16-833 SLAM HW2 EKF

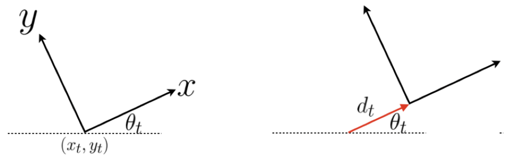
1 Theory

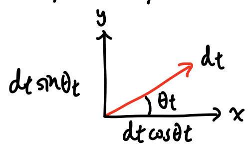
Given: 2D ground plane, t

[move] d_t (meters in x-direction), α_t (rotation in radian)

$p_t = [x_t \ y_t \ \theta_t]^T$, x_t & y_t 2D coordinate of robot's position, θ_t robot's orientation

1. Assume no noise or error, predict the next pose $p_{t+1}(p_t, d_t, \alpha_t)$





$$p_t = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} \xrightarrow{[move]} p_{t+1} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} x_t + d_t\cos(\theta_t) \\ y_t + d_t\sin(\theta_t) \\ \theta_t + \alpha_t \end{bmatrix} \#$$

2. Assume Gaussian errors ($e_x \sim N(0, \sigma_x^2)$, $e_y \sim N(0, \sigma_y^2)$, $e_\alpha \sim N(0, \sigma_\alpha^2)$)

predict uncertainty of the robot at time $t+1$ (Gaussian distribution with zero mean)

$$\Sigma_{t+1}^{robot} = G_t \Sigma_t G_t^T + R_t, \quad R_t = A_t \Sigma_t A_t^T \longrightarrow \Sigma_{t+1}^{robot} = G_t \Sigma_t G_t^T + A_t \Sigma_t A_t^T$$

$$p_t = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}, \quad p_{t+1} = f(p_t, u_t, e) = \begin{bmatrix} x_t + (d_t + e_x)\cos(\theta_t) - e_y\sin(\theta_t) \\ y_t + (d_t + e_x)\sin(\theta_t) + e_y\cos(\theta_t) \\ \theta_t + \alpha_t + e_\alpha \end{bmatrix}, \quad \Sigma_t = \begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_\alpha^2 \end{bmatrix}$$

$$G_t = \frac{\partial f}{\partial p_t} = \begin{bmatrix} 1 & 0 & -(d_t + e_x)\sin(\theta_t) - e_y\cos(\theta_t) \\ 0 & 1 & (d_t + e_x)\cos(\theta_t) - e_y\sin(\theta_t) \\ 0 & 0 & 1 \end{bmatrix}$$

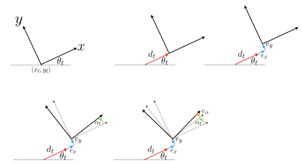


Figure 1: 0) Robot state; 1) robot moves d_t along its x-axis; 2) due to noise it shifted e_x and e_y along its x and y axises respectively; 3) robot rotates α_t after shifting; 4) due to noise the rotation is disturbed by e_α.

$$G_t^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(d_t + e_x)\sin(\theta_t) - e_y\cos(\theta_t) & (d_t + e_x)\cos(\theta_t) - e_y\sin(\theta_t) & 1 \end{bmatrix}$$

$$A_t = \frac{\partial f}{\partial e} = \begin{bmatrix} \cos(\theta_t) & -\sin(\theta_t) & 0 \\ \sin(\theta_t) & \cos(\theta_t) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_t^T = \begin{bmatrix} \cos(\theta_t) & \sin(\theta_t) & 0 \\ -\sin(\theta_t) & \cos(\theta_t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

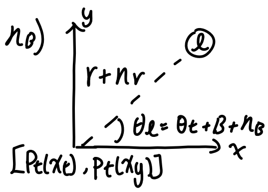
$$\Sigma_{t+1}^{robot} = \begin{bmatrix} 1 & 0 & -(d_t+\ell_x)\sin(\theta_t)-\ell_y\cos(\theta_t) \\ 0 & 1 & (d_t+\ell_x)\cos(\theta_t)-\ell_y\sin(\theta_t) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_\alpha^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(d_t+\ell_x)\sin(\theta_t)-\ell_y\cos(\theta_t) & (d_t+\ell_x)\cos(\theta_t)-\ell_y\sin(\theta_t) & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} \cos(\theta_t) & -\sin(\theta_t) & 0 \\ \sin(\theta_t) & \cos(\theta_t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_\alpha^2 \end{bmatrix} \begin{bmatrix} \cos(\theta_t) & \sin(\theta_t) & 0 \\ -\sin(\theta_t) & \cos(\theta_t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \#$$

3. Consider landmarks ℓ observed by laser sensor with measurement of bearing angle $\beta [-\pi, \pi]$, range r, with noise $n_\beta \sim \mathcal{N}(0, \sigma_\beta^2)$ & $n_r \sim \mathcal{N}(0, \sigma_r^2)$.

Write down (ℓ_x, ℓ_y) in global coordinates as a function of P_t, β, r.

$$P_t = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}, \quad \begin{aligned} \theta_\ell &= P_t(\theta_t) + \beta + n_\beta \\ \ell_x &= P_t(x_t) + (r+n_r)\cos(\theta_\ell) \\ \ell_y &= P_t(y_t) + (r+n_r)\sin(\theta_\ell) \end{aligned} \rightarrow \begin{aligned} \ell_x &= P_t(x_t) + (r+n_r)\cos(P_t(\theta_t) + \beta + n_\beta) \\ \ell_y &= P_t(y_t) + (r+n_r)\sin(P_t(\theta_t) + \beta + n_\beta) \end{aligned}$$



$$\begin{bmatrix} \ell_x \\ \ell_y \end{bmatrix} = \begin{bmatrix} P_t(x_t) + (r+n_r)\cos(P_t(\theta_t) + \beta + n_\beta) \\ P_t(y_t) + (r+n_r)\sin(P_t(\theta_t) + \beta + n_\beta) \end{bmatrix} \#$$

4. From 1.3, predict the measurement β & r based on $\ell_x, \ell_y, P_t, n_r, n_\beta$

$$r + n_r = [(\ell_x - P_t(x_t))^2 + (\ell_y - P_t(y_t))^2]^{\frac{1}{2}} \rightarrow r = [(\ell_x - P_t(x_t))^2 + (\ell_y - P_t(x_y))^2]^{\frac{1}{2}} - n_r$$

$$\beta = \theta_\ell - P_t(\theta_t) - n_\beta, \quad \theta_\ell = \arctan\left(\frac{\ell_y - P_t(y_t)}{\ell_x - P_t(x_t)}\right), \quad \beta = \arctan\left(\frac{\ell_y - P_t(y_t)}{\ell_x - P_t(x_t)}\right) - P_t(\theta_t) - n_\beta$$

$$\beta = \text{warp2pi}(\text{np.artan2}(\ell_y - P_t(y_t), \ell_x - P_t(x_t)) - P_t(\theta_t) - n_\beta)$$

$$\begin{bmatrix} \hat\beta \\ \hat r \end{bmatrix} = \begin{bmatrix} \text{warp2pi}(\text{np.artan2}(\ell_y - P_t(y_t), \ell_x - P_t(x_t)) - P_t(\theta_t) - n_\beta) \\ ((\ell_x - P_t(x_t))^2 + (\ell_y - P_t(y_t))^2)^{\frac{1}{2}} - n_r \end{bmatrix} \#$$

5. From 1.4, derive the analytical form of the measurement Jacobian H_p (2×3 matrix)

$$H_p = \frac{\partial h(\bar{u}_p)}{\partial x_p} \rightarrow H_p = \begin{bmatrix} \frac{\partial h(\beta)}{\partial P_t(x_t)} & \frac{\partial h(\beta)}{\partial P_t(y_t)} & \frac{\partial h(\beta)}{\partial P_t(\theta_t)} \\ \frac{\partial h(r)}{\partial P_t(x_t)} & \frac{\partial h(r)}{\partial P_t(y_t)} & \frac{\partial h(r)}{\partial P_t(\theta_t)} \end{bmatrix},$$

$$\beta = \text{warp2pi}(\text{np.artan2}(\ell_y - P_t(y_t), \ell_x - P_t(x_t)) - P_t(\theta_t) - n_\beta) = \arctan\left(\frac{\ell_y - P_t(y_t)}{\ell_x - P_t(x_t)}\right) - P_t(\theta_t) - n_\beta$$

$$r = [(\ell_x - P_t(x_t))^2 + (\ell_y - P_t(y_t))^2]^{\frac{1}{2}} - n_r$$

$$H_p = \begin{bmatrix} \frac{-\ell_y + P_t(y_t)}{(\ell_x - P_t(x_t))^2 + (\ell_y - P_t(y_t))^2} & \frac{\ell_x - P_t(x_t)}{(\ell_x - P_t(x_t))^2 + (\ell_y - P_t(y_t))^2} & -1 \\ \frac{-\ell_x + P_t(x_t)}{\sqrt{(\ell_x - P_t(x_t))^2 + (\ell_y - P_t(y_t))^2}} & \frac{-\ell_y + P_t(y_t)}{\sqrt{(\ell_x - P_t(x_t))^2 + (\ell_y - P_t(y_t))^2}} & 0 \end{bmatrix} \#$$

$$\frac{d}{dx}\left(\arctan\left(\frac{y}{x}\right)\right) = -\frac{y}{y^2+x^2}$$

$$\frac{d}{dy}\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{x}{y^2+x^2}$$

$\ell_x - P_t(x_t) \cdot (-1)$

$\ell_y - P_t(y_t) \cdot (-1)$

6. Derive the analytical form of the measurement Jacobian H_ℓ (2×2 matrix)

$$H_\ell = \begin{bmatrix} \frac{\partial h(\beta)}{\partial \ell x} & \frac{\partial h(\beta)}{\partial \ell y} \\ \frac{\partial h(r)}{\partial \ell x} & \frac{\partial h(r)}{\partial \ell y} \end{bmatrix}, \quad \beta = \arctan\left(\frac{\ell y - P_t(y_t)}{\ell x - P_t(x_t)}\right) - P_t(\theta_t) - n_\beta,$$

$$r = [(\ell x - P_t(x_t))^2 + (\ell y - P_t(y_t))^2]^{\frac{1}{2}} - n_r$$

$$H_\ell = \begin{bmatrix} \frac{-\ell y + P_t(y_t)}{(\ell x - P_t(x_t))^2 + (\ell y - P_t(y_t))^2} & \frac{\ell x - P_t(x_t)}{(\ell x - P_t(x_t))^2 + (\ell y - P_t(y_t))^2} \\ \frac{-\ell x + P_t(x_t)}{\sqrt{(\ell x - P_t(x_t))^2 + (\ell y - P_t(y_t))^2}} & \frac{-\ell y + P_t(y_t)}{\sqrt{(\ell x - P_t(x_t))^2 + (\ell y - P_t(y_t))^2}} \end{bmatrix} \#$$

$$\frac{d}{dx}\left(\arctan\left(\frac{y}{x}\right)\right) = -\frac{y}{y^2+x^2}$$

$$\frac{d}{dy}\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{x}{y^2+x^2}$$

$\ell x - P_t(x_t) \cdot (-1)$

$\ell y - P_t(y_t) \cdot (-1)$

Due to the independence assumption in EKF-SLAM, we do not need to calculate the measurement Jacobian with respect to other landmarks except for itself. Measurements from different landmarks are conditionally independent from each other.