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MATH3401

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Submission Topic	Assignment 2		

Due date of submission	4/04/2022			Marked
Due time of submission	11:50 AM	Name of tutor (if appropriate)	Marcus Flook	**Yes / No**
Group	05	Session attended (if appropriate)	Thursday (day) 8am (time)	
Actual submission date	1/04/2022			

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MATH3401 Assignment 2

Ryan White s4499039

4th of April, 2022

Question 1

We want a mobius transform, $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that satisfies three key mappings. First, note the general form of a mobius transform:

$$T(z) = \frac{az+b}{cz+d} \tag{1}$$

Now, the three conditions will narrow down the general form to match the required criteria:

- The first condition is that $T(2) = 0 \Rightarrow 2a + b = 0 \Rightarrow b = -2a$ and so the general form becomes

$$T(z) = \frac{a(z-2)}{cz+d}$$

- The second condition necessitates that $T(i) = \infty \Rightarrow ic + d = 0 \Rightarrow d = -ic$ and so the transform becomes

$$T(z) = \frac{a(z-2)}{c(z-i)}$$

- The third and final condition says that $T(0) = -2i$, which means that

$$\Rightarrow \frac{-2a}{-ic} = -2i$$
$$\frac{a}{ic} = -i$$
$$a = -i^2 c = c$$
$$\Rightarrow \frac{a}{c} = 1$$

  And so the transform (finally) becomes

$$T(z) = \frac{z-2}{z-i}$$

Question 2

Let T be a mapping from $\mathbb{C} \to \mathbb{C}$.

a. Given the general form of the mobius transform by equation 1, the implicit form, for some fixed point $T(z) = z$, is given by:

$$Az^2 + (B+C)z + D = 0 \tag{2}$$

where $A = c$, $B = -a$, $C = d$, and $D = -b$ from the general form. Therefore, the implicit form can be rewritten as

$$cz^2 + (d-a)z - b = 0 \tag{3}$$

Of course, this is in the form of the quadratic equation, with solutions (fixed points in this case) found by:

$$z = \frac{a - d + ((d-a)^2 + 4bc)^{1/2}}{2c} \tag{4}$$

And so:

- There will be no fixed points if the constant $c = 0$ in the mobius transform. Also notice the case for $z = 0$, in which case b must not be 0 so that some shift occurs away from 0 (meaning that $z = 0$ does not correspond to a fixed point).
- There will be one solution when $(d-a)^2 + 4bc = 0$, $c \neq 0$ since that the root in equation 4 is single valued when this is the case.
- There will be two solutions when $(d-a)^2 + 4bc \neq 0$, $c \neq 0$, as the root in equation 4 is double valued when this is the case.
- Naturally, the identity transformation $z \to z$ will correspond to infinite fixed points.

The above shows that, due to the nature of solutions to fixed points of the mobius transform, that there can be at most two fixed points for a non-identity mapping.

b. i. There are two solutions when $(d-a)^2 + 4bc \neq 0$. As such, take $a = b = 1$, $c = 2$ and $d = 5 \Rightarrow (d-a)^2 + 4bc = 24 \neq 0$. The corresponding mobius transform is

$$T(z) = \frac{z+1}{2z+5}$$

which has fixed points at:

$$\begin{aligned} z_1 &= \frac{a-d+((d-a)^2+4bc)^{1/2}}{2c} = -1+6 = 5 \\ z_2 &= \frac{a-d+((d-a)^2+4bc)^{1/2}}{2c} = -1-6 = -7 \end{aligned}$$

ii. There is one solution when $(d-a)^2 + 4bc = 0$. Take $a = b = 2$, $c = -2$, and $d = 6 \Rightarrow (d-a)^2 + 4bc = 0$. The transform is then

$$T(z) = \frac{2z+2}{-2z+6}$$

which has fixed point at

$$z = \frac{a-d+0}{2c} = \frac{-4}{-4} = 1$$

iii. There are no fixed point solutions when $c = 0$ and $b \neq 0$. Therefore, take the transform

$$T(z) = \frac{2z+1}{3}$$

Question 3

a. Recall that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

The complex conjugate of this is then found by

$$\begin{aligned} \overline{\sin z} &= \overline{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)} \\ &= \frac{e^{-i\bar{z}} - e^{i\bar{z}}}{-2i} \\ &= \frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i} \\ &= \sin \overline{z} \end{aligned}$$

b. Recall that

$$\begin{aligned} \cosh z &= \frac{e^{z} - e^{-z}}{2} \\ \Rightarrow \overline{\cosh z} &= \overline{\left(\frac{e^{z} - e^{-z}}{2}\right)} \\ &= \frac{e^{\overline{z}} - e^{-\overline{z}}}{2} \\ &= \cosh \overline{z} \end{aligned}$$

Question 4

a. Beginning with $\log z = 4i$, first exponentiate both sides:

$$\begin{aligned} z &= e^{4i} \\ &= \cos(4) + i\sin(4) \end{aligned}$$

However, since we began with the lowercase log, there are an infinite family of solutions for additional $2n\pi$, $n \in \mathbb{Z}$ rotations in the complex plane. As such,

$$z = \cos(4 + 2n\pi) + i\sin(4 + 2n\pi)$$

b. Starting with $z^i = i$, first log both sides;

$$\begin{aligned} \log(z^i) &= \log i \\ i\log z &= \ln|i| + i\arg(i) \\ &= 0 + i\left(\frac{\pi}{2} + 2n\pi\right) \qquad n \in \mathbb{Z} \\ \Rightarrow \log z &= \frac{\pi}{2} + 2n\pi \\ \Rightarrow z &= e^{\pi/2 + 2n\pi} \end{aligned}$$

And so z has no imaginary component.

Question 5

a. Firstly, note that $\cot z = \cos z / \sin z$, and define

$$\begin{aligned} w &= \cot^{-1} z \\ \Rightarrow z &= \cot w \\ &= \frac{\cos w}{\sin w} \\ &= \frac{e^{iw} + e^{-iw}}{2} \cdot \frac{2i}{e^{iw} - e^{-iw}} \\ &= i\left(\frac{e^{iw} + e^{-iw}}{e^{iw} - e^{-iw}}\right) \\ &= i\left(\frac{e^{iw} + e^{-iw}}{e^{iw} - e^{-iw}} \cdot \frac{e^{iw}}{e^{iw}}\right) \\ &= i\left(\frac{e^{2iw} + 1}{e^{2iw} - 1}\right) \\ \Rightarrow -iz &= \frac{e^{2iw} + 1}{e^{2iw} - 1} \\ \Rightarrow -ize^{2iw} + iz &= e^{2iw} + 1 \\ \Rightarrow e^{2iw}(1 + iz) &= (iz - 1) \\ \Rightarrow e^{2iw} &= \frac{iz - 1}{1 + iz} \\ &= \frac{iz - 1}{1 + iz} \cdot \frac{-i}{-i} \\ &= \frac{z + i}{z - i} \end{aligned}$$

Taking the logarithm of both sides gives

$$\begin{aligned} 2iw &= \log\left(\frac{z + i}{z - i}\right) \\ w &= \frac{1}{2i}\log\left(\frac{z + i}{z - i}\right) = -\frac{i}{2}\log\left(\frac{z + i}{z - i}\right) \end{aligned}$$

But remember that $w = \cot^{-1} z$, and so

$$\cot^{-1} z = -\frac{i}{2} \log\left(\frac{z+i}{z-i}\right)$$

Now, note that for $\cot^{-1} z$ to be defined, $z \neq \pm i$. This is because if $z = i$, a forbidden division by 0 occurs. In the other case that $z = -i$, the value within the logarithm is 0 in which case a computation of $\log(0)$ would take place which is undefined.

b. Notice from the previous part that

$$\cot z = i\left(\frac{e^{2iz}+1}{e^{2iz}-1}\right)$$

Now, we want to find solutions of the above such that $\cot z = 1$;

$$\begin{aligned}
1 &= i\left(\frac{e^{2iz}+1}{e^{2iz}-1}\right) \\
-i &= \frac{e^{2iz}+1}{e^{2iz}-1} \\
-ie^{2iz} + i &= e^{2iz} + 1 \\
i - 1 &= e^{2iz} + ie^{2iz} \\
e^{2iz} &= \frac{i-1}{i+1}
\end{aligned}$$

Once again, taking the logarithm of each side gives

$$\begin{aligned}
2iz &= \log\left(\frac{i-1}{i+1}\right) \\
z &= -\frac{i}{2}\log\left(\frac{i-1}{i+1}\right) \\
&= -\frac{i}{2}\left(\ln\left|\frac{i-1}{i+1}\right| + i\arg\left(\frac{i-1}{i+1}\right)\right) \\
&= -\frac{i}{2}\left(0 + i\left(\frac{3\pi}{4} - \frac{\pi}{4} + 2n\pi\right)\right) \qquad n \in \mathbb{Z} \\
&= -\frac{i}{2}\left(i\left(\frac{\pi}{2} + 2n\pi\right)\right) \\
&= \frac{1}{2}\left(\frac{\pi}{2} + 2n\pi\right) \\
z &= \frac{\pi}{4} + n\pi
\end{aligned}$$

And so the solution of $\cot z = 1$ is $z = \pi/4 + n\pi$, $n \in \mathbb{Z}$. Note that there is no imaginary component for this solution.

Question 6

Let Ω_1 and Ω_2 be two non-empty, closed sets in $\mathbb{C}$.

a. Consider the case in which Ω_1 and Ω_2, two non-empty and closed sets, are disjoint:

- Both $\partial\Omega_1 \subseteq \Omega_1$ and $\partial\Omega_2 \subseteq \Omega_2$, since they are each defined as closed sets.
- As a result, $\partial(\Omega_1 \cup \Omega_2) = \partial\Omega_1 \cup \partial\Omega_2 \subseteq (\Omega_1 \cup \Omega_2)$, and so $\Omega_1 \cup \Omega_2$ is closed in the disjoint case.

Now, consider the case where there is some intersection of Ω_1 and Ω_2, $\Omega_1 \cap \Omega_2$.

- A portion of the boundaries of each of Ω_1 and Ω_2 are now within the opposite region. That is,

$$\partial(\Omega_1 \cup \Omega_2) \subseteq \partial\Omega_1 \cup \partial\Omega_2$$

However, the region $\Omega_1 \cup \Omega_2$ is still completely enclosed by $\partial(\Omega_1 \cup \Omega_2)$, which is separate from $(\Omega_1 \cup \Omega_2)^c$, and since $\partial\Omega_1 \cup \partial\Omega_2 \subseteq \Omega_1 \cup \Omega_2$, then:

$$\partial(\Omega_1 \cup \Omega_2) \subseteq \Omega_1 \cup \Omega_2 \tag{5}$$

where the above holds by transitivity. Thus, equation (5) shows that $\Omega_1 \cup \Omega_2$ is closed by definition. Therefore, the union of two closed sets is always closed.

b. For the following, Ω_2 now denotes a non-empty, open set.

i. $\Omega_1 \cup \Omega_2$ *could* still be a closed set in the case that $\Omega_2 \subseteq \Omega_1$, where

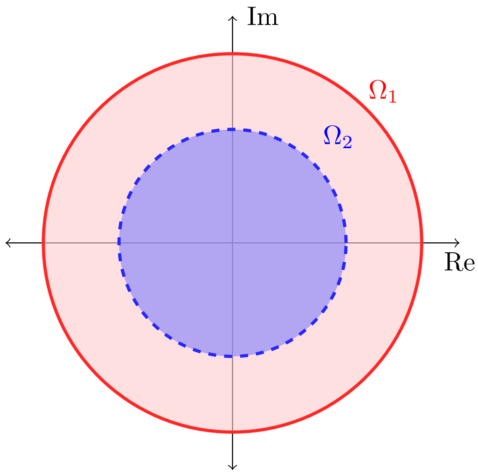
$$\partial(\Omega_1 \cup \Omega_2) = \partial\Omega_1 \cup \partial\Omega_2 \subseteq \partial\Omega_1 \subseteq \Omega_1$$

But Ω_1 is closed, and therefore so is $\Omega_1 \cup \Omega_2$.
As an example of how these sets could be interpreted is shown in Fig 1a

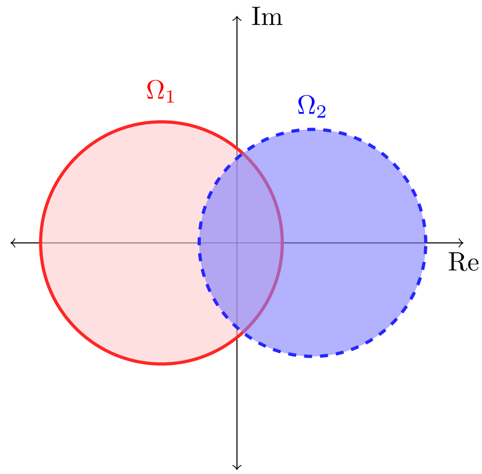
ii. The union $\Omega_1 \cup \Omega_2$ isn't necessarily closed in the case that Ω_2 is non-empty and open. Take the example $\Omega_2 \nsubseteq \Omega_1$, shown in Fig 1b. Therefore,

$$\partial(\Omega_1 \cup \Omega_2) = \partial\Omega_1 \cup \partial\Omega_2 \nsubseteq \Omega_1 \cup \Omega_2$$

And so this example does not fit the definition of a closed region.



(a) Case where $\Omega_2 \subseteq \Omega_1$



(b) Case where $\Omega_2 \nsubseteq \Omega_1$

Fig. 1 Valid Examples of Closed and not Closed $\Omega_1 \cup \Omega_2$