

$$\delta R_{\alpha\beta} = -\Box h_{\alpha\beta}(x) = 0. \tag{12}$$

2 Gravitational Waves

Consider the metric perturbation:

$$h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t-z) \tag{13}$$

where $f(t-z)$ is any function of $t-z$ provided $|f(t-z)| << 1$. It is easily shown that Eq.13 satisfies Eq.11 and 12. The line element for the metric is thus

$$ds^2 = -dt^2 + (1 + f(t-z))dx^2 + (1 - f(t-z))dy^2 + dz^2. \tag{14}$$

This represents a plane wave of curvature propagating in the $+z$ direction with velocity $= 1$ (i.e. c).

For example, consider the choice

$$f(t-z) = Ae^{-\frac{(t-z)^2}{\sigma^2}}. \tag{15}$$

This is a solitary Gaussian wave of amplitude A and width σ. Along the line $z = t$ there will be maximum stretch in the x direction and maximum contraction in the y direction. Away from the line $z = t$ the amplitude is exponentially suppressed and spacetime becomes flat.
A second example is the choice

$$f(t-z) = A\sin\omega(t-z). \tag{16}$$

This represents a plane gravitational wave of definite frequency ω and amplitude A.
The gravitational wave metric (Eq.14) has the following symmetries

$$x \to x + k_1 \quad y \to y + k_2 \quad z,\ t \to z + k_3,\ t + k_3 \tag{17}$$

where the k_i are constants.

Detecting Gravitational Waves

Consider two test particles, A and B, at rest in flat space-time such that:

$$u^\alpha_A = u^\alpha_B = (1, 0, 0, 0) \tag{18}$$

and

$$x^i_A = (x_A, y_A, z_A) \quad x^i_B = (x_B, y_B, z_B). \tag{19}$$

To predict the motion when the gravitational wave arrives we consider the 1st order corrections to the particle positions, δx^i_A and δx^i_B. For either particle

$$\frac{d^2\delta x^i}{d\tau^2} = -\delta\Gamma^i_{\alpha\beta}u^\alpha u^\beta = -\delta\Gamma^i_{tt}, \tag{20}$$

where we have used Eq.18. However, because $h_{tt} = 0$ for the gravitational wave metric, so $\delta\Gamma^i_{tt} = 0$, thus

$$\frac{d^2\delta x^i}{d\tau^2} = 0 \tag{21}$$

and the particle positions are unchanged, i.e. $\delta x^i_A = \delta x^i_B = 0$.

Even though the coordinate separation is constant, the proper distance separation is not. The proper distance in the x-direction is given by $ds_x = ds|_{dt=dy=dz=0}$ via

$$L(t) = \int_{x_A}^{x_B} ds_x = \int_0^{L^*} dx\sqrt{g_{xx}} = \int_0^{L^*} dx\sqrt{1 + h_{xx}(t)} = L^*(1 + \frac{1}{2}h_{xx}(t)) \tag{22}$$

where $L^* = (x_B - x_A)$. For a gravitational wave of definite frequency, ω, and amplitude, A, i.e. $f(t-z) = A\sin\omega(t-z)$, the fractional change in length is

$$\frac{\delta L(t)}{L^*} = \frac{1}{2}A\sin\omega t. \tag{23}$$

The fractional change in length in the y direction is similar but π out of phase. Notice there is no change in the z direction indicating that they are transverse waves.

Polarisation of Gravitational Waves

A second, linearly independent solution to the linearised Einstein vacuum equations is:

$$h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} f(t-z) \tag{24}$$

is referred to as the $\times$ (cross) polarisation, whilst the original solutions the $+$ (plus) polarisation. Thus the most general gravitational plane wave propagating in the z-direction is of the form

$$h_{\alpha\beta}(t,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_+(t-z) & f_x(t-z) & 0 \\ 0 & f_x(t-z) & -f_+(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{25}$$

for different functions $f_+(t-z)$ and $f_x(t-z)$. A superposition of metric perturbations with different functions f_+ and f_x and different directions of propagation produces the most general linearised gravitational wave.

Gravitational Wave Interferometers

Optical interferometers are a very precise way to measure small fluctuations in length. Consider a Michelson interferometer (see Fig. 1). If the input amplitude is α, then the output amplitude at the dark port is

$$\alpha_d = \frac{1}{2}\alpha e^{-i\omega L^* 2(1+\delta L(t))} - \frac{1}{2}\alpha e^{-i\omega L^* 2(1-\delta L(t))}. \tag{26}$$

Then the output intensity is:

$$\begin{aligned} |\alpha_d|^2 &= |\alpha|^2 |\frac{1}{4}(e^{i\omega L^* 2\delta L(t)} - e^{-i\omega L^* 2\delta L(t)})^2 && (27) \\ &= |\alpha|^2 \sin^2 \omega L^* 2\delta L(t) && (28) \\ &\approx |\alpha|^2 \omega^2 L^{*2} 4\delta L(t)^2. && (29) \end{aligned}$$

If we assume that the incoming gravitational wave is a $+$ polarised plane wave propagating in the z-direction of definite frequency ω_g, then

$$|\alpha_d|^2 = |\alpha|^2 \omega^2 L^{*2} A^2 \sin^2 \omega_g t. \tag{30}$$

The signal can be enhanced by high laser intensity, $|\alpha|^2$, higher optical frequency, ω, and longer arm length, L^*. It is expected that typical sources such as neutron star or black-hole mergers will have $A^2 \sim 10^{-21}$, but these will be rare events. LIGO has $|\alpha|^2 \sim 200W$, $\lambda = 1064nm$ and $L^* = 4km$. The effective arm length is extended by a factor of over 75 times using cavities in the interferometer arms. Advanced LIGO achieved a sensitivity of $\sim 1 : 10^{21}$ and observed two confirmed and one suspected black-hole in-spirals in its 2015 - 2016 data run.

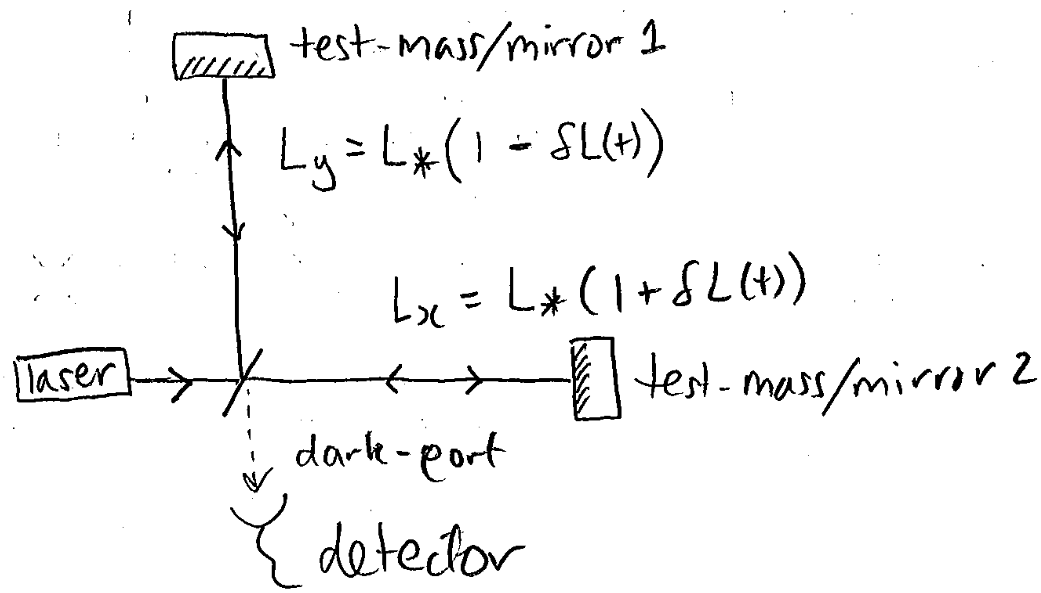

Figure 1: The passage of a gravitational wave (propagating into the page) causes the arm lengths of a Michelson interferometer to oscillate out of phase. This couples light out of the dark port of the interferometer producing a signal at the detector.

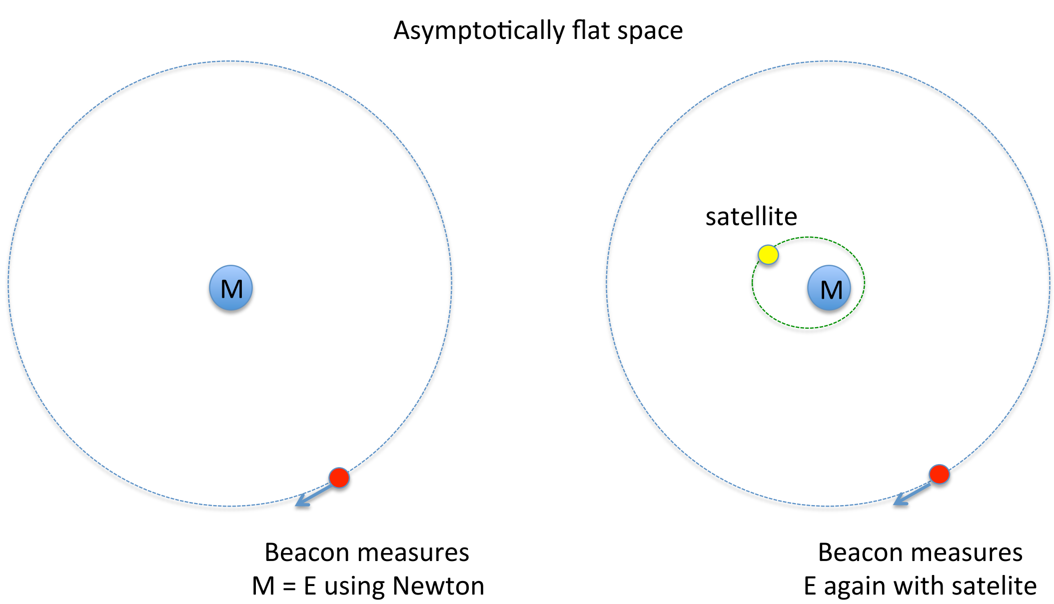

Figure 2: Example of determining energy in general relativity from an asymptotically flat vantage point. Initially, a beacon determines mass $\equiv$ energy of the source of curvature from a far away vantage point where the curvature is sufficiently small that Newtonian physics applies approximately. The measurement is repeated with the satellite in place. The difference between the first and second measurement is the energy of the satellite.

Energy in Gravitational Waves

There is no local gravitational potential energy in general relativity. Recall that an object on a geodesic in Schwarzschild has constant energy – c.f Newton, where the objects potential energy would change along the path. Total energy can be defined from an asymptotically flat "vantage point" (see Fig. 2 for example).

The short wavelength energy density, ϵ_{GW} or flux, f_{GW}, of gravitational radiation can be written:

$$\epsilon_{GW} = f_{GW} = \frac{\omega_g^2 A^2}{32\pi} \tag{31}$$

where plane, single polarisation, single frequency (ω_g) is assumed and $\frac{\lambda_g}{R} << 1$, with R the background curvature. For the first black-hole merger observed by LIGO the total energy released as gravitational radiation is estimated to have been 3 solar masses.

Cosmology: the Friedmann-Lemaître-Robertson-Walker metric

Summary

- The cosmological principle and Robertson-Walker spacetime geometries
- Matter, radiation, and cosmological constant contributions to the energy density

These notes are mainly based on selected parts of Ch. 18 in Hartle.

1 The cosmological principle and Robertson-Walker spacetime geometries

Various observations show that the distribution of matter and radiation in our universe is to a very good approximation both isotropic and homogeneous on large enough spatial distance scales.[1] Spatial isotropy means that the universe looks the same in all directions. Spatial homogeneity means that the universe looks the same in all places. In the following we'll discuss cosmological models in which these properties of space are built in from the outset. That the universe has these properties of spatial isotropy and homogeneity on large scales is often referred to as the **cosmological principle**.

The line element for a spacetime geometry that satisfies the cosmological principle can be written

$$ds^2 = -dt^2 + a^2(t)d\mathcal{L}^2 \tag{1}$$

where t is the time coordinate, $a(t)$ is a quantity with units of length called the **scale factor**, and $d\mathcal{L}^2$ is the line element of a three-dimensional **space** that is homogeneous and isotropic. It turns out that there are only 3 different types of possible spaces of this kind. Due to the homogeneity, all of them have a constant **spatial** curvature (NB! not spacetime curvature, but spatial curvature); the 3 spaces differ in whether this curvature is positive, zero, or negative. The line element for these three-dimensional spaces can be written

$$d\mathcal{L}^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2}$$

where the parameter k takes the value $+1$ for the space with positive spatial curvature, 0 for the space with zero spatial curvature, and -1 for the space with negative spatial curvature.[2]

[1] Here "large enough" means scales larger than several hundred Mpc (1 parsec (pc) = 3.086×10^{13} km = 3.262 light years). To put this number in perspective, note that the distance from us to the nearest large galaxy cluster (the Virgo cluster of several thousand galaxies) is ≈ 20 Mpc.

[2] Of course, spaces with an arbitrary positive or negative value of the spatial curvature may exist, but it is always possible to redefine the coordinates to absorb this into the scale factor $a(t)$ s.t. the curvature parameter k in those cases is always normalized to ± 1.

We see immediately that the case of zero spatial curvature, i.e. $k = 0$, is, not unexpectedly, just **flat** three-dimensional space, with the line element here written in standard spherical coordinates (r, θ, ϕ). This flat space is infinite in extent. This space is obviously the 3D generalization of a flat plane (2D flat space). Similarly, the two spaces corresponding to $k = \pm 1$ can also be thought of as 3D generalizations of certain 2D surfaces, although the nonzero curvature means we can no longer visualize these 3D spaces. The space with $k = +1$ is the 3D generalization of a spherical surface (the spherical surface has constant positive curvature) and is thus referred to as a spherical geometry. The space with $k = -1$ is the 3D generalization of a 2D "potato chip" surface (a "saddle"-shaped surface with constant negative curvature) and is known as a hyperbolic geometry. These two 3D spaces are respectively finite and infinite in extent (i.e. volume) just like their 2D counterparts. A summary of various properties of the three constant-curvature spaces is given in Table 1.

Geometry	Spatial curvature	k	Volume	Also called
Spherical	Positive	+1	Finite	Closed
Flat	Zero	0	Infinite	Critical
Hyperbolic	Negative	-1	Infinite	Open

Table 1: Defining characteristics and terminology for the 3 types of three-dimensional isotropic and homogenous spaces.

Spacetime geometries of the form (1), with the spatial part given by (2), are referred to as **Robertson-Walker (RW) metrics**. The scale factor $a(t)$ will be determined by the 00 component of the Einstein equation (see later), which in the context of RW metrics is known as the **Friedmann equation**. Cosmological models of this type are thus called **Friedmann-Robertson-Walker (FRW) models**. We will find that these models predict that $a(t)$ grows with time, which means that space is expanding.

Interestingly, among the 3 alternatives of homogeneous and isotropic spaces, i.e. closed/flat/open, observations indicate that our universe seems to be best described by the flat space ($k = 0$). Let us therefore assume flat space for the moment. Then we can introduce Cartesian coordinates x, y, z as usual with spatial line element $d\mathcal{L}^2 = dx^2 + dy^2 + dz^2$.

In FRW models the distributions of matter (galaxies) and radiation are modeled by perfect **fluids**. An individual galaxy can be thought of as a particle in this fluid, with spatial coordinates x, y, and z. In FRW models the coordinates x, y, z of a galaxy do not vary with the time coordinate t, so the velocity dx^i/dt vanishes: one says that the galaxy coordinates (x, y, z) are **co-moving** (i.e. moving with the expansion of the universe).[3] A decent (albeit only 2D) analogy is to think of the universe as the surface of a balloon. Focus on a particular point on the balloon (representing some galaxy). Imagine drawing a coordinate grid on the balloon in the vicinity of this point. As the balloon is inflated (representing the expansion of the universe) the coordinates of the galaxy do not change, because the grid expands with

[3] If $dx^i = 0$ at one instant of time, it follows from the RW metric that $dx^i = 0$ at all times. This can be seen from the geodesic equation and the fact that the Christoffel symbol $\Gamma^i_{00} = 0$ for these metrics. Evaluating the geodesic equation at the instant when $dx^i = 0$ gives

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2 = 0. \tag{3}$$

In other words, the acceleration is also zero, and thus the velocities will not change, hence $dx^i = 0$ at all times.

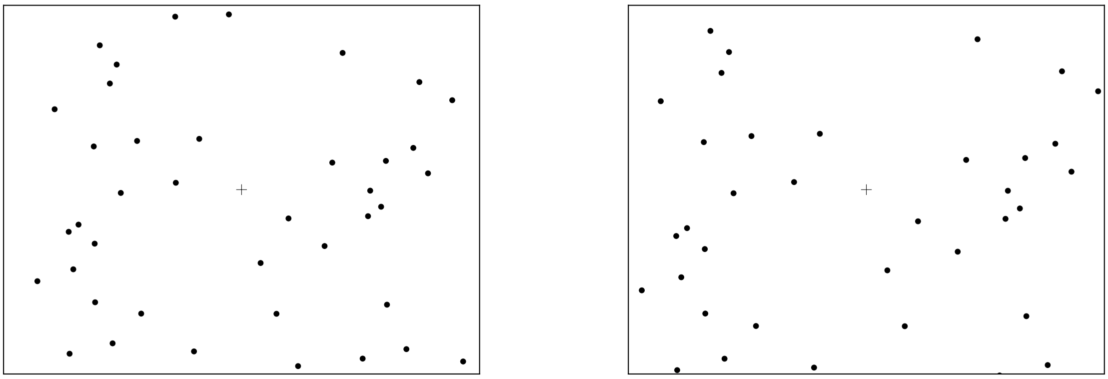

Figure 1: The positions of galaxies are represented by dots, and the observer by the cross. As the universe expands, all the distances between the galaxies are dilated equally, so there is no centre to the expansion.

the balloon. This is illustrated in figure 1.

The constancy of the spatial coordinates also means that the coordinate t measures proper time τ for galaxies. To see this, use that the proper time is obtained from $d\tau^2 = -g_{\alpha\beta}dx^\alpha dx^\beta$ with $g_{\alpha\beta}$ being the RW metric. The fact that $dx^i = 0$ then implies that $d\tau^2 = -g_{00}dt^2 = dt^2$, i.e. $d\tau = dt$. The distribution of radiation is also at rest in the frame defined by the co-moving coordinate system (x, y, z).

Consider two galaxies whose coordinates are respectively (x_1, y_1, z_1) and (x_2, y_2, z_2). We can find the **physical distance** $D(t)$ between these galaxies at a given time t from the RW metric. Since the time is fixed, we have $dt = 0$, so the line element is

$$ds^2 = a^2(t)(\Delta x^2 + \Delta y^2 + \Delta z^2). \tag{4}$$

Taking the square root gives $D(t) = a(t)\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. The square root here is the **coordinate distance** and does not depend on time t since the galaxies' coordinates are stationary. However, if $a(t)$ varies with t, the physical distance will depend on time. Thus the physical distance increases with time if $a(t)$ does. This is the situation in our expanding universe, as we will see later. Defining the relative recession velocity V of the two galaxies as the derivative of the instantaneous physical distance $D(t)$ wrt t, we get $V \equiv \frac{dD}{dt} = \dot{a}\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \frac{\dot{a}}{a}D$, i.e.

$$V = HD \tag{5}$$

where H is the **Hubble constant**

$$H = H(t) \equiv \frac{\dot{a}(t)}{a(t)} \tag{6}$$

with $\dot{a} \equiv da/dt$. Actually, calling H a constant is a little misleading since it depends on time, but this has become standard terminology. Eq. (5) is called **Hubble's law** (after Edwin Hubble who deduced it in 1929 from empirical observations): galaxies "move" away from each other at a rate that is proportional to the distance between them. This proportionality implies that V can exceed the speed of light for two galaxies that are sufficiently far away from each other. This is however not a contradiction of relativity, since the speed-of-light speed limit of relativity only pertains to the velocity of objects moving **through space**, while the recession velocity V is due to the expansion **of space**.[4] Put differently, only the former type of velocity is a local quantity associated with a point in spacetime, at which the local light cone structure imposes the speed limit.

To get some further intuition about the meaning of Hubble's law it is again helpful to consider the surface of a balloon as an analogy for the universe. Imagine galaxies being drawn as dots on the balloon. As the balloon expands the space between the dots increases, so dots will effectively "move" away from each other at a rate given by Hubble's law: faraway galaxies will recede from each other faster than galaxies closer together. Although from the vantage point of the Earth the fact that all (sufficiently distant[5]) galaxies move away from us might at first seem to suggest that we are "at the center of the universe", this idea is wrong (as also seen from the balloon analogy): Hubble's law applies for all pairs of (sufficiently distant) galaxies; thus no galaxies are at "the center of the universe"; indeed there is no such center as that would violate the cosmological principle.

2 Matter, radiation, and cosmological constant contributions to the energy density

As we will see in the next section, the equation that determines the time evolution of the scale factor $a(t)$ is the 00 component of the Einstein field equation. Let us first consider the rhs of this equation, which involves T_{00}, the 00 component of the energy-momentum tensor. As noted in the previous section, the matter and energy in the universe will be modelled by a (perfect) fluid that is at rest in the co-moving coordinate system. In this case $T_{00} = \rho$, where ρ is the total energy density of the fluid. In the "standard model" of cosmology, ρ is a sum of contributions from three different sources: matter (density ρ_m), radiation (ρ_r), and cosmological constant (ρ_Λ):

$$\rho = \rho_m + \rho_r + \rho_v. \tag{7}$$

As we will see, these contributions ρ_i depend differently on the scale factor a. To find these dependencies we will use the Bianchi identity, such that

$$\nabla_\beta T^{\alpha\beta} = 0 \tag{8}$$

where ∇_β is the covariant derivative. This relation is the *local conservation of energy-momentum*, because it reduces to the conservation law $\partial T^{\alpha\beta}/\partial x^\beta = 0$ in flat spacetime. In a homogenous, isotropic cosmology such as FRW, the density and pressure of a fluid are function only of time, so can be written as $\rho(t)$ and $p(t)$. Writing the stress-energy tensor as:

$$T^{00} = \rho(t)\,, \quad T^{ij} = g^{ij} p(t) = \delta^{ij}[p(t)/a(t)^2]\,. \tag{9}$$

Since these quantities only depend on t, we only need the t-component of the Bianchi identity,

$$\nabla_\beta T^{t\beta} = \frac{\partial T^{t\beta}}{\partial x^\beta} + \Gamma^t_{\beta\gamma} T^{\gamma\beta} + \Gamma^t_{\beta\beta} T^{t\gamma} \tag{10}$$

[4]For a more detailed discussion, see the very nice article "Misconceptions about the Big Bang" by Charles Lineweaver and Tamara Davis, Scientific American, March 2005.

[5]The qualifier "sufficiently distant" means that the galaxy must be far enough away from us that the recession velocity dominates over the "local" velocity of the galaxy acquired from gravitational attraction to other galaxies nearby.

The only relevant Christoffel symbols are

$$\Gamma^t_{ij} = a\dot{a}\delta_{ij}\,, \quad \Gamma^i_{jt} = \frac{\dot{a}}{a}\delta^i_j\,, \tag{11}$$

and so equation 10 becomes

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)\,. \tag{12}$$

This is sometimes called the **continuity equation**, and we now apply it to each of the three sources of energy density separately.

Matter. The distribution of matter (galaxies) is well described by a pressureless gas (i.e. the pressure $p_m \approx 0$). Thus (12) becomes $d\log(\rho_m) = -3d\log(a)$, which has the solution

$$\rho_m \propto a^{-3}. \tag{13}$$

Radiation. For a gas of blackbody radiation the pressure p_r and energy density ρ_r are related by

$$p_r = \rho_r/3. \tag{14}$$

Thus (12) becomes $d\log(\rho_r) = -3\cdot(1+1/3)d\log(a)$, which has the solution

$$\rho_r \propto a^{-4}. \tag{15}$$

We also note that for a gas of blackbody radiation, the relation between its energy density ρ_r and temperature T_r is $\rho_r \propto T_r^4$. Combining this with Eq. (15) then gives

$$T_r \propto a^{-1}, \tag{16}$$

i.e. the temperature of the photon gas is inversely proportional to the scale factor. This is relevant to the **cosmic background radiation** (discovered in 1964 by Penzias and Wilson) which when it was released about 400,000 years after the Big Bang had a temperature of about 3000 K, but which now has a much lower temperature of about 2.7 K: this temperature decrease is due to the increase of the scale factor in the intervening time period.

Cosmological constant. The cosmological constant is still very poorly understood theoretically, but is considered to be the leading candidate for what drives the **acceleration** of the expansion of the universe (which was discovered in 1998). In this context cosmological constant is exists under the general term **dark energy**. It was realized by Einstein that because the covariant derivative of the metric tensor vanishes, the covariant derivative of the lhs of the Einstein equation would still vanish if the equation was modified by adding an additional term $\Lambda g_{\mu\nu}$ to the lhs, so that the equation reads

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{17}$$

In more modern treatments the cosmological constant term is usually moved over to the rhs and interpreted as a vacuum contribution to the energy-momentum tensor of the form $T^{(\mathrm{vac})}_{\mu\nu} = -\kappa^{-1}\Lambda g_{\mu\nu}$. This gives the 00 component $T^{(\mathrm{vac})}_{00} = -\kappa^{-1}g_{00}\Lambda = \kappa^{-1}\Lambda \equiv \rho_v$. The cosmological constant is intrinsic property of spacetime, with an associated energy density ρ_Λ that is simply a constant (for now assumed positive) independent of the scale factor, i.e.

$$\rho_\Lambda \propto a^0. \tag{18}$$

From (12) it then follows that the associated pressure is *negative*:

$$p_\Lambda = -\rho_\Lambda. \tag{19}$$

Since a positive pressure means resistance to compression, a negative pressure must mean resistance to expansion, and is thus a bit like the tension in a stretched rubber band.

General fluids. One can image a general fluid (which we call 'X'), with different physical properties to either of these types of material. Such a fluid is normally described by an **equation of state** w, that relates the preesure to the density, i.e.

$$p_X = w\rho_X\,. \tag{20}$$

So for pressureless matter, $w = 0$, radiation $w = 1/3$ and for a cosmological constant $w = -1$. Solving equation 12 for such a general fluid, we find

$$\rho_X \propto a^{-3(1+w)}. \tag{21}$$

Cosmology: The Expanding Universe

Summary

- The Friedmann and acceleration equations
- Cosmological redshift

These notes are mainly based on selected parts of Ch. 18 in Hartle.

1 The Friedmann and Raychaudhuri equations

The Robertson-Walker metric is defined to be

$$ds^2 = -dt^2 + \frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1}$$

Given this metric, one can calculate the Christoffel symbols, the Riemann curvature tensor, the Ricci tensor, and the Ricci scalar (in that order). In order to write down the 00 component of the Einstein equation, we only need R_{00} in addition to R. We state without proof that

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] \tag{2}$$

where $\dot{a} \equiv da/dt$ and $\ddot{a} = d^2a/dt^2$. Thus the lhs of the 00 component of the Einstein equation becomes (note that $g_{00} = -1$)

$$R_{00} - \frac{1}{2}Rg_{00} = -3\frac{\ddot{a}}{a} - \frac{1}{2}(-1)\cdot 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]. \tag{3}$$

Using also that $T_{00} = \rho$ (see previous section), the 00 component of the Einstein equation can be written

$$\dot{a}^2 - \frac{8\pi\rho}{3}a^2 = -k. \tag{4}$$

This is called the **Friedmann equation**. We see that the evolution of the scale factor $a(t)$ is affected both by the energy density ρ and by the constant $k = 0, \pm 1$ that reflects the spatial curvature of the universe.

Let t_0 refer to the present time. Let the current values of the scale factor, Hubble constant and energy density be $a_0 = a(t_0)$, $H_0 = H(t_0)$ and $\rho_0 = \rho(t_0)$, respectively. Furthermore, define the "critical" density ρ_{crit} as

$$\rho_{\text{crit}} \equiv \frac{3H_0^2}{8\pi} \tag{5}$$

and the dimensionless parameter

$$\Omega \equiv \frac{\rho_0}{\rho_{\text{crit}}}. \tag{6}$$

The Friedmann equation, evaluated at the present time, can then be written as

$$\Omega - 1 = \frac{k}{(H_0 a_0)^2}. \tag{7}$$

We thus have the following correspondences:

$$\begin{array}{llllllll}
\text{if } \rho_0 > \rho_{\rm crit} & : & \Omega > 1 & \Rightarrow & k = +1 & \Rightarrow & \text{space is positively curved} & (8) \\
\text{if } \rho_0 = \rho_{\rm crit} & : & \Omega = 1 & \Rightarrow & k = 0 & \Rightarrow & \text{space is flat} & (9) \\
\text{if } \rho_0 < \rho_{\rm crit} & : & \Omega < 1 & \Rightarrow & k = -1 & \Rightarrow & \text{space is negatively curved.} & (10)
\end{array}$$

Thus the spatial curvature of the universe is related to the current values of the density and Hubble constant. As already mentioned, current observations are most consistent with a spatially flat universe.

It is also standard to introduce the parameters

$$\Omega_m \equiv \frac{\rho_m(t_0)}{\rho_{\rm crit}}, \quad \Omega_r \equiv \frac{\rho_r(t_0)}{\rho_{\rm crit}}, \quad \Omega_\Lambda \equiv \frac{\rho_\Lambda(t_0)}{\rho_{\rm crit}} \quad (= \frac{\rho_\Lambda}{\rho_{\rm crit}} \quad \text{since } \rho_\Lambda \text{ is time-indep.}) \tag{11}$$

so that one can write

$$\Omega = \Omega_m + \Omega_r + \Omega_\Lambda. \tag{12}$$

Furthermore, one defines the parameter

$$\Omega_k \equiv -\frac{k}{(H_0 a_0)^2}. \tag{13}$$

Then the present-time Friedmann equation (7) can be written

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1. \tag{14}$$

Note that Ω_k will be negative in a universe with positive spatial curvature.

We can differentiate the Friedmann equation to see how the universe accelerates or decelerates. We get

$$2\ddot{a}\dot{a} - \frac{16\pi\rho}{3}\dot{a}a - \frac{8\pi\dot{\rho}}{3}a^2 = 0 \tag{15}$$

(Since the curvature is constant). Rearranging this, we get

$$\frac{\ddot{a}}{a} = \frac{8\pi\rho}{3} + \frac{4\pi\dot{\rho}}{3}\frac{a}{\dot{a}}. \tag{16}$$

Now we need to add matter back in. The continuity equation is defined as

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p). \tag{17}$$

Using this, we find

$$\begin{aligned}
\frac{\ddot{a}}{a} &= \frac{8\pi\rho}{3}\left(\rho - \frac{3}{2}(\rho + P)\right) \\
&= \frac{4\pi\rho}{3}(2\rho - 3\rho - 3p) \\
&= -\frac{4\pi\rho}{3}(\rho + 3p)
\end{aligned} \tag{18}$$

So the universe will decelerate in its expansion, unless the pressure of the dominant fluid $p < -\rho/3$, or alternatively the equation of state $w < -1/3$. Of the three specific fluids we have considered, this is only satisfied by the cosmological constant.

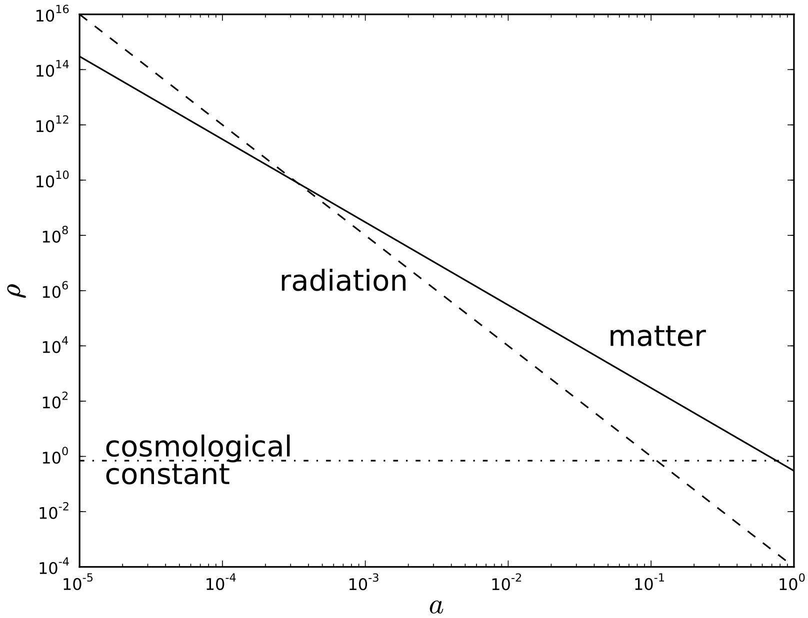

Figure 1: The energy densities for matter (ρ_m), radiation (ρ_r), and cosmological constant (ρ_Λ) as function of the scale factor a. As a grows, the dominant contribution to ρ changes from radiation, to matter, and finally to cosmological constant.

2 Solving the Friedmann equation

We will not discuss in detail the general solution of the Friedmann equation, but will limit ourselves to some simplified treatments. As established in the previous section, the separate contributions ρ_m, ρ_r, and ρ_Λ to the energy density ρ depend differently on the scale factor $a(t)$:

$$\rho_r \propto a^{-4}, \quad \rho_m \propto a^{-3}, \quad \rho_\Lambda \propto a^0. \tag{19}$$

It follows that when the scale factor (and thus the universe) is sufficiently small, the dominant contribution to the energy density comes from radiation (ρ_r). On the other hand, for sufficiently large values of a, the dominant contribution will come from cosmological constant (ρ_Λ). For intermediate values of the scale factor, matter (ρ_m) may give the dominant contribution; whether this is the case will depend on the prefactors in (19): it was the case for our own universe. Thus different contributions to the energy density dominated in different eras of our universe, as shown in Fig. 1 where also the approximate transition times between the three different eras (**radiation-dominated**, **matter-dominated**, and **Λ-dominated**) have been indicated.

In order to get a semi-quantitative understanding of the evolution of the scale factor, it is then reasonable to consider each era separately and approximate the total energy density ρ in each era with the dominant contribution only: this will give an approximate result for the time dependence of the scale factor in each era. Furthermore, since observations indicate that our universe is spatially flat, we take $k = 0$ in the Friedmann equation, which then can be written $\dot{a}^2 = \frac{8\pi}{3}\rho a^2$, i.e.

$$\dot{a} = \sqrt{8\pi/3}\,\rho^{1/2} a. \tag{20}$$

Now consider a given era and set $\rho \approx \rho_i \propto a^{-n_i}$ where ρ_i is the dominant contribution to ρ in that era, with n_i being the appropriate exponent given in (19). Inserting this into (20)

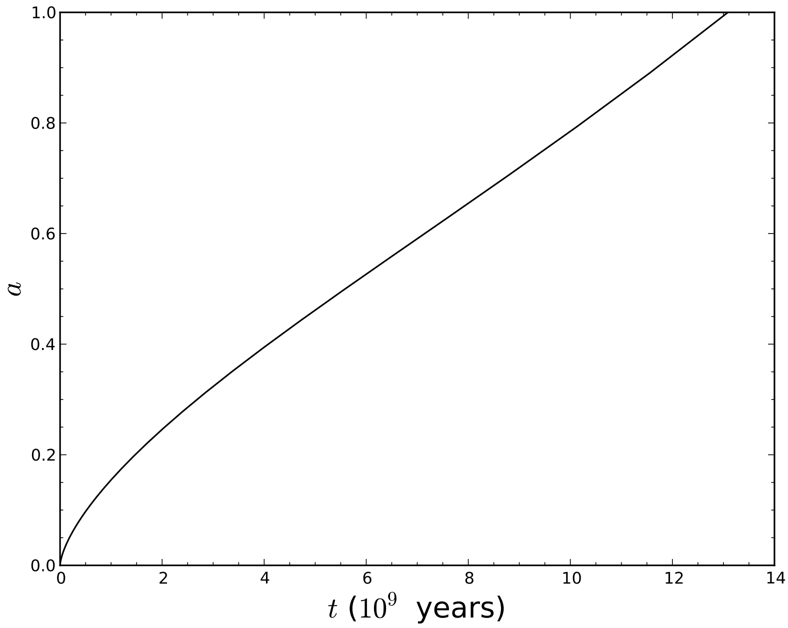

Figure 2: A plot of the evolution of the scale factor $a(t)$ as a function of time for our universe.

gives

$$\dot{a} \propto a^{1-n/2}. \tag{21}$$

Trying an ansatz $a(t) \propto t^p$ where p is an exponent to be determined, we get $\dot{a} \propto pt^{p-1}$ and $a^{1-n/2} \propto t^{p(1-n/2)}$. Equating the powers of t on both sides gives $p - 1 = p(1 - n/2)$, i.e. $p = 2/n$. This gives $p = 2/3$ for a matter-dominated universe and $p = 2/4 = 1/2$ for a radiation-dominated universe. Our ansatz breaks down in the cosmological constant-dominated case that has $n = 0$, for which the solution can instead be seen to be an exponential. Thus in summary, one finds

$$\text{Radiation-dominated universe}: \qquad a(t) \propto t^{1/2}, \tag{22}$$

$$\text{Matter-dominated universe}: \qquad a(t) \propto t^{2/3}, \tag{23}$$

$$\Lambda\text{-dominated universe}: \qquad a(t) \propto e^{Ht} \tag{24}$$

where in the last case the exponent H equals the Hubble constant, which is a true constant (i.e. time-independent) when the only contribution to the energy density comes from the cosmological constant, as can be seen from the Friedmann equation. Combining these results for the different eras of our universe leads to the (qualitative) plot of $a(t)$ as a function of time for our universe shown in Fig. 2. Note that of this plot had been drawn linearly, then it would have been impossible to see the very brief early radiation-dominated era on the plot. One sees that the solution has $a = 0$ for time $t = 0$, which is a point of infinite energy density in this model (since both ρ_r and ρ_m diverge as $a \to 0$, with ρ_r diverging most strongly), i.e. a singularity: this is the **"Big Bang."**[1]

From the Friedmann equation one can also predict the age of the universe. Let us again consider the case of a flat universe ($k = 0$) and furthermore assume a universe with only

[1]Some more sophisticated models of the evolution of the universe involve an extremely brief period of so-called **inflation** before the radiation-dominated era, during which the universe is predicted to have expanded by a colossal amount (by a factor of order e^{50} or so).

matter. This latter assumption is not too bad since our universe was matter-dominated during most of its history so far (cf. Fig. 2). Thus the solution to the Friedmann equation is $a(t) = Qt^{2/3}$ where $Q > 0$ is some constant. Since $t = 0$ corresponds to the Big Bang, the age of this universe is just the current time t_0. The Hubble constant at the current time is

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = \frac{(2/3)Qt_0^{2/3-1}}{Qt_0^{2/3}} = \frac{2}{3t_0}. \tag{25}$$

Thus the age of the matter-dominated universe is $t_0 = 2/(3H_0)$. If we use the current value of the Hubble constant which is $H_0 \approx 72$ (km/s)/Mpc, one finds $t_0 \approx 9$ billion years. Although this answer is a bit less than the best current estimate of 13.7 billion years for the age of our universe, it is in the right ballpark.

To solve the Friedmann equation in its full generality requires the input of 4 cosmological parameters: H_0, Ω_m, Ω_r, and Ω_Λ. Properties of the universe (both present, past and future) can then be predicted from these by solving the Friedmann equation. For our universe $\Omega_r \sim 10^{-5}$ is much smaller than Ω_m and Ω_Λ. Fig. 2 shows a "phase diagram" for such a universe as a function of the parameters Ω_m and Ω_Λ. Spatially flat universes ($k = 0$) lie along the line $\Omega_\Lambda = 1 - \Omega_m$ (this follows from Eq. (14) with $\Omega_k = 0$ and $\Omega_r \approx 0$) that separates regions of closed and open universes that have respectively positive and negative spatial curvature. Our universe lies on, or approximately on, this line with $\Omega_m \approx 0.3$ and $\Omega_\Lambda \approx 0.7$. Thus the cosmological constant (dark energy) gives by far the largest contribution to the energy density of the universe today. Also note that the matter contribution $\Omega_m = \Omega_\mathrm{b} + \Omega_\mathrm{CDM}$ where the first term is the contribution from ordinary 'baryonic' matter and the second term is the contribution from non-baryonic Cold Dark Matter, with $\Omega_\mathrm{b} \approx 0.04$ and $\Omega_\mathrm{CDM} \approx 0.25$, i.e. most of the matter in the universe appears to be dark matter of unknown origin. In Fig. 2 our universe is seen to be in a region of solutions that will expand forever, in contrast to regions that have the universe first expanding and then collapsing ("recollapse") (note that in most of the "recollapse" region shown the cosmological constant is negative, in contrast to our assumption in these notes). There is also a region of "bounce" solutions to the Friedmann equation that don't have a Big Bang.

3 The cosmological redshift

Consider the RW metric for a spatially flat universe:

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)]. \tag{26}$$

Pick the origin $r = 0$ to coincide with our location. Imagine an observer in a galaxy at $r = R$. Suppose this observer emits a photon with frequency ω_e at time t_e, and suppose further that we receive the photon at our location at the present time t_0. What is the frequency ω_0 of the received photon?

The photon travels on a radial null curve, for which $ds^2 = 0$ and $d\theta = d\phi = 0$. Thus

$$ds^2 = 0 = -dt^2 + a^2(t)dr^2. \tag{27}$$

Therefore $dr = -dt/a(t)$ ($dr < 0$ since the photon is moving towards smaller r). Integrating gives

$$\int_R^0 dr = -\int_{t_e}^{t_0} \frac{dt}{a(t)}, \tag{28}$$

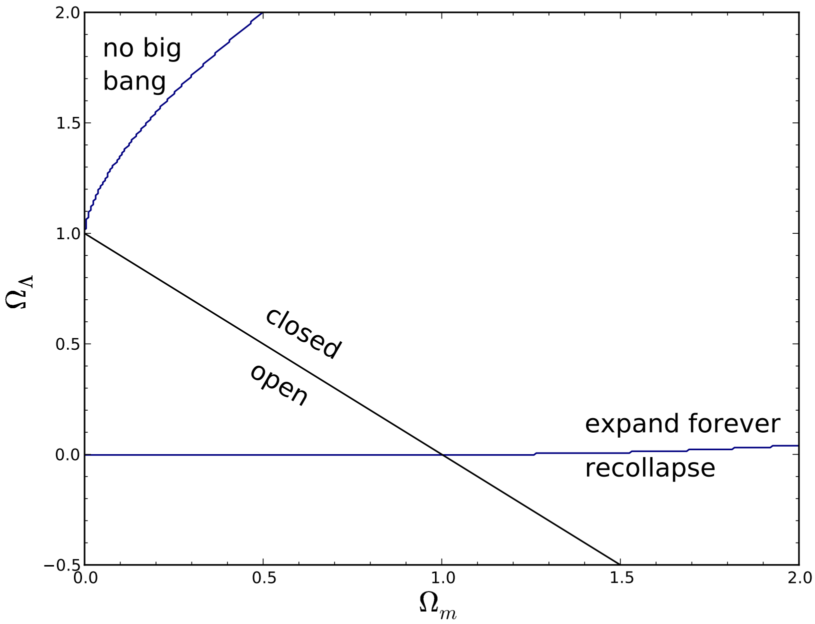

Figure 3: "Phase diagram" for FRW cosmological models with $\Omega_r \ll \Omega_m, |\Omega_\Lambda|$.

i.e.

$$R = \int_{t_e}^{t_0} \frac{dt}{a(t)}. \tag{29}$$

Suppose that the distant observer emits a series of pulses spaced by a short time interval δt_e. Since all pulses travel the distance R to get to us, the corresponding time interval δt_0 between the pulses when received by us must satisfy (by the same argument as above)

$$\int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a(t)} = R, \tag{30}$$

Inserting (29) for R and moving it to the other side gives

$$0 = \left(\int_{t_e+\delta t_e}^{t_0+\delta t_0} - \int_{t_e}^{t_0} \right) \frac{dt}{a(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{dt}{a(t)} - \int_{t_e}^{t_e+\delta t_e} \frac{dt}{a(t)} \approx \frac{\delta t_0}{a(t_0)} - \frac{\delta t_e}{a(t_e)} \tag{31}$$

where the last step follows from the smallness of the time intervals. Defining angular frequencies $\omega = 2\pi/\delta t$ for the emitted and received light, we thus see that

$$\frac{\omega_0}{\omega_e} = \frac{a(t_e)}{a(t_0)}. \tag{32}$$

Although this has been derived assuming flat space ($k = 0$), the result holds also for spatially curved universes ($k = \pm 1$). In an expanding universe (like ours) which thus has $a(t_0) > a(t_e)$, we see that (32) implies $\omega_0 < \omega_e$ and thus the wavelengths λ_e and λ_0 of the emitted and received light satisfies $\lambda_0 > \lambda_e$. In other words, the light becomes redshifted due to the expansion of the universe. This effect is called the **cosmological redshift**. Actually, one defines a parameter z called the redshift to be the ratio of the wavelength change and the original wavelength,

$$z \equiv \frac{\Delta\lambda}{\lambda_e} = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1. \tag{33}$$

Thus

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{\omega_e}{\omega_0} = \frac{a(t_0)}{a(t_e)}. \tag{34}$$

If the time difference $t_0 - t_e$ between emission and reception is sufficiently small that $a(t)$ changes little over this time interval, we have

$$R = \int_{t_e}^{t_0} \frac{dt}{a(t)} \approx \frac{t_0 - t_e}{a(t_0)}, \tag{35}$$

i.e.

$$t_e = t_0 - Ra(t_0) = t_0 - D(t_0), \tag{36}$$

where the quantity $D(t)$ is the physical distance between the two galaxies at time t, as introduced earlier in the derivation of Hubble's law. Thus our assumption that $t_0 - t_e$ be sufficiently small implies that $D(t_0)$ be sufficiently small. In the following we write $D(t_0) \equiv D_0$. We get

$$a(t_e) = a(t_0 - D_0) \approx a(t_0) - \dot{a}(t_0)D_0 \tag{37}$$

where we Taylor-expanded $a(t)$ around $t = t_0$ to leading order. Thus

$$\frac{a(t_e)}{a(t_0)} \approx 1 - \frac{\dot{a}(t_0)}{a(t_0)}D_0 = 1 - H_0 D_0. \tag{38}$$

This gives

$$z = \frac{a(t_0)}{a(t_e)} - 1 = \frac{1}{1 - H_0 D_0} - 1 \overset{H_0 D_0 \ll 1}{\approx} (1 + H_0 D_0) - 1 = H_0 D_0 = V_0 \tag{39}$$

where we used Hubble's law in the last step. Thus for a distant galaxy that is not too far away, satisfying $H_0 D_0 \ll 1$, the redshift z that we observe for light emitted from that galaxy is given by the galaxy's recession velocity V_0 (note that $z = V_0 \ll 1$ in this case). Note that although this result is superficially reminiscent of the well-known Doppler shift due to a moving source of light, the cosmological redshift z is fundamentally *not* a Doppler shift. Instead it is due to the expansion of space, which increases the wavelength of the photons *during* the journey from emitter to receiver (see the 2005 Sci Am article by Lineweaver and Davis referenced earlier).