Part A:

Problem 1.1:

The height of an area of land relative to some reference point is given by the equation

$$h(x,y) = 10e^{-\frac{x^2}{10}} + \frac{y^2}{10}$$

Part A:

Figure 2 shows the height map of some area of land relative to an undefined reference point.

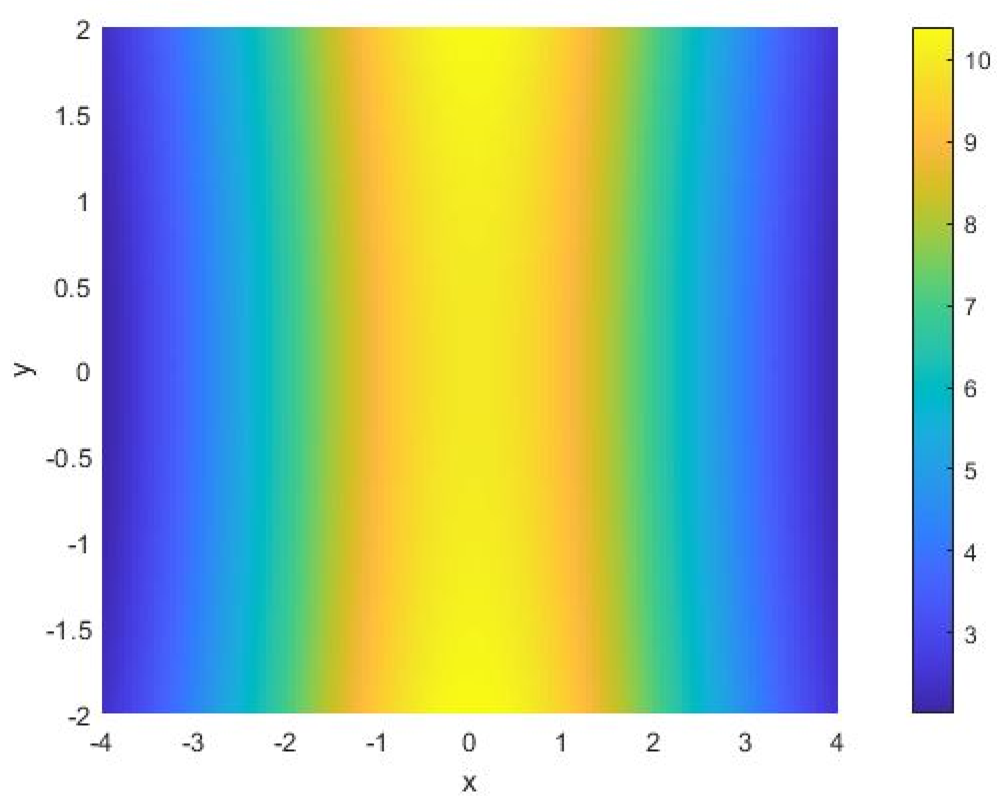

Figure 1: Height of Land Area Relative to Reference Point
Domain was chosen as -4≤x≤4 to account for a suitable variation in height values. A range of -2≤y≤2 was chosen for a similar reason, and to show a significant effect of height variation as a result of y value change.

It can be seen that the height of the land increases as x tends to 0, up to a total height of just over 10 units. Also worth noting is that as the absolute magnitude of y increases and x is kept constant, height increases slowly. As this happens, the statement that as x tends to 0, height increases, still holds.

Part B:
Figure 2 shows the gradient of the height of some area of land.

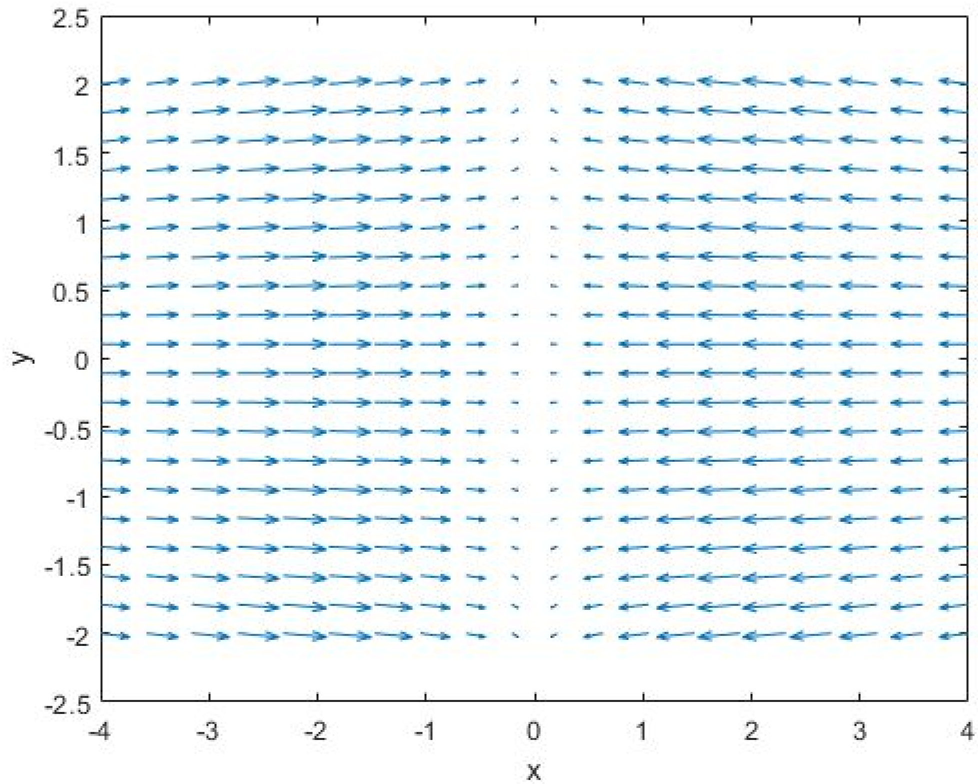

Figure 2: Gradient of the Height of Land Area
Domain was chosen as $-4 \leq x \leq 4$ to account for a suitable variation in slope values. A range of $-2 \leq y \leq 2$ was chosen for a similar reason, and to show a significant effect of slope variation as a result of y value change.

It can be seen that the slope of the land gradually tapers off as 0 is approached on the x-axis. It also seems that, based on visual analysis only, the highest inclination on the land occurs at around $x = \pm 2$. As claimed in part (a), it can be seen easily in Figure 2 that as the absolute magnitude of y increases, the height increases for constant x. This is shown by the vector arrows gradually pointing away from $y = 0$ as y gets further from 0.

Part C:
Figure 3 shows the divergence of the slope field for some area of land.

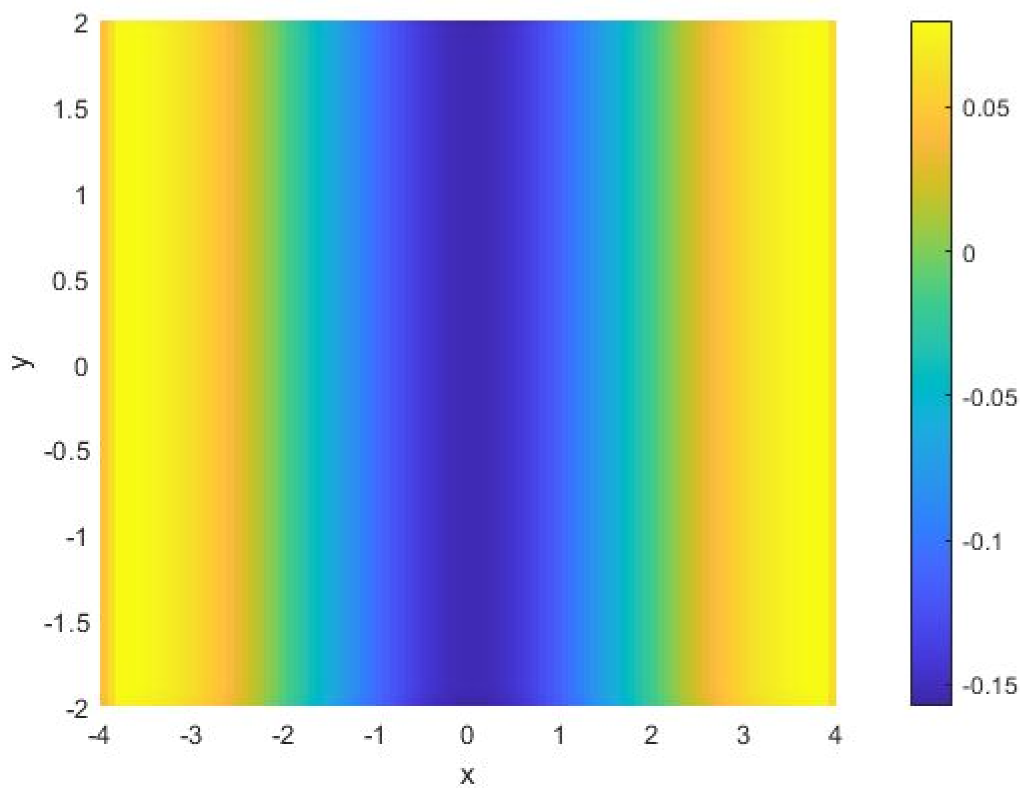

Figure 3: Divergence of Area Height Gradient Field
Domain was chosen as $-4 \le x \le 4$ to account for a suitable variation in divergence values. A range of $-2 \le y \le 2$ was chosen for the same reason.

The figure supports the claim made in part (b) that the maximum slope of the field occurs at around $x = \pm 2$. While $x > 2$, and $x < -2$, the gradient field is essentially 'created', meaning that the slope is increasing for this domain. The inverse of this is true, in that for $-2 < x < 2$, the gradient field is 'destroyed', meaning that the gradient gradually tapers off to 0 at $x = 0$. Interestingly, the aforementioned height change as a result of y-value variance doesn't seem to significantly affect the divergence at points close to x=0. This implies that the magnitude of the x-component of each gradient vector is unchanged as a result of y value variation.

Problem 1.2:

Part A:
Show that a temperature profile of the form

$$T(x,t) \; = \; T_A \sin(\tfrac{2\pi x}{\lambda}) \; e^{-t/\tau} + T_0$$

is a solution of the temperature diffusion equation when the diffusion constant, D, is a function of one or more of the constants T_A, T_0, λ and τ . Determine this relationship.

Firstly, we may differentiate the desired equation with respect to time,

$$\frac{\delta T(x,t)}{\delta t} = (T_A \sin(\tfrac{2\pi x}{\lambda}) \; e^{-t/\tau} + T_0) \; \frac{\delta}{\delta t}$$
$$\frac{\delta T(x,t)}{\delta t} = T_A \sin(\tfrac{2\pi x}{\lambda}) \; \times \frac{e^{-t/\tau}}{(-\tau)}$$
$$\frac{\delta T(x,t)}{\delta t} = -\, 1/\tau \; T_A \sin(\tfrac{2\pi x}{\lambda}) \; e^{-t/\tau}$$

Now, differentiating with respect to x,

$$\frac{\delta^2 T(x,t)}{\delta x^2} = (T_A \sin(\tfrac{2\pi x}{\lambda}) \; e^{-t/\tau} + T_0) \frac{\delta^2}{\delta x^2}$$
$$\frac{\delta^2 T(x,t)}{\delta x^2} = (\tfrac{2\pi}{\lambda} \; T_A \cos(\tfrac{2\pi x}{\lambda}) \; e^{-t/\tau}) \frac{\delta}{\delta x}$$
$$\frac{\delta^2 T(x,t)}{\delta x^2} = -\, \frac{2\pi^2}{\lambda^2} \; T_A \sin(\tfrac{2\pi x}{\lambda}) \; e^{-t/\tau}$$

Both of the differentiated functions have common terms, so we may find them in terms of said common terms, which gives:

$$T_A \sin(\tfrac{2\pi x}{\lambda}) \; e^{-t/\tau} \; = \; -\, \frac{\lambda^2}{2\pi^2} \frac{\delta^2 T(x,t)}{\delta x^2}$$
$$T_A \sin(\tfrac{2\pi x}{\lambda}) \; e^{-t/\tau} \; = \; -\, \tau \frac{\delta T(x,t)}{\delta t}$$

Equating both sides gives:

$$-\, \tau \frac{\delta T(x,t)}{\delta t} \; = \; -\, \frac{\lambda^2}{2\pi^2} \frac{\delta^2 T(x,t)}{\delta x^2}$$
$$\frac{\delta T(x,t)}{\delta t} \; = \; \frac{\lambda^2}{2\pi^2 \tau} \frac{\delta^2 T(x,t)}{\delta x^2}$$

Letting $D = \frac{\lambda^2}{2\pi^2 \tau}$,

$$\frac{\delta T(x,t)}{\delta t} = D \frac{\delta^2 T(x,t)}{\delta x^2}$$

Which is the form of the one-dimensional temperature diffusion equation. Thus the aforementioned temperature profile holds as a possible solution.

Part B:
Using the numbers provided below, and the constant-temperature boundary conditions, find numerical values for T_A, T_0 and λ.

Firstly, at $x = 0$, $T = 293K$. So,

$$T(0,t) = T_A \sin(\frac{2\pi \times 0}{\lambda})e^{-t/\tau} + T_0$$
$$293K = T_A \times 0 \times e^{-t/\tau} + T_0$$
$$T_0 = 293K$$

With this value for the rod,the temperature equation can now be written as:

$$T(x,t) = T_A \sin(\frac{2\pi x}{\lambda})e^{-t/\tau} + 293$$

The maximum temperature of $T = 353K$ occurs at $t = 0$ when $\frac{2\pi x}{\lambda} = \frac{\pi}{2}$, or when $x = \frac{\lambda}{4}$. So,

$$T(\frac{\lambda}{4}, 0) = T_A \sin(\frac{\pi}{2})e^{-0/\tau} + 293$$
$$353K = T_A + 293K$$
$$T_A = 60K$$

And, the temperature equation can now be written as:

$$T(x,t) = 60 \sin(\frac{2\pi x}{\lambda})e^{-t/\tau} + 293$$

Now, assuming that the hottest temperature of 353K occurs in the middle of the rod, the wavelength of the sine wave will be equal to $2L$, where L is the length of the rod as there are “zeros” of the wave at $x = 0$, and $x = 0.2$:

$$\lambda = 2L$$
$$\lambda = 2 \times 0.2$$
$$\lambda = 0.4 \text{ metres}$$

And finally, the temperature equation for the rod may be written as:

$$T(x,t) = 60 \sin(5\pi x)e^{-t/\tau} + 293$$

With the values of the original coefficients being $T_A = 60K$, $T_0 = 293K$, and $\lambda = 0.4$ metres.

Part C:
The diffusion constant of a material can be obtained from

$$D = \frac{k}{c\rho}$$

where k is the thermal conductivity of the material, c is the specific heat, and ρ is the density. Using literature values of these constants for type 304 stainless steel (quote your source), and your expression for D, calculate a numerical value for τ . What does τ mean, and does your value seem reasonable for this material? (Type 304 stainless steel is commonly used for BBQ plates)

From Part A of this question, we have that the diffusion constant can be expressed by $D = \frac{\tau\lambda^2}{2\pi^2}$. Equating the two expressions for the diffusion constant and finding a final expression for τ gives:

$$\frac{\lambda^2}{2\pi^2\tau} = \frac{k}{c\rho}$$

$$\tau = \frac{c\rho\lambda^2}{2k\pi^2}$$

As per Austral Wright, the physical properties of 304 stainless steel are: $k = 16.2\frac{J}{m \cdot s \cdot K}$, $c = 500\frac{J}{kg \cdot K}$, and $\rho = 7900\frac{kg}{m^3}$. It was previously determined that $\lambda = 0.4$ metres, so the equation, including units, becomes:

$$\tau = \frac{500 \times 7900 \times 0.4^2}{2 \times 16.2\pi^2}(\frac{J}{kg \cdot K})(\frac{kg}{m^3})(m^2)(\frac{m \cdot s \cdot K}{J})$$

$$\tau \approx 1976s$$

This value of τ represents how much impact time has on the diffusion of temperature of the material. i.e. it can describe how quickly the material reaches thermal equilibrium. For one time increment of τ seconds, the temperature peaks of the material will approach equilibrium by about 37% of their values (Cooper, I, n.d.).

Considering that 304 stainless steel is used for cookware and hotplates, a large value for τ sounds reasonable, considering that an ideal hotplate would stay at a hot temperature for a long time.

Part B:

Problem 1.3:

In this problem you will develop a model for the gravitational field inside a globular cluster. A globular cluster is a spherical collection of stars that are gravitationally bound. A simple model of the density function of a globular cluster is

$$\rho(r) = \begin{cases} A(\frac{1}{r} - \frac{1}{r_0})^2 & \text{if}\, r \leq r_0 \\ 0, & \text{otherwise} \end{cases}$$

where r_0 is the radius of the cluster and A is a constant. You will need to use the divergence relation for acceleration to do your calculations, assuming spherical symmetry. This can be written as

$$\nabla \cdot \mathbf{a} = \frac{1}{r^2}\frac{\partial(r^2 a)}{\partial r} = -4\pi G\rho$$

where r is the distance from the centre of the body, a is the radial acceleration, G is the gravitational constant, and ρ is the density.

Part A:
Find an expression for the radial acceleration inside the globular cluster.

The divergence due of some gravitational field due to a mass distribution may be described by

$$\frac{1}{r^2}\frac{\partial(r^2 a)}{\partial r} = -4\pi G\rho$$

For the example globular cluster described in the question, the mass density of the cluster may be substituted into the divergence expression:

$$\frac{1}{r^2}\frac{\partial(r^2 a)}{\partial r} = -4\pi GA(\frac{1}{r} - \frac{1}{r_0})^2$$

$$\frac{1}{r^2}\frac{\partial(r^2 a)}{\partial r} = -4\pi GA(\frac{1}{r^2} - \frac{2}{r_0 r} + \frac{1}{r_0^2})$$

To find an expression for the radial acceleration due to the mass density, the equation may be rearranged to find a solution for $a(r)$,

$$\frac{\partial(r^2 a)}{\partial r} = -4\pi GA(1 - \frac{2r}{r_0} + \frac{r^2}{{r_0}^2})$$

$$\int_0^r \partial(r^2 a) = -4\pi GA \int_0^r (1 - \frac{2r}{r_0} + \frac{r^2}{{r_0}^2})\partial r$$

$$r^2 a = -4\pi GA(r - \frac{r^2}{r_0} + \frac{r^3}{3{r_0}^2})$$

$$a = -4\pi GA(\frac{1}{r} - \frac{1}{r_0} + \frac{r}{3{r_0}^2})$$

And so the radial acceleration inside the globular cluster may be given by the above expression, where $r \leq r_0$.

Part B:

Using the mass and radius of a real globular cluster (you will need to look up some literature to find this, quote your source), determine the constant A, and hence find a numerical value for the acceleration inside the globular cluster at $r = r_0/2$. Compare this number with the acceleration at the edge of the globular cluster ($r = r_0$).

According to J. Boyles, et al. (2011), the globular cluster NGC 6325 has a mass $m = 2.23 \times 10^5 M_\odot \approx 4.44 \times 10^{35} kg$, and a radius $r_0 = 15.8pc \approx 4.88 \times 10^{17} m$.

Using Gauss' Law, the gravitational flux through the surface of the cluster's radius is found as

$$\int_A a \cdot dA = -4\pi Gm$$

Rearranging to find an expression for just a gives

$$a \int_A dA = -4\pi Gm$$

$$a(4\pi {r_0}^2) = -4\pi Gm$$

Equating this acceleration for the formula for the acceleration in the cluster,

$$a = -4\pi GA(\frac{1}{r} - \frac{1}{r_0} + \frac{r}{3{r_0}^2})$$

$$\Rightarrow -4\pi GA(\frac{1}{r} - \frac{1}{r_0} + \frac{r}{3{r_0}^2})(4\pi r_0^2) = -4\pi Gm$$

Since the equation is being evaluated at $r = r_0$,

$$-4\pi GA(\frac{1}{r_0} - \frac{1}{r_0} + \frac{r_0}{3{r_0}^2})(4\pi r_0^2) = -4\pi Gm$$

Finally, rearranging to solve for A gives,

$$A(\frac{4\pi r_0}{3}) = m$$

$$A = \frac{3m}{4\pi r_0}$$

To find the acceleration at half the radius of NGC 6325, let $r = \frac{r_0}{2}$ in the original acceleration equation with the formula for A substituted in:

$$a = -4\pi G(\frac{3m}{4\pi r_0})(\frac{1}{(\frac{r_0}{2})} - \frac{1}{r_0} + \frac{(\frac{r_0}{2})}{3{r_0}^2})$$

Substituting in all values for the constants, and the sourced properties of NGC 6325 gives:

$$a_{r_0/2} \approx -4.36 \times 10^{-10} m/s^2$$

Thus, at half of the radius of NGC 6325, the acceleration that a mass would experience towards the centre would be approximately $4.36 \times 10^{-10} m/s^2$. The acceleration a mass would experience at the edge of the cluster may be given by the acceleration formula with $r = r_0$:

$$a = -4\pi G(\frac{3m}{4\pi r_0})(\frac{1}{r_0} - \frac{1}{r_0} + \frac{r_0}{3{r_0}^2})$$

$$a = -G(\frac{m}{{r_0}^2})$$

Substituting in the values for G and the properties of NGC 6325 gives:

$$a_{r_0} \approx -1.24 \times 10^{-10} m/s^2$$

Therefore, at the edge of the cluster NGC 6325, the acceleration that a mass would experience towards the centre would be approximately $-1.24 \times 10^{-10} m/s^2$. The acceleration at half the radius of the cluster is about 3.52 times that of the acceleration at the edge of the cluster. Given that gravitational acceleration is proportional to the inverse square law of the radius from some mass, one might expect the acceleration at $\frac{r_0}{2}$ to be 4 times that of the acceleration at r_0. However, as distance from the middle of the cluster is decreased, there is more mass outwards (away from the centre) which influences the field in the opposite direction. Therefore, it is reasonable that the acceleration between the centre and the edge of the cluster is less than 4 times that at the edge.

Appendices:

Appendix 1: Problem 1.1a MATLAB Scalar Field Script

```
x = linspace(-4, 4, 100);
y = linspace(-2, 2, 100);
[X, Y] = meshgrid(x, y);
h = (10 * exp(-1 * (1 / 10) * X.^2)) + ((1 / 10) * Y.^2);

surf(X, Y, h, 'EdgeColor', 'none');
colorbar;
xlabel('x');
ylabel('y');
```

Appendix 2: Problem 1.1b MATLAB Vector Field Script

```
X = linspace(-4, 4, 20);
Y = linspace(-2, 2, 20);
[x, y] = meshgrid(X, Y);
h = (10 * exp(-1 * (1 / 10) * x.^2)) + ((1 / 10) * y.^2);

[dfdx, dfdy] = gradient(h);
quiver(X, Y, dfdx, dfdy);
xlabel('x');
ylabel('y');
```

Appendix 3: Problem 1.1c MATLAB Divergence Field Script

```
X = linspace(-4, 4, 100);
Y = linspace(-2, 2, 100);
[x, y] = meshgrid(X, Y);
h = (10 * exp(-1 * (1 / 10) * x.^2)) + ((1 / 10) * y.^2);

[dfdx, dfdy] = gradient(h);
div = divergence(X, Y, dfdx, dfdy);
surf(X, Y, div, 'EdgeColor', 'none');
colorbar;
xlabel('x');
ylabel('y');
```

Bibliography

1. Austral Wright Metals, n.d., *Product Data Sheet – Stainless Steel,* Accessed 28/03/2020, <https://www.australwright.com.au/304-stainless-steel/>
2. Boyles, J.; et al., 2011, *Young Radio Pulsars in Galactic Globular Clusters,* The Astrophysical Journal, 742: 51, Accessed 28/03/2020, doi:10.1088/0004-637X/742/1/51
3. Cooper, I, n.d., *Energy Transfer by Conduction Through Composite Materials*, Accessed 28/03/2020, <http://www.physics.usyd.edu.au/teach_res/mp/doc/tp_conduction.pdf>