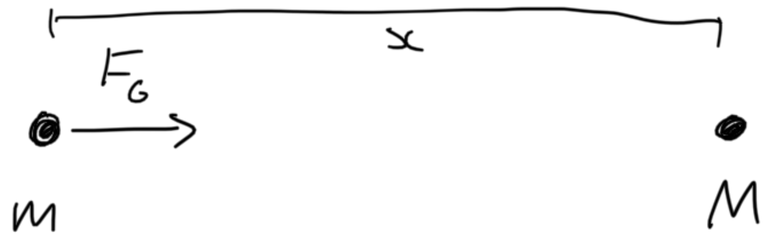
1.



Assuming no other forces in the system, m is attracted to M with force

$$F_G = -\frac{GMm}{x^2}$$

Thus,

$$V(x) = -\int F \, dx$$
$$= -\frac{GMm}{x} \quad (\text{choosing } c = 0).$$

By cons. of energy,

$$T + V = \frac{1}{2} m v^2 - \frac{GMm}{x} = E$$

is constant.

Given $x \to \infty$ implies $v \to 0$, $E = 0$. Thus,

$$\frac{1}{2} \cancel{m} v^2 = \frac{GM\cancel{m}}{x}$$

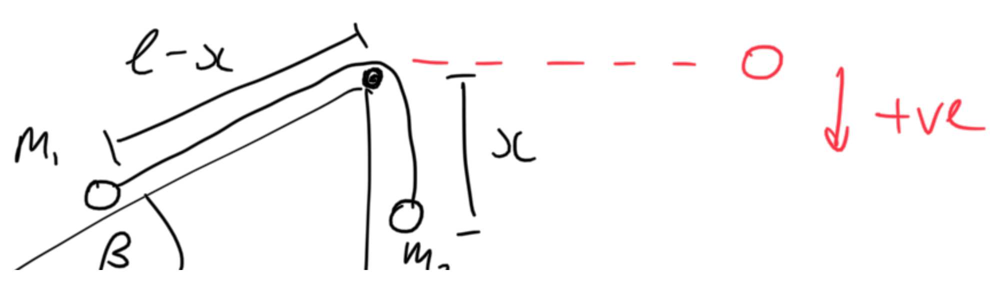
$$v^2 = \frac{2GM}{x}$$

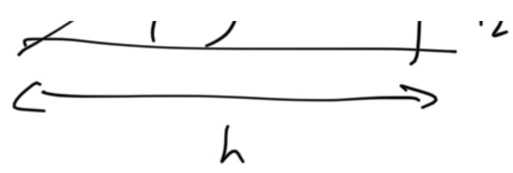
$$v^2 = \frac{Rc^2}{x}$$

$$v = -c\sqrt{\frac{R}{x}}$$

where we take the negative branch because the star is moving to the origin.

2. a)





We have one degree of freedom, with gen. coord x. The kinetic energy is given by

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2$$

The only forces are due to gravity. This means on m_2, we have

$$F = m_2 g$$

On m_1, the force is scaled by the incline:

$$F = m_1 g \sin\beta$$

m_1 is also situated $(\ell - x)\sin\beta$ below the pulley. Therefore, the potential is

$$V = -g\left(m_1 (\ell - x)\sin^2\beta + m_2 x\right)$$

The Lagrangian is

$$L = T - V$$
$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + g\left(m_1(\ell - x)\sin^2\beta + m_2 x\right)$$

b) We compute

$$\frac{\partial L}{\partial x} = -m_1 g \sin^2\beta + m_2 g$$

$$\frac{\partial L}{\partial \dot{x}} = m_1 \dot{x} + m_2 \dot{x}$$

So by Euler-Lagrange,

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}$$

$$= \frac{d}{dt}\left(m_1 x + m_2 x\right) + m_1 g \sin^2\beta - m_2 g$$

$$= m_1\left(\ddot{x} + g\sin^2\beta\right) + m_2\left(\ddot{x} - g\right)$$

c) The system will be in equilibrium when $\ddot{x} = 0$.
That is,

$$0 = m_1 \cancel{g} \sin^2\beta - m_2 \cancel{g}$$

$$\frac{m_2}{m_1} = \sin^2\beta$$

3. a) $F(x,y) = -k_1 x\,\hat{i} - k_2 y\,\hat{j}$

$$= (-k_1 x, -k_2 y)$$

Let V be s.t. $F = -\nabla V$. Thus

$$\nabla V = (k_1 x, k_2 y)$$

$$\Rightarrow \quad \frac{\partial V}{\partial x} = k_1 x \Rightarrow V = \frac{k_1 x^2}{2} + g(y)$$

$$\frac{\partial V}{\partial y} = g'(y) = k_2 y$$

$$\Rightarrow g(y) = \frac{k_2 y^2}{2} + c$$

so $V = \dfrac{k_1 x^2}{2} + \dfrac{k_2 y^2}{2}$ (choosing $c = 0$).

and $T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right)$

Thus $H = T + V$

$$= \frac{1}{2}\left(m(\dot{x}^2 + \dot{y}^2) + k_1 x^2 + k_2 y^2\right)$$

Also, $L = T - V$

$$= \frac{1}{2}\left(m(\dot{x}^2 + \dot{y}^2) - k_1 x^2 - k_2 y^2\right)$$

Now,

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x},$$

$$\partial x$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

Finally,

$$H = \frac{1}{2}\left(\frac{p_x^2 + p_y^2}{m} + k_1 x^2 + k_2 y^2 \right)$$

b) Hamilton's equations of motion are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -k_1 x$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -k_2 y$$

c) Solving,

$$\ddot{x} = \frac{\dot{p}_x}{m} = \frac{-k_1 x}{m}, \quad \ddot{y} = \frac{\dot{p}_y}{m} = \frac{-k_2 y}{m}$$

Thus $\lambda^2 + \frac{k_1}{m} = 0 \Rightarrow \lambda = \pm\sqrt{\frac{k_1}{m}}\, i$

So the solutions are

$$x = \left(A_1 \cos\left(\sqrt{\tfrac{k_1}{m}}\, t\right) + B_1 \sin\left(\sqrt{\tfrac{k_1}{m}}\, t\right)\right),$$

$$y = \left(A_2 \cos\left(\sqrt{\tfrac{k_2}{m}}\, t\right) + B_2 \sin\left(\sqrt{\tfrac{k_2}{m}}\, t\right)\right)$$

Initially we have $x = 1$, $y = 0$. Thus,

$$1 = A_1, \quad 0 = A_2$$

$$\dot{x} = \left(\text{sin term} + \sqrt{\tfrac{k_1}{m}}\, B_1 \cos\left(\sqrt{\tfrac{k_1}{m}}\, t\right)\right)$$

$$\dot{y} = \left(\text{sin term} + \sqrt{\tfrac{k_2}{m}}\, B_2 \cos\left(\sqrt{\tfrac{k_2}{m}}\, t\right)\right)$$

We have $\dot{x} = -1$, $\dot{y} = 0$. Thus,

$$B_1 = -\sqrt{\frac{m}{k_1}} \quad , \quad B_2 = 0$$

So

$$x = \cos\left(\sqrt{\frac{k_1}{m}}\, t\right) - \sqrt{\frac{m}{k_1}} \sin\left(\sqrt{\frac{k_1}{m}}\, t\right), \quad y = 0$$

4. $$H = \frac{p^2}{2} + V(q), \quad V(q) = q^2(|q| - a)$$

a) Hamilton's equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p} = p$$

$$\begin{aligned} \dot{p} = -\frac{\partial H}{\partial q} &= -V'(q) \\ &= 2q(|q|-a) + q^2 \cdot \frac{q}{|q|} \\ &= 2q(|q|-a) + \frac{q^3}{|q|} \end{aligned}$$

Fixed points occur when $(\dot{q}, \dot{p}) = (0, 0)$ i.e.

$$p = 0, \quad 2q(|q| - a) + \frac{q^3}{|q|} = 0$$

$$2q(q^2 - a|q|) + q^3 = 0$$

$$3q^{\cancel{3}2} - a\cancel{q}|q| = 0$$

$$3q^2 - a|q| = 0$$

$q = 0$ is obviously a fixed point.

$q > 0$

$$3q^{\cancel{2}} = a\cancel{q}$$

$$3q = a$$

$$q = \frac{a}{3}$$

$q < 0$

$$3q^{\cancel{2}} = -a\cancel{q}$$

$$3q = -a$$

$$q = \frac{-a}{3}$$

only holds for $a > 0$.

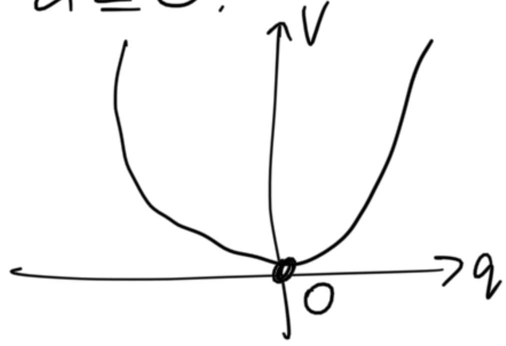
$\underline{a \leq 0}$: $(0,0)$ as a fixed point

$\underline{a > 0}$: $(0,0)$, $\left(\pm\frac{a}{3}, 0\right)$ as fixed points.

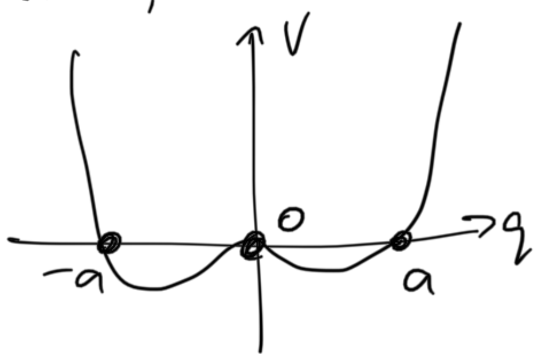
At $(0,0)$, $H = 0$

At $\left(\pm\frac{a}{3}, 0\right)$, $H = \frac{a^2}{9}\left(\frac{a}{3} - a\right) = \frac{-2a^3}{27}$

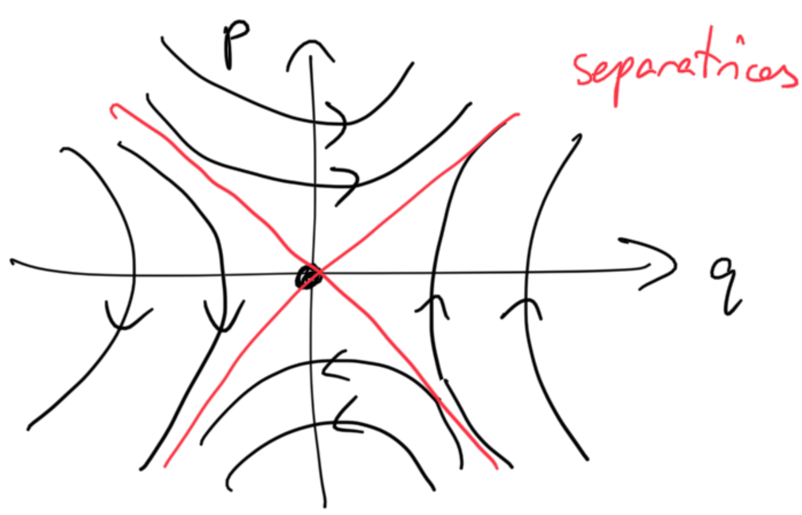
b) (i) For $a \leq 0$.



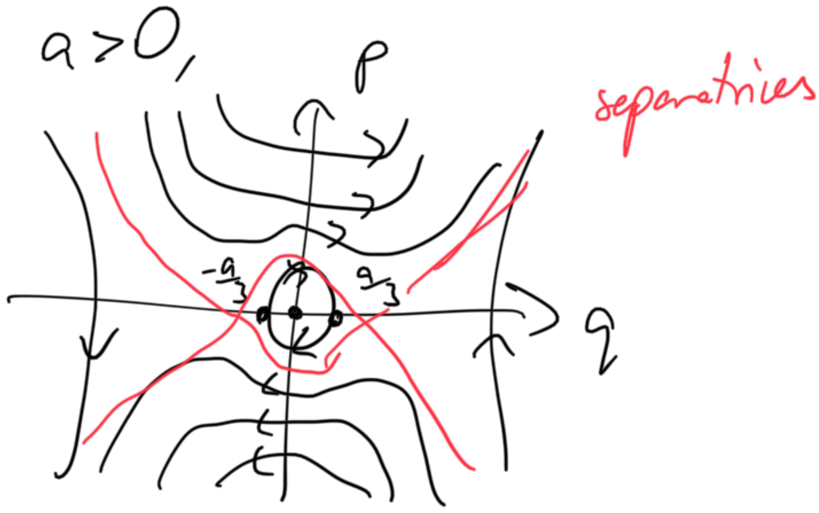
For $a > 0$,



(ii) For $a \leq 0$,



For $a > 0$,



When $a \leq 0$ we only get a single saddle (unstable) point. If $a > 0$, we get oscillating motion with stable fixed points.

c) We get a potential similar to the SHO when $V(q)$ looks like a quadratic, which occurs near $a = 0$.

5. $$x_{n+1} = f(x_n)$$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left| \frac{df^n(x)}{dx}\bigg|_{x=x_0} \right|$$

$$f^n = \underbrace{f(f(\cdots f(x))\cdots))}_{n \text{ times}}$$

a) $$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)|$$

Notice that

$$\frac{df^n(x)}{dx} = f'(x_{n-1}) \cdot \frac{df^{n-1}(x)}{dx} = f'(x_{n-1}) \cdot f'(x_{n-2}) \cdots$$

so

$$= \prod_{i=0}^{n-1} f'(x_i)$$

Thus $$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left| \prod_{i=0}^{n-1} f'(x_i) \right|$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)|$$

b) Stable: $\lambda < 1$

p-cycle: $\lambda < 1$

chaos: $\lambda > 1$

c) $$f(x) = \begin{cases} rx, & 0 \le x \le \frac{1}{2} \\ r(1-x), & \frac{1}{2} < x \le 1 \end{cases}$$

$$f'(x) = \begin{cases} r & , 0 \le x \le \frac{1}{2} \\ -r & , \frac{1}{2} < x \le 1 \end{cases}$$

$$\Rightarrow |f'(x)| = r$$

$$\therefore \lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln r$$

$$= \ln r$$

We expect chaos for $r > e$.

6. a) Alice has the proper time:

$$\tau = t\sqrt{1 - v^2}$$

Setting $t = 4y$, $v = 0.8$,

$$\tau = 4\sqrt{1 - 0.8^2} = 2.4\,y$$

b) While time dilation means Alice experiences less time, length contraction means Alice measures lengths as longer than Bob, by the same scaling factor γ. These factors cancel out so we have no superluminal speeds.

7. a) Let $x_1 = (t, \vec{x}_1)$, $x_2 = (t, \vec{x}_2)$.

We have $u = \frac{dx}{d\tau}$, $d\tau = \gamma^{-1} dt$.

Thus, $$u = \frac{dx}{dt}\frac{dt}{d\tau} = \gamma \frac{dx}{dt}$$

i.e. $u_1 = (\gamma, \gamma \vec{v}_1)$, $u_2 = (\gamma, \gamma \vec{v}_2)$.

b) $$(u_1 \cdot u_2)^{-2} = (-\gamma_1 \gamma_2 + \gamma_1 v_1 \gamma_2 v_2)^{-2}$$

$$= \frac{1}{\quad}$$

$\gamma_1^2\gamma_2^2 - 2\gamma_1^2\gamma_2^2 v_1 v_2 + \gamma_1^2 v_1^2 \gamma_2^2 v_2^2$

$$V_{rel}^2 = 1 - (u_1 \cdot u_2)^{-2}$$

$$= 1 - \frac{1}{(\gamma_1\gamma_2)^2 (v_1 v_2 - 1)^2}$$

I ain't doing this computation lol

8. $E = \sqrt{m^2 + p^2}$

$\vec{p} = m\gamma\vec{v}, \quad \gamma = (1 - v^2)^{-1/2} \quad v^2 << 1$

Binomial approx $\gamma \approx 1 + \frac{v^2}{2}$

$$\Rightarrow \vec{p} \approx m(1 + \underbrace{\frac{v^2}{2}}_{\approx 0})\vec{v}$$

$$\approx m\vec{v}$$

so $E \approx \sqrt{m^2 + m^2 v^2}$

$$= \sqrt{m^2(1 + v^2)}$$

$$= m(1+v^2)^{1/2}$$

$$= m\left(1 + \frac{v^2}{2}\right)$$

binomial approx.

$$= m + \frac{mv^2}{2}.$$