

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 4.2 DIRECTED GRAPHS

---

- ▶ *introduction*
- ▶ *digraph API*
- ▶ *digraph search*
- ▶ *topological sort*
- ▶ *strong components*

# Algorithms

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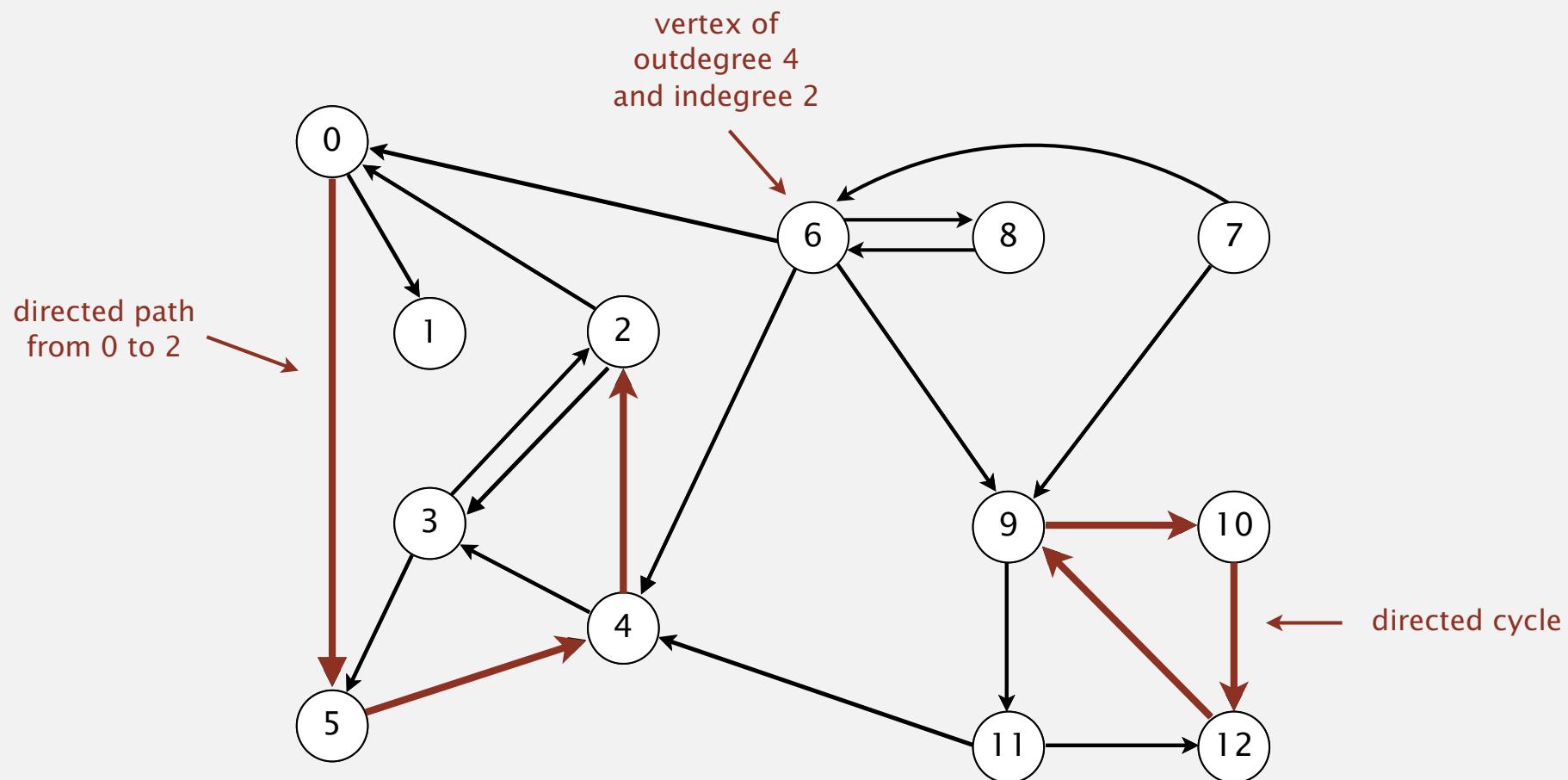
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# Directed graphs

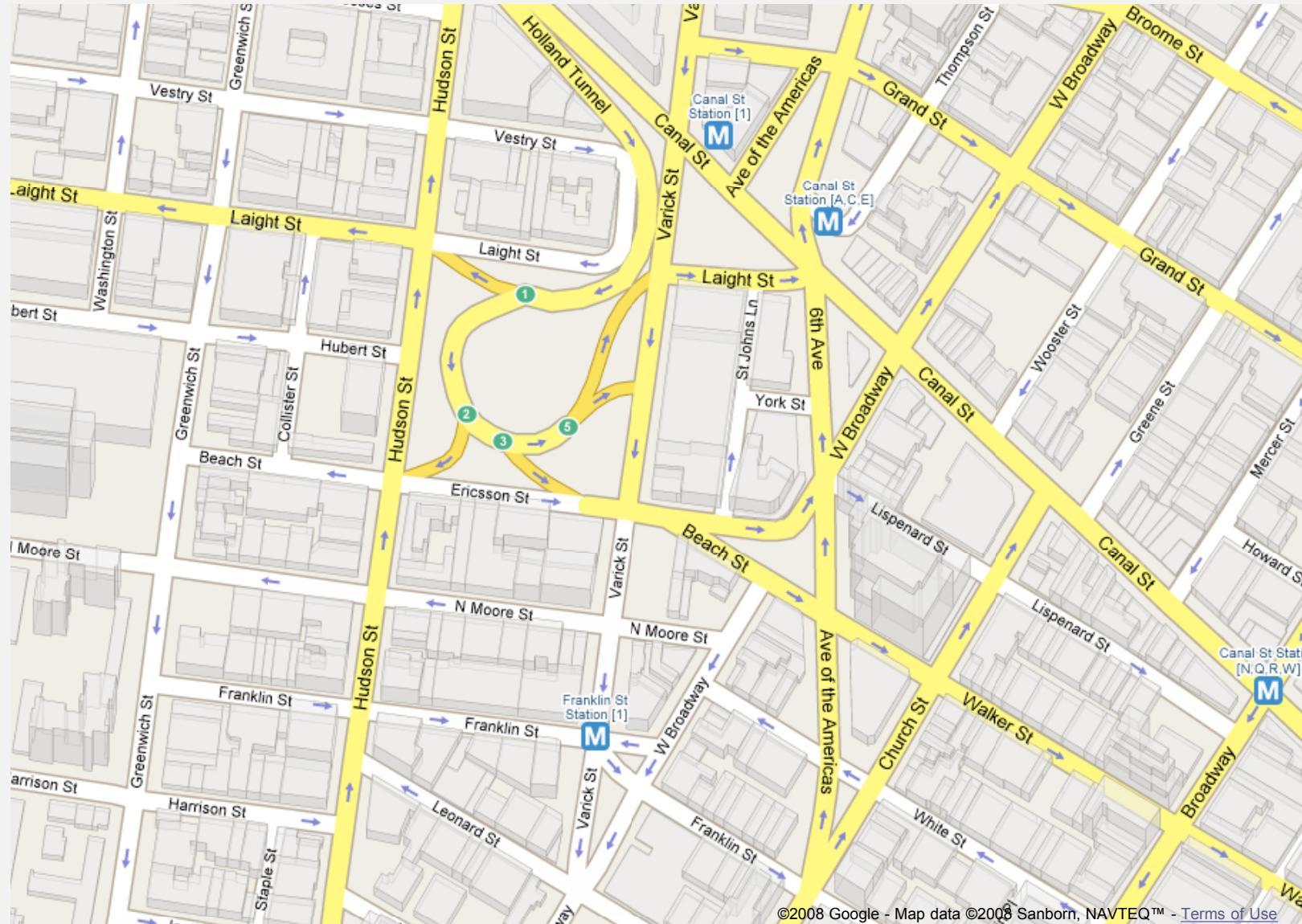
Digraph. Set of vertices connected pairwise by **directed edges**.



# Road network

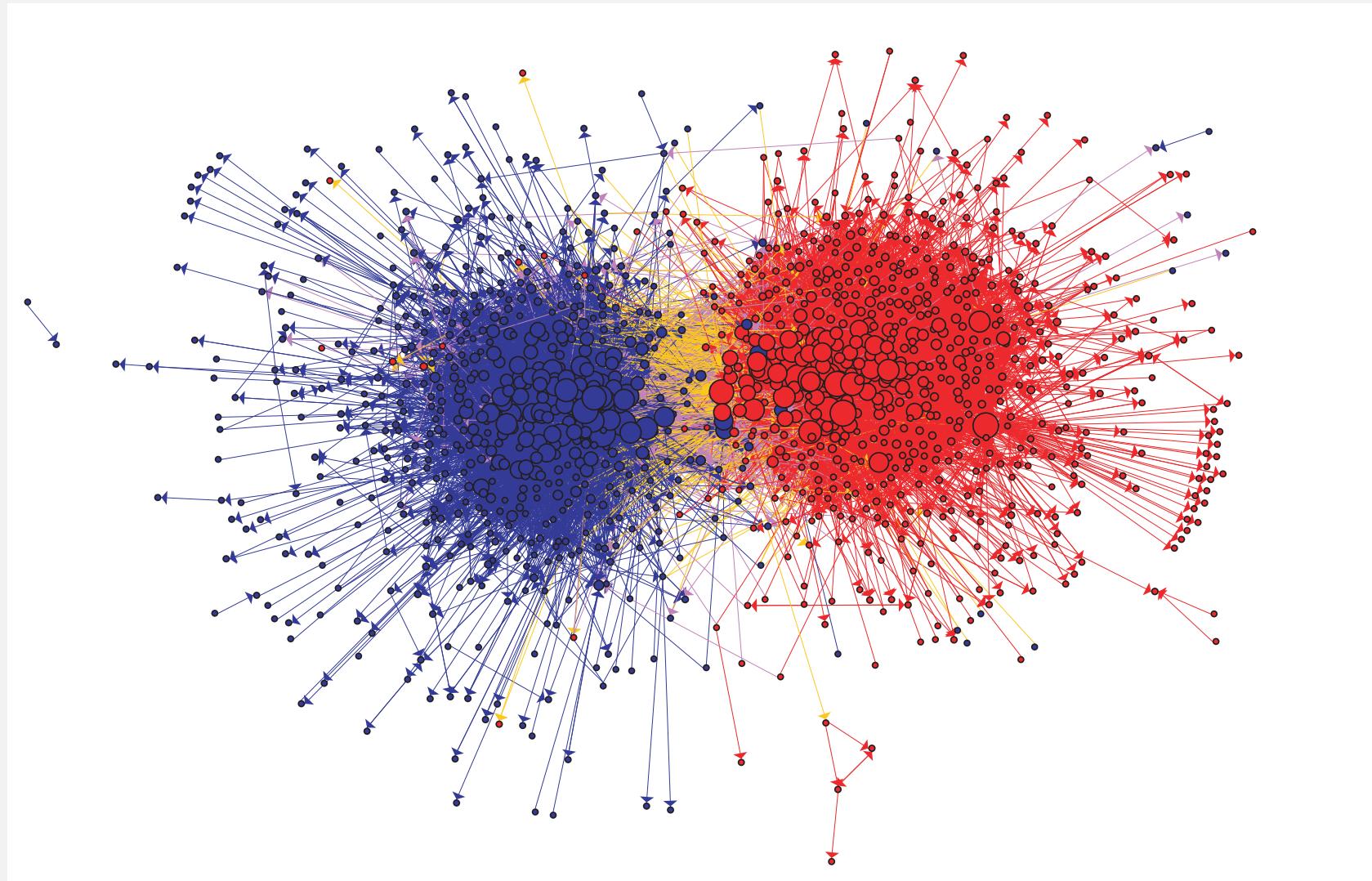
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Vertex = intersection; edge = one-way street.



# Political blogosphere graph

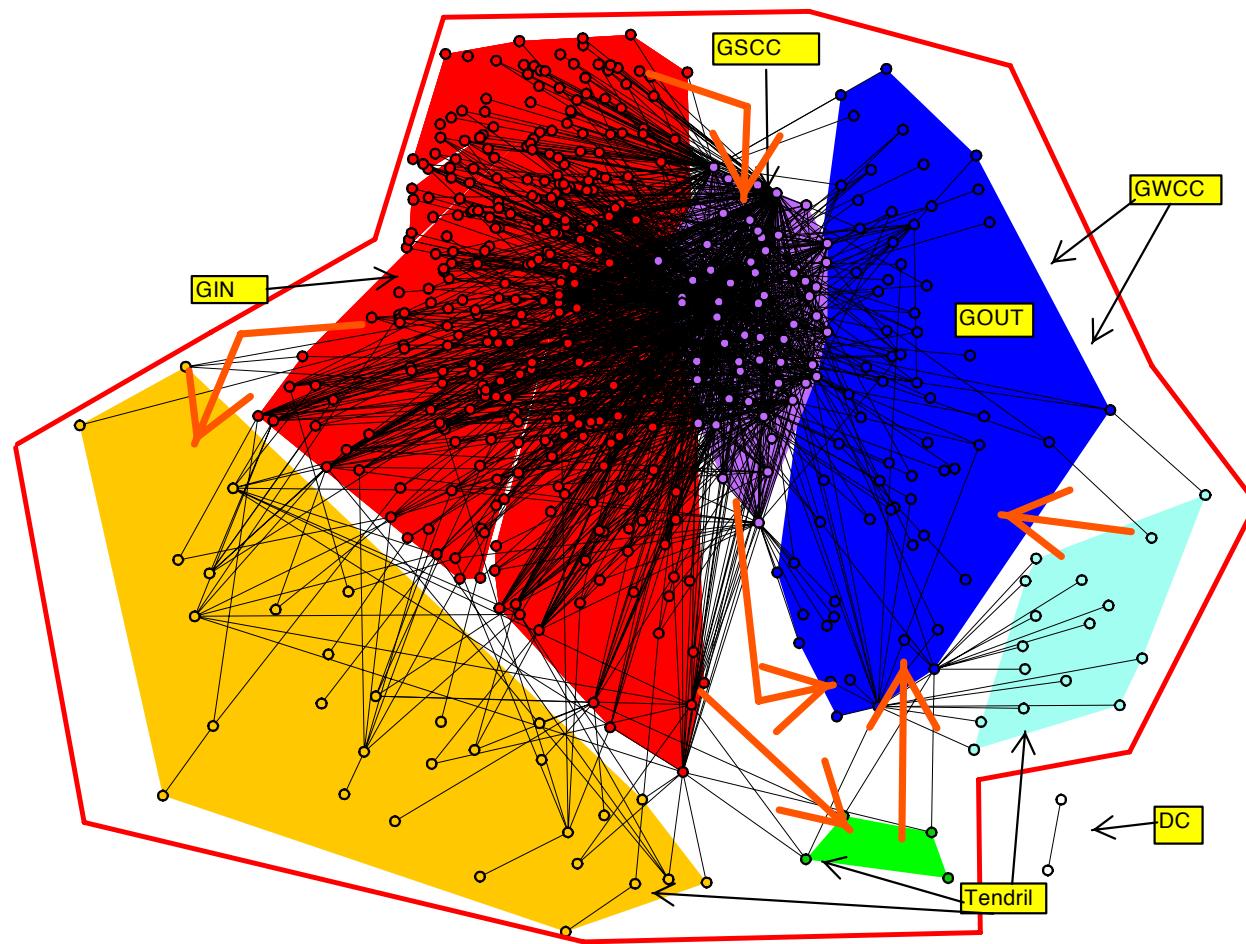
Vertex = political blog; edge = link.



The Political Blogosphere and the 2004 U.S. Election: Divided They Blog, Adamic and Glance, 2005

# Overnight interbank loan graph

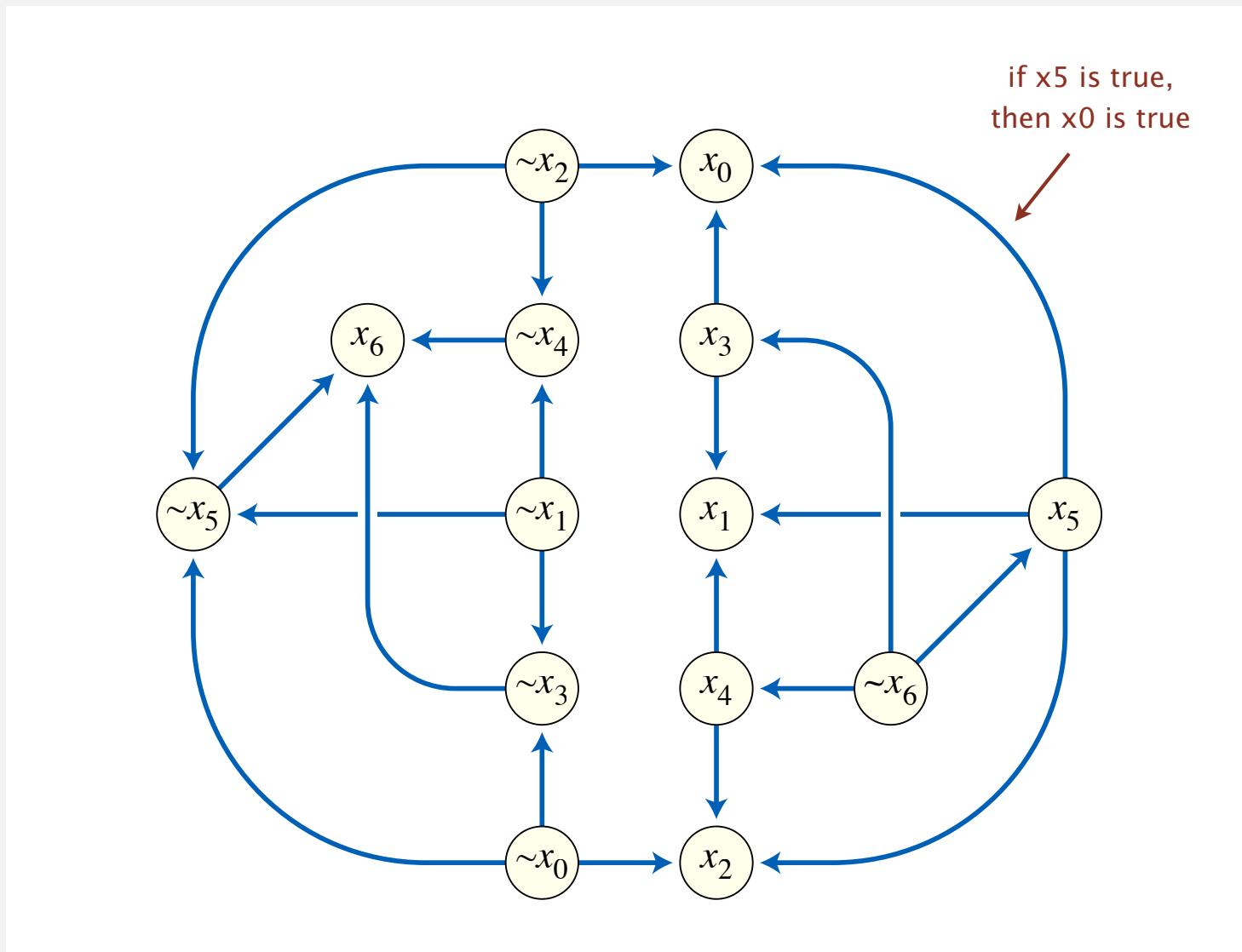
Vertex = bank; edge = overnight loan.



The Topology of the Federal Funds Market, Bech and Atalay, 2008

# Implication graph

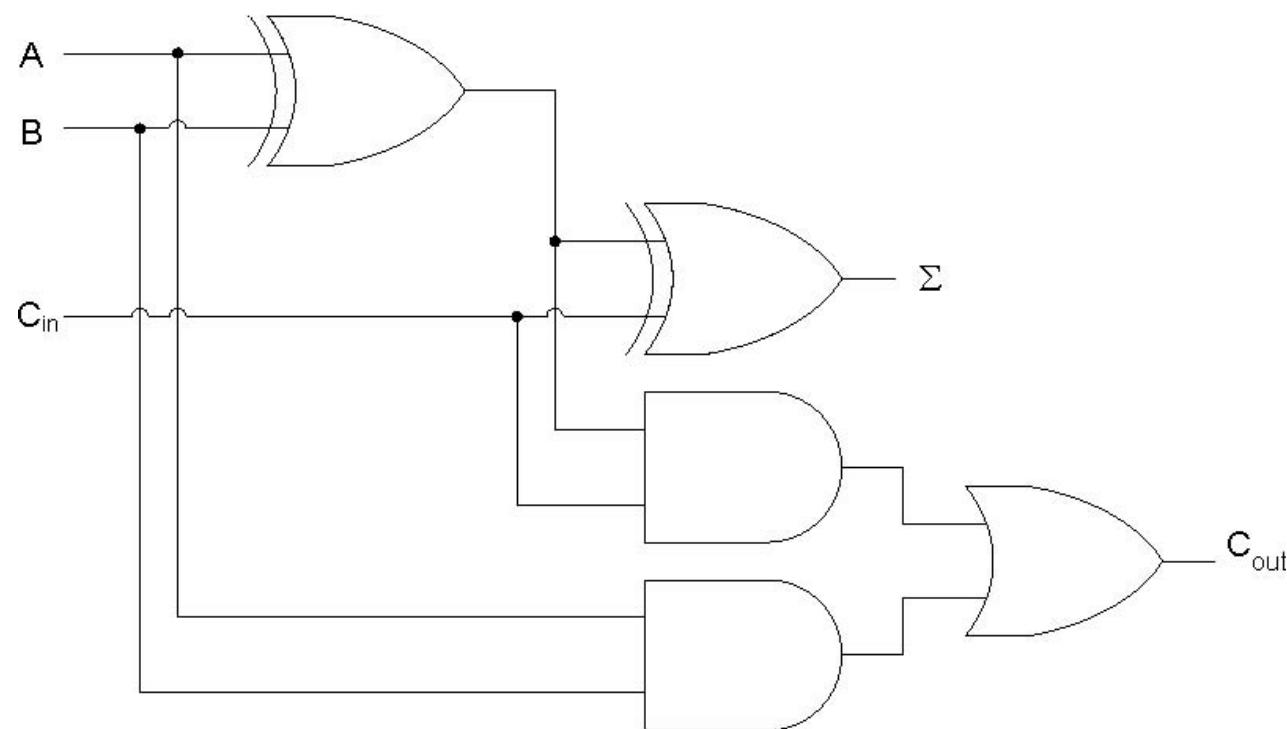
Vertex = variable; edge = logical implication.



# Combinational circuit

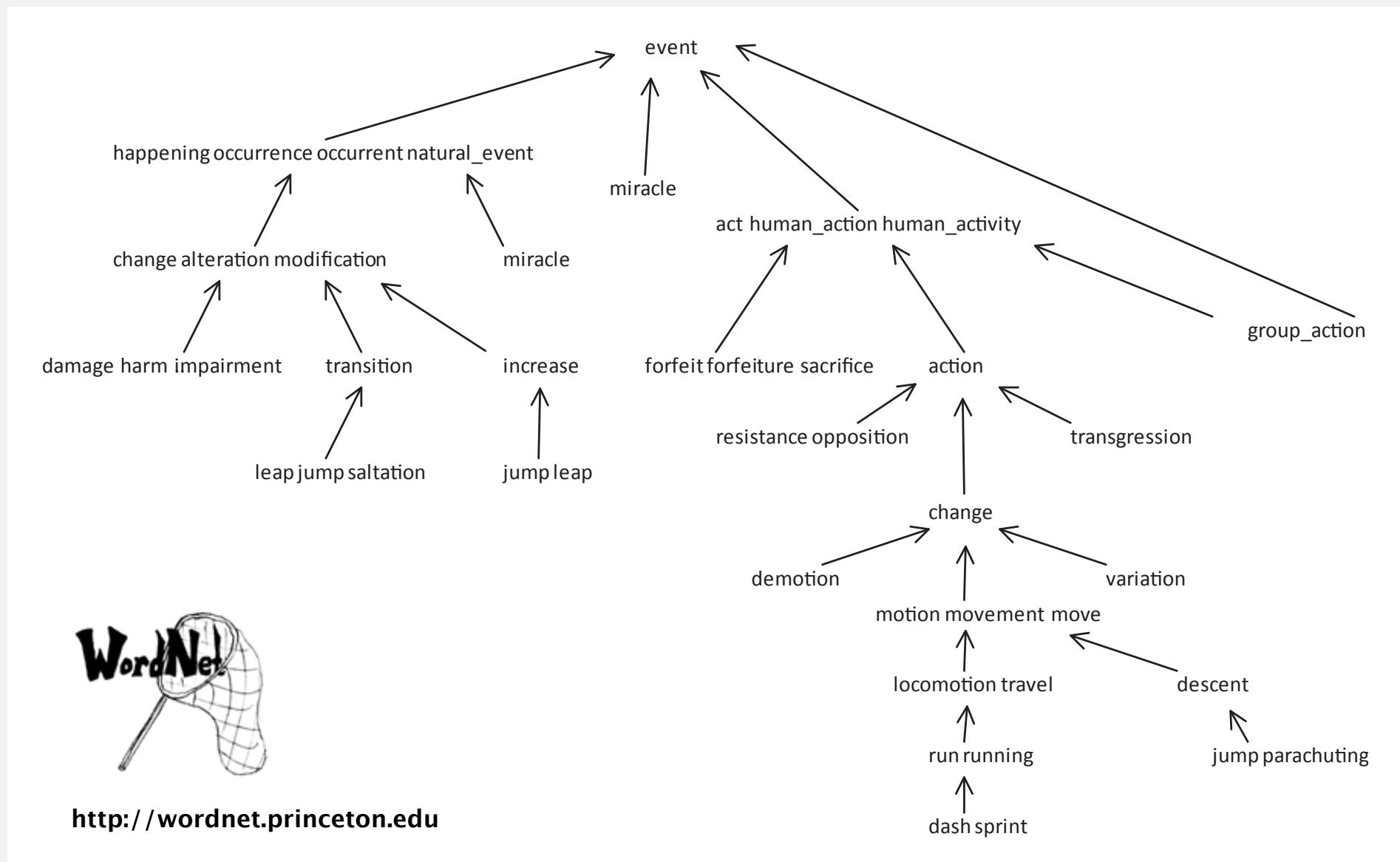
---

Vertex = logical gate; edge = wire.



# WordNet graph

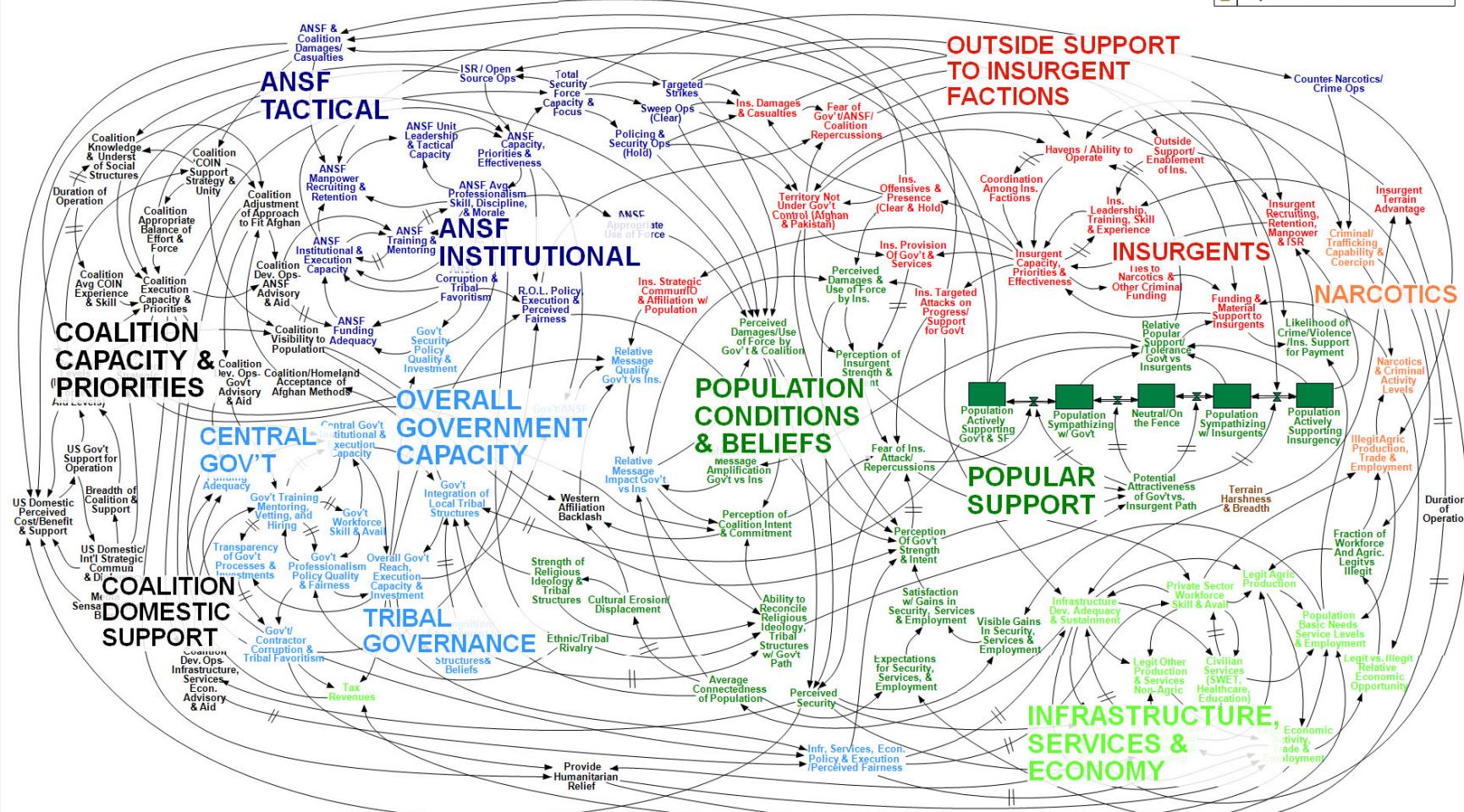
Vertex = synset; edge = hypernym relationship.



# The McChrystal Afghanistan PowerPoint slide

# Afghanistan Stability / COIN Dynamics

 = Significant Delay



**WORKING DRAFT – V3**

# Digraph applications

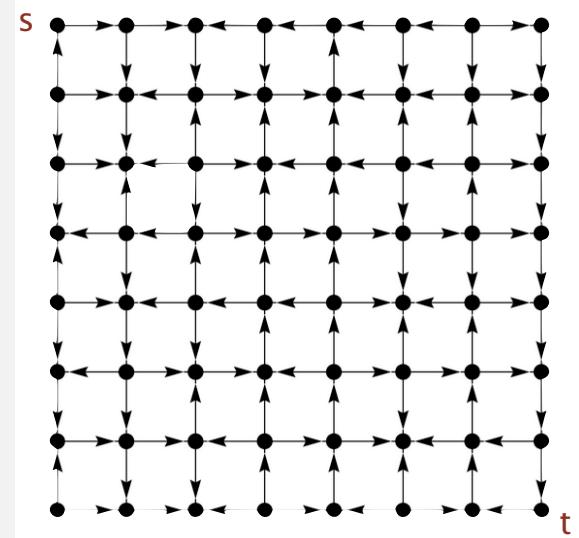
---

digraph	vertex	directed edge
transportation	street intersection	one-way street
web	web page	hyperlink
food web	species	predator-prey relationship
WordNet	synset	hypernym
scheduling	task	precedence constraint
financial	bank	transaction
cell phone	person	placed call
infectious disease	person	infection
game	board position	legal move
citation	journal article	citation
object graph	object	pointer
inheritance hierarchy	class	inherits from
control flow	code block	jump

# Some digraph problems

---

**Path.** Is there a directed path from  $s$  to  $t$ ?



**Shortest path.** What is the shortest directed path from  $s$  to  $t$ ?

**Topological sort.** Can you draw a digraph so that all edges point upwards?

**Strong connectivity.** Is there a directed path between all pairs of vertices?

**Transitive closure.** For which vertices  $v$  and  $w$  is there a path from  $v$  to  $w$ ?

**PageRank.** What is the importance of a web page?

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# Digraph API

public class Digraph	
Digraph(int V)	<i>create an empty digraph with V vertices</i>
Digraph(In in)	<i>create a digraph from input stream</i>
void addEdge(int v, int w)	<i>add a directed edge <math>v \rightarrow w</math></i>
Iterable<Integer> adj(int v)	<i>vertices pointing from v</i>
int V()	<i>number of vertices</i>
int E()	<i>number of edges</i>
Digraph reverse()	<i>reverse of this digraph</i>
String toString()	<i>string representation</i>

```
In in = new In(args[0]);
Digraph G = new Digraph(in);
```

```
for (int v = 0; v < G.V(); v++)
    for (int w : G.adj(v))
        StdOut.println(v + "->" + w);
```

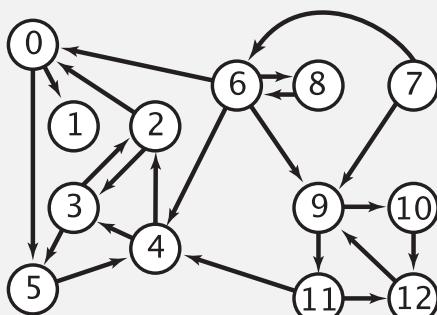
read digraph from  
input stream

print out each  
edge (once)

# Digraph API

**tinyDG.txt**  
V → 13  
E ← 22

4 2  
2 3  
3 2  
6 0  
0 1  
2 0  
11 12  
12 9  
9 10  
9 11  
7 9  
10 12  
11 4  
4 3  
3 5  
6 8  
8 6  
:



```
% java Digraph tinyDG.txt
0->5
0->1
2->0
2->3
3->5
3->2
4->3
4->2
5->4
:
11->4
11->12
12->9
```

```
In in = new In(args[0]);
Digraph G = new Digraph(in);
```

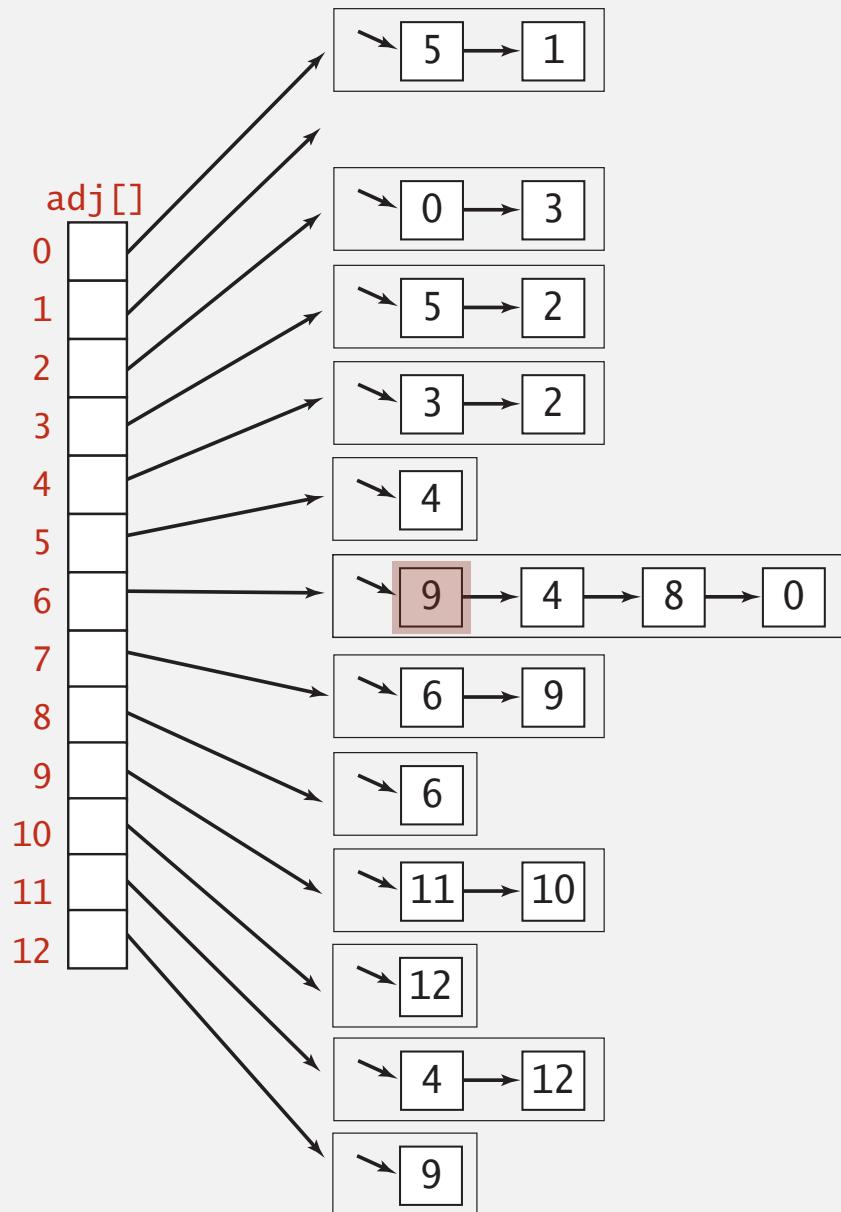
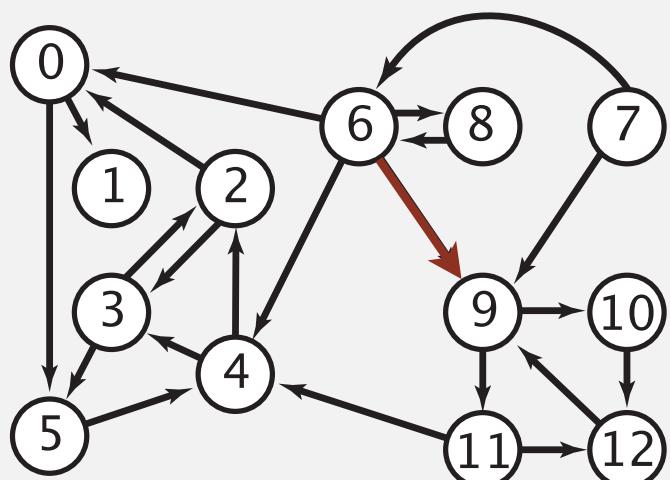
read digraph from  
input stream

```
for (int v = 0; v < G.V(); v++)
    for (int w : G.adj(v))
        StdOut.println(v + "->" + w);
```

print out each  
edge (once)

# Adjacency-lists digraph representation

Maintain vertex-indexed array of lists.



# Adjacency-lists graph representation (review): Java implementation

```
public class Graph
{
    private final int V;
    private final Bag<Integer>[] adj; ← adjacency lists

    public Graph(int V)
    {
        this.V = V;
        adj = (Bag<Integer>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Integer>();
    }

    public void addEdge(int v, int w) ← add edge v-w
    {
        adj[v].add(w);
        adj[w].add(v);
    }

    public Iterable<Integer> adj(int v) ← iterator for vertices
    {   return adj[v];  }
}
```

# Adjacency-lists digraph representation: Java implementation

```
public class Digraph
{
    private final int V;
    private final Bag<Integer>[] adj;           ← adjacency lists

    public Digraph(int V)
    {
        this.V = V;
        adj = (Bag<Integer>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<Integer>();
    }

    public void addEdge(int v, int w)           ← add edge v→w
    {
        adj[v].add(w);
    }

    public Iterable<Integer> adj(int v)         ← iterator for vertices
    {   return adj[v];  }
}
```

# Digraph representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices pointing from  $v$ .
- Real-world digraphs tend to be sparse.

huge number of vertices,  
small average vertex degree

representation	space	insert edge from $v$ to $w$	edge from $v$ to $w$ ?	iterate over vertices pointing from $v$ ?
list of edges	$E$	1	$E$	$E$
adjacency matrix	$V^2$	$1^\dagger$	1	$V$
adjacency lists	$E + V$	1	outdegree( $v$ )	outdegree( $v$ )

$\dagger$  disallows parallel edges

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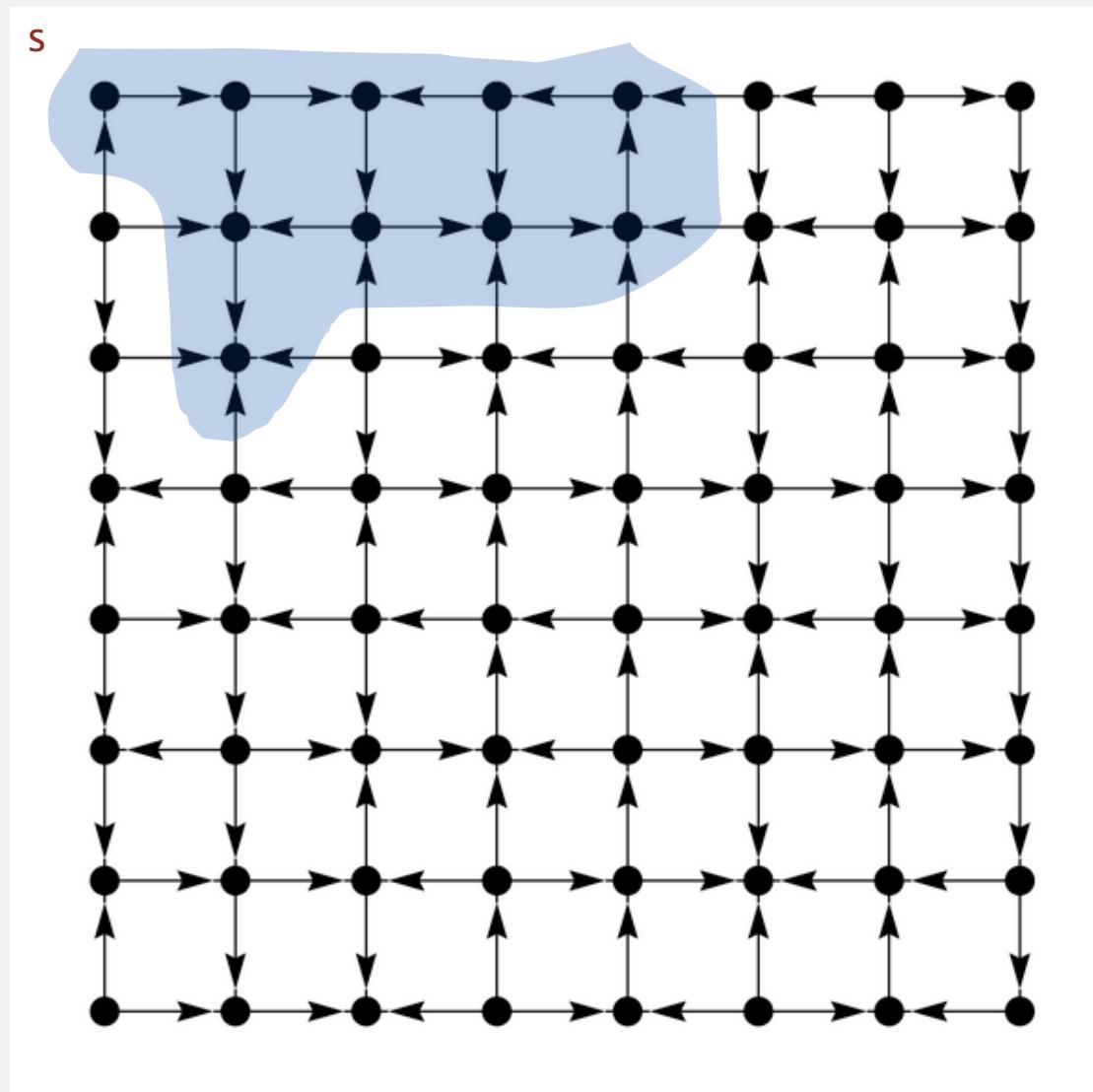
---

- ▶ *introduction*
- ▶ *digraph API*
- ▶ ***digraph search***
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- ▶ *strong components*

# Reachability

---

**Problem.** Find all vertices reachable from  $s$  along a directed path.



# Depth-first search in digraphs

---

Same method as for undirected graphs.

- Every undirected graph is a digraph (with edges in both directions).
- DFS is a **digraph** algorithm.

**DFS (to visit a vertex v)**

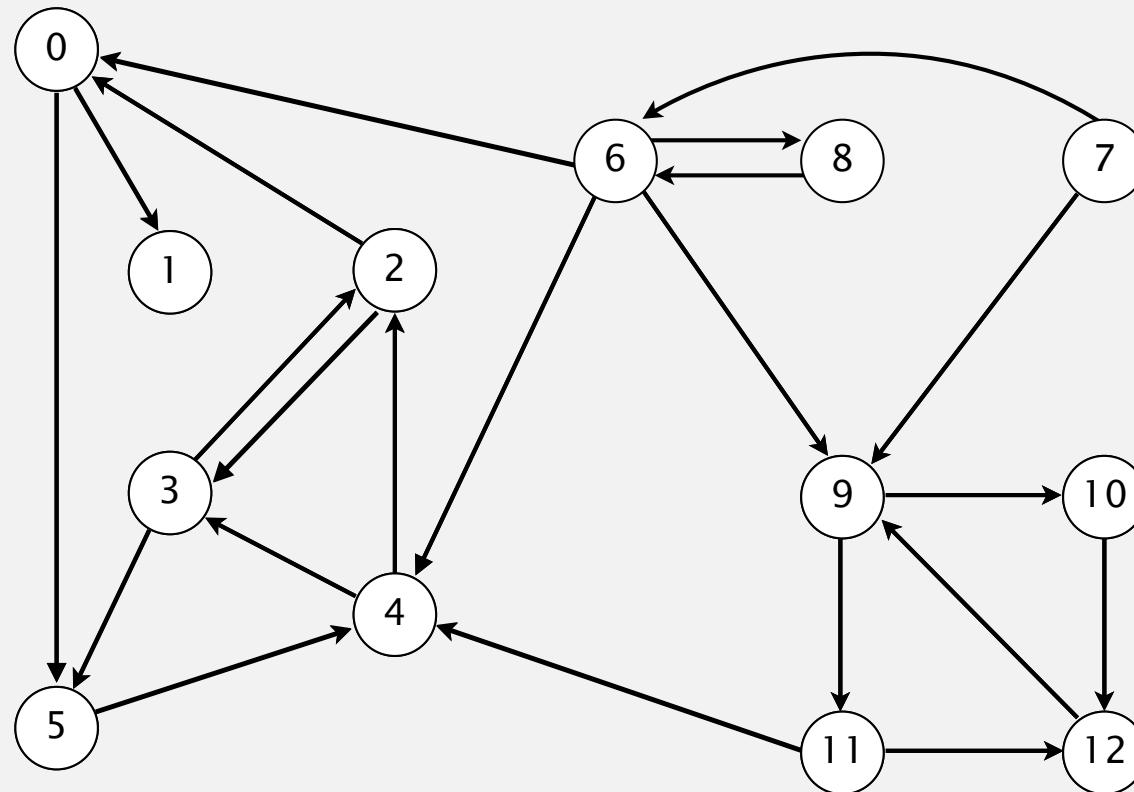
Mark v as visited.

Recursively visit all unmarked  
vertices w pointing from v.

# Depth-first search demo

To visit a vertex  $v$ :

- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices pointing from  $v$ .



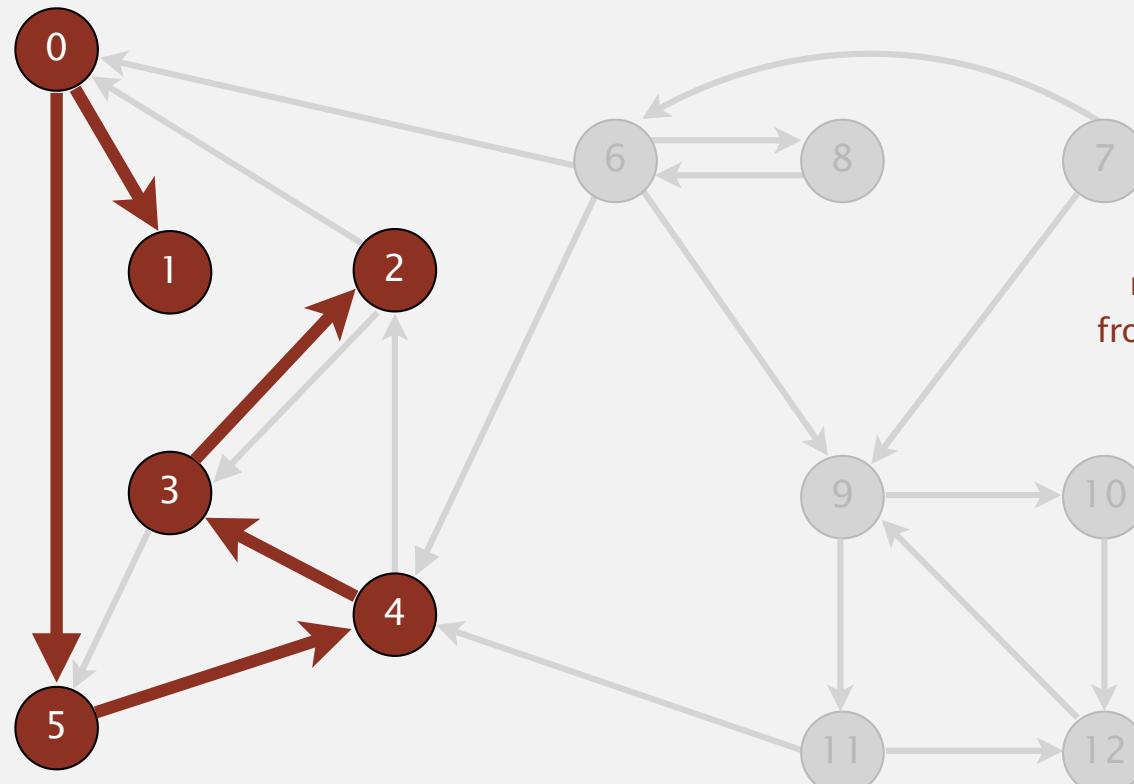
a directed graph

4→2  
2→3  
3→2  
6→0  
0→1  
2→0  
11→12  
12→9  
9→10  
9→11  
8→9  
10→12  
11→4  
4→3  
3→5  
6→8  
8→6  
5→4  
0→5  
6→4  
6→9  
7→6

# Depth-first search demo

To visit a vertex  $v$ :

- Mark vertex  $v$  as visited.
- Recursively visit all unmarked vertices pointing from  $v$ .



reachable from 0

$v$	marked[]	edgeTo[]
0	T	-
1	T	0
2	T	3
3	T	4
4	T	5
5	T	0
6	F	-
7	F	-
8	F	-
9	F	-
10	F	-
11	F	-
12	F	-

# Depth-first search (in undirected graphs)

Recall code for **undirected** graphs.

```
public class DepthFirstSearch
{
    private boolean[] marked;           ← true if path to s

    public DepthFirstSearch(Graph G, int s)
    {
        marked = new boolean[G.V()];
        dfs(G, s);
    }

    private void dfs(Graph G, int v)     ← recursive DFS does the work
    {
        marked[v] = true;
        for (int w : G.adj(v))
            if (!marked[w]) dfs(G, w);
    }

    public boolean visited(int v)        ← client can ask whether any
    {   return marked[v]; }             vertex is connected to s
}
```

# Depth-first search (in directed graphs)

Code for **directed** graphs identical to undirected one.  
[substitute Digraph for Graph]

```
public class DirectedDFS
{
    private boolean[] marked;           ← true if path from s

    public DirectedDFS(Digraph G, int s)
    {
        marked = new boolean[G.V()];
        dfs(G, s);
    }

    private void dfs(Digraph G, int v)
    {
        marked[v] = true;
        for (int w : G.adj(v))
            if (!marked[w]) dfs(G, w);
    }

    public boolean visited(int v)           ← client can ask whether any
    {   return marked[v]; }               vertex is reachable from s
}
```

true if path from s

constructor marks  
vertices reachable from s

recursive DFS does the work

client can ask whether any  
vertex is reachable from s

# Reachability application: program control-flow analysis

Every program is a digraph.

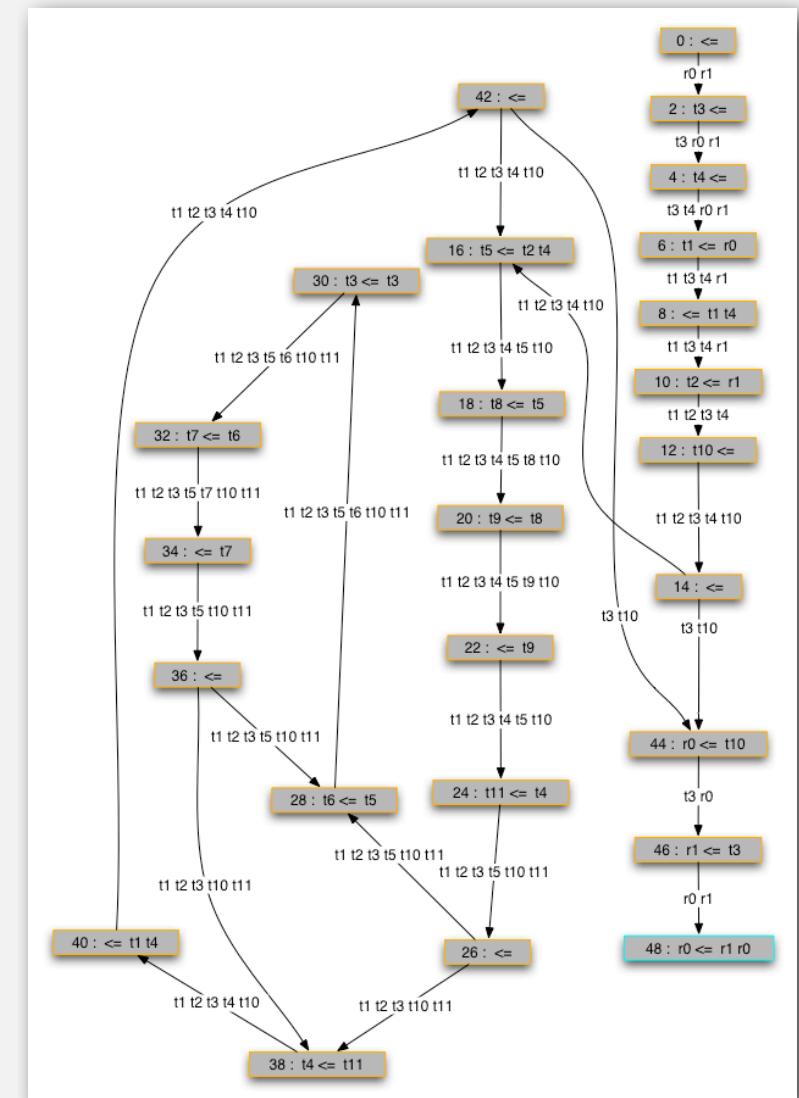
- Vertex = basic block of instructions (straight-line program).
- Edge = jump.

Dead-code elimination.

Find (and remove) unreachable code.

Infinite-loop detection.

Determine whether exit is unreachable.



# Reachability application: mark-sweep garbage collector

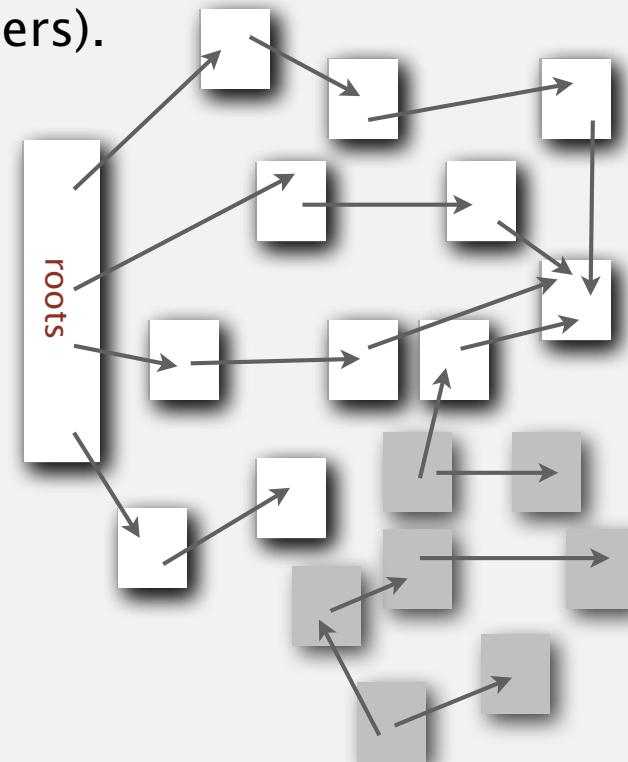
---

Every data structure is a digraph.

- Vertex = object.
- Edge = reference.

**Roots.** Objects known to be directly accessible by program (e.g., stack).

**Reachable objects.** Objects indirectly accessible by program (starting at a root and following a chain of pointers).



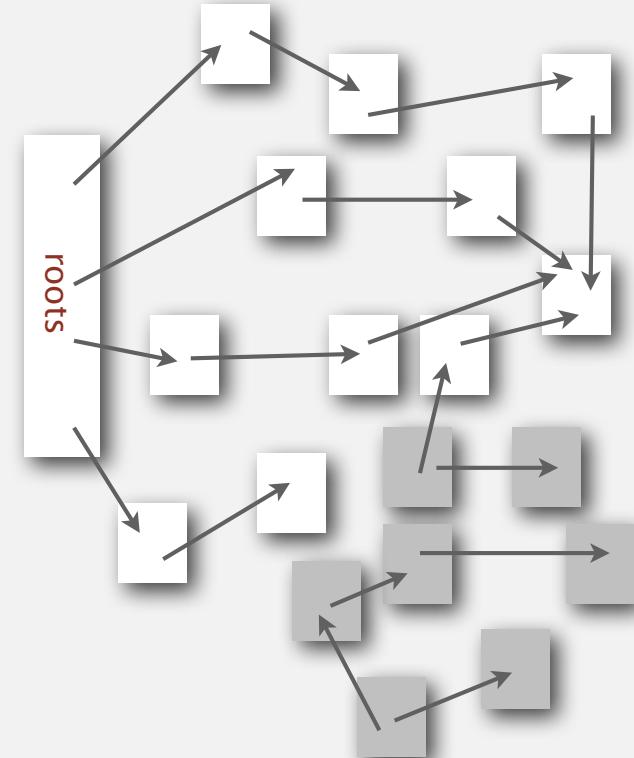
# Reachability application: mark-sweep garbage collector

---

Mark-sweep algorithm. [McCarthy, 1960]

- Mark: mark all reachable objects.
- Sweep: if object is unmarked, it is garbage (so add to free list).

Memory cost. Uses 1 extra mark bit per object (plus DFS stack).



# Depth-first search in digraphs summary

---

DFS enables direct solution of simple digraph problems.

- ✓ • Reachability.
- Path finding.
- Topological sort.
- Directed cycle detection.

Basis for solving difficult digraph problems.

- 2-satisfiability.
- Directed Euler path.
- Strongly-connected components.

SIAM J. COMPUT.  
Vol. 1, No. 2, June 1972

**DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS\***

ROBERT TARJAN†

**Abstract.** The value of depth-first search or “backtracking” as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirected graph are presented. The space and time requirements of both algorithms are bounded by  $k_1V + k_2E + k_3$  for some constants  $k_1$ ,  $k_2$ , and  $k_3$ , where  $V$  is the number of vertices and  $E$  is the number of edges of the graph being examined.

# Breadth-first search in digraphs

Same method as for undirected graphs.

- Every undirected graph is a digraph (with edges in both directions).
- BFS is a **digraph** algorithm.

## **BFS (from source vertex s)**

**Put s onto a FIFO queue, and mark s as visited.**

**Repeat until the queue is empty:**

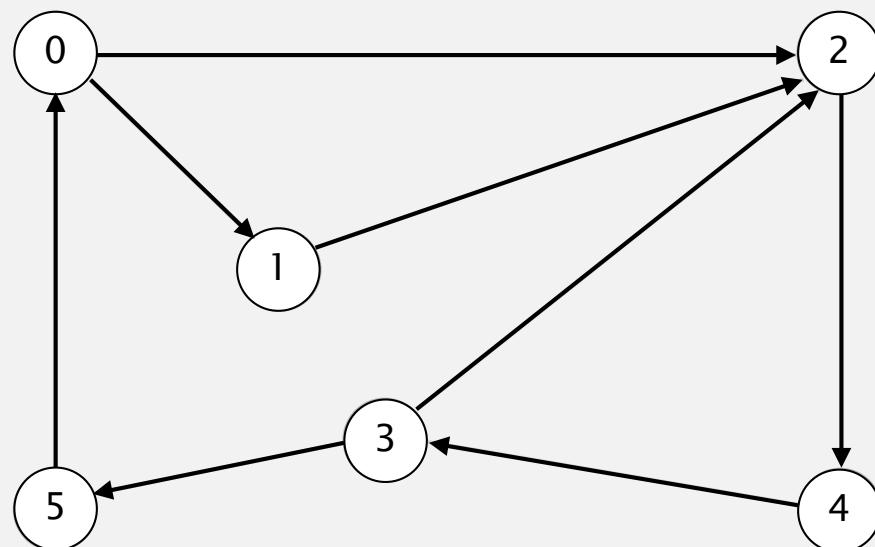
- **remove the least recently added vertex v**
- **for each unmarked vertex pointing from v:**  
**add to queue and mark as visited.**

**Proposition.** BFS computes shortest paths (fewest number of edges) from  $s$  to all other vertices in a digraph in time proportional to  $E + V$ .

# Directed breadth-first search demo

Repeat until queue is empty:

- Remove vertex  $v$  from queue.
- Add to queue all unmarked vertices pointing from  $v$  and mark them.



**tinyDG2.txt**

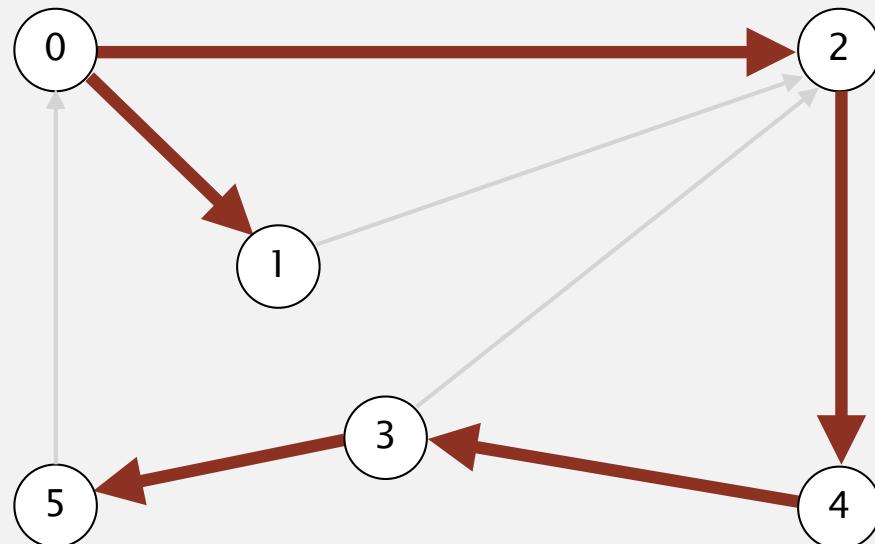
V	E
6	8
8	5 0
5	2 4
2	3 2
4	1 2
3	0 1
1	4 3
0	3 5
2	0 2

graph G

# Directed breadth-first search demo

Repeat until queue is empty:

- Remove vertex  $v$  from queue.
- Add to queue all unmarked vertices pointing from  $v$  and mark them.



$v$	edgeTo[]	distTo[]
0	-	0
1	0	1
2	0	1
3	4	3
4	2	2
5	3	4

done

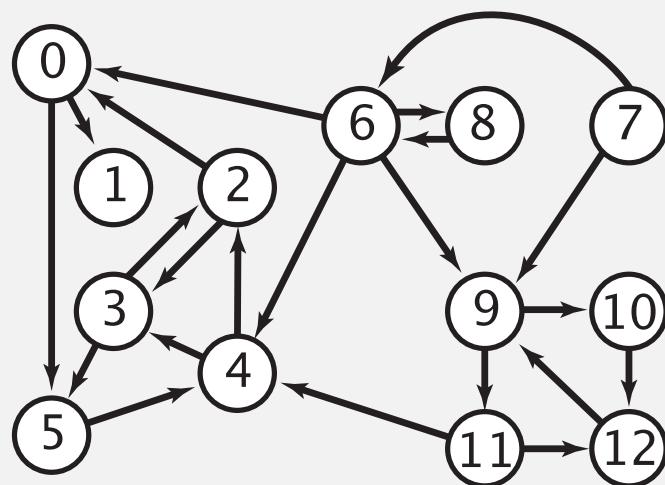
## Multiple-source shortest paths

---

Multiple-source shortest paths. Given a digraph and a **set** of source vertices, find shortest path from any vertex in the set to each other vertex.

Ex.  $S = \{ 1, 7, 10 \}$ .

- Shortest path to 4 is  $7 \rightarrow 6 \rightarrow 4$ .
- Shortest path to 5 is  $7 \rightarrow 6 \rightarrow 0 \rightarrow 5$ .
- Shortest path to 12 is  $10 \rightarrow 12$ .
- ...



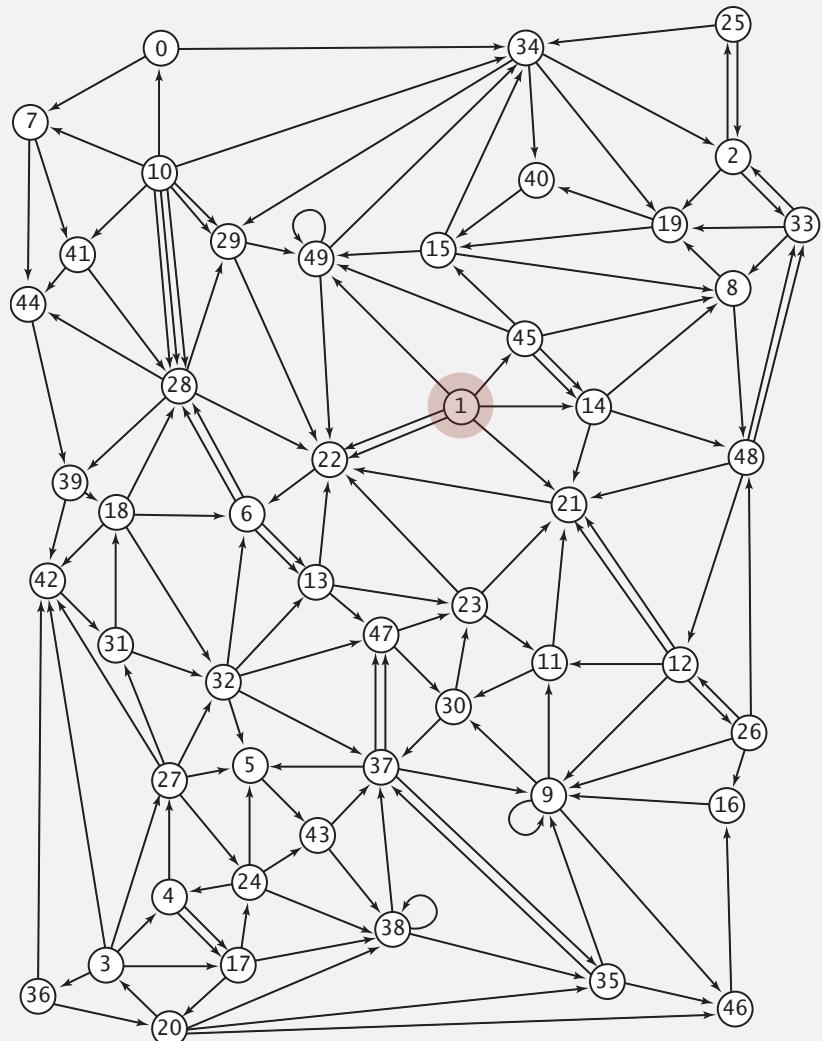
- Q. How to implement multi-source shortest paths algorithm?  
A. Use BFS, but initialize by enqueueing all source vertices.

# Breadth-first search in digraphs application: web crawler

**Goal.** Crawl web, starting from some root web page, say `www.princeton.edu`.

**Solution.** [BFS with implicit digraph]

- Choose root web page as source  $s$ .
- Maintain a Queue of websites to explore.
- Maintain a SET of discovered websites.
- Dequeue the next website and enqueue websites to which it links  
(provided you haven't done so before).



**Q.** Why not use DFS?

# Bare-bones web crawler: Java implementation

```
Queue<String> queue = new Queue<String>();           ← queue of websites to crawl
SET<String> marked = new SET<String>();             ← set of marked websites

String root = "http://www.princeton.edu";
queue.enqueue(root);
marked.add(root);

while (!queue.isEmpty())
{
    String v = queue.dequeue();
    StdOut.println(v);
    In in = new In(v);
    String input = in.readAll();

    String regexp = "http://(\w+\.\w+)*(\w+)";
    Pattern pattern = Pattern.compile(regexp);
    Matcher matcher = pattern.matcher(input);           ← use regular expression to find all URLs
                                                       in website of form http://xxx.yyy.zzz
                                                       [crude pattern misses relative URLs]

    while (matcher.find())
    {
        String w = matcher.group();
        if (!marked.contains(w))
        {
            marked.add(w);
            queue.enqueue(w);                           ← if unmarked, mark it and put
                                                       on the queue
        }
    }
}
```

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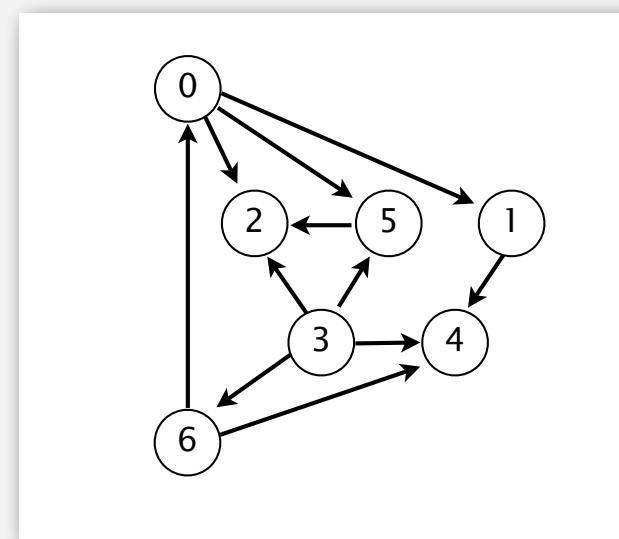
# Precedence scheduling

**Goal.** Given a set of tasks to be completed with precedence constraints, in which order should we schedule the tasks?

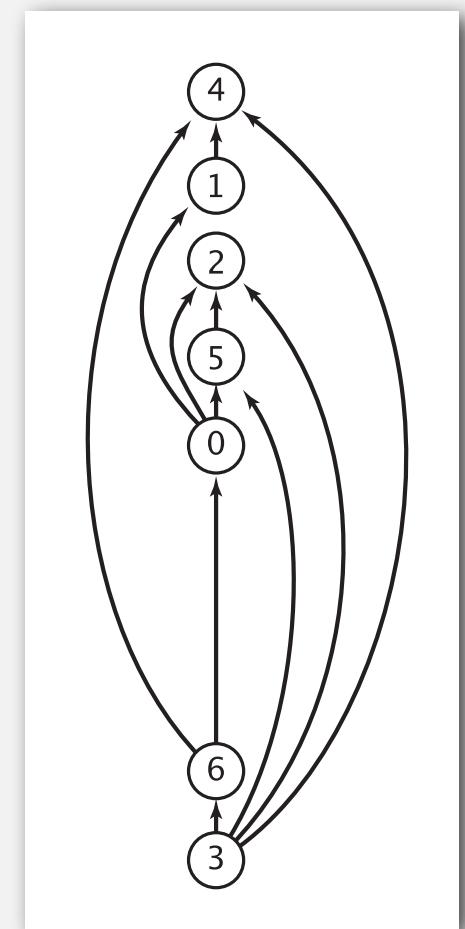
**Digraph model.** vertex = task; edge = precedence constraint.

- 0. Algorithms
- 1. Complexity Theory
- 2. Artificial Intelligence
- 3. Intro to CS
- 4. Cryptography
- 5. Scientific Computing
- 6. Advanced Programming

tasks



precedence constraint graph



feasible schedule

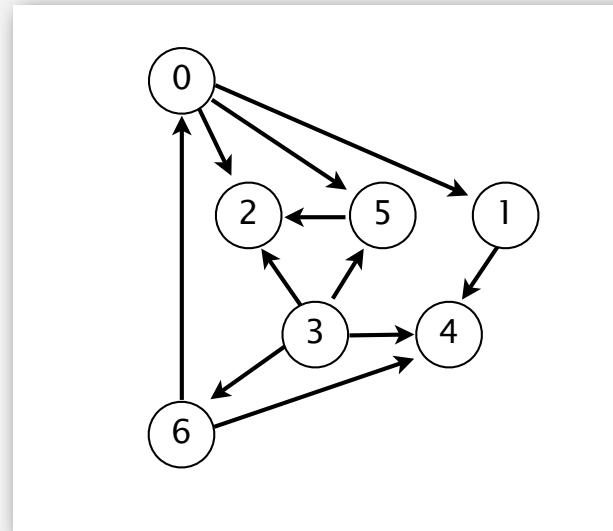
# Topological sort

DAG. Directed acyclic graph.

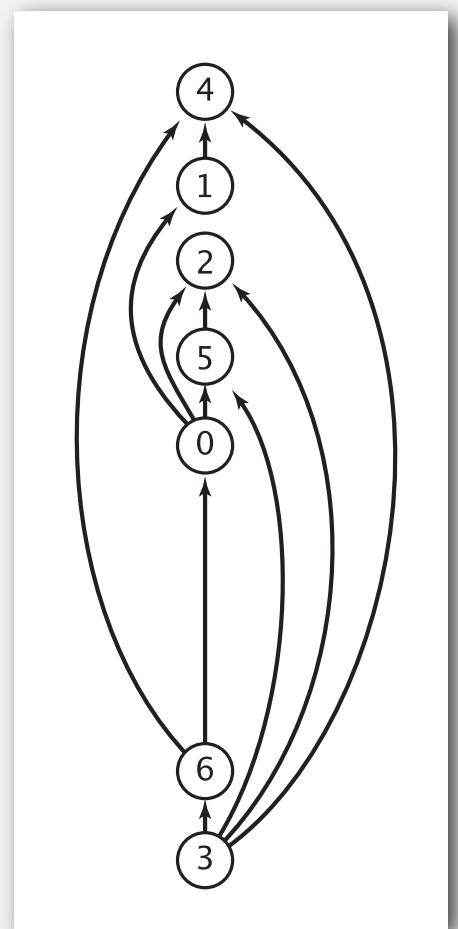
Topological sort. Redraw DAG so all edges point upwards.

$0 \rightarrow 5$	$0 \rightarrow 2$
$0 \rightarrow 1$	$3 \rightarrow 6$
$3 \rightarrow 5$	$3 \rightarrow 4$
$5 \rightarrow 2$	$6 \rightarrow 4$
$6 \rightarrow 0$	$3 \rightarrow 2$
$1 \rightarrow 4$	

directed edges



DAG

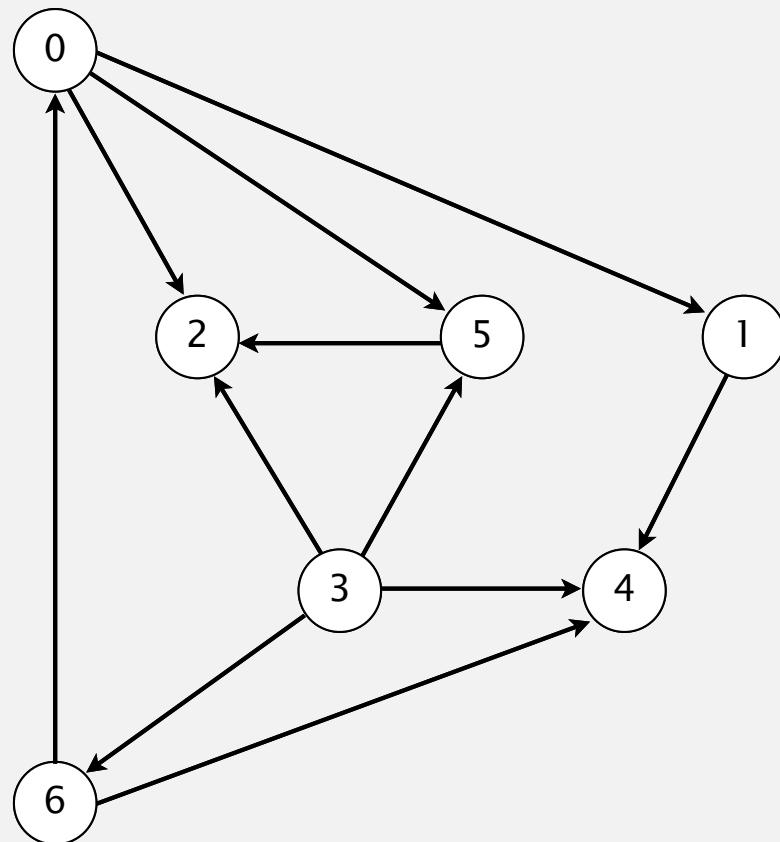


topological order

Solution. DFS. What else?

# Topological sort demo

- Run depth-first search.
- Return vertices in reverse postorder.

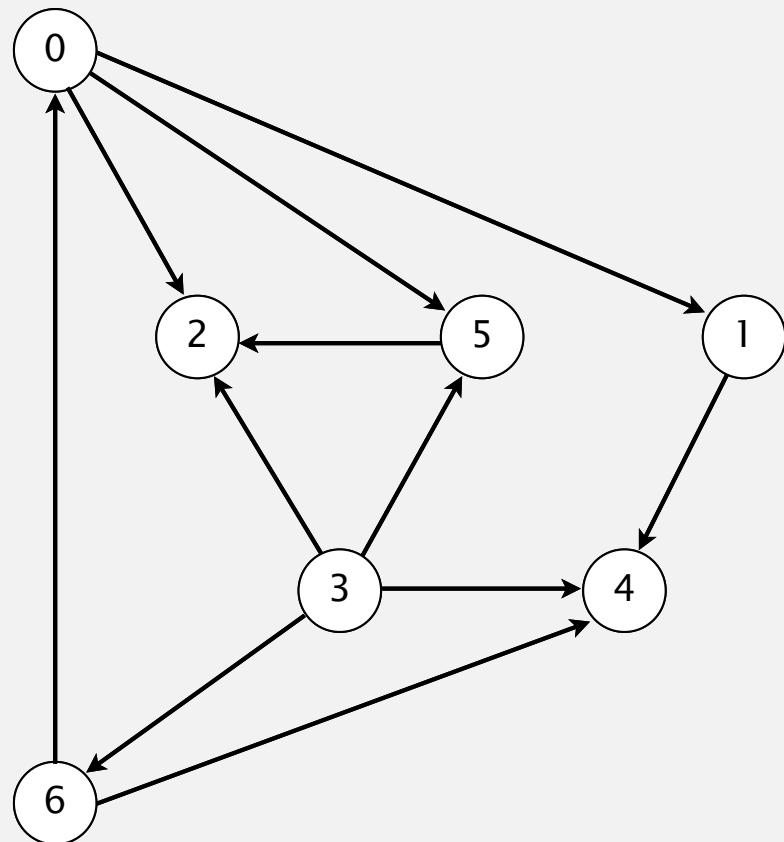


$0 \rightarrow 5$   
 $0 \rightarrow 2$   
 $0 \rightarrow 1$   
 $3 \rightarrow 6$   
 $3 \rightarrow 5$   
 $3 \rightarrow 4$   
 $5 \rightarrow 2$   
 $6 \rightarrow 4$   
 $6 \rightarrow 0$   
 $3 \rightarrow 2$   
 $1 \rightarrow 4$

a directed acyclic graph

# Topological sort demo

- Run depth-first search.
- Return vertices in reverse postorder.



**postorder**

4 1 2 5 0 6 3

**topological order**

3 6 0 5 2 1 4

**done**

# Depth-first search order

```
public class DepthFirstOrder
{
    private boolean[] marked;
    private Stack<Integer> reversePost;

    public DepthFirstOrder(Digraph G)
    {
        reversePost = new Stack<Integer>();
        marked = new boolean[G.V()];
        for (int v = 0; v < G.V(); v++)
            if (!marked[v]) dfs(G, v);
    }

    private void dfs(Digraph G, int v)
    {
        marked[v] = true;
        for (int w : G.adj(v))
            if (!marked[w]) dfs(G, w);
        reversePost.push(v);
    }

    public Iterable<Integer> reversePost() ←
    {   return reversePost;   }
}
```

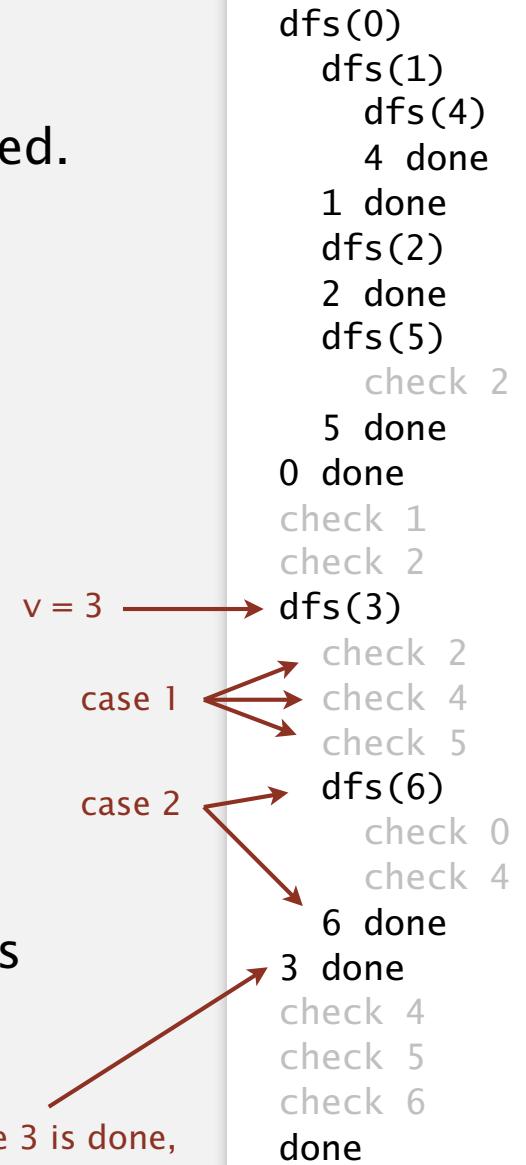
returns all vertices in  
“reverse DFS postorder”

# Topological sort in a DAG: correctness proof

**Proposition.** Reverse DFS postorder of a DAG is a topological order.

**Pf.** Consider any edge  $v \rightarrow w$ . When  $\text{dfs}(v)$  is called:

- Case 1:  $\text{dfs}(w)$  has already been called and returned.  
Thus,  $w$  was done before  $v$ .
- Case 2:  $\text{dfs}(w)$  has not yet been called.  
 $\text{dfs}(w)$  will get called directly or indirectly  
by  $\text{dfs}(v)$  and will finish before  $\text{dfs}(v)$ .  
Thus,  $w$  will be done before  $v$ .
- Case 3:  $\text{dfs}(w)$  has already been called,  
but has not yet returned.  
Can't happen in a DAG: function call stack contains  
path from  $w$  to  $v$ , so  $v \rightarrow w$  would complete a cycle.

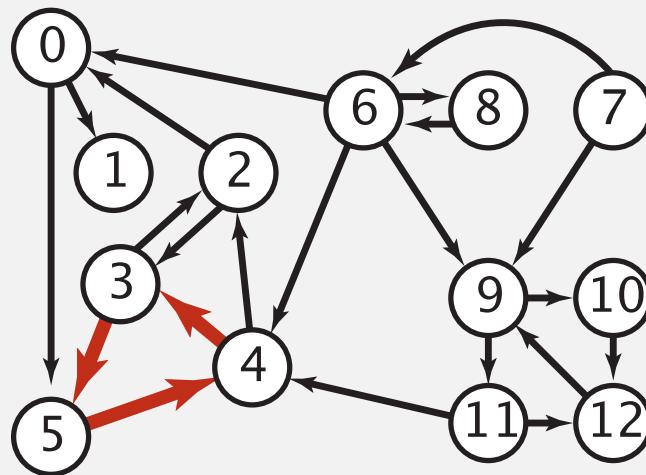


# Directed cycle detection

**Proposition.** A digraph has a topological order iff no directed cycle.

Pf.

- If directed cycle, topological order impossible.
- If no directed cycle, DFS-based algorithm finds a topological order.



**Goal.** Given a digraph, find a directed cycle.

**Solution.** DFS. What else? See textbook.

# Directed cycle detection application: precedence scheduling

---

Scheduling. Given a set of tasks to be completed with precedence constraints, in what order should we schedule the tasks?

PAGE 3

DEPARTMENT	COURSE	DESCRIPTION	PREREQS
COMPUTER SCIENCE	CPSC 432	INTERMEDIATE COMPILER DESIGN, WITH A FOCUS ON DEPENDENCY RESOLUTION.	CPSC 432

<http://xkcd.com/754>

Remark. A directed cycle implies scheduling problem is infeasible.

## Directed cycle detection application: cyclic inheritance

---

The Java compiler does cycle detection.

```
public class A extends B
{
    ...
}
```

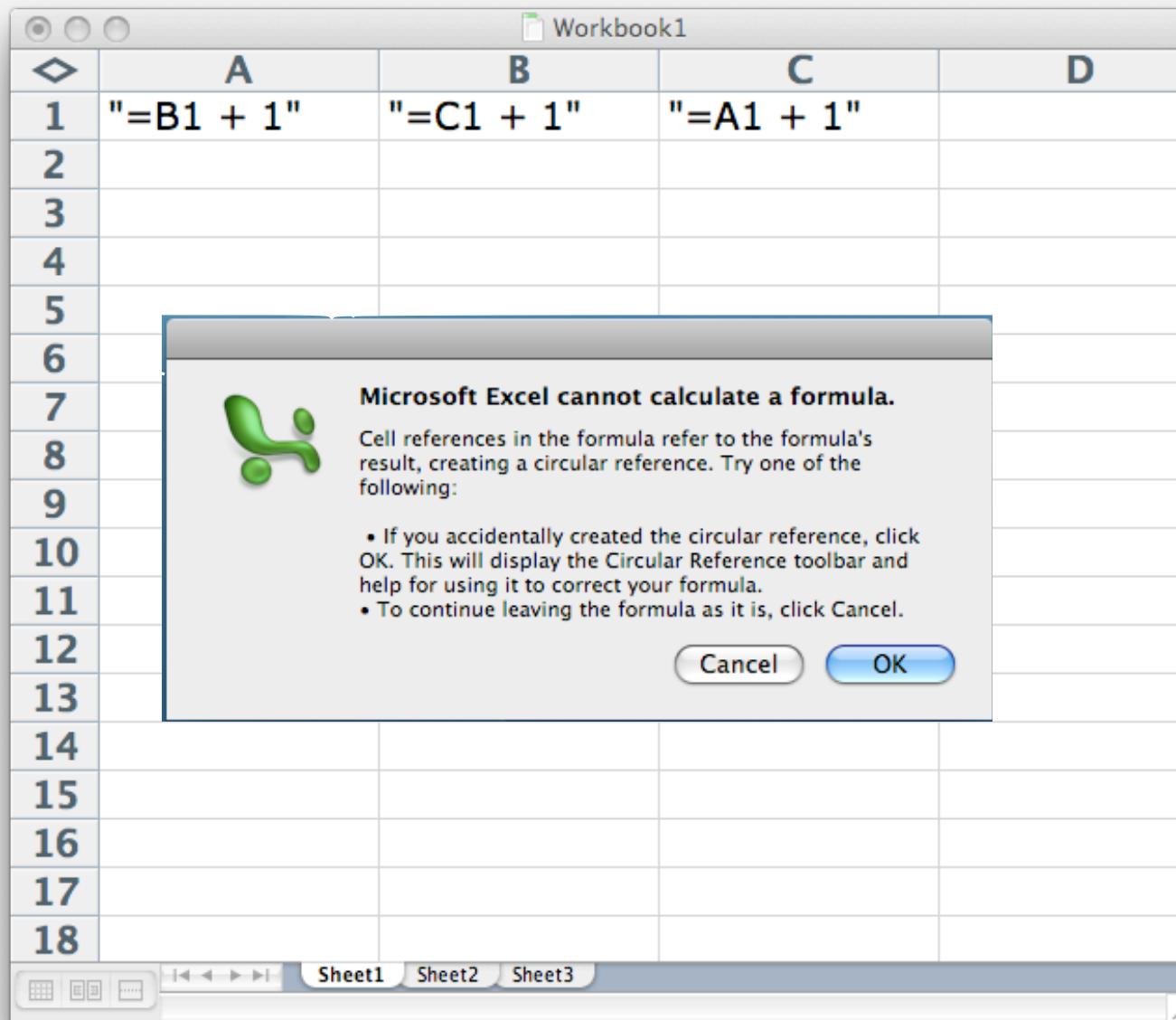
```
public class B extends C
{
    ...
}
```

```
public class C extends A
{
    ...
}
```

```
% javac A.java
A.java:1: cyclic inheritance
involving A
public class A extends B { }
^
1 error
```

# Directed cycle detection application: spreadsheet recalculation

Microsoft Excel does cycle detection (and has a circular reference toolbar!)



# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 4.2 DIRECTED GRAPHS

---

- ▶ *introduction*
- ▶ *digraph API*
- ▶ *digraph search*
- ▶ *topological sort*
- ▶ *strong components*

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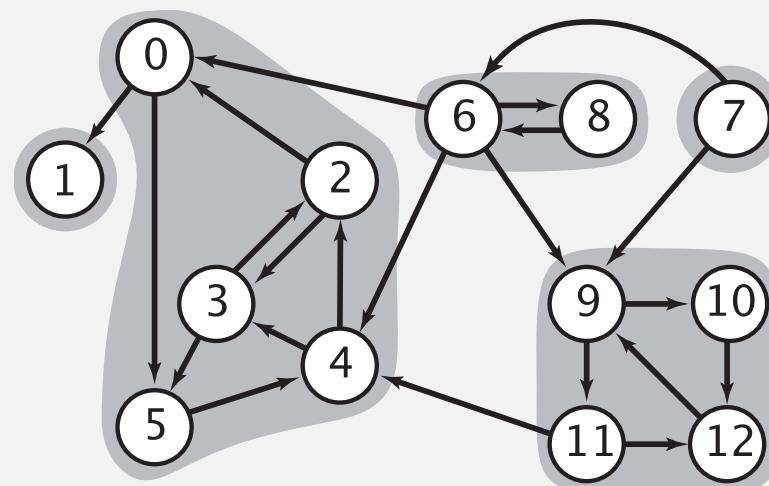
# Strongly-connected components

**Def.** Vertices  $v$  and  $w$  are **strongly connected** if there is both a directed path from  $v$  to  $w$  **and** a directed path from  $w$  to  $v$ .

**Key property.** Strong connectivity is an **equivalence relation**:

- $v$  is strongly connected to  $v$ .
- If  $v$  is strongly connected to  $w$ , then  $w$  is strongly connected to  $v$ .
- If  $v$  is strongly connected to  $w$  and  $w$  to  $x$ , then  $v$  is strongly connected to  $x$ .

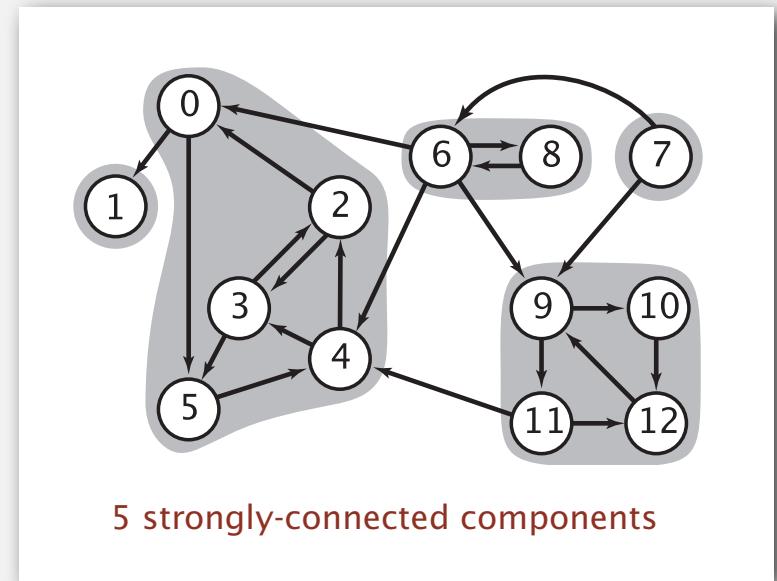
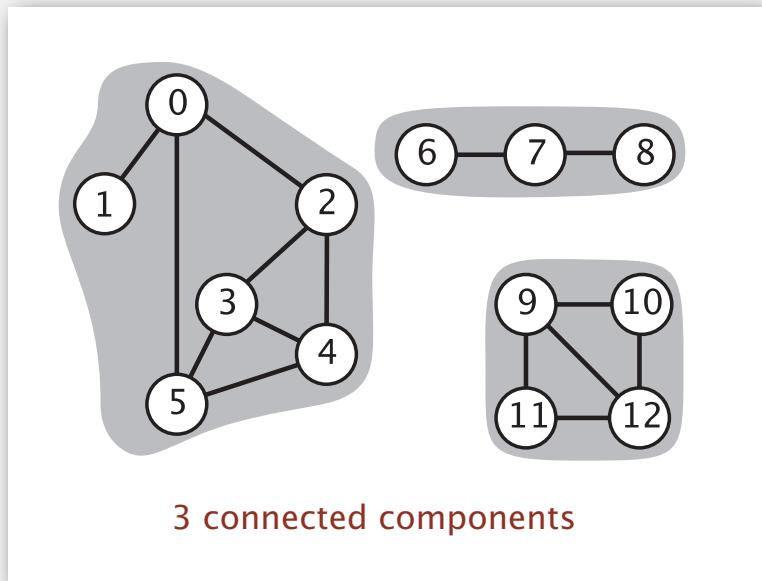
**Def.** A **strong component** is a maximal subset of strongly-connected vertices.



# Connected components vs. strongly-connected components

v and w are **connected** if there is a path between v and w

v and w are **strongly connected** if there is both a directed path from v to w and a directed path from w to v



connected component id (easy to compute with DFS)

0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	1	1	1	2	2	2	2

```
public int connected(int v, int w)
{   return cc[v] == cc[w]; }
```

constant-time client connectivity query

strongly-connected component id (how to compute?)

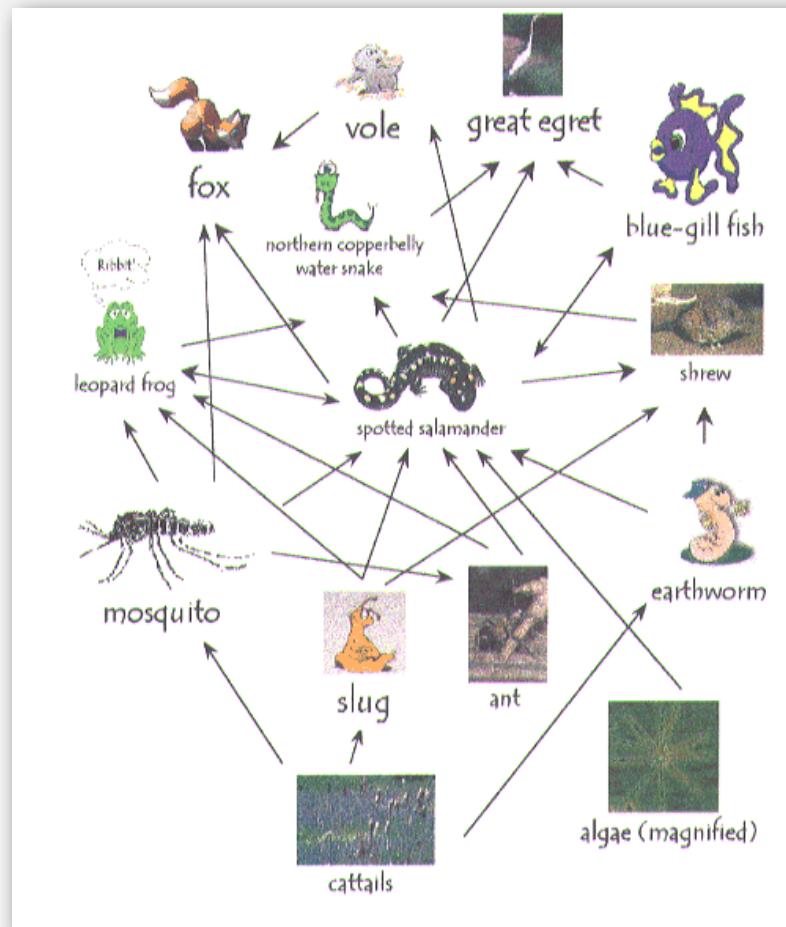
0	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	1	1	1	3	4	3	2	2	2	2

```
public int stronglyConnected(int v, int w)
{   return scc[v] == scc[w]; }
```

constant-time client strong-connectivity query

# Strong component application: ecological food webs

Food web graph. Vertex = species; edge = from producer to consumer.



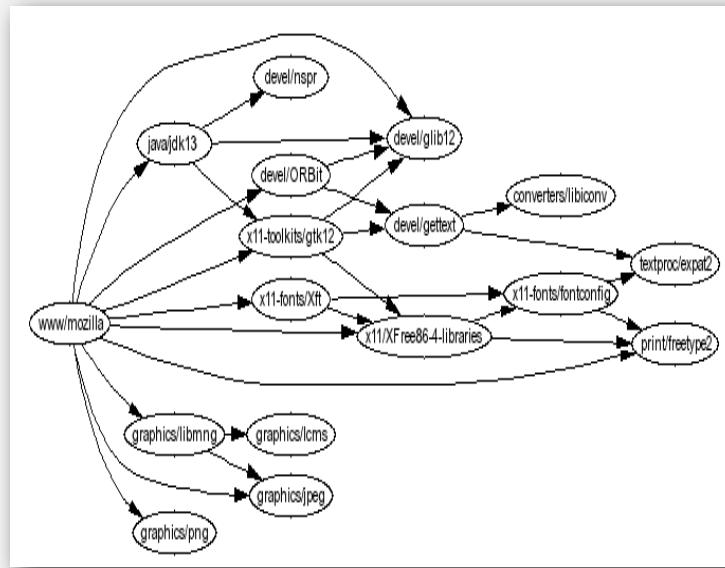
<http://www.twinklives.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.gif>

Strong component. Subset of species with common energy flow.

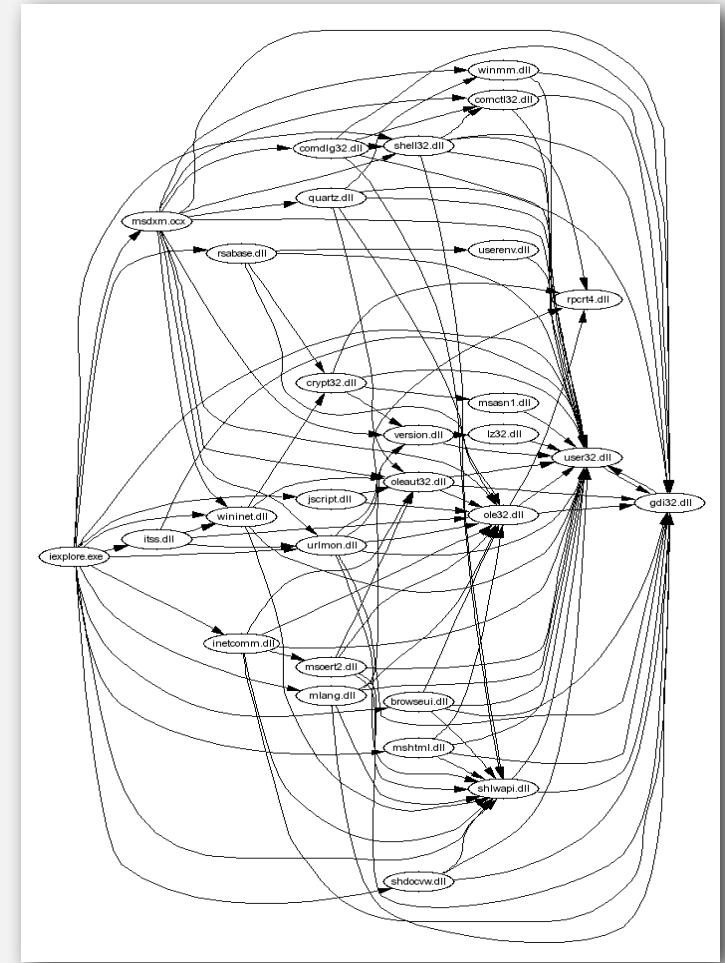
# Strong component application: software modules

# Software module dependency graph.

- Vertex = software module.
  - Edge: from module to dependency.



# Firefox



# Internet Explorer

**Strong component.** Subset of mutually interacting modules.

**Approach 1. Package strong components together.**

**Approach 2. Use to improve design!**

# Strong components algorithms: brief history

---

1960s: Core OR problem.

- Widely studied; some practical algorithms.
- Complexity not understood.

1972: linear-time DFS algorithm (Tarjan).

- Classic algorithm.
- Level of difficulty: Algs4++.
- Demonstrated broad applicability and importance of DFS.

1980s: easy two-pass linear-time algorithm (Kosaraju-Sharir).

- Forgot notes for lecture; developed algorithm in order to teach it!
- Later found in Russian scientific literature (1972).

1990s: more easy linear-time algorithms.

- Gabow: fixed old OR algorithm.
- Cheriyan-Mehlhorn: needed one-pass algorithm for LEDA.

# Kosaraju-Sharir algorithm: intuition

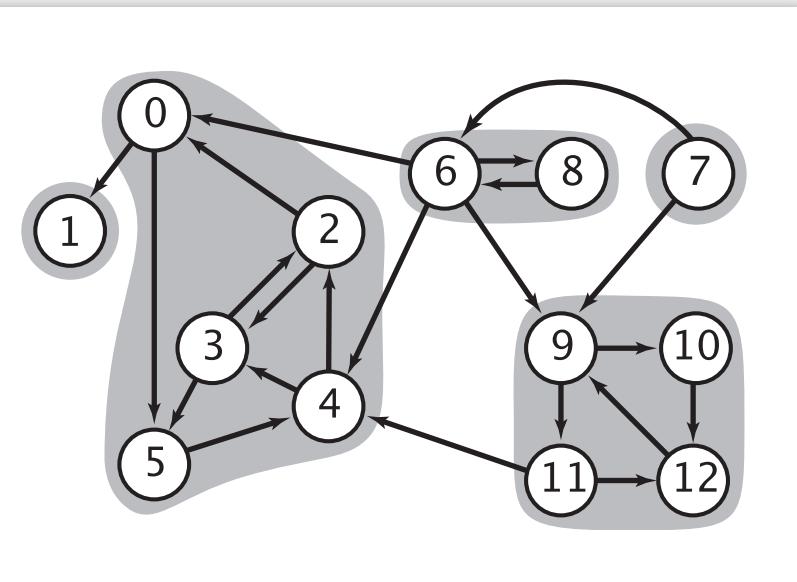
Reverse graph. Strong components in  $G$  are same as in  $G^R$ .

Kernel DAG. Contract each strong component into a single vertex.

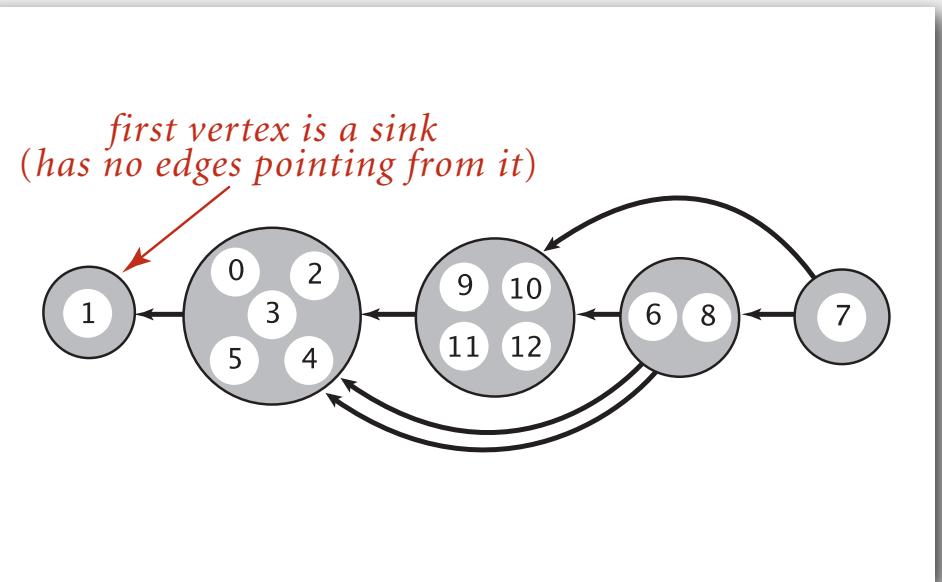
Idea.

- Compute topological order (reverse postorder) in kernel DAG.
- Run DFS, considering vertices in reverse topological order.

how to compute?



digraph G and its strong components

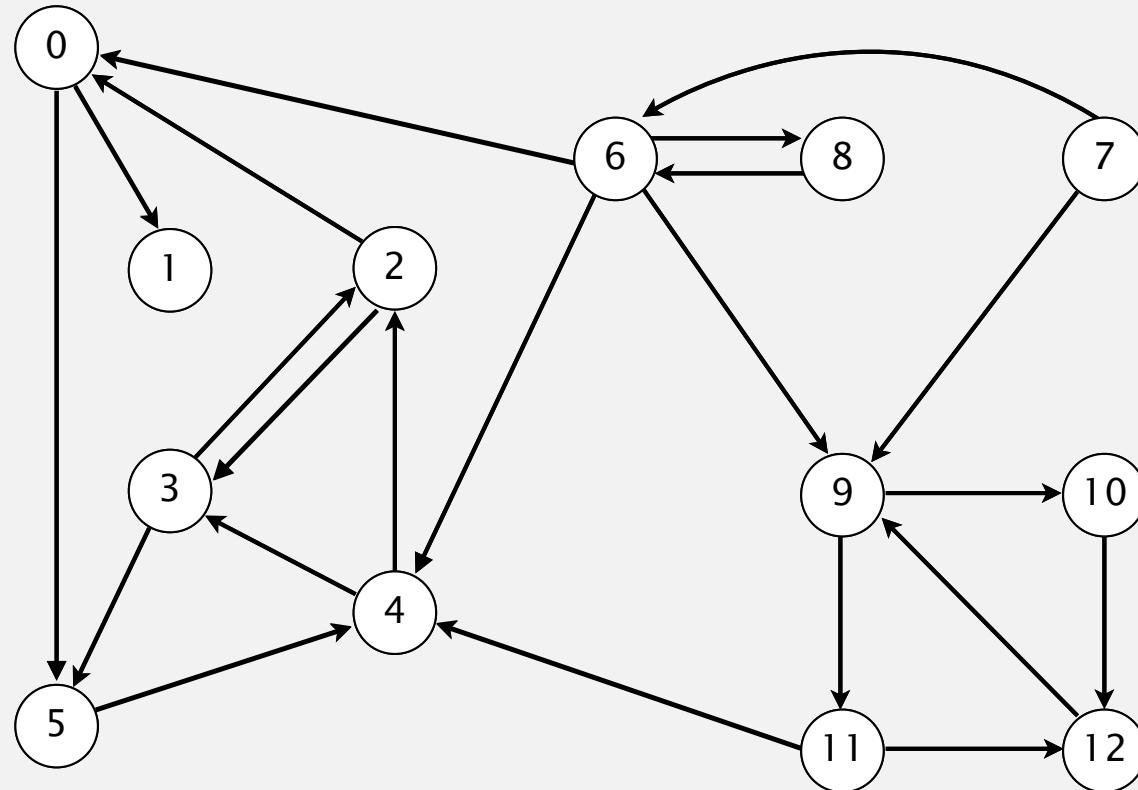


kernel DAG of G (in reverse topological order)

# Kosaraju-Sharir algorithm demo

Phase 1. Compute reverse postorder in  $G^R$ .

Phase 2. Run DFS in  $G$ , visiting unmarked vertices in reverse postorder of  $G^R$ .



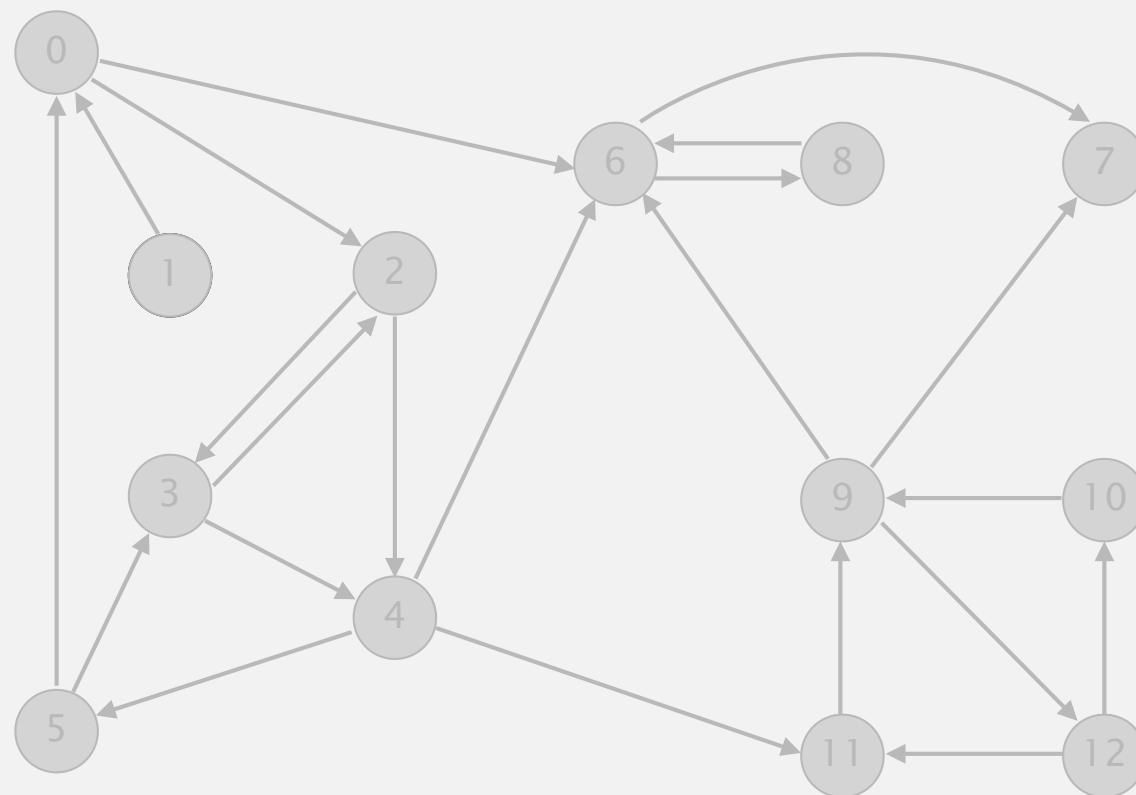
digraph G

# Kosaraju-Sharir algorithm demo

---

Phase 1. Compute reverse postorder in  $G^R$ .

1 0 2 4 5 3 11 9 12 10 6 7 8

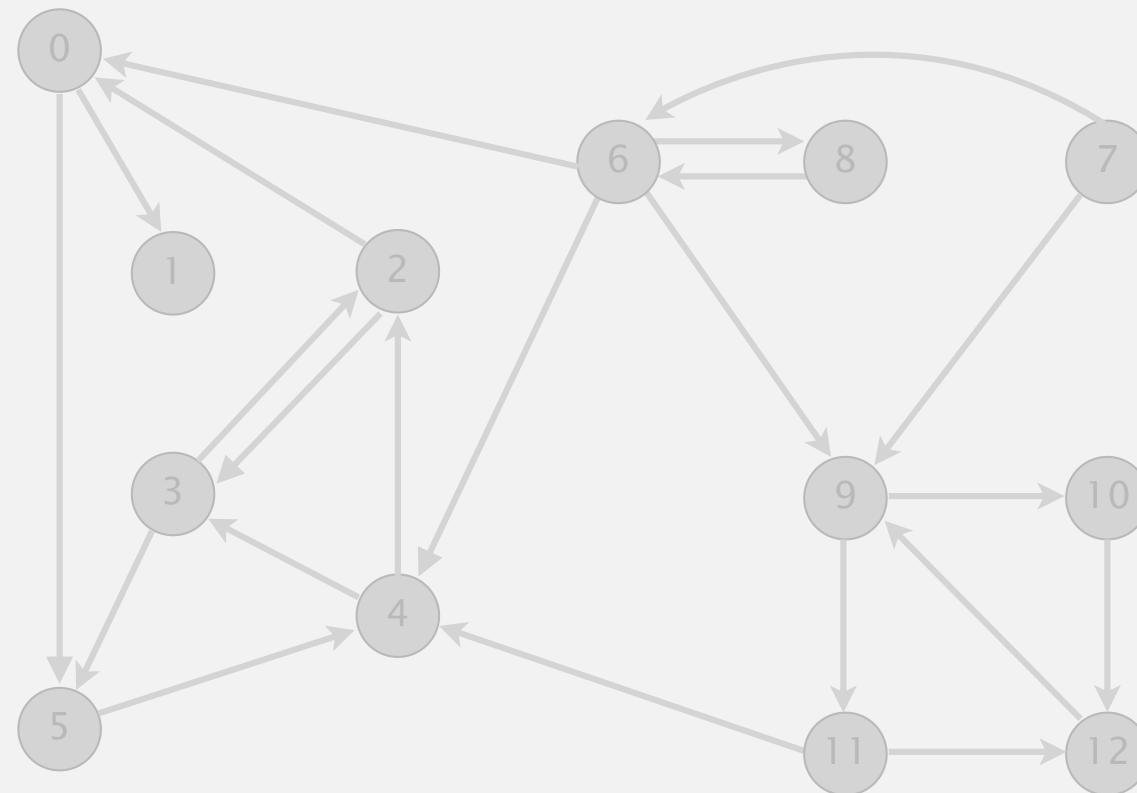


reverse digraph  $G^R$

# Kosaraju-Sharir algorithm demo

Phase 2. Run DFS in  $G$ , visiting unmarked vertices in reverse postorder of  $G^R$ .

1 0 2 4 5 3 11 9 12 10 6 7 8



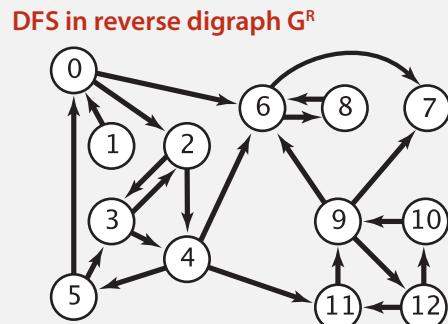
v	scc[]
0	1
1	0
2	1
3	1
4	1
5	3
6	4
7	4
8	3
9	2
10	2
11	2
12	2

done

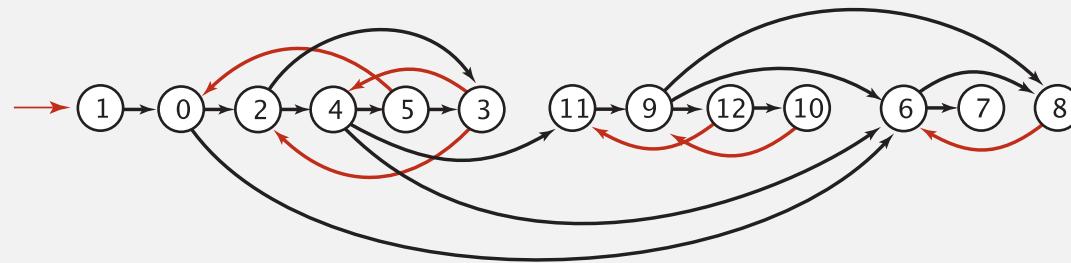
## Kosaraju-Sharir algorithm

Simple (but mysterious) algorithm for computing strong components.

- Phase 1: run DFS on  $G^R$  to compute reverse postorder.
  - Phase 2: run DFS on  $G$ , considering vertices in order given by first DFS.



*check unmarked vertices in the order*



*reverse postorder for use in second dfs()*

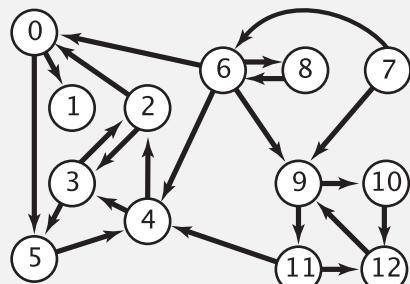
```
dfs(0)
  |
  dfs(6)
    |
    dfs(8)
      |
      check 6
      8 done
      dfs(7)
      7 done
    6 done
    dfs(2)
      |
      dfs(4)
        |
        dfs(11)
          |
          dfs(9)
            |
            dfs(12)
              |
              check 11
              dfs(10)
                |
                check 9
                10 done
              12 done
              check 7
              check 6
```

# Kosaraju-Sharir algorithm

Simple (but mysterious) algorithm for computing strong components.

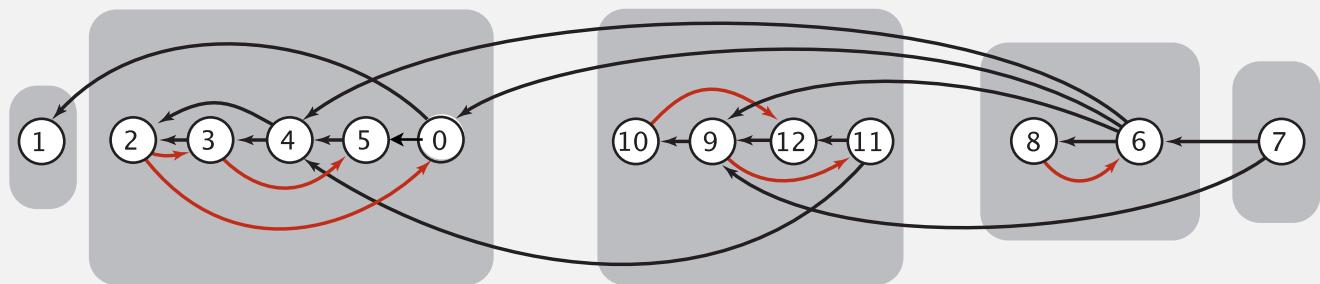
- Phase 1: run DFS on  $G^R$  to compute reverse postorder.
- Phase 2: run DFS on  $G$ , considering vertices in order given by first DFS.

DFS in original digraph  $G$



check unmarked vertices in the order

1 0 2 4 5 3 11 9 12 10 6 7 8



dfs(1)  
1 done

dfs(0)  
dfs(5)  
dfs(4)  
dfs(3)  
check 5  
dfs(2)  
check 0  
check 3  
2 done  
3 done  
check 2  
4 done  
5 done  
check 1  
0 done  
check 2  
check 4  
check 5  
check 3

dfs(11)  
check 4  
dfs(12)  
dfs(9)  
check 11  
dfs(10)  
check 12  
10 done  
9 done  
12 done  
11 done  
check 9  
check 12  
check 10

dfs(6)  
check 9  
check 4  
dfs(8)  
check 6  
8 done  
check 0  
6 done

dfs(7)  
check 6  
check 9  
7 done  
check 8

## Kosaraju-Sharir algorithm

---

**Proposition.** Kosaraju-Sharir algorithm computes the strong components of a digraph in time proportional to  $E + V$ .

Pf.

- Running time: bottleneck is running DFS twice (and computing  $G^R$ ).
- Correctness: tricky, see textbook (2<sup>nd</sup> printing).
- Implementation: easy!

# Connected components in an undirected graph (with DFS)

---

```
public class CC
{
    private boolean marked[];
    private int[] id;
    private int count;

    public CC(Graph G)
    {
        marked = new boolean[G.V()];
        id = new int[G.V()];

        for (int v = 0; v < G.V(); v++)
        {
            if (!marked[v])
            {
                dfs(G, v);
                count++;
            }
        }
    }

    private void dfs(Graph G, int v)
    {
        marked[v] = true;
        id[v] = count;
        for (int w : G.adj(v))
            if (!marked[w])
                dfs(G, w);
    }

    public boolean connected(int v, int w)
    { return id[v] == id[w]; }
}
```

# Strong components in a digraph (with two DFSs)

```
public class KosarajuSharirSCC
{
    private boolean marked[];
    private int[] id;
    private int count;

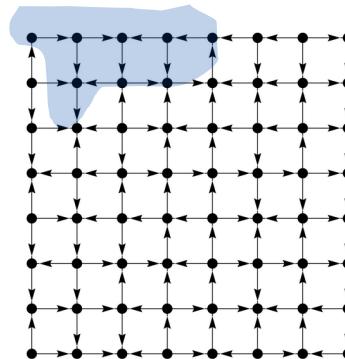
    public KosarajuSharirSCC(Digraph G)
    {
        marked = new boolean[G.V()];
        id = new int[G.V()];
        DepthFirstOrder dfs = new DepthFirstOrder(G.reverse());
        for (int v : dfs.reversePost())
        {
            if (!marked[v])
            {
                dfs(G, v);
                count++;
            }
        }
    }

    private void dfs(Digraph G, int v)
    {
        marked[v] = true;
        id[v] = count;
        for (int w : G.adj(v))
            if (!marked[w])
                dfs(G, w);
    }

    public boolean stronglyConnected(int v, int w)
    {   return id[v] == id[w];  }
}
```

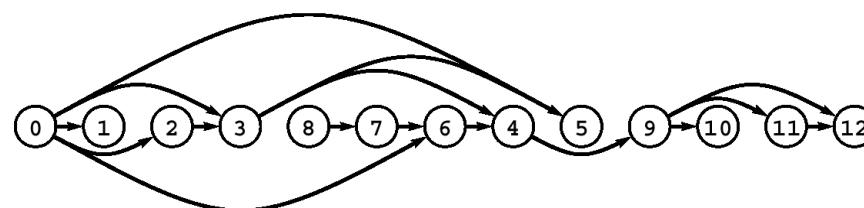
# Digraph-processing summary: algorithms of the day

single-source  
reachability  
in a digraph



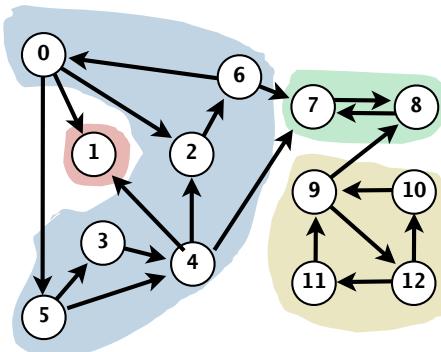
DFS

topological sort  
in a DAG



DFS

strong  
components  
in a digraph



Kosaraju-Sharir  
DFS (twice)

# Algorithms

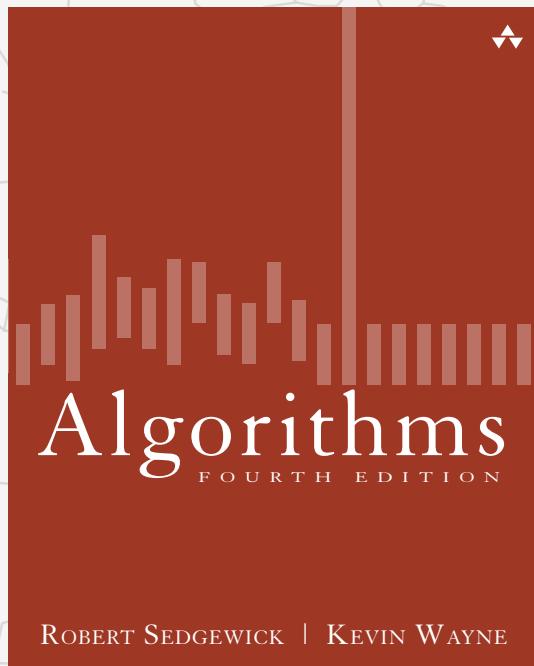
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