

## Exercise I (Asymptotic stability and Lyapunov stability)

Given:

$$a) \quad x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$

$$b) \quad \dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

Find: determine Lyapunov stable, asymptotic stable

Solution:

(a) Condition:  $\bar{x}=0$  for  $x(k+1)=Ax(k)$  is stable

all eigenvalues of A satisfy  $|\lambda_i| \leq 1$  and all  $|\lambda_i|=1$  are non-defective

$\forall \lambda_i, r_i < 1 \Rightarrow$  Asymptotic Stable ;  $\exists \lambda_i, r_i=1 \neq m=0 \Rightarrow$  stable i.s.L

$$\det(A-\lambda I) = \det \begin{pmatrix} 1-\lambda & 0 \\ -0.5 & 0.5-\lambda \end{pmatrix} = (1-\lambda)(0.5-\lambda) \rightarrow \lambda_1=1, \lambda_2=0.5$$

Check  $\lambda_1=1$  is defective or not  $\lambda_1=1, v_1=\begin{bmatrix} -1 \\ 1 \end{bmatrix}; \lambda_2=0.5, v_2=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$  not defective  $\rightarrow$  stable

$\because \lambda_2=0.5 < 1$  is asymptotic stable, but  $\lambda_1=1$  (not defective) is only stable i.s.L

$\therefore x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$  system overall is stable i.s.L # (a)

(b) Condition:  $\bar{x}$  for  $\dot{x}=Ax$  is stable  $\Leftrightarrow$

all eigenvalues of A have non-positive real parts, and those with zero real parts are n.d.

$\forall \lambda_i, \operatorname{Re} \lambda_i < 0 \Rightarrow$  Asymptotic stable ;  $\exists \lambda_i, \operatorname{Re} \lambda_i = 0, m=0 \Rightarrow$  stable i.s.L

$$\det(A-\lambda I) = \det \begin{pmatrix} -7-\lambda & -2 & 6 \\ 2 & -3-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{pmatrix} \rightarrow \lambda_1=-1, \lambda_2=-3, \lambda_3=-5 \rightarrow \text{stable}$$

$\det \begin{bmatrix} abc \\ def \\ ghi \end{bmatrix} = a \begin{vmatrix} ef \\ hi \end{vmatrix} - b \begin{vmatrix} df \\ gi \end{vmatrix} + c \begin{vmatrix} de \\ fg \end{vmatrix}$

$\because$  all eigVs ( $\lambda_1=-1, \lambda_2=-3, \lambda_3=-5$ ) are negative poles

$\therefore \dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$  is asymptotic stable # (b)

Exercise 2. (Stabilizability)

Given:  $\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}u$ ,  $y = [1 \ 1 \ 1]x$

Find: (a) decompose the state equation to a controllable form

(b) Is the reduced state observable, stabilizable, detectable?

Solution:

$$(a) P = [B \ AB \ A^2B], A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, AB = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{\text{ref}} P = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{rank}(P)=2$$

$\text{rank}(P)=2 < n=3 \rightarrow \text{not controllable}$

$$\det(\lambda I - A) = 0 \rightarrow \det \begin{pmatrix} \lambda - 0 & 1 & -1 \\ -1 & \lambda + 2 & -1 \\ 0 & -1 & \lambda + 1 \end{pmatrix} = 0 \rightarrow \lambda(\lambda+1)(\lambda+2) = 0$$

$$\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 0$$

$$V_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}, \hat{A} = M^{-1}AM = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\hat{B} = M^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \hat{C} = CM = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u, \quad y = [3 \ 3 \ 1] \hat{x} + [0] u$$

$$\dot{\hat{x}}_c = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \hat{x}_c + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u, \quad y = [C_c \ C_{\bar{c}}] \hat{x}_c + D u$$

$$\dot{\hat{x}}_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u, \quad y [3 \ 3] \hat{x}_c + [0] u$$

Check

$$\hat{P} = [\hat{B} \ \hat{A}\hat{B}] , \quad \hat{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad \hat{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} , \quad \hat{A}\hat{B} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \rightarrow \text{rank } \hat{P} = 2 = n \rightarrow \text{controllable}$$

$$\because \text{rank } \hat{P} = n = 2$$

$$\therefore \hat{x}_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u, \quad y [3 \ 3] \hat{x}_c + [0] u \text{ is a reduced controllable form} \#(a)$$

(b)

Observable

$$\hat{Q} = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \end{bmatrix}, \quad \hat{C} = [3 \ 3], \quad \hat{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \hat{C}\hat{A} = [3 \ 3] \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = [0 \ 0]$$

$$\hat{Q} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{ref}} \hat{Q} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \text{rank } \hat{Q} = 1 < n \rightarrow \text{not observable}$$

$$\because \text{rank } \hat{Q} = 1 < n \therefore \hat{x}_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u, \quad y [3 \ 3] \hat{x}_c + [0] u \text{ is not observable} \#(b)$$

Stabilizable

Condition: A system is stabilizable if all unstable states are Lyapunov stable

$\because$  Since the system is already in reduced controllable form

$$\therefore \hat{x}_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u, \quad y [3 \ 3] \hat{x}_c + [0] u \text{ is stabilizable} \#(b)$$

Detectable

Condition: A system is detectable if the unobservable states are Lyapunov stable

$$A_C = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda + 1 \end{vmatrix} \rightarrow \lambda_1 = -1 \rightarrow \text{asymptotically stable}$$
$$\lambda_2 = 0 \rightarrow \text{stable i.s.l}$$

$\therefore$  Since  $\lambda_1 = -1$  is asymptotically stable  $\nless \lambda_2 = 0$  is stable i.s.l

$$\therefore \hat{x}_C = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}_C + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u, \quad y [3 \ 3] \hat{x}_C + [0] u \text{ is detectable } \#(b)$$

### Exercise 3 (Stability)

Given:  $m\ddot{x} = -u_1 \sin \theta + Eu_2 \cos \theta$ ,  $m\ddot{y} = u_1 \cos \theta + Eu_2 \sin \theta - mg$ ,  $J\ddot{\theta} = u_2$

$$\dot{x}(t), \dot{y}(t), \dot{\theta}(t) = 0, \quad \ddot{u}_1(t) = mg, \quad \ddot{u}_2(t) = 0$$

$$(\text{States}) \quad \vec{z} = [\theta, \dot{x}, \dot{y}, \dot{\theta}]^T, \quad \vec{u} = [u_1, u_2]^T$$

Find: determine the stability of the linearized model around the equilibrium solution

Solution:

$$\vec{z} = \begin{bmatrix} \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \dot{\vec{z}} = \begin{bmatrix} \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix}, \quad \ddot{\vec{z}} = \begin{bmatrix} \ddot{\theta} \\ \ddot{\dot{x}} \\ \ddot{\dot{y}} \\ \ddot{\dot{\theta}} \end{bmatrix}$$

$\dot{x}(t) = \dot{y}(t) = \dot{\theta}(t) = 0, \quad \ddot{u}_1(t) = mg, \quad \ddot{u}_2(t) = 0$

$m\ddot{x} = -u_1 \sin \theta + Eu_2 \cos \theta \rightarrow \ddot{x} = \frac{Eu_2}{m}$

$m\ddot{y} = u_1 \cos \theta + Eu_2 \sin \theta - mg \rightarrow \ddot{y} = \frac{u_1}{m} - g$

$J\ddot{\theta} = u_2 \rightarrow \ddot{\theta} = \frac{u_2}{J}$

$$\dot{\vec{z}} = \begin{bmatrix} \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{u_1}{m} \sin \theta_1 + \frac{Eu_2}{m} \cos \theta_1 \\ \frac{u_1}{m} \cos \theta_1 + \frac{Eu_2}{m} \sin \theta_1 - g \\ \frac{u_2}{J} \end{bmatrix}, \quad f(\vec{z}, \vec{u}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial \vec{z}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial f}{\partial \vec{u}} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{E}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix}$$

$$\dot{\vec{z}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\vec{z}} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{E}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix} \dot{\vec{u}}$$

$\therefore \text{Since } \lambda=0 \notin m>0$

$\therefore \text{This is an unstable system}$

Exercise 4 (Lyapunov's direct method)

Given:  $\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix}x$ ,  $V = x_1^2 + x_2^2$

Find: use Lyapunov's direct method to determine the range of  $a$  for which system is asymptotically stable.

Solution:

Conditions: The origin of  $\dot{x} = f(x)$  is asymptotically stable if

(1)  $V(x)=0$  if and only if  $x=0$

(2)  $V(x)>0$  if and only if  $x\neq 0$

(3)  $\dot{V}(x) = \frac{d}{dt}V(x) = \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(x) = \nabla V \cdot f(x) < 0$  for all values of  $x \neq 0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \dot{x}_1 = ax_1, \quad \dot{x}_2 = x_1 - x_2, \quad V = x_1^2 + x_2^2$$

Check (1) & (2)  $\rightarrow$  clearly  $V(x)=0$  if and only if  $x=0$  &  $V(x)>0$  if and only if  $x\neq 0$

Check (3)  $\rightarrow \dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \rightarrow$  plug in given  $\dot{x}_1$  &  $\dot{x}_2 \rightarrow \dot{V}(x) = 2x_1(ax_1) + 2x_2(x_1 - x_2)$

$$\dot{V}(x) = 2ax_1^2 + 2x_1x_2 - 2x_2^2 < 0$$

Analyze  $2ax_1^2 + 2x_1x_2 - 2x_2^2 < 0 \rightarrow ax_1^2 + x_1x_2 - x_2^2 < 0$

$$\rightarrow ax_1^2 < x_2^2 - x_1x_2 \rightarrow a < \frac{x_2^2 - x_1x_2}{x_1^2}$$

$\therefore$  the reasoning in Condition (1), (2), (3)

$\therefore a < \frac{x_2^2 - x_1x_2}{x_1^2}$  is the range when system is asymptotically stable #

### Exercise 5 (Stability of Non-Linear Systems)

Given:  $\dot{x}_1 = x_2 - x_1 x_2^2$ ,  $\dot{x}_2 = -x_1^3$

Find: (a) is the approximated linearized system stable? Based on Lyapunov's Indirect Method, is the system stable?

(b) based on Lyapunov's Direct Method?  $V(x_1, x_2) = x_1^4 + 2x_2^2$

(c) plot the Phase Portrait plot of the original system & linearized system in (a)

(d) generate 3D plot showing the variation of  $V$  with respect to  $x_1$  &  $x_2$

Solution:

$$(a) \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 x_2^2 \\ -x_1^3 \end{bmatrix}$$

(1) Find equilibria

$$\begin{cases} x_2 - x_1 x_2^2 = 0 \rightarrow x_2 = 0 \\ -x_1^3 = 0 \rightarrow x_1 = 0 \end{cases}$$

(2) Linearization

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 - x_2^2 & 1 - 2x_1 x_2 \\ -3x_1^2 & 0 \end{bmatrix}$$

$$\text{Plug in } x_1 = x_2 = 0 \quad (0,0): \dot{\delta_x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \delta_x$$

The equilibria points are at origin  $(0,0)$ , it's hard to get a firm answer on system stability # (a)

$$(b) V(x_1, x_2) = x_1^4 + 2x_2^2$$

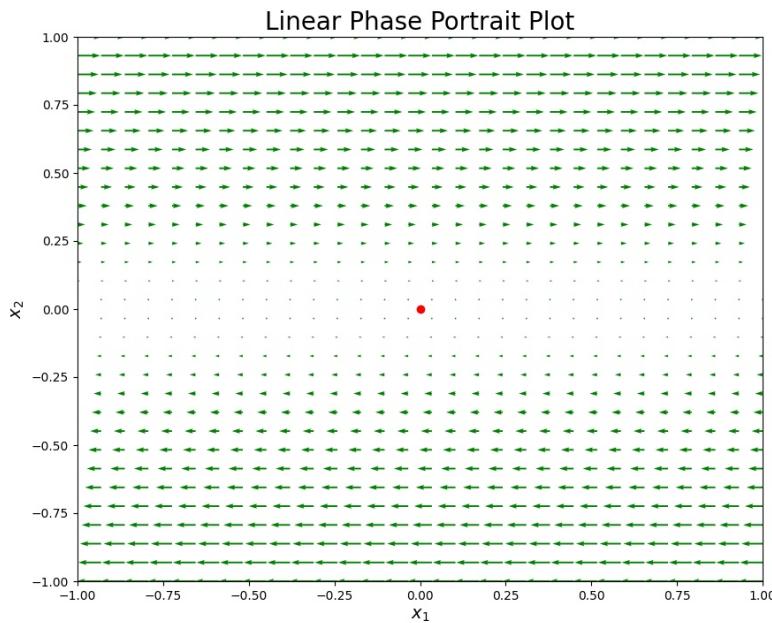
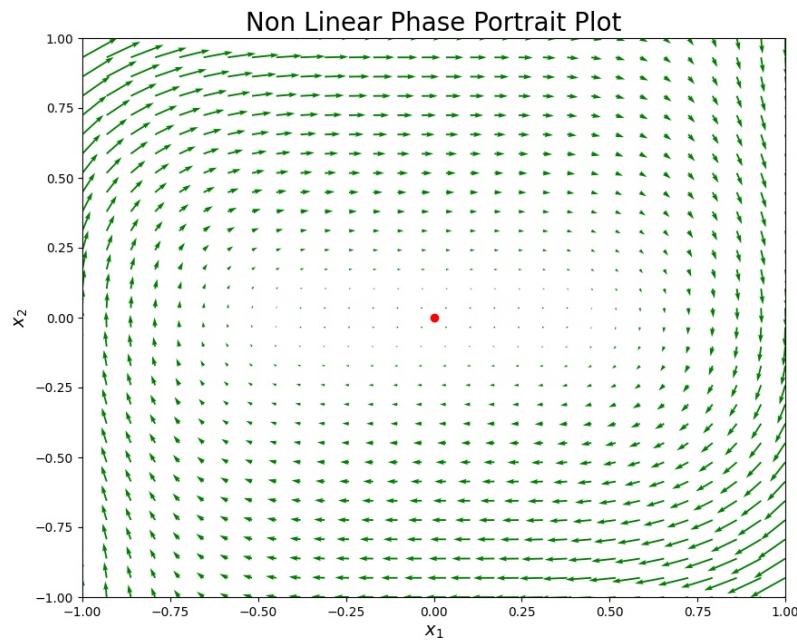
$$\dot{V}(x_1, x_2) = 4x_1^3 \dot{x}_1 + 4x_2 \dot{x}_2 \text{ plug } \rightarrow (\dot{x}_1 = x_2 - x_1 x_2^2, \dot{x}_2 = -x_1^3)$$

$$\dot{V}(x_1, x_2) = 4x_1^3(x_2 - x_1 x_2^2) + 4x_2(-x_1^3)$$

$$\dot{V}(x_1, x_2) = \cancel{4x_1^3} x_2 - \cancel{4x_1^4} x_2^2 - \cancel{4x_1^3} x_2 \rightarrow \dot{V}(x_1, x_2) = -4x_1^4 x_2^2 < 0$$

$\therefore \dot{V}(x_1, x_2) = -4x_1^4 x_2^2 < 0 \quad \therefore \text{the linearized system is stable. } # (b)$

(c)



Figures above is the phase portrait plot of the original system & linearized system in (a)

(d)

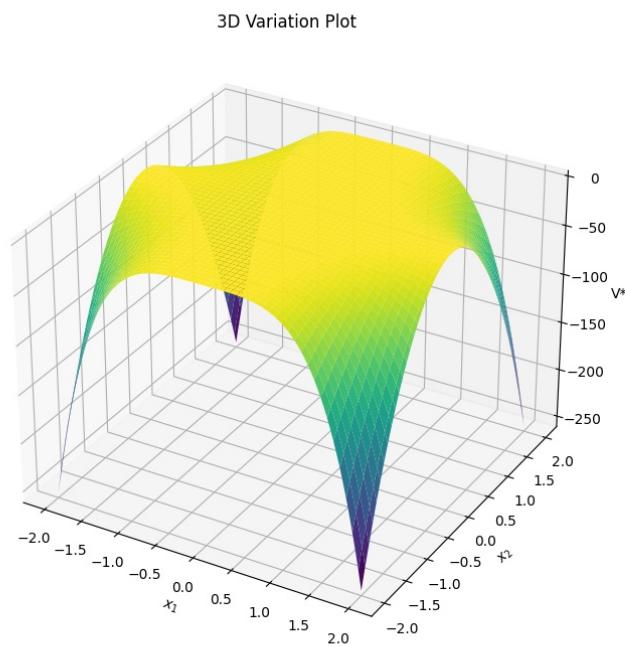


Figure above is a 3D plot showing the variation of  $v$  with respect to  $x_1$  &  $x_2$

### Exercise 6 (BIBO Stability)

Given:

$$(a) \quad x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k), \quad y(k) = [s \ s] x(k)$$

$$(b) \quad \dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u, \quad y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

Find: determine whether it is BIBO stable.

Solution:

(a) Condition: let  $G_D(z) = C(zI - A)^{-1}B + D$

DT LTI system is BIBO stable  $\Leftrightarrow$  every pole of every  $G_{Dij}$  is inside the unit circle

$$A = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = [s \ s], \quad D = 0$$

$$G_D(z) = [s \ s] \begin{bmatrix} z-1 & 0 \\ 0.5 & z-0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow G_D(z) = [s \ s] \begin{bmatrix} \frac{z-0.5}{z^2-1.5z+0.5} & 0 \\ 0.5 & \frac{z-1}{z^2-1.5z+0.5} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow G_D(z) = \left[ \frac{s(z-0.5)}{z^2-1.5z+0.5} - \frac{2s}{z^2-1.5z+0.5} \quad 0 + \frac{s(z-1)}{z^2-1.5z+0.5} \right] \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow G_p(z) = \frac{0}{z^2-1.5z+0.5} = 0$$

$\therefore$  The system has no poles with zero as final state

$$\therefore x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k), \quad y(k) = [s \ s] x(k) \text{ is BIBO stable}$$

(b) condition: let  $G_c(s) = C(sI - A)^{-1}B + D$

A DT LTI system is BIBO stable  $\Leftrightarrow$  every pole of every  $G_{cij}$  have negative real parts

$$A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$D = 0$$

$$G_c(s) = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s+7 & 2 & -6 \\ -2 & s+3 & 2 \\ 2 & 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\rightarrow G_c(s) = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{s^2 + 2s - 7}{(s+1)(s+3)(s+5)} & \frac{-2}{(s+1)(s+3)} & \frac{s+5}{(s+1)(s+3)} - \frac{1}{s+5} \\ \frac{2}{(s+3)(s+5)} & \frac{1}{s+3} & \frac{1}{s+5} - \frac{1}{s+3} \\ \frac{-2}{(s+1)(s+3)} & \frac{-2}{(s+1)(s+3)} & \frac{s+5}{(s+1)(s+3)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

From the above matrix we can observe that the poles will be  $\lambda_1 = -1$ ,  $\lambda_2 = -3$ ,  $\lambda_3 = -5$

$\therefore$  All the poles are  $< 0$  ( $\lambda_1 = -1$ ,  $\lambda_2 = -3$ ,  $\lambda_3 = -5$ )

$\therefore x = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u, \quad y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x \text{ is BIBO Stable # (b)}$

### Exercise 7 (BIBO Stability)

Given:  $V_C \frac{dT_C}{dt} = f_C(T_{Ci} - T_C) + B(T_H - T_C)$ ,  $V_H \frac{dT_H}{dt} = f_H(T_{Hi} - T_H) + B(T_C - T_H)$

inputs  $u_1 = T_{Ci}$ ,  $u_2 = T_{Hi}$ , outputs  $y_1 = T_C$ ,  $y_2 = T_H$

assume  $f_C = f_H = 0.1 \left[ \frac{m^3}{min} \right]$ ,  $B = 0.2 \left[ \frac{m^3}{min} \right]$ ,  $V_H = V_C = 1 \text{ [m}^3\text{]}$

Find: (1) write the state space & output equations for the system

(2) in the absence of input, determine  $y_1(t)$  &  $y_2(t)$

(3) is the system BIBO stable? show why or why not

Solution:

(1) → plug given #s into  $V_C \frac{dT_C}{dt}$  &  $V_H \frac{dT_H}{dt}$

$$V_C \frac{dT_C}{dt} = f_C(T_{Ci} - T_C) + B(T_H - T_C) \rightarrow \frac{dT_C}{dt} = 0.1(u_1 - y_1) + 0.2(y_2 - y_1)$$

$$V_H \frac{dT_H}{dt} = f_H(T_{Hi} - T_H) + B(T_C - T_H) \rightarrow \frac{dT_H}{dt} = 0.1(u_2 - y_2) + 0.2(y_1 - y_2)$$

let  $\dot{x}_1 = \frac{dT_C}{dt}$ ,  $\dot{x}_2 = \frac{dT_H}{dt}$ ,  $y_1 = x_1$ ,  $y_2 = x_2$

Rewrite both Eqs. →  $\dot{x}_1 = 0.1(u_1 - x_1) + 0.2(x_2 - x_1)$ ,  $\dot{x}_2 = 0.1(u_2 - x_2) + 0.2(x_1 - x_2)$

$$\dot{x}_1 = 0.1u_1 - 0.1x_1 + 0.2x_2 - 0.2x_1 \rightarrow \dot{x}_1 = -0.3x_1 + 0.2x_2 + 0.1u_1$$

$$\dot{x}_2 = 0.1u_2 - 0.1x_2 + 0.2x_1 - 0.2x_2 \rightarrow \dot{x}_2 = 0.2x_1 - 0.3x_2 + 0.1u_2$$

Rewrite both Eqs. in S.S.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \#(1)$$

(2)  $y(t) = (e^{A(t-t_0)} x(t_0) + C \int_{t_0}^t e^{\lambda(t-\tau)} Bu(\tau) d\tau + Du(t))$ , disregard input  $[B]$

$$y(t) = e^{At} x(0), y_1 = x_1, y_2 = x_2$$

$$A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}, \lambda_1 = -0.5, \lambda_2 = -0.1, \text{ Apply C.H. (n=2)} \rightarrow f(\lambda) = B_1 \lambda + B_0$$

$$(\lambda_1 = -0.5) \quad f(\lambda_1) = e^{\lambda_1 t} = B_1 \lambda_1 + B_0 \rightarrow e^{-0.5t} = -0.5 B_1 + B_0 \quad (1)$$

$$(\lambda_2 = -0.1) \quad f(\lambda_2) = e^{\lambda_2 t} = B_1 \lambda_2 + B_0 \rightarrow e^{-0.1t} = -0.1 B_1 + B_0 \quad (2)$$

$$\text{Eqn (2) - (1)} \quad \begin{aligned} e^{-0.1t} &= -0.1\beta_1 + \beta_0 \\ \rightarrow e^{-0.5t} &= -0.5\beta_1 + \beta_0 \\ \frac{e^{-0.1t}}{e^{-0.5t}} &= 0.4\beta_1 \end{aligned} \quad \rightarrow \beta_1 = \frac{e^{-0.1t} - e^{-0.5t}}{0.4}$$

$$\text{Plug } \beta_1 \text{ into (2)} \quad \beta_0 = e^{-0.1t} + 0.1\beta_1 \quad \rightarrow \beta_0 = e^{-0.1t} + 0.25(e^{-0.1t} - e^{-0.5t})$$

$$P(A) = e^{At} = \beta_1 A + \beta_0 I = \frac{e^{-0.1t} - e^{-0.5t}}{0.4} \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} + \left[ e^{-0.1t} + 0.25(e^{-0.1t} - e^{-0.5t}) \right] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} -0.75(e^{-0.1t} - e^{-0.5t}) + e^{-0.1t} + 0.25(e^{-0.1t} - e^{-0.5t}) & 0.5(e^{-0.1t} - e^{-0.5t}) \\ 0.5(e^{-0.1t} - e^{-0.5t}) & -0.75(e^{-0.1t} - e^{-0.5t}) + e^{-0.1t} + 0.25(e^{-0.1t} - e^{-0.5t}) \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 0.5e^{-0.1t} + 0.5e^{-0.5t} & 0.5(e^{-0.1t} - e^{-0.5t}) \\ 0.5(e^{-0.1t} - e^{-0.5t}) & 0.5e^{-0.1t} + 0.5e^{-0.5t} \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} \frac{e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} & -\frac{e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} \\ -\frac{e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} & \frac{e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} \end{bmatrix}$$

$$y(t) = e^{At}x(0) \rightarrow \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} \\ -\frac{e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} \end{bmatrix} = \begin{bmatrix} \frac{x_1(0)}{2} + \frac{x_2(0)}{2} \\ \frac{-x_1(0)}{2} + \frac{x_2(0)}{2} \end{bmatrix}$$

$$y_1(t) = \frac{x_1(0)[e^{-\frac{t}{2}} + e^{-\frac{t}{10}}] + x_2(0)[e^{-\frac{t}{10}} - e^{-\frac{t}{2}}]}{2}, \quad y_2(t) = \frac{x_1(0)[e^{-\frac{t}{2}} - e^{-\frac{t}{10}}] + x_2(0)[e^{-\frac{t}{2}} + e^{-\frac{t}{10}}]}{2} \quad \#(2)$$

(3)  $\because \lambda_1 = -0.5, \lambda_2 = -0.1$  for the S.S system, two negative poles

$\therefore$  The system is BIBO Stable  $\#(3)$