Exercise 1 (Asymptotic Stability and Lyapunov stability)

Given:

a) $x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$

b) $\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$

Find: determine Lyapunov stable, asymptotic stable

Solution:

(a) Condition: $\bar{x} = 0$ for $x(k+1) = Ax(k)$ is stable

all eigenvalues of A satisfy $|\lambda_i| \leq 1$ and all $|\lambda_i| = 1$ are non-defective

$\forall \lambda_i,\ r_i < 1 \Rightarrow$ Asymptotic Stable ; $\exists \lambda_i,\ r_i = 1\ \&\ m = 0 \Rightarrow$ stable i.s.L

$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ -0.5 & 0.5-\lambda \end{bmatrix}\right) = (1-\lambda)(0.5-\lambda) \rightarrow \lambda_1 = 1,\ \lambda_2 = 0.5$

Check $\lambda_1 = 1$ is defective or not $\lambda_1 = 1,\ v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; $\lambda_2 = 0.5,\ v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$ not defective $\rightarrow$ stable

$\because$ $\lambda_2 = 0.5 < 1$ is asymptotic stable, but $\lambda_1 = 1$ (not defective) is only stable i.s.L

$\therefore$ $x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$ system overall is stable i.s.L # (a)

(b) Condition: $\bar{x}$ for $\dot{x} = Ax$ is stable $\Leftrightarrow$

all eigenvalues of A have non-positive real parts, and those with zero real parts are n.d.

$\forall \lambda_i,\ Re < 0 \Rightarrow$ Asymptotic stable ; $\exists \lambda_i,\ Re = 0,\ m = 0 \Rightarrow$ stable i.s.L.

$\det(A - \lambda I) = \det\left(\begin{bmatrix} -7-\lambda & -2 & 6 \\ 2 & -3-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{bmatrix}\right) \rightarrow \lambda_1 = -1,\ \lambda_2 = -3,\ \lambda_3 = -5 \rightarrow$ stable

$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} - c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$\because$ all eigVs ($\lambda_1 = -1, \lambda_2 = -3, \lambda_3 = -5$) are negative poles

$\therefore$ $\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$ is asymptotic stable #(b)

Exercise 2. (Stabilizability)

Given: $\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} u$, $\quad y = [1 \ 1 \ 1] x$

Find: (a) decompose the state equation to a controllable form

(b) Is the reduce state observable, stabilizable, detectable ?

Solution:

(a) $P = [B \ AB \ A^2B]$, $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, $AB = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$

$A^2B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{ref} P = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ rank(P)=2

rank(P) = 2 < n = 3 $\rightarrow$ not controllable

$\det(\lambda I - A) = 0 \rightarrow \det\left(\begin{bmatrix} \lambda - 0 & 1 & -1 \\ -1 & \lambda+2 & -1 \\ 0 & -1 & \lambda+1 \end{bmatrix}\right) = 0 \rightarrow \lambda(\lambda+1)(\lambda+2) = 0$ $\quad \lambda_1 = -2$, $\lambda_2 = -1$, $\lambda_3 = 0$

$V_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, $M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$, $\hat{A} = M^{-1}AM = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$\hat{B} = M^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\hat{C} = CM = [1 \ 1 \ 1] \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = [3 \ 3 \ 1]$

$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u$, $\quad y = [3 \ 3 \ 1] \hat{x} + [0] u$

$\dot{\hat{x}}_c = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \hat{x}_c + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u$, $\quad y = [C_c \ C_{\bar{c}}] \hat{x}_c + Du$

$$\dot{\hat{x}}_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \ , \quad y \begin{bmatrix} 3 & 3 \end{bmatrix} \hat{x}_c + [0] u$$

Check

$$\hat{P} = [\hat{B} \ \hat{A}\hat{B}] \ , \quad \hat{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ , \quad \hat{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \ , \quad \hat{A}\hat{B} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\hat{P} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \rightarrow \text{rank}\,\hat{P} = 2 = n \rightarrow \text{controllable}$$

$\because$ rank $\hat{P} = n = 2$

$\therefore$ $\dot{\hat{x}}_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \ , \ y \begin{bmatrix} 3 & 3 \end{bmatrix} \hat{x}_c + [0] u$ is a reduced controllable form

#(a)

(b)

Observable

$$\hat{Q} = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \end{bmatrix} \ , \quad \hat{C} = [3 \ 3] \ , \quad \hat{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \ , \quad \hat{C}\hat{A} = [3 \ 3] \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = [0 \ 0]$$

$$\hat{Q} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{ref}} \hat{Q} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \text{rank}\,\hat{Q} = 1 < n \rightarrow \text{not observable}$$

$\because$ rank $\hat{Q} = 1 < n$ $\therefore$ $\dot{\hat{x}}_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u, \ y \begin{bmatrix} 3 & 3 \end{bmatrix} \hat{x}_c + [0] u$ is not observable

#(b)

Stabilizable

Condition: A system is stabilizable if all unstable states are Lyapunov stable

$\because$ Since the system is already in reduced controllable form

$\therefore$ $\dot{\hat{x}}_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \ , \ y \begin{bmatrix} 3 & 3 \end{bmatrix} \hat{x}_c + [0] u$ is stabilizable #(b)

Detectable

Condition: A system is detectable if the unobservable states are Lyapunov stable

$A_C = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, $\det(\lambda I - A) = \begin{vmatrix} s & -1 \\ 0 & s+1 \end{vmatrix}$ $\rightarrow$ $\lambda_1 = -1 \rightarrow$ asymptotically stable

$\lambda_2 = 0 \rightarrow$ stable i.s.L

$\because$ Since $\lambda_1 = -1$ is asymptotically stable & $\lambda_2 = 0$ is stable i.s.L

$\therefore \dot{\hat{x}}_c = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \hat{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$, $y [\,3 \;\; 3\,] \hat{x}_c + [0] u$ is detectable # (b)

Exercise 3 (Stability)

Given: $m\ddot{x} = -u_1 \sin\theta + \epsilon u_2 \cos\theta$, $m\ddot{y} = u_1\cos\theta + \epsilon u_2 \sin\theta - mg$, $J\ddot{\theta} = u_2$

$\hat{x}(t), \hat{y}(t), \hat{\theta}(t) = 0$, $\tilde{u}_1(t) = mg$, $\tilde{u}_2(t) = 0$

(States) $z = [\theta, \dot{x}, \dot{y}, \dot{\theta}]^T$, $u = [u_1, u_2]^T$

Find: determine the stability of the linearized model around the equilibrium solution

Solution:

$$z = \begin{bmatrix} \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \dot{z} = \begin{bmatrix} \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix},$$

$\hat{x}(t) = \hat{y}(t) = \hat{\theta}(t) = 0$, $\hat{u}_1(t) = mg$, $\hat{u}_2(t) = 0$

$m\ddot{x} = -u_1 \sin\theta + \epsilon u_2 \cos\theta \longrightarrow \ddot{x} = \frac{\epsilon u_2}{m}$

$m\ddot{y} = u_1 \cos\theta + \epsilon u_2 \sin\theta - mg \rightarrow \ddot{y} = \frac{u_1}{m} - g$

$J\ddot{\theta} = u_2 \rightarrow \ddot{\theta} = \frac{u_2}{J}$

$$\dot{z} = \begin{bmatrix} z_4 \\ -\frac{u_1}{m}\sin z_1 + \frac{\epsilon u_2}{m}\cos z_1 \\ \frac{u_1}{m}\cos z_1 + \frac{\epsilon u_2}{m}\sin z_1 - g \\ \frac{u_2}{J} \end{bmatrix}, \quad f(\hat{z}, \hat{u}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\epsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \partial z + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\epsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix} \partial u$$

$\because$ Since $\lambda = 0$ & $m > 0$

$\therefore$ This is an unstable system

Exercise 4 (Lyapunov's direct method)

Given: $\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$, $V = x_1^2 + x_2^2$

Find: use Lyapunov's direct method to determine the range of a for which system is asymptotically stable.

Solution:

Conditions: The origin of $\dot{x} = f(x)$ is asymptotically stable if

(1) $V(x) = 0$ if and only if $x = 0$

(2) $V(x) > 0$ if and only if $x \neq 0$

(3) $\dot{V}(x) = \frac{d}{dt} V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \nabla V \cdot f(x) < 0$ for all values of $x \neq 0$

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \dot{x}_1 = a x_1$, $\dot{x}_2 = x_1 - x_2$, $V = x_1^2 + x_2^2$

Check (1) & (2) $\rightarrow$ clearly $V(x) = 0$ if and only if $x = 0$ & $V(x) > 0$ if and only if $x \neq 0$

Check (3) $\rightarrow \dot{V}(x) = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2 \rightarrow$ plug in given $\dot{x}_1$ & $\dot{x}_2 \rightarrow \dot{V}(x) = 2x_1(ax_1) + 2x_2(x_1 - x_2)$

$\dot{V}(x) = 2ax_1^2 + 2x_1x_2 - 2x_2^2 < 0$

Analyze $2ax_1^2 + 2x_1x_2 - 2x_2^2 < 0 \rightarrow ax_1^2 + x_1x_2 - x_2^2 < 0$

$\rightarrow ax_1^2 < x_2^2 - x_1x_2 \rightarrow a < \frac{x_2^2 - x_1x_2}{x_1^2}$

$\because$ the reasoning in Condition (1), (2), (3)

$\therefore a < \frac{x_2^2 - x_1x_2}{x_1^2}$ is the range when system is asymptotically stable #

Exercise 5 (Stability of Non-Linear Systems)

Given: $\dot{x}_1 = x_2 - x_1 x_2^2$, $\dot{x}_2 = -x_1^3$

Find: (a) is the approximated linearized system stable? Based on Lyapunov's Indirect Method, is the system stable?

(b) based on Lyapunov's Direct Method? $V(x_1, x_2) = x_1^4 + 2x_2^2$

(c) plot the Phase Portrait plot of the original system & linearized system in (a)

(d) generate 3D plot showing the variation of $\dot{V}$ with respect to x_1 & x_2

Solution:

(a) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 x_2^2 \\ -x_1^3 \end{bmatrix}$$

(1) Find equilibria

$$\begin{cases} x_2 - x_1 x_2^2 = 0 & \rightarrow x_2 = 0 \\ -x_1^3 = 0 & \rightarrow x_1 = 0 \end{cases}$$

(2) Linearization

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 - x_2^2 & 1 - 2x_1 x_2 \\ -3x_1^2 & 0 \end{bmatrix}$$

Plug in $x_1 = x_2 = 0$ $(0,0)$: $\delta \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \delta x$

The equilibria points are at origin $(0,0)$, it's hard to get a firm answer on system stability # (a)

(b) $V(x_1, x_2) = x_1^4 + 2x_2^2$

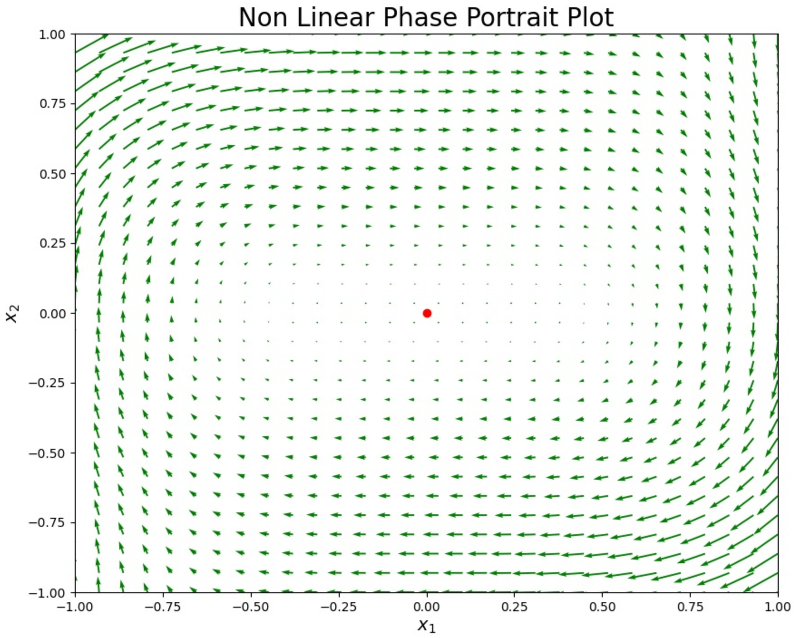
$\dot{V}(x_1, x_2) = 4x_1^3 \dot{x}_1 + 4x_2 \dot{x}_2$ plug $\rightarrow$ $(\dot{x}_1 = x_2 - x_1 x_2^2 , \ \dot{x}_2 = -x_1^3)$

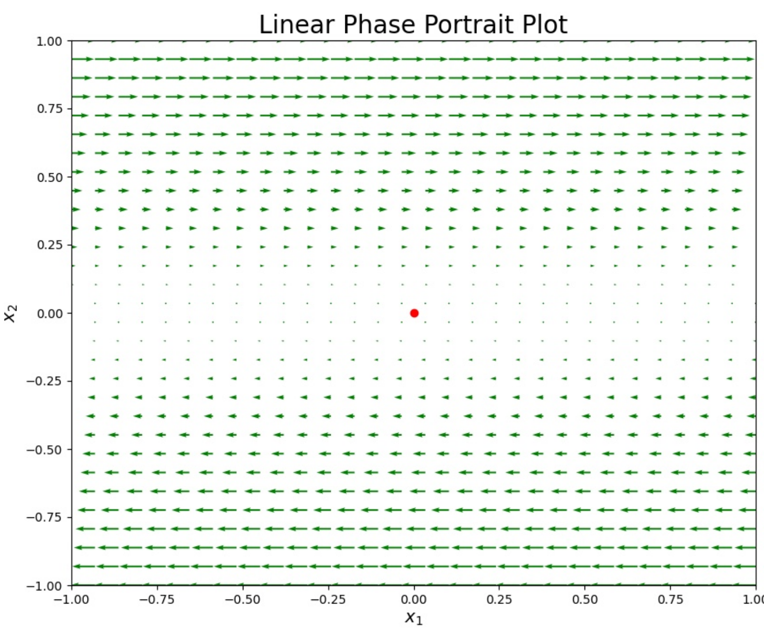
$\dot{V}(x_1, x_2) = 4x_1^3 (x_2 - x_1 x_2^2) + 4x_2(-x_1^3)$

$\dot{V}(x_1, x_2) = \cancel{4x_1^3} x_2 - 4x_1^4 x_2^2 - \cancel{4x_1^3} x_2 \Rightarrow \dot{V}(x_1, x_2) = -4x_1^4 x_2^2 < 0$

$\because \dot{V}(x_1, x_2) = -4x_1^4 x_2^2 < 0$ $\therefore$ the linearized system is stable. # (b)

(c)





Figures above is the phase portrait plot of the original system & linearized system in (a)

(d)

3D Variation Plot

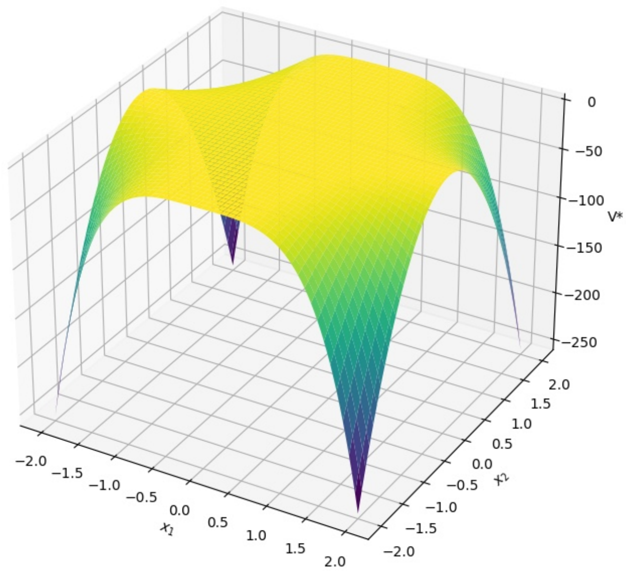


Figure above is a 3D plot showing the variation of $\dot{V}$ with respect to x_1 & x_2

Exercise 6 (BIBO Stability)

Given:

(a) $x(k+1)=\begin{bmatrix}1 & 0\\ -0.5 & 0.5\end{bmatrix}x(k)+\begin{bmatrix}1\\ -1\end{bmatrix}u(k)$, $y(k)=[5\ \ 5]\,x(k)$

(b) $\dot{x}=\begin{bmatrix}-7 & -2 & 6\\ 2 & -3 & -2\\ -2 & -2 & 1\end{bmatrix}x+\begin{bmatrix}1 & 1\\ 1 & -1\\ 1 & 0\end{bmatrix}u$, $y=\begin{bmatrix}-1 & -1 & 2\\ 1 & 1 & -1\end{bmatrix}x$

Find: determine whether it is BIBO stable.

Solution:

(a) Condition: let $G_D(z)=C(zI-A)^{-1}B+D$

A DT LTI system is BIBO stable $\Leftrightarrow$ every pole of every G_{Dij} is inside the unit circle

$A=\begin{bmatrix}1 & 0\\ -0.5 & 0.5\end{bmatrix}$, $B=\begin{bmatrix}1\\ -1\end{bmatrix}$, $C=[5\ \ 5]$, $D=0$

$G_D(z)=[5\ \ 5]\begin{bmatrix}z-1 & 0\\ 0.5 & z-0.5\end{bmatrix}^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}$

$\rightarrow G_D(z)=[5\ \ 5]\begin{bmatrix}\frac{z-0.5}{z^2-1.5z+0.5} & 0\\ \frac{0.5}{-z^2-1.5z+0.5} & \frac{z-1}{z^2-1.5z+0.5}\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}$

$\rightarrow G_D(z)=\begin{bmatrix}\frac{5(z-0.5)}{z^2-1.5z+0.5}-\frac{2.5}{z^2-1.5z+0.5} & 0+\frac{5(z-1)}{z^2-1.5z+0.5}\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}$

$\rightarrow G_D(z)=\dfrac{0}{z^2-1.5z+0.5}=0$

$\because$ The system has no poles with zero as final state

$\therefore$ $x(k+1)=\begin{bmatrix}1 & 0\\ -0.5 & 0.5\end{bmatrix}x(k)+\begin{bmatrix}1\\ -1\end{bmatrix}u(k)$, $y(k)=[5\ \ 5]\,x(k)$ is BIBO stable

(b) condition: let $G_c(s) = C(sI-A)^{-1}B + D$

A DT LTI system is BIBO stable $\Leftrightarrow$ every pole of every G_{cij} have negative real parts

$$A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$D = 0$

$$G_c(s) = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s+7 & 2 & -6 \\ -2 & s+3 & 2 \\ 2 & 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\to G_c(s) = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{s^2+2s-7}{(s+1)(s+3)(s+5)} & \frac{-2}{(s+1)(s+3)} & \frac{s+5}{(s+1)(s+3)} - \frac{1}{s+5} \\ \frac{2}{(s+3)(s+5)} & \frac{1}{s+3} & \frac{1}{s+5} - \frac{1}{s+3} \\ \frac{-2}{(s+1)(s+3)} & \frac{-2}{(s+1)(s+3)} & \frac{s+5}{(s+1)(s+3)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

From the above matrix we can observe that the poles will be $\lambda_1 = -5$, $\lambda_2 = -3$, $\lambda_3 = -1$

$\because$ All the poles are < 0 $(\lambda_1 = -5, \lambda_2 = -3, \lambda_3 = -1)$

$$\therefore \dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u, \quad y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

is BIBO Stable # (b)

Exercise 7 (BIBO Stability)

Given: $V_C \frac{dT_C}{dt} = f_C (T_{Ci} - T_C) + \beta (T_H - T_C)$, $V_H \frac{dT_H}{dt} = f_H (T_{Hi} - T_H) + \beta (T_C - T_H)$

inputs $u_1 = T_{Ci}$, $u_2 = T_{Hi}$, outputs $y_1 = T_C$, $y_2 = T_H$

assume $f_C = f_H = 0.1 \left[\frac{m^3}{min}\right]$, $\beta = 0.2 \left[\frac{m^3}{min}\right]$, $V_H = V_C = 1 \, [m^3]$

Find: (1) write the state space & output equations for the system

(2) in the absence of input, determine $y_1(t)$ & $y_2(t)$

(3) is the system BIBO stable? Show why or why not

Solution:

(1) $\rightarrow$ plug given #s into $V_C \frac{dT_C}{dt}$ & $V_H \frac{dT_H}{dt}$

$V_C \frac{dT_C}{dt} = f_C (T_{Ci} - T_C) + \beta (T_H - T_C) \rightarrow \frac{dT_C}{dt} = 0.1 (u_1 - y_1) + 0.2 (y_2 - y_1)$

$V_H \frac{dT_H}{dt} = f_H (T_{Hi} - T_H) + \beta (T_C - T_H) \rightarrow \frac{dT_H}{dt} = 0.1 (u_2 - y_2) + 0.2 (y_1 - y_2)$

let $\dot{x}_1 = \frac{dT_C}{dt}$, $\dot{x}_2 = \frac{dT_H}{dt}$, $y_1 = x_1$, $y_2 = x_2$

Rewrite both Eqs. $\rightarrow$ $\dot{x}_1 = 0.1 (u_1 - x_1) + 0.2 (x_2 - x_1)$, $\dot{x}_2 = 0.1 (u_2 - x_2) + 0.2 (x_1 - x_2)$

$\dot{x}_1 = 0.1 u_1 - 0.1 x_1 + 0.2 x_2 - 0.2 x_1 \rightarrow \dot{x}_1 = -0.3 x_1 + 0.2 x_2 + 0.1 u_1$

$\dot{x}_2 = 0.1 u_2 - 0.1 x_2 + 0.2 x_1 - 0.2 x_2 \rightarrow \dot{x}_2 = 0.2 x_1 - 0.3 x_2 + 0.1 u_2$

Rewrite both Eqs in S.S.

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0.1 \;\; 0.1] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ # (1)

Cancel

(2) $y(t) = Ce^{A(t-t_0)} x(t_0) + C \cancel{\int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau + Du(t)}$, disregard input [B]

$y(t) = Ce^{At} x(0)$, $y_1 = x_1$, $y_2 = x_2$

$A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}$, $\lambda_1 = -0.5$, $\lambda_2 = -0.1$, Apply C.H. (n=2) $\rightarrow f(\lambda) = B_1 \lambda + B_0$

$(\lambda_1 = -0.5)$ $f(\lambda_1) = e^{\lambda_1 t} = B_1 \lambda_1 + B_0 \rightarrow e^{-0.5t} = -0.5 B_1 + B_0$ (1)

$(\lambda_2 = -0.1)$ $f(\lambda_2) = e^{\lambda_2 t} = B_1 \lambda_2 + B_0 \rightarrow e^{-0.1t} = -0.1 B_1 + B_0$ (2)

Eqn (2) − (1)

$$\begin{aligned} e^{-0.1t} &= -0.1B_1 + B_0 \\ -)\ e^{-0.5t} &= -0.5B_1 + B_0 \\ \hline e^{-0.1t} - e^{-0.5t} &= 0.4B_1 \end{aligned} \quad \rightarrow \quad B_1 = \frac{e^{-0.1t} - e^{-0.5t}}{0.4}$$

Plug B_1 into (2)

$$B_0 = e^{-0.1t} + 0.1B_1 \quad \rightarrow \quad B_0 = e^{-0.1t} + 0.25\left(e^{-0.1t} - e^{-0.5t}\right)$$

$$P(A) = e^{At} = B_1A + B_0I = \frac{e^{-0.1t} - e^{-0.5t}}{0.4}\begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix} + \left[e^{-0.1t} + 0.25\left(e^{-0.1t} - e^{-0.5t}\right)\right]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} -0.75\left(e^{-0.1t} - e^{-0.5t}\right) + e^{-0.1t} + 0.25\left(e^{-0.1t} - e^{-0.5t}\right) & 0.5\left(e^{-0.1t} - e^{-0.5t}\right) \\ 0.5\left(e^{-0.1t} - e^{-0.5t}\right) & -0.75\left(e^{-0.1t} - e^{-0.5t}\right) + e^{-0.1t} + 0.25\left(e^{-0.1t} - e^{-0.5t}\right) \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 0.5e^{-0.1t} + 0.5e^{-0.5t} & 0.5\left(e^{-0.1t} - e^{-0.5t}\right) \\ 0.5\left(e^{-0.1t} - e^{-0.5t}\right) & 0.5e^{-0.1t} + 0.5e^{-0.5t} \end{bmatrix} \rightarrow e^{At} = \begin{bmatrix} \frac{e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} & \frac{-e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} \\ \frac{-e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} & \frac{e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} \end{bmatrix}$$

$y(t) = Ce^{At}x(0)$

$x(0) = [x_1(0), x_2(0)]^T$

$$\rightarrow \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} & \frac{-e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} \\ \frac{-e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} & \frac{e^{-\frac{t}{2}} + e^{-\frac{t}{10}}}{2} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$y_1(t) = \frac{x_1(0)\left[e^{-\frac{t}{2}} + e^{-\frac{t}{10}}\right] + x_2(0)\left[e^{-\frac{t}{10}} - e^{-\frac{t}{2}}\right]}{2}, \quad y_2(t) = \frac{x_1(0)\left[e^{-\frac{t}{10}} - e^{-\frac{t}{2}}\right] + x_2(0)\left[e^{-\frac{t}{2}} + e^{-\frac{t}{10}}\right]}{2}$$

\# (2)

(3) ∵ $\lambda_1 = -0.5$, $\lambda_2 = -0.1$ for the S.S system, two negative poles

∴ The system is BIBO Stable # (3)

```python
# 24-677 Linear Control Systems
# Homework 5 Exercise 5
# Ryan Wu (weihuanw)

import matplotlib.pyplot as plt
from scipy.integrate import odeint
from mpl_toolkits.mplot3d import Axes3D
import numpy as np

# define non linear state space function
def stateSpace(x, t):
    d_dot = [x[1] - x[0] * (x[1] * x[1]), -x[0] * x[0] * x[0]]
    return d_dot

# # define linear state space function
# def stateSpace(x, t):
#     A = np.array([[0, 1], [0, 0]])
#     return np.dot(A, x)

# grid setup
x0 = np.linspace(-1, 1, 30)
x1 = np.linspace(-1, 1, 30)
X0, X1 = np.meshgrid(x0, x1)

dX0 = np.zeros(X0.shape)
dX1 = np.zeros(X1.shape)

shape1, shape2 = X1.shape

# looping through each index
for indexShape1 in range(shape1):
    for indexShape2 in range(shape2):
        dxdt = stateSpace([X0[indexShape1, indexShape2], X1[indexShape1, indexShape2]], 0)
        dX0[indexShape1, indexShape2] = dxdt[0]
        dX1[indexShape1, indexShape2] = dxdt[1]

# phase trajectory lines
initialState = np.array([0, 0])
simulationStep = np.linspace(0, 2, 200)
```

```python
finalState = odeint(stateSpace, initialState, simulationStep)



# define three dimension function
def threeDimension(x1_3d, x2_3d):
    v_dot = -4 * x1_3d**4 * x2_3d**2
    return v_dot

x1_3d = np.linspace(-2, 2, 100)
x2_3d = np.linspace(-2, 2, 100)

x1_3d, x2_3d = np.meshgrid(x1_3d, x2_3d)
v_dot = threeDimension(x1_3d, x2_3d)

# plot and figure features (Phase Portraits)
plt.figure(figsize=(10, 8))
plt.quiver(X0, X1, dX0, dX1, color='g')
plt.plot(0, 0, marker='o', color='r')
plt.plot(finalState[:, 0], finalState[:, 1])
plt.xlim(-1, 1)
plt.ylim(-1, 1)
plt.title('Non Linear Phase Portrait Plot', fontsize=20)
# plt.title('Linear Phase Portrait Plot', fontsize=20) # for linear case
plt.xlabel('$x_{1}$', fontsize=14)
plt.ylabel('$x_{2}$', fontsize=14)
plt.savefig('NonlinerPhasePortraitPlot.png')
# plt.savefig('linearPhasePortraitPlot.png') # for linear case
plt.show()

# plot and figure features (3 Dimensional)
fig = plt.figure(figsize=(10, 8))
d_plot = fig.add_subplot(111, projection='3d')
d_plot.plot_surface(x1_3d, x2_3d, v_dot, cmap='viridis')
d_plot.set_xlabel('$x_{1}$')
d_plot.set_ylabel('$x_{2}$')
```

```
d_plot.set_zlabel('V*')
d_plot.set_title('3D Variation Plot')
plt.savefig('3D plot.png')
plt.show()

```