24 - 677
Fall 2023
Mid-term Exam
10/24/23
Time: 24 Hours

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Print your initials on each page that has your answers

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use your equation sheet and calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- You are allowed to use course slides, homework solution sheets as references. You are allowed to search for the knowledge needed on the internet.
- You must conduct the exam independently. Discussion or seeking help from others, online or in-person, is prohibited.
- All answers need to be derived by hand to get points. You are allowed to use a calculator for basic calculation of scalars. You can use calculate/-computer programs to verify your answers but the effort does not account as credits.
- You can ask questions on campuswire but only to the TAs and instructors.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	10	
6	10	
7	20	
Total:	100	

Do not write in the table to the right.

Problem 1

Determine true / false of each statement

(a) False

(b) True

(c) False

(d) True

(e) True

Problem 2

Given: $\dot{x}_1 = (r_1 - x_2)x_1$ $\quad r_1 = 10$, $r_2 = 25$

$\dot{x}_2 = (r_2 - x_3)x_2$ $\quad x_1$: population level of prey species

$\dot{x}_3 = u$ $\quad x_2$: population level of predator species

$y = x_2$ $\quad x_2$: effort expended by humans in fishing the predator

u: input , y: the measurement of the predator

Find: (a) Find the equilibrium point if the prey species population $\bar{x}_1 = 20$

(b) Linearize the model using the equilibrium point from (a)

(c) Find the transfer function of the linearized state model from (b)

Solution:

(a)
$$\begin{aligned}\dot{x}_1 &= r_1x_1 - x_2x_1 \\ \dot{x}_2 &= r_2x_2 - x_3x_2 \\ \dot{x}_3 &= u\end{aligned} \rightarrow \left(\begin{matrix}\text{Plug} \\ r_1 = 10 \\ r_2 = 25 \\ \bar{x}_1 = 20\end{matrix}\right) \rightarrow \begin{aligned}\dot{x}_1 &= 200 - 20x_2 \\ \dot{x}_2 &= 25x_2 - x_3x_2 \\ \dot{x}_3 &= u = 0\end{aligned} \rightarrow \begin{aligned}\dot{x}_1 &= 200 - 20\bar{x}_2 = 0 \rightarrow \bar{x}_2 = 10 \\ \dot{x}_2 &= 25\bar{x}_2 - \bar{x}_3\bar{x}_2 = 0 \rightarrow \bar{x}_3 = 25 \\ \dot{x}_3 &= \bar{u} = 0 \rightarrow \bar{u} = 0\end{aligned}$$

equilibrium point @ $\bar{x}_1 = 20$

$$\bar{x} = \begin{bmatrix} 20 \\ 10 \\ 25 \end{bmatrix} \quad \#(a)$$

(b)
$$\frac{\partial f}{\partial x} = \begin{bmatrix} r_1 - x_2 & -x_1 & 0 \\ 0 & r_2 - x_3 & -x_2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \left(\begin{matrix}\text{Plug} \\ r_1 = 10 \\ r_2 = 25 \\ \bar{x}_1 = 20 \\ \bar{x}_2 = 10 \\ \bar{x}_3 = 25\end{matrix}\right) \rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -20 & 0 \\ 0 & 0 & -10 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \#(b)$$

(c) $G(s) = C(sI - A)^{-1}B$

$$A = \begin{bmatrix} 0 & -20 & 0 \\ 0 & 0 & -10 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s & 20 & 0 \\ 0 & s & 10 \\ 0 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow G(s) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & \frac{-20}{s^2} & \frac{200}{s^3} \\ 0 & \frac{1}{s} & -\frac{10}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$G(s) = \left[-\frac{10}{s^2} \right] \quad \#C$$

Problem 3

Given: $\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

Find: $x(0)$ when $u(t)=0$ & $x(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution:

$x(t) = e^{A(t-t_0)} x(t_0) + \cancel{\int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau}$ 0 since $u(t)=0$ $\rightarrow x(t) = e^{A(t-t_0)} x(t_0)$

apply C-H $(n=2)$

$\Delta(\lambda) = \det(\lambda I - A)$

$= \det \begin{bmatrix} \lambda-1 & 0 \\ -3 & \lambda-1 \end{bmatrix} = (\lambda-1)(\lambda-1) \rightarrow \lambda_1 = \lambda_2 = 1$ repeated

$f(\lambda) = e^{\lambda t} = \beta_1 \lambda + \beta_0$

$\frac{df(\lambda)}{d\lambda} = \frac{dg(\lambda)}{d\lambda} = te^{\lambda t} = \beta_1$

$\Rightarrow \begin{pmatrix} \text{Plug} \\ \lambda_1 = 1 \\ \lambda_2 = 1 \end{pmatrix} \Rightarrow \begin{matrix} e^t = \beta_1 + \beta_0 \\ te^t = \beta_1 \end{matrix} \Rightarrow \begin{cases} \beta_0 = e^t - te^t \\ \beta_1 = te^t \end{cases}$

$f(A) = e^{At} = \beta_1 A + \beta_0 I = te^t \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} + e^t - te^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow f(A) = e^{At} = \begin{bmatrix} e^t & 0 \\ 3te^t & e^t \end{bmatrix}$

evaluate @ $x(0)$

$x(0) = e^{A(-2)} x(2) = \begin{bmatrix} e^{-2} & 0 \\ -6e^{-2} & e^{-2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow x(0) = \begin{bmatrix} e^{-2} \\ -6e^{-2} \end{bmatrix}$ #

Problem 4

Given: LTI system

$$\dot{x}(t) = \begin{bmatrix} -1 & -\alpha \\ 0 & 1-\alpha \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u(t) \quad , \quad y(t) = [1 \ \ \alpha]\, x(t) + u(t)$$

Find: (a) Find the range of α for which the system is exponentially stable

(b) For the infimum (largest lower bound) of the range of α determined in (a), check whether the given system is BIBO stable.

Solution:

(a) condition: every asymptotically stable LTI system is exponetially stable if & only if the system has eigenvalues with strictly negative real parts

$\det(\lambda I - A)$ $(\lambda < 0)$

$$A = \begin{bmatrix} -1 & -\alpha \\ 0 & 1-\alpha \end{bmatrix} , \ \det \begin{bmatrix} \lambda+1 & \alpha \\ 0 & \lambda-1+\alpha \end{bmatrix} = (\lambda+1)(\lambda-1+\alpha) \rightarrow \begin{matrix} \lambda_1 = -1 < 0 \\ \lambda_2 = 1-\alpha < 0 \end{matrix}$$

$1-\alpha < 0 \rightarrow \alpha > 1$

$\because$ all λ must be < 0 to satisfy the exponetially stable condition.

$\therefore$ $\alpha > 1$ is the range for which the system is exponentially stable # (a)

(b) From (a) the infimum is $\alpha = 1$ & plug back to the given system

$$\dot{x}(t) = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \quad , \quad y(t) = [1 \ \ 1]\, x(t) + u(t)$$

Condition: let $G_c(s) = C(sI-A)^{-1}B + D$

A CT LTI system is BIBO stable $\Leftrightarrow$ every pole of every $G_{c_{ij}}$ have negative real part

$$A = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} , \ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \ C = [1 \ \ 1] , \ D = [1]$$

$$G_c(s) = [1 \ \ 1] \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + [1] \rightarrow G_c(s) = \begin{bmatrix} \frac{2}{s+1} + 1 \end{bmatrix} \rightarrow \text{pole of } G_c(s) = -1$$

$\because$ Since the pole of $G_c(s) = -1 < 0$ which satisfy the BIBO stable condition

$$\therefore \dot{x}(t) = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \quad , \quad y(t) = [1 \ \ 1]\, x(t) + u(t)$$

is BIBO stable # (b)

Problem 5

Given: nonlinear system

$$\dot{x}_1 = -\frac{x_2}{1+x_1^2} - 2x_1 \quad , \quad \dot{x}_2 = \frac{x_1}{1+x_1^2}$$

Find: (a) Using $V(x) = x_1^2 + x_2^2$, find the equilibrium point & the stability of the system at the equilibrium point

(b) Linearize the system about the equilibrium point & find the stability of the linearized system using Lyapunov indirect method

Solution:

(a)

(find equilibrium point)

$$\left.\begin{array}{l} \dot{x}_1 = -\dfrac{x_2}{1+x_1^2} - \cancel{2x_1}^{\,x_1=0} = 0 \rightarrow x_2 = 0 \\ \\ \dot{x}_2 = \dfrac{x_1}{1+x_1^2} = 0 \rightarrow x_1 = 0 \end{array}\right\} \quad \text{equilibrium point } \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \# (a)$$

Conditions: The origin of $\dot{x} = f(x)$ is asymptotically stable if

(1) $V(x) = 0$ if and only if $x = 0$

(2) $V(x) > 0$ if and only if $x \neq 0$

(3) $\dot{V}(x) = \frac{d}{dt}V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \nabla V \cdot f(x) < 0$ for all values of $x \neq 0$

Check (1) & (2) $\longrightarrow$ clearly $V(x) = 0$ if & only if $x = 0$ & $V(x) > 0$ if & only if $x \neq 0$

check (3) $\longrightarrow$ $\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \rightarrow$ plug in given $\dot{x}_1$ & $\dot{x}_2$

$$\dot{V}(x) = 2x_1\left(-\frac{x_2}{1+x_1^2} - 2x_1\right) + 2x_2\left(\frac{x_1}{1+x_1^2}\right) \rightarrow \dot{V}(x) = -\cancel{\frac{2x_1x_2}{1+x_1^2}} - 4x_1^2 + \cancel{\frac{2x_1x_2}{1+x_1^2}} \rightarrow \dot{V}(x) = -4x_1^2 < 0$$

$\because$ the reasoning in conditions (1), (2), (3)

$\therefore$ the system is asymptotically stable @ equilibrium point # (a)

(b)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{2x_1x_2}{(1+x_1^2)^2} - 2 & -\frac{1}{1+x_1^2} \\ \frac{1-x_1^2}{(1+x_1^2)^2} & 0 \end{bmatrix} \rightarrow \begin{pmatrix} \text{Plug} \\ \bar{x}_1 = 0 \\ \bar{x}_2 = 0 \end{pmatrix} \rightarrow \frac{\partial f}{\partial x} \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \#(b)$$

$\det(\lambda I - A)$

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \qquad \det(\lambda I - A) = \begin{vmatrix} \lambda+2 & 1 \\ -1 & \lambda \end{vmatrix} = (\lambda+2)(\lambda)+1 = \lambda^2+2\lambda+1 = (\lambda+1)(\lambda+1) \Big\} \lambda_1 = \lambda_2 = -1 < 0$$

Condition: Let $\dot{x} = f(x)$. Linearize the system, we have

- The origin is locally Asymptotically Stable if $Re(\lambda_i) < 0$, $\forall \lambda_i$ of A

$\because$ $\lambda_1 = \lambda_2 = -1 < 0$ satisfied the above condition

$\therefore$ the linearized system is Asymptotically stable #(b)

Problem 6

Given: $G(s)=\begin{bmatrix} \frac{s}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix}$

Find: The minimal realization

Solution:

$G(\infty)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $G_{sp}=\begin{bmatrix} \frac{s-1(s+1)}{s+1} \\ \frac{1-0[s(s+1)]}{s(s+1)} \end{bmatrix} \rightarrow G_{sp}=\begin{bmatrix} \frac{-1}{s+1} \\ \frac{1}{s(s+1)} \end{bmatrix}$

$d(s)=(s)(s+1) \rightarrow ds=s^2+s \rightarrow \alpha_1=1$, $\alpha_2=0$

$G_{sp}=\frac{1}{s^2+s}\begin{bmatrix} -1(s) \\ 1 \end{bmatrix} \rightarrow G_{sp}=\frac{1}{s^2+s}\begin{bmatrix} -s \\ 1 \end{bmatrix}$

$N_1(s)=\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $N_2(s)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A=\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $D=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $(n=2)$

$P=[B \vdots AB]$

$A=\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $AB=\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$P=\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \rightarrow \text{rank}(P)=n=2 \rightarrow$ controllable

$Q=\begin{bmatrix} C \\ CA \end{bmatrix}$, $C=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $A=\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$, $CA=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$Q=\begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{ref}} Q=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \text{rank}(Q)=n=2 \rightarrow$ observable

$\because$ rank(P) & rank(Q) $=n=2$

$\therefore \dot{x}=\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}x+\begin{bmatrix} 1 \\ 0 \end{bmatrix}u$, $y=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}x+\begin{bmatrix} 1 \\ 0 \end{bmatrix}u$ is the minimal realization

Problem 7

Given: $x = [p\ r\ \beta\ \phi]^T$, $u = [\delta_a\ \delta_r]^T$

$$A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\quad B = \begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\quad \dot{x} = Ax + Bu$$

$(n=4)$

Find:

(a) Is the linearized aircraft model asymptotically stable? Is it stable i.s.L.?

(b) Is the aircraft controllable with just δ_r? With both δ_r & δ_a?

(c) Malfunction with the rudder angle δ_r, is it possible to control with only δ_a?

(d) Which one state $\{p, r, \beta, \phi\}$ so the whole system is observable?

Solution:

$$\dot{x} = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ r \\ \beta \\ \phi \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

(a)
$$A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \det(\lambda I - A) = \begin{vmatrix} \lambda+10 & 0 & 1 & 0 \\ 0 & \lambda+1 & -1 & 0 \\ 0 & 1 & \lambda & 0 \\ -1 & 0 & 0 & \lambda \end{vmatrix} = \lambda(\lambda+10)(\lambda^2+\lambda+1)$$

$\lambda_1 = -10$, $\lambda_2 = \frac{1}{2}(-1+i\sqrt{3})$, $\lambda_3 = \frac{1}{2}(-1-i\sqrt{3})$, $\lambda_4 = 0$

Condition: Let $\dot{x} = f(x)$. Linearize the system, we have

- The origin is locally Asymptotically Stable if $Re(\lambda_i) < 0$, $\forall \lambda_i$ of A
- $\exists \lambda_i$, $Re = 0$, $m = 0$ $\Rightarrow$ stable i.s.L

$\because$ Since real parts of $\lambda_1\,(-10)$, $\lambda_2\,(-\frac{1}{2})$, $\lambda_3\,(-\frac{1}{2})$ are negative but $\lambda_4 = 0$ which is not negative (also non-defective)

$\therefore$ The linearized model is not asymptotically stable

The linearized model is stable i.s.L. # (a)

(b) δ_r, $P=[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, A^3B]$

$$B=\begin{bmatrix}0\\-1\\0\\0\end{bmatrix},\quad A=\begin{bmatrix}-10&0&-1&0\\0&-1&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix},\quad AB=\begin{bmatrix}-10&0&-1&0\\0&-1&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}0\\-1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

$$A^2B=\begin{bmatrix}-10&0&-1&0\\0&-1&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}^2\begin{bmatrix}0\\-1\\0\\0\end{bmatrix}=\begin{bmatrix}-1\\0\\-1\\0\end{bmatrix}\qquad A^3B=\begin{bmatrix}-10&0&-1&0\\0&-1&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}^3\begin{bmatrix}0\\-1\\0\\0\end{bmatrix}=\begin{bmatrix}11\\-1\\0\\-1\end{bmatrix}$$

$$P=\begin{bmatrix}0&0&-1&11\\-1&1&0&-1\\0&1&-1&0\\0&0&0&-1\end{bmatrix}\xrightarrow{\text{ref}} P=\begin{bmatrix}1&-1&0&1\\0&1&-1&0\\0&0&1&-11\\0&0&0&1\end{bmatrix}\rightarrow \text{rank}(P)=n=4\rightarrow \text{controllable}$$

δ_a & δ_r $\quad P=[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, A^3B]$

$$B=\begin{bmatrix}10&0\\0&-1\\0&0\\0&0\end{bmatrix},\quad A=\begin{bmatrix}-10&0&-1&0\\0&-1&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix},\quad AB=\begin{bmatrix}-10&0&-1&0\\0&-1&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}10&0\\0&-1\\0&0\\0&0\end{bmatrix}=\begin{bmatrix}-100&0\\0&1\\0&1\\10&0\end{bmatrix}$$

$$A^2B=\begin{bmatrix}-10&0&-1&0\\0&-1&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}^2\begin{bmatrix}10&0\\0&-1\\0&0\\0&0\end{bmatrix}=\begin{bmatrix}1000&-1\\0&0\\0&-1\\-100&0\end{bmatrix},\quad A^3B=\begin{bmatrix}-10&0&-1&0\\0&-1&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}^3\begin{bmatrix}10&0\\0&-1\\0&0\\0&0\end{bmatrix}=\begin{bmatrix}-10000&11\\0&-1\\0&0\\1000&-1\end{bmatrix}$$

$$P=\begin{bmatrix}10&0&-100&0&1000&-1&-10000&11\\0&-1&0&1&0&0&0&-1\\0&0&0&1&0&-1&0&0\\0&0&10&0&-100&0&1000&-1\end{bmatrix}\xrightarrow{\text{ref}} P=\begin{bmatrix}1&0&-10&0&100&-\frac{1}{10}&-1000&\frac{11}{10}\\0&1&0&-1&0&0&0&1\\0&0&1&0&-10&0&100&-\frac{1}{10}\\0&0&0&1&0&-1&0&0\end{bmatrix}$$

$\text{rank}(P)=n=4\rightarrow$ controllable

$\because$ Since $\text{rank}(P)=n=4$ for both δ_r & δ_r and δ_a cases

$\therefore$ The aircraft is controllable for both δr only & δr and δa cases. #(b)

(c) δa, $P = [B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, A^3B]$

$$B = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad AB = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -100 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ -100 \end{bmatrix}, \quad A^3B = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^3 \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -10000 \\ 0 \\ 0 \\ 1000 \end{bmatrix}$$

$$P = \begin{bmatrix} 10 & -100 & 1000 & -10000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & -100 & 1000 \end{bmatrix} \xrightarrow{\text{ref}} P = \begin{bmatrix} 1 & -10 & 100 & -1000 \\ 0 & 1 & -10 & 100 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank}(P) = 2 \neq n$$

$\because$ Since $\text{rank}(P) = 2 \neq n(4)$

$\therefore$ The aircraft is not controllable using only the aileron angle δa. #(c)

(d)

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}, \quad A = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 100 & 1 & 10 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -10 & 0 & -1 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} -1000 & -11 & -99 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix}$$

P, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

$$CA = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 & -1 & 0 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 100 & 1 & 10 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -10 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 100 & 1 & 10 & 0 \end{bmatrix}$$

$$CA^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1000 & -11 & -99 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix} = \begin{bmatrix} -1000 & -11 & -99 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -10 & 0 & -1 & 0 \\ 100 & 1 & 10 & 0 \\ -1000 & -11 & -99 & 0 \end{bmatrix} \xrightarrow{\text{ref}} Q = \begin{bmatrix} 1 & \frac{11}{1000} & \frac{99}{1000} & 0 \\ 0 & 1 & \frac{-1}{11} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{rank}(Q) = 3 \neq n(4)$$

not observable

r, $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

$$CA = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 100 & 1 & 10 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -10 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}$$

$$CA^3=[0\ 1\ 0\ 0]\begin{bmatrix} -1000 & -11 & -99 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix}=[0\ 1\ 0\ 0]$$

$$Q=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{ref} Q=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank}(Q)=2 \neq n(4)$$

not observable

B, $[0\ 0\ 1\ 0]$

$$CA=[0\ 0\ 1\ 0]\begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}=[0\ -1\ 0\ 0]$$

$$CA^2=[0\ 0\ 1\ 0]\begin{bmatrix} 100 & 1 & 10 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -10 & 0 & -1 & 0 \end{bmatrix}=[0\ 1\ -1\ 0]$$

$$CA^3=[0\ 0\ 1\ 0]\begin{bmatrix} -1000 & -11 & -99 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix}=[0\ 0\ 1\ 0]$$

$$Q=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{ref} Q=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank}(Q)=2 \neq n(4)$$

not observable

ϕ, $[0\ 0\ 0\ 1]$

$$CA = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -10 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 100 & 1 & 10 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -10 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 & -1 & 0 \end{bmatrix}$$

$$CA^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1000 & -11 & -99 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix} = \begin{bmatrix} 100 & 1 & 10 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -10 & 0 & -1 & 0 \\ 100 & 1 & 10 & 0 \end{bmatrix} \xrightarrow{ref} Q = \begin{bmatrix} 1 & \frac{1}{100} & \frac{1}{100} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{rank}(Q) = n = 4$$

observable

$\because$ Since only ϕ's rank(Q) $= n = 4$ (observable)

$\therefore$ The ϕ state should be measured so that the whole system is observable #(d)