Chaos Assignment 2

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Q1 $H = \frac{p^2}{2} + V(q) \qquad V(q) = |q|^r$

$r > 0$, constant

a. $\dot{q} = \frac{\partial H}{\partial p} = p$

$\dot{p} = -\frac{\partial H}{\partial q} = -\partial(|q|^r)/\partial q$

$= -(r|q|^{r-1} \cdot q \cdot |q|^{-1})$

$= -rq|q|^{r-2} = -r\frac{|q|^r}{q}$

$\left(\text{since } \frac{1}{|q|^2} = \frac{1}{q^2} \Rightarrow \frac{q}{q^2} = \frac{1}{q}\right)$

Fixed points when $(\dot{q}, \dot{p}) = (0,0)$

$\dot{q} = 0 \Rightarrow p = 0$

$\dot{p} = 0 \Rightarrow q = 0 \quad (r \geq 2)$

Define

$$A = \begin{pmatrix} \frac{\partial \dot{q}}{\partial q} & \frac{\partial \dot{q}}{\partial p} \\ \frac{\partial \dot{p}}{\partial q} & \frac{\partial \dot{p}}{\partial p} \end{pmatrix}_{(q,p)=(0,0)}$$

$$= \begin{pmatrix} 0 & 1 \\ -\frac{(r-1)r|q|^r}{q^2} & 0 \end{pmatrix}_{(q,p)=(0,0)}$$

$$= \begin{pmatrix} 0 & 1 \\ -0 & 0 \end{pmatrix} \; [r>2] \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ (r-1)r & 0 \end{pmatrix} \; [r=2]$$

at $r = 2$, $\det(A) = 2$

$\Rightarrow \Delta = -8$

$\Rightarrow (0,0)$ spiral or elliptic fixed

Since $\mathrm{Tr}(A) = 0$,

$\mu = 0 \Rightarrow$ elliptic fixed

for $r > 2$ $\det(A) = 0$

$\Rightarrow (0,0)$ is a degenerate node with $\lambda = 0$

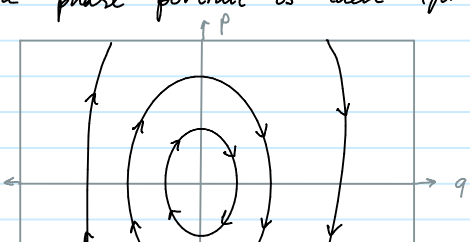
and so there is a line of unstable fixed points with no stable fixed points.

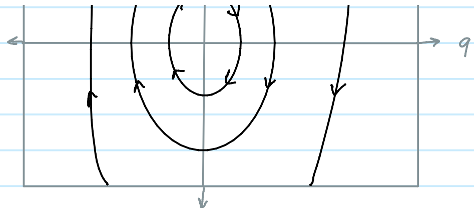
For $r = 2$, when $q > 0$, $\dot{p} < 0$

$p > 0$, $\dot{q} > 0$

And so the phase portrait consists of clockwise motion about the fixed point.

The phase portrait is then (for $r = 2$)





b. For some H, the positive turning q coordinate, q_+ is

$$H = \frac{p^2}{2} + q_+^r$$
$$q_+^r = H - \frac{p^2}{2} \Rightarrow q_+ = \left(H - \frac{p^2}{2}\right)^{1/r}$$

But the turning point occurs at $p=0$, so

$$q_+ = H^{1/r}$$

Rearranging H to give an expression of p:

$$H = \frac{p^2}{2} + |q|^r$$
$$\Rightarrow p = \pm\sqrt{2H - 2|q|^r}$$

The action variable is then

$$I(H) = \frac{1}{2\pi}\oint p(q,H)\,dq$$
$$= \frac{1}{2\pi}\left[\int_{-q_+}^{q_+}\sqrt{2H-2|q|^r}\,dq - \int_{q_+}^{-q_+}\sqrt{2H-2|q|^r}\,dq\right]$$
$$= \frac{\sqrt{2}}{\pi}\int_{-q_+}^{q_+}(H-|q|^r)^{1/2}\,dq$$

Since q is oscillatory, make the substitution

$$q = H^{1/r}\sin\phi \Rightarrow dq = H^{1/r}\cos\phi\,d\phi$$

over the bounds $\frac{\pi}{2} \to -\frac{\pi}{2}$

Then,

$$I(H) = \frac{\sqrt{2}}{\pi}\int_{-\pi/2}^{\frac{\pi}{2}}(H - H\sin^r\phi)^{1/2}\,H^{1/r}\cos\phi\,d\phi$$
$$= \frac{\sqrt{2}}{\pi}\left(H^{\frac{1}{2}+\frac{1}{r}}\right)\int_{-\pi/2}^{\pi/2}(1-\sin^r\phi)^{1/2}\cos\phi\,d\phi$$

but

$$I(1) = \frac{\sqrt{2}}{\pi}\int_{-\pi/2}^{\pi/2}(1-\sin^r\phi)^{1/2}\cos\phi\,d\phi$$
$$\Rightarrow I(H) = H^{\frac{r+2}{2r}}\,I_1$$

c. $$H^{\frac{r+2}{2r}} = \frac{I(H)}{I_1}$$
$$H = \left(\frac{I(H)}{I_1}\right)^{\frac{2r}{r+2}}$$

The frequency of oscillation is found by

$$\omega = \frac{\partial H}{\partial I(H)}$$
$$= \frac{\partial}{\partial I}\left(\frac{I(H)}{I_1}\right)^{\frac{2r}{r+2}}$$
$$\frac{2r}{}\left(\frac{I(H)}{}\right)^{\frac{2r}{r+2}-1}\cdot\frac{1}{r}$$

$$= \frac{\partial}{\partial I}\left(\frac{I(H)}{I_1}\right)^{\frac{2r}{r+2}}$$

$$= \frac{2r}{r+2}\left(\frac{I(H)}{I_1}\right)^{\frac{2r}{r+2}-1} \cdot \frac{1}{I_1}$$

$$= \frac{2r\left(\frac{I(H)}{I_1}\right)^{\frac{2r}{r+2}}}{(r+2)\ I(H)}$$

d. $q_+ = H^{1/r} \Rightarrow H = q_+^{\ r}$

$$\Rightarrow I(H) = H^{\frac{r+2}{2r}} I_1$$

$$= (q_+^{\ r})^{\frac{r+2}{2r}} I_1 = q_+^{\frac{r+2}{2}} I_1$$

$$\Rightarrow \omega = \frac{2r\left(q_+^{\frac{r+2}{2}}\right)^{\frac{2r}{r+2}}}{(r+2)\, q_+^{\frac{r+2}{2}} I_1}$$

$$= \frac{2r}{(r+2) I_1} \frac{q_+^{\ r}}{q_+^{\frac{r+2}{2}}}$$

$$= \frac{2r}{(r+2) I_1} q_+^{\ r - \frac{r+2}{2}}$$

$$= \frac{2r}{(r+2) I_1} q_+^{\frac{r-2}{2}}$$

$$\frac{\partial \omega}{\partial q_+} = \frac{2r}{(r+2) I_1} \frac{(r-2)}{2} q_+^{\frac{r-2}{2}-1}$$

$$= \frac{2r^2 - 4r}{I_1(2r+4)} q_+^{\frac{r-4}{2}}$$

$$\frac{\partial \omega}{\partial q_+} = 0 \Rightarrow 2r^2 - 4r = 0$$

$$\Rightarrow r^2 = 2r$$

$$r = 2$$

This corresponds to the only stable fixed point solution. Also note that $r=2 \Rightarrow H = \frac{1}{2} p^2 + |q|^2$ which is of the form of the simple harmonic oscillator!

Q2

$$H = \frac{p^2}{2m} - mgq$$

a. $\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$

$$\dot{p} = -\frac{\partial H}{\partial q} = mg$$

$$\Rightarrow q = \int \frac{p}{m}\, dt$$

$$= \frac{p}{m} t + c$$

$$p = \int mg\, dt$$

$$= mgt + d$$

b. Take a rectangle in phase space described by points

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$p_2 > p_1$, $q_2 > q_1$

$p_1 = mgt + d_1 \quad p_2 = mgt + d_2$

$q_1 = \frac{p}{m}t + c_1 \quad q_2 = \frac{p}{m}t + c_2$
$= gt^2 + \frac{d_1}{m}t + c_1 \quad = gt^2 + \frac{d_2}{m}t + c_2$

And so the area at $t=0$

$$A(t=0) = \Delta p \Delta q$$
$$= (p_2 - p_1)(q_2 - q_1)$$
$$= (d_2 - d_1)(c_2 - c_1)$$

which of the form of a rectangle, as expected.

After some t

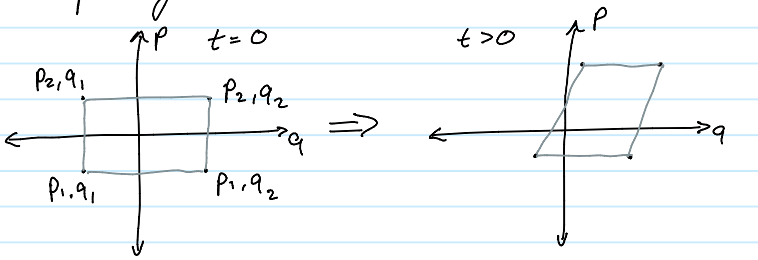
$p_2 = mgt + d_2 \quad p_1 = mgt + d_2$

$q_2 = \frac{p}{m}t + c_2 \quad q_1 = \frac{p}{m}t + c_1$

Recall

$\dot{q} = \frac{p}{m} \quad \dot{p} = mg$

And so p_2 and p_1 are always equidistant.
q scales as a function of p and so q_1 and q_2 will change separation
Graphically, this is



For all t, an area of a parallelogram is $\Delta q \Delta p$
$\forall t > 0$, $\Delta p = (d_2 - d_1)$ (constant $\forall t$)

$$\Delta q = q_2 - q_1$$
$$= \frac{p}{m}t + c_2 - \left(\frac{p}{m}t + c_1\right)$$
$$= c_2 - c_1$$

$$\Rightarrow A = (c_2 - c_1)(d_2 - d_1)$$

which is the same for $t=0$, so the top and bottom sides remain equal length $\forall t$.

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Therefore the four points trace a parallelogram with constant area $\forall t$

Q3

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) = \frac{\vec{p}^{\,2}}{2m}$$

$$L_z = x p_y - y p_x$$

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m} \qquad \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$\frac{\partial L}{\partial p_x} = -y \qquad \frac{\partial L}{\partial p_y} = x$$

a.
$$\begin{aligned}\{p_x, H\} &= \sum_{i=1}^{2}\left(\frac{\partial p_x}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial p_x}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \\ &= \frac{\partial p_x}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial p_x}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial p_x}{\partial y}\frac{\partial H}{\partial p_y} \\ &\quad - \frac{\partial p_x}{\partial p_y}\frac{\partial H}{\partial y} \\ &= 0\cdot\frac{p_x}{m} - 1\cdot 0 + 0\cdot\frac{p_y}{m} - 0\cdot 0 \\ &= 0\end{aligned}$$

$$\begin{aligned}\{p_y, H\} &= \sum_{i=1}^{2}\left(\frac{\partial p_y}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial p_y}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \\ &= \frac{\partial p_y}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial p_y}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial p_y}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial p_y}{\partial p_y}\frac{\partial H}{\partial y} \\ &= 0\cdot\frac{p_x}{m} - 0\cdot 0 + 0\cdot\frac{p_y}{m} - 1\cdot 0 \\ &= 0\end{aligned}$$

$$\begin{aligned}\{L_z, H\} &= \sum_{i=1}^{2}\left(\frac{\partial L_z}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial L_z}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \\ &= \frac{\partial L_z}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial L_z}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial L_z}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial L_z}{\partial p_y}\frac{\partial H}{\partial y} \\ &= p_y\cdot\frac{p_x}{m} + y\cdot 0 - p_x\cdot\frac{p_y}{m} - x\cdot 0 \\ &= p_y\cdot\frac{p_x}{m} - p_y\cdot\frac{p_x}{m} \\ &= 0\end{aligned}$$

Therefore, p_x, p_y and L_z are all constants of motion.

b. If $\{c(\vec{q},\vec{p}), H\} = 0$

and $F(\vec{q},\vec{p}) := F(c(\vec{q},\vec{p}))$, then

$$\begin{aligned}\{F, H\} &= \sum_{i=1}^{2}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \\ &= \sum_{i=1}^{2}\left(\frac{\partial F}{\partial c}\frac{\partial c}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial c}\frac{\partial c}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \\ &= \frac{\partial F}{\partial c}\sum_{i}^{2}\left(\frac{\partial c}{\partial}\frac{\partial H}{\partial} - \frac{\partial c}{\partial}\frac{\partial H}{\partial}\right)\end{aligned}$$

$$= \sum_{i=1}^{2} \left(\frac{\partial F}{\partial c} \frac{\partial c}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial c} \frac{\partial c}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$= \frac{\partial F}{\partial c} \sum_{i=1}^{2} \left(\frac{\partial c}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial c}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

but, since $\{c, H\} = 0$

$$\sum_{i=1}^{2} \left(\frac{\partial c}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial c}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = 0$$

by definition of the Poisson bracket with two degrees of freedom.

Therefore,

$$\{F, H\} = \frac{\partial F}{\partial c} \cdot 0 = 0$$

and so every F is a constant of motion.