Final Exam Math 3403 Ryan White

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Question 1:

$$\begin{cases} \partial_t u - x\,\partial_x u = x^2 \partial_{xx} u & 1<x<e \quad t>0 \\ u(t, x=1) = u(t, x=e) = 0 & t>0 \\ u(t=0, x) = 1 & 1<x<e \end{cases}$$

Looking at the PDE, this is

$$\partial_t u = x^2 \partial_{xx} u + x\,\partial_x u$$
$$= x\left(x\,\partial_x u\right)_x$$

Check: $x(x\partial_x u)_x = x\partial_x u + x^2 \partial_{xx} u$

Therefore this holds!

Using separation of variables, $u(x,t) = T(t)X(x)$ and so the problem is

$$\partial_t TX = x\,(x\,\partial_x TX)_x$$
$$\Rightarrow T_t X = x(x\,TX_x)_x$$

Dividing both sides by $T(t)X(x)$ gives

$$\frac{T_t}{T} = \frac{x(xX_x)_x}{X}$$

And we want to find a λ such that

$$\frac{T_t}{T} = \lambda = \frac{x(xX_x)_x}{X}$$

Beginning with the right side,

$$\frac{x(xX)_x}{X} = \lambda$$
$$\Rightarrow x(xX)_x = \lambda X$$
$$\Rightarrow x^2 X_{xx} + xX_x = \lambda X$$

Guess a solution $X = x^\alpha$

$$\Rightarrow \lambda x^\alpha = x\,(x^\alpha)_x + x^2 (x^\alpha)_{xx}$$
$$= x\alpha x^{\alpha-1} + x^2 \alpha (x^{\alpha-1})_x$$
$$= x\alpha x^{\alpha-1} + x^2 \alpha(\alpha-1)\, x^{\alpha-2}$$
$$= \alpha x^\alpha + \alpha(\alpha-1)\, x^\alpha$$

(since $x^{\alpha-1} = x^\alpha / x \Rightarrow x x^{\alpha-1} = x^\alpha$, etc)

$$\lambda x^\alpha = x^\alpha(\alpha + \alpha(\alpha-1))$$

Dividing both sides by x^α gives

$$\lambda = \alpha + \alpha^2 - \alpha$$
$$= \alpha^2$$
$$\Rightarrow \alpha = \pm\sqrt{\lambda}$$

And so there are two solutions:

$$X(x) = Ax^{\sqrt{\lambda}} + Bx^{-\sqrt{\lambda}}$$

but $e^{a \ln x} = x^a$

$$\Rightarrow X(x) = Ae^{\ln x \sqrt{\lambda}} + Be^{-\ln x \sqrt{\lambda}}$$

but $e^{a \ln x} = x^a$

$\Rightarrow X(x) = Ae^{\ln x \sqrt{\lambda}} + Be^{-\ln x \sqrt{\lambda}}$

which has general solution:

$$X(x) = A\sinh(\ln x \sqrt{\lambda}) + B\cosh(\ln x \sqrt{\lambda})$$

By the ~~boundary~~ conds, when $x=1$,

$u(t, x=1) = 0 = X(1)$

$\Rightarrow 0 = A\sinh(0) + B\cosh(0)$

$= 0 + B \quad \Rightarrow B = 0$

and when $x = e$, $X(e) = 0$

$\Rightarrow 0 = A\sinh(\ln e \sqrt{\lambda})$

$= A\sinh(\sqrt{\lambda}) \quad \Rightarrow \sqrt{\lambda} < 0 \Rightarrow \lambda < 0$

so either $A = 0$, or $\sinh(\sqrt{\lambda}) = 0$

but $\sinh(x) = -i\sin(ix)$

$0 = -iA\sin(i\sqrt{\lambda}) \Rightarrow \frac{0}{-i} = A\sin(i\sqrt{\lambda})$

$\Rightarrow 0 = A\sin(i\sqrt{\lambda})$

$= A\sin(\sqrt{-\lambda})$

which is zero when $\sqrt{-\lambda_n} = n\pi$ for $n \in \mathbb{Z}$

$\Rightarrow \lambda_n = -n^2\pi^2$

substituting this into $X_n = A\sinh(\ln x \sqrt{\lambda})$ gives

$$X_n(x) = A_n \sin(n\pi \ln x)$$

Now, looking at $\frac{T_t}{T} = \lambda_n$:

$T_t = T\lambda_n$

$= -n^2\pi^2 T$

Which has solution $T_n(t) = e^{-n^2\pi^2 t}$

Check: $\partial_t e^{-n^2\pi^2 t} = -n^2\pi^2 e^{-n^2\pi^2 t}$

So this holds.

The general solution of the PDE is then:

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} A_n \sin(n\pi \ln x) e^{-n^2\pi^2 t}$$

And by the <u>initial</u> condition, $u(x, t=0) = 1$

$\Rightarrow 1 = \sum_{n=1}^{\infty} A_n \sin(n\pi \ln x)$

Integrating this against $\sin(m\pi \ln x)$ for $m \in \mathbb{Z}$ with respect to x:

$$\int_1^e 1 \cdot \sin(m\pi \ln x)\,dx = \sum_{n=1}^{\infty} A_n \int_1^e \sin(n\pi \ln x)\sin(m\pi \ln x)\,dx$$

Since $\int_a^b \sin(n\pi x)\sin(m\pi x) =$

– I can't spend more time on this. My answer is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi \ln x) e^{-n^2\pi^2 t}$$

Question 2:

a. $-2u_{xx} + u_{xy} + yu_x = 0$

This is a linear 2^{nd} order equation with $A = -2$, $B = 1$, $D = y$, and $G(x,y) = 0$

Classifying this:

$$\Delta = B^2 - 4AC$$
$$= 1^2 - 4 \cdot -2 \cdot 0$$
$$= 1^2$$

And so the PDE is Hyperbolic, with characteristics:

$$y'(x) = \frac{B \pm \sqrt{\Delta}}{2A} = \frac{1^2 \pm 1}{-4}$$

$$\Rightarrow \quad y_1(x) = \frac{1+1}{-4} x + \xi \Rightarrow \xi = y + \tfrac{1}{2}x$$

$$y_2(x) = \frac{1-1}{-4} x + \eta \Rightarrow \eta = y$$

With partial derivatives:

$$\xi_x = \tfrac{1}{2} \quad \xi_y = 1 \quad \xi_{xx} = \xi_{xy} = \xi_{yy} = 0$$

$$\eta_x = 0 \quad \eta_y = 1 \quad \eta_{xx} = \eta_{xy} = \eta_{yy} = 0$$

Hyperbolic equations have $w_{\xi\xi} = w_{\eta\eta} = 0$, and so the canonical transform is

$$0 = w_{\xi\eta}(2A\xi_x\eta_x + 2C\eta_y\xi_y + B(\eta_y\xi_x + \xi_y\eta_x))$$
$$+ w_\xi(D\xi_x) + w_\eta(D\eta_x)$$

Substituting values in gives

$$0 = w_{\xi\eta}(-4 \cdot 0 + 0 + (\tfrac{1}{2} \cdot 1 + 1 \cdot 0)$$
$$+ w_\xi(y \cdot 1) + w_\eta(y \cdot 0)$$
$$= \tfrac{1}{2}w_{\xi\eta} + yw_\xi$$
$$= \tfrac{1}{2}w_{\xi\eta} + \eta w_\xi$$
$$= (\tfrac{1}{2}w_\eta + \eta w)_\xi$$

Check: $(\tfrac{1}{2}w_\eta + \eta w)_\xi = \tfrac{1}{2}w_{\xi\eta} + \eta_\xi w + \eta w_\xi$

and since $\eta_\xi = 0$, this holds

$$\Rightarrow \quad 0 = (\tfrac{1}{2}w_\eta + \eta w)_\xi$$

$$\Rightarrow \tfrac{1}{2}w_\eta + \eta w = A(\eta) \quad \Rightarrow w_\eta + 2\eta w = 2A(\eta) = A(\eta)$$

Multiply both sides by e^{η^2}:

$$e^{\eta^2}w_\eta + \eta e^{\eta^2}w = e^{\eta^2}A(\eta)$$

Where $(e^{\eta^2}w)_\eta = 2\eta e^{\eta^2}w + e^{\eta^2}w_\eta$

$$\Rightarrow \quad (e^{\eta^2}w)_\eta = B(\eta) \qquad \text{where } B(\eta) = e^{\eta^2}A(\eta)$$

$$e^{\eta^2}w = G(\eta) + F(\xi) \qquad (G(\eta) = \textstyle\int B(\eta)\, d\eta)$$

$$\Rightarrow \quad (e^{\eta^2} w)_\eta = B(\eta) \qquad \text{where } B(\eta) = e^{\eta^2} A(\eta)$$

$$e^{\eta^2} w = G(\eta) + F(\xi) \qquad \left(G(\eta) = \int B(\eta)\, d\eta\right)$$

$$\Rightarrow w = G(\eta) + e^{-\eta^2} F(\xi)$$

Given that $u(x,y) = w(\xi(x,y), \eta(x,y))$
and $\xi = y + \frac{1}{2}x$ and $\eta = y$,

$$u(x,y) = G(y) + e^{-y^2} F(y + \tfrac{1}{2}x)$$

for some functions G and F.

b. From part a, we had that $0 = (w_\eta + 2\eta w)_\xi$
But now,

$$y = (w_\eta + 2\eta w)_\xi$$

and since $y = \eta$,

$$\eta = (w_\eta + 2\eta w)_\xi$$

$$\Rightarrow w_\eta + 2\eta w = \eta\xi + A(\eta)$$

As before, multiplying both sides by e^{η^2}:

$$(e^{\eta^2} w)_\eta = B(\eta)\xi + C(\eta)$$

where $B(\eta) = \eta e^{\eta^2}$ and $C(\eta) = e^{\eta^2} A(\eta)$

$$\Rightarrow e^{\eta^2} w = D(\eta)\xi + E(\eta) + F(\xi)$$

where $D(\eta) = \int B(\eta)\, d\eta$ and $E(\eta) = \int C(\eta)\, d\eta$

$$\Rightarrow w = G(\eta)\xi + H(\eta) + e^{-\eta^2} F(\xi)$$

where $G(\eta) = e^{-\eta^2} D(\eta)$ and $H(\eta) = e^{-\eta^2} E(\eta)$

As before, $u(x,y) = w(\xi(x,y), \eta(x,y))$
and $\xi = y + \frac{1}{2}x$, $\eta = y$

$$\Rightarrow u(x,y) = (y + \tfrac{1}{2}x) G(y) + H(y) + e^{-y^2} F(y + \tfrac{1}{2}x)$$

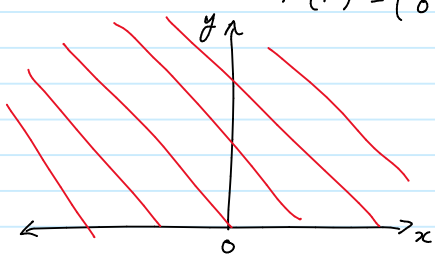
which is the general solution.

<u>Question 3:</u>

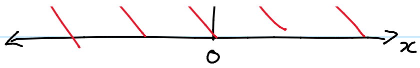
$$\begin{cases} u_x + (2-x) u_y + u = 0 & \text{on } \{y > 0, x \in \mathbb{R}\} \\ u(x, y=0) = e^x & \text{for } x < x_0 \end{cases}$$

This has $\vec{V} = \begin{pmatrix} 1 \\ 2-x \end{pmatrix}$

$$\gamma(r) = \begin{pmatrix} r \\ 0 \end{pmatrix}$$



Looking for change of coordinates such that



Looking for change of coordinates such that
$$(s,r) \mapsto (x(s,r), y(s,r))$$

Take
$$\begin{cases} x_s = 1 \\ x(s=0,r) = r \end{cases} \Rightarrow \left.\begin{array}{l} x(s,r) = s + A(r) \\ x(0,r) = r = A(r) \end{array}\right\} \Rightarrow x(s,r) = s+r$$
and
$$\begin{cases} y_s = (2-x) \\ y(s=0,r) = 0 \end{cases} \Rightarrow \left.\begin{array}{l} y(s,r) = s^2 + rs + B(r) \\ y(0,r) = 0 = B(r) \end{array}\right\} \Rightarrow y(s,r) = s^2 + rs$$

??

Question 4:

We have
$$\begin{cases} u_t - u_{xx} = -au & x \in \mathbb{R}^+ \\ u(x,0) = \delta(x-b) \\ u_x(0,t) = 0 \end{cases}$$

Define $v(x,t) = e^{at} u(x,t)$

$$\Rightarrow \quad v_t = a e^{at} u(x,t) + e^{at} u_t(x,t)$$
$$v_{xx} = e^{at} u_{xx}(x,t)$$
$$\Rightarrow \quad u_{xx} = e^{-at} v_{xx}$$
$$e^{-at} v_t = au + u_t$$
$$\Rightarrow \quad u_t - u_{xx} + au = 0 = e^{-at} v_t - e^{-at} v_{xx}$$
$$= v_t - v_{xx}$$

Therefore, $v(x,t)$ solves the heat equation.
Now, take the even extension of $\delta(x-b) = f(x)$ defined by
$$f_0(x) = \begin{cases} f(x) = \delta(x-b) & x \geq 0 \\ f(-x) = \delta(-x-b) & x < 0 \end{cases}$$
Also note that the derivative of the even extension is odd, with $f_{0,x}(x) = 0 = u_x(0,t)$ (as shown in lectures)
Then, the system is
$$\begin{cases} v_t - v_{xx} = 0 & x \in \mathbb{R}^+ \\ v(x,0) = f_0(x) \end{cases}$$

We know that $v_x(x,t) = 0$ $\forall t$ since the solution to the initial value problem with even data is itself even.

Therefore the solution is:
$$v(x,t) = \int_0^\infty \Phi(x-y,t) f_0(y)\,dy + \int_{-\infty}^0 \Phi(x-y,t) f_0(-y)\,dy$$
$$= \int^\infty \Phi(x-y,t)\,\delta(y-b)\,dy + \int^\infty \Phi(x+y,t)\,\delta(y-b)\,dy$$

$$v(x,t) = \int_0^{\infty} \Phi(x-y,t)\, g_0(y)\,dy + \int_{-\infty}^{0} \Phi(x-y,t)\, g_0(-y)\,dy$$

$$= \int_0^{\infty} \Phi(x-y,t)\,\delta(y-b)\,dy + \int_0^{\infty} \Phi(x+y,t)\,\delta(y-b)\,dy$$

And, since $\int_0^{\infty} \Phi(x-y,t)\,\delta(y-b)\,dy = \Phi(x-b,t)$

$$v(x,t) = \Phi(x-b,t) + \Phi(x+b,t)$$

and, since $v(x,t) = e^{at}u(x,t)$,

$$u(x,t) = e^{-at}\big(\Phi(x-b,t) + \Phi(x+b,t)\big)$$

Question 5:

We have

$$\begin{cases} -u_{xx} = f(x) \\ u(0) = u(L) = 0 \end{cases}$$

choose a Green's Function, $G(x)$, such that

$$\begin{cases} -\Delta G(x) = \delta(x-x_0) & \text{in } D=[0,L] \\ G(x) = 0 = g(x) & \text{on } \partial D \end{cases}$$

with $G(x,x_0) = \Phi_L(x) - \varphi^x(x_0)$

With representation formula:

$$u(x_0) = -\int_{\partial D} g(y)\,\frac{\partial G(y,x_0)}{\partial n}\,dS_y + \int_D f(y)\,G(y,x_0)\,dy$$

but boundary conditions necessitate $g(y) = 0$

$$\Rightarrow u(x_0) = \int_D f(y)\,G(y,x_0)\,dy$$

$$= \int_0^L f(y)\,G(y,x_0)\,dy$$

$$= \int_0^L f(y)\big(\Phi_L(y-x_0) - \varphi^x(y)\big)\,dy$$

Question 6:

a. Consider the system

$$\begin{cases} \Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

and assume that there are two solutions, u_1 and u_2, such that

$u_1 - u_2 = w$

solves the Dirichlet problem:

$$\begin{cases} \Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

solves the Dirichlet problem:

$$\begin{cases} \Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

Green's First identity (Divergence Theorem) says

$$\iint_{\partial U} w \frac{\partial w}{\partial n} \, dS = \iiint_U \nabla w \cdot \nabla w \, dx + \iiint_U w \Delta w \, dx$$

but $w = 0$ on ∂U (Dirichlet BC), and $\Delta w = 0$ in U. Therefore,

$$\iiint_U \nabla w^2 \, dx = 0$$

and since ∇w^2 is negative nowhere

$$\iiint_U \nabla w^2 \, dx = 0 \Rightarrow w = 0 \quad \text{in } U$$

$$\Rightarrow \quad u_1 - u_2 = 0 \Rightarrow u_1 = u_2$$

and so there is only one unique solution to the system

$$\begin{cases} \Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

b. Since u_{xx} and u_{yy} are equally weighted, this corresponds to radial symmetry about the center of the disk, with

$$r = \sqrt{x^2 + y^2}$$

and so it has solution (by lecture 10)

$$(r u_r)_r = 5$$

$$\Rightarrow \quad r u_r = 5r + c_1$$

$$u_r = 5 + \frac{c_1}{r}$$

$$\Rightarrow u(r) = 5r + c_1 \ln(r) + c_2$$

with boundary condition $u(a) = 0$:

$$u(a) = 0 = 5a + c_1 \ln(a) + c_2$$

$$\Rightarrow c_2 = -5a - c_1 \ln(a)$$

$$\Rightarrow u(r) = 5(r - a) + c_1 (\ln(r) - \ln(a))$$

$$= 5(r - a) + c_1 \ln\left(\frac{r}{a}\right)$$

but since $r = \sqrt{x^2 + y^2}$,

$$u(x, y) = 5\left(\sqrt{x^2 + y^2} - a\right) + c_1 \ln\left(\frac{\sqrt{x^2 + y^2}}{a}\right)$$

How was this meant to take 2 hours?

I found many of the questions took 2 hours each

Also the fact that many of these problems were harder than assignment questions confuses me greatly.