Exercise 1 (Cayley-Hamilton Theorem)

Given:

$$A=\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}$$

Find: A^{10} & e^{At}

Solution:

(A^{10}) $A=\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}$ A^{10} $n=3$

$\det(A-\lambda I) = -\lambda^3+2\lambda^2-\lambda=0$ $\lambda_1=0$, $\lambda_2=1$

$f(\lambda)=\lambda^{10}=B_2\lambda^2+B_1\lambda+B_0$, $\frac{f(\lambda)}{d\lambda}=10\lambda^9=2B_2\lambda+B_1$

$\lambda_1=0 \rightarrow 0=B_0$

$\lambda_2=1 \rightarrow 1^{10}=B_2+B_1$

$B_1=1^{10}-B_2 \rightarrow B_1=-8$

$\lambda_2=1 \rightarrow 10\cdot 1^9=2B_2+B_1$

$\rightarrow 1^{10}=B_2+B_1$

$9=B_2$

$A^k=B_2A^2+B_1A+\cancel{B_0}^{\,0}I \rightarrow A^{10}=B_2A^2+B_1A$

$$A^2=\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{pmatrix} \rightarrow A^2=\begin{pmatrix}1+0+0 & 1+0+0 & 0+1+0\\ 0 & 0 & 0+0+1\\ 0 & 0 & 1\end{pmatrix} \rightarrow A^2=\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}$$

$$A=\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}$$

$$A^{10}=9\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}-8\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix} \rightarrow A^{10}=\begin{bmatrix}1 & 1 & 9\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix} \# A^{10}$$

(e^{At})

$$\Delta(\lambda) = |A-\lambda I| = \left| \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \right| = (1-\lambda)(1-\lambda)(0-\lambda) = 0 \qquad \lambda_1 = 0,\ \lambda_2 = \lambda_3 = 1$$

$n = 3$

$f(\lambda) = g(B,\lambda) = B_{n-1}\lambda^{n-1} + \dots + B_1\lambda + B_0$

For $(n=3) \rightarrow f(\lambda) = g(B,\lambda) \rightarrow e^{\lambda t} = B_2\lambda^2 + B_1\lambda + B_0 \rightarrow e^{At} = B_2A^2 + B_1A + B_0 I$

For $(\lambda_1 = 0)$

$f(\lambda) = e^{\lambda t} = \cancel{B_2\lambda^2} + \cancel{B_1\lambda} + B_0 \rightarrow B_0 = 1$

For $(\lambda_2 = \lambda_3 = 1)$

$$\left.\begin{aligned} f(\lambda) &= e^{\lambda t} = B_2\lambda^2 + B_1\lambda + B_0 \\ \frac{df(\lambda)}{d\lambda} &= \frac{dg(\lambda)}{d\lambda} \rightarrow te^{\lambda t} = 2B_2\lambda + B_1 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} e^t &= B_2 + B_1 + B_0 \\ te^t &= 2B_2 + B_1 \end{aligned}\right.$$

$$\begin{aligned} te^t &= 2B_2 + B_1 \\ -)\ e^t &= B_2 + B_1 + 1 \\ \hline te^t - e^t &= B_2 - 1 \end{aligned}$$

$B_2 = te^t - e^t + 1$

$B_1 = e^t - B_2 - 1 \rightarrow B_1 = e^t - te^t + e^t - 1 - 1 \rightarrow B_1 = 2e^t - te^t - 2$

Combining all λs and plug into $f(A)$ $\qquad A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$f(A) = e^{At} = B_2A^2 + B_1A + B_0I,$

$$f(A) = e^{At} = (te^t - e^t + 1)\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + (2e^t - te^t - 2)\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} te^t - e^t + 1 + 2e^t - te^t - 2 + 1 & te^t - e^t + 1 + 2e^t - te^t - 2 + 0 & te^t - e^t + 1 + 0 + 0 \\ 0 + 0 + 0 & 0 + 0 + 1 & te^t - e^t + 1 + 2e^t - te^t - 2 + 0 \\ 0 + 0 + 0 & 0 + 0 + 0 & te^t - e^t + 1 + 2e^t - te^t - 2 + 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix} \# \ e^{At}$$

Exercise 2. (Linear dynamics solution)

Given: $\frac{dx_1}{dt} = -\alpha x_1 + u$ $\quad \alpha = 0.1$, $\beta = 0.2$, $u = 1$

$x_1(0) = 2$, $x_2(0) = 1$

$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2$

Find: water level in both tanks after 5s (using C-H theorem)

Solution:

$\dot{x}_1 = -0.1\,x_1 + 1$ $\qquad \dot{x}_2 = 0.1x_1 - 0.2x_2$

$\dot{x} = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ $\qquad n=2 \qquad \lambda_1 = -0.1 \quad \lambda_2 = -0.2$

Apply C-H

$f(\lambda) = g(B,\lambda) = B_1\lambda + B_0 \rightarrow f(\lambda) = e^{\lambda t} = B_1\lambda + B_0$

$(\lambda_1 = -0.1)\ f(-0.1) = e^{-0.1t} = -0.1B_1 + B_0$ $\qquad e^{-0.1t} = -0.1B_1 + B_0$

$(\lambda = -0.2)\ f(-0.2) = e^{-0.2t} = -0.2B_1 + B_0$ $\qquad -)\ \underline{e^{-0.2t} = -0.2B_1 + B_0}$

$B_1 = 10e^{-0.1t} - 10e^{-0.2t}$ $\qquad e^{-0.1t} - e^{-0.2t} = 0.1B_1$

$B_0 = e^{-0.2t} + 0.2B_1 \rightarrow B_0 = e^{-0.2t} + 2e^{-0.1t} - 2e^{-0.2t} \rightarrow B_0 = 2e^{-0.1t} - e^{-0.2t}$

$e^{At} = B_1A + B_0I = (10e^{-0.1t} - 10e^{-0.2t})\begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix} + (2e^{-0.1t} - e^{-0.2t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$e^{At} = \begin{bmatrix} -e^{-0.1t} + \cancel{e^{-0.2t}} + 2e^{-0.1t} - \cancel{e^{-0.2t}} & 0 \\ e^{-0.1t} - e^{-0.2t} & -\cancel{2e^{-0.1t}} + 2e^{-0.2t} + \cancel{2e^{-0.1t}} - e^{-0.2t} \end{bmatrix}$

$e^{At} = \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix}$

$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$

$x(t) = e^{At}\ \cancel{e^{At_0}}^{\,1}\ x(t_0) + e^{At}\int_{t_0}^{t} e^{-A\tau}Bu(\tau)\,d\tau$

$\int_{t_0}^{t} e^{-A\tau}Bu(\tau)\,d\tau \rightarrow \int_{t_0}^{t} \begin{bmatrix} e^{0.1\tau} & 0 \\ e^{0.1\tau} - e^{0.2\tau} & e^{0.2\tau} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau$

$$\rightarrow \int_{t_0}^{t} \begin{bmatrix} e^{0.1t} \\ e^{0.1t} - e^{0.2t} \end{bmatrix} d\tau \xrightarrow{\text{Calculator}} \begin{bmatrix} 6.48721 \\ -2.10420 \end{bmatrix}$$

$$e^{5A} = \begin{bmatrix} 0.60653 & 0 \\ 0.23865 & 0.36788 \end{bmatrix}, \quad \begin{matrix} x_1(0) = 2 \\ x_2(0) = 1 \end{matrix}, \quad \int_{t_0}^{t} e^{-A\tau} Bu(\tau)\, d\tau = \begin{bmatrix} 6.48721 \\ -2.10420 \end{bmatrix}$$

$$x(5) = \begin{bmatrix} 0.60653 & 0 \\ 0.23865 & 0.36788 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.60653 & 0 \\ 0.23865 & 0.36788 \end{bmatrix} \begin{bmatrix} 6.48721 \\ -2.10420 \end{bmatrix}$$

$$x(5) = \begin{bmatrix} 5.14775 \\ 1.61926 \end{bmatrix} \qquad \begin{matrix} x_1(5) = 5.14775 \\ x_2(5) = 1.61926 \ \# \end{matrix}$$

Exercise 3 (Jordan form, decomposition)

Given:

$$A_1=\begin{bmatrix}1&4&8\\0&2&0\\0&0&3\end{bmatrix}\quad A_2=\begin{bmatrix}0&1&0\\0&0&1\\-2&-4&-3\end{bmatrix}\quad A_3=\begin{bmatrix}1&0&-1\\0&1&0\\0&0&2\end{bmatrix}\quad A_4=\begin{bmatrix}0&4&3\\0&20&16\\0&-25&-20\end{bmatrix}$$

Find: Derive Jordan-form J & M & M^{-1} matrices

Solution: $A = MJM^{-1}$

(A_1)

Step: (1) Eigenvalues of A (λ)

(2) Algebraic $\alpha(\lambda)$ and geometric $\gamma(\lambda)$ multiplicities

(3) Calculate the eigenspace and generalised eigenspaces

(4) Transformation matrix M

$\det\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} = a\begin{vmatrix}e&f\\h&i\end{vmatrix} - b\begin{vmatrix}d&f\\g&i\end{vmatrix} - c\begin{vmatrix}d&e\\g&h\end{vmatrix}$

<u>(1) Eigenvalues: $\det(A-\lambda I)$</u>

$$\det\begin{bmatrix}1-\lambda&4&8\\0&2-\lambda&0\\0&0&3-\lambda\end{bmatrix} = (1-\lambda)\begin{vmatrix}2-\lambda&0\\0&3-\lambda\end{vmatrix} - 4\overbrace{\begin{vmatrix}0&0\\0&3-\lambda\end{vmatrix}}^{0} - 8\overbrace{\begin{vmatrix}0&2-\lambda\\0&0\end{vmatrix}}^{0}$$

$= (1-\lambda)(2-\lambda)(3-\lambda) \rightarrow \lambda_1=1,\ \lambda_2=2,\ \lambda_3=3$

<u>(2) Multiplicities</u>

(For $\lambda_1=1$) $\alpha(\lambda_1)=1 \rightarrow$ Jordan block size 1×1

$$\ker(A-\lambda_1 I) = \ker\begin{bmatrix}1-1&4&8\\0&2-1&0\\0&0&3-1\end{bmatrix} = \ker\begin{bmatrix}0&4&8\\0&1&0\\0&0&2\end{bmatrix} \overset{\text{G.E}}{\underset{\mathrm{II}-\frac{1}{4}\mathrm{I}}{=}} \ker\begin{bmatrix}0&4&8\\0&0&-2\\0&0&2\end{bmatrix} \overset{\mathrm{III}+\mathrm{II}}{=} \ker\begin{bmatrix}0&4&8\\0&0&-2\\0&0&0\end{bmatrix}$$

x_1 = free varible

$\gamma(\lambda_1) = \dim(\ker(A-\lambda_1 I)) = 1$

(For $\lambda_2=2$) $\alpha(\lambda_2)=1 \rightarrow$ Jordan block size 1×1

$$\ker(A-\lambda_2 I) = \ker\begin{bmatrix}1-2&4&8\\0&2-2&0\\0&0&3-2\end{bmatrix} = \ker\begin{bmatrix}-1&4&8\\0&0&0\\0&0&1\end{bmatrix}$$

x_2 = free varible

$\gamma(\lambda_2) = \dim(\ker(A-\lambda_2 I)) = 1$

(For $\lambda_3=3$) $\alpha(\lambda_3)=1 \rightarrow$ Jordan block Size 1×1

$$\ker(A-\lambda_3 I) = \ker\begin{bmatrix}1-3&4&8\\0&2-3&0\\0&0&3-3\end{bmatrix} = \ker\begin{bmatrix}-2&4&8\\0&-1&0\\0&0&0\end{bmatrix}$$

x_3 = free varible

$\gamma(\lambda_3) = \dim(\ker(A - \lambda_3 I)) = 1$

Algebraic multiplicities $\alpha(\lambda_1, \lambda_2, \lambda_3) = 1$ $\rightarrow$ Jordan block size 1×1 for each λ

Geometric multiplicities $\gamma(\lambda_1, \lambda_2, \lambda_3) = 1$

(3) Eigenspace and generalised eigenspaces

(For $\lambda_1 = 1$)

$$(A - \lambda_1 I) V_1 = 0 \rightarrow \begin{bmatrix} 0 & 4 & 8 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} V_1 = 0 \rightarrow V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(For $\lambda_2 = 2$)

$$(A - \lambda_2 I) V_2 = 0 \rightarrow \begin{bmatrix} -1 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V_2 = 0 \rightarrow V_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

$$x_3 = 0, \quad -x_1 + 4x_2 = 0 \rightarrow x_1 = 4, \ x_2 = 1$$

(For $\lambda_3 = 3$)

$$(A - \lambda_3 I) V_3 = 0 \rightarrow \begin{bmatrix} -2 & 4 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V_3 = 0 \rightarrow V_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$-x_2 = 0, \quad -2x_1 + 8x_3 = 0 \rightarrow x_1 = 4, \ x_3 = 1$$

(4) Transformation matrix M

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \# (A_1)$$

(A₂)

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

(1) Eigenvalues: $\det(A-\lambda I)$

$$\det\begin{bmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ -2 & -4 & -3-\lambda \end{bmatrix} = (-\lambda)\begin{vmatrix} 0-\lambda & 1 \\ -4 & -3-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ -2 & -3-\lambda \end{vmatrix} - 0\begin{vmatrix} 0 & 0-\lambda \\ -2 & -4 \end{vmatrix} \quad 0$$

$$= (-\lambda)[(-\lambda)(-3-\lambda)+4] - 1[2] \rightarrow -\lambda^3 - 3\lambda^2 - 4\lambda - 2$$

$$\lambda_1 = -1+i, \quad \lambda_2 = -1-i, \quad \lambda_3 = -1$$

(2) Multiplicities

(For $\lambda_1, \lambda_2, \lambda_3$) $\quad \alpha(\lambda_{all}) = 1 \rightarrow$ Jordan size 1×1

$\gamma(\lambda_{all}) = 1 \rightarrow J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1-i & 0 \\ 0 & 0 & -1+i \end{bmatrix}$

(3) Eigenspace and generalised eigenspaces

$(A-\lambda I)V = 0$

For ($\lambda_1 = -1+i$)

$$\begin{bmatrix} 1-i & 1 & 0 \\ 0 & 1-i & 1 \\ -2 & -4 & -2-i \end{bmatrix} V_1 = 0 \rightarrow V_1 \begin{bmatrix} i \\ -1-i \\ 2 \end{bmatrix}$$

For ($\lambda_2 = -1-i$)

$$\begin{bmatrix} 1+i & 1 & 0 \\ 0 & 1+i & 1 \\ -2 & -4 & -2+i \end{bmatrix} V_2 = 0 \rightarrow V_2 \begin{bmatrix} -i \\ -1+i \\ 2 \end{bmatrix}$$

For ($\lambda_3 = -1$)

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix} V_3 = 0 \rightarrow V_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(4) Transformation matrix M

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1-i & 0 \\ 0 & 0 & -1+i \end{bmatrix}; \quad M = \begin{bmatrix} i & -i & 1 \\ -1-i & -1+i & -1 \\ 2 & 2 & 1 \end{bmatrix}; \quad M^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ -\frac{1}{2}-\frac{i}{2} & -\frac{1}{2}-i & -\frac{i}{2} \\ -\frac{1}{2}+\frac{i}{2} & -\frac{1}{2}+i & \frac{i}{2} \end{bmatrix}$$

#A₂

(A3)

$$A_3=\begin{bmatrix}1&0&-1\\0&1&0\\0&0&2\end{bmatrix}$$

<u>(1) Eigenvalues: $\det(A-\lambda I)$</u>

$$\det\begin{bmatrix}1-\lambda&0&-1\\0&1-\lambda&0\\0&0&2-\lambda\end{bmatrix}=(1-\lambda)\begin{vmatrix}1-\lambda&0\\0&2-\lambda\end{vmatrix}-0\cancel{\begin{vmatrix}0&0\\0&2-\lambda\end{vmatrix}}-(-1)\cancel{\begin{vmatrix}0&1-\lambda\\0&0\end{vmatrix}}$$

$$=(1-\lambda)(1-\lambda)(2-\lambda)\rightarrow(1-\lambda)^2(2-\lambda)\rightarrow\lambda_1=\lambda_2=1,\ \lambda_3=2$$

<u>(2) Multiplicities</u>

(For $\lambda_1=\lambda_2=1$) $\alpha(\lambda_1,\lambda_2)=2\rightarrow$ Jordan block size 2x2

$$\ker(A-\lambda_1 I)=\ker\begin{bmatrix}1-1&0&-1\\0&1-1&0\\0&0&2-1\end{bmatrix}\rightarrow\ker\begin{bmatrix}0&0&-1\\0&0&0\\0&0&1\end{bmatrix}\xrightarrow{\text{III}+\text{I}}\ker\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}$$

$\gamma(\lambda_1,\lambda_2)=\dim(\ker(A-\lambda_1 I))=2$ $\quad\gamma=2\quad\cancel{\gamma=1}$ $\quad x_1$ & x_2 free varible

(For $\lambda_3=2$) $\alpha(\lambda_3)=1\rightarrow$ Jordan block size 1x1

$$\ker(A-\lambda_3 I)=\ker\begin{bmatrix}1-2&0&-1\\0&1-2&0\\0&0&2-2\end{bmatrix}\rightarrow\ker\begin{bmatrix}-1&0&-1\\0&-1&0\\0&0&0\end{bmatrix}$$

$\gamma(\lambda_3)=\dim(\ker(A-\lambda_3 I))=1$ $\quad x_3$ free varible

<u>(3) Eigenspace and generalised eigenspaces</u>

(For $\lambda_1=\lambda_2=1$)

$$(A-\lambda_{12}I)V_{1,2}=0\rightarrow\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}V_{1,2}\quad V_1=\begin{bmatrix}1\\0\\0\end{bmatrix}\quad V_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

(For $\lambda_3=2$)

$$(A-\lambda_3 I)V_3=0\rightarrow\begin{bmatrix}-1&0&-1\\0&-1&0\\0&0&0\end{bmatrix}V_3=0\quad V_3=\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$-x_2=0$

$-x_1=x_3$

<u>(4) Transformation matrix M</u>

$$J=\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix},\quad M=\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix},\quad M^{-1}=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$$ #A3

(A4)

$$A4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

(1) Eigenvalues: $\det(A-\lambda I)$

$$\det \begin{bmatrix} 0-\lambda & 4 & 3 \\ 0 & 20-\lambda & 16 \\ 0 & -25 & -20-\lambda \end{bmatrix} = (0-\lambda)\begin{vmatrix} 20-\lambda & 16 \\ -25 & -20-\lambda \end{vmatrix} - 4\begin{vmatrix} 0 & 16 \\ 0 & -20-\lambda \end{vmatrix} - 3\begin{vmatrix} 0 & 20-\lambda \\ 0 & -25 \end{vmatrix}$$

$$= (-\lambda)\left[(20-\lambda)(-20-\lambda) - [16(-25)]\right] = -\lambda^3 \rightarrow \lambda_1 = 0$$

(2) Multiplicities

(For $\lambda_1 = 0$) $\quad \alpha(\lambda_1) = 3 \longrightarrow$ Jordan block 3×3

$$\ker(A-\lambda_1 I) = \ker \begin{bmatrix} 0-0 & 4 & 3 \\ 0 & 20-0 & 16 \\ 0 & -25 & -20-0 \end{bmatrix} \rightarrow \ker \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

$$\xrightarrow{II-5I} \ker \begin{bmatrix} 0 & 4 & 3 \\ 0 & 0 & 1 \\ 0 & -25 & -20 \end{bmatrix} \xrightarrow{III+20II} \ker \begin{bmatrix} 0 & 4 & 3 \\ 0 & 0 & 1 \\ 0 & -25 & 0 \end{bmatrix}$$

x_1 free varible

$$\gamma(\lambda_1) \dim(\ker(A-\lambda_1 I)) = 1$$

$$\begin{bmatrix} \boxed{0} & & \\ & \boxed{0} & \\ & & \boxed{0} \end{bmatrix} \quad \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ & 0 \end{matrix}} & \\ & 0 \end{bmatrix} \quad \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{matrix}} \end{bmatrix}$$

$\gamma = 3 \qquad \gamma = 2 \qquad \gamma = 1$

(3) Eigenspace and generalised eigenspaces

(For $\lambda_1 = 0$)

$$(A-\lambda_1 I)V_{1,2,3} = 0 \rightarrow \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} V_{1,2,3} = 0 \qquad V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ 4 \\ -5 \end{bmatrix}, V_3 = \begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix}$$

(4) Transformation matrix M

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} ; M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -5 & 4 \end{bmatrix} ; M^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 5 & 4 \end{bmatrix}$$

#A4

Exercise 4 (CT and DT Dynamics)

Given: $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$, $y(t) = [2 \; 3] \, x(t)$

$x(0) = 0$, u is unit step input

Find: i) $y(5)$ for CT system

ii) discretized state space representation using sample time $T = 1\,s$

iii) $y(5)$ for discrete time system & plot signal $y(t)$ for CT & DT

Solution:

(i)
$$y(t) = Ce^{A(t-t_0)} x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau + Du(t)$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Apply C-H $(n=2)$

$\Delta(\lambda) = \lambda^2 + 2\lambda + 2$, $\lambda_1 = -1+i$, $\lambda_2 = -1-i$

$f(\lambda) = g(\beta, \lambda) = \beta_1 \lambda + \beta_0$

For $(\lambda_1 = -1+i)$

$f(\lambda_1) = e^{(-1+i)t} = \beta_1(-1+i) + \beta_0$

For $(\lambda_2 = -1-i)$

$f(\lambda_2) = e^{(-1-i)t} = \beta_1(-1-i) + \beta_0$

$$\begin{aligned} & e^{(-1-i)t} = \beta_1(-1-i) + \beta_0 \\ - \; & e^{(-1+i)t} = \beta_1(-1+i) + \beta_0 \\ \hline & e^{(-1-i)t} - e^{(-1+i)t} = -2\beta_1 i \end{aligned}$$

$$\beta_1 = \frac{e^{(-1-i)t} - e^{(-1+i)t}}{-2i} \rightarrow \beta_1 = \frac{(e^{2it} - 1)(e^{(-i-1)t})}{2i}$$

$$\beta_0 = e^{(-1+i)t} - \beta_1(-1+i) \rightarrow \beta_0 = e^{(-1+i)t} - \left(\frac{e^{(-1-i)t} - e^{(-1+i)t}}{-2i}\right)(-1+i)$$

$$e^{At} = \beta_1 A + \beta_0 I$$

$$e^{At} = \left[\frac{e^{(-1-i)t} - e^{(-1+i)t}}{-2i}\right]\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} + \left[e^{(-1+i)t} - \left[\left(\frac{e^{(-1-i)t} - e^{(-1+i)t}}{-2i}\right)(-1+i)\right]\right]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \dfrac{((i+1)e^{2it}+i-1)e^{(-i-1)t}}{2i} & \dfrac{-(e^{t}-1)e^{(i-1)t}}{2i} \\ \dfrac{(e^{t}-1)e^{(i-1)t}}{i} & \dfrac{((2e^{t}+i-1)e^{2it}+i-1)e^{(-i+1)t}}{2i} \end{bmatrix}$$

$$e^{At}(t=5) = \begin{bmatrix} -0.0046 & -0.0065 \\ 0.0129 & 0.0084 \end{bmatrix}, \quad C = [2, 3]$$

$$\int_{t_0}^{t} e^{-A\tau} Bu(\tau)\, d\tau \rightarrow \int_0^5 e^{-A\tau} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d\tau = \begin{bmatrix} -\dfrac{e^{-A\tau}}{A} \\ -\dfrac{e^{-A\tau}}{A} \end{bmatrix}\Bigg|_0^5$$

$$\int_{t_0}^{t} e^{-A\tau} Bu(\tau)\, d\tau = \begin{bmatrix} \quad \end{bmatrix}$$

$$y(t) = \cancel{Ce^{A(t-t_0)}}^{\,0} x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau + \cancel{Du(t)}^{\,0}$$

$$y(t) = Ce^{At}\int_{t_0}^{t} e^{-A\tau} Bu(\tau)\, d\tau$$

$$y(5) = [2\ 3] \begin{bmatrix} -0.0046 & -0.0065 \\ 0.0129 & 0.0084 \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} \# \text{ CT}$$

unable to solve
due to not understanding
the integral

(ii) $T = 1s$

$$\dot{x} = Ax + Bu \longrightarrow x(k+1) = A_d x(k) + B_d u(k)$$

$$y = Cx + Du \qquad y(k) = Cx(k) + Du(k)$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \end{bmatrix}, \quad D = 0$$

$$A_d = e^{AT}(T=1) = \begin{bmatrix} 0.50833 & 0.30956 \\ -0.61912 & -0.11079 \end{bmatrix}$$

$$B_d = A^{-1}(A_d - I)B$$

$$B_d = \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.50833 & 0.30956 \\ -0.61912 & -0.11079 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow B_d = \begin{bmatrix} -0.45294 \\ 0.81789 \end{bmatrix}$$

$$x(k+1) = \begin{bmatrix} 0.50833 & 0.30956 \\ -0.61912 & -0.11079 \end{bmatrix} x(k) + \begin{bmatrix} -0.45294 \\ 0.81789 \end{bmatrix} u(k) \;\#$$

$$y(k) = [2, 3]\, x(k) \;\#$$

(iii)

$$y(k) = \cancel{CA_d^k x(0)}^{\,0} + \sum_{m=0}^{k-1} CA_d^{k-m-1} B_d u(m) + \cancel{Du(k)}^{\,0}$$

$$C = [2 \;\; 3] \qquad A_d = e^{AT}(t=5) = \begin{bmatrix} -0.0046 & -0.0065 \\ 0.0129 & 0.0084 \end{bmatrix}$$

$$B_d = A^{-1}(A_d - I)B$$

$$B_d = \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.50833 & 0.30956 \\ -0.61912 & -0.11079 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow B_d = \begin{bmatrix} -0.45294 \\ 0.81789 \end{bmatrix}$$

$$u(m) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y(5) = [2 \;\; 3] \begin{bmatrix} -0.0046 & -0.0065 \\ 0.0129 & 0.0084 \end{bmatrix}^4 \begin{bmatrix} -0.45294 \\ 0.81789 \end{bmatrix} = 3.13004 \times 10^{-9}$$

DT

Makes no sense

Exercise 5 (Diagonalization)

Given: Fibonacci number: 0, 1, 1, 2, 3, 5, 8, 13, ...

Fibonacci equation: $F_{k+2} = F_{k+1} + F_k$

Find: construct a discrete linear time invariant system to find 20th Fibonacci number.

Solution:

$$x(k) = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x(k+1) = \begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_k + F_{k+1} \end{bmatrix} \rightarrow x(k+1) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(k)$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 0-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = 0 \rightarrow -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$$

$$\lambda = \frac{1 \pm \sqrt{1^2+4}}{2} \rightarrow \lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \begin{bmatrix} -\left(\frac{1+\sqrt{5}}{2}\right) & 1 \\ 1 & 1-\left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix} v_1 = 0 \quad v_1 = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2}, \quad \begin{bmatrix} -\left(\frac{1-\sqrt{5}}{2}\right) & 1 \\ 1 & 1-\left(\frac{1-\sqrt{5}}{2}\right) \end{bmatrix} v_2 = 0 \quad v_2 = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$A = MJM^{-1}$$

$$M = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}, \quad J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} \frac{-\lambda_2}{\lambda_1-\lambda_2} & \frac{1}{\lambda_1-\lambda_2} \\ \frac{\lambda_1}{\lambda_1-\lambda_2} & \frac{-1}{\lambda_1-\lambda_2} \end{bmatrix}$$

$$A^k = MJ^kM^{-1}$$

$$A^k = \begin{bmatrix} \frac{\lambda_1\lambda_2^2}{\lambda_1-\lambda_2} - \frac{\lambda_1^k\lambda_2}{\lambda_1-\lambda_2} & \frac{\lambda_1^k}{\lambda_1-\lambda_2} - \frac{\lambda_2^k}{\lambda_1-\lambda_2} \\ \frac{\lambda_1\lambda_2\lambda_2^k}{\lambda_1-\lambda_2} - \frac{\lambda_1^{k+1}\lambda_2}{\lambda_1-\lambda_2} & \frac{\lambda_1^{k+1}}{\lambda_1-\lambda_2} - \frac{\lambda_2^{k+1}}{\lambda_1-\lambda_2} \end{bmatrix}$$

$$F_1 = Ax(0), \quad F_2 = AAx(0), \quad F_3 = AAAx(0) \quad \ldots \quad F_k = A^k x(0)$$

$$F_k = A^k X(0) \rightarrow F_k = \begin{bmatrix} \frac{\lambda_1^k}{\lambda_1-\lambda_2} - \frac{\lambda_2^k}{\lambda_1-\lambda_2} \\ \frac{\lambda_1^{k+1}}{\lambda_1-\lambda_2} - \frac{\lambda_2^{k+1}}{\lambda_1-\lambda_2} \end{bmatrix}$$

F_{20} ($\lambda_1 \approx 1.618$, $\lambda_2 = -0.618$)

$$\begin{bmatrix} F_{20} \\ F_{21} \end{bmatrix} = \begin{bmatrix} \frac{1.618^{20}}{1.618+0.618} - \frac{-0.618^{20}}{1.618+0.618} \\ \frac{1.618^{21}}{1.618+0.618} - \frac{-0.618^{21}}{1.618+0.618} \end{bmatrix} \rightarrow F_{20} \approx 6762.3641$$

Better approximation

$$F_{20} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{20}}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}} - \frac{\left(\frac{1-\sqrt{5}}{2}\right)^{20}}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}} \approx 6765.00002956$$

or 6765 #