Exercise 1 (Controllability and Observability)

Given:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x$$

Find: controllable? observable?

Solution:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{bmatrix} \xrightarrow{R.E.F} \begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{array}{l} n=3 \\ \text{rank}(A) = 3 \end{array}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$P = [B \vdots AB \vdots \ldots \vdots A^{k-1}B]$, if $\text{rank}(P) = n \rightarrow$ controllable

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -6 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -3 \\ 0 & -1 & -6 \end{bmatrix} \xrightarrow{R.E.F} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{rank}(P) = 3 = n \rightarrow \text{controllable}$$

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}, \quad \text{if rank}(Q) = n \rightarrow \text{observable}$$

$$C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 5 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & 5 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{R.E.F} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank}(Q) = 2 \neq n \Rightarrow \text{not observable}$$

The system is controllable but not observable #

Exercise 2 (Jordan form test)

Given:

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} x$$

Find: controllable? obserable?

Solution:

$\lambda_1 = 2$

Jordan Block rank = 3

$$B_{\lambda_1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{R.E.F} \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{rank}(B_{\lambda_1}) = 3 \ \checkmark \rightarrow \text{controllable}$$

$$C_{\lambda_1} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R.E.F} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank}(C_{\lambda_1}) = 2 \ \times \rightarrow \text{not observable}$$

$\lambda_2 = 1$

Jordan Block rank = 2

$$B_{\lambda_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R.E.F} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{rank}(B_{\lambda_2}) = 2 \ \checkmark \rightarrow \text{controllable}$$

$$C_{\lambda_2} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \xrightarrow{R.E.F} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \text{rank}(C_{\lambda_2}) = 2 \ \checkmark \rightarrow \text{observable}$$

The system is controllable but not observable #

Exercise 3 (Controllability)

Given:

Case 1 $\frac{dx_1}{dt} = -\alpha x_1 + u$, $\frac{dx_2}{dt} = \alpha x_1 - \beta x_2$, $\alpha = 0.1$, $\beta = 0.2$, $u = 1$

$$\dot{x} = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad \text{(From H2E2)}$$

Case 2. $\frac{dx_1}{dt} = -\alpha x_1$, $\frac{dx_2}{dt} = \alpha x_1 - \beta x_2 + u$ $\quad \dot{x} = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

Find: determine controllability for both cases.

Solution:

(Case 1)

$$A = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix} \xrightarrow{\text{R.E.F.}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{matrix} n = 2 \\ \text{rank}(A) = 2 \end{matrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$P = [B \vdots AB \vdots \ldots \vdots A^{k-1}B]$, if $\text{rank}(P) = n \rightarrow$ controllable

$$AB = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

$$P = [B \vdots AB] = \begin{bmatrix} 1 & -0.1 \\ 0 & 0.1 \end{bmatrix} \xrightarrow{\text{R.E.F.}} \begin{bmatrix} 1 & -0.1 \\ 0 & 1 \end{bmatrix} \rightarrow \text{rank}(P) = 2 = n \rightarrow \text{controllable}$$

$\because$ the A (input matrix) and P (controllability matrix) has the same rank ($n=2$)

$\therefore$ the system is controllable. # Case 1

(Case 2)

$$A = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix} \rightarrow \begin{matrix} k = 2 \\ \text{rank}(A) = n = 2 \end{matrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 \\ 1 & -0.2 \end{bmatrix} \rightarrow \text{rank}(P) = 1 \neq n \rightarrow \text{not controllabe}$$

$\because$ the A (input matrix) & P (controllability matrix) has varying ranks

$\therefore$ the system is not controllable. # case 2

Exercise 4 (Gauss Elimination and LU Decomposition)

1. Given:

a) $x+y+z=3$
$x+2y+3z=0$
$x+3y+2z=3$

b) $x+2y-z=1$
$2x+5y-z=3$
$x+3y+2z=6$

c) $x_1+x_2-x_3+x_4=1$
$2x_1+3x_2+x_3 \quad =4$
$3x_1+5x_2+3x_3-x_4=5$

Find: solve using Gauss Elimination Method

Solution:

a)
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 1 & 3 & 2 & 3 \end{array}\right] \xrightarrow{R_3-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -3 & 6 \end{array}\right] \rightarrow \begin{array}{l} x+y+z=3 \rightarrow x=3-y-z \rightarrow x=4 \\ y+2z=-3 \rightarrow y=-3-(2z) \rightarrow y=1 \\ -3z=6 \rightarrow z=-2 \end{array}$$

$x=4,\ y=1,\ z=-2$ (a) #

b)
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 \end{array}\right] \xrightarrow{R_3-R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right] \rightarrow \begin{array}{l} x+2y-z=1 \rightarrow x=1+z-2y \rightarrow x=5 \\ y+z=1 \rightarrow y=1-z \rightarrow y=-1 \\ 2z=4 \rightarrow z=2 \end{array}$$

$x=5,\ y=-1,\ z=2$ (b) #

c)
$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 3 & 5 & 3 & -1 & 5 \end{array}\right] \xrightarrow{R_3-3R_1} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 2 & 6 & -4 & 2 \end{array}\right]$$

$$\xrightarrow{R_2-\frac{1}{2}R_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 6 & -4 & 2 \end{array}\right] \rightarrow 0x_1+0x_2+0x_3+0x_4=1 \rightarrow \text{not possible}$$

matrix with no solution (c) #

2. Given:

$$x_1 + 2x_2 + 4x_3 = 3$$
$$3x_1 + 8x_2 + 14x_3 = 13$$
$$2x_1 + 6x_2 + 13x_3 = 4$$

Find: solve using LU Decomposition Method

Solution:

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

Let $A = LU$, solve $LUX = B$ for X

Let $UX = Y$, $LY = B$, $UX = Y$

Solve $LY = B$ for Y & $UX = Y$ for X

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 2 & 6 & 13 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = U \; ; \quad \left(L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \right)$$

(Y) $$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} \rightarrow$$

$y_1 = 3$

$3y_1 + y_2 = 13 \rightarrow y_2 = 13 - 3y_1 \rightarrow y_2 = 4$

$2y_1 + y_2 + y_3 = 4 \rightarrow y_3 = 4 - 2y_1 - y_2 \rightarrow y_3 = -6$

(x) $$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} \rightarrow$$

$x_1 + 2x_2 + 4x_3 = 3 \rightarrow x_1 = 3 - 2x_2 - 4x_3 \rightarrow x_1 = 3$

$2x_2 + 2x_3 = 4 \rightarrow x_2 = \frac{4 - 2x_3}{2} \rightarrow x_2 = 4$

$3x_3 = -6 \rightarrow x_3 = -2$

$x_1 = 3,\ x_2 = 4,\ x_3 = -2$ #

Exercise 5 (SVD)

Original image storage compressed to 50% using singular value of 338

Original image storage compressed to 10% using singular value of 68

Original image storage compressed to 5 % using singular value of 34

Exercise 6 (Design for Controllability and Observability)

Given: $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ r & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Find: 1. r values for system to be controllable but not observable

2. r values for system to be observable but not controllable

Solution:

$A = \begin{bmatrix} -3 & 3 \\ r & -4 \end{bmatrix} \longrightarrow n=2, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} -3 & 3 \\ r & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ r \end{bmatrix}$

$P = [B \vdots AB] = \begin{bmatrix} 1 & -3 \\ 0 & r \end{bmatrix}$

$CA = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ r & -4 \end{bmatrix} = \begin{bmatrix} r-3 & -1 \end{bmatrix}$

$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ r-3 & -1 \end{bmatrix}$

$A = \begin{bmatrix} -3 & 3 \\ r & -4 \end{bmatrix}$

$P = \begin{bmatrix} 1 & -3 \\ 0 & r \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 1 \\ r-3 & -1 \end{bmatrix}$

$\because$ system is controllable & observable when $n(A) = \text{rank}(P) = \text{rank}(Q)$

(1) controllable but not observable $\rightarrow n(A) = \text{rank}(P)$; $n(A) \neq \text{rank}(Q)$

$\det\left|\begin{bmatrix} 1 & 1 \\ r-3 & -1 \end{bmatrix}\right| = 0 \rightarrow [1 \times (-1)] - (r-3) = 0 \rightarrow -1 - r + 3 = 0 \rightarrow r = 2$

$n(A) \neq \text{rank}(Q) \rightarrow r = 2$

(2) observable but not controllable $\rightarrow n(A) = \text{rank}(Q)$; $n(A) \neq \text{rank}(P)$

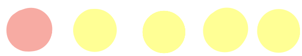
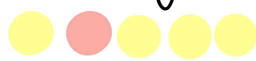
$\det\left|\begin{bmatrix} 1 & -3 \\ 0 & r \end{bmatrix}\right| = 0 \rightarrow r = 0$

$n(A) \neq \text{rank}(P) \rightarrow r = 0$

$\therefore$ For (1) $r = 4$, For (2) $r = 0$ #

Exercise 7 (States Space Representation , Controllability)

Given: 5 red LEDs with tunable brightness

$t=n$  $t=n+1$  $\dot{x} = Ax + Bu$

Find: 1. State equations in matrix form

2. determind system's controllability & explain intuitvely the meaning of controllability in this system

Solution:

(1)

Discrete system, $u(t)$ as brightness command to the left-most LED

State is the brightness of LEDs, output eqauls to the state

Consider t & $t+1$

$$x(t+1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \quad \#(1)$$

(2)

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \rightarrow n=5$$

$$P = [B \vdots AB \vdots A^2B \vdots A^3B \vdots A^4B]$$

$$AB = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AB^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$AB^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$AB^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{rank}(P) = 5$$

$\because$ the P matric is full rank & $= n(A) = 5$

$\therefore$ the system is controllable

Contollability, intuitively, means the system can achieve any desire pattern of brightness over time. #(2)