MATH4105 Assignment 2

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Question 2

We have $\delta^n(x - x_0)$ satisfying

$$\int d^n x\ \delta^n(x - x_0)\Phi(x) = \Phi(x_0)$$

for an arbitrary scalar field $\Phi(x)$.
We know that, under a change of coordinates $x \to x'$, $\Phi'(x') = \Phi(x)$ for an arbitrary scalar field in a (pseudo) Riemannian manifold. Hence

$$\begin{aligned}\int d^n x\ \delta^n(x - x_0)\Phi(x) &= \Phi'(x_0') \\ &= \int d^n x'\ \delta'^n(x' - x_0')\Phi'(x')\end{aligned}$$

Which is the transformation law for a scalar density. Further,

$$\begin{aligned}\int d^n x'\ \delta'^n(x' - x_0')\Phi'(x') &= \int \left[\frac{\partial(x)}{\partial(x')}\right]^{-1} d^n x\ \delta'^n(x' - x_0')\Phi'(x') \\ &= \int d^n x\ \delta^n(x - x_0)\Phi(x)\end{aligned}$$

where $\left[\frac{\partial(x)}{\partial(x')}\right]^{-1}$ has weight $W = -1$. Since the left hand side and right hand side both equal $\Phi(x_0)$, and the scalar field has weight $W = 0$ (by definition), we know that the above equation holds only if $\delta^n(x - x_0)\Phi(x)$ has weight $W = 1$. Hence $\delta^n(x - x_0)\Phi(x)$ is a scalar density of weight 1. Since $\Phi(x)$ is an arbitrary scalar field, we also know that $\delta^n(x - x_0)$ is itself a scalar density of weight 1 (as we could set $\Phi(x) = 1\ \forall x$).

From the lectures, we know $\sqrt{|g(x)|} = \sqrt{g}$ is a scalar density of weight 1. Therefore, $(\sqrt{g})^{-1}$ will be a scalar density of weight -1, and so

$$\frac{1}{\sqrt{g}}\delta^n(x - x_0)$$

will be a scalar density of weight $W = 1 - 1 = 0$, and therefore is a scalar.

Question 3

From iii), we know $g_{\mu\nu;\lambda} = 0$, so

$$\begin{aligned}0 &= g_{\mu\nu,\lambda} - \Gamma_{\mu\nu}{}^{\sigma} g_{\sigma\mu} - \Gamma_{\lambda\mu}{}^{\sigma} g_{\sigma\nu} \\ \implies \Gamma_{\lambda\mu}{}^{\sigma} g_{\sigma\nu} &= g_{\mu\nu,\lambda} - \Gamma_{\lambda\nu}{}^{\sigma} g_{\sigma\mu}\end{aligned}$$

Relabelling $\lambda \longleftrightarrow \nu$, we get

$$\Gamma_{\nu\mu}{}^{\sigma} g_{\sigma\lambda} = g_{\mu\lambda,\nu} - \Gamma_{\lambda\nu}{}^{\sigma} g_{\sigma\mu}$$

From i), we have $\Gamma_{\nu\mu}{}^{\sigma} = \Gamma_{\mu\nu}{}^{\sigma}$, and from ii) we have that $g_{\sigma\lambda} = g_{\lambda\sigma}$. With these in mind, we have

$$\begin{aligned}g_{\lambda\sigma}\Gamma_{\mu\nu}{}^{\sigma} &= g_{\mu\lambda,\nu} - \Gamma_{\lambda\nu}{}^{\sigma} g_{\sigma\mu} \\ \Gamma_{\mu\nu\lambda} &= g_{\mu\lambda,\nu} - g^{\sigma\chi}\Gamma_{\lambda\nu\chi} g_{\sigma\mu}\end{aligned}$$

$$
\begin{aligned}
&= g_{\mu\lambda,\nu} - \delta^{\chi}_{\mu}\Gamma_{\lambda\nu\chi} \\
&= g_{\mu\lambda,\nu} - \Gamma_{\lambda\nu\mu} \\
&= g_{\mu\lambda,\nu} - \frac{1}{2}\left(g_{\nu\mu,\lambda} + g_{\mu\lambda,\nu} - g_{\lambda\nu,\mu}\right) \\
&= \frac{1}{2}\left(g_{\lambda\nu,\mu} + g_{\mu\lambda,\nu} - g_{\nu\mu,\lambda}\right)
\end{aligned}
$$

From ii), we can swap the first two indices in each term on the right hand side, and so

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2}\left(g_{\nu\lambda,\mu} + g_{\lambda\mu,\nu} - g_{\mu\nu,\lambda}\right)$$

Question 4

We begin by expanding the metric tensor term:

$$
\begin{aligned}
g^{\mu\nu}_{;\lambda} &= g^{\alpha\mu}g^{\beta\nu}g_{\alpha\beta;\lambda} \\
&= g^{\alpha\mu}g^{\beta\nu}\left(g_{\alpha\beta,\lambda} - \Gamma_{\lambda\alpha}{}^{\eta}g_{\eta\beta} - \Gamma_{\lambda\beta}{}^{\eta}g_{\alpha\eta}\right) \\
&= g^{\alpha\mu}g^{\beta\nu}\left(g_{\alpha\beta,\lambda} - g^{\varphi\eta}\Gamma_{\lambda\alpha\varphi}g_{\eta\beta} - g^{\varphi\eta}\Gamma_{\lambda\beta\varphi}g_{\alpha\eta}\right) \\
&= g^{\alpha\mu}g^{\beta\nu}\left(g_{\alpha\beta,\lambda} - \delta^{\varphi}_{\beta}\Gamma_{\lambda\alpha\varphi} - \delta^{\varphi}_{\alpha}\Gamma_{\lambda\beta\varphi}\right) \\
&= g^{\alpha\mu}g^{\beta\nu}\left(g_{\alpha\beta,\lambda} - \Gamma_{\lambda\alpha\beta} - \Gamma_{\lambda\beta\alpha}\right) \\
&= g^{\alpha\mu}g^{\beta\nu}\left(g_{\alpha\beta,\lambda} - \frac{1}{2}\left(g_{\alpha\beta,\lambda} + g_{\beta\lambda,\alpha} - g_{\lambda\alpha,\beta}\right) - \frac{1}{2}\left(g_{\beta\alpha,\lambda} + g_{\alpha\lambda,\beta} - g_{\lambda\beta,\alpha}\right)\right) \\
&= g^{\alpha\mu}g^{\beta\nu}\left(g_{\alpha\beta,\lambda} - \tfrac{1}{2}g_{\alpha\beta,\lambda} - \tfrac{1}{2}g_{\alpha\beta,\lambda} - \tfrac{1}{2}g_{\beta\lambda,\alpha} + \tfrac{1}{2}g_{\beta\lambda,\alpha} + \tfrac{1}{2}g_{\lambda\alpha,\beta} - \tfrac{1}{2}g_{\lambda\alpha,\beta}\right) \\
&= g^{\alpha\mu}g^{\beta\nu}\left(g_{\alpha\beta,\lambda} - g_{\alpha\beta,\lambda}\right) \\
&= 0
\end{aligned}
$$

Question 5

i. We begin with the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial x^\lambda} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial p^\lambda}\right) = 0$$

We can evaluate this term by term:

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x^\lambda} &= \frac{\partial\mathcal{L}}{\partial W}\frac{\partial W}{\partial x^\lambda} \\
&= \mathcal{L}'\frac{\partial W}{\partial x^\lambda}
\end{aligned}
$$

Where $\mathcal{L} = \mathcal{L}(W)$, and $W = \frac{1}{2}g_{\mu\nu}p^\mu p^\nu$. So,

$$
\begin{aligned}
\frac{\partial W}{\partial x^\lambda} &= \frac{\partial}{\partial x^\lambda}\left(\frac{1}{2}g_{\mu\nu}p^\mu p^\nu\right) \\
&= \frac{1}{2}g_{\mu\nu,\lambda}p^\mu p^\nu + 0 + 0 \\
&= \frac{1}{2}g_{\alpha\beta,\lambda}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}
\end{aligned}
$$

Hence

$$\frac{\partial\mathcal{L}}{\partial x^\lambda} = \mathcal{L}'\left(\frac{1}{2}g_{\alpha\beta,\lambda}\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}$$

Now, for the momentum derivative,

$$\frac{\partial\mathcal{L}}{\partial p^\lambda} = \frac{\partial\mathcal{L}}{\partial W}\frac{\partial W}{\partial p^\lambda}$$

$$
\begin{aligned}
&= \mathcal{L}' \frac{\partial}{\partial p^\lambda}\left(\frac{1}{2} g_{\mu\nu} p^\mu p^\nu\right) \\
&= \mathcal{L}' \cdot \frac{1}{2}\left(g_{\mu\nu}\delta^\nu_\lambda p^\mu + g_{\mu\nu} p^\mu \delta^\nu_\lambda\right) \\
&= \mathcal{L}' \cdot g_{\lambda\mu} p^\mu \\
\implies \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial p^\lambda}\right) &= \frac{d}{dt}\left(\mathcal{L}' \cdot g_{\lambda\mu} p^\mu\right) \\
&= \frac{d}{dt}(\mathcal{L}') g_{\lambda\mu} p^\mu + \mathcal{L}' \frac{d}{dt}\left(g_{\lambda\mu} p^\mu\right) \\
&= \frac{\partial \mathcal{L}'}{\partial W}\frac{dW}{dt} g_{\lambda\mu} p^\mu + \mathcal{L}' g_{\lambda\mu}\frac{d^2 x^\mu}{dt^2} + \mathcal{L}' \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{\partial}{\partial x^\nu} g_{\lambda\mu} \\
&= \mathcal{L}'' \frac{dW}{dt} g_{\lambda\mu}\frac{dx^\mu}{dt} + \mathcal{L}' g_{\lambda\mu}\frac{d^2 x^\mu}{dt^2} + \mathcal{L}' g_{\lambda\beta,\alpha}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}
\end{aligned}
$$

Putting these together, we get

$$
\frac{\partial \mathcal{L}}{\partial x^\lambda} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial p^\lambda}\right) = 0
$$

$$
\begin{aligned}
0 &= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial p^\lambda}\right) - \frac{\partial \mathcal{L}}{\partial x^\lambda} \\
&= \mathcal{L}' g_{\lambda\mu}\frac{d^2 x^\mu}{dt^2} + \mathcal{L}'' g_{\lambda\mu}\frac{dW}{dt}\frac{dx^\mu}{dt} + \mathcal{L}'\left(g_{\lambda\beta,\alpha} - \tfrac{1}{2} g_{\alpha\beta,\lambda}\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}
\end{aligned}
$$

ii. Along the extremal curve, we have $t \to s$ and $dW/ds = 0$. So,

$$
\begin{aligned}
0 &= \mathcal{L}' g_{\lambda\mu}\frac{d^2 x^\mu}{dt^2} + \mathcal{L}'\left(g_{\lambda\beta,\alpha} - \tfrac{1}{2} g_{\alpha\beta,\lambda}\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} \\
&= g_{\lambda\mu}\frac{d^2 x^\mu}{dt^2} + \left(g_{\lambda\beta,\alpha} - \tfrac{1}{2} g_{\alpha\beta,\lambda}\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} \\
&= g^{\lambda\mu} g_{\lambda\mu}\frac{d^2 x^\mu}{dt^2} + g^{\lambda\mu}\left(g_{\lambda\beta,\alpha} - \tfrac{1}{2} g_{\alpha\beta,\lambda}\right)\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}
\end{aligned}
$$

Now, the Christoffel symbol is defined as

$$
\begin{aligned}
\Gamma_{\alpha\beta}{}^\mu &= g^{\lambda\mu}\Gamma_{\alpha\beta\lambda} \\
&= \frac{1}{2} g^{\lambda\mu}\left(g_{\beta\lambda,\alpha} + g_{\lambda\alpha,\beta} - g_{\alpha\beta,\lambda}\right) \\
\Gamma_{\alpha\beta}{}^\mu \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} &= g^{\lambda\mu}\left(\tfrac{1}{2} g_{\beta\lambda,\alpha} + \tfrac{1}{2} g_{\lambda\alpha,\beta} - \tfrac{1}{2} g_{\alpha\beta,\lambda}\right)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}
\end{aligned}
$$

Since we're contracting over both α and β indices,

$$
\left(\tfrac{1}{2} g_{\beta\lambda,\alpha} + \tfrac{1}{2} g_{\lambda\alpha,\beta}\right)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = g_{\lambda\beta,\alpha}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}
$$

and so finally,

$$
\begin{aligned}
\Gamma_{\alpha\beta}{}^\mu \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} &= g^{\lambda\mu}\left(g_{\lambda\beta,\alpha} - \tfrac{1}{2} g_{\alpha\beta,\lambda}\right)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \\
\implies \frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}{}^\mu \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} &= 0
\end{aligned}
$$

as desired.

Question 6

We're given that the metric of a 2-torus is given by

$$
ds^2 = (R + a\cos\theta)^2 d\phi^2 + a^2 d\theta^2
$$

where R, $a > 0$ are constants and $0 \le \phi \le 2\pi$ and $0 \le \theta \le 2\pi$.

i. From the metric above, we can deduce that

$$\begin{aligned} g_{\phi\phi} &= (R + a\cos\theta)^2; \qquad g_{\phi\theta} = g_{\theta\phi} = 0; \qquad g_{\theta\theta} = a^2 \\ \implies g^{\phi\phi} &= \frac{1}{(R+a\cos\theta)^2}; \qquad g^{\theta\theta} = \frac{1}{a^2} \end{aligned}$$

Now we use the expanded form of the Christoffel symbol to find the non-zero terms:

$$\begin{aligned} \Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(g_{\nu\lambda,\mu} + g_{\lambda\mu,\nu} - g_{\mu\nu,\lambda}) \\ \Rightarrow \Gamma_{\mu\nu\theta} &= \begin{cases} -\frac{1}{2}g_{\phi\phi,\theta} & \mu = \nu = \phi \\ 0 & \text{otherwise} \end{cases} \\ \Gamma_{\mu\nu\phi} &= \begin{cases} \frac{1}{2}g_{\phi\phi,\theta} & \nu = \phi,\ \mu = \theta \text{ or } \mu = \theta,\ \nu = \phi \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} \Gamma_{\phi\phi\theta} &= -\frac{1}{2}\frac{d}{d\theta}(R + a\cos\theta)^2 \\ &= -\frac{1}{2}\left[-2aR\sin\theta - 2a^2\sin\theta\ \cos\theta\right] \\ &= aR\sin\theta + a^2\sin\theta\ \cos\theta \end{aligned}$$

and

$$\Gamma_{\theta\phi\phi} = \Gamma_{\phi\theta\phi} = -aR\sin\theta - a^2\sin\theta\ \cos\theta$$

where all others are zero. Finally,

$$\begin{aligned} \Gamma_{\phi\phi}{}^{\theta} &= g^{\theta\theta}\Gamma_{\phi\phi\theta} \\ &= \frac{1}{a^2}(aR\sin\theta + a^2\sin\theta\ \cos\theta) \\ &= \frac{1}{a}\sin\theta(R + \cos\theta) \\ \Gamma_{\theta\phi}{}^{\phi} &= \Gamma_{\phi\theta}{}^{\phi} = g^{\phi\phi}\Gamma_{\theta\phi\phi} = g^{\phi\phi}\Gamma_{\phi\theta\phi} \\ &= \frac{1}{(R+a\cos\theta)^2}(-aR\sin\theta - a^2\sin\theta\ \cos\theta) \\ &= \frac{-a\sin\theta(R+a\cos\theta)}{(R+a\cos\theta)^2} \\ &= -\frac{a\sin\theta}{R+a\cos\theta} \end{aligned}$$

with all other Christoffel symbols being 0.

ii. We have a contravariant vector field a^i undergoing parallel transport along the curve

$$\theta = \pi/4 \qquad \text{from } \phi = 0 \to 2\pi$$

That is, $(\theta(t), \phi(t)) = (\pi/4,\ t)$. We have that $a^\theta = 0$, and $a^\phi = 1$ at $\phi = 0$.
Our parallel transport equation is

$$0 = \frac{Da^i}{Dt} = \frac{da^i}{dt} + \Gamma_{jk}{}^{i}a^j\frac{dx^k}{dt}$$

Hence for a^θ,

$$\begin{aligned} 0 = \frac{Da^\theta}{Dt} &= \frac{da^\theta}{dt} + \Gamma_{\phi\phi}{}^{\theta}a^\phi\frac{d\phi}{dt} \\ &= \frac{da^\theta}{dt} + \frac{1}{a}\sin\theta(R + a\cos\theta)a^\phi \\ \implies \frac{da^\theta}{dt} &= \frac{1}{a\sqrt{2}}\left(R + \frac{a}{\sqrt{2}}\right)a^\phi \end{aligned}$$

$$= -a^\phi \left(\frac{2R + a\sqrt{2}}{2a\sqrt{2}} \right)$$

For a^ϕ,

$$\begin{aligned}
0 = \frac{Da^\phi}{Dt} &= \frac{da^\phi}{dt} + \Gamma_{\theta\phi}{}^\phi a^\theta \frac{d\phi}{dt} + \Gamma_{\phi\theta}{}^\phi a^\phi \frac{d\theta}{dt} \\
\implies \frac{da^\phi}{dt} &= \frac{a\sin\theta}{R + a\cos\theta} a^\theta \\
&= \left(\frac{a}{R\sqrt{2} + a} \right) a^\theta \\
\implies \frac{d^2 a^\phi}{dt^2} &= \frac{d}{dt} \left(- \left[\frac{a}{R\sqrt{2} + a} \right] a^\theta \right) \\
&= - \left(\frac{a}{R\sqrt{2} + a} \right) \frac{da^\theta}{dt} \\
&= - \left(\frac{2aR + a^2\sqrt{2}}{4aR + 2a^2\sqrt{2}} \right) a^\phi \\
&= -\frac{1}{2} a^\phi
\end{aligned}$$

Which has solution

$$\begin{aligned}
a^\phi(t) &= A \sin\left(\tfrac{1}{\sqrt{2}} t\right) + B \cos\left(\tfrac{1}{\sqrt{2}} t\right) \\
\implies a^\theta(t) &= \frac{R\sqrt{2} + a}{a} \frac{da^\phi}{dt} \\
&= \left(\frac{R\sqrt{2}}{a} + 1 \right) \left[\frac{A}{\sqrt{2}} \cos\left(\tfrac{1}{\sqrt{2}} t\right) - \frac{B}{\sqrt{2}} \sin\left(\tfrac{1}{\sqrt{2}} t\right) \right]
\end{aligned}$$

When $t = 0$, $a^\theta = 0$, and so

$$0 = \left(\frac{R\sqrt{2}}{a} + 1 \right) \left(\frac{A}{\sqrt{2}} \right) \implies A = 0$$

When $t = 0$, we have that $a^\phi = 1$, and so $1 = B$. Hence,

$$\begin{aligned}
a^\theta(t) &= - \left(\frac{R\sqrt{2}}{a} + 1 \right) \cdot \frac{1}{\sqrt{2}} \sin\left(\frac{1}{\sqrt{2}} t \right) \\
a^\phi(t) &= \cos\left(\frac{1}{\sqrt{2}} t \right)
\end{aligned}$$

Finally, when $\phi = 2\pi = t$, we have

$$\begin{aligned}
a^\theta(2\pi) &= -\frac{1}{\sqrt{2}a} (R\sqrt{2} + a) \sin(\sqrt{2}\pi) \\
a^\phi(2\pi) &= \cos(\sqrt{2}\pi)
\end{aligned}$$

as desired.

iii. The Riemann-Christoffel tensors are

$$\begin{aligned}
R^\theta{}_{\phi\theta\phi} &= \frac{\partial \Gamma_{\phi\phi}{}^\theta}{\partial\theta} - \frac{\partial \Gamma_{\phi\theta}{}^\theta}{\partial\phi} + \Gamma_{\phi\phi}{}^\rho \Gamma_{\rho\theta}{}^\theta - \Gamma_{\phi\theta}{}^\rho \Gamma_{\rho\phi}{}^\theta \\
&= \frac{\partial}{\partial\theta} \left(\frac{1}{a} \sin\theta (R + a\cos\theta) - 0 + 0 - \left(\Gamma_{\phi\theta}{}^\theta \Gamma_{\theta\phi}{}^\theta + \Gamma_{\phi\theta}{}^\phi \Gamma_{\phi\phi}{}^\theta \right) \right) \\
&= \frac{R\cos\theta}{a} - \sin^2\theta + \cos^2\theta - \left(-\frac{a\sin\theta}{R + a\cos\theta} \cdot \frac{1}{a} \sin\theta (R + a\cos\theta) \right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{R\cos\theta}{a} - \sin^2\theta + \cos^2\theta + \sin^2\theta \\
&= \frac{R\cos\theta}{a} + \cos^2\theta \\
&= \frac{1}{a}\cos\theta(R + a\cos\theta)
\end{aligned}
$$

and

$$
\begin{aligned}
R^{\phi}{}_{\theta\phi\theta} &= \frac{\partial \Gamma_{\theta\theta}{}^{\phi}}{\partial\phi} - \frac{\partial \Gamma_{\theta\phi}{}^{\phi}}{\partial\phi} + \Gamma_{\theta\theta}{}^{\rho}\Gamma_{\rho\phi}{}^{\phi} - \Gamma_{\theta\phi}{}^{\rho}\Gamma_{\rho\theta}{}^{\phi} \\
&= 0 - \frac{\partial}{\partial\theta}\left(-\frac{a\sin\theta}{R + a\cos\theta}\right) + 0 - \left(\Gamma_{\theta\phi}{}^{\theta}\Gamma_{\theta\theta}{}^{\phi} + \Gamma_{\theta\phi}{}^{\phi}\Gamma_{\phi\theta}{}^{\phi}\right) \\
&= \frac{a(a\sin^2\theta + a\cos^2\theta + R\cos\theta)}{(R + a\cos\theta)^2} - \left(0 + \left(-\frac{a\sin\theta}{R + a\cos\theta}\right)^2\right) \\
&= \frac{a^2\sin^2\theta + a^2\cos^2\theta + aR\cos\theta - a^2\sin^2\theta}{(R + a\cos\theta)^2} \\
&= \frac{a^2\cos^2\theta + aR\cos\theta}{(R + a\cos\theta)^2} \\
&= \frac{a\cos\theta}{R + a\cos\theta}
\end{aligned}
$$

The Ricci tensors are then

$$
\begin{aligned}
R_{\theta\phi\theta\phi} &= g_{\theta i}R^{i}{}_{\phi\theta\phi} = g_{\theta\theta}R^{\theta}{}_{\phi\theta\phi} + 0 \\
&= a^2\left(\frac{1}{a}\cos\theta(R + a\cos\theta)\right) \\
&= a\cos\theta(R + a\cos\theta)
\end{aligned}
$$

and

$$
\begin{aligned}
R_{\phi\theta\phi\theta} &= g_{\phi i}R^{i}{}_{\theta\phi\theta} = g_{\phi\phi}R^{\phi}{}_{\theta\phi\theta} \\
&= (R + a\cos\theta)^2\left(\frac{a\cos\theta}{R + a\cos\theta}\right) \\
&= a\cos\theta(R + a\cos\theta) \\
&= R_{\theta\phi\theta\phi}
\end{aligned}
$$

So,

$$
\begin{aligned}
R_{\theta\theta} &= g^{ji}R_{i\theta j\theta} \\
&= g^{\phi\phi}R_{\phi\theta\phi\theta} \\
&= \frac{a\cos\theta}{R + a\cos\theta} \\
R_{\phi\phi} &= g^{ji}R_{i\phi j\phi} \\
&= g^{\theta\theta}R_{\theta\phi\theta\phi} \\
&= \frac{1}{a}\cos\theta(R + a\cos\theta)
\end{aligned}
$$

where all others are equal to zero. Finally, the scalar tensor is

$$
\begin{aligned}
R &= g^{ij}R_{ij} \\
&= g^{\theta\theta}R_{\theta\theta} + 0 + 0 + g^{\phi\phi}R_{\phi\phi} \\
&= \frac{1}{a^2}\cdot\frac{a\cos\theta}{R + a\cos\theta} + \frac{1}{(R + a\cos\theta)^2}\cdot\frac{1}{a}\cos\theta(R + a\cos\theta) \\
&= \frac{\cos\theta}{aR + a^2\cos\theta} + \frac{\cos\theta}{aR + a^2\cos\theta} \\
R &= \frac{2\cos\theta}{aR + a^2\cos\theta}
\end{aligned}
$$

Question 7

To check that the Weyl tensor satisfies $g^{\lambda\sigma} C_{\lambda\mu\nu\sigma} = 0$, we expand and multiply through by the metric:

$$\begin{aligned}
g^{\lambda\sigma} C_{\lambda\mu\nu\sigma} &= g^{\lambda\sigma} R_{\lambda\mu\nu\sigma} - \frac{1}{n-2}\left(g^{\lambda\sigma} g_{\lambda\nu} R_{\mu\sigma} - g^{\lambda\sigma} g_{\lambda\sigma} R_{\mu\nu} - g^{\lambda\sigma} g_{\mu\nu} R_{\lambda\sigma} + g^{\lambda\sigma} g_{\mu\sigma} R_{\lambda\nu}\right) + \frac{R}{(n-1)(n-2)}\left(g^{\lambda\sigma} g_{\lambda\nu} g_{\mu\sigma} - g^{\lambda\sigma} g_{\lambda\sigma} g_{\mu\nu}\right) \\
&= -g^{\lambda\sigma} R_{\mu\lambda\nu\sigma} - \frac{1}{n-2}\left(\delta^{\sigma}_{\nu} R_{\mu\sigma} - \sigma^{\lambda}_{\lambda} R_{\mu\nu} - g_{\mu\nu} R + \delta^{\lambda}_{\mu} R_{\lambda\nu}\right) + \frac{R}{(n-1)(n-2)}\left(\delta^{\sigma}_{\nu} g_{\mu\sigma} - \delta^{\lambda}_{\lambda} g_{\mu\nu}\right) \\
&= -R_{\mu\nu} - \frac{1}{n-2}\left(R_{\mu\nu} - nR_{\mu\nu} - g_{\mu\nu} R + R_{\mu\nu}\right) + \frac{R}{(n-1)(n-2)}\left(g_{\mu\nu} - n g_{\mu\nu}\right) \\
&= -R_{\mu\nu} - \frac{1}{n-2}\left((2-n) R_{\mu\nu} - g_{\mu\nu} R\right) - \frac{g_{\mu\nu} R}{n-2} \\
&= -R_{\mu\nu} - \frac{(2-n) R_{\mu\nu}}{n-2} \\
&= -R_{\mu\nu} + R_{\mu\nu} \\
&= 0
\end{aligned}$$

And so we conclude that the Weyl tensor does indeed satisfy the desired identity.