THE UNIVERSITY OF QUEENSLAND
SCHOOL OF MATHEMATICS AND PHYSICS
PHYS2041 – Quantum Physics

Tutorial 8 Solutions

Problem 8.1

An operator $\hat{A}$ is Hermitian if it is equal to its own Hermitian conjugate (or complex transpose, written $\hat{A}^\dagger$). An equivalent definition, that we will use extensively in this assignment is

$$\langle f|\hat{A}g\rangle = \langle \hat{A}f|g\rangle, \tag{1}$$

where $|\hat{A}g\rangle = \hat{A}|g\rangle$ and $\langle \hat{A}f| = \left(\hat{A}|f\rangle\right)^\dagger$.

(a) If $\hat{A}$ and $\hat{B}$ are Hermitian, then we can write their sum as

$$\langle f|\left(\hat{A}+\hat{B}\right)g\rangle = \langle f|\hat{A}g\rangle + \langle f|\hat{B}g\rangle = \langle \hat{A}f|g\rangle + \langle \hat{B}f|g\rangle = \langle f\left(\hat{A}+\hat{B}\right)|g\rangle \tag{2}$$

which by Eq. (1), shows their sum is Hermitian.

(b) If $\hat{A}$ and $\hat{B}$ are Hermitian, then we can write their product as

$$\langle f|\hat{A}\hat{B}g\rangle = \langle \hat{A}f|\hat{B}g\rangle = \langle \hat{B}\hat{A}f|g\rangle \tag{3}$$

Thus, if the product $\hat{A}\hat{B}$ is Hermitian we require $\langle f|\hat{A}\hat{B}g\rangle = \langle \hat{B}\hat{A}f|g\rangle$, which is true if the operators commute: $\hat{A}\hat{B} = \hat{B}\hat{A}$.

If we assume that $\hat{A}\hat{B}$ commute, then Eq. (3) can be re-written $\langle f|\hat{A}\hat{B}g\rangle = \langle f|\hat{B}\hat{A}g\rangle = \langle \hat{B}\hat{A}f|g\rangle$, which implies that the product $\hat{A}\hat{B} = \hat{B}\hat{A}$ is Hermitian [by Eq. (1)].

We have proved that the product $\hat{A}\hat{B}$ is Hermitian if $\hat{A}$ and $\hat{B}$ commute, and also that if they commute their product is Hermitian. Thus they commute if and only if they're Hermitian.

(c) This result obviously follows from the previous question. However just for fun, let's show it directly. Suppose $\hat{Q}$ is Hermitian, i.e. $\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$. This implies that $\langle f|\hat{Q}^2 g\rangle = \langle f|\hat{Q}\hat{Q}g\rangle = \langle \hat{Q}f|\hat{Q}g\rangle = \langle \hat{Q}^2 f|g\rangle$. Applying this process n times gives the result that $\hat{Q}^n$ is Hermitian.

Problem 8.2*

(a)

Let's denote the eigenstates and eigenvalues of $\hat{P}$ *according to* $\hat{P}|f_n\rangle = \lambda_n|f_n\rangle$ *, and because we are saying that the eigenstates of* $\hat{Q}$ *are the same ones (but the respective eigenvalues would have to be different), we can write the eigenvalue problem for* $\hat{Q}$ *as* $\hat{Q}|f_n\rangle = \mu_n|f_n\rangle$*.*

Now, $[\hat{P},\hat{Q}] = 0$ *means* $[\hat{P},\hat{Q}]\,|\Psi\rangle = 0$ *or* $\hat{P}\hat{Q}\,|\Psi\rangle - \hat{Q}\hat{P}\,|\Psi\rangle = 0$ *for <u>any</u> state* $|\Psi\rangle$ *.*

This "any state" $|\Psi\rangle$ *can be expanded in terms of the basis eigenstates* $|f_n\rangle$ *(because of their completeness, as for the eigenstates of any Hermitian operator):* $|\Psi\rangle = \sum_n c_n|f_n\rangle$ *.*

So, we want to check if $\hat{P}\hat{Q}\,|\Psi\rangle - \hat{Q}\hat{P}\,|\Psi\rangle = 0$ *:*

$$\hat{P}\hat{Q}\,|\Psi\rangle - \hat{Q}\hat{P}\,|\Psi\rangle = \hat{P}\hat{Q}\sum_n c_n|f_n\rangle - \hat{Q}\hat{P}\sum_n c_n|f_n\rangle$$

$$= \hat{P}\sum_n c_n\mu_n|f_n\rangle - \hat{Q}\sum_n c_n\lambda_n|f_n\rangle$$

$$= \sum_n c_n\mu_n\lambda_n|f_n\rangle - \sum_n c_n\lambda_n\mu_n|f_n\rangle = 0 \qquad \text{Q.E.D}$$

(b)

$$\textbf{\textit{If}}\ [\hat{P},\hat{Q}] = 0 \implies |f_n\rangle = |g_n\rangle, \text{ where } \hat{P}|f_n\rangle = \lambda_n|f_n\rangle \text{ and } \hat{Q}|g_n\rangle = \mu_n|g_n\rangle$$

$$[\hat{Q}|f_n\rangle = \mu_n|f_n\rangle \text{ – must return } |f_n\rangle, \text{ with some other eigenvalue}]$$

Use the method of contradiction, i.e., assume the opposite is true and arrive at contradiction with $[\hat{P},\hat{Q}] = 0$*. I.e., suppose that* $\hat{Q}|f_n\rangle \neq \mu_n|f_n\rangle$, but instead it returns $\hat{Q}|f_n\rangle = |f_n'\rangle$ *– some other state* $|f_n'\rangle$*, and then show that* $[\hat{P},\hat{Q}]$ *cannot be zero.*

Again, $[\hat{P},\hat{Q}] = 0$ *means* $[\hat{P},\hat{Q}]\,|\Psi\rangle = 0$ *or* $\hat{P}\hat{Q}\,|\Psi\rangle - \hat{Q}\hat{P}\,|\Psi\rangle = 0$ *for <u>any</u> state* $|\Psi\rangle$*.*

Let's expand in terms of the basis eigenstates $|f_n\rangle$ *of the operator* $\hat{P}$*:* $|\Psi\rangle = \sum_n c_n|f_n\rangle$*.*

Lets now see what happens to

$$0 = \hat{P}\hat{Q}\,|\Psi\rangle - \hat{Q}\hat{P}\,|\Psi\rangle = \hat{P}\hat{Q}\sum_n c_n|f_n\rangle - \hat{Q}\hat{P}\sum_n c_n|f_n\rangle$$

$$= \hat{P}\sum_n c_n|f_n'\rangle - \hat{Q}\sum_n c_n\lambda_n|f_n\rangle$$

[not an eigenstate of $\hat{P}$, so $\hat{P}|f_n'\rangle$ returns some new state $|f_n''\rangle$]

$$= \sum_n c_n|f_n''\rangle - \sum_n c_n\lambda_n|f_n'\rangle =$$

$$= \sum_n c_n\left(|f_n''\rangle - \lambda_n|f_n'\rangle\right) \neq 0 \implies \textit{contradiction} \qquad \text{Q.E.D}$$

Problem 8.3 (a) We will need to start off working in position space, and convert the integral to momentum space. Remember that in position space, the position operator is simply $\hat{x} = x$, so the expectation value of x with respect to the arbitrary state $\Psi(x,t)$ is

$$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t) x \Psi(x,t) dx. \tag{4}$$

Recall that we can write $\Psi(x,t)$ in terms of the momentum-space wavefunction $\Phi(p,t)$,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Phi(p,t) e^{ixp/\hbar} dp \tag{5}$$

For the purposes of this question, it doesn't actually matter what $\Phi(p,t)$ is. For a free particle these are stationary states, but what follows is general so we'll keep the time dependence arbitrary. Substituting Eq. (5) into Eq. (4) (being careful to use different variables p and p'),

$$\langle \hat{x} \rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \Phi^*(p',t) e^{-ixp'/\hbar} dp' \right) x \left(\int_{-\infty}^{\infty} \Phi(p,t) e^{ixp/\hbar} dp \right) dx \tag{6}$$

Let's just consider the second term for a moment. We are free to bring the x inside the p integral, and because

$$x e^{ixp/\hbar} = -i\hbar \frac{\partial}{\partial p} \left(e^{ixp/\hbar} \right) \tag{7}$$

we can use integration by parts to show (remember that $\Phi(p,t)$ must be normalisable, and therefore vanish as $p \to \pm\infty$),

$$\begin{aligned} x \left(\int_{-\infty}^{\infty} \Phi(p,t) e^{ixp/\hbar} dp \right) &= \int_{-\infty}^{\infty} \Phi(p,t) x e^{ixp/\hbar} dp \qquad (8) \\ &= -i\hbar \int_{-\infty}^{\infty} \Phi(p,t) \frac{\partial}{\partial p} \left(e^{ixp/\hbar} \right) dp \qquad (9) \\ &= \cancel{\left[\Phi(p,t) e^{ixp/\hbar} \right]_{\infty}^{\infty}} + i\hbar \int_{-\infty}^{\infty} e^{ixp/\hbar} \frac{\partial \Phi(p,t)}{\partial p} dp. \qquad (10) \end{aligned}$$

Substituting this back into Eq. (6),

$$\begin{aligned} \langle \hat{x} \rangle &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \Phi^*(p',t) e^{-ixp'/\hbar} dp' \right) \left(i\hbar \int_{-\infty}^{\infty} e^{ixp/\hbar} \frac{\partial \Phi(p,t)}{\partial p} dp \right) dx \qquad (11) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar} dx \right) \Phi^*(p',t) \left(i\hbar \frac{\partial}{\partial p} \right) \Phi(p,t) dp dp'. \qquad (12) \end{aligned}$$

The step is to recognise (introducing a new variable $y = x/\hbar$)

$$\begin{aligned} \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar} dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(p-p')y} dy \qquad (13) \\ &= \delta(p-p'). \qquad (14) \end{aligned}$$

This integral is a bit tricky, but you can think of is as the inverse Fourier transform of a plane-wave $\exp(-ip'x/\hbar)$, which is a Dirac-delta function (since it only contains a single frequency component). Substituting this back in, the mean position becomes

$$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(p - p')\Phi^*(p', t) \left(i\hbar \frac{\partial}{\partial p} \right) \Phi(p, t) dp dp' \tag{15}$$

$$= \int_{-\infty}^{\infty} \Phi^*(p, t) \left(i\hbar \frac{\partial}{\partial p} \right) \Phi(p, t) dp, \tag{16}$$

where we have made use of the delta function property $\int_{-\infty}^{\infty} f(p)\delta(p - a)dp = f(a)$. Examining the final line, it makes sense to identify $\hat{x} = i\hbar\frac{\partial}{\partial p}$ as the momentum-space position operator.

(b) To show it is Hermitian, we need to show $\langle f|\hat{x}g\rangle = \langle \hat{x}f|g\rangle$. We are free to continue to work in momentum space, f and g are just arbitrary, square-integrable functions which can be written in any basis (position, momentum or anything else), you can imagine them to be Φ as in the previous question. Writing the inner-product explicitly as an integral and using integration by parts (remember that for f, g in Hilbert space, the boundary terms must vanish as $p \to \pm\infty$),

$$\langle f(p)|\hat{x}g(p)\rangle = \int_{-\infty}^{\infty} f^*(p) \left(i\hbar \frac{\partial}{\partial p} \right) g(p) dp \tag{17}$$

$$= i\hbar \int_{-\infty}^{\infty} f^*(p) \frac{\partial g(p)}{\partial p} dp \tag{18}$$

$$= \cancel{[f(p)g(p)]_{\infty}^{\infty}} - i\hbar \int_{-\infty}^{\infty} \frac{\partial f^*(p)}{\partial p} g(p) dp \tag{19}$$

$$= \int_{-\infty}^{\infty} \left(i\hbar \frac{\partial}{\partial p} f(p) \right)^* g(p) dp \tag{20}$$

$$= \langle \hat{x} f(p)|g(p)\rangle \tag{21}$$

which proves that $\hat{x} = i\hbar\frac{\partial}{\partial p}$ is Hermitian.

(c) Using the result of Problem 8.1 (a), let's split the Hamiltonian up. The simpler of the two terms is the potential,

$$\langle f|V(x)g\rangle \tag{22}$$

$$= \int_{\infty}^{\infty} f^*(x) \left(V(x)g(x) \right) dx \tag{23}$$

$$= \int_{\infty}^{\infty} \left(V(x)f(x) \right)^* g(x) dx \tag{24}$$

$$= \langle V(x)f|g\rangle \tag{25}$$

which follows because $V(x)$ is a real function in the coordinate representation; the operator $\hat{V}(x) = V(\hat{x})$ in coordinate representation simply corresponds to multiplication by $V(x)$.

For the kinetic energy term (dropping the x dependence from f and g for brevity)

$$\langle f| - \frac{\hbar^2}{2m}\frac{d^2}{dx^2} g\rangle \tag{26}$$

$$= -\frac{\hbar^2}{2m}\int_{\infty}^{\infty} f^* \frac{d^2 g}{dx^2} dx \tag{27}$$

$$= -\frac{\hbar^2}{2m}\int_{\infty}^{\infty} \frac{d^2 f^*}{dx^2} g dx \tag{28}$$

$$= \langle -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} f|g\rangle \tag{29}$$

We have used $\int_{\infty}^{\infty} f^* \frac{d^2 g}{dx^2} dx = \int_{\infty}^{\infty} \frac{d^2 f^*}{dx^2} g dx$, which we can show using integration by parts twice

$$\int_{\infty}^{\infty} f^* \frac{d^2 g}{dx^2} dx = \left[f^* \frac{dg}{dx} \right]_{-\infty}^{\infty} - \int_{\infty}^{\infty} \frac{df^*}{dx}\frac{dg}{dx} dx \tag{30}$$

$$= \left[f^* \frac{dg}{dx} \right]_{-\infty}^{\infty} - \left[g \frac{df^*}{dx} \right]_{-\infty}^{\infty} + \int_{\infty}^{\infty} \frac{d^2 f^*}{dx^2} g dx \tag{31}$$

$$= \int_{\infty}^{\infty} \frac{d^2 f^*}{dx^2} g dx. \tag{32}$$

The boundary terms vanish at $\pm\infty$ because f, g are in Hilbert space and must be normalisable.

Because the sum of two Hermitian operators is Hermitian [Problem 8.1 (a)], we have shown that the Hamiltonian operator is Hermitian.

Problem 8.4 **[FOR ASSIGNMENT 2; max 10 points]**

(a) Recall the commutator of any two operators is $\left[\hat{A},\hat{D}\right] = \hat{A}\hat{D} - \hat{D}\hat{A}$, which means we can use the commutator to re-order $\hat{D}\hat{A} = \hat{A}\hat{D} - \left[\hat{A},\hat{D}\right]$. With this in mind, we have

$$\begin{aligned}
\left[\hat{A},\hat{B}\hat{C}\right] &= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} && (33)\\
&= \hat{A}\hat{B}\hat{C} - \hat{B}\left(\hat{A}\hat{C} - \left[\hat{A},\hat{C}\right]\right) && (34)\\
&= \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \hat{B}\left[\hat{A},\hat{C}\right] && (35)\\
&= \hat{A}\hat{B}\hat{C} - \left(\hat{A}\hat{B} - \left[\hat{A},\hat{B}\right]\right)\hat{C} + \hat{B}\left[\hat{A},\hat{C}\right] && (36)\\
&= \cancel{\hat{A}\hat{B}\hat{C}} - \cancel{\hat{A}\hat{B}\hat{C}} + \left[\hat{A},\hat{B}\right]\hat{C} + \hat{B}\left[\hat{A},\hat{C}\right] && (37)\\
&= \left[\hat{A},\hat{B}\right]\hat{C} + \hat{B}\left[\hat{A},\hat{C}\right] && (38)
\end{aligned}$$

which is the result. Notice we are always careful with the ordering of operator multiplication.

(b) By definition (and as mentioned in Problem 8.1), an operator $\hat{A}$ is Hermitian if it is equal to its own Hermitian conjugate. An equivalent definition is $\langle f|\hat{A}g\rangle = \langle \hat{A}f|g\rangle$, which is true if and only if $\hat{A} = \hat{A}^\dagger$, where $|\hat{A}g\rangle = \hat{A}|g\rangle$ and $\langle \hat{A}f| = \left(\hat{A}|f\rangle\right)^\dagger$.

Like all observables, $\hat{x}$ and $\hat{p}$ are Hermitian, i.e. $\hat{x} = \hat{x}^\dagger$ and $\hat{p} = \hat{p}^\dagger$ (we showed that $\hat{x}$ is Hermitian in the previous question, the proof for $\hat{p}$ is similar). This means that

$$\begin{aligned}
(\hat{a}_+)^\dagger &= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x} - i\hat{p}\right)^\dagger && (39)\\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x}^\dagger - [i\hat{p}]^\dagger\right) && (40)\\
&= \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x} + i\hat{p}\right) && (41)\\
&= \hat{a}_-. && (42)
\end{aligned}$$

The Hermitian conjugate of the raising operator is the lowering operator! The converse is also true, $(\hat{a}_-)^\dagger = ((\hat{a}_+)^\dagger)^\dagger = \hat{a}_+$

(c) What about the sum of the raising and lowering operators? Define $\hat{X} = \hat{a}_+ + \hat{a}_-$,

$$\begin{aligned}
\hat{X}^\dagger &= (\hat{a}_+ + \hat{a}_-)^\dagger && (43)\\
&= (\hat{a}_+)^\dagger + (\hat{a}_-)^\dagger && (44)\\
&= \hat{a}_- + \hat{a}_+ && (45)\\
&= \hat{X} && (46)
\end{aligned}$$

The sum is Hermitian! Generally, the sum of any operator $\hat{A}$ and its Hermitian conjugate $\hat{A}^\dagger$ is Hermitian. You may remember this as the position operator (from the quantum harmonic oscillator assignment), just without the units.

What about the product? First let's think about what happens when we conjugate a product of non-Hermitian operators, say $\hat{A}$ and $\hat{B}$. We have $\langle f|\hat{A}\hat{B}g\rangle = \langle \hat{A}^\dagger f|\hat{B}g\rangle = \langle \hat{B}^\dagger\hat{A}^\dagger f|g\rangle$, which implies $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$. With this in mind, let's define $\hat{n} = \hat{a}_+\hat{a}_-$,

$$\begin{aligned}
\hat{n}^\dagger &= (\hat{a}_+\hat{a}_-)^\dagger && (47)\\
&= (\hat{a}_-)^\dagger(\hat{a}_+)^\dagger && (48)\\
&= \hat{a}_+\hat{a}_- && (49)\\
&= \hat{n} && (50)
\end{aligned}$$

The product is Hermitian too! Generally, the product of any operator and its Hermitian conjugate is Hermitian.

<u>Problem 8.5</u> [FOR ASSIGNMENT 2; max 10 points]

We can re-write $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}_\pm$ (Eq. (2.69) in Griffifths, check these for yourself!)

$$\begin{aligned}
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-) && (51)\\
\hat{p} &= i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a}_+ - \hat{a}_-)\,, && (52)
\end{aligned}$$

and substituting them into the harmonic oscillator Hamiltonian gives,

$$\begin{aligned}
\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 && (53)\\
&= \frac{1}{2m}\left(-\frac{\hbar m\omega}{2}\right)(\hat{a}_+ - \hat{a}_-)^2 + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)(\hat{a}_+ + \hat{a}_-)^2 && (54)\\
&= \frac{\hbar\omega}{4}\left(-\cancel{(\hat{a}_+)^2} + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ - \cancel{(\hat{a}_-)^2} + \cancel{(\hat{a}_+)^2} + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \cancel{(\hat{a}_-)^2}\right) && (55)\\
&= \frac{\hbar\omega}{2}\left(\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+\right) && (56)
\end{aligned}$$

Notice that we were careful with the ordering of $\hat{a}_\pm$, which don't commute. Now, to tidy up the final line we need to make use of the commutator $[\hat{a}_-, \hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = \hat{1}$, which implies

$\hat{a}_-\hat{a}_+ = \hat{a}_+\hat{a}_- + \hat{1}$. Substituting this back into the final line gives

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}_+\hat{a}_- + \hat{a}_+\hat{a}_- + \hat{1}\right) \tag{57}$$

$$= \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right), \tag{58}$$

which is the desired result.

(a) The Heisenberg equation of motion for any operator $\hat{Q}(t)$ is

$$\frac{d\hat{Q}(t)}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{Q}(t)\right]. \tag{59}$$

Let's evaluate the commutator of $\hat{a}_+$ with the Hamiltonian,

$$\left[\hat{H}, \hat{a}_+\right] = \left[\hbar\omega\hat{a}_+\hat{a}_-, \hat{a}_+\right] \tag{60}$$

$$= \hbar\omega\hat{a}_+\left[\hat{a}_-, \hat{a}_+\right] \tag{61}$$

$$= \hbar\omega\hat{a}_+ \tag{62}$$

The second line follows because $\hat{a}_+$ commutes with itself, and we are always free to pull constants out the front of a commutator (if you're unsure about this, write out the commutator and factorise the constants). In the final line we used $[\hat{a}_-, \hat{a}_+] = 1$.

The Heisenberg equation of motion for $\hat{a}_+(t)$ is therefore

$$\frac{d\hat{a}_+(t)}{dt} = i\omega\hat{a}_+(t). \tag{63}$$

Treating this as a regular differential equation (don't be scared by the operators!), we can solve this by rearranging $d\hat{a}_+(t)/\hat{a}_+(t) = i\omega dt$ (its separable) and integrating both sides, resulting in

$$\hat{a}_+(t) = \hat{A}e^{i\omega t}. \tag{64}$$

The operator $\hat{A}$ is determined by the initial condition, which we can find by setting $t = 0$, $\hat{a}_+(0) = \hat{A}$, or

$$\hat{a}_+(t) = \hat{a}_+(0)e^{i\omega t}. \tag{65}$$

We could repeat this process to find $\hat{a}_-(t)$, however we have already proved that $\hat{a}_- = (\hat{a}_+)^\dagger$. It turns out that working in the Heisenberg picture conserves this property, so we can immediately identify

$$\hat{a}_-(t) = (\hat{a}_+(t))^\dagger \tag{66}$$

$$= \hat{a}_-(0)e^{-i\omega t} \tag{67}$$

(note the negative sign in the exponential). As a sanity check, its easy to see $t = 0$ reduces to the Schrödinger picture operators $\hat{a}_\pm(0)$.

(b) From $\hat{H} = \hbar\omega(\underbrace{\hat{a}_+\hat{a}_-}_{\equiv \hat{n}} + \frac{1}{2})$

$$\hat{H} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$$

$$[\hat{H}, \hat{n}] = \left[\hbar\omega\left(\hat{n}+\frac{1}{2}\right), \hat{n}\right] = \hbar\omega \underbrace{\cancelto{0}{[\hat{n}, \hat{n}]}}_{\substack{\text{any operator} \\ \text{commutes with itself.}}} + \frac{\hbar\omega}{2}\cancelto{0}{[1, \hat{n}]} = 0$$

$\hat{n}|n\rangle$ — ?

$$\hat{n} = \hat{a}_+\hat{a}_- \Rightarrow \hat{n}|n\rangle = \hat{a}_+\hat{a}_-|n\rangle \text{ — ?}$$

$$\begin{cases} \hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle \\ \hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle \end{cases}$$ (from Lecture 10)

$$\Rightarrow \hat{n}|n\rangle = \hat{a}_+\underbrace{\hat{a}_-|n\rangle} = \hat{a}_+ \cdot \sqrt{n}\,|n-1\rangle =$$

$$= \sqrt{n}\,\underbrace{\hat{a}_+|n-1\rangle}_{\sqrt{n}\,|n-1+1\rangle = \sqrt{n}\,|n\rangle} = \sqrt{n}\cdot\sqrt{n}\,|n\rangle = n|n\rangle$$

$\Rightarrow \hat{n}|n\rangle = n|n\rangle$ ($\hat{n}$ acting on $|n\rangle$ returns the same state $|n\rangle$, ∴ eigenstate)

eigenvalue: n