Two-view geometry

What have we learned in this class so far:

2. Convolution, Filters, Canny Edge Detector

Original image

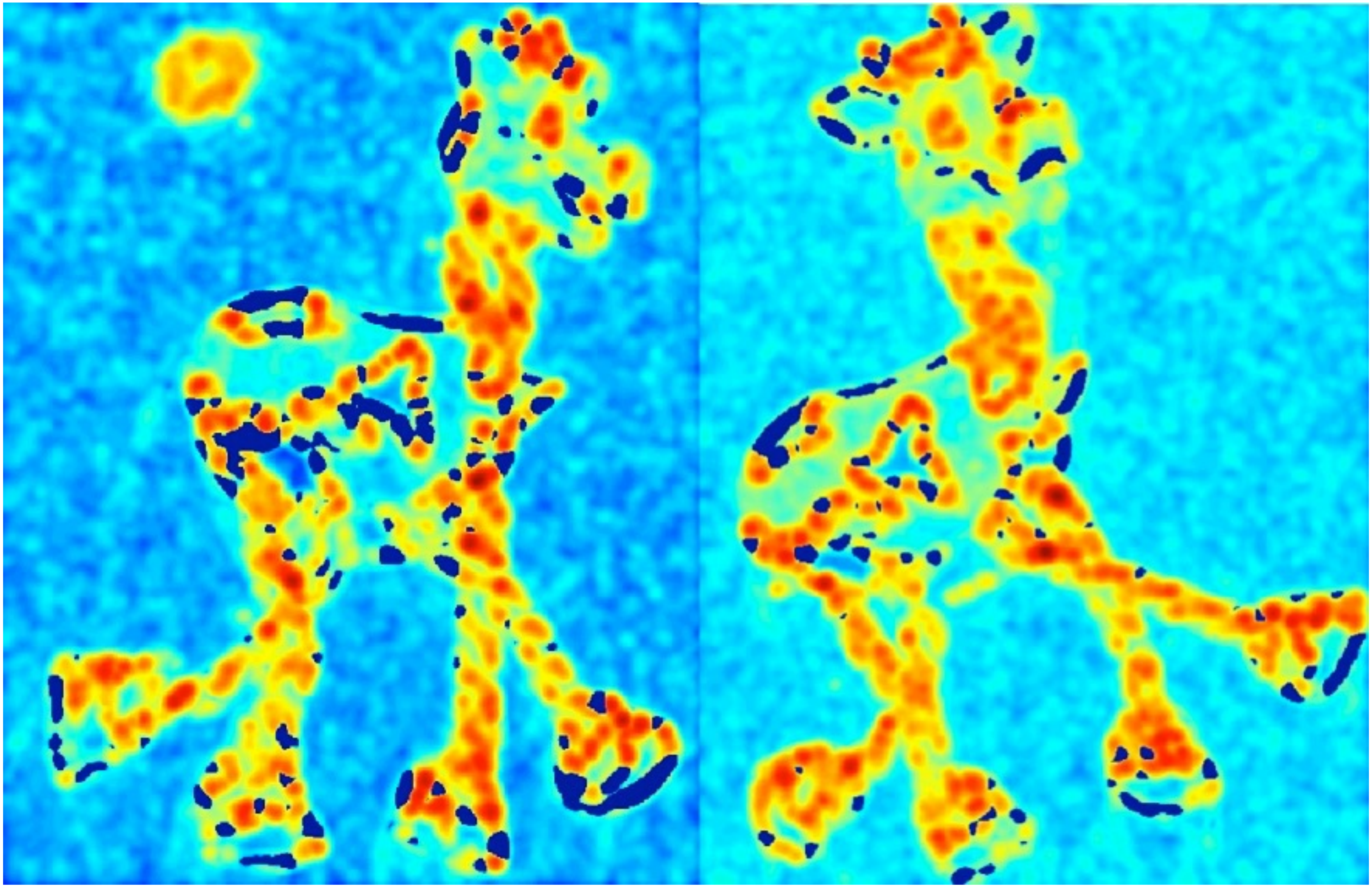
Edge image / gradient image

Non-maximum suppression

- Smooth the image (Gaussian filter)
- Edge / derivative filter
  - In practice combine the above with “Derivative of Gaussian” filter
- Compute norm of the gradient
- Threshold
- Non-maximum suppression to find the single-pixel edges
- Hysteresis thresholding

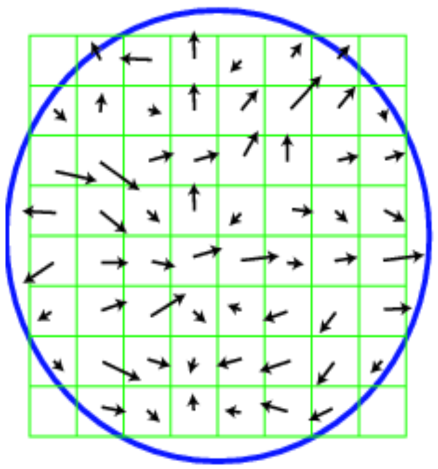
3. Cornerness Score (Harris Corners)

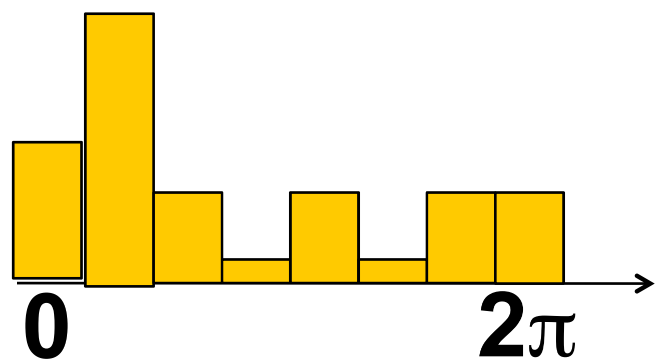
$$R = Det(A) - \alpha Trace(A)^2$$



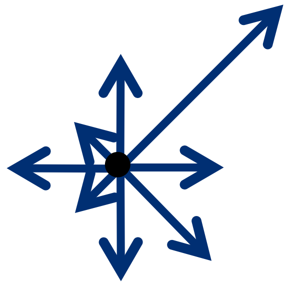
3. Patch Descriptors (e.g. SIFT):

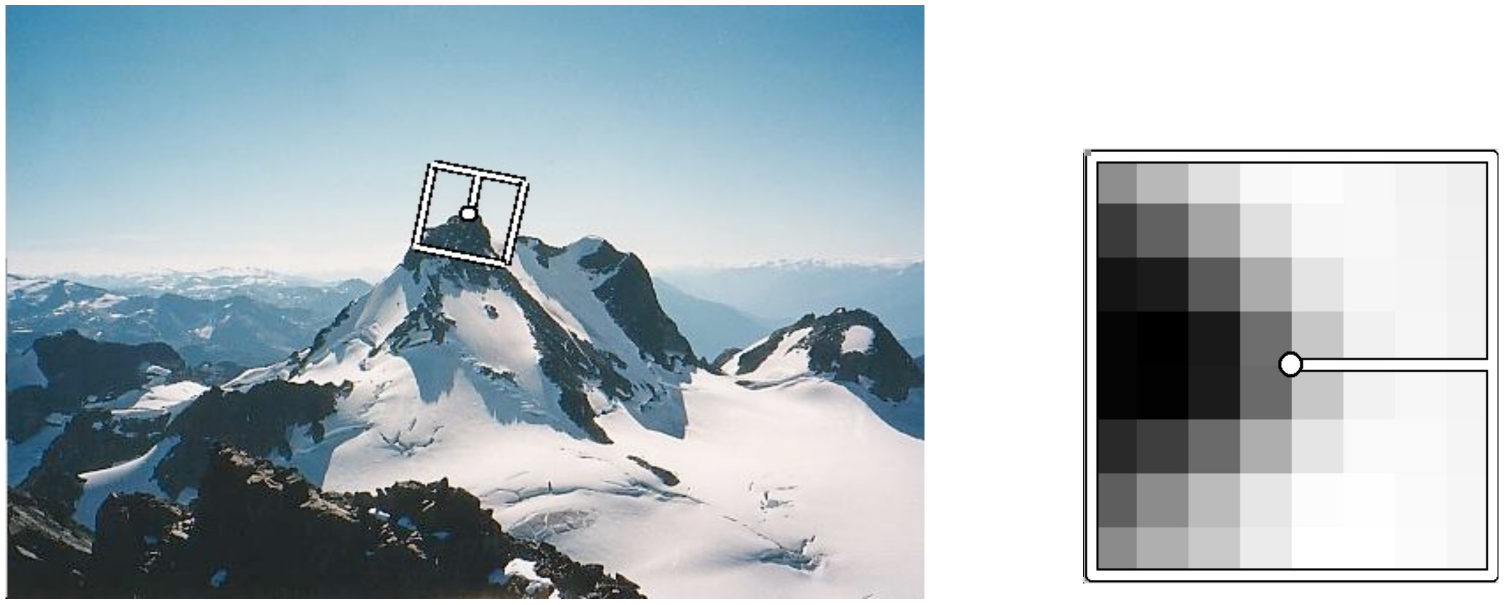
1. Compute gradients for all pixels in patch.
2. Histogram (bin) gradients by orientation.
3. Rotate patch so that dominant orientation is bin "0"





Alternative visualization of histogram:



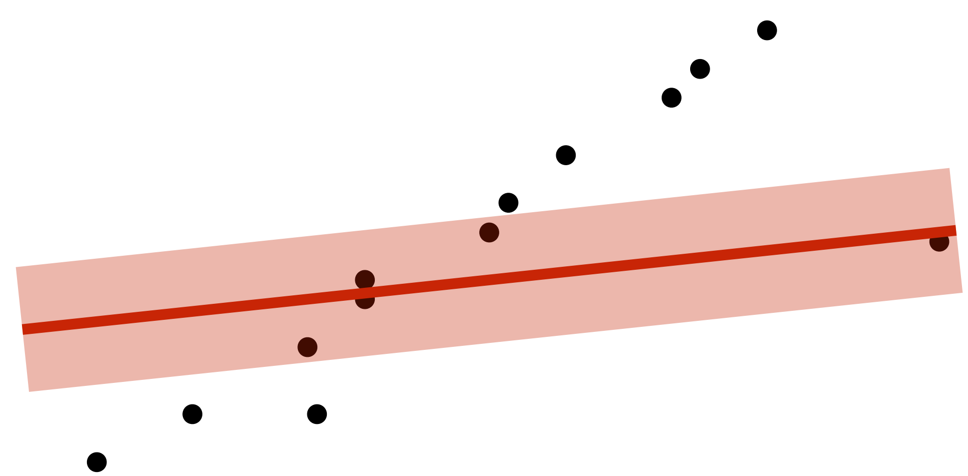


3. RANSAC

Repeat ***k*** times:

- Sample **n** points uniformly at random
- Fit model to these points (line, plane, sphere, etc)
- Count inliers to this model (i.e., points whose distance from the model is less than a threshold ***t***)

Return model with largest inlier count



Optional post processing:

Take the inliers of the returned model and perform a least-squares fit (using just the inliers)

Source: M. Pollefeys

4. Camera Projection

$$\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & s_\theta & o_x \\ 0 & s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

2D point in image coordinates (homogenous)

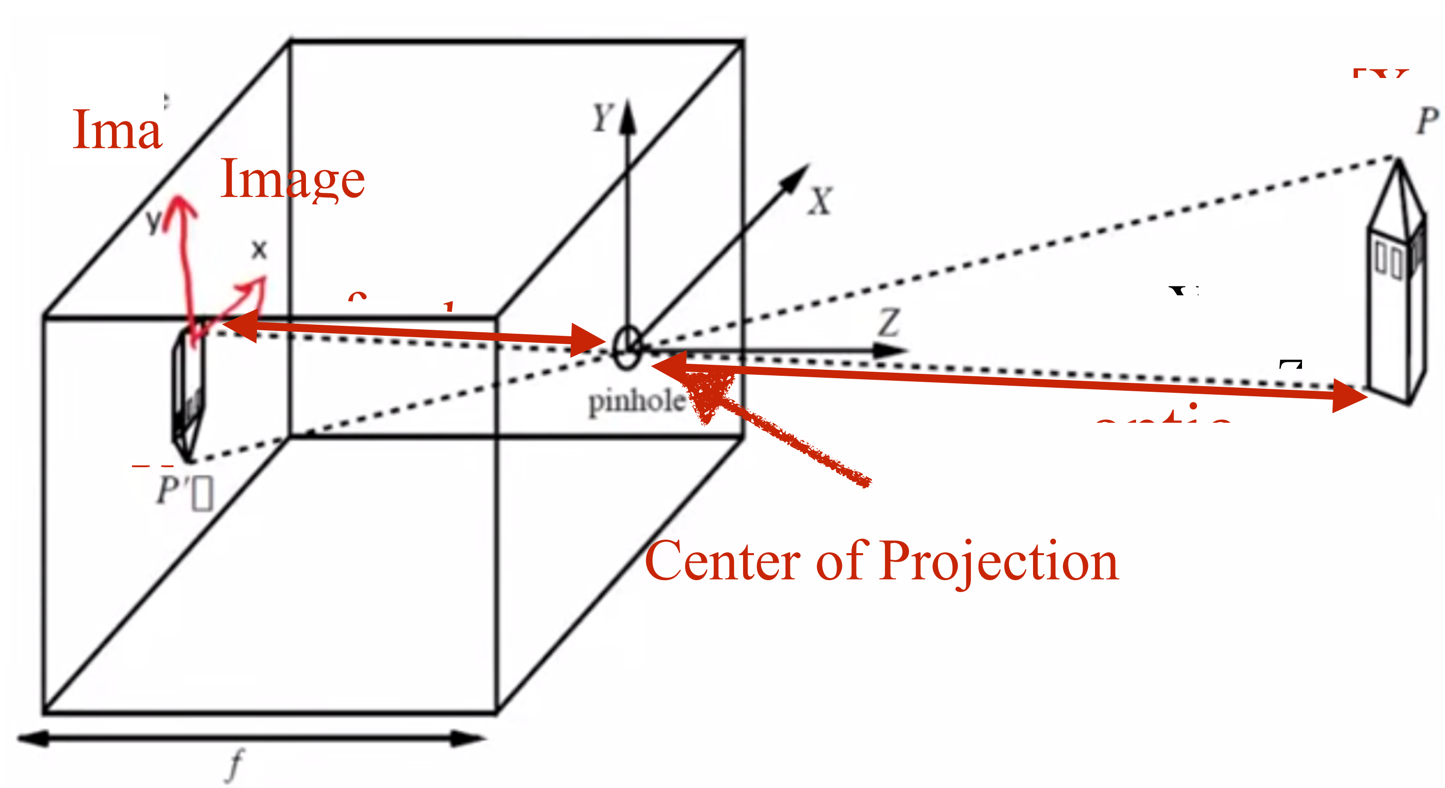
Camera instrinsic matrix K

Camera extrinsics (rotation and translation)

3D point in world coordinates (homogenous)

$$= \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Camera projection matrix M

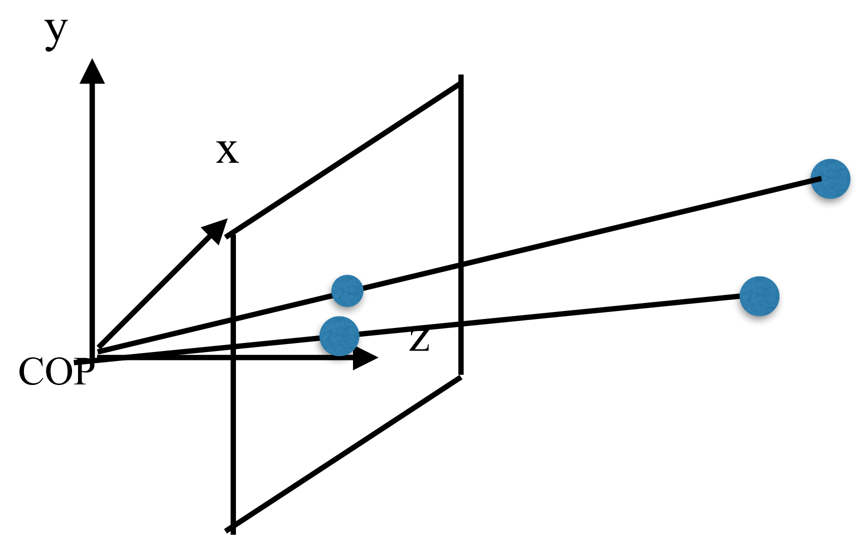


4. Camera Calibration + SVD

For a given camera, how can we find the the camera matrix M_{3X4}?
We need a set of scene points $P_1,..,P_N$
and the corresponding projections onto the image plane $p_1,..,p_N$

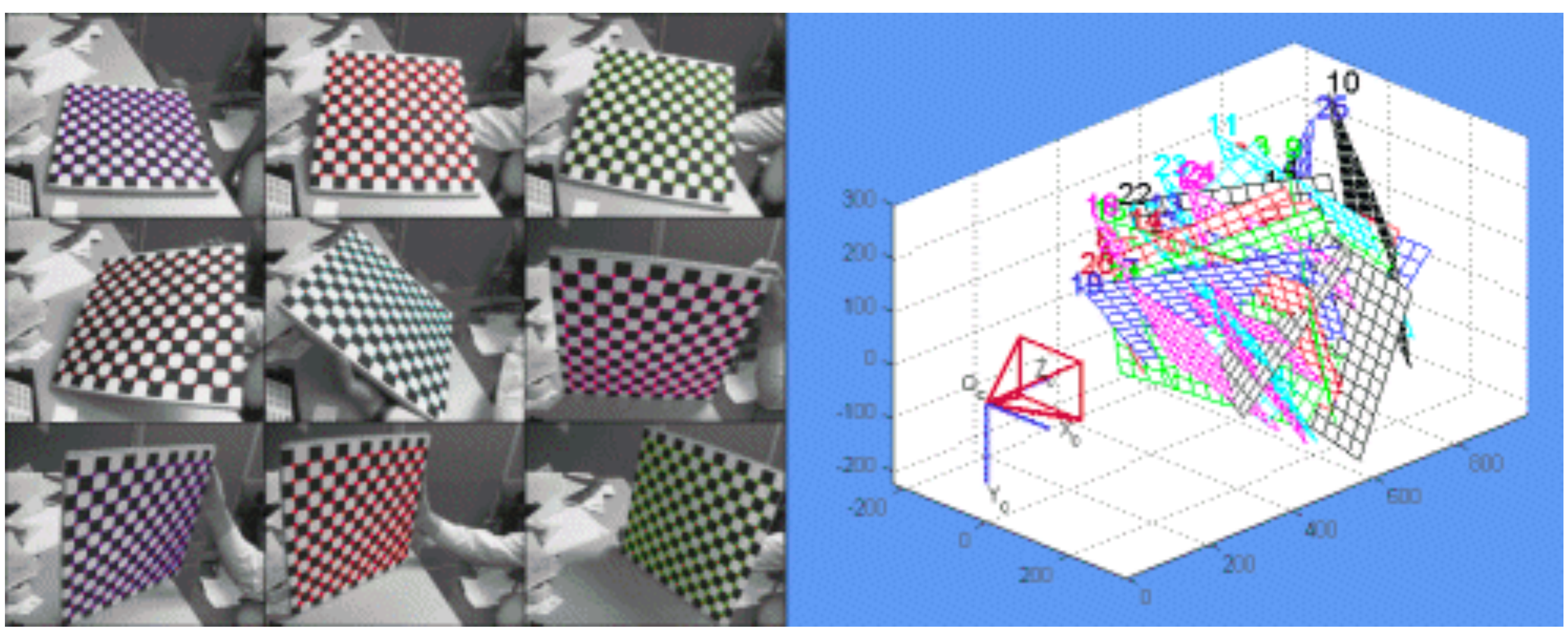
2D point in image coordinates

$$\mathbf{p}_i = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$



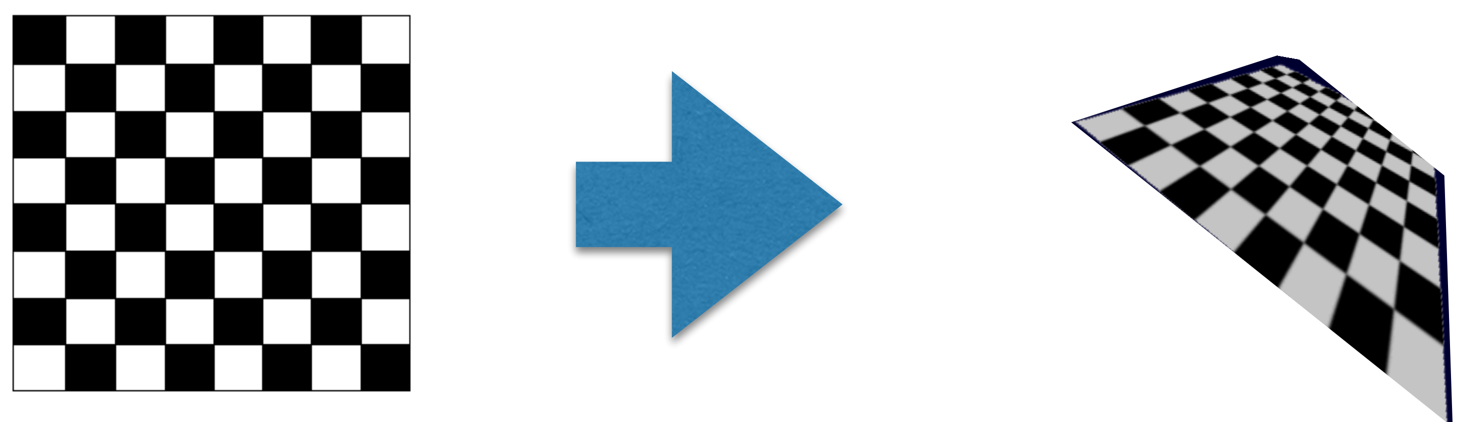
$$\mathbf{P}_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix}$$

3D point in world coordinates

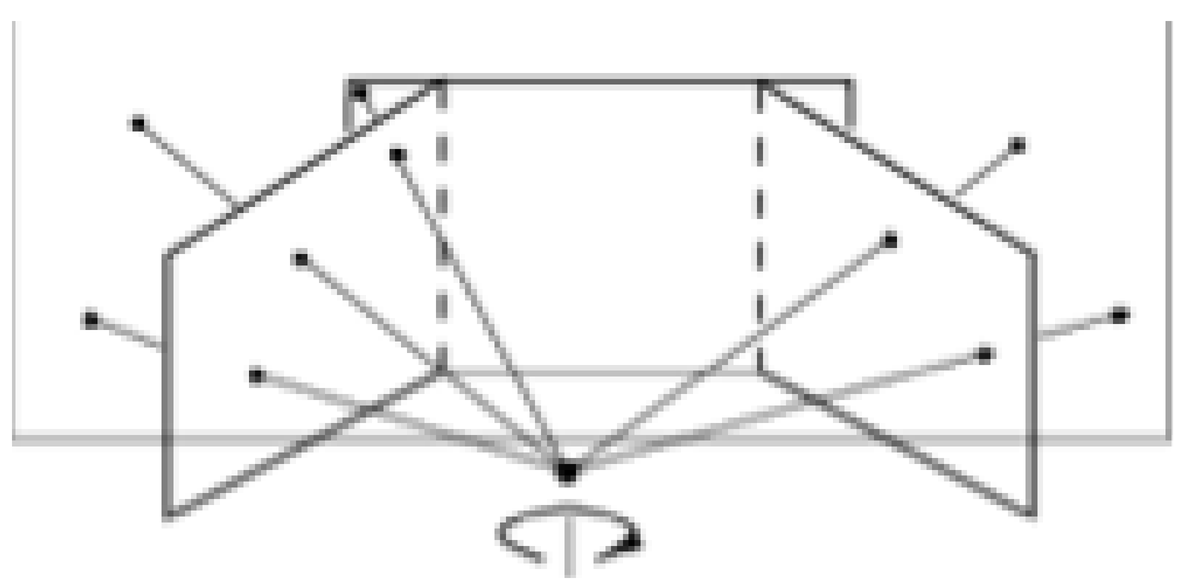


4. Homography transformations

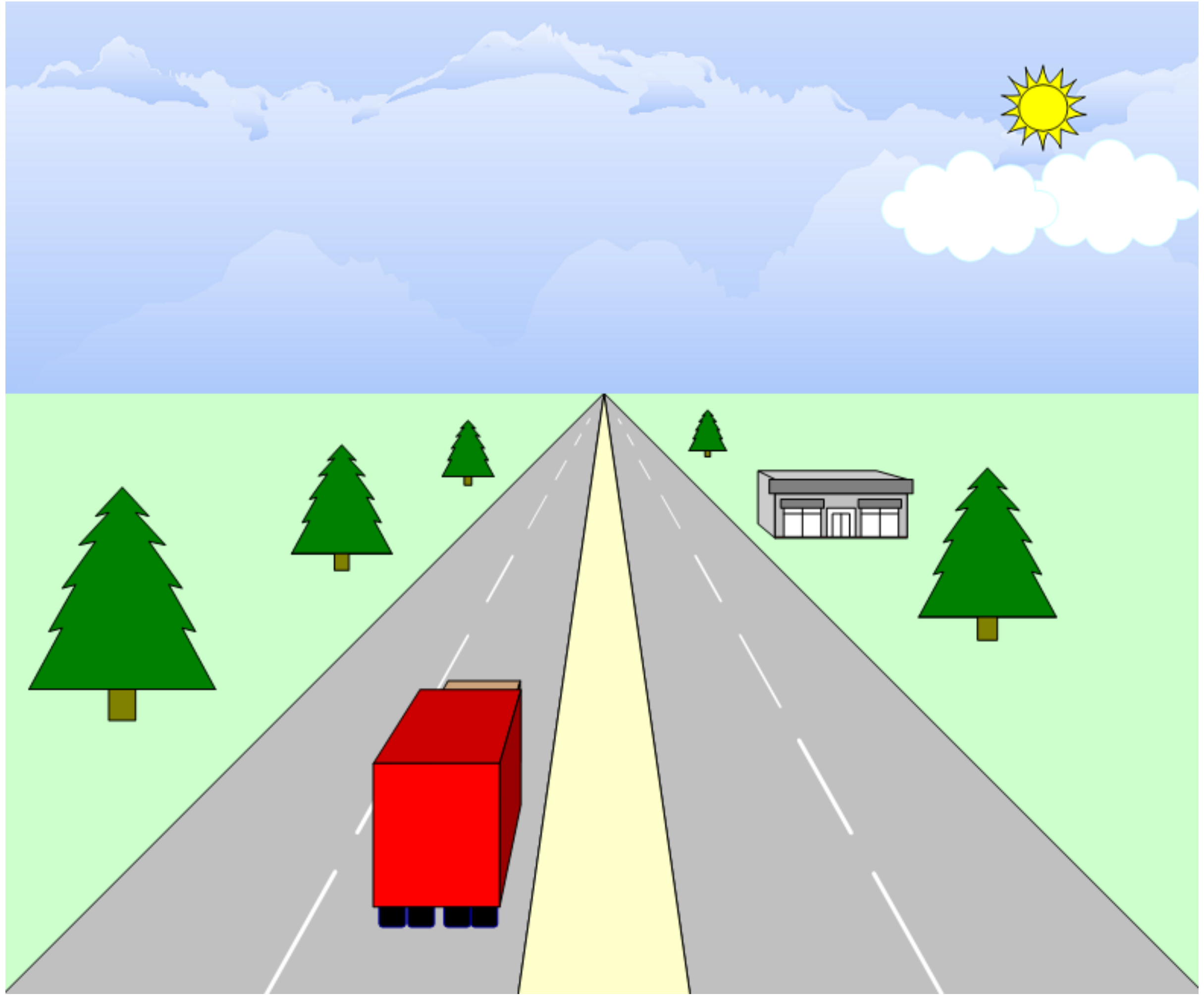
1. Models image projection of an arbitrary planar scene



2. Models warping (perspective) effects from camera rotations



5. Perspective projection



Closer objects appear larger

Closer objects are lower in the image

Parallel lines meet

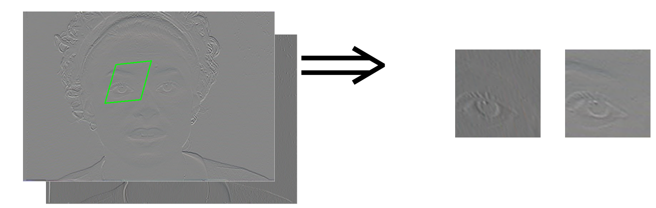
All these can be simply derived with $x = X\dfrac{f}{Z} \quad y = Y\dfrac{f}{Z}$

6. Object Tracking (Lucas-Kanade)

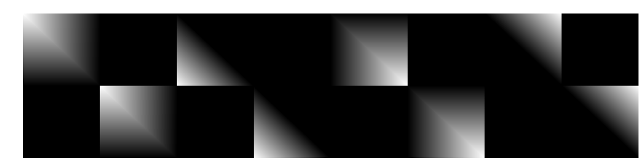
1. Warp I with $\mathbf{W}(\mathbf{x};\mathbf{p}) \Rightarrow I(\mathbf{W}(\mathbf{x};\mathbf{p}))$
   current warp   warped image

2. Compute error image $T(\mathbf{x}) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))$
   template   warped image

3. Warp gradient of I to compute ∇I



4. Evaluate Jacobian $\frac{\partial \mathbf{W}}{\partial \mathbf{p}}$



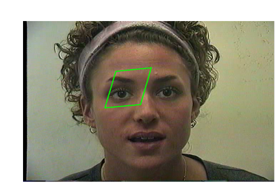
5. Compute approximate Hessian

$$H = \sum_{\mathbf{x}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}}\right]^{\mathrm{T}} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}}\right]$$

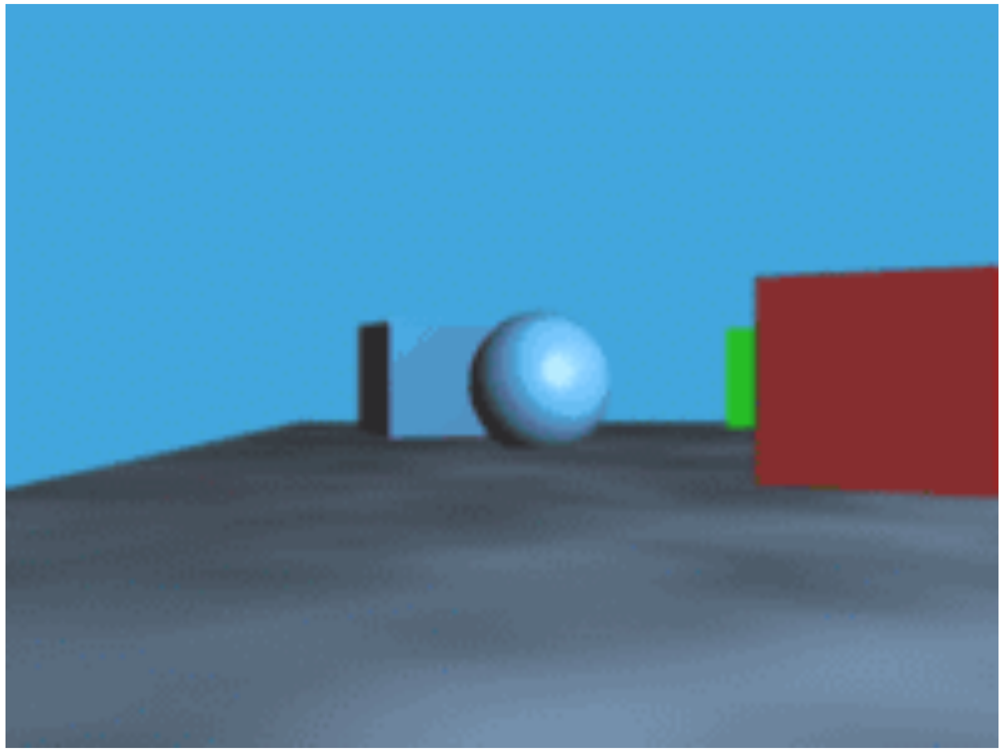
6. Compute $\Delta\mathbf{p}$

$$\Delta\mathbf{p} = \sum_{\mathbf{x}} H^{-1} \left[\nabla I \frac{\partial \mathbf{W}}{\partial \mathbf{p}}\right]^{\mathrm{T}} [T(\mathbf{x}) - I(\mathbf{W}(\mathbf{x};\mathbf{p}))]$$

7. Update warp parameters $\mathbf{p} \leftarrow \mathbf{p} + \Delta\mathbf{p}$



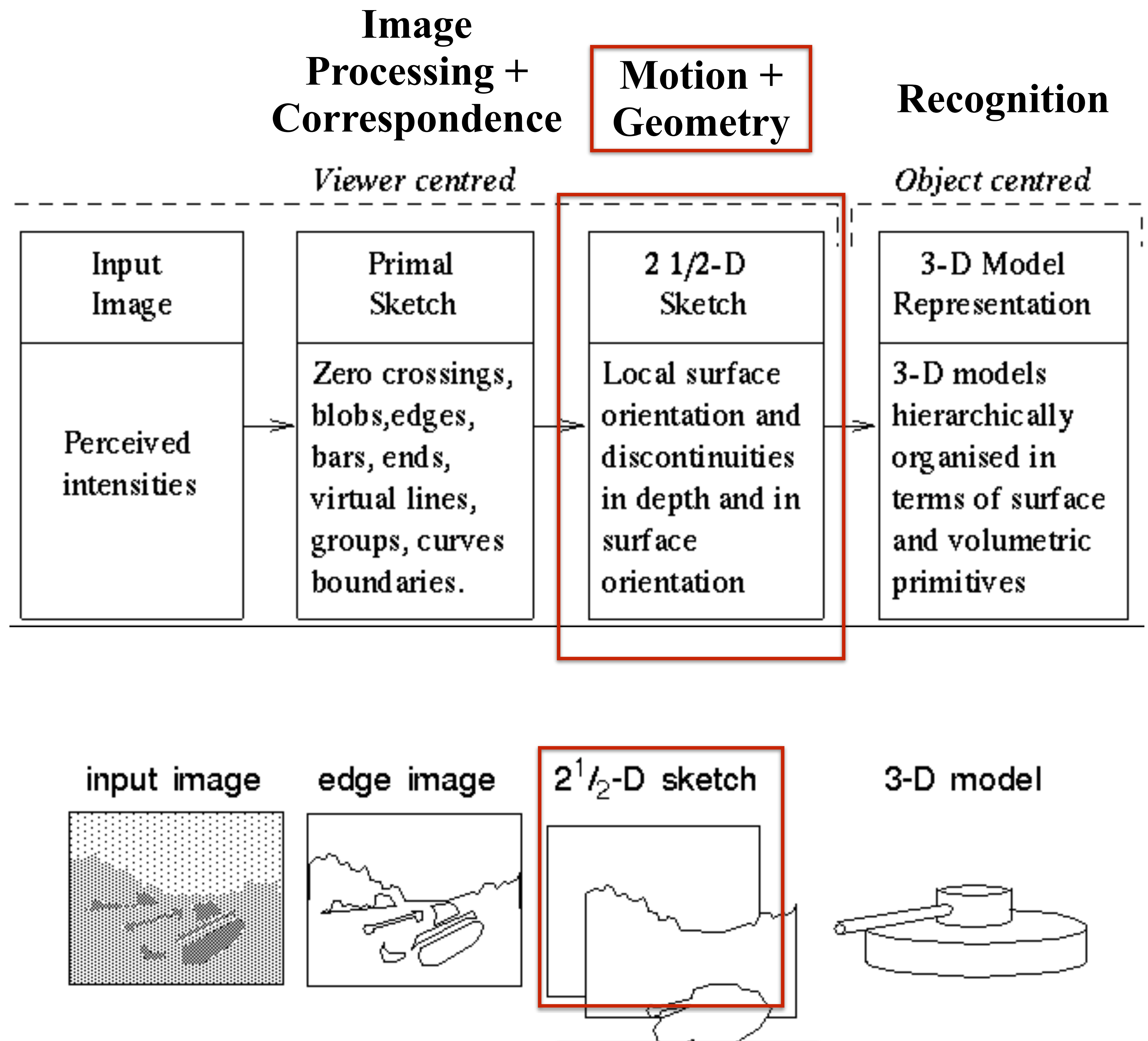
7. Ego-motion and Flow



- If we can compute the 2D optical flow, then we can:
    - compute the time to contact
    - estimate the rotation of the camera
    - estimate the translation of the camera (if scene depth is known)
    - estimate the scene depth if camera translation is known

Brief Review from Last Time

David Marr’s Taxonomy of Vision

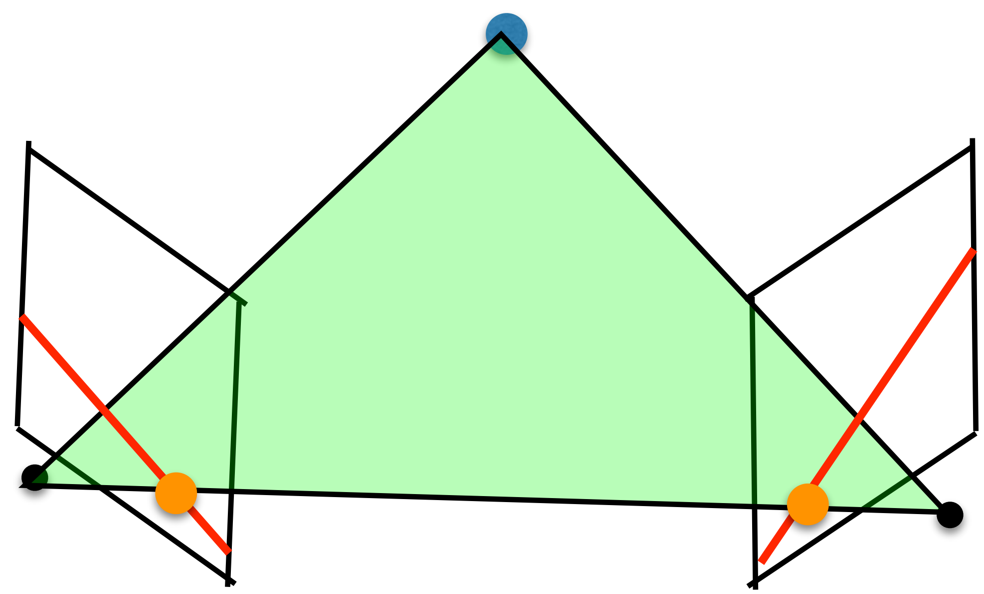


Low-level Mid-level High-level

Today: Multi-view geometry

8. Two-view Geometry

Definitions



Epipolar plane: plane defined by 2 camera centers & candidate 3D point (green)

(also defined by 2 camera centers any 1 points in either image plane)

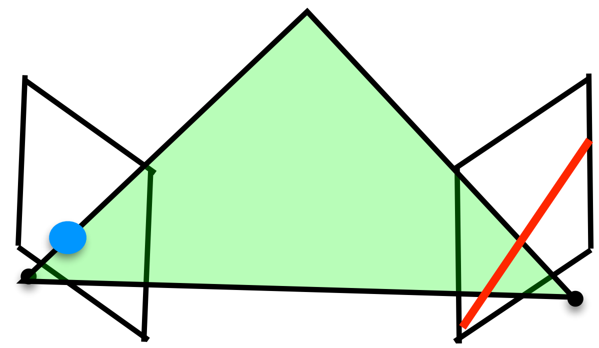
Epipolar lines (red): intersection of epipolar plane and image planes

Epipoles (orange): projection of camera center 1 in camera 2 (& vice versa)

All epipolar lines include the epipole; **the set of all epipolar lines intersect at the epipoles**

Essential matrix

$$\boxed{\boldsymbol{x}_2^\top E \boldsymbol{x}_1 = 0}$$



$$\begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \overset{Ex_1}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = 0$$

$$a x_2 + b y_2 + c = 0$$ (equation of a line!)

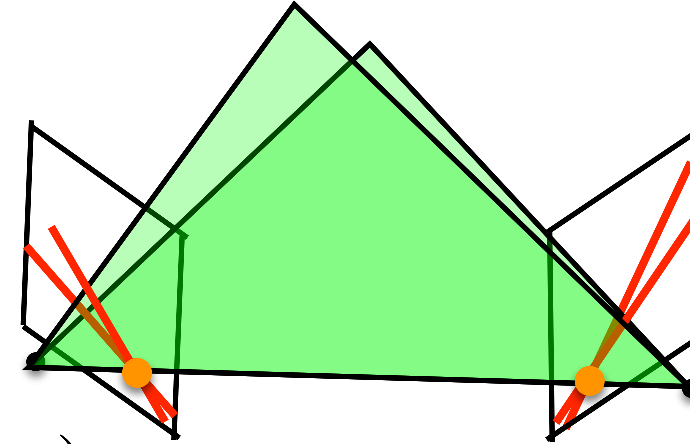
Maps point (x_1,y_1) from left image to line (a,b,c) in right image… and vice versa.

This is the line of points that might correspond to the point (x_1, y_1)

This is an “epipolar line”

Epipoles

$$\boxed{\boldsymbol{x}_2^\top E \boldsymbol{x}_1 = 0}$$



Thus for any point on the left (x_1, y_1),
the point e_2 = right epipole will satisfy this equation

$$\begin{bmatrix} e_x & e_y & 1 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} = [0 \quad 0 \quad 0]$$

$$A = U\Sigma V^T = \sum_i^{\min(m,n)} u_i \sigma_i v_i^T$$

$$e_2^T E = \sum_i e_2^T u_i \sigma_i v_i^T = 0$$

What does this imply? Recall that all of the vectors $u_1, \ldots, u_N$ are orthonormal

This implies that e_2 is equal to one of the singular vectors u_N
with an associated singular value $\sigma_N = 0$

Thus, the **right** epipole e_2 is equal to the smallest **left** singular vector u_N

Thus, the **left** epipole e_1 is equal to the smallest **right** singular vector v_N

Equations for corresponding points x_1 and x_2

Essential matrix (normalized cameras: K = I):

$$\mathbf{x}_2^T \hat{\mathbf{T}} R \mathbf{x}_1 = 0$$

$$\boldsymbol{x}_2^\top E \boldsymbol{x}_1 = 0$$

Fundamental matrix (unnormalized cameras: K ≠ I):

$$\mathbf{x_2}^T F \mathbf{x_1} = 0$$

$$\mathbf{x_2}^T K_2{}^{-T} \hat{\mathbf{T}} R K_1^{-1} \mathbf{x_1} = 0$$

Three questions:

(i) **Correspondence geometry:** Given two camera views and an image point x in one image, what points x’ can this correspond to in the second image (geometrically)?

(ii) **Camera geometry (motion):** Given a set of corresponding image points $\{x_i \leftrightarrow x'_i\}$, i=1,…,n across two camera views, what are the camera poses P and P’ for the two views?

(iii) **Scene geometry (structure):** Given corresponding image points $x_i \leftrightarrow x'_i$ and camera poses P, P’, what is the position of the 3D location X_i of these points?

Roadmap

- logistics
- review of motion segmentation
- two-view
    - geometric intuition
    - essential matrix E, fundamental matrix F
    - **properties of E, F**
    - estimating E,F from correspondences
    - inferring R,T from E,F

Roadmap

- logistics
- review of motion segmentation
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    - properties of E,F
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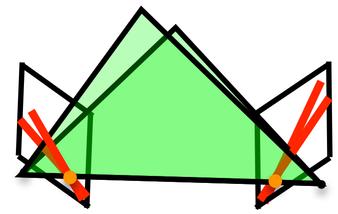
    - inferring R,T from E,F

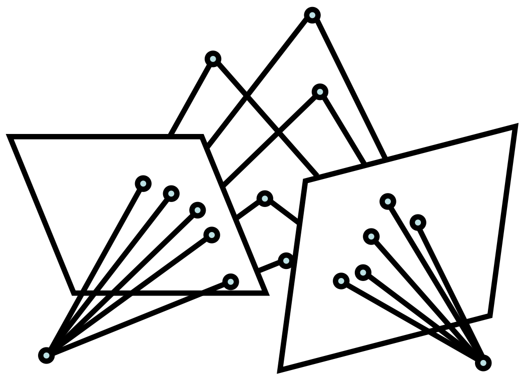
Estimation (fundamental matrix)



Assume we have a corresponding pair of points: in noise-free case….

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0 \iff \begin{bmatrix} xx' & xy' & x & yx' & yy' & y & x' & y' & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

Estimation (fundamental matrix)



Given m point correspondences (x_i, y_i) and (x'_i, y'_i):

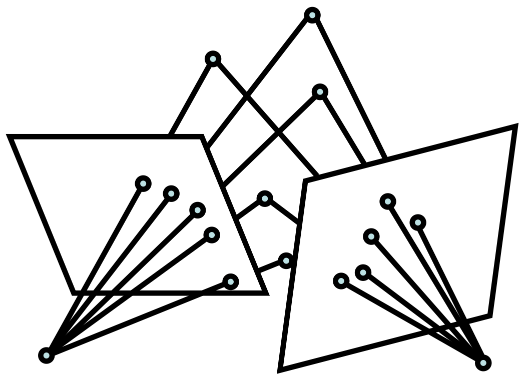
$$\begin{bmatrix} x_1x'_1 & x_1y'_1 & x_1 & y_1x'_1 & y_1y'_1 & y_1 & x'_1 & y'_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_mx'_m & x_my'_m & x_m & y_mx'_m & y_my'_m & y_m & x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

$$AF(:) = 0$$

What is the trivial solution for F?

How do I prevent this?

Estimation (fundamental matrix)



Given m point correspondences (x_i,y_i) and (x'_i,y'_i):

$$\begin{bmatrix} x_1x'_1 & x_1y'_1 & x_1 & y_1x'_1 & y_1y'_1 & y_1 & x'_1 & y'_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_mx'_m & x_my'_m & x_m & y_mx'_m & y_my'_m & y_m & x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

$$\min_{||F||=1} ||AF(:)||^2$$

How do I solve this? **SVD!**

How many pairs of points are needed?

m = 8 points ("8 point algorithm" from Longuet-Higgens)

Issue:

Imagine an image that is 100x100 pixels

$$
\begin{array}{c}
\left[\begin{array}{ccccccccc}
x_1x_1' & x_1y_1' & x_1 & y_1x_1' & y_1y_1' & y_1 & x_1' & y_1' & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_mx_m' & x_my_m' & x_m & y_mx_m' & y_my_m' & y_m & x_m' & y_m' & 1
\end{array}\right] \\
\begin{array}{ccccccccc}
\sim 10000 & \sim 10000 & \sim 100 & \sim 10000 & \sim 10000 & \sim 100 & \sim 100 & \sim 100 & 1
\end{array}
\end{array}
\begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0
$$

!

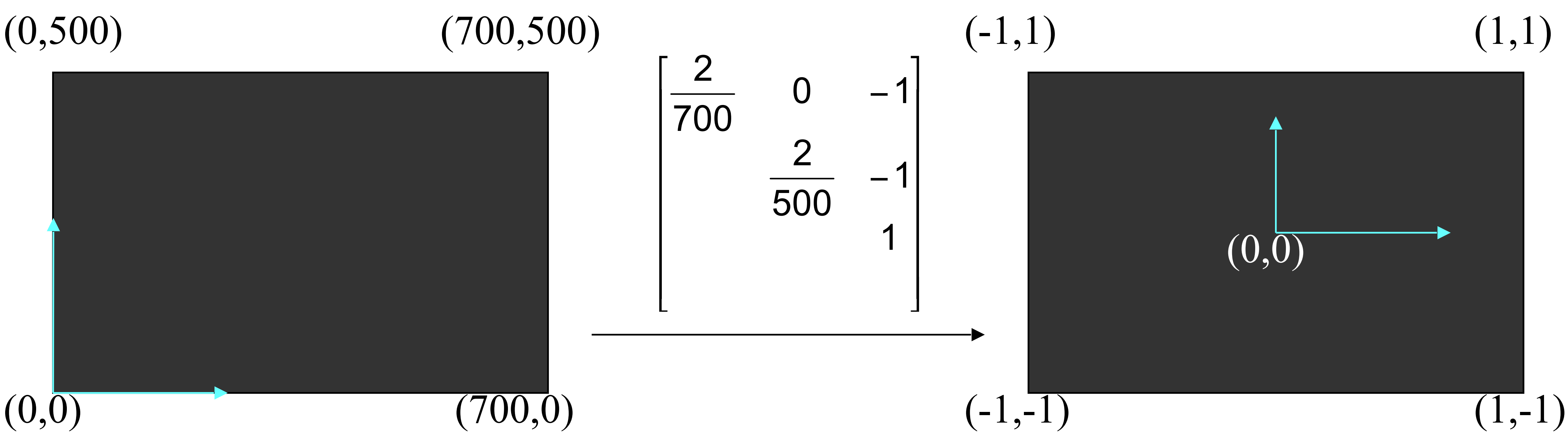
Orders of magnitude difference
Between column of data matrix
$\rightarrow$ least-squares yields poor results

How do I fix this?

“In Defense of the 8-point Algorithm”

(Hartley, PAMI´97)

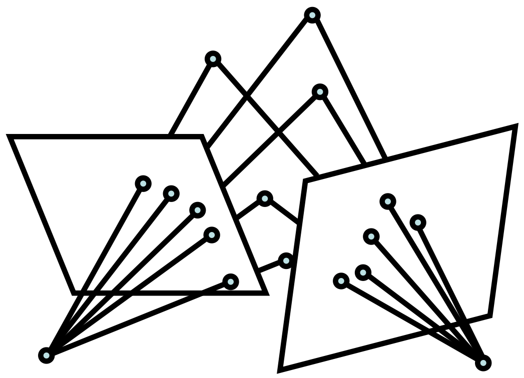
Transform image to [-1,1]x[-1,1]



$$\begin{bmatrix} \frac{2}{700} & 0 & -1 \\ & \frac{2}{500} & -1 \\ & & 1 \\ & & \end{bmatrix}$$

SVD now produces good results

Estimation (fundamental matrix)



Given m point correspondences (x_i,y_i) and (x'_i,y'_i):

$$\begin{bmatrix} x_1x'_1 & x_1y'_1 & x_1 & y_1x'_1 & y_1y'_1 & y_1 & x'_1 & y'_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_mx'_m & x_my'_m & x_m & y_mx'_m & y_my'_m & y_m & x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

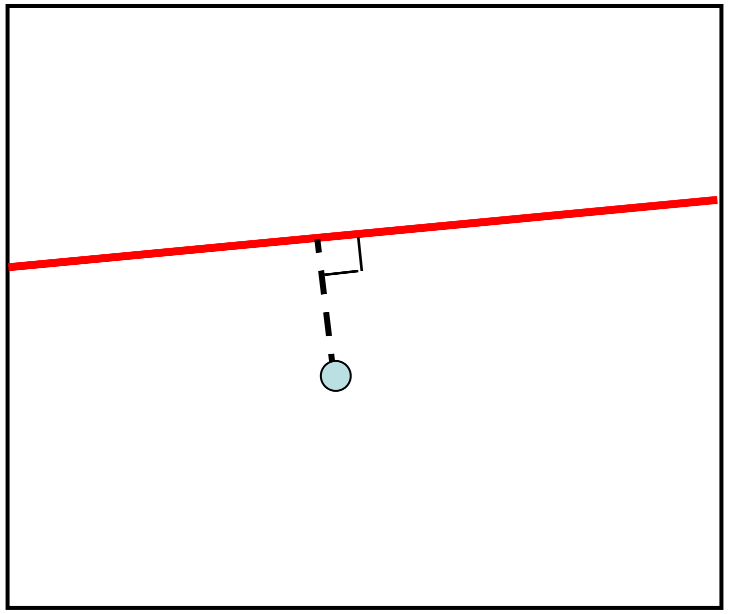
$$\min_{||F||=1} ||AF(:)||^2 = \min_F \sum_i (\mathbf{x'}_i^T F \mathbf{x}_i)^2$$

Is this a reasonable error to minimize?
What does this geometrically correspond to?

Recall: distance of point from a line

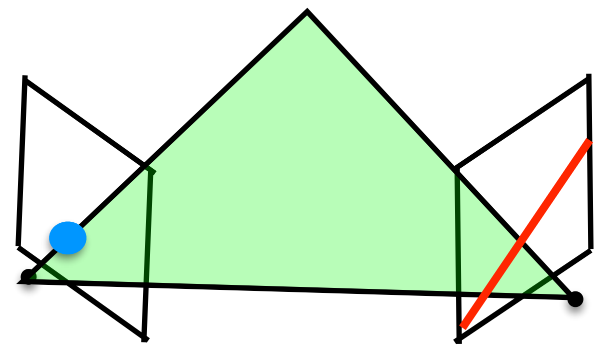
https://en.wikipedia.org/wiki/Distance_from_a_point_to_a_line

$$\operatorname{distance}(ax+by+c=0,(x_0,y_0)) = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}.$$



Recall:

$$x_2^T E x_1 = 0 \qquad\qquad x_2^T F x_1 = 0$$



$$\begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} . & . & . \\ . & . & . \\ . & . & . \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{Ex_1} = 0$$

$$ax_2 + by_2 + c = 0$$ (equation of a line!)

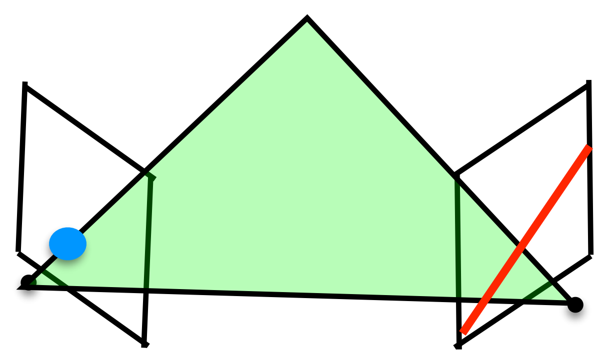
Maps point (x_1, y_1) from left image to line (a,b,c) in right image… and vice versa.

(a,b,c) is the line of points that might correspond to the point (x_1, y_1)

This is an "epipolar line"

Recall:

$$(x_i')^T E x_i = 0 \qquad\qquad (x_i')^T F x_i = 0$$



Ex_1

$$\begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$ax_2 + by_2 + c = 0 \qquad \text{(equation of a line!)}$$

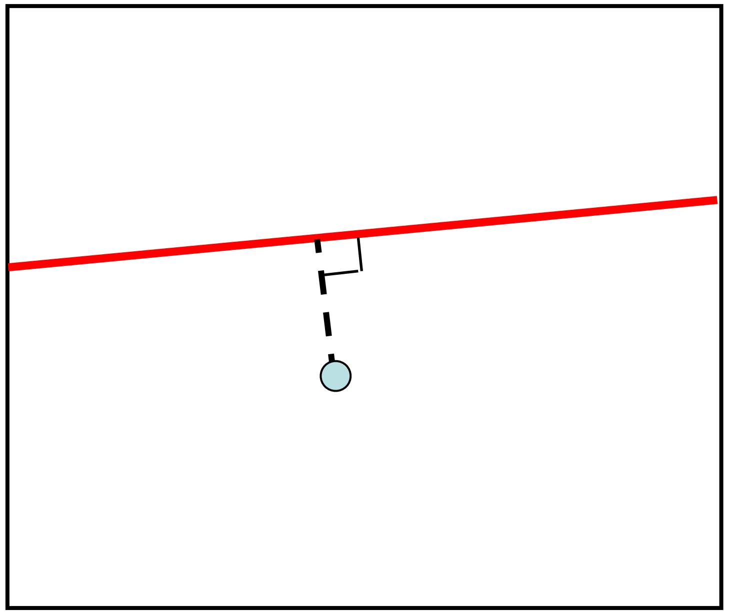
(a,b,c) is the epipolar line of points that might correspond to the point (x_1, y_1)

$$\min_F \sum_i (\mathbf{x'}_i^T F \mathbf{x}_i)^2 \qquad\qquad (x_i')^T F x_i = ax_2 + by_2 + c$$

Recall: distance of point from a line

https://en.wikipedia.org/wiki/Distance_from_a_point_to_a_line

$$\text{distance}(ax + by + c = 0, (x_0, y_0)) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$



$$\min_F \sum_i (\mathbf{x'}_i^T F \mathbf{x}_i)^2$$

$$(x_i')^T F x_i = a x_2 + b y_2 + c$$

this is *almost* the Euclidean distance between (x'_i, y'_i) and the epipolar line defined by (x_i, y_i)

Correct error requires minimizing Euclidean distance: $$\sum_i d^2(\mathbf{x'}_i, F\mathbf{x}_i) + d^2(\mathbf{x}_i, F^T\mathbf{x'}_i)$$

How do we optimize this?

Initialize with SVD from the previous problem, then use a nonlinear optimization solver

Recall: distance of point from a line

Step 1: Solve with SVD:

$$\begin{bmatrix} x_1 x'_1 & x_1 y'_1 & x_1 & y_1 x'_1 & y_1 y'_1 & y_1 & x'_1 & y'_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_m x'_m & x_m y'_m & x_m & y_m x'_m & y_m y'_m & y_m & x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0 \qquad \min_{||F||=1} ||AF(:)||^2$$

Step 2:

Minimize Euclidean distance: $$\sum_i d^2(\mathbf{x}'_i, F\mathbf{x}_i) + d^2(\mathbf{x}_i, F^T \mathbf{x}'_i)$$

Are we done? Is the resulting F a valid Fundamental Matrix?

$$\boldsymbol{x}_2^\top E \boldsymbol{x}_1 = 0 \qquad \mathbf{x_2}^T F \mathbf{x_1} = 0$$

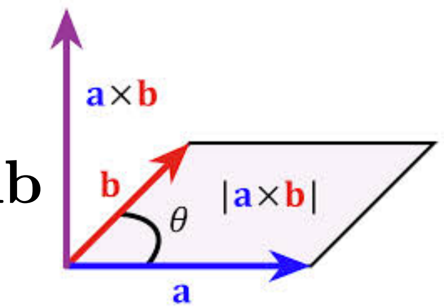
$$\mathbf{x}_2^T \hat{\mathbf{T}} R \mathbf{x}_1 = 0 \qquad \mathbf{x_2}^T K_2^{-T} \hat{\mathbf{T}} R K_1^{-1} \mathbf{x_1} = 0$$

Background: SVDs of skew symmetric matrices

Any skew-symmetric matrix (A = -A^T) can be thought of as a cross-product

Cross product: $\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta \, \mathbf{n}$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \equiv \hat{\mathbf{a}}\mathbf{b}$$



SVD of a skew-symmetric matrix:

$$\hat{\mathbf{a}} = \begin{bmatrix} -\mathbf{e}_2 & \mathbf{e}_1 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} ||\mathbf{a}|| & 0 & 0 \\ 0 & ||\mathbf{a}|| & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \\ \mathbf{e}_3^T \end{bmatrix}$$

where $\mathbf{e}_3 = \mathbf{a} \,/\, ||\mathbf{a}||$

Note that the singular values are $(\|a\|, \|a\|, 0)$

Properties (essential matrix)

https://en.wikipedia.org/wiki/Essential_matrix#Properties_of_the_essential_matrix

Q. How many DOFs are needed to specify an essential matrix?

$$\boldsymbol{x}_2^\top E \boldsymbol{x}_1 = 0$$

3 (rotation) + 2 (translation direction)

$$\mathbf{x}_2^T \hat{\mathbf{T}} R \mathbf{x}_1 = 0$$

Q. Can any 3x3 matrix be an essential matrix?

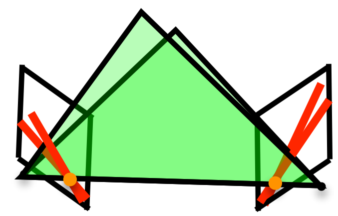
No…

E is the product of a rotation and skew-symmetric matrix $\hat{\mathbf{T}}$

Singular values of E = singular values of $\hat{\mathbf{T}}$ = (σ,σ,0)

[rotations do not effect singular values]

Properties (fundamental matrix)



$$\mathbf{x_2}^T K_2^{-T} \hat{\mathbf{T}} R K_1^{-1} \mathbf{x_1} = 0$$

$$\boxed{\mathbf{x_2}^T F \mathbf{x_1} = 0}$$

Q. Can any 3x3 matrix be a fundamental matrix?

Let $\mathbf{e}_2 = K_2\mathbf{T}$

$e_2^T F = ?$ $\mathbf{e}_2^T F = 0$

similar argument for $\mathbf{e}_1$

What does this imply about the answer to this question?

No! epipoles are in the null space, implying rank(F) = 2

Q. How many DOFs are needed to specify F?

7 = 9 - 1 (for scale) - 1 (for 0-determinant)

Formal characterizations

Ma et al, An Invitation to 3D Vision

Theorem 5.1 (Characterization of the essential matrix). *A non-zero matrix $E \in \mathbb{R}^{3\times 3}$ is an essential matrix* if and only if *E has a singular value decomposition (SVD): $E = U\Sigma V^T$ with*

$$\Sigma = diag\{\sigma, \sigma, 0\}$$

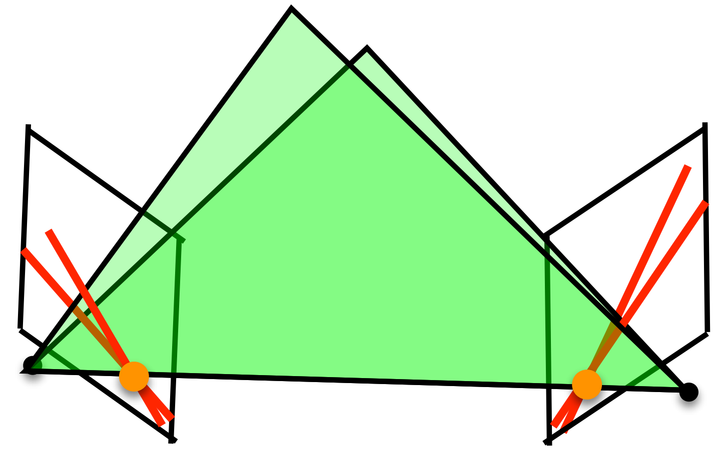
for some $\sigma \in \mathbb{R}_+$ and $U, V \in SO(3)$.

Remark 6.1. *Characterization of the fundamental matrix. A non-zero matrix $F \in \mathbb{R}^{3\times 3}$ is a fundamental matrix if F has a singular value decomposition (SVD): $E = U\Sigma V^T$ with*

$$\Sigma = diag\{\sigma_1, \sigma_2, 0\}$$

for some $\sigma_1, \sigma_2 \in \mathbb{R}_+$.

Essential and Fundamental Matrices



$$E = \hat{\mathbf{T}} R$$

$$\mathbf{x}_2^T E \mathbf{x}_1 = 0$$

$$E = \begin{bmatrix} \mathbf{u}_0 & \mathbf{u}_1 & \mathbf{e}_2 \end{bmatrix} \begin{bmatrix} \sigma & & \\ & \sigma & \\ & & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_0^T \\ \mathbf{v}_1^T \\ \mathbf{e}_1^T \end{bmatrix}$$

$$F = {K_2}^{-T} E K_1^{-1}$$

$$\mathbf{x}_2^T F \mathbf{x}_1 = 0$$

$$F = \begin{bmatrix} \mathbf{u}_0 & \mathbf{u}_1 & \mathbf{e}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_0^T \\ \mathbf{v}_1^T \\ \mathbf{e}_1^T \end{bmatrix}$$

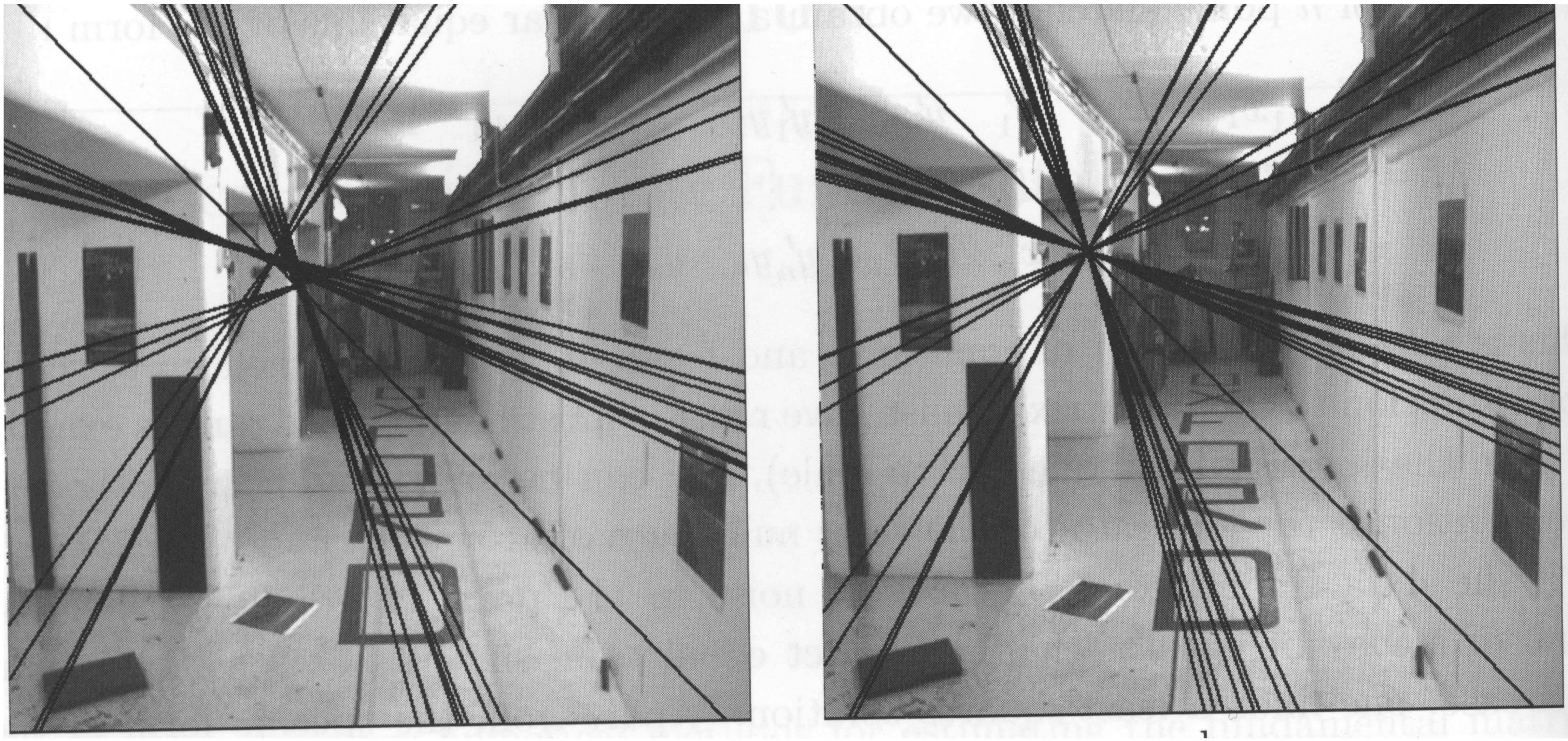
*where **e_1, e_2** are epipoles in right and left images and singular values can be arbitrarily scaled*

Once we solve for E or F using SVD + nonlinear optimization,
how do we ensure that we have a valid Essential or Fundamental Matrix?

F: zero out smallest singular value — set $\sigma_3 = 0$

E: zero out smallest singular value and average the over 2 — set $\sigma_3 = 0$, $\sigma = .5*(\sigma_1+\sigma_2)$

Rank-2 Fundamental Matrix



Closest fundamental matrix: set $\sigma_3 = 0$

7-point algorithm

Since F is rank-difficient (det = 0), we can estimate it with only m=7 correspondences

$$\begin{bmatrix} x_1x'_1 & x_1y'_1 & x_1 & y_1x'_1 & y_1y'_1 & y_1 & x'_1 & y'_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_mx'_m & x_my'_m & x_m & y_mx'_m & y_my'_m & y_m & x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{bmatrix} = 0$$

AF(:)=0

Idea: search for null vector of A_{7x9} such that F (after being reshaped into a 3x3 matrix) is rank 2)

1) Find 2 vectors that span *null space* of A_{7x9}: F_1 and F_2.
2) Find alpha such that Determinant$(\alpha F_1 + (1-\alpha)F_2) = 0$

[3rd order polynomial in α with at least one real solution]

Aside: what if cameras are calibrated?

Then we know K_1 and K_2, only need to solve for Essential Matrix
Has only 5 DOF

Turns out we only need 5 pairs of corresponding points,
but need to find roots of 10th degree polynomial

[Nister 04]

Incorrect Matches

What if you use an automatic method to find pairs of corresponding points
Then some of the matches will be incorrect (not just noisy)
How do we handle this?

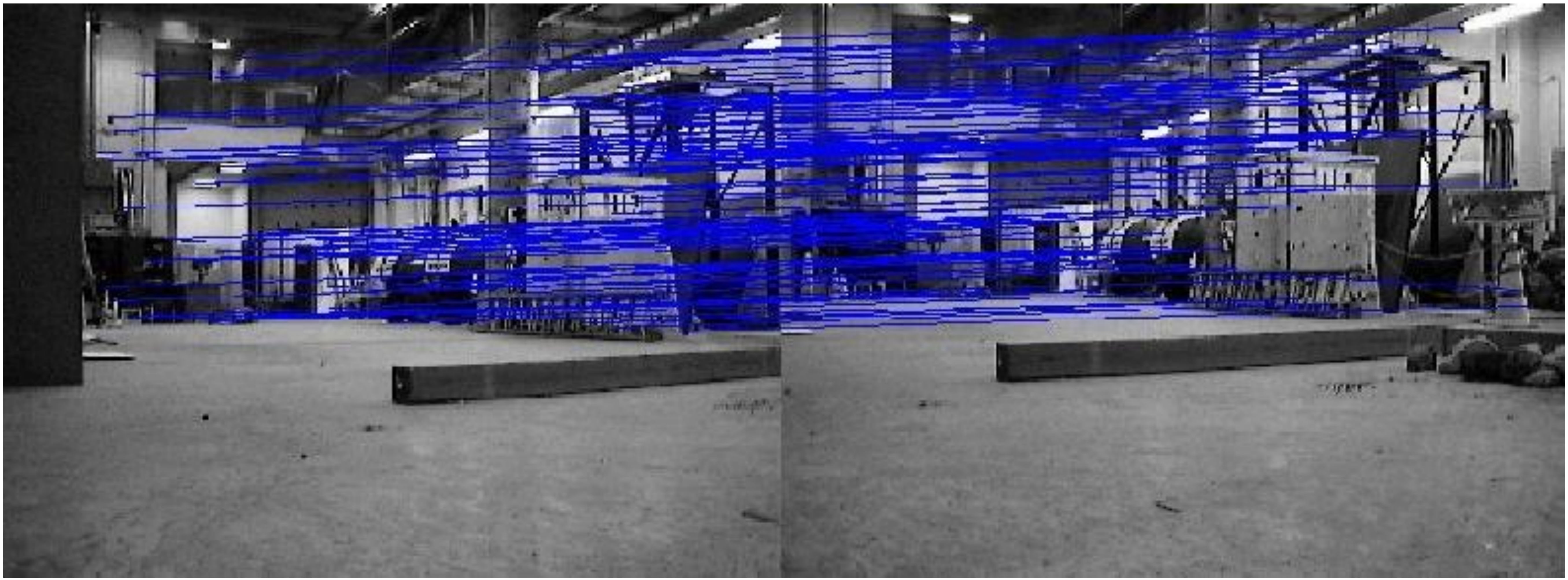
RANSAC loop:

1. Select 8 feature pairs (at random)
2. Compute Fundamental Matrix F
3. Compute *inliers, e.g.* point matches where $d(\mathbf{x}_i', F\mathbf{x}_i)^2 \leq \epsilon$
   Keep largest set of inliers

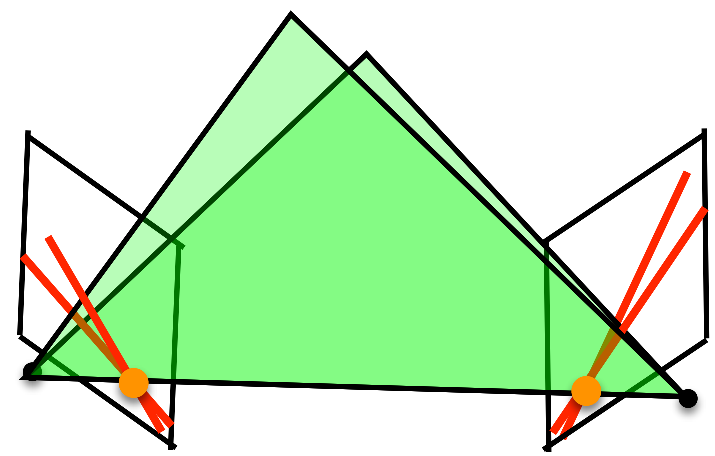
Re-compute least-squares estimate of Fundamental Matrix F using all of the inliers

Fundamental matrix estimation with RANSAC

Candidate correspondences:



Recovering T,R from E



1. Universal scale ambiguity

Doubling **T** results in same epipolar lines

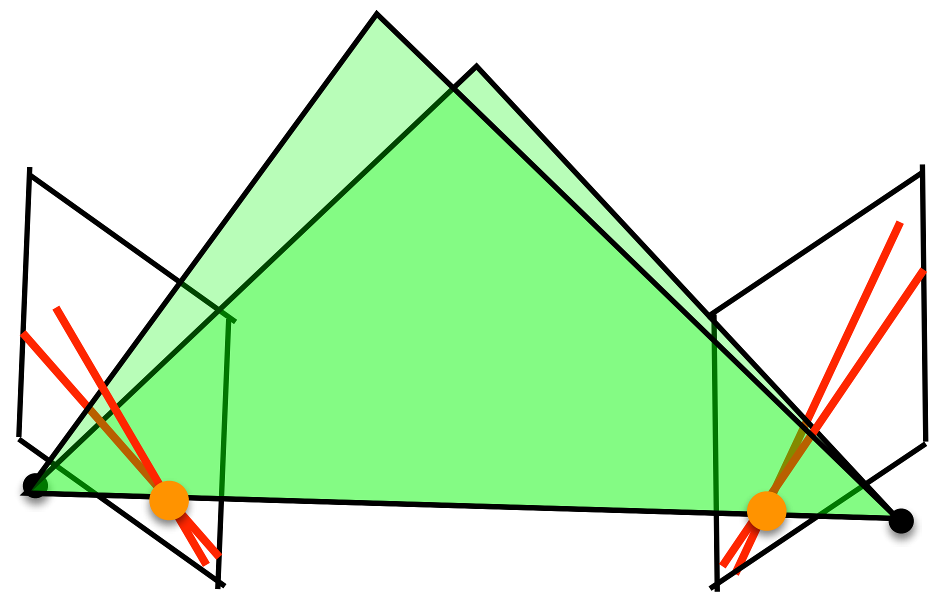
Let's fix $||\mathbf{T}|| = 1$

Numerous methods for recovering **T**,R from E exist:
SVD, Lounget-Higgen's alg, etc.

Recovering T from E

SVD-based approach for noise-free E (Szeliski Chap 7.2)

$$\boxed{\boldsymbol{x}_2^\top E \boldsymbol{x}_1 = 0}$$



Take (left-handside) product of E = $\hat{\mathbf{T}}$R with **T**

$$T^\top E = T^\top (T \times R) = 0$$

Implies that translation vector = left singular vector of E associated with smallest singular value

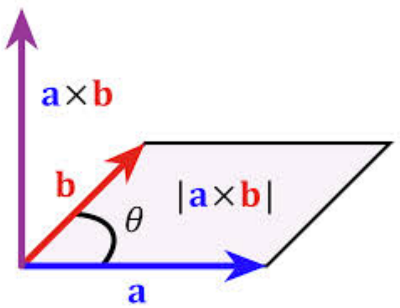
(but we only know T up to a scale factor)

Background: SVDs of skew symmetric matrices

Any skew-symmetric matrix (A = -A^T) can be thought of as a cross-product

Cross product: $\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta \; \mathbf{n}$

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \equiv \hat{\mathbf{a}}\mathbf{b}$$



SVD of a skew-symmetric matrix:

$$\hat{\mathbf{a}} = \begin{bmatrix} -\mathbf{e}_2 & \mathbf{e}_1 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} ||\mathbf{a}|| & 0 & 0 \\ 0 & ||\mathbf{a}|| & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \\ \mathbf{e}_3^T \end{bmatrix}$$

where $\mathbf{e}_3 = \mathbf{a} \,/\, ||\mathbf{a}||$

$$\hat{\mathbf{a}} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} ||\mathbf{a}|| & 0 & 0 \\ 0 & ||\mathbf{a}|| & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_{90}} \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \\ \mathbf{e}_3^T \end{bmatrix}$$

Recovering R from E

SVD-based approach (Szeliski Chap 7.2)

Recall skew-symmetric decomposition

$$\hat{\mathbf{T}} = \underbrace{\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{T}_3 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{Z} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_{90}} \underbrace{\begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \\ \mathbf{T}_3^T \end{bmatrix}}_{S^T}$$

$$\mathrm{SVD}(\hat{\mathbf{T}}R) = U\Sigma V^T$$

$$SZR_{90}S^T R = U\Sigma V^T$$

Match orthogonal and diagonal matrices (S = U, Z = Σ) and solve for R:

$$SZR_{90}S^T R = SZV^T \qquad\qquad R_{90}S^T R = V^T$$

$$R_{90^\circ} U^T R = V^T$$

$$\begin{aligned} R &= U R_{90^\circ}^T V^T \\ &= \pm U R_{\pm 90^\circ}^T V^T \end{aligned}$$

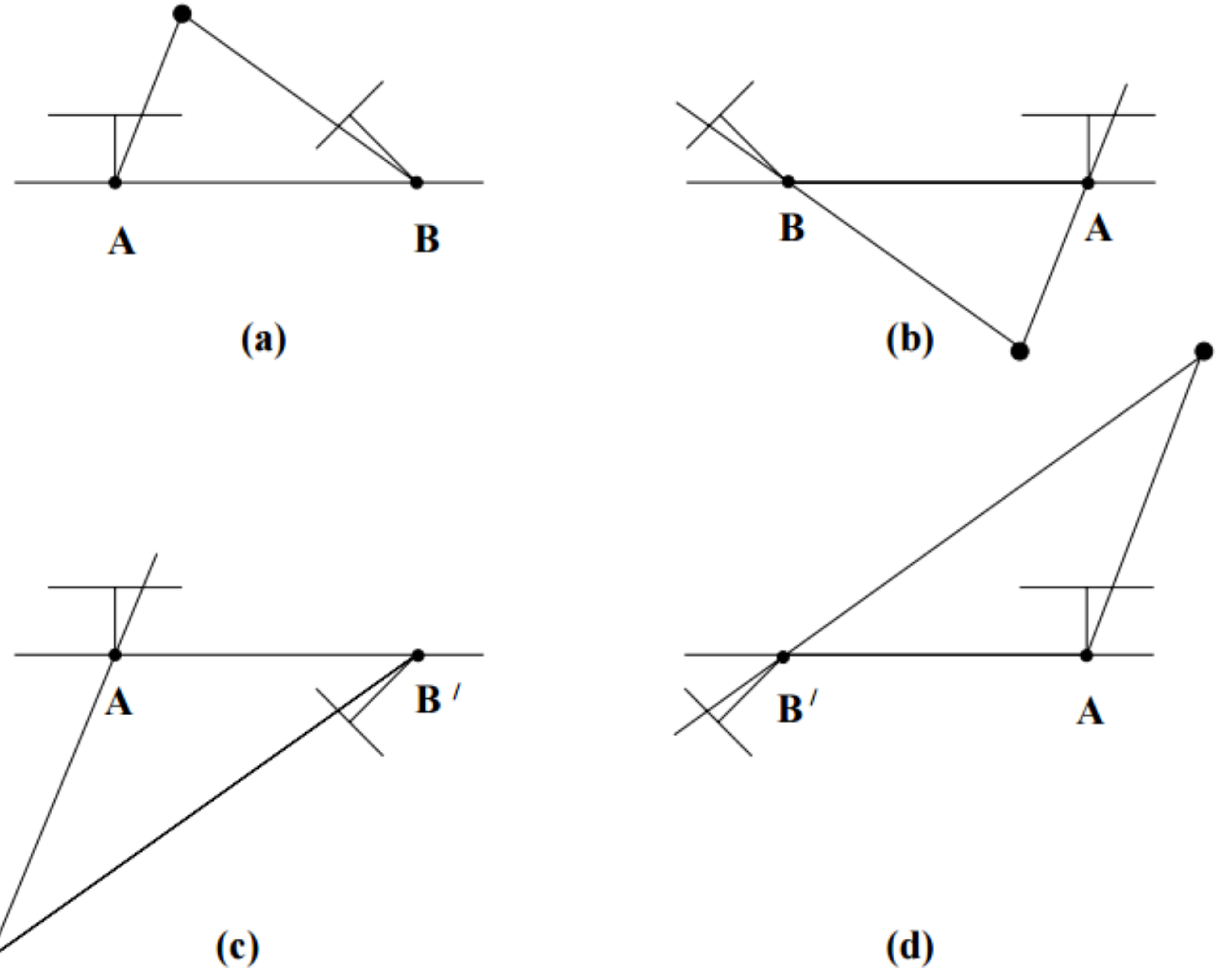


Fig. 8.12. **The four possible solutions for calibrated reconstruction from E.** *Between the left and right sides there is a baseline reversal. Between the top and bottom rows camera B rotates* 180° *about the baseline. Note, only in (a) is the reconstructed point in front of both cameras.*

Roadmap

- logistics
- review of motion segmentation
- two-view
  - geometric intuition
  - essential matrix E, fundamental matrix F
  - properties of E,F
  - estimating E,F from correspondences
  - Computing T and R from E