Financial Engineering & Risk Management

Review of the Binomial Model for Option Pricing

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Why is There a Skew?

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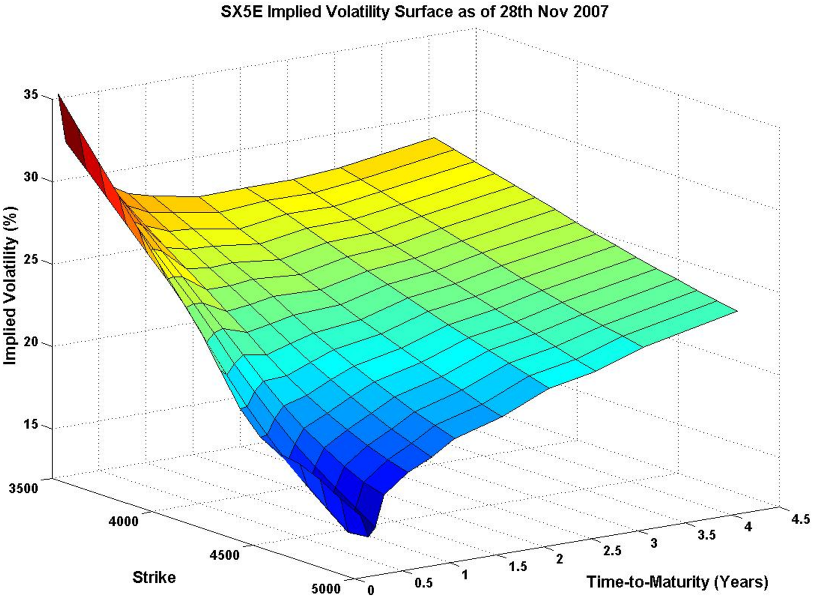
SX5E Implied Volatility Surface as of 28th Nov 2007
Implied Volatility (%)
35
30
25
20
15
3500
4000
4500
5000
Strike
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
Time-to-Maturity (Years)

Why is There a Skew?

Shape of the volatility surface is always changing but in general there is typically a skew in the case of stocks and stock indices

– that is more pronounced at shorter expirations.

As stated earlier, there was little or no skew before Wall street crash of 1987.

There are at least two principal explanations for the skew:

Risk aversion: This explanation can appear in many guises:

1. Security prices often jump and jumps to the downside tend to be larger and more frequent than jumps to the upside.
2. As markets go down, fear sets in and volatility goes up.
3. Supply and demand. Investors like to protect their portfolio by purchasing out-of-the-money puts and so there is more demand for options with lower strikes.

The leverage effect: The total value of company assets, i.e. debt $+$ equity, is a more natural candidate to follow GBM

– in this case equity volatility should increase as the equity value decreases.

The Leverage Effect

Let V, E and D denote the total value of a company, the company's equity and the company's debt, respectively.

Then fundamental accounting equation states that

$$V = D + E. \tag{1}$$

Equation (1) is the basis for classical structural models

– sometimes used to price **risky debt** and credit default swaps.

Merton (1970's) recognized that equity could be viewed as a **call option** on V with strike equal to D

– valid since debt-holders get paid **before** equity holders.

The Leverage Effect

Let ΔV, ΔE and ΔD be the change in values of V, E and D, respectively.

Then $V + \Delta V = (E + \Delta E) + (D + \Delta D)$ so that

$$\begin{aligned} \frac{V + \Delta V}{V} &= \frac{E + \Delta E}{V} + \frac{D + \Delta D}{V} \\ &= \frac{E}{V}\left(\frac{E + \Delta E}{E}\right) + \frac{D}{V}\left(\frac{D + \Delta D}{D}\right). \end{aligned} \tag{2}$$

If equity component is substantial so that it absorbs **almost all losses** and the debt is not very risky, then ΔD will be very small and (2) implies

$$\sigma_V \approx \frac{E}{V}\,\sigma_E$$

where σ_V and σ_E are the firm value and equity volatilities, respectively.

Therefore have

$$\sigma_E \approx \frac{V}{E}\,\sigma_V = \left(1 + \frac{D}{E}\right)\sigma_V \tag{3}$$

If σ_V a constant, then σ_E will **increase** as E **decreases**.

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What the Volatility Surface Tells Us

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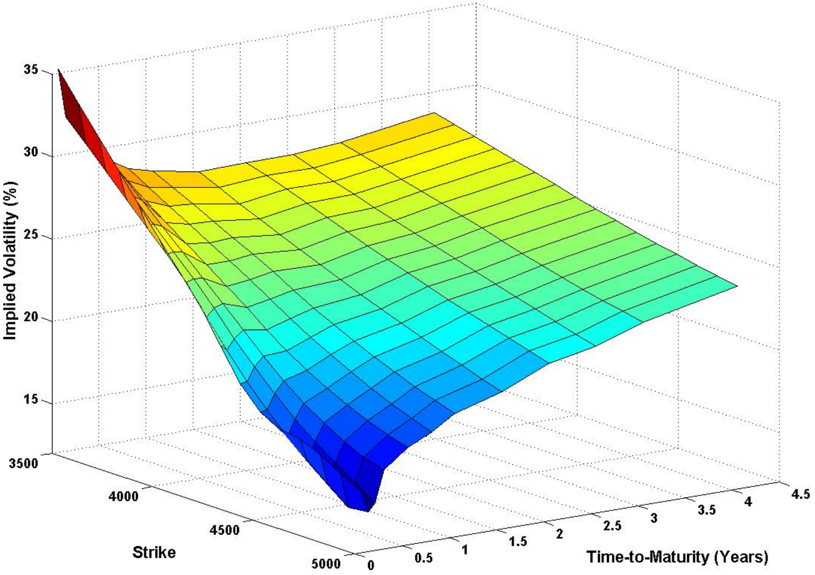
SX5E Implied Volatility Surface as of 28th Nov 2007



What the Volatility Surface Tells Us

We continue to assume that volatility surface has been constructed from European option prices.

Consider a butterfly strategy centered at K where you are:

1. long a call option with strike $K - \Delta K$
2. long a call with strike $K + \Delta K$
3. short 2 call options with strike K.

Value of butterfly, B_0, at time $t = 0$, satisfies

$$B_0 \;=\; C(K - \Delta K, T) \;-\; 2C(K, T) \;+\; C(K + \Delta K, T) \tag{4}$$

What the Volatility Surface Tells Us

But we also have

$$\begin{aligned} B_0 &\approx e^{-rT}\, \mathbb{Q}\left(K - \Delta K \leq S_T \leq K + \Delta K\right) \times \Delta K/2 \\ &\approx e^{-rT}\, f_T(K) \times 2\Delta K \times \Delta K/2 \\ &= e^{-rT}\, f_T(K) \times (\Delta K)^2 \end{aligned} \tag{5}$$

where $f_T(K)$ is the **risk-neutral** PDF of S_T evaluated at K.

Equating (4) and (5) yields

$$f_T(K) \approx e^{rT}\, \frac{C(K-\Delta K, T) - 2C(K,T) + C(K+\Delta K, T)}{(\Delta K)^2}. \tag{6}$$

and now letting $\Delta K \to 0$ in (6) we obtain

$$f_T(K) = e^{rT}\, \frac{\partial^2 C}{\partial K^2}.$$

Volatility surface therefore gives the **marginal** risk-neutral distribution of the stock price, S_T, for any fixed time, T.

What the Volatility Surface Tells Us

This means that given the implied volatility surface, $\sigma(K, T)$, we can compute the price,

$$P_0 = \mathsf{E}_0^{\mathbb{Q}}\left[e^{-rT} f(S_T)\right]$$

of **any** derivative security whose payoff, $f(\cdot)$, only depends on the underlying stock price at a **single** and **fixed** time T.

It tells us **nothing** about the joint distribution of the stock price at multiple times, $T_1, \ldots, T_n$

– not surprising since the volatility surface is constructed from European option prices

– and European option prices only depend on **marginal** distributions of S_T.

Example. Suppose we wish to compute the price of a knockout put option with time T payoff

$$\max\left(K - S_T, 0\right) 1_{\{\min_{0 \leq t \leq T} S_t \geq B\}}.$$

We **cannot** compute the price of this option given only the volatility surface

– will return to this issue soon.

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Pricing Derivatives Using the Volatility Surface*

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Pricing a Digital Option

Wish to price a digital option which pays $\$1$ if the time T stock price, S_T, is greater than K and 0 otherwise

- we know (why?) that we can price this security given the implied volatility surface.

Easy to see that the digital price, $D(K, T)$, is given by

$$\begin{aligned} D(K, T) &= \lim_{\Delta K \to 0} \frac{C_{\mathsf{mkt}}(K, T) - C_{\mathsf{mkt}}(K + \Delta K, T)}{\Delta K} \\ &= -\lim_{\Delta K \to 0} \frac{C_{\mathsf{mkt}}(K + \Delta K, T) - C_{\mathsf{mkt}}(K, T)}{\Delta K} \\ &= -\frac{\partial C_{\mathsf{mkt}}(K, T)}{\partial K}. \end{aligned}$$

Pricing a Digital Option

Recall that $C_{\mathsf{mkt}}(K, T) = C_{\mathsf{BS}}(K, T, \sigma(K, T))$.

Chain rule now implies

$$
\begin{aligned}
D(K,T) &= -\frac{\partial C_{\mathsf{BS}}(K, T, \sigma(K,T))}{\partial K} \\
&= -\frac{\partial C_{\mathsf{BS}}}{\partial K} - \frac{\partial C_{\mathsf{BS}}}{\partial \sigma} \frac{\partial \sigma}{\partial K} \\
&= -\frac{\partial C_{\mathsf{BS}}}{\partial K} - \text{vega} \times \text{skew}. \qquad (7)
\end{aligned}
$$

We can calculate $\frac{\partial C_{\mathsf{BS}}}{\partial K}$ and $\frac{\partial C_{\mathsf{BS}}}{\partial \sigma}$ using the Black-Scholes formula.
And the skew, $\frac{\partial \sigma}{\partial K}$, can be estimated from the implied volatility surface.

This is an example of how the Black-Scholes **"technology"** is used in practice

– even though the Black-Scholes model is known to be wrong.

An Example from Gatheral's "The Volatility Surface"

Suppose $r = c = 0$, $T = 1$ year, $S_0 = 100$ and $K = 100$ so the digital is at-the-money.

Suppose also that the skew is 2.5% per 10% change in strike and $\sigma_{atm} = 25\%$.

Letting $\phi(\cdot)$ be the PDF of a standard normal random variable we then have

$$\begin{aligned}
D(100, 1) &= \; = \mathsf{N}\left(-\frac{\sigma_{atm}}{2}\right) \; - \; S_0 \; \phi\left(\frac{\sigma_{atm}}{2}\right) \times \frac{-.025}{.1 S_0} \\
&= \; \mathsf{N}\left(-\frac{\sigma_{atm}}{2}\right) \; + \; .25 \; \phi\left(\frac{\sigma_{atm}}{2}\right) \\
&\approx \; .45 \; + \; .25 \times .4 \\
&= \; .55.
\end{aligned}$$

Therefore the digital price $= 55\%$ of notional when priced correctly.

If we ignored the skew and just computed the Black-Scholes digital price using the ATM implied volatility, the price would have been 45% of notional

– which is significantly less than the correct price.

Pricing a Range Accrual

Consider a 3-month range accrual on the S&P 500 index with range $1,500$ to $1,550$.

After 3 months the product pays $X\%$ of notional where

$$X = \% \text{ of days over the } 3 \text{ months that index is inside the range}$$

For example, if the notional is $\$10M$ and the index is **inside** the range 70% of the time, then the payoff is $\$7M$.

Question: Is it possible to calculate the price of this range accrual using the volatility surface?

Answer: Yes. Consider a portfolio consisting of a pair of digitals for each date between now and the expiration.

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Beyond the Volatility Surface and Black-Scholes

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Same Marginals But Different Joint Distributions

Suppose there are two time periods, T_1 and T_2, of interest and that a non-dividend paying security A has **risk-neutral** distributions given by

$$\begin{aligned} S_{T_1}^A &= e^{(r-\sigma^2/2)\,T_1+\sigma\sqrt{T_1}\,Z_1^A} \\ S_{T_2}^A &= e^{(r-\sigma^2/2)\,T_2+\sigma\sqrt{T_2}\,\left(\rho_A Z_1^A+\sqrt{1-\rho_A^2}Z_2^A\right)} \end{aligned}$$

where Z_1^A and Z_2^A are independent $\mathsf{N}(0,1)$ random variables.

Note that a value of $\rho_A > 0$ can capture a momentum effect and a value of $\rho_A < 0$ can capture a mean-reversion effect.

Suppose now that there is another non-dividend paying security B with **risk-neutral** distributions given by

$$\begin{aligned} S_{T_1}^B &= e^{(r-\sigma^2/2)\,T_1+\sigma\sqrt{T_1}\,Z_1^B} \\ S_{T_2}^B &= e^{(r-\sigma^2/2)\,T_2+\sigma\sqrt{T_2}\,\left(\rho_B Z_1^B+\sqrt{1-\rho_B^2}Z_2^B\right)} \end{aligned}$$

where Z_1^B and Z_2^B are again independent $\mathsf{N}(0,1)$ random variables.

Same Marginals But Different Joint Distributions

Observation: If Z_1 and Z_2 are independent $\mathsf{N}(0,1)$ random variables then for any $\rho \in [-1,1]$

$$\rho Z_1 + \sqrt{1-\rho^2} Z_2 \ \sim \mathsf{N}(0,1)$$

Therefore see that:

$S^A_{T_1}$ and $S^B_{T_1}$ have the same **marginal** risk-neutral distributions.

$S^A_{T_2}$ and $S^B_{T_2}$ have the same **marginal** risk-neutral distributions.

Therefore follows that European options on A and B with the same strike and maturity must have the **same** price

– A and B therefore have identical "volatility surfaces".

Same Marginals But Different Joint Distributions

But now consider a knock-in put option with strike 1 and expiration T_2.

In order to knock-in, the stock price at time T_1 must exceed the barrier price of 1.2.

The payoff function is then given by

$$\text{Payoff} \ = \ \max\left(1 - S_{T_2}, \ 0\right) \ 1_{\{S_{T_1} \geq 1.2\}}.$$

Question: Would the knock-in put option on A have the same price as the knock-in put option on B?

Question: How does your answer depend on ρ_A and ρ_B?

Question: What does this say about the ability of the volatility surface to price barrier options?

Derivatives Pricing in Practice

The **dynamic replication** theory of Black-Scholes-Merton is very elegant

- but it is not possible to dynamically replicate (and therefore price) derivative securities in practice.

Instead **supply and demand** sets derivative security prices

- particularly for the most liquid securities like European and American options
  - indeed **volatility** is itself an **asset class**!

This is also true of derivative securities in other markets, e.g., fixed income derivatives, FX derivatives, credit derivatives, commodity derivatives etc.

- (most) derivative prices in these markets are determined by supply and demand.

Derivatives Pricing in Practice

But derivatives pricing **models** are still needed to:

1. **price** exotic and other less liquid derivative securities
2. **risk-manage** derivatives portfolios via the Greeks or scenario analysis.

The models are arbitrage-free by construction and **calibrated** to liquid security prices

- but note that the models are only an approximation to reality
- and generally not a great approximation
  - witness how often they need to be re-calibrated
  - and they generally completely ignore the endogeneity of markets
    - an almost fatal flaw in some circumstances.

But the ideas of dynamic replication have not been abandoned and are still useful

- these ideas are still used to (partially) **hedge** derivatives portfolios
- e.g. recall that we could still partially hedge a vanilla European option even when we assumed the wrong volatility parameter, σ.