PHYS4080 Content Notes

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Semester 2, 2023

1 Week 1

Ch 8.7 – Cosmological Parameters

8.7.1 – The Standard Cosmological Model from CMB Measurements

The base model. According to the model of inflation in the early universe, our Universe is expected to be spatially flat. Such a model is described by a minimum of six parameters. Two of them characterise the initial density fluctuations (namely an amplitude A and the power-law slope n_s). As a third parameters, the post-reionisation scattering optical depth τ needs to be chosen. The remaining three parameters describe the energy contents and scale of the universe. The three typically used are $\omega_b \equiv \Omega_b h^2$ (subscript b for baryonic matter), $\omega_c \equiv \Omega_c h^2$ (where $\Omega_c = \Omega_m - \Omega_b$ is the density parameter of cold dark matter), and Ω_Λ.

8.7.2 – Consistency and Discrepancies with Other Measurements

Baryonic Acoustic Oscillations. The results from BAO studies of galaxy distributions is perfectly compatible with the cosmological model parameters as determined by WMAP. Furthermore, one can also compare the observed power spectrum of galaxies with the predictions from the standard model (using the best-fit parameters). The agreement again is excellent on large scales; on smaller scales, slight discrepancies appear which mostly likely are due to non-linear effects in structure formation, and thus also in the clustering of galaxies.

8.7.3 – Extensions of the Standard Model

Although the standard model, defined by the six basic parameters mentioned earlier, provides an excellent fit to the CMB data – temperature fluctuation power spectrum out to $l = 2500$, low-l polarisation fluctuations, and the CMB lensing effects – it is worth generalising this model by considering extensions in various ways.

Curvature. The standard model assumes spatial flatness. The CMB data, together with the lensing information and BAO studies, gives a spatial flatness estimate of $(\Omega_m + \Omega_\Lambda - 1) = 0.10^{+0.65}_{-0.62}\%$. Hence, there is very little room for generalising the model away from flatness.

Number of Neutrino Families and their Masses. The standard model assumes that there are three families of neutrinos, and that they are essentially massless. Relaxing one or both of these assumptions, we can see whether the CMB data preferences a non-standard picture of the neutrinos. If the sum of the neutrino masses is significantly larger than the minimum mass required from neutrino oscillations, then they would contribute to the current matter density in the Universe in the form of hot dark matter, and thus affect the shape of the power spectrum. If the number of neutrino families is larger than three, there would be a larger radiation component in the early universe, changing the expansion law and the epoch of matter-radiation equality. All of this affects the CMB fluctuations.
The CMB and BAO data together yield $\sum m_\nu < 0.28$ eV, and $N_{\text{eff}} = 3.32^{+0.54}_{-0.52}$. This result confirms our picture of particle physics, according to which there are three families of leptons, and thus three kinds of neutrinos.

Big Bang Nucleosynthesis. The standard model assumes that the helium abundance following the first few minutes after the Big Bang is given by the theory of nucleosynthesis, and is therefore a function of $\omega_b \equiv \Omega_b h^2$. This helium abundance then determines the evolution of the number density of free electrons before recombination, since helium recombines earlier than hydrogen because of its higher ionization potential. The value of ω_b, as determined from CMB anisotropies, is in excellent agreement with the results obtained from the observed abundance of helium and deuterium in the Universe. Leaving the helium abundance as a free parameter and using the CMB data to directly predict it, one obtains a result in excellent agreement to BBN, and hence very strongly rules out any exotic model that wants to explain the current helium contents of the Universe solely by nuclear fusion in stars.

8.7.4 – Cosmic Harmony

Hubble Constant. H_0 was determined by means of the distance ladder, particularly using Cepheids. All of the methods of measuring H_0 suggest that the Hubble constant is within 5% of 71 km s^{-1}Mpc^{-1}, or $h \approx 1/\sqrt{2}$.
Contribution of Baryons to the Total Matter Density. The ratio Ω_b/Ω_m is determined from the baryon fraction in clusters of galaxies, from redshift surveys of galaxies, and from the CMB fluctuations, all yielding $\Omega_b/\Omega_m \approx 0.15$.

Ch 8.8 – Dark Energy: Cosmological Constant, or Something Else?

The cosmological constant and Einstein's field equation. Einstein's field equation of general relativity can be expressed as

$$G = T \tag{8.43}$$

where G (called the Einstein tensor) describes the curvature of spacetime and thus the effects of gravity, whereas T (the so-called energy-momentum tensor) contains information about the matter and energy density. As it stands, equation (8.43) does not allow a static cosmos, and so Einstein modified his equation to include the cosmological constant which then reads

$$G - \Lambda = T \tag{8.44}$$

With this modification, one can construct a static model. A modern interpretation of the cosmological constant is obtained by slightly rewriting (8.44) as

$$G = T + \Lambda \tag{8.45}$$

where now the Λ-term is seen as a contribution to the source of the gravitational field; it has the same structure as one would get from a constant, uniform vacuum energy density. The difference of the interpretation is that instead of Λ

being a modification of the laws of gravity, it adds a new energy component as a source of gravity.

Time-varying dark energy. There are ideas that the value of the vacuum energy density is actually not a constant in time, but that it may vary. In that case, the dark energy density would be a function of cosmic epoch, $\rho_{\rm DE}(a)$. The question of whether dark energy is compatible with a cosmological constant, or has more complicated properties, is a very interesting one: if it were not constant in time, then it must have a more dynamical origin which would clearly argue against it being another fundamental constant of nature. One therefore considers it as a possible variant of the cosmological constant an equation-of-state of dark energy of the form

$$P_{\rm DE} = w\rho_{\rm DE}c^2 \tag{8.46}$$

where $w = -1$ corresponds to the cosmological constant. In order for this component to potentially lead to an accelerated expansion, the second Friedmann equation requires $w < -1/3$.

In fact, the equation-of-state parameter w does not need to be a constant, and can vary with cosmic epoch. Can one observationally distinguish between the case $w = -1$ for a cosmological constant (or a vacuum energy density that is indistinguishable from a cosmological constant) and the more interesting, dynamical case of $w \neq -1$?. The first major impact of $w \neq -1$ on cosmology would be a change of the expansion rate $H(a)$ of the universe. Second, the growth rate of structure would be affected for a dynamical w.

Lecture 2

Dark matter is the only way to consistently explain rotation curves of galaxies (and the velocity dispersions in elliptical galaxies), the magnitude of gravitational lensing, and cosmological data (which says that $\Omega_{\rm matter} \simeq 0.27$, where big bang nucleosynthesis says $\Omega_{\rm baryonic} \approx 0.04$ and hence $\Omega_{\text{non-baryonic}} \sim 5\Omega_{\rm baryons}$).

What we know about dark matter

Dark matter as we know it must be massive (because of its gravitational interaction), unable to interact via electromagnetism (and hence it's dark), non-baryonic (lest BBN would have accounted for it), cold-ish (in order to allow large scale structure formation), stable on cosmological timescales (otherwise it would have decayed by now), and produced in the right relic abundance in the early universe.

Primordial black holes, massive compact halo objects, and standard model neutrinos all come with their own problems as descriptors of dark matter (MACHOs would be baryonic, and SM neutrinos are too warm). Hence, we turn mainly to Weakly Interacting Massive Particles and axions to describe it.

We don't know what mass these particles are (the possible mass range spans 90 orders of magnitude), their interactions, coupling, etc although many of these properties have been constrained by observations (or lack thereof).

WIMPs

WIMPs as we hypothesise them are dark (no electromagnetic interactions), cold (as they're very massive [$\sim$ 10 GeV to $\sim$ 10 TeV]), non-baryonic and stable (and so no problems with BBN or the CMB). The weak force interaction implies some scattering with atomic nuclei, and many WIMPs are Majorana particles (i.e. their own antiparticle) and so their self-annihilation provides a detection channel.

Dark Matter Density Profiles

Rotation curves place good constraints on the density profiles of dark matter within galaxies and clusters, and essentially tell us that dark matter is gravitationally bound, more or less frictionless, and comprise dark matter halos within galaxies. N-body simulations of dark matter halo formation suggesta universal Navarro-Frenk-White profile

$$\rho(r) = \frac{\delta_c\rho_c}{\frac{r}{r_s}(1+\frac{r}{r_s})^2}$$

or Einasto profile

$$\rho(r) = \rho_s \exp\left(-2n\left[(r/r_s)^{1/n} - 1\right]\right)$$

These profiles may be steepened in the innermost regions by adiabatic contraction or softened by baryonic effects. For low mass galaxies in particular, there is some discrepancy between simulations and the data with sims implying a "cusp" (higher density DM) at small radius, but data implies a "core" (flattening profile) at low radius.

The galactic dark matter halo is *isothermal* to a first approximation and so a WIMP wind is predicted as the solar system moves through the randomly oriented DM particle motion.

Thermal Production

A particle with a larger interaction cross section, $\langle\sigma v\rangle$, can withstand more universe expansion before its abundance 'freezes out' (i.e. its abundance stabilises).

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Ch 7.5 – Non-Linear Structure Evolution

7.5.1 – Model of Spherical Collapse

Assumptions. We consider a spherical region in an expanding universe, with its density $\rho(t)$ enhanced compared to the mean cosmic density $\overline{\rho}(t)$,

$$\rho(t) = [1+\delta(t)]\overline{\rho}(t) \tag{7.46}$$

For reasons of simplicity we assume that the density within the sphere is homogeneous although, as we'll later see, this is

not really a restriction. The density perturbation is assumed to be small for small t so that it will grow linearly at first, $\delta(t) \propto D_+(t)$, as long as $\delta \ll 1$. If we consider a time t_i which is sufficiently early such that $\delta(t_i) \ll 1$, then according to the definition of the growth factor D_+, $\delta(t_i) = \delta_0 D_+(t_i)$, where δ_0 is the density contrast linearly extrapolated to the present day. It should be mentioned once again that $\delta_0 \neq \delta(t_0)$, because the latter is in general affected by the non-linear evolution.
Let R be the initial *comoving* radius of the overdense sphere; as long as $\delta \ll 1$, the comoving radius will change only marginally. The mass within the sphere is

$$M = \frac{4\pi}{3} R^3 \rho_0 (1 + \delta_i) \approx \frac{4\pi}{3} R^3 \rho_0 \tag{7.47}$$

Evolution. Due to the enhanced gravitational force, the sphere will expand slightly more slowly than the universe as a whole, which will lead to an increase in its density contrast. This then decelerates the expansion rate (relative to the cosmic rate) even further. If the initial mass density within the sphere is sufficiently large, the expansion of the sphere will come to a halt, i.e., its proper radius $R_{\rm phys}(t)$ will reach a maximum; after this, the sphere will recollapse. If $t_{\rm max}$ is the time of maximum expansion, then the sphere will, theoretically, collapse to a single point at time $t_{\rm coll} = 2t_{\rm max}$. This relation follows from the time reversal symmetry of the equation of motion.

Special case: Einstein-de Sitter model. In the special case of $\Omega_m = 1$, $\Omega_\Lambda = 0$, the Friedmann equations can be solved to yield a recollapse time at the present epoch corresponding to an overdensity of

$$\delta_c \simeq 1.69 \tag{7.48}$$

More generally, for a recollapse before redshift z, one needs $\delta_0 \geq \delta_c(1+z)$ (although the critical overdensity is cosmology dependent).

Violent Relaxation and Virial Equilibrium. Of course, the sphere will not really collapse to a single point. In reality, small scale density and gravitational fluctuation will exist within such a sphere which lead to derivations in the particles' tracks from perfectly radial orbits. The particles will scatter on these fluctuations in the gravitational field and will virialize; this process occurs on short time-scales – roughly the dynamical time-scale (i.e. the time it takes for particles to fully cross the sphere). In this case, the virialization is essentially complete at $t_{\rm coll}$. After that, the sphere will be in virial equilibrium and its average density will be $(1 + \delta_{\rm vir}) \sim 178$. A conclusion from this consideration is that a massive galaxy cluster with a virial radius of $1.5h^{-1}$Mpc must have formed from the collapse of a region that originally had a comoving radius larger by about an order of magnitude. Such a virialized mass concentration of dark matter is called a ***dark matter halo***.

7.5.2 – Number Density of Dark Matter Halos

The Mass Spectrum. We consider the number density of relaxed dark matter halos in the universe, $n(M, z)$, which is a function of mass M and redshift z.
We find that $n(M, z)$ is a decreasing function of halo mass. For large M, $n(M, z)$ decreases exponentially because sufficiently high peaks (of the initial mass function) become very rare for large distance scales. Therefore, *very* few clusters with $M \gtrsim 2 \times 10^{15} M_\odot$ exist today. At higher redshift, the cut-off in the abundance is at smaller masses, so that massive clusters are expected to be increasingly rare at higher z.

Hierarchical Structure Formation. The Press-Schechter model (i.e. the spherical collapse formation of DM halos) describes a very general property of structure formation in a CDM model, namely that low-mass structure (like galaxy-mass dark halos) format at early times, whereas large mass accumulations evolve only later. This is an example of a ***bottom-up scenario***, in which small structures that form early merge to form large structures.

Ch 7.6 – Properties of Dark Matter Halos

7.6.1 – Profile of Dark Matter Halos

Dark matter halos can be identified in mass distributions generated by numerical simulations. If we define a halo as a spherical region within which the average density is ~ 200 times the critical density at the respective redshift, the mass M of the halo is related to its (virial) radius r_{200} by

$$M = \frac{100 r_{200}^3 H^2(z)}{G} \tag{7.56}$$

so that each redshift, a unique relation exists between the halo mass and its radius. We can also define the virial velocity V_{200} of a halo as the circular velocity at the virial radius

$$V_{200}^2 = \frac{GM}{r_{200}} \tag{7.57}$$

We can combine the two above equations to express the halo mass and virial radius as a function of the virial velocity,

$$M = \frac{V_{200}^3}{10GH(z)}; \qquad r_{200} = \frac{V_{200}}{10H(z)} \tag{7.58}$$

Since the Hubble function $H(z)$ increases with redshift, the virial radius at fixed virial velocity decreases with redshift. We also see that r_{200} decreases with redshift at fixed halo mass. Hence, halos at a given mass (or given virial velocity) are more compact at higher redshift than they are today because the critical density was higher in the past.

The NFW Profile. The ***density profile*** of halos averaged over spherical shells seems to have a universal functional form described by

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2} \tag{7.59}$$

where r_s specifies a characteristic radius, and $\rho_s = 4\rho(r_s)$ determines the amplitude of the density profile. For $r \ll r_s$ we find $\rho \propto r^{-1}$, whereas for $r \gg r_s$ the profile follows $\rho \propto r^{-3}$. Therefore, r_s is the radius at which the slope of the density profile changes. We define the ***concentration index*** as

$$c \equiv \frac{r_{200}}{r_s} \tag{7.61}$$

where the larger the value of c, the more strongly the mass is concentrated towards the inner regions. We can write the density profile amplitude factor as

$$\rho_s = \frac{200}{3}\rho_{\rm cr}(z)\frac{c^3}{\ln(1+c) - c/(1+c)}$$

where $\rho_{\rm cr}(z) = 3H^2(z)/(8\pi G)$ is the critical density at redshift z.

Comparison with Observations. A complication of modelling profiles with NFW (or other) profiles is that baryonic matter is present in the inner regions of galaxies (and clusters), thus contributing to the density, but also that these baryons have modified the density profile of dark matter halos in the course of cosmic evolution. Baryons are dissipative, they can cool, form a disk, and accrete inwards. Conversely, SNe can push gas to larger radii or even drive it out of the halo. The changes in the resulting density distribution of baryons by dissipative processes cause a change of the gravitational potential over time, to which dark matter also reacts. The dark matter profile in real galaxies is thus modified compare to pure dark matter simulations.

7.6.2 – The Shape and Spin of Halos

Halo Shapes. There is no reason for halos to have spherical symmetry. Approximating the surfaces of constant density in a halo by an ellipsoid of semi-axes $a_1 \le a_2 \le a_3$, the shape of a halo is characterised by the ratios $s = a_1/a_3$ and $q = a_2/a_3$. If $s = q < 1$, the ellipsoid is prolate, and a halo with $s < 1$ and $q = 1$ is oblate. If all three axes are different, the halo is triaxial.
Numerical simulations show that dark matter halos depend strongly on their formation time and merger history. If two halos of comparable mass collide and merge, the shape of the resulting halo will be strongly prolate. Halos that form early and experience no strong mergers tend to be more spherical.

Week 5

Ch 7.8 – The Substructure of Halos

Sub-halos of galaxies and clusters of galaxies. Numerical simulations of structure formation in the CDM model show that the mass density in halos is not smooth; instead, they reveal that halos contain numerous halos of much lower mass, so called sub-halos. For instance, a halo with the mass of a galaxy cluster contains hundreds or even thousands of halos with masses that are orders of magnitude lower. Similarly, a galaxy-mass halo will have sub-halos in the simulations. The presence of substructure over a wide range in mass is a direct consequence of hierarchical structure formation, in which objects of higher mass each contain smaller structure that have been formed earlier in the cosmic evolution.
Such simulations show that of order $\sim 10\%$ of the mass of halos is contained in sub-halos, with galaxy-mass halos having about $\sim 7\%$.

The spatial distribution of the sub-halos is less centrally concentrated than the total mass. The reason for this lies in the fact that the sub-halos whose orbits bring them deep into the potential well of the host halo are subject to strong tidal forces and get disrupted in the course of evolution. Simulations that include gas physics find that disruption becomes weaker if baryons are included – their dissipational nature leads to more compact (and thus more tightly bound) sub-halos which can resist tidal forces for longer.

The 'substructure' problem. The apparent deficit in the number of observed sub-halos within a galaxy-mass halo is considered to be a potential problem of CDM models. One possibility is that many of the sub-halos are extremely non-luminous and have just not yet been found. Indeed, the currently identified sub-halos (such as satellite galaxies) generally have a high velocity dispersion and thus high mass for low luminosities. Their extremely low metallicity argues for a very early epoch of star formation; this is confirmed by the colour-magnitude diagrams for some of the dwarf galaxies, which assign them an age of ~ 13 Gyr.

Warm dark matter as an alternative. The apparent conflict between the abundance of sub-halos and the observed satellite galaxies in the Milky Way can be potentially avoided if the initial power spectrum of density fluctuations has less power on small spatial scales – corresponding to masses of satellite galaxies. In particular, if a WDM particle has a mass of ~ 2 keV, the cut-off in the power spectrum would correspond to the halo mass of dwarf galaxies and all small-scale sub-halos effectively disappear.
However, observations of the Lyman-α forest strongly constrain the allowed mass range of WDM particles, and lower limits on the mass of WDM particles obtained from these studies come exceedingly close to the mass needed to substantially reduce the abundance of sub-halos.

Evidence for the presence of CDM substructure in galaxies. Gravitational lensing studies have provided strong constraints on the number and mass of sub-halos within lensing galaxies. This generally favours complex sub-halo structure, rather than a 'smooth', simple mass model consisting of few and sparse halos.

The 'disk of satellite galaxies'. Whereas the abundance of dark matter sub-halos in galaxies no longer presents a serious problem for CDM models of structure formation, the spatial distribution of satellite galaxies around the Milky Way requires more explanation. The 11 classical satellites

of the Milky Way seem to form a planar distribution. Such a distribution would be extremely unlikely if the satellite population was drawn from a near-isotropic probability distribution.
Using models of galaxy formation, the accretion of smaller mass halos onto a high-mass halo occurs predominantly in the direction of the filaments that galaxies preferentially form in within large scale structure. The most massive sub-halos therefore tend to form in a planar distribution, not unlike the one seen in the Milky Way's satellite distribution – solving the problem.

Ch 10.1 – Galaxy Evolution

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In the cold dark matter universe, small density structures formed first, which means that low-mass dark matter halos preceded those of higher mass. This 'bottom-up' scenario of structure formation follows from the shape of the power spectrum of density fluctuations, which itself is determined by the nature of dark matter—namely cold dark matter. The gas in these halos is compressed and heated, the source of heat being the potential energy. If the gas is able to cool by radiative processes, i.e., to get rid of some of its thermal energy and thus pressure, it can collapse into denser structures, and eventually form stars. In order for this to happen, the potential wells have to have a minimum depth, so the resulting kinetic energy of atoms is sufficient to excite the lowest-lying energy levels whose de-excitation then leads to the emission of a photon which yields the radiative cooling. We shall see that this latter aspect is particularly relevant for the first stars to form, since they have to be made of gas of primordial composition, i.e., only of hydrogen and helium.

Once the first stars form in the Universe, the baryons in their cosmic neighborhood get ionized. This reionization at first happens locally around the most massive dark matter halos that were formed; later on, the individual ionized regions begin to overlap, the remaining neutral regions become increasingly small, until the process of reionization is completed, and the Universe becomes largely transparent to radiation, i.e., photons can propagate over large distances in the Universe. The gas in dark matter halos is denser than that in intergalactic space; therefore, the recombination rate is higher there and the gas in these halos is more difficult to ionize. Probably, the ionizing intergalactic radiation has a small influence on the gas in halos hosting a massive galaxy. However, for lower-mass halos, the gas not only maintains a higher ionization fraction, but the heating due to ionization can be appreciable. As a result, the gas in these low-mass halos finds it more difficult to cool and to form stars. Thus, the star-formation efficiency—or the fraction of baryons that is turned into stars—is expected to be smaller in low-mass galaxies.

The mass of halos grows, either by merging processes of smaller-mass halos or by accreting surrounding matter through the filaments of the large-scale density field. The behavior of the baryonic matter in these halos depends on the interplay of various processes. If the gas in a halo can cool, it will sink towards the center. One expects that the gas, having a finite amount of angular momentum like the dark matter halo itself, will initially accumulate in a disk perpendicular to the angular momentum of the gas, as a consequence of gas friction—provided a sufficiently long time of quiescent evolution for this to happen. The gas in the disk then reaches densities at which efficient star formation can set in. In this way, the formation of disk galaxies, thus of spirals, can be understood qualitatively.

As soon as star formation sets in, it has a feedback on the gas: the most massive stars very quickly explode as supernovae, putting energy into the gas and thereby heating it. This feedback then prevents that all the gas turns into stars on a very short time-scale, providing a self-regulation mechanism of the star-formation rate. In the accretion of additional material from the surrounding of a dark matter halo, also additional gas is accreted as raw material for further star formation.

When two dark matter halos with their embedded galaxies merge, the outcome depends mainly on the mass ratio of the halos: if one of them is much lighter than the other, its mass is simply added to the more massive halo; the same is true for their stars. More specific, the small-mass galaxy is disrupted by tidal forces, in the same way as the Sagittarius dwarf galaxy is currently destroyed in our Milky Way, with the stars being added to the Galactic halo. If, on the other hand, the masses of the two objects are similar, the kinematically cold disks of the two galaxies are expected to be disrupted, the stars in both objects obtain a large random velocity component, and the resulting object will be kinematically hot, resembling an elliptical galaxy. In addition, the merging of gas-rich galaxies can yield strong compression of the gas, triggering a burst of star formation, such as we have seen in the Antennae galaxies. Merging should be particularly frequent in regions where the galaxy density is high, in galaxy groups for instance. A large number of such merging and collision processes are detected in galaxy clusters at high redshift.

In parallel, the supermassive black holes in the center of galaxies must evolve, as clearly shown by the tight scaling relations between black hole mass and the properties of the stellar component of galaxies. The same gas that triggers star formation, say in galaxy mergers, can be used to 'feed' the central black hole. If, for example, a certain fraction of infalling gas is accreted onto the black hole, with the rest being transformed into stars, the parallel evolution of black hole mass and stellar mass could be explained. In those phases where the black hole accretes, the galaxy turns into an active galaxy; energy from the active galactic nucleus, e.g., in the form of kinetic energy carried by the jets, can be transmitted to the gas of the galaxy, thereby heating it. This provides another kind of feedback regulating the cooling of gas and star formation.

When two halos merge, both hosting a galaxy with a central black hole, the fate of the black holes needs to be considered. At first they will be orbiting in the resulting merged galaxy. In this process, they will scatter off stars, transmitting a small fraction of their kinetic energy to these stars. As a result, the velocity of the stars on average increases and many of them will be 'kicked out' of the galaxy. Through these scattering events, the black holes lose orbital energy and sink towards the center of the potential. Finally, they form a tight binary black hole system which loses energy through the emission of gravitational waves until they merge.

The more massive halos corresponding to groups and clusters only form in the more recent cosmic epoch. In those regions of space where at a later cosmic epoch a cluster will form, the galaxy-mass halos form first—the larger-scale overdensity corresponding to the proto-cluster promotes the formation of galaxy-mass halos, compared to the average density region in the Universe; this is the physical origin of galaxy bias. Therefore, one expects the oldest massive galaxies to be located in clusters nowadays, explaining why most massive cluster galaxies are red. In addition, the large-scale environment provided by the cluster affects the evolution of galaxies, e.g., through tidal stripping of material.

Ch 10.2 – Gas in Dark Matter Halos

10.2.1 – The Infall of Gas During Halo Collapse

Gas Heating. As long as the fractional overdensities are small, the spatial distribution of baryons and dark matter are expected to be very similar in the very early halo formation. However, when the spherical halo collapses, the behaviour of both components must be very different: dark matter is collisionless and baryons are collisional (meaning that friction prevents gas from crossing through a gas distribution). Thus, as the halo collapses, the potential energy of the gas is transformed into heat through the frictional processes. Furthermore, the pressure of the gas can prevent it from falling into the dark matter potential well.
In the case of (approximate) spherical symmetry, the gas had already settled down in the inner part of the halo and gas pressure balances gravitational force. As the outer part of the halo collapses, gas falls onto this gas distribution. The infall speed is much higher than the sound velocity of the (cold) infalling gas, and a shock front develops. Inside this front, the gas is hot and almost all of its kinetic energy gets converted into heat.

Hence, a collapsed halo should contain hot gas with a temperature depending on the halo radius and mass. Such hot gas is seen in galaxy groups and clusters through X-ray emission. Galaxy mass halos have a characteristic gas temperature of $\sim 10^6$K and so are much more difficult to observe (since they're too cool to emit X-rays and most atoms are fully ionized at that temperature). Nevertheless, some significant fraction of the hot gas in galaxy-mass halos does not stay hot and must cool to form stars.

10.2.2 – Cooling of Gas

Optically thin gas can cool by emitting radiation, e.g. by scattering between electrons and nuclei in an ionized gas (bremsstrahlung) or collisional excitations (and relaxation and emission of photons).

Cooling function. Although elements heavier than helium have a small abundance in number, they can dominate the gas cooling due to the rich energy spectrum of many-electron atoms. Hence, more enriched gas finds it easier to cool.
Atomic gas cannot cool efficiently for temperatures $T \lesssim 10^4$K due to the lack of charged particles in the gas. Molecules, on the other hand, have a rich spectrum of energy levels at low temperatures and can lead to efficient cooling. This is why star formation occurs in molecular clouds, where gas can efficiently cool and thereby compress to high densities.

Cooling time. Once we know the rate at which gas loses its energy, we can calculate the cooling time (the time it takes the gas [at constant cooling rate] to lose all of its energy). If this cooling time is longer than the age of the Universe, then the gas essentially stays at the same temperature and is unable to collapse towards a halo center. For most regions within galaxy clusters, this is the case, and cooling is only efficient towards cluster centers.

Conditions for efficient cooling. If the cooling time is shorter than the free-fall time (given by equation (10.5)), then gas falls freely towards the center of a halo essentially unaffected by gas pressure.

$$t_{\rm ff} = \sqrt{\frac{3\pi}{32G\rho}} \tag{10.5}$$

If the cooling time is much longer than the free-fall time, the gas at *best* sinks to the center at a rate given by the cooling rate.

Difference between galaxies and groups/clusters. In sufficiently massive halos with $M_g \gtrsim 10^{12}M_\odot$, the small efficiency of gas cooling prevents gas from collapsing to the center and forming stars there. At smaller masses, cooling is effective to enable rapid gas collapse. This dividing line in mass is about the mass which distinguishes galaxies from groups and clusters. In the latter, only a small fraction of the baryons is turned into stars (and this is generally contained within the galaxies of the group). In contrast, a large fraction of baryons in galaxies is concentrated towards the center, as visible in their stellar distribution. Thus, the difference between galaxies and groups/clusters is their efficiency to turn baryons into stars, and this difference is explained with the different cooling efficiencies of matter halos (as in figure 10.4).

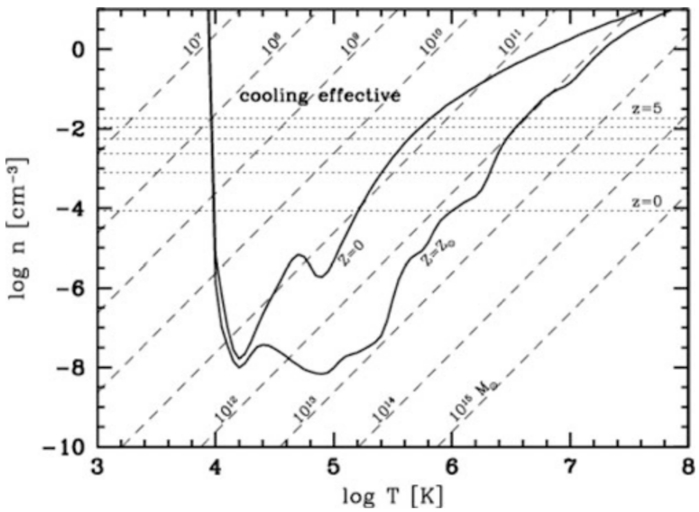

Figure 10.4: Solid curves are for gas metallicities of $Z = 0$ and $Z = Z_\odot$. Dotted lines show the density of halos formed at the specified redshifts. For gas densities and temperatures above the curve, cooling is efficient, whereas cooling time is longer than the free-fall time below the curves.

Low-mass halos. A halo with gas mass $\sim 10^{7.5} M_\odot$ lies inside the cooling curve only at *very* high redshift, i.e. when the corresponding density in a halo is very high. Therefore, gas can cool, and stars form, in halos of this mass only if they formed early enough. We therefore expect that the stars in such low-mass halos are very old.

Week 6

Ch 10.3 – Reionization of the Universe

10.3.1 – The First Stars

Understanding reionization (currently understood to have occurred around $z \sim 10$) is directly linked to studying the first generation of stars.

The Jeans Mass. There exists a certain mass of a halo for which pressure forces are unable to prevent the infall of gas into a potential well,

$$M > M_J \equiv \frac{\pi^{5/2}}{6} \left(\frac{c_s^2}{G}\right)^{3/2} \frac{1}{\sqrt{\overline{\rho}}} \tag{10.6}$$

where $\overline{\rho}$ is the mean cosmic matter density, and c_s is the speed of sound in the gas. The Jeans mass depends on the temperature of the gas, expressed through the sound speed c_s. As a function of redshift, the Jeans mass can be calculated as

$$M_J = 5.7 \times 10^3 \left(\frac{\Omega_m h^2}{0.15}\right)^{-1/2} \left(\frac{\Omega_b h^2}{0.022}\right)^{-3/5} \left(\frac{1+z}{10}\right)^{3/2} M_\odot \tag{10.8}$$

Cooling of the Gas. The Jeans criterion is a necessary condition for the formation of proto-galaxies, i.e. dark matter halos which contain baryons. In order to form stars, the gas in the halos needs to be able to cool further. Here, we are dealing with the particular situation of the first galaxies, whose gas is metal-free, so metal lines cannot contribute to the cooling. The cooling function of primordial gas is much smaller than that of enriched material; in particular, the absence of metals means that even slow cooling through excitation of fine-structure lines cannot occur, as there are no atoms with such transitions present. Thus, cooling by the primordial gas is efficient only above $T \gtrsim 2 \times 10^4$K. Therefore, atomic hydrogen is a very inefficient coolant for these first halos, insufficient to initiate the formation of stars.

The importance of molecular hydrogen. Besides atomic hydrogen and helium, the primordial gas contained a small fraction of molecular hydrogen. Whereas in enriched gas, molecular hydrogen is formed on dust particles, and the primordial gas had no dust, and so H_2 must have formed in the gas phase itself (rendering its abundance very small). Despite its very small density, H_2 dominated the cooling rate of primordial gas at temperatures below $T \sim 10^4$K.
By means of H_2, the gas can cool in halos with a mass of $M \gtrsim 5 \times 10^4 M_\odot$ at $z \sim 20$. In these halos, stars may then be able to form. Such stars, which at the same mass presumably have a much higher temperature and luminosity (due to their zero metallicity) and thus a shorter lifetime, are called ***population III stars***. Due to their high temperature, they are much more efficient sources of ionizing photons than stars with 'normal' metallicity.

10.4.5 – The Formation and Evolution of Supermassive Black Holes

Black holes grow in mass by accreting material, a process we witness through the radiation from accreting black holes in AGN. There is no firm conclusion on how SMBHs initially formed, but three plausible formation scenarios have been studied in detail. We do know, however, that the first SMBHs must have formed early in the universe, as indicated by the presence of very luminous QSOs at $z > 6$.

Remnants of Population III Stars. The first stars formed out of primordial gas, i.e. gas with zero metallicity. The cooling properties of this gas are different from that of enriched material, since no metal lines are available for radiating energy away. It is expected that many stars can form with very high masses, well above $100 M_\odot$; if the mass of a star is above $\sim 250 M_\odot$, it will leave behind a black hole of mass $\gtrsim 100 M_\odot$. Since these stars burn their nuclear fuel very quickly (exploding within a few million years), and are expected to first form at $z \gtrsim 20$, this formation mechanism would yield a very early population of seed black holes.

Gas-Dynamical Processes. Another route for the formation of SMBHs arises if the primordial gas in a high-redshift dark matter halo manages to concentrate in its center, through global dynamical instabilities (e.g. related to the formation of bar-like structures) that are able to transport angular momentum outwards. This angular momentum transport is needed since, otherwise, the central concentration of gas would be prevented by the angular momentum barrier. Subsequent cooling by molecular hydrogen may then lead to the formation of a rapidly rotating supermassive star with up to $10^6 M_\odot$, provided the accumulation of gas occurs rapidly enough. Once the inner core of the supermassive star has burned its hydrogen, a black hole with mass of a few tens of $M_\odot$ would form (depending on the initial angular velocity of the star), which then subsequently accretes material from the outer layers of the star. Since quasi-spherical accretion has a low radiative efficiency, the BH can grow in mass quickly until it exceeds the Eddington luminosity and the remaining gas is expelled, leaving behind a SMBH with $\sim 10^5 M_\odot$.

Stellar-Dynamical Processes. In the inner part of a forming galaxy, dense nuclear star clusters may form. Because of the high density, star-star collisions can occur which can lead to the formation of very massive stars in excess of $10^3 M_\odot$. This has to happen very quickly (before the first stars explode in SNe) since otherwise the massive star would be polluted with metals, its opacity increased, and it would no longer be stable under its mass. The fate of this supermassive star is then similar to the scenario above, resulting in a black hole remnant of several hundred solar masses.

The above three scenarios are not mutually exclusive, and the current understanding of these processes is not sufficient to establish their likelihood and frequency.

Mass growth. Once the seek black holes have formed, they can grow in mass by accreting material. The characteristic time-scale for mass growth is the time by which the mass can double i.e.

$$\epsilon t_{\rm gr} = \frac{\epsilon M_\bullet c^2}{L_{\rm edd}} \approx 5\epsilon \times 10^8 \text{ yr}$$

With $\epsilon \sim 0.1$, a $10^4 M_\odot$ seed black hole formed at $z \sim 20$ could grow to a few $\times 10^8 M_\odot$ by $z \sim 7$ if accreted continuously at the Eddington rate. The situation is more difficult for seed black holes formed from population III stars; they probably require super-Eddington rates to be able to power the luminous QSOs at $z > 6$, which is possible but likely not for long time-scales or by large factors.

Ch 4.4 – Thermal History of the Universe

4.4.8 – Summary

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- Our Universe originated from a very dense, very hot state in the ***Big Bang***. Shortly afterwards, it consisted of a mix of various elementary particles, all interacting with each other.
- We are able to examine the history of the universe in detail, starting at an early epoch where it cooled down by expansion such as to leave only those particle species known to us, and probably a dark matter particle.
- Because of their weak interaction and the decreasing density, the neutrinos experience only little interaction at temperatures below $\sim 10^{10}$K, their decoupling temperature.
- at $T \sim 10^9$K, electrons and positrons annihilate into photons. At this low temperature, pair production ceases to take place.
- Protons and neutrons interact and form deuterium nuclei. As soon as $T \sim 10^9$K, deuterium is no longer efficiently destroyed by energetic photons. Further, nuclear reactions produce mainly helium nuclei. About 25% of the mass in nucleons is transformed into helium, and traces of lithium are produced, but no heavier elements.
- At about $T \sim 3000$K, some 400,000 years after the Big Bang, the protons and helium nuclei combine with the electrons, and the universe becomes essentially neutral (we say that is ***recombines***). From then on, photons can travel without further interactions. At recombination, the photons follow a blackbody distribution. By the ongoing cosmic expansion, the temperature of the spectral distribution decreases, $T \propto (1 + z)$, though its Planck property remains.
- After recombination, the matter is the Universe is almost completely neutral. However, we know from the observation of sources at very high redshift that the intergalactic medium is essentially fully ionized at $z \lesssim 6$. Before $z > 6$, our Universe must there have experienced a phase of reionization. This effect cannot be explained in the context of the *strictly homogeneous* world models; rather it must be examined in the context of structure formation in the Universe and the formation of the first stars and AGN.

Ch 4.5 – Achievements and Problems of the Standard Model

4.5.2 – Problems

Despite the achievements of the standard model, there are some aspects which require further consideration.

The Horizon Problem: Since no signal can travel faster than light, the CMB radiation from two directions separated by more than about one degree originates in regions that were not in causal contact before recombination, i.e. the time when the CMB photons interacted with matter the last time. Therefore, these two regions have never been able to exchange information, for example about their temperature. Nevertheless, their temperature is the same, as seen from the high degree of isotropy of the CMB (which shows relative fluctuations of only $\Delta T/T \sim 10^{-5}$).

The Flatness Problem: For the total density parameter to be of order unity today, it must have been extremely close to 1 at earlier times, which means that a very precise 'fine tuning' of this parameter was necessary.

The above also suggests that the curvature term has never had a dominant effect on the universe.

4.5.3 – Extension of the Standard Model: Inflation

For all of the cosmological parameters to be the value that they are today (and with them, the universe not collapsing/rapidly expanding so that we see it in its current state today) is unlikely from a random standpoint. Different parameters could have led to the universe recollapsing billions of years ago or expanding so rapidly that no structure could be formed. These parameters come down to the initial conditions of the universe just after the Big Bang. The only answer to why these parameters are the values that they are today lies in some processes that must have taken place even earlier in the universe, and so the initial conditions of the normal Friedmann-Lemaitre expansion have a physical origin. Cosmologists believe they have found such a physical reason in the inflationary model.

Inflation. Physical laws and properties of elementary particles are well known up to energies of $\sim$ 100 GeV because they have been experimentally tested in particle accelerators. For higher energies, particles and their interactions are unknown. This means that the history of the Universe can only be considered secure up to this 100 GeV energy. The extrapolation to earlier times, up to the Big Bang, is considerably less certain. From particle physics, we expect new phenomena to occur at an energy scale of the Grand Unified Theories (GUTs), at about 10^{14} GeV, corresponding to $t \sim 10^{-34}$s.
In the inflationary scenario, it is presumed that at very early times the vacuum energy density was much higher than today, and so it dominated the Hubble expansion. Because of this, there would have been exponential expansion (inflationary phase), but only for a brief period. We assume that a phase transition took place in which the vacuum energy density is transformed into normal matter and radiation (a process called reheating), which ends the inflationary phase and after which the normal Friedmann evolution of the Universe begins.

Inflation solves the Horizon Problem. During inflation, the Hubble parameter is constant and so the horizon may become arbitrarily large in the inflationary phase (dependent on the duration of the exponential expansion). According to this scenario, the whole universe visible today was in causal contact prior to inflation, so that the homogeneity of the physical conditions at recombination, and with it the nearly perfect isotropy of the CMB, is provided by causal processes.

Inflation solves the Flatness Problem. Due to the tremendous expansion, any initial curvature is straightened out. Formally, during the inflationary phase we have

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} = 1$$

and since it is assumed that the inflationary phase lasts long enough for the vacuum energy to be completely dominant, when it ends we then have $\Omega_0 = 1$. Hence the universe is flat to an extremely good approximation.

Week 7

Ch 7.2 – Gravitational Instability

7.2.1 – Overview

The smallness of the CMB anisotropy suggests that the density inhomogeneities at redshift $z \sim 1000$ (the epoch where most CMB photons interacted with matter for the last time) must have had very small amplitudes. Today, the amplitudes of the density inhomogeneities are considerably larger (due to structure growth and contraction), and there are generally no longer small density fluctuations.
The universe becomes more inhomogeneous in the course of its evolution, since density perturbations grow over time. We can define the ***relative density contrast*** as

$$\delta(\mathbf{r}, t) \equiv \frac{\rho(\mathbf{r}, t) - \overline{\rho}(t)}{\overline{\rho}(t)} \tag{7.1}$$

where $\overline{\rho}(t)$ denotes the mean cosmic matter density at time t. The smallness of the CMB anisotropy suggests that $|\delta| \ll 1$ at $z \sim 1000$. The dynamics of the cosmic Hubble expansion is controlled by the gravitational field of the average matter density $\overline{\rho}(t)$, whereas the density fluctuations generate an additional gravitational field.

Density fluctuations grow over time due to their self-gravity; overdense regions increase their density contrast over the course of time, while underdense regions decrease their density contrast. In both cases, $|\delta|$ increases. Hence, this effect of ***gravitational instability*** leads to an increase of density fluctuations with increasing time. The evolution of structure in the universe is described by this effect of gravitational instability.

7.2.2 – Linear Perturbation Theory

The Hubble Expansion. The Hubble expansion of

$$\mathbf{v}(\mathbf{r}, t) = H(t)\mathbf{r}$$

satisfies the Poisson equation for the gravitational potential if ρ is homogeneous and the Friedmann equation for the scale factor applies. As long as the density contrast $|\delta| \ll 1$, the deviations of the velocity field from the Hubble expansion will be small.
It is convenient to consider the problem in comoving coordinates, hence define

$$\mathbf{r} = a(t)\mathbf{x}$$

In a homogeneous cosmos, $\mathbf{x}$ is a constant for every matter particle, and its spatial position $\mathbf{r}$ changes only due to the Hubble expansion. Likewise, the velocity field is then written in the form

$$\mathbf{v}(\mathbf{r}, t) = \frac{\dot{a}}{a}\mathbf{r} + \mathbf{u}\left(\frac{\mathbf{r}}{a}, t\right) \tag{7.5}$$

where $\mathbf{u}(\mathbf{x}, t)$ is a function of the comoving coordinate $\mathbf{x}$. The first term represents the homogeneous Hubble expansion, whereas the second term describes the deviations from this homogeneous expansion. For this reason, $\mathbf{u}$ is called the *peculiar velocity*.

The Growth Factor. The density contrast at some comoving coordinate $\mathbf{x}$ as a function of time can be represented in terms of a growth factor,

$$\delta(\mathbf{x}, t) = D_+(t)\delta_0(\mathbf{x}) \tag{7.16}$$

This tells us that the spatial shape of density fluctuations are frozen in comoving coordinates, and only their amplitude increases with time.
For any cosmological model, the growth factor can be found by

$$D_+(a) \propto \frac{H(a)}{H_0} \int_0^a \frac{da'}{[\Omega_m a'^{-1} + \Omega_\Lambda a'^2 - (\Omega_m + \Omega_\Lambda - 1)]^{3/2}} \tag{7.17}$$

where the factor of proportionality is determined from the condition $D_+(t_0) = 1$. In accordance with this, $\delta_0(\mathbf{x})$ would be the distribution of density fluctuations today if the evolution was linear until the present epoch. Therefore, $\delta_0(\mathbf{x})$ is denoted as the *linearly extrapolated density fluctuation field.*

Evidence for Dark Matter on Cosmic Scales. At the present epoch, $\delta \gg 1$ on scales of clusters of galaxies ($\sim$ 2Mpc), and $\delta \sim 1$ on scales of superclusters ($\sim$ 10Mpc). Hence, the law of linear structure growth (eq. 7.16) and the behaviour of $D_+(t)$ would mean that $\delta \gtrsim 10^{-3}$ at $z = 1000$ for these structures to be able to grow to non-linear structures at the present epoch. For this reason, we should also expect CMB fluctuations to be of comparable magnitude, $\Delta T/T \gtrsim 10^{-3}$. The observed fluctuation amplitude is much smaller, $\Delta T/T \sim 10^{-5}$, however, and so the corresponding density fluctuations cannot have grown sufficiently strongly up to today to form non-linear structures.
This contradiction can be resolved by the dominance of dark matter. Since photons interact with baryonic matter only, the CMB anisotropies basically provide information on the density contrast of *baryons*. Dark matter may have had a higher density contrast at recombination, but the baryons, which are strongly coupled to the radiation field before recombination, are prevented from strong clustering due to the radiation pressure. Only after recombination, when the electrons have combined with the atomic nuclei and essentially no free electrons remain, the coupling to the radiation field ends, after which the baryons may fall into the potential wells formed by the dark matter.

Ch 7.3 – Description of Density Fluctuations

We can at best hope to predict the statistical properties of the mass distribution in the universe, rather than having an analytical solution at every spatial point. How can the statistical properties of a density field best be described?
Two universe are considered equivalent if their density fields δ have the same statistical properties. One may then imagine considering a large (statistical) ensemble of universes whose density fields all have the same statistical properties, but for which the individual functions $\delta(\mathbf{x})$ can all be different. This statistical ensemble is called a *random field*, and any individual distribution with the respective statistical properties is called a realization of the random field.

7.3.1 – Correlation Functions

Galaxies are not randomly distributed in space, but rather they gather in groups, clusters, or large scale structure. This means that the probability of finding a galaxy at location $\mathbf{x}$ is not independent of whether there is a galaxy at a neighbouring point $\mathbf{y}$. It is more probable to find a galaxy in the vicinity of another that at an arbitrary location. Consider two points $\mathbf{x}$ and $\mathbf{y}$, and the two volume elements dV around these points. If $\overline{n}$ is the average number density of galaxies, the probability of finding a galaxy in the volume element dV around $\mathbf{x}$ is then

$$P_1 = \overline{n}\, dV$$

independent of $\mathbf{x}$ if we assume that the universe is statistically homogeneous. We choose dV such that $P_1 \ll 1$, so that the probability of finding two or more galaxies in the volume is negligible.
The probability of finding a galaxy in the volume element dV at location $\mathbf{x}$ and at the same time finding a galaxy in the volume element dV at location $\mathbf{y}$ is then

$$P_2 = (\overline{n}\, dV)^2[1 + \xi_g(\mathbf{x}, \mathbf{y})] \tag{7.26}$$

If the distribution of galaxies was uncorrelated, the probability P_2 would simply be the product of the probabilities of finding a galaxy at each of the locations $\mathbf{x}$ and $\mathbf{y}$ in a volume element dV (i.e. $P_2 = P_1^2$). But since the distribution is correlated, the relation needs to be modified to include the ***two-point correlation function*** of galaxies $\xi_g(\mathbf{x}, \mathbf{y})$.
By analogy to this, the correlation function for the total matter density can be defined as

$$\langle \rho(\mathbf{x})\rho(\mathbf{y})\rangle \equiv \overline{\rho}^2\ [1 + \xi(\mathbf{x}, \mathbf{y})] \tag{7.27}$$

where the angular brackets denote averaging over an ensemble of distributions that have the same statistical properties.

Since the universe is considered statistically homogeneous, ξ can only depend on the difference $\mathbf{x} - \mathbf{y}$ and not on $\mathbf{x}$ and $\mathbf{y}$ individually. Further, ξ can only depend on the separation $r = |\mathbf{x} - \mathbf{y}|$ and not on the direction of the separation vector because of the assumed statistical isotropy of the universe. Thus, $\xi = \xi(r)$ is simply a function of the separation between two points.

By spatial averaging, the galaxy correlation function was found as

$$\xi_g(r) = \left(\frac{r}{r_0}\right)^{-\gamma} \tag{7.28}$$

for galaxies of luminosity $\sim L^*$, where $r_0 \simeq 5h^{-1}$Mpc is the correlation length, and the slope is $\gamma \simeq 1.7$. This relation is valid over a range $0.2h^{-1}\text{Mpc} \lesssim r \lesssim 30h^{-1}\text{Mpc}$.

The correlation function provides a means to characterise the structure of the cosmological matter distribution.

7.3.2 – The Power Spectrum

An alternative (and equivalent) description of the statistical properties of a random field, and thus of the matter distribution in a universe, is the power spectrum $P(k)$. Roughly

speaking, the power spectrum describes the level of structure as a function of the length-scale $L \simeq 2\pi/k$; the larger $P(k)$ is, the larger the amplitude of the fluctuations on a length scale of $2\pi/k$. Here, k is a comoving wave number.
Phrased differently, the density fluctuations are decomposed into a sum of plane waves of the form

$$\delta(\mathbf{x}) = \sum a_{\mathbf{k}} \cos(\mathbf{x} \cdot \mathbf{k})$$

with a wave vector $\mathbf{k}$ and an amplitude $a_{\mathbf{k}}$. The power spectrum $P(k)$ then describes the mean of the squares of the amplitudes, $|a_{\mathbf{k}}|^2$, averaged over all wave vectors with equal length $k = |\mathbf{k}|$.
The power spectrum and correlation function are related through a Fourier transform, and both are used interchangeably in cosmology.

Gaussian Random Fields. In general, knowing the power spectrum is not sufficient to unambiguously describe the statistical properties of any random field. However, for certain classes of random fields, this is different. In fact, Gaussian random fields are uniquely characterised by $P(k)$. Among the properties which characterise them, the probability distribution of the density fluctuations $\delta(\mathbf{x})$ at any point is a Gaussian. Such Gaussian random fields play an important role in cosmology because it is assumed that at very early epochs, the density field obeyed Gaussian statistics (an assumption backed by the anisotropies in the CMB).

Ch 7.4 – Evolution of Density Fluctuations

$P(k)$ and $\xi(r)$ both depend on cosmological time or redshift, because the density field in the universe evolves over time. Therefore, the dependence on t is explicitly written as $P(k,t)$ and $\xi(r,t)$. Note that $P(k,t)$ is linearly related to $\xi(r,t)$, and ξ in turn depends quadratically on the density contrast δ. If $\mathbf{x}$ is the comoving separation vector, we then know the time dependence of the density fluctuations, $\delta(\mathbf{x},t) = D_+(t)\delta_0(\mathbf{x})$. So,

$$\xi(r,t) = D_+^2(t)\xi(r,t_0) \tag{7.30}$$

$$P(k,t) = D_+^2(t)P(k,t_0) \equiv D_+^2(t)P_0(k) \tag{7.31}$$

These relations are only valid in the Newtonian, linear perturbation theory in the matter dominated era of the universe.

7.4.1 – The Initial Power Spectrum

The Harrison-Zeldovich Spectrum. At early times, the expansion of the universe follows a power law, $a(t) \propto t^{1/2}$ in the radiation dominated era. At that time, no natural length-scale existed in the universe to which one might compare a wavelength. The only mathematical function that depends on a length but does not contain any characteristic scale is a power law; hence for very early times one should expect

$$P(k) \propto k^{n_s} \tag{7.32}$$

Harrison, Zeldovich, Peebles and others argued that $n_s = 1$, where the amplitude of the fluctuations of the gravitational potential are constant, i.e. preferring neither small nor large scales. For this reason, the spectrum with $n_s = 1$ is called a scale-invariant spectrum. With such a spectrum, we may choose a time t_i after the inflationary epoch and write

$$P(k,t_i) = D_+^2(t_i)Ak^{n_s} \tag{7.33}$$

where A is a normalisation constant fixed by observations. This, however, needs to be modified to account for the different growth of the amplitude of density fluctuations in the radiation-dominated epoch (compared to the later cosmic epochs from which it was derived).

Cold and Hot Dark Matter. The aforementioned modifications depend on the nature of dark matter. We differentiate ***cold dark matter*** (CDM) and ***hot dark matter*** (HDM) based on their characteristic velocities. CDM has a velocity dispersion that is negligible compared to astrophysically relevant velocities. Therefore, their initial velocity dispersion can well be approximated by zero, and all dark matter particles have the bulk velocity $\mathbf{u}$ of the cosmic 'fluid'. In contrast, the velocity dispersion of hot dark matter is appreciable. Neutrinos are the best candidate for HDM, and their characteristic velocity is specified by their rest mass, which prevents them from forming matter concentrations at all mass scales except for the most massive ones, as their velocity is larger than the corresponding escape velocities.

7.4.2 – Growth of Density Perturbations and the Transfer Function

The Transfer Function. The power spectrum $P(k)$ is affected by the nature of dark matter, and so we modify the power spectrum by including a transfer function $T(k)$, in the form

$$P(k,t) = D_+^2(t)Ak^{n_s}T^2(k) \tag{7.35}$$

where $T(k)$ depends on the nature of dark matter.

CDM and HDM. In HDM models, small-scale fluctuations are washed out by free-streaming of relativistic particles, i.e. the power spectrum is completely suppressed for large k, which is expressed by the transfer function $T(k)$ decreasing exponentially for large k. In the context of such a theory, large structures will form first, and galaxies can form only later by fragmentation of large structures. This scenario is in clear contradiction to observations, since we observe galaxies and QSOs at $z > 6$ so that small-scale structure is already present at times when the universe was only 10% of its current age. In addition, both in the local universe and at high redshift, the observed correlation function of galaxies is incompatible with cosmological models in which the dark matter is composed mainly of HDM.
Therefore, we can exclude HDM as the dominant constituent of dark matter, and we assume that the dark matter is 'cold'.

Qualitative Behaviour of the Transfer Function. The behaviour of the growth of a density perturbation on a scale L for $z < z_{\text{center}}(L)$ depends on z_{center} itself. If a perturbation enters the horizon in the radiation-dominated phase, $z_{\rm eq} \lesssim z_{\rm center}(L)$, it ceases to grow during the epoch $z_{\rm eq} \lesssim z \lesssim z_{\rm center}(L)$. In this period, the energy density in the universe is dominated by radiation, and the resulting expansion rate prevents an efficient perturbation growth. At later epochs, when $z \lesssim z_texteq$, the growth of density perturbations continues. If $z_{\rm center}(L) \lesssim z_{\rm eq}$, that is if the perturbation enters the horizon during the matter-dominated epoch of the universe, these perturbations will grow as described earlier with $\delta \propto D_+(t)$. This implies that a length scale L_0 is singled out, namely the one for which

$$z_{\rm eq} = z_{\rm center}(L_0) \tag{7.37}$$

so that L_0 is the comoving horizon size at matter-radiation equality. This length scale is

$$L_0 \approx 16(\Omega_m h^2)^{-1}\ \mathrm{Mpc} \tag{7.39}$$

Density fluctuations with $L > L_0$ enter the horizon after matter started to dominate the energy density of the universe; hence their growth is not impeded by a phase of radiation dominance. In contrast, density fluctuations with $L < L_0$ enter the horizon at a time when radiation was still dominating. These then cannot grow further as long as $z > z_{\rm eq}$, and only in the matter dominated epoch will their amplitudes proceed to grow again. Their relative amplitude up to the present time has therefore grown by a smaller factor than that of fluctuations with $L > L_0$.
In the limiting cases of $L \gg L_0$ and $L \ll L_0$, we get

$$T(k) \approx \begin{cases} 1 & k \ll 1/L_0 \\ (kL_0)^{-2} & k \gg 1/L_0 \end{cases} \tag{7.40}$$