Problem 1

Determine true / false of each statement

(a) False

(b) True

(c) True

(d) True

(e) False

Problem 2

Given: $\dot{x}_1 = (r_1 - x_2)x_1$ $\quad r_1 = 10$, $r_2 = 25$

$\dot{x}_2 = (r_2 - x_3)x_2$ $\quad x_1$: population level of prey species

$\dot{x}_3 = u$ $\quad x_2$: population level of predator species

$y = x_2$ $\quad x_2$: effort expended by humans in fishing the predator

u = input , y = the measurement of the predator

Find: (a) Find the equilibrium point if the prey species population $\bar{x}_1 = 20$

(b) Linearize the model using the equilibrium point from (a)

(c) Find the transfer function of the linearized state model from (b)

Solution:

(a) $$\begin{aligned}\dot{x}_1 &= r_1x_1 - x_2x_1 \\ \dot{x}_2 &= r_2x_2 - x_3x_2 \\ \dot{x}_3 &= u\end{aligned} \rightarrow \begin{pmatrix}\text{Plug} \\ r_1 = 10 \\ r_2 = 25 \\ \bar{x}_1 = 20\end{pmatrix} \rightarrow \begin{aligned}\dot{x}_1 &= 200 - 20x_2 \\ \dot{x}_2 &= 25x_2 - x_3x_2 \\ \dot{x}_3 &= u = 0\end{aligned} \rightarrow \begin{aligned}&\dot{x}_1 = 200 - 20\bar{x}_2 = 0 \rightarrow \bar{x}_2 = 10 \\ &\dot{x}_2 = 25\bar{x}_2 - \bar{x}_3\bar{x}_2 = 0 \rightarrow \bar{x}_3 = 25 \\ &\dot{x}_3 = \bar{u} = 0 \rightarrow \bar{u} = 0\end{aligned}$$

equilibrium point @ $\bar{x}_1 = 20$

$$\boxed{\bar{x} = \begin{bmatrix} 20 \\ 10 \\ 25 \end{bmatrix}} \quad \# (a)$$

(b) $$\frac{\partial f}{\partial x} = \begin{bmatrix} r_1 - x_2 & -x_1 & 0 \\ 0 & r_2 - x_3 & -x_2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{pmatrix}\text{Plug} \\ r_1 = 10 \\ r_2 = 25 \\ \bar{x}_1 = 20 \\ \bar{x}_2 = 10 \\ \bar{x}_3 = 25\end{pmatrix} \rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -20 & 0 \\ 0 & 0 & -10 \\ 0 & 0 & 0 \end{bmatrix}, \ \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \# (b)$$

(c) $G(s) = C(sI - A)^{-1}B$

$$A = \begin{bmatrix} 0 & -20 & 0 \\ 0 & 0 & -10 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s & 20 & 0 \\ 0 & s & 10 \\ 0 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow G(s) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{-20}{s^2} & \frac{200}{s^3} \\ 0 & \frac{1}{s} & -\frac{10}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{G(s) = \begin{bmatrix} -\frac{10}{s^2} \end{bmatrix}} \quad \# C$$

Problem 3

Given: $\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

Find: $x(0)$ when $u(t)=0$ & $x(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution:

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} \cancel{e^{A(t-\tau)} B u(\tau)\, d\tau}^{\,0 \text{ since } u(t)=0} \rightarrow x(t) = e^{A(t-t_0)} x(t_0)$$

apply C-H $(n=2)$

$$\Delta(\lambda) = \det(\lambda I - A)$$

$$= \det \begin{bmatrix} \lambda-1 & 0 \\ -3 & \lambda-1 \end{bmatrix} = (\lambda-1)(\lambda-1) \rightarrow \lambda_1 = \lambda_2 = 1 \quad \text{repeated}$$

$$f(\lambda) = e^{\lambda t} = \beta_1 \lambda + \beta_0$$

$$\frac{df(\lambda)}{d\lambda} = \frac{dg(\lambda)}{d\lambda} = te^{\lambda t} = \beta_1$$

$$\Rightarrow \begin{pmatrix} \text{Plug} \\ \lambda_1 = 1 \\ \lambda_2 = 1 \end{pmatrix} \Rightarrow \begin{matrix} e^t = \beta_1 + \beta_0 \\ te^t = \beta_1 \end{matrix} \Rightarrow \begin{cases} \beta_0 = e^t - te^t \\ \beta_1 = te^t \end{cases}$$

$$f(A) = e^{At} = \beta_1 A + \beta_0 I = te^t \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} + (e^t - te^t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow f(A) = e^{At} = \begin{bmatrix} e^t & 0 \\ 3te^t & e^t \end{bmatrix}$$

evaluate @ $x(0)$

$$x(0) = e^{A(-2)} x(2) = \begin{bmatrix} e^{-2} & 0 \\ -6e^{-2} & e^{-2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow x(0) = \begin{bmatrix} e^{-2} \\ -6e^{-2} \end{bmatrix} \#$$

Problem 4

Given: LTI system

$$\dot{x}(t) = \begin{bmatrix} -1 & -\alpha \\ 0 & 1-\alpha \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u(t) \quad , \quad y(t) = [1 \;\; \alpha] \, x(t) + u(t)$$

Find: (a) Find the range of α for which the system is exponentially stable

(b) For the infimum (largest lower bound) of the range of α determined in (a), check whether the given system is BIBO stable.

Solution:

(a) condition: every asymptotically stable LTI system is exponetially stable if & only if the system has eigenvalues with strictly negative real parts

$\det(\lambda I - A)$ $(\lambda < 0)$

$$A = \begin{bmatrix} -1 & -\alpha \\ 0 & 1-\alpha \end{bmatrix}, \quad \det \begin{bmatrix} \lambda+1 & \alpha \\ 0 & \lambda-1+\alpha \end{bmatrix} = (\lambda+1)(\lambda-1+\alpha) \rightarrow \begin{matrix} \lambda_1 = -1 < 0 \\ \lambda_2 = 1-\alpha < 0 \end{matrix}$$

$1-\alpha < 0 \rightarrow \alpha > 1$

$\because$ all λ must be < 0 to satisfy the exponetially stable condition.

$\therefore$ $\alpha > 1$ is the range for which the system is exponentially stable # (a)

(b) From (a) the infimum is $\alpha = 1$ & plug back to the given system

$$\dot{x}(t) = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \quad , \quad y(t) = [1 \;\; 1] \, x(t) + u(t)$$

Condition: let $G_C(s) = C(sI-A)^{-1}B + D$

A CT LTI system is BIBO stable $\Leftrightarrow$ every pole of every $G_{C_{ij}}$ have negative real part

$$A = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = [1 \;\; 1], \quad D = [1]$$

$$G_C(s) = [1 \;\; 1] \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + [1] \rightarrow G_C(s) = \left[\frac{2}{s+1} + 1 \right] \rightarrow \text{pole of } G_C(s) = -1$$

$\because$ Since the pole of $G_C(s) = -1 < 0$ which satisfy the BIBO stable condition

$$\therefore \dot{x}(t) = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \quad , \quad y(t) = [1 \;\; 1] \, x(t) + u(t)$$

is BIBO stable # (b)

Problem 5

Given: nonlinear system

$$\dot{x}_1 = -\frac{x_2}{1+x_1^2} - 2x_1 \quad , \quad \dot{x}_2 = \frac{x_1}{1+x_1^2}$$

Find: (a) Using $V(x) = x_1^2 + x_2^2$, find the equilibrium point & the stability of the system at the equilibrium point

(b) Linearize the system about the equilibrium point & find the stability of the linearized system using Lyapunov indirect method

Solution:

(a)

(find equilibrium point)

$$\left.\begin{aligned} \dot{x}_1 &= -\frac{x_2}{1+x_1^2} - \cancel{2x_1}^{\,x_1=0} = 0 \rightarrow x_2 = 0 \\ \dot{x}_2 &= \frac{x_1}{1+x_1^2} = 0 \rightarrow x_1 = 0 \end{aligned}\right\} \quad \text{equilibrium point } \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ \# (a)}$$

Conditions: The origin of $\dot{x} = f(x)$ is asymptotically stable if

(1) $V(x) = 0$ if and only if $x = 0$

(2) $V(x) > 0$ if and only if $x \neq 0$

(3) $\dot{V}(x) = \frac{d}{dt}V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \nabla V \cdot f(x) < 0$ for all values of $x \neq 0$

Check (1) & (2) $\longrightarrow$ clearly $V(x) = 0$ if & only if $x = 0$ & $V(x) > 0$ if & only if $x \neq 0$

check (3) $\longrightarrow$ $\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \rightarrow$ plug in given $\dot{x}_1$ & $\dot{x}_2$

$$\dot{V}(x) = 2x_1\left(-\frac{x_2}{1+x_1^2} - 2x_1\right) + 2x_2\left(\frac{x_1}{1+x_1^2}\right) \rightarrow \dot{V}(x) = -\cancel{\frac{2x_1x_2}{1+x_1^2}} - 4x_1^2 + \cancel{\frac{2x_1x_2}{1+x_1^2}} \rightarrow \dot{V}(x) = -4x_1^2 < 0$$

$\because$ the reasoning in conditions (1), (2), (3)

$\therefore$ the system is asymptotically stable @ equilibrium point # (a)

(b)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{2x_1x_2}{(1+x_1^2)^2} - 2 & -\frac{1}{1+x_1^2} \\ \frac{1-x_1^2}{(1+x_1^2)^2} & 0 \end{bmatrix} \rightarrow \begin{pmatrix} \text{Plug} \\ \bar{x}_1 = 0 \\ \bar{x}_2 = 0 \end{pmatrix} \rightarrow \frac{\partial f}{\partial x} \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$$ # (b)

$\det(\lambda I - A)$

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \qquad \det(\lambda I - A) = \begin{vmatrix} \lambda+2 & 1 \\ -1 & \lambda \end{vmatrix} = (\lambda+2)(\lambda)+1 = \lambda^2+2\lambda+1 = (\lambda+1)(\lambda+1) \Big\} \ \lambda_1 = \lambda_2 = -1 < 0$$

Condition: Let $\dot{x} = f(x)$. Linearize the system, we have

- The origin is locally Asymptotically stable if $Re(\lambda_i) < 0$, $\forall \lambda_i$ of A

$\because$ $\lambda_1 = \lambda_2 = -1 < 0$ satisfied the above condition

$\therefore$ the linearized system is Asymptotically stable #(b)

Problem 6

Given: $G(s) = \begin{bmatrix} \frac{s}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix}$

Find: The minimal realization

Solution:

$G(\infty) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $G_{sp} = \begin{bmatrix} \frac{s-1(s+1)}{s+1} \\ \frac{1-0[s(s+1)]}{s(s+1)} \end{bmatrix} \rightarrow G_{sp} = \begin{bmatrix} \frac{-1}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix}$

$d(s) = (s)(s+1) \rightarrow ds = s^2+s \rightarrow \alpha_1 = 1$, $\alpha_2 = 0$

$G_{sp} = \frac{1}{s^2+s} \begin{bmatrix} -1(s) \\ 1 \end{bmatrix} \rightarrow G_{sp} = \frac{1}{s^2+s} \begin{bmatrix} -s \\ 1 \end{bmatrix}$

$N_1(s) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $N_2(s) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $(n=2)$

$P = [B \vdots AB]$

$A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $AB = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \rightarrow \text{rank}(P) = n = 2 \rightarrow$ controllable

$Q = \begin{bmatrix} C \\ CA \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$, $CA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$Q = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \xrightarrow{ref} Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \text{rank}(Q) = n = 2 \rightarrow$ observable

$\because \text{rank}(P) \,\&\, \text{rank}(Q) = n = 2$

$\therefore \dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} x \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$, $y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ is the minimal realization

Problem 7

Given: $x = [p \ r \ \beta \ \phi]^T$, $u = [\delta_a \ \delta_r]^T$

$$A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \dot{x} = Ax + Bu \quad (n=4)$$

Find:

(a) Is the linearized aircraft model asymptotically stable? Is it stable i.s.L.?

(b) Is the aircraft controllable with just δ_r? With both δ_r & δ_a?

(c) Malfunction with the rudder angle δ_r, is it possible to control with only δ_a?

(d) Which one state $\{p, r, \beta, \phi\}$ so the whole system is observable?

Solution:

$$\dot{x} = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ r \\ \beta \\ \phi \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

(a) $$A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \det(\lambda I - A) = \begin{vmatrix} \lambda+10 & 0 & 1 & 0 \\ 0 & \lambda+1 & -1 & 0 \\ 0 & 1 & \lambda & 0 \\ -1 & 0 & 0 & \lambda \end{vmatrix} = \lambda(\lambda+10)(\lambda^2+\lambda+1)$$

$\lambda_1 = -10$, $\lambda_2 = \frac{1}{2}(-1+i\sqrt{3})$, $\lambda_3 = \frac{1}{2}(-1-i\sqrt{3})$, $\lambda_4 = 0$

Condition: Let $\dot{x} = f(x)$. Linearize the system, we have

- The origin is locally Asymptotically Stable if $Re(\lambda_i) < 0$, $\forall \lambda_i$ of A
- $\exists \lambda_i$, $Re = 0$, $m = 0 \Rightarrow$ stable i.s.L

$\because$ Since real parts of $\lambda_1 (-10)$, $\lambda_2 (-\frac{1}{2})$, $\lambda_3 (-\frac{1}{2})$ are negative but $\lambda_4 = 0$ which is not negative (also non-defective)

$\therefore$ The linearized model is not asymptotically stable

The linearized model is stable i.s.L. # (a)

(b) δ_r, $P = [B \vdots AB \vdots A^2B \vdots A^3B]$

$$B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad AB = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \qquad A^3B = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^3 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & -1 & 11 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \xrightarrow{ref} P = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{rank}(P) = n = 4 \rightarrow \text{controllable}$$

δ_a & δ_r $\quad P = [B \vdots AB \vdots A^2B \vdots A^3B]$

$$B = \begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad AB = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -100 & 0 \\ 0 & 1 \\ 0 & 1 \\ 10 & 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 \begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1000 & -1 \\ 0 & 0 \\ 0 & -1 \\ -100 & 0 \end{bmatrix}, \quad A^3B = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^3 \begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -10000 & 11 \\ 0 & -1 \\ 0 & 0 \\ 1000 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 10 & 0 & -100 & 0 & 1000 & -1 & -10000 & 11 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 10 & 0 & -100 & 0 & 1000 & -1 \end{bmatrix} \xrightarrow{ref} P = \begin{bmatrix} 1 & 0 & -10 & 0 & 100 & -\frac{1}{10} & -1000 & \frac{11}{10} \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -10 & 0 & 100 & -\frac{1}{10} \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

rank(P) $= n = 4 \rightarrow$ controllable

$\because$ Since rank(P)$=n=4$ for both δ_r & δ_r and δ_a cases

$\therefore$ The aircraft is controllable for both δr only & δr and δa cases. #(b)

(c) δa, $P = [B \vdots AB \vdots A^2B \vdots A^3B]$

$$B = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad AB = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -100 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ -100 \end{bmatrix}, \quad A^3B = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^3 \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -10000 \\ 0 \\ 0 \\ 1000 \end{bmatrix}$$

$$P = \begin{bmatrix} 10 & -100 & 1000 & -10000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & -100 & 1000 \end{bmatrix} \xrightarrow{ref} P = \begin{bmatrix} 1 & -10 & 100 & -1000 \\ 0 & 1 & -10 & 100 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow rank(P) = 2 \neq n$$

$\because$ Since $rank(P) = 2 \neq n(4)$

$\therefore$ The aircraft is not controllable using only the aileron angle δa. #(c)

(d)

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}, \quad A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 100 & 1 & 10 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -10 & 0 & -1 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} -1000 & -11 & -99 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix}$$

P, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

$$CA = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 & -1 & 0 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 100 & 1 & 10 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -10 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 100 & 1 & 10 & 0 \end{bmatrix}$$

$$CA^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1000 & -11 & -99 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix} = \begin{bmatrix} -1000 & -11 & -99 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -10 & 0 & -1 & 0 \\ 100 & 1 & 10 & 0 \\ -1000 & -11 & -99 & 0 \end{bmatrix} \xrightarrow{\text{ref}} Q = \begin{bmatrix} 1 & \frac{11}{1000} & \frac{99}{1000} & 0 \\ 0 & 1 & \frac{-1}{11} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{rank}(Q) = 3 \neq n(4)$$

not observable

r, $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

$$CA = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 100 & 1 & 10 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -10 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}$$

$$CA^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1000 & -11 & -99 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{ref} Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank}(Q) = 2 \neq n(4)$$

not observable

B. $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$

$$CA = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 100 & 1 & 10 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -10 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$$

$$CA^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1000 & -11 & -99 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{ref} Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank}(Q) = 2 \neq n(4)$$

not observable

D. $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$

$$CA = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 100 & 1 & 10 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -10 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 & -1 & 0 \end{bmatrix}$$

$$CA^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1000 & -11 & -99 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix} = \begin{bmatrix} 100 & 1 & 10 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -10 & 0 & -1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix} \xrightarrow{ref} Q = \begin{bmatrix} 1 & \frac{1}{100} & \frac{1}{100} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \operatorname{rank}(Q) = n = 4$$

observable

$\because$ Since only ϕ's $\operatorname{rank}(Q) = n = 4$ (observable)

$\therefore$ The ϕ state should be measured so that the whole system is observable #(d)