Homework 5

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24-677 Special Topics: Linear Control Systems

Due: Oct 13, 2023, 11:59 pm. Submit within deadline.

- All assignments will be submitted through Gradescope. Your online version and its timestamp will be used for assessment. Gradescope is a tool licensed by CMU and integrated with Canvas for easy access by students and instructors. When you need to complete a Gradescope assignment, here are a few easy steps you will take to prepare and upload your assignment, as well as to see your assignment status and grades. Take a look at Q&A about Gradescope to understand how to submit and monitor HW grades. https://www.cmu.edu/teaching//gradescope/index.html
- You will need to upload your solution in .pdf to Gradescope (either scanned handwritten version or LaTeX or other tools). If you are required to write Python code, upload the code to Gradescope as well.
- Grading: The score for each question or sub-question is discrete with three outcomes: fully correct (full score), partially correct/unclear (half the score), and totally wrong (zero score).
- Regrading: please review comments from TAs when the grade is posted and make sure no error in grading. If you find a grading error, you need to inform the TA as soon as possible but no later than a week from when your grade is posted. The grade may NOT be corrected after 1 week.
- At the start of every exercise you will see topic(s) on what the given question is about and what will you be learning.
- We advise you to start with the assignment early. All the submissions are to be done before the respective deadlines of each assignment. For information about the late days and scale of your Final Grade, refer to the Syllabus in Canvas.

Exercise 1. *Asymptotic stability and Lyapunov stability.* ***(10 points)***

For each of the systems given below, determine whether it is Lyapunov stable, whether it is asymptotic stable.

(a) **(5 points)**

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$

(b) **(5 points)**

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

Solution:

(a) We have $|A - \lambda I| = (1-\lambda)(0.5-\lambda) = 0$, therefore the eigenvalues are 1 and 0.5. It is Lyapunov stable but it is not asymptotically stable.

(b) We have $\lambda_A = -1, -5, -3$, thus it is both Lyapunov stable and asymptotically stable.

Exercise 2. *Stabilizability* ***(20 points)***

Decompose the state equation

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

to a controllable form. Is the reduced state equation observable, stabilizable, detectable?

Solution:
A system is state stabilizable if all unstable modes are state controllable. A system is state detectable if all unstable modes are state observable.

Let $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$

We wish to reduce the system to the standard form with a controllable part and an uncontrolable part. Here,

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

and hence $\operatorname{rank}(P) = q = 2 < 3 = n$
Thus, $\dim(\mathcal{R}\{P\}) = 2$ and a basis $\{v_1, v_2\}$ for the range is found by taking two linearly independent columns of P, say the first two, to obtain:

$$M = \begin{bmatrix} & & 0 \\ v_1 & v_2 & 0 \\ & & 1 \end{bmatrix}$$

where the last column is selected to make M nonsingular. We have,

$$\begin{aligned} \tilde{A} = M^{-1}AM &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} =: \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \\ \tilde{B} = M^{-1}B &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\tilde{C} = CM = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix}$$

The controllable system, therefore is:

$$\dot{x_c} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 3 & 3 \end{bmatrix} x_c$$

The above equations represent the reduced form.
Since the reduced form is always controllable, it is always stabilizable.
Checking Observability:
$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$
rank(Q) = 1<2
Hence the system is not observable.
Eigen values of $\tilde{(A_{11})}$: λ = 0, -1
Corresponding Eigen Vectors:
$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
For $v_1, y = \begin{bmatrix} 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3 \rightarrow$ Observable
For $v_2, y = \begin{bmatrix} 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \rightarrow$ Not Observable
Therefore the system is not observable.
Checking stability of Non observable mode:
λ = -1<0; Hence stable i.s.L.
Therefore, the non observable mode is stable; hence detectable.
Therefore, the reduced system is stabilizable, non observable and detectable.

Exercise 3. *Stability* ***(15 points)***

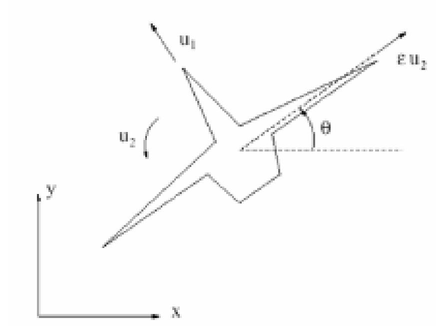

Figure 1: The VTOL. aircraft

The following is a planar model of a Vertical Take-off and Landing (VTOL) aircraft such as Lock-heed's F35 Joint Strike fighter around hover (cf. Figure 1):

$$
\begin{aligned}
m\ddot{x} &= -u_1 \sin\theta + \epsilon u_2 \cos\theta \\
m\ddot{y} &= u_1 \cos\theta + \epsilon u_2 \sin\theta - mg \\
J\ddot{\theta} &= u_2
\end{aligned}
$$

where x, y are the position of the center of mass of the aircraft in the vertical plane and θ is the roll angle of the aircraft. $u1$ and $u2$ are the thrust forces (control inputs). The thrust is generated by a powerful fan and is vectored into two forces $u1$ and $u2$. J is the moment of inertia, and ϵ is a small coupling constant. Determine the stability of the linearized model around the equilibrium solution

$$\tilde{x}(t), \tilde{y}(t), \tilde{\theta}(t) = 0, \tilde{u}_1(t) = mg; \tilde{u}_2(t) = 0.$$

The linearized model should be time invariant. The state $z = [\theta, \dot{x}, \dot{y}, \dot{\theta}]^T$, $u = [u_1, u_2]^T$

Solution:

Choose θ, $\dot{x}$, $\dot{y}$ and $\dot{\theta}$ as the states of the system. The system is

$$f(\theta, \dot{x}, \dot{y}, \dot{\theta}, u) = \begin{bmatrix} \dot{\theta} \\ \frac{-u_1 \sin\theta + \epsilon u_2 \cos\theta}{m} \\ \frac{u_1 \cos\theta + \epsilon u_2 \sin\theta}{m} - g \\ \frac{u_2}{J} \end{bmatrix}$$

the Jacobian is given by

$$J_1 = \frac{\partial f(\theta, \dot{x}, \dot{y}, \dot{\theta}, u)}{\partial z} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{-u_1 \cos\theta - \epsilon u_2 \sin\theta}{m} & 0 & 0 & 0 \\ \frac{-u_1 \sin\theta + \epsilon u_2 \cos\theta}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

On the equilibrium we have $u_1(t) \equiv mg$, $u_2(t) \equiv 0$, $x(t) \equiv 0$, $y(t) \equiv 0$, and $\theta(t) = 0$, and so at the equilibrium the Jacobian evaluates to

$$J_1 = \frac{\partial f(\theta,\dot{x},\dot{y},\dot{\theta},u)}{\partial z}|_{ref} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, the Jacobian of f with respect to u is

$$J_2 = \frac{\partial f(\theta,\dot{x},\dot{y},\dot{\theta},u)}{\partial u} = \begin{bmatrix} 0 & 0 \\ \frac{-\sin\theta}{m} & \frac{\epsilon\cos\theta}{m} \\ \frac{\cos\theta}{m} & \frac{\epsilon\sin\theta}{m} \\ 0 & \frac{1}{J} \end{bmatrix}$$

and so

$$\frac{\partial f(\theta,\dot{x},\dot{y},\dot{\theta},u)}{\partial u}|_{ref} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\epsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix}$$

Therefore, the linearized equation about the equilibrium is

$$\begin{bmatrix} \frac{d}{dt}\delta\theta \\ \frac{d}{dt}\delta\dot{x} \\ \frac{d}{dt}\delta\dot{y} \\ \frac{d}{dt}\delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\theta \\ \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\epsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix}$$

The eigen values of the A matrix of the linearized system are 0.

Doing Jordan decomposition on A, we get J matrix as $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ As we can see, for $\lambda = 0$, there exist terms in J(of the form $t^m e^{R_e t}(\cos(I_m t) + j\sin(I_m t))$ for which m > 0. Hence the system is unstable.

Exercise 4. *Lyapunov's direct method* ***(10 points)***

An LTI system is described by the equations

$$\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$$

Use Lyapunov's direct method to determine the range of variable a for which the system is asymptotically stable. Consider the Lyapunov function,

$$V = x_1^2 + x_2^2$$

Solution:

Let $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$, and the Lyapunov function be $V = x_1^2 + x_2^2$, and we have

$$\dot{V} = 2(ax_1^2 + x_1x_2 - x_2^2)$$

and we have

$$\begin{aligned} ax_1^2 + x_1x_2 - x_2^2 &< 0 \\ a(\frac{x_1}{x_2})^2 + \frac{x_1}{x_2} &< 1 \end{aligned}$$

Let $\dfrac{x_1}{x_2} = k$ and we have

$$\begin{aligned} ak^2 + k - 1 &< 0 \\ 1 + 4a &< 0 \\ a < -0.25 \end{aligned}$$

thus we have $a \in (-\infty, -0.25)$

Exercise 5. *Stability of Non-Linear Systems* ***(20 points)***

Consider the following system:

$$\dot{x_1} = x_2 - x_1 x_2^2$$
$$\dot{x_2} = -x_1^3$$

(a) Is the approximated linearized system stable? Based on Lyapunov's Indirect method, is the system stable? [Hint: can you get a firm answer in this case?] **(5 points)**

(b) Based on Lyapunov's Direct method? **(5 points)** Consider the Lyapunov function:

$$V(x_1, x_2) = x_1^4 + 2x_2^2$$

(c) Plot the Phase Portrait plot of the original system and linearized system in a. **(5 points)**. Submit the code to Gradescope.

(d) Generate a 3D plot showing the variation of $\dot{V}$ with respect to x_1 and x_2. **(5 points)** [Hint: Use Axes3D python library]. Submit the code to Gradescope.

Note: For (c) and (d), include the code along with the plot in the pdf to be submitted. No need to submit .py file.

Solution:

1. The equilibrium point of the system is $[0, 0]$.
   Computing the Jacobian Matrix:

   $$J = \begin{bmatrix} -x_2^2 & 1 - 2x_1 x_2 \\ -3x_1^2 & 0 \end{bmatrix}$$

   The linearized system:

   $$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

   Hence the linearized system is unstable. Based on Lyapunov's Indirect method, the system's stability is inconclusive as we have eigen value(s) at 0.

2. If we show that $\dot{V} < 0$, then the system is stable.

   $$\begin{aligned} \dot{V} &= 4x_1^3 \dot{x}_1 + 4x_2 \dot{x}_2 \\ &= 4x_1^3(x_2 - x_1 x_2^2) + 4x_2(-x_1^3) \\ &= -4x_1^4 x_2^2 \end{aligned}$$

   Since $-4x_1^4 x_2^2 < 0$, the energy of the system is decreasing and hence the system is stable, based on Lyapunov's direct method.

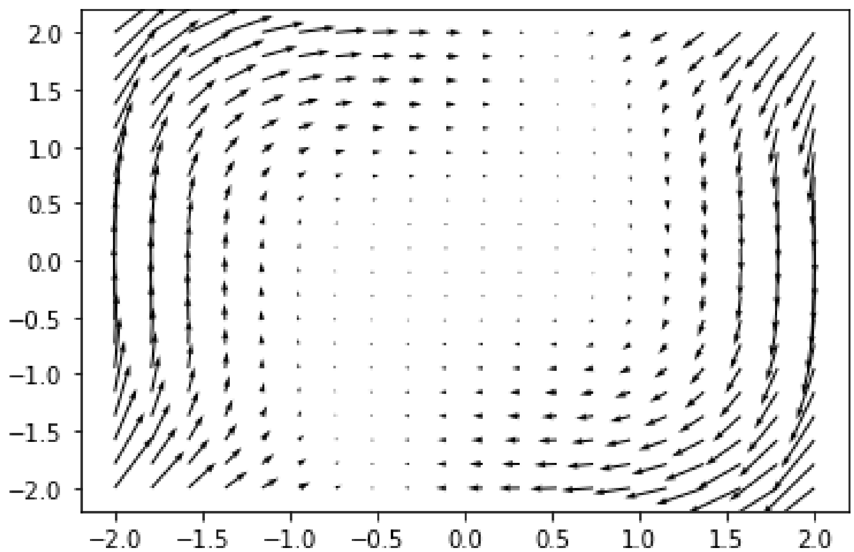

Figure 2: Phase Portrait Plot of Original System

3.
```
import numpy as np
import matplotlib.pyplot as plt

# Defining the range of inputs x1,x2
x1=np.linspace(-2,2,20)
x2=np.linspace(-2,2,20)

# Formlating meshgrid from x1,x2
X1,X2=np.meshgrid(x1,x2)

dot_X1=X2-X1*X2**2
dot_X2=-X1**3
# Plotting the phase plot
plt.figure()
plt.quiver(X1,X2,dot_X1,dot_X2)
plt.show()

# Equations of linearized system:
#                  x1_dot=x2
#                  x2_dot=0
dot_X1=X2
dot_X2=np.zeros(X1.shape)
# Plotting the phase plot
plt.figure()
plt.quiver(X1,X2,dot_X1,dot_X2)
plt.show()
```

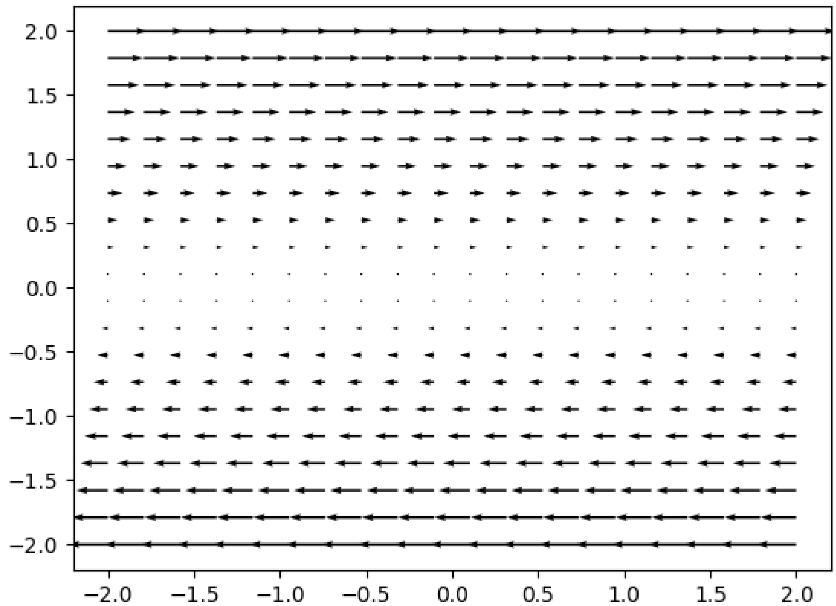

Figure 3: Phase Portrait Plot of Linearized System

4.
```python
import numpy as np
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
from matplotlib import cm

# Defining a range of inputs x1 and x2
X1=np.arange(-100,100)
X2=np.arange(-100,100)

# Computing a meshgrid from the 2 inputs
x1,x2=np.meshgrid(X1,X2)

# Computing V_dot=-4*(x1^4)*(x2^2)
v=-4*np.power(x1,4)*np.power(x2,2)

# Plottting v,x1,x2
fig=plt.figure()
ax=fig.gca(projection='3d')

ax.plot_surface(x1,x2,v,cmap=cm.coolwarm, linewidth=0, antialiased=False)
plt.show()
```

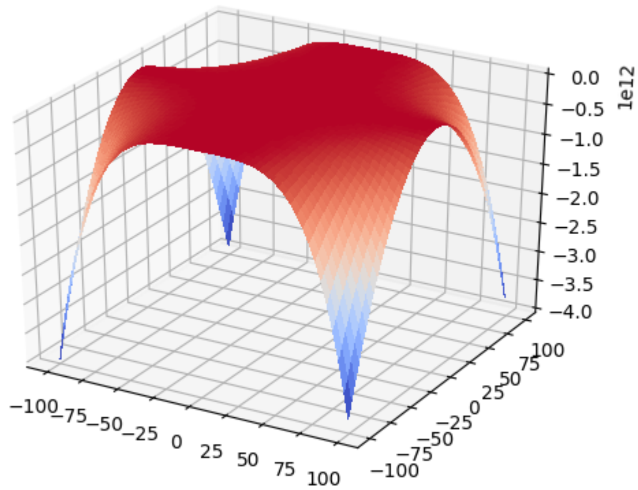

Figure 4: $\dot{V}$ vs x_1 and x_2

Exercise 6. *BIBO Stability* ***(10 points)***

For each of the systems given below, determine whether it is BIBO stable.

(a) **(5 points)**

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 5 & 5 \end{bmatrix} x(k)$$

(b) **(5 points)**

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

Solution:

(a)

$G_D(z) = C(zI - A)^{-1}B + D$

For the given system, we get $G_D(z) = 0$

Since the transfer function is zero, the system is stable.

(b) $H(s) = C(sI - A)^{-1}B$

$$H(s) = \begin{bmatrix} 0 & 0 \\ \dfrac{1}{s+3} & 0 \end{bmatrix}$$

We have the poles at $s = -3$, thus it is BIBO stable.

Exercise 7. *BIBO Stability* ***(15 points)***

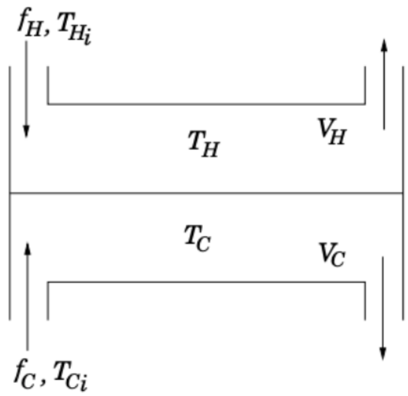

Figure 5: A simple heat exchanger

Consider a simplified model for a heat exchanger shown in Figure 5, in which f_C and f_H are the flows (assumed constant) of cold water and hot water, T_H and T_C represent the temperatures in the hot and cold compartments, respectively, T_{Hi} and T_{Ci} denote the temperature of the hot and cold inflow, respectively, and V_H and V_C are the volumes of hot and cold water. The temperatures in both compartments evolve according to:

$$V_C \frac{dT_C}{dt} = f_C(T_{Ci} - T_C) + \beta(T_H - T_C) \tag{1}$$

$$V_H \frac{dT_H}{dt} = f_H(T_{Hi} - T_H) + \beta(T_C - T_H) \tag{2}$$

Let the inputs to the system be $u_1 = T_{Ci}$, $u_2 = T_{Hi}$, the outputs are $y1 = T_C$ and $y2 = T_H$, and assume that $f_C = f_H = 0.1(m^3/min)$, $\beta = 0.2(m^3/min)$ and $V_H = V_C = 1(m^3)$.

1. Write the state space and output equations for this system. **(5 points)**
2. In the absence of any input, determine $y_1(t)$ and $y_2(t)$. **(5 points)**
3. Is the system BIBO stable? Show why or why not. **(5 points)**

Solution:

1. Choosing $x_1 = T_C$ and $x_2 = T_H$ we have $\dot{x} = Ax + Bu$ and $y = x$, where

$$A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}, B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

after plugging in numerical values for the constants.

2. We find that the eigenvalues are $\lambda_1 = -0.1, \lambda_2 = -0.5$. Recall:
   $f(\lambda) = g(\beta, \lambda) = \beta_{n-1}\lambda^{n-1} + \ldots + \beta_1\lambda + \beta_0$

   $$\begin{array}{l} f(\lambda) = e^{\lambda_1 t} = \beta_1\lambda_1 + \beta_0 \\ f(\lambda) = e^{\lambda_2 t} = \beta_1\lambda_2 + \beta_0 \end{array} \Rightarrow \begin{array}{l} e^{-0.1t} = -0.1\beta_1 + \beta_0 \\ e^{-0.5t} = -0.5\beta_1 + \beta_0 \end{array} \Rightarrow \left\{ \begin{array}{l} \beta_0 = 1.25e^{-0.1t} - 0.25e^{-0.5t} \\ \beta_1 = 2.5e^{-0.1t} - 2.5e^{-0.5t} \end{array} \right.$$

   $$f(A) = e^{At} = \beta_1 A + \beta_0 I = \begin{pmatrix} 0.5e^{-0.1t} + 0.5e^{-0.5t} & 0.5e^{-0.1t} - 0.5e^{-0.5t} \\ 0.5e^{-0.1t} - 0.5e^{-0.5t} & 0.5e^{-0.1t} + 0.5e^{-0.5t} \end{pmatrix}$$

   We can then calculate $y(t)$ with

   $$\begin{aligned} y(t) &= Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t) \\ &= Ce^{A(t-t_0)}x(t_0) \quad \text{(since in the absence of any input)} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 0.5e^{-0.1t} + 0.5e^{-0.5t} & 0.5e^{-0.1t} - 0.5e^{-0.5t} \\ 0.5e^{-0.1t} - 0.5e^{-0.5t} & 0.5e^{-0.1t} + 0.5e^{-0.5t} \end{pmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \begin{bmatrix} (0.5e^{-0.1t} + 0.5e^{-0.5t})x_1(0) + (0.5e^{-0.1t} - 0.5e^{-0.5t})x_2(0) \\ (0.5e^{-0.1t} - 0.5e^{-0.5t})x_1(0) + (0.5e^{-0.1t} + 0.5e^{-0.5t})x_2(0) \end{bmatrix} \end{aligned}$$

3. The system has both eigenvalues in the open left half plane, and thus is internally exponentially stable. Therefore it is also BIBO stable.