Assignment 1

Wednesday, 2 August 2023 4:08 PM

Q1

a. $F = G\dfrac{m_1 m_2}{d^2}$

$[F] = M^1 L^1 T^{-2}$ - dimensions of force

$[m_1] = [m_2] = M^1$ - dimensions of mass

$[d] = L^1$ - dimensions of distance

And so Newton's law of gravitation has dimensions:

$\Rightarrow M^1 L^1 T^{-2} = [G]\cdot M^2 \cdot L^{-2}$

$\Rightarrow [G] = L^3 M^{-1} T^{-2}$

which are the dimensions of the gravitational constant.

b. We expect orbital period to be proportional to
$P \propto M^a r^b G^c$

$\Rightarrow [P] = M^a L^b \quad ([G])^c$

$[P] = M^a L^b \quad (L^3 M^{-1} T^{-2})^c$

$= M^{a-c} L^{b+3c} T^{-2c}$

$= T^1$ - what we need

Now we can solve simultaneous equations for the powers.

$\Rightarrow -2c = 1 \Rightarrow c = -\frac{1}{2}$

$b + 3c = 0 \Rightarrow b = \frac{3}{2}$

$a - c = 0 \Rightarrow a = -\frac{1}{2}$

$P \propto \sqrt{\dfrac{r^3}{MG}}$

Q2

We have the advection diffusion equation

$\dfrac{\partial C}{\partial t} + \vec{v}(\vec{r}, t)\cdot \nabla C = D\Delta C$

define
$C = c_0 c'$
$t = t_0 t'$
$x = x_0 x' = wx'$
$y = y_0 y'$
$z = z_0 z'$
$v = v_0 v'$

and so, in these non-dimensionalised variables

$\dfrac{\partial c}{\partial t} = \dfrac{\partial(c_0 c')}{\partial(t_0 t')} = \dfrac{c_0}{t_0}\dfrac{\partial c'}{\partial t'}$

$\nabla C = \left(\dfrac{\partial c}{\partial x}, \dfrac{\partial c}{\partial y}, \dfrac{\partial c}{\partial z}\right) = \left(\dfrac{c_0}{w}\dfrac{\partial c'}{\partial x'}, \dfrac{c_0}{y_0}\dfrac{\partial c'}{\partial y'}, \dfrac{c_0}{z_0}\dfrac{\partial c'}{\partial y'}\right)$

$\Delta C = \dfrac{\partial^2 c}{\partial x^2} + \dfrac{\partial^2 c}{\partial y^2} + \dfrac{\partial^2 c}{\partial z^2}$

$= \dfrac{c_0^2}{w^2}\dfrac{\partial^2 c'}{\partial x'^2} + \dfrac{c_0^2}{y_0^2}\dfrac{\partial^2 c'}{\partial y'^2} + \dfrac{c_0^2}{z_0^2}\dfrac{\partial^2 c'}{\partial z'^2}$

$$\Delta C = \partial x^2 + \partial y^2 + \partial z^2$$

$$= \frac{c_0^2}{w^2}\frac{\partial^2 c'}{\partial x'^2} + \frac{c_0^2}{y_0^2}\frac{\partial^2 c'}{\partial y'^2} + \frac{c_0^2}{z_0^2}\frac{\partial^2 c'}{\partial z'^2}$$

$$\Rightarrow \quad \frac{\partial C}{\partial t} + \vec{v}(\vec{r},t)\cdot\nabla C = D\Delta C$$

$$\rightarrow \quad \frac{c_0}{t_0}\frac{\partial c'}{\partial t'} + v_0\vec{v}'([w x', y_0 y', z_0 z'], t_0 t')\cdot\left(\frac{c_0}{w}\frac{\partial c'}{\partial x'}, \frac{c_0}{y_0}\frac{\partial c'}{\partial y'}, \frac{c_0}{z_0}\frac{\partial c'}{\partial z'}\right)$$

$$= D\left(\frac{c_0^2}{w^2}\frac{\partial^2 c'}{\partial x'^2} + \frac{c_0^2}{y_0^2}\frac{\partial^2 c'}{\partial y'^2} + \frac{c_0^2}{z_0^2}\frac{\partial^2 c'}{\partial z'^2}\right)$$

Simplifying:

$$\frac{1}{t_0}\frac{\partial c'}{\partial t'} + v_0\vec{v}'(\vec{r}', t_0 t')\cdot\nabla' c' = c_0 D\left(\frac{1}{w^2}\frac{\partial^2 c'}{\partial x'^2} + \frac{1}{y_0^2}\frac{\partial^2 c'}{\partial y'^2} + \frac{1}{z_0^2}\frac{\partial^2 c'}{\partial z'^2}\right)$$

where $\vec{r}' = [x', y', z']$

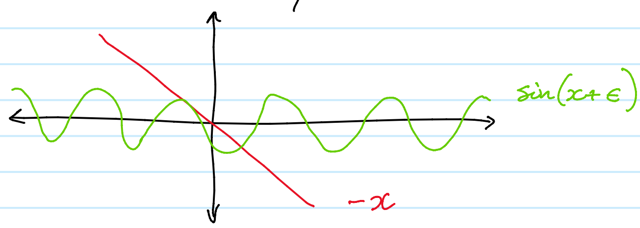
and $\nabla' = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)$

Q3

a. We have the function

$$\sin(x+\epsilon) = x \Rightarrow \sin(x+\epsilon) - x = 0$$

The functions in this equation are



To obtain the asymptotic expansion of this, we first approximate the sine term with the Taylor series:

$$\sin(x+\epsilon) \approx x+\epsilon - \frac{(x+\epsilon)^3}{3!} + \frac{(x+\epsilon)^5}{5!} + \dots$$

$$\Rightarrow \sin(x+\epsilon) - x \simeq \epsilon - \frac{(x+\epsilon)^3}{3!} + \frac{(x+\epsilon)^5}{5!} + \dots = 0$$

We want a solution of the form

$$x(\epsilon) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$$

Now, make a change of variable $x = \epsilon^\alpha z$

$$\Rightarrow \quad \epsilon - \frac{1}{3!}(\epsilon^\alpha z + \epsilon)^3 = 0$$

$$\Rightarrow \quad 6\epsilon - (\epsilon^{3\alpha}z^3 + 3\epsilon^{2\alpha+1}z^2 + 3\epsilon^{\alpha+2}z + \epsilon^3) = 0$$

$$\Rightarrow \quad 6 - (\epsilon^{3\alpha-1}z^3 + 3\epsilon^{2\alpha}z^2 + 3\epsilon^{\alpha+1}z + \epsilon^2) = 0$$

We want the z^3 term to dominate, so try

$$3\alpha - 1 = 0 \Rightarrow \alpha = 1/3$$

$$\Rightarrow \quad 6 - (z^3 + 3\epsilon^{2/3}z^2 + 3\epsilon^{4/3}z + \epsilon^2) = 0$$

Set $\mu = \epsilon^{2/3}$, and so we want a solution of the form

$$z(\mu) = z_0 + \mu z_1 + \mu^2 z_2 + \dots$$

$$\Rightarrow \quad 6 - \left((z_0+\mu z_1 + \dots)^3 + 3\mu(z_0+\mu z_1+\dots)^2 + 3\mu^2(z_0+\mu z_1+\dots) + \mu^3\right) = 0$$

$$\Rightarrow 6 - \left((z_0 + \mu z_1 + \ldots)^3 + 3\mu (z_0 + \mu z_1 + \ldots)^2 + 3\mu^2 (z_0 + \mu z_1 + \ldots) + \mu^3 \right) = 0$$

Largest μ^0 terms:

$$6 - z_0^3 = 0 \Rightarrow z_0 = \sqrt[3]{6}$$

Largest μ^1 terms:

$$-(3z_0^2 z_1 + 3z_0^2) = 0$$

$$\Rightarrow 3z_0^2 z_1 = -3z_0^2 \Rightarrow z_1 = -1$$

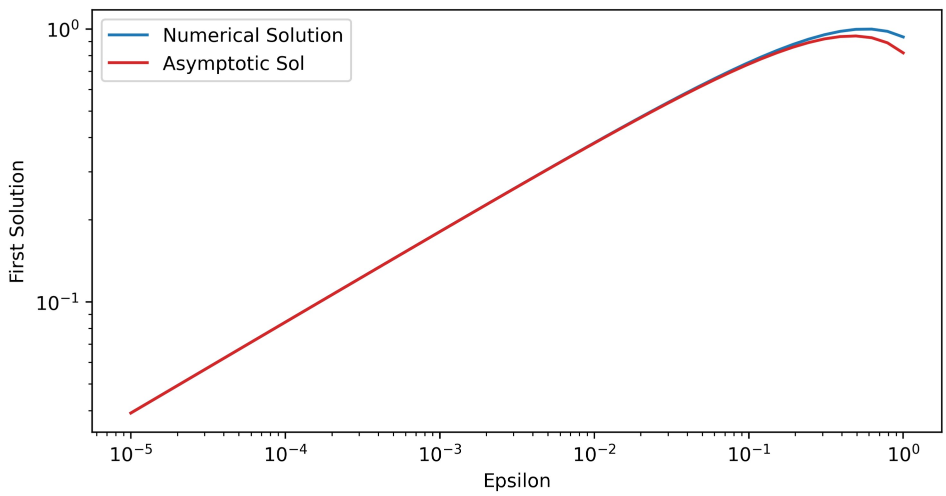
Hence $z(\mu) = \sqrt[3]{6} - \mu$

$\Rightarrow z(\epsilon) \approx \sqrt[3]{6} - \epsilon^{2/3}$

But $x = \epsilon^{\alpha} z$ with $\alpha = 1/3$, and so

$$x(\epsilon) = \epsilon^{1/3}(\sqrt[3]{6} - \epsilon^{2/3}) = \sqrt[3]{6}\,\epsilon^{1/3} - \epsilon$$

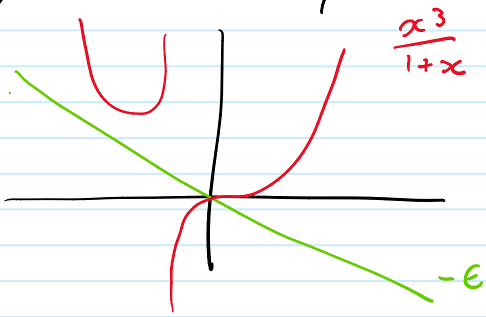
6. The numerical solution compared to this two term solution is



Question 4:

$$\frac{x^3}{1+x} - \epsilon = 0$$

The functions in this equation are



$$\frac{x^3}{1+x} - \epsilon = 0$$

$$x^3 - \epsilon x - \epsilon = 0$$

Want solution of the form

$$x(\epsilon) = \epsilon^{\alpha} x_0 + \epsilon^{\beta} x_1 + \epsilon^{\gamma} x_2 + \dots$$

$$\Rightarrow (\epsilon^{\alpha} x_0 + \epsilon^{\beta} x_1 + \dots)^3 - \epsilon(\epsilon^{\alpha} x_0 + \epsilon^{\beta} x_1 + \dots) - \epsilon = 0$$

$$\Rightarrow (\epsilon^{\alpha} x_0 + \epsilon^{\beta} x_1 + \dots)^3 - \epsilon(1 + \epsilon^{\alpha} x_0 + \epsilon^{\beta} x_1 + \dots) = 0$$

Want $\epsilon^{3\alpha} z_0^3$ to balance with $-\epsilon^{1} \Rightarrow \alpha = 1/3$

ϵ^0 terms:

$$x_0^3 - 1 = 0 \Rightarrow x_0 = 1$$

Now, want $3\epsilon^{2\alpha+\beta} x_0^2 x_1$ to balance with $-\epsilon^{\alpha+1} x_0$

$$\Rightarrow 2\alpha + \beta = \alpha + 1 \Rightarrow \alpha + \beta = 1 \Rightarrow \beta = 2/3$$

$\epsilon^{4/3}$ terms:

$$3x_0^2 x_1 - x_0 = 0$$

$$3x_1 = 1 \Rightarrow x_1 = 1/3$$

Now, want $3\epsilon^{2\alpha+\gamma} x_0^2 x_2$ to balance with $-\epsilon^{\beta+1} x_1$

$$\Rightarrow 2\alpha + \gamma = \beta + 1 \Rightarrow \gamma = 1$$

$$3x_0^2 x_2 = x_1 \quad \Rightarrow x_2 = 1/9$$

$$\Rightarrow x(\epsilon) = \epsilon^{1/3} + 1/3\,\epsilon^{2/3} + 1/9\,\epsilon$$

<u>Q5</u> We have the differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{1}{(1+\epsilon y)^2} = 0$$

with $y(0) = 0$ and $y'(0) = 1$

a. When $\epsilon = 0$, $\frac{d^2y}{dt^2} = -\frac{dy}{dt} - 1$

We want a solution of the form

$$y(t) = y_0 + \epsilon y_1 + \epsilon_2 y_2 + \dots$$

$$\Rightarrow \frac{d^2}{dt^2}(y_0 + \epsilon y_1 + \dots) + \frac{d}{dt}(y_0 + \epsilon y_1 + \dots) = -\frac{1}{(1+\epsilon(y_0+\epsilon y_1+\dots))^2}$$

$$\Rightarrow \left(\frac{d^2}{dt^2}(y_0 + \epsilon y_1 + \dots) + \frac{d}{dt}(y_0 + \epsilon y_1 + \dots)\right)(1 + \epsilon(y_0 + \epsilon y_1 + \dots))^2 = -1$$

<u>ϵ^0 terms:</u>

$$\frac{d^2y_0}{dt^2} + \frac{dy_0}{dt} = -1 \qquad y_0(0) = 0 \,;\, y_0'(0) = 1$$

(as before with $\epsilon = 0$)

Wolfram alpha gives a solution

$$y_0(t) = -2e^{-t} - t + 2$$

<u>ϵ^1 terms:</u>

$$\frac{d^2y_1}{dt^2} + \frac{dy_1}{dt} + 2y_0\underbrace{\left(\frac{d^2y_0}{dt^2} + \frac{dy_0}{dt}\right)}_{=-1} = 0$$

$$\Rightarrow \frac{d^2y_1}{dt^2} + \frac{dy_1}{dt} + 2(2e^{-t} + t - 2) = 0 \qquad \text{with } y_1(0) = 0 \,;\, y_1'(0) = 1$$

$$\Rightarrow \frac{d^2y_1}{dt^2} + \frac{dy_1}{dt} + 2(2e^{-t} + t - 2) = 0 \quad \text{with } y_1(0) = 0;\ y_1'(0) = 1$$

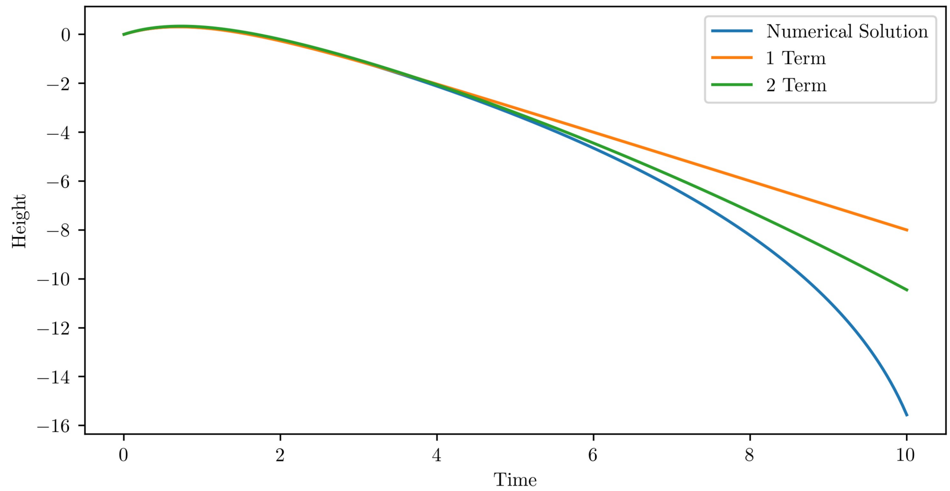
Wolfram alpha gives a solution of

$$y_1(t) = e^{-t}(-e^{t}(t-3)^2 + 4t + 9)$$

Hence the two term solution is

$$y(t) = y_0(t) + \epsilon y_1(t)$$
$$= -2e^{-t} - t + 2 + \epsilon e^{-t}(e^{t}(t-3)^2 + 4t + 9)$$

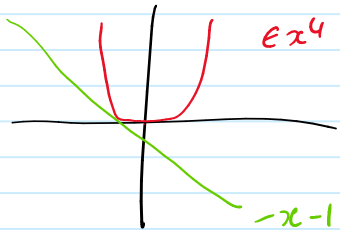
b. The numerical solution, found with scipy's odeint function was compared against the one term expansion y_0, and two term expansion $y_0 + \epsilon y_1$ for $\epsilon = 0.05$ in the domain $t \in [0, 10]$:



Question 6:

$$\epsilon x^4 - x - 1 = 0$$

a. The functions in the equation are



Hence, when $\epsilon > 0$, we'd expect two real solutions since the ϵx^4 will eventually dominate for $x \gg 0$ and is convex. If $\epsilon = 0$, we'd expect one real valued solution at $x = -1$.
For small, non-zero values of ϵ, we'd expect a root near $x = -1$ (with error on the order of ϵ), and another at increasingly large $x > 0$ (which tends to ∞ as $\epsilon \to 0$).

b. We want a solution of the form

$$x(\epsilon) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$$

Since we expect 2 solutions, make the change of variables

$$x = \epsilon^{k} z$$

Since we expect 2 solutions, make the change of variables

$$x = \epsilon^{\kappa} z$$

$$\Rightarrow \quad \epsilon^{4\kappa+1} z^4 - \epsilon^{\kappa} z - 1 = 0$$

We want z^4 to be dominant, hence choose

$$4\alpha + 1 = \alpha \Rightarrow \alpha = -1/3$$

$$\Rightarrow \epsilon^{-4/3+1} z^4 - \epsilon^{-1/3} z - 1 = 0$$

$$\Rightarrow z^4 - z - \epsilon^{1/3} = 0$$

Make substitution $\mu = \epsilon^{1/3}$, and we want a solution of the form

$$z(\mu) = z_0 + \mu z_1 + \mu^2 z_2 + \dots$$

$$\Rightarrow (z_0 + \mu z_1 + \dots)^4 - (z_0 + \mu z_1 + \dots) - \mu = 0$$

μ^0 terms:

$$z_0^4 - z_0 = 0 \quad \Rightarrow z_0 = 0 \text{ is a solution}$$

or,

$$z_0^3 = 1 \Rightarrow z_0 = 1 \text{ is a sol.}$$

μ^1 terms:

$$4 z_0^3 z_1 - z_1 - 1 = 0$$

when $z_0 = 0$,

$$-z_1 = 1 \Rightarrow z_1 = -1$$

when $z_0 = 1$

$$3 z_1 = 1 \Rightarrow z_1 = 1/3$$

$\therefore$ two, two-term solutions are

$$z(\mu) = 0 + -\mu$$

$$z(\mu) = 1 + 1/3\,\mu$$

but $\mu = \epsilon^{1/3}$, so

$$z(\epsilon) = -\epsilon^{1/3}$$

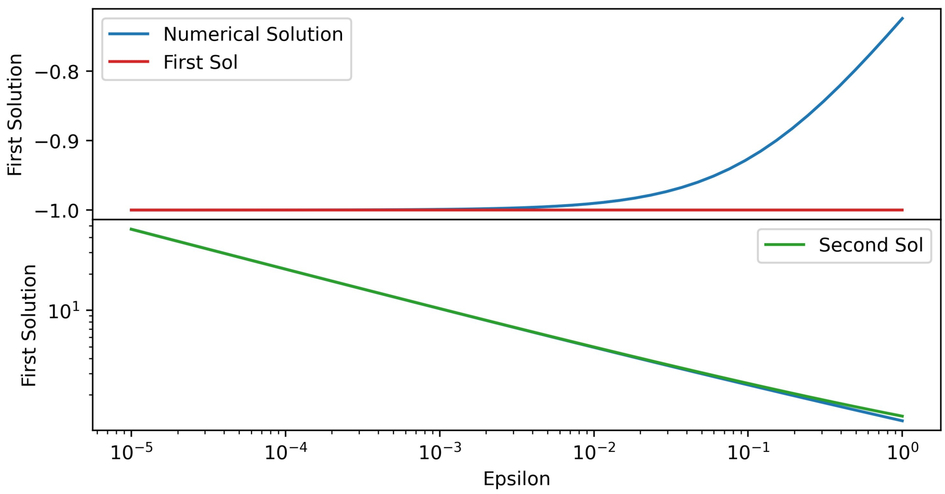
$$z(\epsilon) = 1 + 1/3\,\epsilon^{1/3}$$

and $x = \epsilon^{\alpha} z$, with $\alpha = -1/3$

$$\Rightarrow x(\epsilon) = \epsilon^{-1/3} \cdot -\epsilon^{1/3} = -1$$

$$x(\epsilon) = \epsilon^{-1/3} + 1/3\,\epsilon^{1/3} \cdot \epsilon^{-1/3} = \epsilon^{-1/3} + 1/3$$

On comparison to the numerical solution,



Question 7:

$$\epsilon \frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y^4 = 0$$

with boundary layer at $x=0$, and initial conditions $y(0)=1$, $y(1)=1$

We want a solution of the form

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \dots$$

$$\Rightarrow \epsilon(y_0(x)+\epsilon y_1(x)+\dots)'' + 3(y_0(x)+\epsilon y_1(x)+\dots)' - (y_0(x)+\epsilon y_1(x)+\dots)^4 = 0$$

at $\epsilon = 0$,
largest ϵ^0 terms:

$$3y_0'(x) - y_0(x)^4 = 0$$

Wolfram alpha gives the solution

$$y(x) = \frac{1}{\sqrt[3]{c-x}}$$

We want to use $y(1)=1$ since this solution is valid away from the boundary layer

$$\Rightarrow 1 = \frac{1}{\sqrt[3]{c-1}} \Rightarrow \sqrt[3]{c-1} = 1 \Rightarrow c-1=1 \Rightarrow c=2$$

$\therefore$ we have the outer solution

$$y_{out}(x) = \frac{1}{\sqrt[3]{2-x}}$$

Now to find the boundary solution
Rescale x such that $x = \epsilon^\alpha z$

$$\Rightarrow y(x) \to Y(z)$$

$$\Rightarrow Y'(z) = \frac{dY}{dz} = \frac{dY}{dx}\frac{dx}{dz} = y'(x)\epsilon^\alpha$$

$$\Rightarrow Y''(z) = y''(x)\epsilon^{2\alpha}$$

And so the problem is now about solving

$$\epsilon^{1-2\alpha}Y''(z) + 3\epsilon^{-\alpha}Y'(z) - Y(z)^4 = 0$$

We want $Y''(z)$ to be in the dominant balance...
Try $1-2\alpha = -a \Rightarrow \alpha = 1$

$$\Rightarrow \epsilon^{-1}Y''(z) + 3\epsilon^{-1}Y'(z) - Y(z)^4 = 0$$

or,

$$Y''(z) + 3Y'(z) - \epsilon Y(z)^4 = 0$$

Expect a solution of the form

$$Y(z) = Y_0(z) + \epsilon Y_1(z) + \epsilon^2 Y_2(z) + \dots$$

$$\Rightarrow (Y_0(z)+\epsilon Y_1(z)+\dots)'' + 3(Y_0(z)+\epsilon Y_1(z)+\dots)' - \epsilon(Y_0(z)+\epsilon Y_1(z)+\dots)^4 = 0$$

Largest ϵ^0 terms:

$$Y_0(z)'' + 3Y_0(z)' = 0$$

Expect sol. of form $Y_0 = e^{\lambda z}$

$$\Rightarrow \lambda^2 + 3\lambda = 0 \Rightarrow \lambda_{1,2} = 0, -3$$

$$\Rightarrow Y_0(z) = c_1 + c_2 e^{-3z}$$

$$Y_0(0) = 1 \Rightarrow 1 = c_1 + c_2 \Rightarrow c_1 = 1 - c_2$$

$$\Rightarrow Y_0(z) = 1 + c(e^{-3z} - 1)$$

Hence $Y_{BL}(z) = 1 + c(e^{-3z} - 1)$

Matching:

$$y_{BL}(x; \epsilon \to 0) \to 1 - c$$

$$y_{out}(x \to 0) \to \frac{1}{3\sqrt{2}}$$

$$\Rightarrow 1 - c = \frac{1}{3\sqrt{2}} \Rightarrow c = 1 - \frac{1}{3\sqrt{2}}$$

$$\therefore y(x) = y_{BL}(x) + y_{out}(x) + \text{common limit}$$

$$= 1 + \left(1 - \frac{1}{3\sqrt{2}}\right)\left(e^{-3x/\epsilon} - 1\right) + \frac{1}{3\sqrt{2-x}} + \frac{1}{3\sqrt{2}}$$

For a few different values of ϵ, the composite solution is