Financial Engineering and Risk Management

Implementation difficulties with mean-variance

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Implementation details

Parameter estimation: mean return $\boldsymbol{\mu}$ and covariance $\boldsymbol{V}$

- Statistical errors in estimation.
- Never have enough data: $\boldsymbol{V}$ has $d(d+1)/2$ independent parameters.
- Portfolio is very sensitive to estimation errors.

How does one get negative exposures in optimal mean-variance portfolios?

- Short the assets: unlimited downside! And often not feasible.
- leveraged Exchange Traded Funds (ETFs): Be very careful!

Is variance a good risk measure in practice?

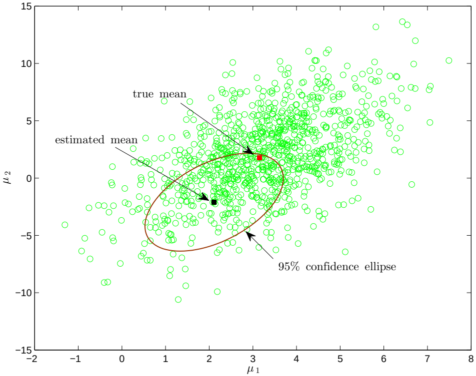
- Works for Normal (or elliptic) random variables
- Need higher moments to capture deviation from Normality
- Tail risk measures

Parameter estimation

The true parameters $(\boldsymbol{\mu}, \boldsymbol{V})$ are never known!

Use Historical returns $\boldsymbol{r}^{(1)}$, $\boldsymbol{r}^{(2)}$, ..., $\boldsymbol{r}^{(N)}$ to compute estimates for

- mean: $\boldsymbol{\mu}^{(\text{est})} = \frac{1}{N} \sum_{t=1}^{N} \boldsymbol{r}^{(t)}$
- covariance matrix: $\boldsymbol{V}^{(\text{est})} = \frac{1}{N-1} \sum_{t=1}^{N} (\boldsymbol{r}^{(t)} - \boldsymbol{\mu}^{(\text{est})})(\boldsymbol{r}^{(t)} - \boldsymbol{\mu}^{(\text{est})})^\top$

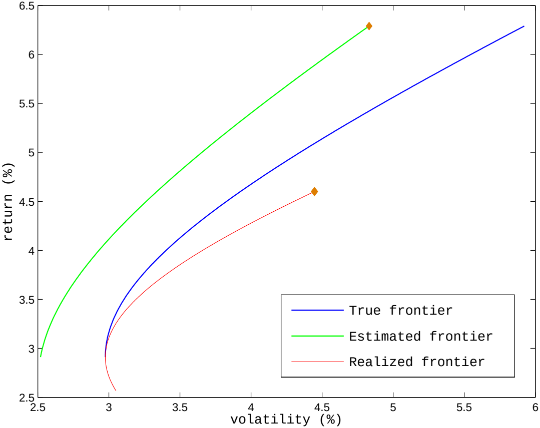


Each dot is an estimate for the mean vector with $N = 60$ months of simulated data.

Estimated mean can be quite "far" from the true mean.

True mean lies in the 95% confidence ellipse with probability 0.95.

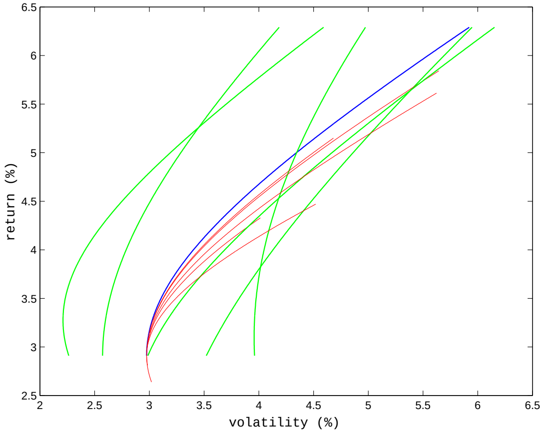
Does parameter error matter?



- Estimated mean and covariance from $N = 60$ months of simulated data
- Estimated frontier = frontier for estimated parameters
- Realized frontier = true mean and volatility of the estimated frontier portfolios

Why is parameter error so serious in portfolio selection?

Many independent samples!



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The efficient frontier is extremely unstable.

Why is parameter error so serious?

Two identical assets with mean μ, variance σ^2, and correlation $\rho = 0$

Optimal investment $\boldsymbol{x}^* = (0.5, 0.5)$

Suppose estimates

- $\mu_1^{\text{(est)}} = \mu + \epsilon$ (positive error)
- $\mu_2^{\text{(est)}} = \mu - \epsilon$ (negative error)

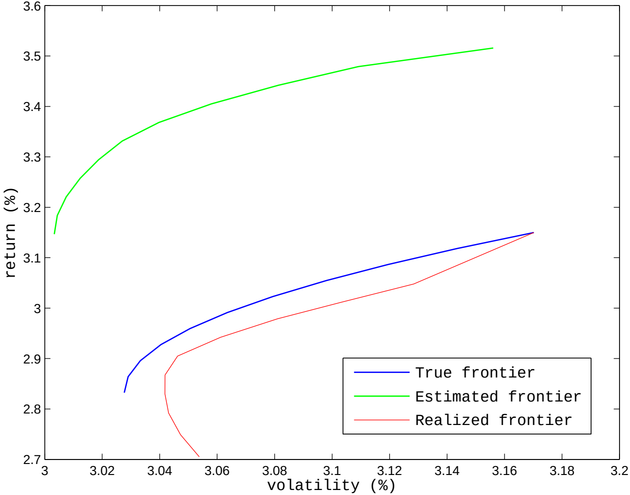
Average error in the estimates = 0 ... estimator is good on average

Optimal investment: overweight asset 1. Precisely the wrong thing to do!

Realized performance is lower than expected for the overweighted asset and better than expected for the underweighted asset! Performance becomes worse as more shorting is allowed.

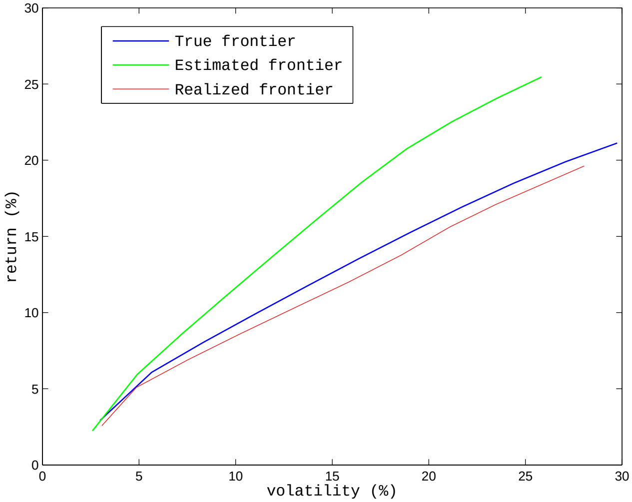
Ron Michaud: "Mean-variance results in **error-maximizing investment-irrelevant** portfolios."

Efficient frontier with no-short sales constraint



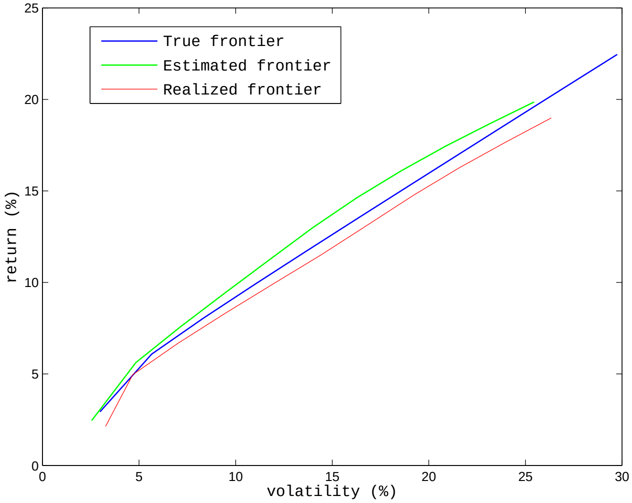
The "corner" in the set $\{\boldsymbol{x} : \boldsymbol{x} \geq \mathbf{0}\}$ makes solutions more sensitive to estimates!

Efficient frontier with leverage constraint



Leverage constraint: $\sum_{i=1}^{d} |x_i| \leq 2$

Efficient frontier with robust constraints



Let S_m = confidence region

Robust target return constraint: $\min_{\boldsymbol{\mu} \in S_m} \{\boldsymbol{\mu}^\top \boldsymbol{x}\} \geq r$.

Methods to improve parameter estimates

Shrinkage methods: James and Stein (1961), Ledoit and Wolf (2004)

- "Shrink" to the global mean: $\mu_i^{(\mathsf{sh})} = \alpha\mu_i^{(\mathsf{est})} + (1-\alpha)\left(\frac{1}{d}\sum_{j=1}^{d}\mu_j\right)^{(\mathsf{est})}$

- "Shrink" to identity matrix: $\boldsymbol{V}^{(\mathsf{sh})} = \alpha\,\boldsymbol{V}^{(\mathsf{est})} + (1-\alpha)\left(\frac{1}{d}\sum_{i=1}^{d}\sigma_i^2\right)^{(\mathsf{est})}\boldsymbol{I}$

Use subjective views to improve estimates: Black-Litterman method

Non-parametric nearest neighbor-like methods

- Let r denote the currently observed return
- Let $S = \{t : \|r - r_t\| \le \beta\}$ = times when historical returns were close to r
- Predict that future return belongs to the set $\{r_{t+1} : t \in S\}$

Financial Engineering and Risk Management

Negative exposures and leveraged ETFs

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Short positions

Short positions can result in superior returns.

But short positions are very risky!

- Long positions have limited downside
  - The lowest price of an asset is zero.
  - The largest amount one can lose is the amount invested.
- Short positions have unlimited downside
  - Short positions are created by selling assets borrowed from a broker.
  - The asset has to be re-purchased and returned to the broker at a later date.
  - The price of the asset can become arbitrarily large.
  - Potential loss from the short position can be arbitrarily large

Need a product that has negative exposure and limited liability.

Exchange Traded Funds

Exchange Traded Funds (ETFs) are exchange traded products that track the returns on stock indices, bond indices, commodities, currencies, etc.

World ETFs

Americas | Europe, Africa and Middle East | Asia-Pacific

NAME (SYMBOL)	PRICE	CHANGE	% CHANGE	VOLUME	TIME
SPDR S&P 500 ETF (SPY:US)	152.11	-0.18	-0.12%	205,723,923	16:15
ISHARES NAFTRAC (NAFTRAC:MM)	44.18	0.33	0.75%	77,614,068	16:08
ISHARES-EMG MKT (EEM:US)	43.99	-0.16	-0.35%	42,284,343	16:15
IPATH-S&P S/T FU (VXX:US)	21.97	-0.20	-0.88%	37,883,276	16:15
ISHARES-RUS 2000 (IWM:US)	91.74	0.03	0.03%	29,330,024	16:15
ISHARES-JAPAN (EWJ:US)	9.91	-0.03	-0.30%	27,863,174	16:15
POWERSH-QQQ (QQQ:US)	67.75	-0.20	-0.29%	25,321,720	16:30
ISHARES FTSE CHI (FXI:US)	40.32	0.10	0.25%	18,367,753	16:15
PRO ULTRA VIX ST (UVXY:US)	9.60	-0.14	-1.44%	17,534,467	16:15

Source: Bloomberg.com

Leveraged ETFs

Produce daily returns that are a multiple of the daily returns on the index.

- Bull ETFs: return $\beta \times$ the daily return on index. $\beta = 2, 3$
- Inverse ETFs: return $-\beta \times$ the daily return on index. $\beta = 1, 2, 3$

Top U.S. Leveraged ETFs					
NAME	SYMBOL	PRICE	CHANGE	%CHANGE	TIME
PRO ULTRA S&P500	SSO:US	68.58	-0.16	-0.23%	02/15
PRO ULTSH 20+TSY	TBT:US	67.85	0.26	0.38%	02/15
PRO ULTSHRT S&P	SDS:US	47.27	0.14	0.29%	02/15
PRO ULTRA FINCLS	UYG:US	79.60	-0.46	-0.57%	02/15
PRO ULTSHRT RE	SRS:US	21.68	-0.09	-0.41%	02/15
PRO ULTRA QQQ	QLD:US	59.22	-0.29	-0.49%	02/15
PRO ULTSHRT FINL	SKF:US	28.33	0.14	0.50%	02/15
PRO ULT OIL&GAS	DIG:US	54.30	-1.21	-2.18%	02/15
PRO ULTRA DOW30	DDM:US	80.70	0.02	0.02%	02/15
PRO SHORT S&P500	SH:US	31.77	0.04	0.13%	02/15

Source: Bloomberg.com

An inverse ETF is a product that gives negative exposure to the index with limited liability – all one stands to lose is the initial investment.

Details on the return on an ETF

The return on an ETF is the return on the underlying index compounded daily.

$$R_T^{(\mathsf{ETF})} = (1 + r_1)(1 + r_2) \cdots (1 + r_T)$$

where r_t is the net daily return on the underlying index. Just like a stock.

The return on a leveraged ETF with leverage factor β

$$R_T^{(\beta\text{-}\mathsf{ETF})} = (1 + \beta r_1)(1 + \beta r_2) \cdots (1 + \beta r_T)$$

where r_t is the net daily return on the underlying index.

ETFs are constructed by investing in derivatives written on the index.

The daily compounding has consequences that are not immediately obvious.

Impact of daily compounding

Index: Current value of index is 100.

- Day 1: Index goes up to 105. $r_1 = 5\%$
- Day 2: Index goes back down to 100. $r_2 = -4.76\%$

ETF: Current price of an ETF following the index is 100.

- Day 1: ETF price goes up to $100(1 + r_1) = 105$
- Day 2: ETF price goes down to $105(1 + r_2) = 100$

Bull ETF: Current price of a $2\times$ETF following the index is 100.

- Day 1: ETF price goes up to $100(1 + 2r_1) = 110$
- Day 2: ETF price goes down to $105(1 + 2r_2) = 99.52$ ... lose money!

Inverse ETF: Current price of an Inverse ETF following the index is 100.

- Day 1: ETF price goes down to $100(1 - r_1) = 95$
- Day 2: ETF price goes up to $105(1 - r_2) = 99.52$ ... lose money!
- End up in the same place as the $2\times$ETF!

Volatility and ETF returns

The gross return from a static leveraged position in the underlying index is

$$\frac{\beta S_T - (\beta - 1) S_0 (1 + rT)}{S_0}$$

where r is the funding rate and β is the leverage ratio.

The gross return from investing in the β-Leveraged ETF is approximately

$$\left(\frac{S_T}{S_0}\right)^{\beta} \; e^{(1-\beta) rT \; - \; fT \; - \; \frac{(\beta^2 - \beta)}{2} \; \sum_{i=0}^{n-1} \ln\left(\frac{S_{i+1}}{S_i}\right)^2}$$

where f is the expense ratio of the fund and n is the number of daily observations between times 0 and T. (So $S_n = S_T$.)

The term $-\frac{(\beta^2 - \beta)}{2} \sum_{i=0}^{n-1} \ln\left(\frac{S_{i+1}}{S_i}\right)^2$ is the loss due to volatility: short vol

ETFs are generally designed for short-term plays on an index or sector, and should be used that way. Over long-periods leveraged ETFs do not work as one may expect, especially in volatile markets.

Leveraged ETFs during volatile markets

- DIG: ProShares Ultra Oil & Gas ($2\times$ ETF)
- DUG: ProShares UltraShort Oil & Gas ($-2\times$ ETF)

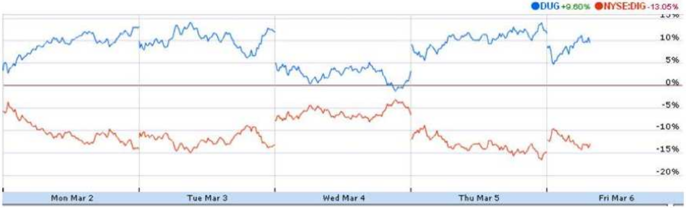

Figure 1: DUG versus DIG (March 2-6, 2009)

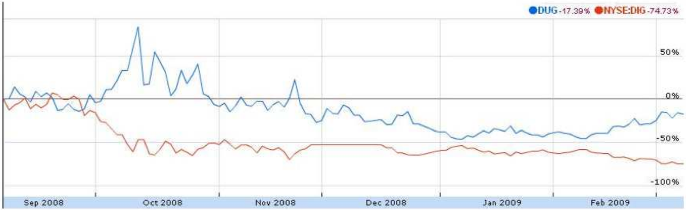

Figure 2: DUG versus DIG (September 2008 - March 2009)

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Beyond variance

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Problems with variance

Appropriate for Normal and other elliptic distributions

Does not capture larger deviations from the mean

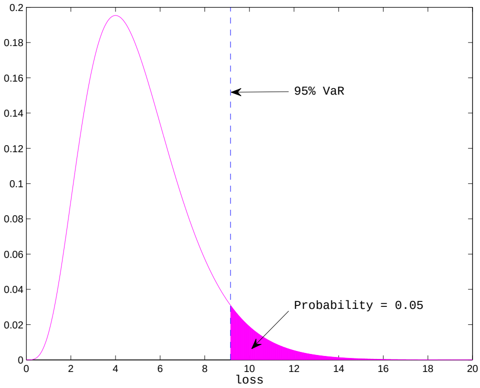
Symmetric measure: equally penalizes the deviation above/below mean

Value at Risk

Definition. The value at risk $\text{VaR}_p(L)$ of random loss L at the confidence level $p \in (0,1)$ is defined as

$$\text{VaR}_p(L) := p^{\text{th}}\text{-quantile of } L \quad \approx \quad F_L^{-1}(p)$$

where F_L is the CDF of the random loss L.



VaR is a "tail" risk measure

VaR_p is increasing in p

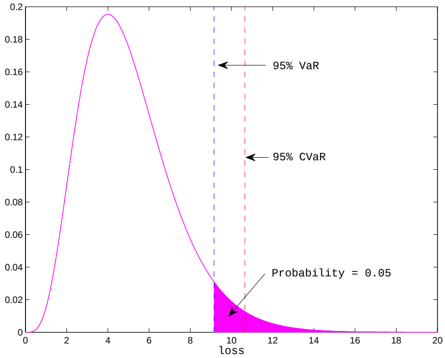
$\text{VaR}_{0.99}(L) \geq \text{VaR}_{0.95}(L)$

Conditional Value at Risk

Definition. The conditional value at risk $\mathsf{CVaR}_p(L)$ of random variable L at the confidence level $p \in (0,1)$ is defined as

$$\mathsf{CVaR}_p(L) = \mathbb{E}\left[L \mid L \geq \mathsf{VaR}_p(L)\right] = \frac{\int_{\mathsf{VaR}_p(L)}^{\infty} x f_L(x)\,dx}{\mathbb{P}(L \geq \mathsf{VaR}_p(L))}$$

where f_L is the density of the random loss L.



CVaR is also a "tail" risk measure

$\mathsf{CVaR}_p(L) \geq \mathsf{VaR}_p(L)$

CVaR_p is increasing in p

Other names: Tail conditional expectation and Expected Shortfall

VaR and CVaR for Normal distribution

Suppose $L \sim N(\mu, \sigma^2)$.

$$\mathsf{VaR}_p(L) = \mu + \sigma\Phi^{-1}(p)$$

where Φ is the CDF of a standard Normal with mean 0 and volatility 1.

$$\mathsf{CVaR}_p(L) = \mu + \sigma\left(\frac{1}{1-p}\int_p^1 \Phi^{-1}(\beta)d\beta\right)$$

where Φ is the CDF of a standard Normal with mean 0 and volatility 1.

VaR_p and CVaR_p for a Normal random variable completely defined by mean μ and volatility σ. Surprise ?

VaR and CVaR from samples

Suppose $L_1, L_2, \ldots, L_N$ are N independently and identically distributed (IID) samples from the Loss L

Let $L_{(1)}, L_{(2)}, \ldots, L_{(N)}$ denote the order-statistics of the N samples

- $L_{(1)}$ = smallest value among all samples
- $L_{(2)}$ = the next larger value ...
- $L_{(N)}$ = the largest value

$L_{(1)}, L_{(2)}, \ldots, L_{(N)}$ are the samples sorted in increasing order.

Let $K_p = \lceil pN \rceil$. E.g. $p = 0.95$ and $N = 1000$ implies $K_p = 950$.

$$
\begin{aligned}
\text{VaR}_p(L) &\approx L_{(K_p)} = K_p\text{-th term in the sorted samples} \\
\text{CVaR}_p(L) &\approx \frac{1}{(1-p)N} \sum_{k=K_p}^{N} L_{(k)} \\
&= \text{sum of the largest } N - K_p + 1 \text{ samples divided by } (1-p)N
\end{aligned}
$$

Impact of return distribution

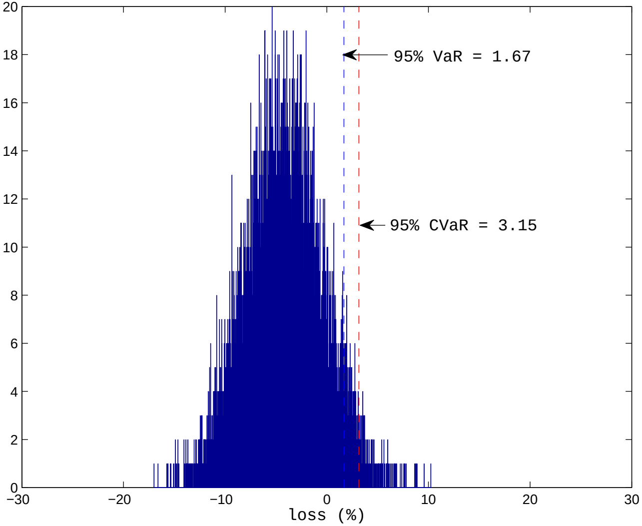
Experimental setup:

- Computed the Sharpe optimal portfolio for the data in the spreadsheet.
- Generated $N = 10,000$ samples of loss $-r_x$ for three different distributions
  (a) Multivariate Normal with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{V}$
  (b) Multivariate t with $\nu = 12$ degrees of freedom, mean $\boldsymbol{\mu}$, covariance $\left(\frac{\nu-2}{\nu}\right) \boldsymbol{V}$,
  (c) Mixture of Normals $0.75N(\boldsymbol{\mu}, (0.76)^2\,\boldsymbol{V}) + 0.25N(\boldsymbol{\mu}, (1.5)^2\,\boldsymbol{V})$
- Used samples to estimate VaR and CVaR

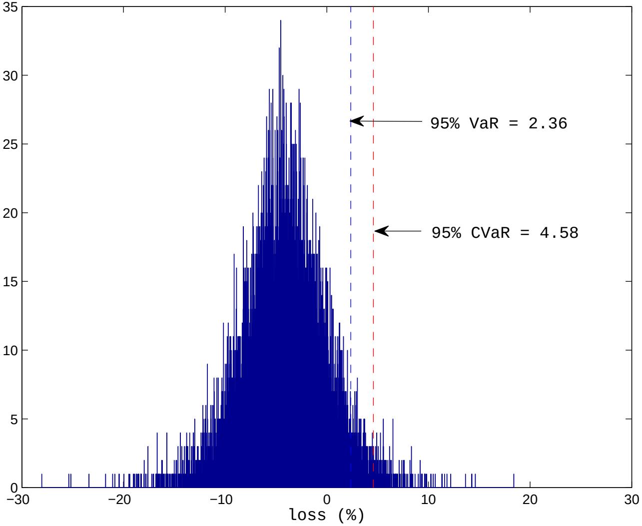
The Students's t distribution has fatter tails than Normal particularly when the degrees of freedom are small. Expect VaR and CVaR to be larger.

The Normal distribution with covariance matrix $(1.5)^2\,\boldsymbol{V}$ has all volatilities 50% higher. The mixture models a situation where there is 25% chance of very high volatility. Expect VaR and CVaR to be larger.

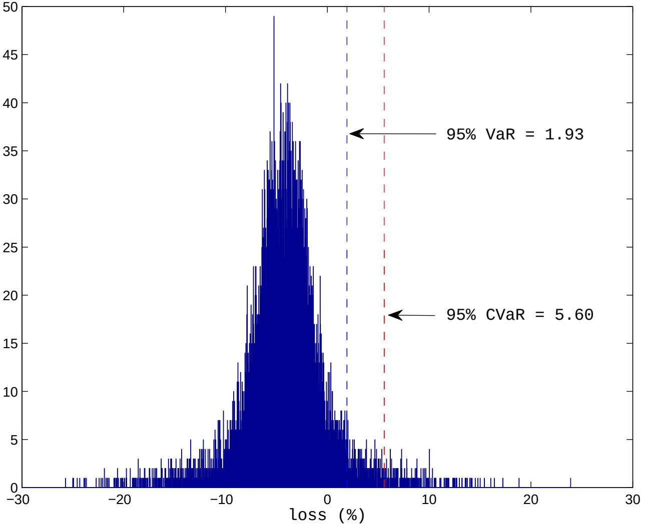
Loss histogram for Normal returns



Loss histogram for t returns



Loss histogram for mixture of Normals



Pros and Cons of VaR

Pros

- Captures the tail behaviour of portfolio losses
- Can be robustly estimated from data: not very susceptible to outliers

Cons:

- Only sensitive to the p-th quantile and not the distribution beyond
- Incentive for "tail stuffing"
- VaR is not sub-additive: diversification can increase VaR

Pros and Cons of CVaR

Pros

- Captures the tail behaviour of portfolio losses beyond the p-th quantile
- Sub-additive: diversification reduces CVaR
- Mean-CVaR portfolio selection problems can be solve very efficiently

Cons:

- CVaR is defined by an expectation: can be sensitive to outliers

We will return to the topic of VaR and CVaR in the risk management module.