Exercise 1 (Canonical form)

Given: $\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$ $m=1$

$n=2 \rightarrow 2$ states

Find: the controllable canonical form state space representation

Solution:

$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$ $m=1$, $n=2$ $\xrightarrow{\text{represent}}$ $\frac{Y(s)}{X_1(s)} \cdot \frac{X_1(s)}{U(s)} = \frac{s+3}{1} \cdot \frac{1}{s^2+3s+2}$

Solve $Y(s)$

$Y(s) = [s+3]\,x_1$

$Y(s) = s x_1 + 3x_1$

Let $\dot{x}_1 = x_2$

$Y = \dot{x}_1 + 3x_1$

$Y = x_2 + 3x_1$

Solve $U(s)$

$U(s) = [s^2+3s+2]\,x_1$

$U(s) = s^2 x_1 + 3s x_1 + 2x_1$

Let $\dot{x}_2 = \ddot{x}_1$

$U = \dot{x}_2 + 3\dot{x}_1 + 2x_1$

$\dot{x}_2 = U - 3\dot{x}_1 - 2x_1$

combine

Let $\dot{x}_1 = x_2$, $\dot{x}_2 = \ddot{x}_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]\,u$$

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Exercise 2 (Realization matrix form of realizable MIMO system)

Given: $\hat{G}_1(s) = \begin{bmatrix} \frac{1}{s} & \frac{s+3}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+1} \end{bmatrix}$

Find: the state-space realization

Solution:

$$G(\infty) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow G_{sp} = \begin{bmatrix} \frac{1-0(s)}{s} & \frac{s+3-1(s+1)}{s+1} \\ \frac{1-0(s)}{s+3} & \frac{s-1(s+1)}{s+1} \end{bmatrix} \rightarrow G_{sp} = \begin{bmatrix} \frac{1}{s} & \frac{2}{s+1} \\ \frac{1}{s+3} & \frac{-1}{s+1} \end{bmatrix}$$

$$d(s) = (s)(s+1)(s+3) \rightarrow d_s = s^3 + 4s^2 + 3s \quad \rightarrow \alpha_1 = 4,\ \alpha_2 = 3,\ \alpha_3 = 0$$

$$G_{sp} = \frac{1}{s^3+4s^2+3s} \begin{bmatrix} 1(s+1)(s+3) & 2(s)(s+3) \\ 1(s)(s+1) & -1(s)(s+3) \end{bmatrix}$$

$$G_{sp} = \frac{1}{s^3+4s^2+3s} \begin{bmatrix} s^2+4s+3 & 2s^2+6s \\ s^2+s & -s^2-3s \end{bmatrix}$$

$$N_1(s) = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix},\quad N_2(s) = \begin{bmatrix} 4 & 6 \\ 1 & -3 \end{bmatrix},\quad N_3(s) \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 4 & 6 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix},\quad D = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} -4 & 0 & 3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u,\quad y = \begin{bmatrix} 1 & 2 & 4 & 6 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} u$$

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Exercise 3 (Minimum Realizations)

Given: $\dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad y = [2 \ 2] x \quad (1)$

$\dot{x} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \quad y = [2 \ 0] x \quad (2)$

Find: (a) are they equivalent; do they have the same transfer function

(b) are they minimal realization

$E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$E^{-1} = \frac{1}{\det(E)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Solution: $G_p(s) = \frac{Y(s)}{U(s)}$

(a)

(1) $s X(s) = A X(s) + B U(s) \rightarrow X(s)[Is - A] = BU(s) \rightarrow X(s) = [Is-A]^{-1} BU(s)$

$Y(s) = C X(s) + D U(s) \rightarrow Y(s) = C[Is-A]^{-1} BU(s) + DU(s)$

$\frac{Y(s)}{U(s)} = C[Is-A]^{-1}B + D = G_p(s)$

$\frac{Y(s)}{U(s)}(1) = [2 \ 2] \begin{bmatrix} s-2 & 0-1 \\ 0-0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \rightarrow \frac{Y(s)}{U(s)}(1) = [2 \ 2] \begin{bmatrix} s-2 & -1 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{Y(s)}{U(s)}(1) = [2 \ 2] \frac{1}{(s-2)(s-1)} \begin{bmatrix} s-1 & 1 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{Y(s)}{U(s)}(1) = [2 \ 2] \frac{1}{s^2-3s+2} \begin{bmatrix} s-1 \\ 0 \end{bmatrix} \rightarrow \frac{Y(s)}{U(s)}(1) = \frac{1}{s^2-3s+2} \begin{bmatrix} 2s-2 \\ 0 \end{bmatrix} \rightarrow G_p(s)_{(1)} = \frac{2s-2}{s^2-3s+2}$

$\frac{Y(s)}{U(s)}(2) = [2 \ 0] \begin{bmatrix} s-2 & 0-0 \\ 0-(-1) & s-(-1) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \rightarrow \frac{Y(s)}{U(s)}(2) = [2 \ 0] \begin{bmatrix} s-2 & 0 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\frac{Y(s)}{U(s)}(2) = [2 \ 0] \frac{1}{(s-2)(s+1)} \begin{bmatrix} s+1 & 0 \\ -1 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

not equivalent

$\frac{Y(s)}{U(s)}(2) = [2 \ 0] \frac{1}{s^2-s-2} \begin{bmatrix} s+1 \\ 2s-5 \end{bmatrix} \rightarrow \frac{Y(s)}{U(s)}(2) = \frac{1}{s^2-s-2} \begin{bmatrix} 2s+2 \\ 0 \end{bmatrix} \rightarrow G_p(s)_{(2)} = \frac{2s+2}{s^2-s-2}$

$G_p(s)_{(1)} = \frac{2s-2}{s^2-3s+2} \neq G_p(s)_{(2)} = \frac{2s+2}{s^2-s-2}$ $\because$ have different TFs.

$\therefore$ they are not equivalent # (a)

(b)

(1) $\dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad y = [2\ 2] x$

$P = [B \vdots AB \vdots \cdots \vdots A^{k-1}B]$, if $\text{rank}(P) = n \rightarrow$ controllable

$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}$, if $\text{rank}(Q) = n \rightarrow$ observable $\qquad (n=2)$

$P = [B \vdots AB]$

$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $AB = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$Q = \begin{bmatrix} C \\ CA \end{bmatrix}$, $C = [2\ 2]$, $CA = [2\ 2]\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = [4\ 4]$, $Q = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$

$\because$ rank$(P) \neq n \rightarrow$ uncontrollable & rank$(Q) \neq n \rightarrow$ unobservable

$\therefore$ not minimal realization

$\dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad y = [2\ 2] x$

is not a minimal realization # (b)(1)

(2) $\dot{x} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \qquad y = [2\ 0] x$

$(n=2)$

$P = [B \vdots AB]$

$B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $AB = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix} \rightarrow$ rank$(P) = n \rightarrow$ controllable

$Q = \begin{bmatrix} C \\ CA \end{bmatrix}$, $C = [2\ 0]$, $CA = [2\ 0]\begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} = [4\ 0]$, $Q = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$

$\because$ rank$(Q) \neq n \rightarrow$ unobservable

$\therefore$ not minimal realization

$\dot{x} = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \qquad y = [2\ 0] x$

is not a minimal realization # (b)(2)

Exercise 4 (Realization)

Given: $g(s) = \dfrac{2s-4}{s^3-7s+6}$

Find: (a) determine the canonical controllable realization

(b) determine the canonical observable realization

(c) determine a minimal realization

Solution:

(a) $g(s) = \dfrac{2s-4}{s^3-7s+6}$ $\quad m=1, \; n=3 \quad \xrightarrow{\text{represent}} \quad \dfrac{Y(s)}{X_1(s)} \cdot \dfrac{X_1(s)}{U(s)} = \dfrac{2s-4}{1} \cdot \dfrac{1}{s^3-7s+6}$

Solve $Y(s)$

$Y(s) = [2s-4]X_1(s)$

$Y(s) = 2sX_1(s) - 4X_1(s)$

Let $\dot{x}_1 = x_2$

$Y = 2x_2 - 4x_1$

Solve $U(s)$

$U(s) = [s^3-7s+6]X_1(s)$

$U(s) = s^3X_1(s) - 7sX_1(s) + 6X_1(s)$

Let $\dot{x}_2 = x_3 = \ddot{x}_1$, $\dot{x}_3 = \ddot{x}_2 = \dddot{x}_1$

$U = \dot{x}_3 - 7x_2 + 6x_1$

$\dot{x}_3 = U + 7x_2 - 6x_1$

combine

Let $\dot{x}_1 = x_2$

$\dot{x}_2 = x_3 = \ddot{x}_1$

$\dot{x}_3 = \ddot{x}_2 = \dddot{x}_1$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & 7 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}u, \quad y = \begin{bmatrix}-4 & 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + [0]u$$

(a) #

(b) $\dfrac{Y(s)}{U(s)} = \dfrac{2s-4}{s^3-7s+6} \rightarrow Y[s^3-7s+6] = U[2s-4] \rightarrow s^3Y = 2sU - 4U + 7sY - 6Y$

$s^3Y = s[2U+7Y] + [-4U-6Y]$

$Y = \underbrace{s^{-2}[2U+7Y] + s^{-3}[-4U-6Y]}_{x_1}$, $\quad Y = au + x_1 \rightarrow \boxed{Y = x_1}$

$x_1 = s^{-2}[2U+7Y] + s^{-3}[-4U-6Y] \rightarrow s^2X_1 = 2U + 7Y + s^{-1}[-4U-6Y]$

Let $\dot{x}_1 = x_2$, $\dot{x}_2 = x_3 = \ddot{x}_1$, $\dot{x}_3 = \ddot{x}_2 = \dddot{x}_1$

$s^2 X_1 = 2U + 7Y + s^{-1}[-4U - 6Y] \rightarrow \dot{X}_2 = 2U + 7Y + \overbrace{s^{-1}[-4U-6Y]}^{X_3}$, $\dot{X}_2 = 2U + 7Y + X_3$

$X_3 = s^{-1}[-4U - 6Y] \rightarrow sX_3 = -4U - 6Y \rightarrow \dot{X}_3 = -4U - 6Y$

$y = x_1$	$\dot{x}_1 = 0$	$\dot{x}_2 = 2U + 7Y + x_3$	$\dot{x}_3 = -4U - 6Y$
		$\dot{x}_2 = 2U + 7x_1 + x_3$	$\dot{x}_3 = -4U - 6x_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 7 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix} u , \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0] u \quad \#$$

(b)

(c) $g(s) = \dfrac{2s - 4}{s^3 - 7s + 6}$ (strictly proper) ($k = 3$)

$D(s) = s^3 - 7s + 6$, $N(s) = 0s^2 + 2s - 4$

$\alpha = [\alpha_1 \ \alpha_2 \ \alpha_3] = [0 \ -7 \ 6]$, $\beta = [\beta_1 \ \beta_2 \ \beta_3] = [0 \ 2 \ -4]$

$$\dot{x}(t) = Ax(t) + Bu(t) = \begin{bmatrix} 0 & 7 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) ,$$

$$y(t) = Cx(t) + u(t) = [0 \ 2 \ -4]\, x(t) + [0]\, u(t)$$

(verify correct realization using Wolfram)

$P = [B \,\vdots\, AB \,\vdots \ldots \vdots\, A^{k-1}B]$, if $\mathrm{rank}(P) = n \rightarrow$ controllable

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} ,$$ if $\mathrm{rank}(Q) = n \rightarrow$ observable ($n = 3$)

$P = [B \,\vdots\, AB \,\vdots\, A^2B]$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \quad AB = \begin{bmatrix} 0 & 7 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \quad A^2B = \begin{bmatrix} 0 & 7 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \mathrm{rank}(P) = n \rightarrow \text{controllable}$$

$$Q=\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

$$C=[0\ 2\ -4]\ ,\ CA=[0\ 2\ -4]\begin{bmatrix} 0 & 7 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}=[2\ -4\ 0]$$

$$CA^2=[0\ 2\ -4]\begin{bmatrix} 0 & 7 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^2=[-4\ 14\ -12]$$

$$Q=\begin{bmatrix} 0 & 2 & -4 \\ 2 & -4 & 0 \\ -4 & 14 & -12 \end{bmatrix} \xrightarrow{REF} Q=\begin{bmatrix} 1 & -\frac{7}{2} & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank}\,Q \neq n \rightarrow \text{unobservable}$$

Factor the TF & try again

Attemp 2

$$g(s)=\frac{2s-4}{s^3-7s+6} \rightarrow g(s)=\frac{2\cancel{(s-2)}}{(s^2+2s+3)\cancel{(s-2)}} \rightarrow g(s)=\frac{2}{s^2+2s+3}$$

$D(s)=s^2+2s+3$, $N(s)=2$

$\alpha=[\alpha_1\ \alpha_2]=[2\ 3]$, $\beta=[\beta_1\ \beta_0]=[0\ 2]$

$$\dot{x}(t)=Ax(t)+Bu(t) = \begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix}x(t)+\begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)$$

$$y(t)=Cx(t)+u(t)$$
$$y(t)=[0\ 2]\,x(t)+[0]\,u(t)$$

(verify correct realization using Wolfram) $(n=2)$

$P=[B \vdots AB]$

$$B=\begin{bmatrix} 1 \\ 0 \end{bmatrix},\ AB=\begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} -2 \\ 1 \end{bmatrix},\ P=\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \rightarrow \text{rank}\,P=n \rightarrow \text{controllable}$$

$$Q=\begin{bmatrix} C \\ CA \end{bmatrix},\ C=[0\ 2],\ CA=[0\ 2]\begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix}=[2\ 0],\ Q=\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \rightarrow \text{rank}\,Q=n$$

observable

$\therefore\ \dot{x}(t)=\begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix}x(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)\ ,\ y(t)=[0\ 2]\,x(t)+[0]\,u(t)$ is observable & controllable

$\therefore$ this is a minimal realization # (c)

Exercise 5 (Controllable decomposition)

Given: $\dot{x} = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \qquad y = [1 \;\; 1]\, x$

$T.F. = \dfrac{2s+10}{s^2+2s-15}$

Find: (a) reduce to a controllable form

(b) is the reduced state equation observable

Solution:

(a) check controllability

$P: [B \vdots AB]$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $AB = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \rightarrow$ not controllable

$A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}$, $\det(A - \lambda I) = 0$, $\left| \begin{bmatrix} -1-\lambda & 4 \\ 4 & -1-\lambda \end{bmatrix} \right| = \lambda^2 + 2\lambda - 15 \rightarrow (\lambda-3)(\lambda+5)$

$\lambda_1 = 3, \quad \lambda_2 = -5$

Eigenvectors: $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, M^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$\hat{A} = M^{-1} A M = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix}$, $A = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}$, $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $M^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$\hat{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix}$

$\hat{B} = M^{-1} B = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$

$\hat{C} = CM = [C_c \;\; C_{\bar{c}}]$, $C = [1 \;\; 1]$, $\hat{C} = [1 \;\; 1]\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = [2 \;\; 0] = [C_c \;\; C_{\bar{c}}]$

$\dot{\hat{x}}_c = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix}\begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u, \quad y = [C_c \;\; C_{\bar{c}}]\begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + Du$

$\dot{\hat{x}}_c = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = [2 \;\; 0]\begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + [0]\, u$

controllable uncontrollable

still not controllable

$\dot{\hat{x}}_c = A_c \hat{x}_c + B_c u, \quad y = C_c \hat{x}_c + Du$

Try $\dot{\hat{x}}_c = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = [2 \;\; 0]\begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + [0]\, u$ #a

controllable

$T.F. = \dfrac{2s+10}{s^2+2s-15}$ check

(b)

$A = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}$ $(n=2)$ $\quad C = \begin{bmatrix} 2 & 0 \end{bmatrix}$

$Q = \begin{bmatrix} C \\ CA \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \end{bmatrix}$, $CA = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 6 & 0 \end{bmatrix}$

$Q = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix} \rightarrow \text{rank}(Q) \neq n$ not observable

$\dot{\hat{x}}_c = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$, $y = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_c \\ \hat{x}_{\bar{c}} \end{bmatrix} + [0] u$ not observable

#b

Exercise 6 (kalman decomposition)

Given:

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u \qquad y = [0\ 1\ 1\ 0\ 1]\, x$$

Find: decompose the equation to a form that is both controllable & observable

Solution:

$$\dot{x} = \left[\begin{array}{cccc:c} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ \hdashline 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right] x + \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \\ \hdashline 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cccc:c} 0 & 1 & 1 & 0 & 1 \end{array}\right] x$$

$$\dot{x} = \left[\begin{array}{c:ccc} \lambda_1 & 1 & 0 & 0 \\ \hdashline 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{array}\right] x + \left[\begin{array}{c} 0 \\ \hdashline 1 \\ 1 \\ 0 \end{array}\right] u$$

Reduced uncontrollable rows & unobservable column 5

$$y = \left[\begin{array}{c:ccc} 0 & 1 & 1 & 0 \end{array}\right] x$$

$$\dot{x} = \left[\begin{array}{cc:c} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ \hdashline 0 & 0 & \lambda_2 \end{array}\right] x + \left[\begin{array}{c} 1 \\ 1 \\ \hdashline 0 \end{array}\right] u$$

Reduced uncontrollable row 1 & unobservable column 1

$$y = \left[\begin{array}{cc:c} 1 & 1 & 0 \end{array}\right] x$$

$$\dot{x} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Reduced uncontrollable row 3 & unobservable column 3

$$y = [1\ \ 1]\, x$$

$P = [B \vdots AB]$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $AB = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$, $P = \begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix} \rightarrow$ full rank controllable

$Q = \begin{bmatrix} C \\ CA \end{bmatrix}$, $C = [1 \quad 1]$, $CA = [1 \quad 1] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = [\lambda_1 \quad \lambda_2]$, $Q = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \rightarrow$ full rank observable

$\dot{x} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$, $y = [1 \quad 1] x$ #

Exercise 7 (Controllable Canonical Form)

Given: $\frac{\theta(s)}{V(s)} = \frac{K_T}{(Ls+R)(Js^2+bs)}$

Find: write the controllable canonical form of this system

Solution:

$\frac{\theta(s)}{V(s)} = \frac{K_T}{JLs^3+[bL+JR]s^2+bRs}$ $m=0$, $n=3$ $\xrightarrow{\text{represent}}$ $\frac{\theta(s)}{X_1(s)} \cdot \frac{X_1(s)}{V(s)} = \frac{K_T}{1} \cdot \frac{1}{JLs^3+[bL+JR]s^2+bRs}$

$\xrightarrow{\text{Divide by } JL}$ $\frac{\theta(s)}{X_1(s)} \cdot \frac{X_1(s)}{V(s)} = \frac{\frac{K_T}{JL}}{1} \cdot \frac{1}{s^3+\frac{[bL+JR]}{JL}s^2+\frac{bR}{JL}s}$

Solve $\theta(s)$

$\theta(s) = \left[\frac{K_T}{JL}\right] X_1(s)$

$\theta = \left[\frac{K_T}{JL}\right] x_1$

Let $\dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{x}_1 = x_3$

$\dot{x}_3 = \ddot{x}_2 = \dddot{x}_1$

Solve $V(s)$

$V(s) = [JLs^3+(bL+JR)s^2+bRs]\, X_1(s)$

$V(s) = JLs^3X_1(s) + (bL+JR)s^2X_1(s) + bRs\,X_1(s)$

Let $\dot{x}_1 = x_2$, $\dot{x}_2 = \ddot{x}_1$, $\dot{x}_3 = \ddot{x}_2 = \dddot{x}_1$

$V = JL\dot{x}_3 + (bL+JR)\dot{x}_2 + bR\dot{x}_1$

$\dot{x}_3 = \frac{V-(bL+JR)\dot{x}_2 - bR\dot{x}_1}{JL}$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -\frac{bR}{JL} & -\frac{(bL+JR)}{JL}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{JL}\end{bmatrix} V, \quad \theta = \begin{bmatrix}\frac{K_T}{JL} & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + [0]\, V \quad \#$$