24 - 677
Fall 2023
Mid-term Exam
10/24/23
Time: 24 Hours

Name: ____________________

Andrew id: ____________________

Print your initials on each page that has your answers

This exam contains 11 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use your equation sheet and calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- You are allowed to use course slides, homework solution sheets as references. You are allowed to search for the knowledge needed on the internet.
- You must conduct the exam independently. Discussion or seeking help from others, online or in-person, is prohibited.
- All answers need to be derived by hand to get points. You are allowed to use a calculator for basic calculation of scalars. You can use calculate/-computer programs to verify your answers but the effort does not account as credits.
- You can ask questions on campuswire but only to the TAs and instructors.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	10	
6	10	
7	20	
Total:	100	

Do not write in the table to the right.

1. Please state whether each of the following statement is **True** or **False**. Explanation is not required.

   (a) (3 points) The system $y(t) = \sin(t)u(3t)$ is linear.

   (b) (3 points) Assume that $\dot{x}(t) = Ax(t)$ is an asymptotically stable continuous-time LTI system. Assuming A^{-1} exists, the system $\dot{x}(t) = A^{-1}x(t)$ is asymptotically stable.

   (c) (3 points) The following continuous time system is BIBO stable.

   $$\dot{x} = u, \quad y = x$$

   (d) (3 points) The following DT system is controllable if $a \neq 0$.

   $$x[k+1] = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]$$

   (e) (3 points) The system given in (d) is stabilizable when $a = 0$.

   Solution:

   (a) **True**
   Given $y(t) = \sin(t)u(3t)$ Consider $u(t) = \alpha u_1(t) + \beta u_2(t)$, with $y_1(t) = \sin(t)u_1(3t)$ and $y_2(t) = \sin(t)u_2(3t)$

   $$\begin{aligned} y(t) &= \sin(t)u(3t) \\ &= \sin(t)\left(\alpha u_1(3t) + \beta u_2(3t)\right) \\ &= \sin(t)\alpha u_1(3t) + \sin(t)\beta u_2(3t) = \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

   $\implies$ Linear

   (b) **True**
   $\dot{x}(t) = Ax(t)$ is asymptotically stable $\implies$ Eigenvalues of A: $\lambda < 0$
   $\implies$ Eigenvalues of A^{-1}: $\frac{1}{\lambda} < 0$
   $\therefore$ $\dot{x}(t) = A^{-1}x(t)$ asymptotically stable

   (c) **False**
   Using Laplace transform,

   $$\begin{aligned} sY(s) &= U(s) \\ \frac{Y(s)}{U(s)} &= \frac{1}{s} \end{aligned}$$

   The poles of system: $s = 0$
   $\therefore$ Not BIBO stable

   (d) **True**
   The controllability matrix

   $$\begin{aligned} P &= [B \ AB] \\ &= \begin{bmatrix} 1 & 1+a \\ 1 & 1 \end{bmatrix} \end{aligned}$$
   $$rank(P) = 2 \text{ when } a \neq 0$$

   $\implies$ The statement is true.

(e) **True**

This is a discrete time linear time invariant system. Based on M1-5 (slide page 21), the system is stable if "all eigenvalues of A satisfy $|\lambda_i| \leqslant 1$ and all $|\lambda_i| = 1$ are non-defective". When $a = 0$, $|\lambda_i| = 1$ and all $|\lambda_i|=1$ are non-defective. So the system is Lyapunov stable. Based on M1-5 (page 31), "A system is stabilizable$\Leftrightarrow$its uncontrollable modes are Lyapunov stable." Since all eigen values are stable, the system is stabilizable regardless whether any state is uncontrollable or not.

2. Consider a model of fisheries management. State x_1 is the population level of a prey species, x_2 is the population level of a predator species, and x_3 is the effort expended by humans in fishing the predator species. The model is

$$\begin{aligned}\dot{x}_1 &= (r_1 - x_2)x_1\\ \dot{x}_2 &= (r_2 - x_3)x_2\\ \dot{x}_3 &= u\\ y &= x_2\end{aligned}$$

where u is the input, y is the measurement of the predator species, and $r_1 = 10$ and $r_2 = 25$

(a) (5 points) Find the equilibrium point if the prey species population is known to be $\bar{x}_1 = 20$.

(b) (5 points) Linearize the model using the equilibrium point from (a)

(c) (5 points) Find the transfer function of the linearized state model from (b)

Solution:

(a) Prey species population given as $x_1 = \bar{x}_1$

Equilibrium point: $\dot{x} = 0$

$$\begin{aligned}&\dot{x}_1 = 0 \rightarrow \bar{x}_2 = r_1\ (\because x_1 = \bar{x}_1 \neq 0)\\ &\dot{x}_2 = 0 \rightarrow \bar{x}_3 = r_2\ (\because \bar{x}_2 = r_1 \neq 0)\\ &\dot{x}_3 = 0 = \bar{u}\end{aligned}$$

$\therefore$ Equilibrium point given by $\bar{x} = \begin{bmatrix}\bar{x}_1\\ r_1\\ r_2\end{bmatrix} = \begin{bmatrix}20\\ 10\\ 25\end{bmatrix},\ \bar{u} = 0$

(b) Let $\dot{x} = f = \begin{bmatrix}f_1\\ f_2\\ f_3\end{bmatrix}$

The linearized state space model is given by:

$$\begin{aligned}\dot{x} &= \begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3}\\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3}\end{bmatrix} x + \begin{bmatrix}\frac{\partial f_1}{\partial u}\\ \frac{\partial f_2}{\partial u}\\ \frac{\partial f_3}{\partial u}\end{bmatrix} u\\ &= \begin{bmatrix}0 & -\bar{x}_1 & 0\\ 0 & 0 & -r_1\\ 0 & 0 & 0\end{bmatrix} x + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} u\\ &= \begin{bmatrix}0 & -20 & 0\\ 0 & 0 & -10\\ 0 & 0 & 0\end{bmatrix} x + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} u\end{aligned}$$

(c) The transfer function is given by

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s & \bar{x}_1 & 0 \\ 0 & s & r_1 \\ 0 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{-\bar{x}_1}{s^2} & \frac{r_1 \bar{x}_1}{s^3} \\ 0 & \frac{1}{s} & \frac{-r_1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{-r_1}{s^2} = \frac{-10}{s^2} \end{aligned}$$

3. (15 points) For the following dynamical system

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

compute $x(0)$ when $u(t) = 0$ and $x(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution:
Solution to CT-LTI system is given by

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Given input $u(t) = 0$

With $t_0 = 0$, x at $t = 2$ given by

$$x(2) = e^{2A}x(0) \Rightarrow x(0) = e^{-2A}x(2)$$

For $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \lambda = 1, 1$ (repeated eigenvalues)

Using C-H theorem,

$$\begin{matrix} f(\lambda) = e^{\lambda t} = \beta_1 \lambda + \beta_0 \\ \frac{df(\lambda)}{d\lambda} = \frac{dg(\lambda)}{d\lambda} \Rightarrow te^{\lambda t} = \beta_1 \end{matrix} \quad \Rightarrow \quad \begin{matrix} e^t = \beta_0 + \beta_1 \\ te^t = \beta_1 \end{matrix} \quad \Rightarrow \quad \begin{matrix} \beta_0 = (1-t)e^t \\ \beta_1 = te^t \end{matrix}$$

$$f(A) = e^{At} = \beta_1 A + \beta_0 I = te^t \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} + (1-t)e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 3te^t & e^t \end{bmatrix}.$$

Therefore

$$\begin{aligned} x(0) &= e^{-2A}x(2) \\ &= \begin{bmatrix} e^{-2} & 0 \\ -6e^{-2} & e^{-2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-2} \\ -6e^{-2} \end{bmatrix} \end{aligned}$$

4. Given an LTI system with state space representation

$$\dot{x}(t) = \begin{bmatrix} -1 & -\alpha \\ 0 & 1-\alpha \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & \alpha \end{bmatrix} x(t) + u(t)$$

where $\alpha \in \mathbb{R}$.

(a) (5 points) Find the range of α for which the system is exponentially stable.

(b) (10 points) For the infimum (largest lower bound) of the range of α determined in (a), check whether the given system is BIBO stable.

Solution:

(a) All asymptotically stable LTI systems are exponentially stable. Therefore, it is sufficient to check for asymptotic stability.
The given system has eigenvalues

$$\lambda = -1, 1-\alpha$$

Asymptotic stability of an LTI system requires that all eigenvalues satisfy $\lambda < 0$.

$$\Rightarrow 1-\alpha < 0 \Rightarrow \alpha > 1$$

Therefore, given system is exponentially stable for $\alpha > 1$.

(b) For $\alpha = 1$, the state space equation is

$$\dot{x}(t) = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) + u(t)$$

here we have

$$A = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix} D = 1$$

The transfer function of the system is given by

$$G(s) = C(sI-A)^{-1}B + D = \frac{s+3}{s+1}$$

The transfer function has pole -1 which has negative real part $\Rightarrow$ BIBO stable.

5. Consider the following nonlinear system

$$\dot{x}_1 = -\frac{x_2}{1+x_1^2} - 2x_1$$
$$\dot{x}_2 = \frac{x_1}{1+x_1^2}$$

(a) (5 points) Using the candidate Lyapunov function $V(x) = x_1^2 + x_2^2$ and Lyapunov Direct method, first find the equilibrium point and then find the stability of the system at the equilibrium point

(b) (5 points) Linearize the system about the equilibrium point and find the stability of the linearized system using Lyapunov indirect method

Solution:

(a) Equilibrium point given by $\dot{x} = \mathbf{0}$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \mathbf{0} \Rightarrow \begin{matrix} -\frac{x_2}{1+x_1^2} - 2x_1 = 0 \\ \frac{x_1}{1+x_1^2} = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}$$

Consider Lyapunov function

$$\begin{aligned} V &= x_1^2 + x_2^2 > 0 \;\forall x \neq 0 \\ \dot{V} &= 2x_1\dot{x_1} + 2x_2\dot{x_2} \\ &= 2x_1(-\frac{x_2}{1+x_1^2} - 2x_1) + x_2(\frac{x_1}{1+x_1^2}) \\ &= -4x_1^2 < 0 \;\forall x \neq 0 \end{aligned}$$

$\therefore$ globally asymptotically stable

(b) Linearized system given by

$$A = \frac{\partial f}{\partial x}|_{x=\bar{x}} = \begin{bmatrix} \frac{2x_1x_2}{(1+x_1^2)^2} - 2 & \frac{-1}{1+x_1^2} \\ \frac{1-x_1^2}{(1+x_1^2)^2} & 0 \end{bmatrix}_{x_1=0, x_2=0} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} \Delta(\lambda) &= \det \begin{bmatrix} -\lambda - 2 & -1 \\ 1 & -\lambda \end{bmatrix} \\ &= \lambda^2 + 2\lambda + 1 \end{aligned}$$

$\Rightarrow$ Eigenvalues $\lambda = -1, -1 \Rightarrow re(\lambda) < 0$
$\therefore$ locally asymptotically stable

6. (10 points) Find the minimal realization for

$$G(s) = \begin{bmatrix} \frac{s}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix}.$$

Solution:

Decompose G into a strictly proper TF matrix and a constant matrix:

$$G(s) = G_{sp}(s) + D = \begin{bmatrix} \frac{-1}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Least common denominator for $G_{sp}(s)$ is $\Delta(s) = s^2 + s$. So $\alpha_1 = 1,\ \alpha_2 = 0$

$$G_{sp}(s) = \frac{1}{\Delta(s)}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Therefore the realization is

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

We could double check the controllability and observability.

So we compute the controllability

$$P = \begin{bmatrix} B, AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow \det P = 1 \neq 0$$

therefore the realization is controllable.

For observability,

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \text{rank} Q = 2$$

therefore the realization is observable.

To conclude, the realization is minimal.

7. Suppose you are invited as a control engineering consultant to investigate a critical safety issue for an airplane company. You are provided with an approximate linear model of the lateral dynamics of the aircraft which has the state and control vectors

$$x = \begin{bmatrix} p & r & \beta & \phi \end{bmatrix}^T \quad \text{and} \quad u = \begin{bmatrix} \delta_a & \delta_r \end{bmatrix}^T$$

where p and r are incremental roll and yaw rates, β is an incremental sideslip angle, and ϕ is an incremental roll angle. The control inputs are the incremental changes in the aileron angle δ_a and in the rudder angle δ_r, respectively. In a consistent set of units, the linearized model is given as $\dot{x} = Ax + Bu$ with

$$\mathbf{A} = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Answer the following questions with derivations. A simple yes or no without explanation will get 0 credit.

(a) (5 points) Is the linearized aircraft model asymptotically stable? Is the linearized aircraft model stable i.s.L.?

(b) (5 points) Is the aircraft controllable with just δ_r? Is the aircraft controllable with both δ_r and δ_a?

(c) (5 points) Suppose a malfunction prevents manipulation of the rudder angle δ_r, is it possible to control the aircraft using only the aileron angle δ_a?

(d) (5 points) If you only have budget to measure one state, which one to measure (choose one from $\{p, r, \beta, \phi\}$) so that the whole system is observable?

Solution:

(a) Characteristic polynomial

$$\begin{aligned}
\Delta(\lambda) &= |A - \lambda I| \\
&= \begin{vmatrix} -10-\lambda & 0 & -1 & 0 \\ 0 & -1-\lambda & 1 & 0 \\ 0 & -1 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} \\
&= (-10-\lambda) \begin{vmatrix} -1-\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & -1-\lambda & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -\lambda \end{vmatrix} \\
&= (-10-\lambda)\left((-1-\lambda)\lambda^2 - \lambda\right) - 1(0) = \lambda(\lambda+10)(\lambda^2+\lambda+1)
\end{aligned}$$

$\Rightarrow$ Eigenvalues $\lambda = 0, -10, -\frac{1}{2} - i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2} \rightarrow Re(\lambda) \leq 0$

The linearized model is not asymptotically stable. Note that the linearized model is stable i.s.L.

(b) When δ_r is the only input, just the second column B_{δ_r} of the B matrix must be used in checking for controllability. The controllability matrix is determined as

$$
\begin{aligned}
P_{\delta_r} &= [B_{\delta_r}\ AB_{\delta_r}\ A^2B_{\delta_r}\ A^3B_{\delta_r}] \\
&= \begin{bmatrix} 0 & 0 & -1 & 11 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
rank(P_{\delta_r}) &= 4
\end{aligned}
$$

Therefore, the aircraft is controllable with just δ_r.
The aircraft has been found to be controllable using just δ_r, with $rank(P_{\delta_r}) = 4$.
$\Rightarrow$ The 4×8 controllability matrix $P = [B\ AB\ A^2B\ A^3B]$ with both the inputs considered shall also have $rank(P) = 4$. Therefore, the system is controllable with both inputs operable.

(c) When δ_a is the only input, the first column B_{δ_a} of B is used. The controllability matrix is

$$
\begin{aligned}
P_{\delta_a} &= [B_{\delta_a}\ AB_{\delta_a}\ A^2B_{\delta_a}\ A^3B_{\delta_a}] \\
&= \begin{bmatrix} 10 & -100 & 1000 & -10000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & -100 & 1000 \end{bmatrix} \\
rank(P_{\delta_a}) &= 2 < n = 4
\end{aligned}
$$

Therefore, the aircraft is uncontrollable with just δ_a

(d) Because we can only measure one state, C is a row vector with only one element is 1 and all others are 0s. To make the system observable, we need rank(Q)=4, where

$$
Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}
$$

Because all elements in the fourth column of A are zeros, so any CA^{k-1} times A, the last element of CA^k will always be 0. So except for the first element, every element of the fourth column of Q will be zero. To make the fourth column of Q non-zero, we need to make the fourth element of C none zero, in other words, we need to choose to measure ϕ, so that $C = [0\ 0\ 0\ 1]$. The observability matrix Q is

$$
\begin{aligned}
Q &= \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -10 & 0 & -1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix} \\
rank(Q) &= 4
\end{aligned}
$$

The aircraft is observable with just ϕ measured.