Assignment 1

Saturday, 13 August 2022 2:03 PM

<u>Q2</u>

$E_q = q\hbar\omega \qquad q \in \{\mathbb{Z}^+, 0\}$

a. Two harmonic oscillators, each with ω and $E_q = q\hbar\omega$. Let

$$U_1 = n_1 \hbar\omega$$

be the total energy of the system.

Since the system is composed of two harmonic oscillators,

$$U_1 = E_i + E_j$$

$$= q_i\hbar\omega + q_j\hbar\omega$$
$$= (q_i + q_j)\hbar\omega$$
$$= n_1 \hbar\omega$$
$$\Rightarrow n_1 = q_i + q_j$$

Therefore there are $n_1 + 1$ possible microstates for the system of n_1 macrostates, as each of the harmonic oscillators could have any number of quanta, provided that their sum equals n_1, plus one to account for the case where one $q = 0$.
Hence, $\Omega_1 = n_1 + 1$ But $U_1 = n_1\hbar\omega$, so

$$n_1 = \frac{U_1}{\hbar\omega}$$

$$\Rightarrow \Omega_1 = \frac{U_1}{\hbar\omega} + 1$$

And since the entropy is given as

$$S_1 = k \ln \Omega_1$$

the entropy for the system (in terms of U_1) is

$$S_1 = k \ln\left(\frac{U_1}{\hbar\omega} + 1\right)$$

b. Two harmonic oscillators, each with 2ω. The total energy of the system is

$$U_2 = n_2 \hbar\omega$$

Given that it's composed of two harmonic oscillators,

$$U_2 = E_i + E_j$$

$$= q_i\hbar 2\omega + q_j \hbar 2\omega$$

$$= 2(q_i + q_j)\hbar\omega$$
$$\Rightarrow n_2 = 2(q_i + q_j)$$

Therefore, there are $q_i + q_j + 1 = \frac{n_2}{2} + 1$ possible microstates for the system, as each harmonic oscillator could have any number of quanta, provided that twice their sum equals n_2, plus 1 to account for the 0 case.

oscillator could have any number of quanta, provided that twice their sum equals n_2, plus 1 to account for the 0 case.

Therefore, $\Omega_2 = \frac{n_2}{2} + 1$

$$= \frac{U_2}{2\hbar\omega} + 1$$

$$\Rightarrow S_2 = k \ln\left(\frac{U_2}{2\hbar\omega} + 1\right)$$

c. Considering both of these systems separated by an impassable divide, the total entropy is just the sum of individual energies:

$$S_{total} = S_1 + S_2$$

$$= k \ln\left(\frac{U_1}{\hbar\omega} + 1\right) + k \ln\left(\frac{U_2}{2\hbar\omega} + 1\right)$$

$$= k \ln\left(\left[\frac{U_1}{\hbar\omega} + 1\right]\left[\frac{U_2}{2\hbar\omega} + 1\right]\right)$$

Q4

a. $2s = N_\uparrow - N_\downarrow \qquad N = N_\uparrow + N_\downarrow \Rightarrow N_\downarrow = N - N_\uparrow$

$$\Rightarrow 2s = N_\uparrow - N + N_\uparrow \qquad 2s = N - N_\downarrow - N_\downarrow$$

$$s = N_\uparrow - \tfrac{1}{2}N \qquad s = \tfrac{1}{2}N - N_\downarrow$$

$$\Rightarrow N_\uparrow = \tfrac{1}{2}N + s \qquad \Rightarrow N_\downarrow = \tfrac{1}{2}N - s$$

$$g(N,s) = \frac{N!}{N_\uparrow!\,N_\downarrow!} = \frac{N!}{(\frac{1}{2}N+s)!(\frac{1}{2}N-s)!}$$

Keep the $N_\uparrow$, $N_\downarrow$ notation. By Stirling's approximation,

$$N! \approx \sqrt{2\pi N}\, N^N e^{-N}$$

$$= \sqrt{2\pi}\, N^{N+\frac{1}{2}} e^{-N}$$

$$\Rightarrow \log N! \approx \tfrac{1}{2}\ln(2\pi) + (N+\tfrac{1}{2})\ln(N) - N$$

$$\Rightarrow \log N_\uparrow! \approx \tfrac{1}{2}\ln(2\pi) + (N_\uparrow + \tfrac{1}{2})\ln(N_\uparrow) - N_\uparrow$$

$$\log N_\downarrow! \approx \tfrac{1}{2}\ln(2\pi) + (N_\downarrow + \tfrac{1}{2})\ln(N_\downarrow) - N_\downarrow$$

But $N = N_\uparrow + N_\downarrow$, so

$$\ln N! = \tfrac{1}{2}\ln(2\pi) + (N_\uparrow + N_\downarrow + \tfrac{1}{2})\ln(N) - (N_\uparrow + N_\downarrow) + \tfrac{1}{2}\ln(N) - \tfrac{1}{2}\ln(N)$$

$$= \frac{1}{2}\ln\left(\frac{2\pi}{N}\right) + (N_\uparrow + \tfrac{1}{2} + N_\downarrow + \tfrac{1}{2})\ln(N) - (N_\uparrow + N_\downarrow)$$

Now,

$$\ln g = \ln N! - \ln N_\uparrow! - \ln N_\downarrow!$$

$$\approx \frac{1}{2}\ln\left(\frac{2\pi}{N}\right) - \ln(2\pi) + (N_\uparrow + \tfrac{1}{2})\big(\ln(N) - \ln(N_\uparrow)\big)$$

$$+ (N_\downarrow + \tfrac{1}{2})\big(\ln(N) - \ln(N_\downarrow)\big)$$

$$= \frac{1}{2}\ln\left(\frac{1}{2\pi N}\right) - (N_\uparrow + \tfrac{1}{2})\ln\left(\frac{N_\uparrow}{N}\right) - (N_\downarrow + \tfrac{1}{2})\ln\left(\frac{N_\downarrow}{N}\right)$$

Now, $\ln\left(\frac{N_\uparrow}{N}\right) = \ln\left(\frac{\frac{1}{2}N + s}{N}\right) = \ln\left(\frac{1}{2} + \frac{s}{N}\right)$

$$= \ln\left(\tfrac{1}{2}\left(1 + \tfrac{2s}{N}\right)\right)$$

$$= \ln(\tfrac{1}{2}) + \ln(1 + \tfrac{2s}{N})$$

$$= \ln\left(\frac{1}{2}\left(1+\frac{2s}{N}\right)\right)$$
$$= \ln\left(\frac{1}{2}\right) + \ln\left(1+\frac{2s}{N}\right)$$
$$= -\ln(2) + \ln\left(1+\frac{2s}{N}\right)$$

Since $|s| \ll N$, $\ln\left(1+\frac{2s}{N}\right) \approx \frac{2s}{N} - \frac{1}{2}\left(\frac{4s^2}{N^2}\right) = \frac{2s}{N} - \frac{2s^2}{N^2}$

$$\Rightarrow \ln\left(\frac{N_\uparrow}{N}\right) \approx -\ln(2) + \frac{2s}{N} - \frac{2s^2}{N^2}$$

Similarly, $\ln\left(\frac{N_\downarrow}{N}\right) \approx -\ln(2) - \frac{2s}{N} - \frac{2s^2}{N^2}$

$$\Rightarrow \ln g \approx \frac{1}{2}\ln\left(\frac{1}{2\pi N}\right) - \left(N_\uparrow + \frac{1}{2}\right)\left(-\ln(2) + \frac{2s}{N} - \frac{2s^2}{N^2}\right)$$
$$- \left(N_\downarrow + \frac{1}{2}\right)\left(-\ln(2) - \frac{2s}{N} - \frac{2s^2}{N^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{1}{2\pi N}\right) + \ln(2)\left(N_\uparrow + \frac{1}{2} + N_\downarrow + \frac{1}{2}\right) - \left(\frac{2s}{N}\right)\left(N_\uparrow + \frac{1}{2} - N_\downarrow - \frac{1}{2}\right)$$
$$+ \left(\frac{2s^2}{N^2}\right)\left(N_\uparrow + \frac{1}{2} + N_\downarrow + \frac{1}{2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{1}{2\pi N}\right) + \ln(2)\left(\frac{1}{2}N + s + \frac{1}{2} + \frac{1}{2}N - s + \frac{1}{2}\right)$$
$$- \left(\frac{2s}{N}\right)\left(\frac{1}{2}N + s + \frac{1}{2} - \frac{1}{2}N + s - \frac{1}{2}\right)$$
$$+ \left(\frac{2s^2}{N^2}\right)\left(\frac{1}{2}N + s + \frac{1}{2} + \frac{1}{2}N - s + \frac{1}{2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{1}{2\pi N}\right) + \ln(2)(N+1) - \left(\frac{2s}{N}\right)(2s) + \left(\frac{2s^2}{N^2}\right)(N+1)$$

$N+1 \approx N$

$$\ln g \approx \frac{1}{2}\ln\left(\frac{1}{2\pi N}\right) + \ln(2) + N\ln(2) - \frac{4s^2}{N} + \frac{2s^2}{N}$$
$$= \frac{1}{2}\ln\left(\frac{4}{2\pi N}\right) + N\ln(2) - \frac{2s^2}{N}$$
$$= \ln\left(\sqrt{\frac{2}{\pi N}}\right) + N\ln(2) - \frac{2s^2}{N}$$

And finally,

$$g(N,s) \approx \sqrt{\frac{2}{\pi N}} \cdot 2^N e^{-\frac{2s^2}{N}}$$
$$= g(N,0)\, e^{-\frac{s^2}{N}}$$

where

$$g(N,0) = \sqrt{\frac{2}{\pi N}}\ 2^N$$

b. $\langle x \rangle = \int_{-\infty}^{\infty} x f(x)\, dx$

$$\Rightarrow \langle s \rangle = \int_{-\infty}^{\infty} s\ 2^N \sqrt{\frac{2}{\pi N}}\, e^{-\frac{2s^2}{N}}\, ds$$

Since this is an odd function $\langle s \rangle = 0$

For the distribution to be normalised to unity,

$$A \int_{-\infty}^{\infty} 2^N \sqrt{\frac{2}{\pi N}}\, e^{-\frac{2s^2}{N}} = 1$$

Since this an even function, $\int_{-\infty}^{0} f(x) = \int_{0}^{\infty} f(x)$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) = 2 \cdot \int_{0}^{\infty} f(x)$$

$$\Rightarrow \frac{1}{A} = 2 \cdot 2^N \sqrt{\frac{2}{\pi N}} \int_0^\infty e^{-\frac{2s^2}{N}}\, ds$$

set $a = \sqrt{\frac{2}{N}}$ $\Rightarrow \int_0^\infty e^{-\frac{2s^2}{N}} = \int_0^\infty e^{-a^2 s^2} = \frac{\sqrt{\pi}}{2a} = \frac{1}{2}\sqrt{\frac{\pi N}{2}}$

$$\Rightarrow \frac{1}{A} = 2^N \quad \Rightarrow \quad A = \frac{1}{2^N}$$

And so

$$\langle s^2 \rangle = \frac{2^N}{2^N} \sqrt{\frac{2}{\pi N}} \int^\infty s^2 e^{-\frac{2s^2}{N}}\, ds$$

And so

$$\langle s^2\rangle = \frac{2^N}{2^N}\sqrt{\frac{2}{\pi N}}\int_{-\infty}^{\infty} s^2 e^{-\frac{2s^2}{N}}\,ds$$

set $a=\sqrt{\frac{2}{N}}$ as before, and this an even function, so

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^{\infty} f(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} s^2 e^{-a^2s^2}\,ds = 2\int_0^{\infty} s^2 e^{-a^2s^2}\,ds = 2\cdot\frac{\sqrt{\pi}}{4a^3}$$

$$= \frac{1}{2}\sqrt{\pi}\cdot\sqrt{\frac{N^3}{2^3}} = \frac{1}{4}\sqrt{\frac{\pi N^3}{2}}$$

$$\Rightarrow \langle s^2\rangle = \sqrt{\frac{2}{\pi N}}\cdot\frac{1}{4}\sqrt{\frac{\pi N^3}{2}} = \frac{N}{4}$$

Now, define the rms width as $\sigma = \sqrt{\langle s^2\rangle - \langle s\rangle^2}$

$$= \sqrt{\frac{N}{4} - 0}$$

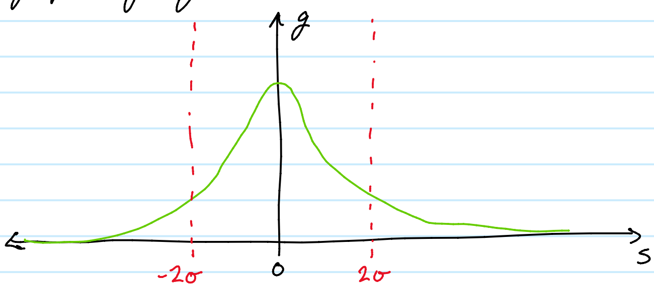
$$= \frac{\sqrt{N}}{2}$$

Define the fractional width as

$$S_w = \frac{2\sigma}{N}$$

$$= \frac{2}{N}\cdot\frac{\sqrt{N}}{2} = \frac{1}{\sqrt{N}}$$

A graph of $g(N,s)$ is shown below:



Q7

i. The number of possible five-card poker hands is given by

$$n = \binom{52}{5} = \frac{52!}{5!\cdot(52-5)!}$$

$$= 2598960$$

ii. The probability of being dealt a royal flush on the first deal is

$$\mathbb{P}(\text{Royal Flush}) = \frac{n_{Ace}}{n_{cards}} \times \frac{n_{King}}{n_{cards}-1} \times \frac{n_{Queen}}{n_{cards}-2} \times \frac{n_{Jack}}{n_{cards}-3} \times \frac{n_{10}}{n_{cards}-4}$$

$$= \frac{4}{52}\times\frac{4}{51}\times\frac{4}{50}\times\frac{4}{49}\times\frac{4}{48}$$

$$= \frac{8}{2436525} \simeq 3.28\times10^{-6}$$

Q8

a. As from Q4a, the multiplicity of a system with spin excess $2s$ is

$$\Omega(N,s) = \frac{N!}{N_\uparrow!\,N_\downarrow}$$

$$\Rightarrow \ln\Omega = \ln N! - \ln N_\uparrow! - \ln N_\downarrow!$$

$$= \ln N! - \ln N_\uparrow! - \ln(N-N_\uparrow)!$$

(since $N_\downarrow = N - N_\uparrow$)

$$\Rightarrow \ln\Omega = \ln N! - \ln N_\uparrow! - \ln N_\downarrow!$$
$$= \ln N! - \ln N_\uparrow! - \ln(N-N_\uparrow)!$$
(since $N_\downarrow = N - N_\uparrow$)

Using Stirling's approximation, this becomes

$$\ln\Omega \approx N\ln N - N - N_\uparrow \ln N_\uparrow + N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow) + (N-N_\uparrow)$$
$$= N\ln N - N_\uparrow \ln N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow)$$
$$S = k\ln\Omega = k\left(N\ln N - N_\uparrow\ln N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow)\right) \quad (1)$$

but $N_\uparrow = \frac{1}{2}N + s$

$$\Rightarrow S = k\left(N\ln(N) - \left(\tfrac{1}{2}N+s\right)\ln\left(\tfrac{1}{2}N+s\right) - \left(\tfrac{1}{2}N-s\right)\ln\left(\tfrac{1}{2}N-s\right)\right)$$

b.

Now, $2s = N_\uparrow - N_\downarrow = N_\uparrow - (N-N_\uparrow) = 2N_\uparrow - N$

and $U = -2smB = (N-2N_\uparrow)mB$

$$\Rightarrow N_\uparrow = \frac{N}{2} - \frac{U}{2mB}$$

And temperature is given by

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,V}$$

It is equivalent to write

$$\frac{\partial S}{\partial U} = \frac{\partial N_\uparrow}{\partial U}\frac{\partial S}{\partial N_\uparrow}$$

So, $$\frac{\partial N_\uparrow}{\partial U} = -\frac{1}{2mB}$$

and, looking at equation (1),

$$\frac{\partial S}{\partial N_\uparrow} = k\frac{\partial}{\partial N_\uparrow}\left(N\ln N - N_\uparrow\ln N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow)\right)$$
$$= k\left(-\ln N_\uparrow - 1 + \ln(N-N_\uparrow) + \frac{N-N_\uparrow}{N-N_\uparrow}\right)$$
$$= -k\ln\left(\frac{N_\uparrow}{N-N_\uparrow}\right)$$
$$= -k\ln\left(\frac{\frac{1}{2}N - U/2mB}{N - \frac{1}{2}N + U/2mB}\right)$$
$$= -k\ln\left(\frac{N - U/mB}{N + U/mB}\right)$$
$$\Rightarrow \frac{1}{T} = \frac{\partial N_\uparrow}{\partial U}\frac{\partial S}{\partial N_\uparrow}$$
$$= -\frac{1}{2mB}\cdot -k\ln\left(\frac{N-U/mB}{N+U/mB}\right)$$
$$= \frac{k}{2mB}\ln\left(\frac{N-U/mB}{N+U/mB}\right) \quad (2)$$

c. Rearrange (2) to yield

$$\frac{2mB}{kT} = \ln\left(\frac{N-U/mB}{N+U/mB}\right)$$
$$\frac{N-U/mB}{N+U/mB} = e^{\frac{2mB}{kT}}$$
$$\Rightarrow \frac{NmB-U}{NmB+U} = e^{\frac{2mB}{kT}}$$
$$NmB - U = e^{\frac{2mB}{kT}}(NmB+U)$$
$$NmB\left(1-e^{\frac{2mB}{kT}}\right) = Ue^{\frac{2mB}{kT}} + U$$
$$= \left(1+e^{\frac{2mB}{kT}}\right)U$$
$$\Rightarrow U = NmB\left(1-e^{\frac{2mB}{kT}}\right)$$

$$= \left(1 + e^{\frac{2mB}{kT}}\right) U$$

$$\Rightarrow \quad U = NmB \left(\frac{1 - e^{\frac{2mB}{kT}}}{1 + e^{\frac{2mB}{kT}}} \right)$$

$$\langle U \rangle = -NmB \left(\frac{e^{\frac{2mB}{kT}} - 1}{e^{\frac{2mB}{kT}} + 1} \right)$$

Since $\tanh(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1}$,

$$\langle U \rangle = -NmB \tanh\left(\frac{mB}{kT} \right)$$

d. In the high temperature limit $kT \gg mB$, $\frac{mB}{kT} \ll 1$
The Taylor series expansion of $\tanh(x)$ is

$$\tanh(x) \simeq x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots$$

So, keeping the first term of the Taylor series expansion of $\langle U \rangle$ gives

$$\langle U \rangle \simeq -NmB \frac{mB}{kT}$$

$$= -N \frac{(mB)^2}{kT} \quad \propto \quad \frac{1}{T}$$

Clearly this agrees (approximately) with Curie's Law which predicts that the net fractional magnetization decreases as $1/T$ in the high temperature limit.

e. Due to the $1/T$ factor, as $T \to 0$ we would expect $\langle U \rangle \to -\infty$, i.e. the system aligns totally with the external magnetic field.