Homework 2

24-677 Special Topics: Linear Control Systems

Prof. D. Zhao

Due: Sept 21, 2023, 11:59 pm. Submit within deadline.

- All assignments will be submitted through Gradescope. Your online version and its timestamp will be used for assessment. Gradescope is a tool licensed by CMU and integrated with Canvas for easy access by students and instructors. When you need to complete a Gradescope assignment, here are a few easy steps you will take to prepare and upload your assignment, as well as to see your assignment status and grades. Take a look at Q&A about Gradescope to understand how to submit and monitor HW grades. https://www.cmu.edu/teaching//gradescope/index.html
- You will need to upload your solution in .pdf to Gradescope (either scanned handwritten version or LaTeX or other tools). If you are required to write Python code, upload the code to Gradescope as well.
- Grading: The score for each question or sub-question is discrete with three outcomes: fully correct (full score), partially correct/unclear (half the score), and totally wrong (zero score).
- Regrading: please review comments from TAs when the grade is posted and make sure no error in grading. If you find a grading error, you need to inform the TA as soon as possible but no later than a week from when your grade is posted. The grade may NOT be corrected after 1 week.
- At the start of every exercise you will see topic(s) on what the given question is about and what will you be learning.
- We advise you to start with the assignment early. All the submissions are to be done before the respective deadlines of each assignment. For information about the late days and scale of your Final Grade, refer to the Syllabus in Canvas.

Exercise 1. *Cayley-Hamilton Theorem* ***(20 points)***

Given

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Find A^{10} and e^{At} uisng C-H.

Solution:

To calculate A^{10}:
The characteristic polynomial of A is $\Delta(\lambda) = \det(\lambda I - A) = \lambda(\lambda - 1)^2$.
Let $h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$.
Substitute the eigenvalues.
For $\lambda = 0$, $0^{10} = \beta_0$.
For $\lambda = 1, f(1) = h(1)$:
$1^{10} = \beta_0 + \beta_1 + \beta_2$.
We also need to use the derivative: $f'(1) = h'(1)$:
$10 \cdot 1^9 = \beta_1 + 2\beta_2$.
Then we have $\beta_0 = 0, \beta_1 = -8, \beta_2 = 9$
Now, as $A^{10} = -8A + 9A^2$

$$= -8 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 9 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

To compute $e^{At} : e^t = \beta_0 + \beta_1 + \beta_2 \Rightarrow \beta_1 = 2e^t - te^t - 2\ te^t = \beta_1 + 2\beta_2$
$\beta_2 = te^t - e^t + 1$

$e^{At} = \beta_0 I + \beta_1 A + \beta_2 A^2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (2e^t - e^t - 2) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + (2e^t - e^t + 1) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$$

When $t = 1$,

$$e^{At} = \begin{bmatrix} e & e - 1 & 1 \\ 0 & 1 & e - 1 \\ 0 & 0 & e \end{bmatrix}$$

Exercise 2. *Linear dynamics solution* ***(20 points)***

Let $x_1(t)$ be the water level in Tank 1 and $x_2(t)$ be the water level in Tank 2 . Let α be the rate of outflow from Tank 1 and β be rate of outflow from Tank 2 . Let u be the supply of water to the system. The system can be modelled into the following differential equations:

$$\frac{dx_1}{dt} = -\alpha x_1 + u$$
$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2$$

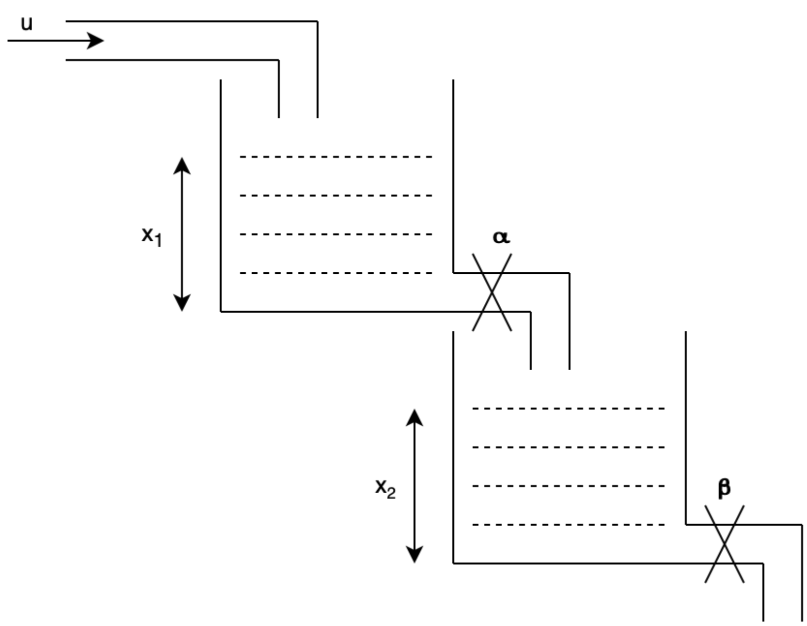

Figure 1: Tank Problem

Given $\alpha = 0.1$, $\beta = 0.2$, $u = 1$, $x_1(0) = 2$, $x_2(0) = 1$, find the water level in both tanks after 5s. Solve with C-H theorem. You may use calculator but do not directly use programming.

Solution:
Using the state space representation, we have

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u$$

The eigenvalues of A are $\lambda = -0.1, -0.2$. Recall

$$\begin{matrix} f(\lambda) = e^{\lambda_1 t} = \beta_1\lambda_1 + \beta_0 \\ f(\lambda) = e^{\lambda_2 t} = \beta_1\lambda_2 + \beta_0 \end{matrix} \Rightarrow \begin{matrix} e^{-0.1t} = -0.1\beta_1 + \beta_0 \\ e^{-0.2t} = -0.2\beta_1 + \beta_0 \end{matrix} \Rightarrow \begin{cases} \beta_0 = 2e^{-0.1t} - e^{-0.2t} \\ \beta_1 = \frac{e^{-0.1t} - e^{-0.2t}}{0.1} \end{cases}$$

Therefore

$$e^{At} = \beta_1 A + \beta_0 I = \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix}$$

$$\begin{aligned}
x(t) &= e^{At} x\left(0\right) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \\
&= \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-0.1(t-\tau)} & 0 \\ e^{-0.1(t-\tau)} - e^{-0.2(t-\tau)} & e^{-0.2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau \\
&= \begin{bmatrix} 2e^{-0.1t} \\ 2e^{-0.1t} - e^{-0.2t} \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-0.1(t-\tau)} \\ e^{-0.1(t-\tau)} - e^{-0.2(t-\tau)} \end{bmatrix} d\tau \\
&= \begin{bmatrix} 2e^{-0.1t} \\ 2e^{-0.1t} - e^{-0.2t} \end{bmatrix} + \begin{bmatrix} \frac{e^0}{0.1} - \frac{e^{-0.1t}}{0.1} \\ \frac{e^0}{0.1} - \frac{e^{-0.1t}}{0.1} - \frac{e^0}{0.2} + \frac{e^{-0.2t}}{0.2} \end{bmatrix} \\
&= \begin{bmatrix} -8e^{-0.1t} + 10 \\ -8e^{-0.1t} + 4e^{-0.2t} + 5 \end{bmatrix}
\end{aligned}$$

When $t = 5$, $x(5) = [5.1478, 1.6193]$

Exercise 3. *Jordan form, decomposition* ***(20 points, 5 for each A)***

Derive the Jordan-form (including the Jordan matrix $\mathbf{J}$, the model matrix $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$) of the following matrices manually.

$$\mathbf{A_1} = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \mathbf{A_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

$$\mathbf{A_3} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \mathbf{A_4} = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

(Write the Jordan form such that eigenvalues should be in ascending order of their absolute values. The absolute value of a complex number is defined as $|a + bi| = \sqrt{a^2 + b^2}$)

Solution:

A1:

The eigenvalues calculated from the characteristic polynomial are: 1, 2 and 3. Now finding linearly independent generalized eigenvectors for each eigenvalue such that $det(A - \lambda I) = 0$

the basis for $\lambda = 1$, 2, 3 are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

Using the Eigen Vectors we form $\mathbf{M} = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

As $\mathbf{A}_1$ is a Type-I matrix, therefore the Jordan Decomposition follows Eigen Decomposition and Jordan form is a diagonal matrix with values as the Eigen Values which can be found by,

$\mathbf{J}_1 = \mathbf{M}^{-1}\mathbf{A}_1\mathbf{M}$

$$\mathbf{M}^{-1} = \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

solving for $\mathbf{J}_1$ we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

A2:

The eigenvalues calculated from the characteristic polynomial are: -1, -1-i and -1+i. Now finding linearly independent generalized eigenvectors for each eigenvalue such that $det(A - \lambda I) = 0$

the basis for λ = -1 , -1-i, -1+i are $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -0.5i \\ -0.5+0.5i \\ 1 \end{bmatrix}, \begin{bmatrix} 0.5i \\ -0.5-0.5i \\ 1 \end{bmatrix}$

Using the Eigen Vectors we form $\mathbf{M} = \begin{bmatrix} 1 & -0.5i & 0.5i \\ -1 & -0.5+0.5i & -0.5-0.5i \\ 1 & 1 & 1 \end{bmatrix}$

$$\mathbf{M}^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ -1-i & -1-2i & -i \\ -1+i & -1+2i & i \end{bmatrix}$$

As $\mathbf{A}_2$ is a Type-I matrix, therefore the Jordan Decomposition follows Eigen Decomposition and Jordan form is a diagonal matrix with values as the Eigen Values as shown,

$$\mathbf{J}_2 = \mathbf{M}^{-1}\mathbf{A}_2\mathbf{M}$$

solving for $\mathbf{J}_2$ we get $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1-i & 0 \\ 0 & 0 & -1+i \end{bmatrix}$.

A3:

The eigenvalues calculated from the characteristic polynomial are: 1, 1 and 2. We have one eigen value as repeating. Now finding the eigenvectors for each eigenvalue such that $det(A - \lambda I) = 0$

the basis for λ = 1 , 1, 2 are $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Using the Eigen Vectors we form $\mathbf{M} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\mathbf{M}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

As $\mathbf{A}_3$ is a Type-II$_1$ matrix(non-defective), therefore the Jordan Decomposition follows Eigen Decomposition and Jordan form is a diagonal matrix with values as the Eigen Values as shown,

$$\mathbf{J}_3 = \mathbf{M}^{-1}\mathbf{A}_3\mathbf{M}$$

solving for $\mathbf{J}_3$ we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

A4:

The eigenvalues calculated from the characteristic polynomial are: 0, 0 and 0. All eigen values are repeating. Now finding the eigenvectors for each eigenvalue such that $det(A - \lambda I) = 0$

the basis for $\lambda = 0$ are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Since it is a Type-II_2 case, to find the remaining generated eigen vectors we do

$(A_4 - \lambda_2 I)v_2 = v_1$

v_2 is $\begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$

$(A_4 - \lambda_3 I)v_3 = v_2$

v_3 is $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Using the Eigen Vectors we form $\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & -5 & -1 \end{bmatrix}$

$$\mathbf{M}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 5 & -4 \end{bmatrix}$$

$$\mathbf{J}_3 = \mathbf{M}^{-1}\mathbf{A}_4\mathbf{M}$$

Solving we get the Jordan form as $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Exercise 4. *CT and DT Dynamics* ***(20 points)***

Given the following system. Assume $x(0) = 0$ and u is a unit step input. Answer the following questions with both manual derivation and Python programming. Include all your derivation. Solutions without derivation will receive zero points. Submit the code to Gradescope.

i) Find $y(5)$ for CT system **(5 points)**

ii) Find the discretized state space representation using sample time $T = 1s$ **(5 points)**

iii) Find $y(5)$ of Discrete Time system. **(5 points)** Also plot signals $y(t)$ for both CT and DT systems in the same figure. **(5 points)**

$$\dot{x(t)} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(t)$$

Solution:

i)

We have the A, B, C, D matrix given to us as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$$

Using the expression of output at time t for a continuous time system we get

$$\begin{aligned}
y(t) &= C \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \\
&= C \int_0^t e^{A(t-\tau)} d\tau B = CA^{-1}(e^{At} - e^0)B \\
&= C \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t}\cos t + 2e^{-t}\sin t - 1 \\ e^{-t}\cos t - 3e^{-t}\sin t - 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t}\cos t + 2e^{-t}\sin t - 1 \\ e^{-t}\cos t - 3e^{-t}\sin t - 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}\cos t + 2e^{-t}\sin t - 1 & -e^{-t}\cos t + 3e^{-t}\sin t + 1 \end{bmatrix} \\
&= 5e^{-t}\sin t, \text{ for } t \geq 0
\end{aligned}$$

Using either method we can get $y(5) = -0.03$, by replacing 5 in place of t.

ii):

Consider the A matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

To calculate the A matrix for a discrete system of sample time T,

$$A_d = e^{AT} = \begin{bmatrix} 0.5083 & 0.3095 \\ -0.6191 & -0.1107 \end{bmatrix}$$

For B_d,

$$B_d = \int_0^T e^{At} dt B = A^{-1}(A_d - I)B$$

$$= \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0.5083 & 0.3095 \\ -0.6191 & -0.1107 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 1.0470 \\ -0.1821 \end{bmatrix}$$

iii):

$$y(k) = CA_d^k x(0) + \sum_{m=0}^{k-1} CA_d^{k-m-1} B_d u(m) + Du(k)$$

As $x(0) = 0$, D is zero, and u is a step input therefore the equation becomes

$$y(k) = \sum_{m=0}^{k-1} CA_d^{k-m-1} B_d$$

Here $k = 5$,

$$y(k) = -0.03230$$

```python
# Import Libraries
import numpy as np
from scipy import signal
import matplotlib.pyplot as plt

# Define the System
A = [[0., 1.], [-2., -2.]]
B = [[1.], [1.]]
C = [[2., 3.]]
D = 0.

system = signal.lti(A, B, C, D)

t = np.linspace(0,5)
u = np.ones_like(t)

# y(5) for CT system
tc, y_c, x_c = signal.lsim(system,u,t)
```

```python
print("***** Continous Time System *****")
print("A: {}".format(system.A))
print("B: {}".format(system.B))

# Discretization by taking T as 1 second
T = 1 # seconds
td = np.arange(6)

dt_sys = system.to_discrete(T)
Ad = dt_sys.A
Bd = dt_sys.B

print("\n***** Discrete Time System *****")
print("Ad: {}".format(dt_sys.A))
print("Bd: {}".format(dt_sys.B))

y_d = []
for k in td:
        ans = 0.0
        for m in range(k):
                ans += (C@np.linalg.matrix_power(Ad, k - m - 1)@Bd)[0,0]
        y_d.append(np.round(ans,4))

# Plots
fig = plt.figure()
ax  = fig.add_subplot()
ax.plot(td, y_d, label = 'Discrete time')
ax.plot(tc, y_c, label = 'Continous time')
plt.legend(loc = 'best')
plt.tight_layout()
plt.show()
```

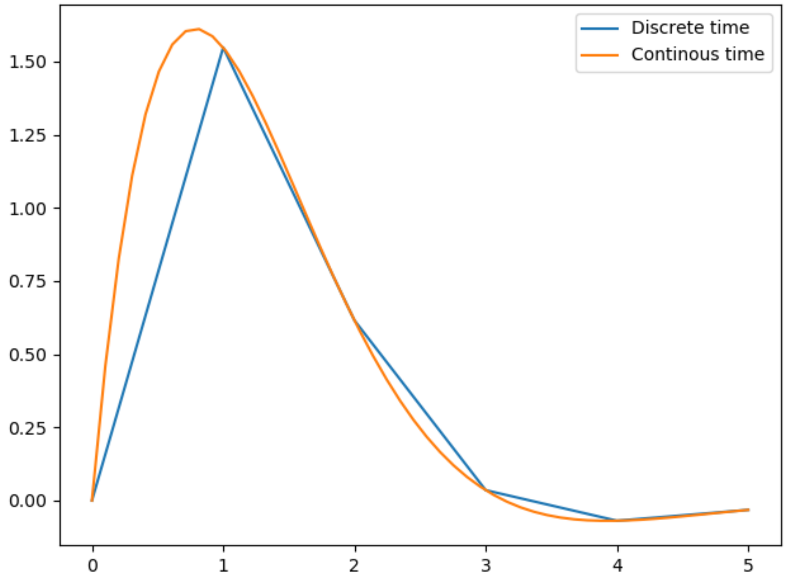

Figure 2: CT vs DT ($T = 1 second$)

Exercise 5. *Diagonalization* ***(20 points)***

In the Fibonacci sequence, a Fibonacci number is the sum of the two previous F's, starting from 0, 1

$$\text{Fibonacci numbers: } 0, 1, 1, 2, 3, 5, 8, 13, ...$$

$$\text{Fibonacci equation: } F_{k+2} = F_{k+1} + F_k$$

How could we find the 20th Fibonacci number, without starting at $F_0 = 0$ and $F_1 = 1$, and add the numbers one by one all the way out to F_{20}? Hint: construct a discrete linear time invariant system.

You are **allowed to use programming in the final matrix multiplication**, but do not use the for loop to directly computer Fibonacci sequence.

Solution:

From the hint, we can generalize:

$$\begin{matrix} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_{k+1} \end{matrix} \quad \text{becomes } x_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = Ax_k$$

where $x_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$.

The solution to the difference equation $x_{k+1} = Ax_k$ is $x_k = A^k x_0$.

Using the Fibonacci matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} F_{20} \\ F_{19} \end{bmatrix} = A^{19} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6765 & 4181 \\ 4181 & 2584 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6765 \\ 4181 \end{bmatrix}$$

$$\text{Procedure to find } A^{19}:$$

$$\text{Finding eigen Values of } A \det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \quad (1-\lambda)(-\lambda) - 1 = 0 \Rightarrow -\lambda + \lambda^2 - 1 = 0 \Rightarrow \quad \lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda_1 \lambda_2 = \tfrac{1+\sqrt{5}}{2}, \tfrac{1-\sqrt{5}}{2}$$

$$\text{Finding corresponding eigen vectors: } (A - \lambda_1 I)\, v_1 = 0 \; (A - \lambda_2 I)\, v_2 = 0$$

$$v_1 = \begin{bmatrix} 2 \\ \sqrt{5} - 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 2 \\ -1 - \sqrt{5} \end{bmatrix}$$

$$M = [v_1, v_2] = \begin{bmatrix} 2 & 2 \\ \sqrt{5} - 1 & -\sqrt{5} - 1 \end{bmatrix}$$

$$M^{-1} = \tfrac{-1}{4\sqrt{5}} \begin{bmatrix} -1 - \sqrt{5} & -2 \\ -\sqrt{5} + 1 & 2 \end{bmatrix}$$

$$A^{19} = M \wedge^{19} M^{-1}$$

$$= \left[\begin{array}{cc} 2 & 2 \\ \sqrt{5}-1 & -\sqrt{5}-1 \end{array} \right] \left[\begin{array}{cc} \left(\frac{1+\sqrt{5}}{2}\right)^{19} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{19} \end{array} \right] \frac{-1}{4\sqrt{5}} \left[\begin{array}{cc} -1-\sqrt{5} & -2 \\ -\sqrt{5}+1 & 2 \end{array} \right]$$

$$F_{20} = 6765$$