Homework 3

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24-677 Special Topics: Linear Control Systems

Due: Sept 29, 2023, 11:59 pm. Submit within deadline.

- All assignments will be submitted through Gradescope. Your online version and its timestamp will be used for assessment. Gradescope is a tool licensed by CMU and integrated with Canvas for easy access by students and instructors. When you need to complete a Gradescope assignment, here are a few easy steps you will take to prepare and upload your assignment, as well as to see your assignment status and grades. Take a look at Q&A about Gradescope to understand how to submit and monitor HW grades. https://www.cmu.edu/teaching//gradescope/index.html

- You will need to upload your solution in .pdf to Gradescope (either scanned handwritten version or LaTeX or other tools). If you are required to write Python code, upload the code to Gradescope as well.

- Grading: The score for each question or sub-question is discrete with three outcomes: fully correct (full score), partially correct/unclear (half the score), and totally wrong (zero score).

- Regrading: please review comments from TAs when the grade is posted and make sure no error in grading. If you find a grading error, you need to inform the TA as soon as possible but no later than a week from when your grade is posted. The grade may NOT be corrected after 1 week.

- At the start of every exercise you will see topic(s) on what the given question is about and what will you be learning.

- We advise you to start with the assignment early. All the submissions are to be done before the respective deadlines of each assignment. For information about the late days and scale of your Final Grade, refer to the Syllabus in Canvas.

Exercise 1. *Controllability and Observability* ***(10 points)***

Is the state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x$$

Controllable?**(5 points)** Observable?**(5 points)** Provide your derivation.

Solution:

We have the controllability matrix as

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

which has rank of 3. Since $rank(P) = 3$ it is controllable.
We have the observability matrix as,

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

which has rank of 1. As $rank(Q) < 3$ i.e $rank(Q) < n$ thus it is not observable.

Exercise 2. *Jordan form test* ***(15 points)***

Is the Jordan-form state equation controllable **(7.5 points)** and observable? **(7.5 points)**

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} x$$

Solution:

For controllability, refer to B matrix. We have,

$$\hat{B}^2 \rightarrow \text{rank}(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}) = 3 \text{ and } \hat{B}^1 \rightarrow \text{rank}(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}) = 2$$

Thus, the state equation is controllable.
However, it is not observable since:

$$\hat{C}^2 \rightarrow \text{rank}(\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}) = 2$$

Exercise 3. *Controllability* ***(10 points)***

Recall the Exercise 2 of Homework 2 from last week. Is that system controllable? **(5 points)** Why?
Now lets move the inlet pipe from tank 1 to tank 2, as shown in the figure. Is this system controllable now? **(5 points)** Why?

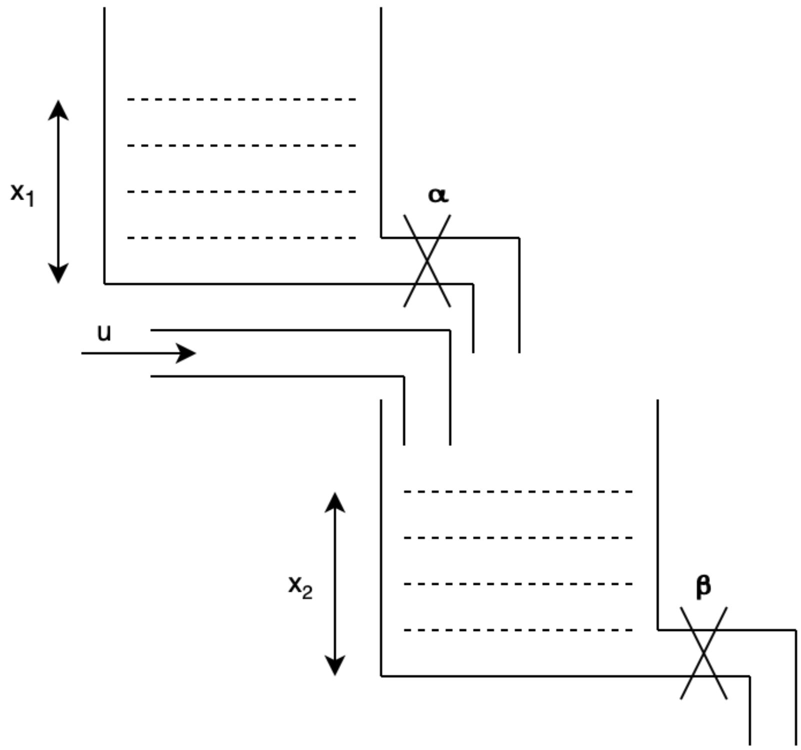

Figure 1: Revised Tank Problem

The system dynamics are

$$\frac{dx_1}{dt} = -\alpha x_1$$
$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2 + u$$

Solution:
The system in Exercise 3 of Homework 2 is controllable, since

$$A = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$rank\left(\begin{bmatrix} B & AB \end{bmatrix}\right) = rank\left(\begin{bmatrix} 1 & -\alpha \\ 0 & \alpha \end{bmatrix}\right) = 2$$

After we move the pipe from tank 1 to tank 2, the system is no longer controllable, since obviously there is no way to influence the water level in tank 1. Also now

$$A = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$rank\left(\begin{bmatrix} B & AB \end{bmatrix}\right) = rank\left(\begin{bmatrix} 0 & 0 \\ 1 & -\beta \end{bmatrix}\right) = 1$$

Exercise 4. *Gauss Elimination and LU Decomposition* ***(20 points)***

1. Solve the following system of linear equations using Gauss Elimination Method

$$
\begin{aligned}
a) x + \ \ y + \ \ z &= 3\\
x + 2y + 3z &= 0\\
x + 3y + 2z &= 3
\end{aligned}
$$

$$
\begin{aligned}
b) \ x + 2y - \ \ z &= 1\\
2x + 5y - \ \ z &= 3\\
x + 3y + 2z &= 6
\end{aligned}
$$

$$
\begin{aligned}
c) \ x_1 + \ \ x_2 - \ \ x_3 + x_4 &= 1\\
2x_1 + 3x_2 + \ \ x_3 \qquad &= 4\\
3x_1 + 5x_2 + 3x_3 - x_4 &= 5
\end{aligned}
$$

2. Solve the following system of linear equations using LU Decomposition Method

$$
\begin{aligned}
x_1 + 2x_2 + \ \ 4x_3 &= 3\\
3x_1 + 8x_2 + 14x_3 &= 13\\
2x_1 + 6x_2 + 13x_3 &= 4
\end{aligned}
$$

Provide your derivation.

Solution:
1.a) Augmented Matrix Form:

$$
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 1 & 2 & 3 & 0\\ 1 & 3 & 2 & 3 \end{array}\right]
$$

Converting Augmented Matrix into Row Echelon Form:

$$
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 1 & 2 & 3 & 0\\ 1 & 3 & 2 & 3 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 2 & -3\\ 1 & 3 & 2 & 3 \end{array}\right] \xrightarrow{R_3 - R_1 \to R_3}
$$

$$
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 2 & -3\\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow{2R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 2 & 4 & -6\\ 0 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3}
$$

$$
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 2 & 4 & -6\\ 0 & 0 & -3 & 6 \end{array}\right]
$$

The equations reduce to:

$$\begin{aligned} x + \; y + \; z &= \;\; 3 \\ 2y + 4z &= -6 \\ -3z &= \;\; 6 \end{aligned}$$

Doing back substitution, we obtain the following solution:

$$x = 4, y = 1, z = -2$$

1.b) Augmented Matrix Form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{array}\right]$$

Converting Augmented Matrix into Row Echelon Form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{array}\right] \xrightarrow[R_1 \Leftrightarrow R_2]{} \left[\begin{array}{ccc|c} 2 & 5 & -1 & 3 \\ 1 & 2 & -1 & 1 \\ 1 & 3 & 2 & 6 \end{array}\right] \xrightarrow[R_1 - 2R_2 \rightarrow R_1]{}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 3 & 2 & 6 \end{array}\right] \xrightarrow[R_2 - R_1 \rightarrow R_2]{} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 2 & 1 & 5 \end{array}\right] \xrightarrow[R_3 - R_2 \longrightarrow R_3]{}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 5 \end{array}\right] \xrightarrow[R_3 - R_1 \rightarrow R_3]{} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 4 \end{array}\right] \frac{R_2 + R_3 \rightarrow R_2}{\frac{R_3}{2} \longrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_1 - R_3 \rightarrow R_1]{} \left[\begin{array}{cccc} 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2 - R_1 \rightarrow R_2]{}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[R_2 \Leftrightarrow R_1]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The equations reduce to which give solution directly:

$$\begin{aligned} x \quad &= \;\; 5 \\ y \;\; &= -1 \\ z &= \;\; 2 \end{aligned}$$

1.c) Augmented Matrix Form:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{array}\right]$$

Converting Augmented Matrix into Row Echelon Form:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{array}\right] \xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 2 & 6 & -4 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_2 \to R_3]{} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

Creating equation of last row, we get,

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = -2$$

As clearly seen, no values of x_i would satisfy this equation and hence the system of equations has no solution.

2) Decomposition of A into L and U:

Say

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = LU$$

$$\text{where } L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

We have LU=A as:

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

Staring from top and comparing matrix element at each location, we obtain the L and U matrices as:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

The given system of linear equations can be written in the form of Ax = B
Replacing A with LU decomposition, we get, LUx = B

In solving Ax = B using LU decomposition, we first assume Ux = y and solve for Ly = B and then solve Ux = y.

We have LUx = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{(Ux = y)}$$

Solving Ly=B

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

Solving it from top to bottom row, we get **y** as

$$y = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

Substituting y in Ly = B, we get:

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

Solving the above equation from the bottom row, we get solution for x as:

$$X = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

Exercise 5. *SVD **(15 points)***

Use SVD to compress the following image to 50%, 10%, and 5% of the original storage required. You will find the image in the Canvas homework folder. For this problem you need to upload code and attached the corresponding compressed images as well as the number of singular value you used for each level of compression. **Note**: you don't need to care about the actual file size after compression since PNG itself is a more complex data structure that contains the grayscale data and other metadata of the image. You only need to care about the storage required in theory, i.e., how many values you need to store the pixel information. For example, for a greyscale image of size $m \times n$, you need $m \times n$ values to store the information if without compression.

You are expected to **attach the images at each compression level in the writeup and also mention the no. of singular values** you used to obtain the compression at each level.

Figure 2: CMU_Grayscale.png

Solution:

Please refer to the code on Google Collaboration Lab https://colab.research.google.com/drive/1ltqoRKIbxEmuYnZsj0nJ2i6uTlDYV82C?usp=sharing. Note that you need to upload the image to the files directory in Google Collaboration Lab, you can do so by selecting the files section in the left sidebar and go to the files section and select upload, after doing so you can upload your local image to Google Collaboration Lab.
Definition of compression rate:
Conventionally, in computer vision, people use (m+n+1)k/(mn) such as the one used in this blog: http://timbaumann.info/svd-image-compression-demo/.
However, this is only an approximation not an exact calculation. The exact formulate to compute the compression ratio is (m+n-k)k/(mn) with derivations recorded in "24677 Lecture" uploaded at Sep 28, 2021. https://cmu.zoom.us/rec/play/qMCACp6yec0avCeTeE5-EtUDuLU445QUzBl8ZAJ6FxcNC7SQHkVVJB9vlQVJ8kRARsyvSQaas2DmXEbM._RlO8fCJQc8oowdR

Exercise 6. *Design for Controllability and Observability* ***(20 points)***

Given the following Linear Time Invariant (LTI) system with a tunable parameter γ,

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ \gamma & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. What values of γ makes the system controllable but not observable? **(10 points)**
2. What values of γ makes the system observable but not controllable? **(10 points)**

Solution:

The controllability matrix for this system is

$$P = [B \quad AB] = \begin{bmatrix} 1 & -3 \\ 0 & \gamma \end{bmatrix} \tag{1}$$

The observability matrix for this system is

$$O = [C \quad CA]^T = \begin{bmatrix} 1 & 1 \\ \gamma - 3 & -1 \end{bmatrix} \tag{2}$$

1. For the system to be controllable but not observable, the matrix P should have full rank while the matrix O should lose rank. For that to happen, the value of γ that makes the matrix O lose rank is $\gamma = 2$ which makes the first and second column of matrix O as identical. The value $\gamma = 2$ does not make the controllability matrix P to lose rank and hence the system is controllable but not observable.
2. For the system to be observable but not controllable, the matrix O should have full rank while the matrix P should lose rank. For that to happen, the value of γ that makes the matrix P lose rank is $\gamma = 0$ which makes the first and second column of matrix P linearly dependant. The value $\gamma = 0$ does not make the observability matrix O to lose rank and hence the system is observable but not controllable.

Exercise 7. *State Space Representation, Controllability* ***(10 points)***

We have an LED strip with 5 red LEDs whose brightnesses we want to set. These LEDs are addressed as a queue: at each time step, we can push a new brightness command between 0 and 255 to the left-most LED. Each of the following LEDs will then take on the brightness previously displayed by the LED immediately to its left.

1. Model the system as a discrete system with input $u(t)$ as the brightness command to the left-most LED. The state to be the brightness of the five LEDs. Output equals to the state. Write out the state equations in matrix form. **(5 points)**
2. Check the system's controllability. Explain intuitively what the controllability means in this system. **(5 points)**

Note: you do NOT need to consider the 0-255 constraints on the input.

Solution:

1. We can use the brightnesses of each LED as our state vector. We can use these values as our state vector since together with the input, they describe everything about our system that we need to know in order to predict what our system will do in the future. Our input is the command to the left-most LED.

   The system is linear as it can be written in the form $x(k+1) = Ax(k) + Bu$. Ordering the LED brightnesses in the state vector from left to right, we get:

$$x_{k+1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(k)$$

   If you chose to put the left-most LED's brightness last in the state vector (so that the LEDs are ordered right to left and the state vector gets flipped upside down), the A matrix gets transposed and the B matrix is flipped upside down.

2. Testing for controllability we see:

$$\begin{bmatrix} A^4B & A^3B & A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   which has full rank. This means that the system is controllable. A system is called controllable if from any initial state, we can reach any final state that we desire at some time in the future. For our LED strip, controllability means that we can display any set of brighnesses that we desire, but it may take a few time steps to get there.