Financial Engineering & Risk Management

Review of the Binomial Model for Option Pricing

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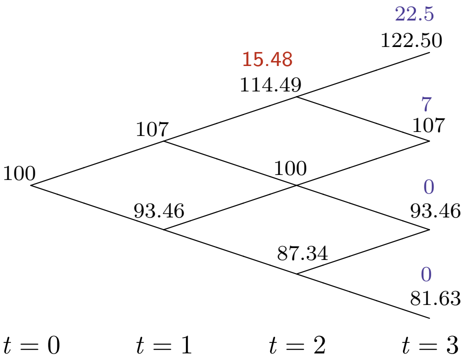
Pricing a European Call Option

Assumptions: expiration at $t = 3$, strike $= \$100$ and $R = 1.01$.

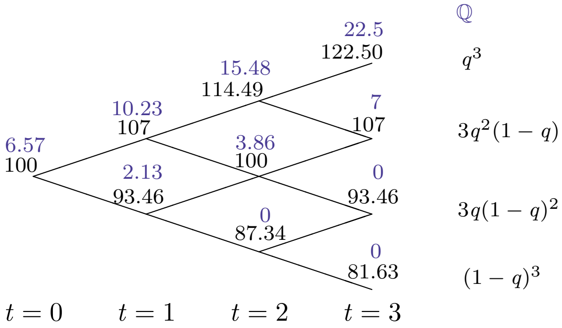


Sample calculation:

$$15.48 \;=\; \frac{1}{1.01}\left[q_u \times 22.5 \;+\; q_d \times 7\right]$$

with $q_u = (R - d)/(u - d)$ and $q_d = 1 - q_u$.

Pricing a European Call Option



- We can also calculate the price as

$$C_0 = \frac{1}{R^3} \mathsf{E}_0^{\mathbb{Q}} \left[\max(S_T - 100,\ 0) \right] \tag{1}$$

  - this is risk-neutral pricing in the binomial model
  - avoids having to calculate the price at every node.

Trading Strategies in the Binomial Model

- Let S_t denote the stock price at time t.
- Let B_t denote the value of the cash-account at time t
  - assume without any loss of generality that $B_0 = 1$ so that $B_t = R^t$
  - so now explicitly viewing the cash account as a security.
- Let x_t denote # of shares held between times $t-1$ and t for $t = 1, \ldots, n$.
- Let y_t denote # of units of cash account held between times $t-1$ and t for $t = 1, \ldots, n$.
- Then $\boldsymbol{\theta}_t := (x_t, y_t)$ is the portfolio held:
  (i) immediately **after** trading at time $t-1$ so it is known at time $t-1$
  (ii) and immediately **before** trading at time t.
- $\boldsymbol{\theta}_t$ is also a random process and in particular, a **trading strategy**.

Self-Financing Trading Strategies

Definition. The value process, $V_t(\boldsymbol{\theta})$, associated with a trading strategy, $\boldsymbol{\theta}_t = (x_t, y_t)$, is defined by

$$V_t = \begin{cases} x_1 S_0 + y_1 B_0 & \text{for } t = 0 \\ \\ x_t S_t + y_t B_t & \text{for } t \geq 1. \end{cases} \tag{2}$$

Definition. A self-financing (s.f.) trading strategy is a trading strategy, $\boldsymbol{\theta}_t = (x_t, y_t)$, where changes in V_t are due entirely to trading gains or losses, rather than the addition or withdrawal of cash funds. In particular, a self-financing strategy satisfies

$$V_t \;=\; x_{t+1} S_t + y_{t+1} B_t \,, \qquad t = 1, \ldots, n-1. \tag{3}$$

The definition states that the value of a s.f. portfolio just before trading is equal to the value of the portfolio just after trading

– so no funds have been deposited or withdrawn.

Self-Financing Trading Strategies

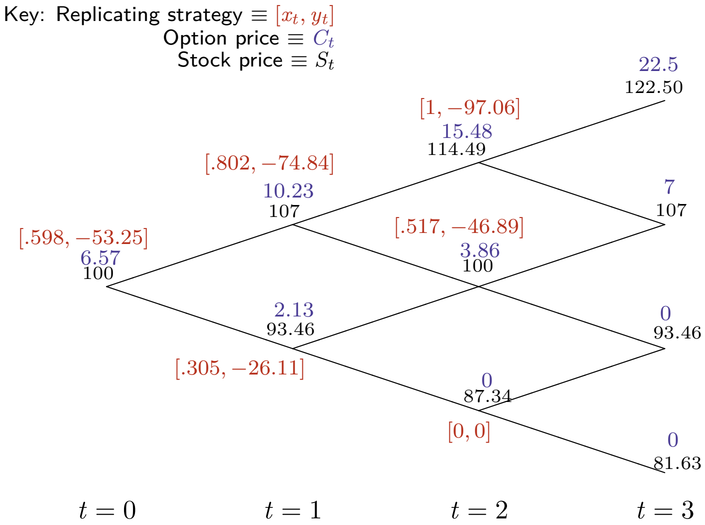
Proposition. If a trading strategy, $\boldsymbol{\theta}_t$, is s.f. then the corresponding value process, V_t, satisfies

$$V_{t+1} - V_t \;=\; x_{t+1}\,(S_{t+1} - S_t) \;+\; y_{t+1}\,(B_{t+1} - B_t)$$

so that changes in portfolio value can only be due to capital gains or losses and not the injection or withdrawal of funds.

- In the multi-period binomial model we can construct a s.f. trading strategy that replicates the payoff of the option
  - this is called dynamic replication.
- The initial cost of this replicating strategy must equal the value of the option
  - otherwise there's an arbitrage opportunity.
- The dynamic replication price is of course equal to the price obtained from using the risk-neutral probabilities and working backwards in the lattice.
- And at any node, the value of the option is equal to the value of the replicating portfolio at that node.

The Replicating Strategy For Our European Option



e.g. $.802 \times 107 + (-74.84) \times 1.01 = 10.23$ at upper node at time $t = 1$

Financial Engineering & Risk Management

The Black-Scholes Model

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The Black-Scholes Model

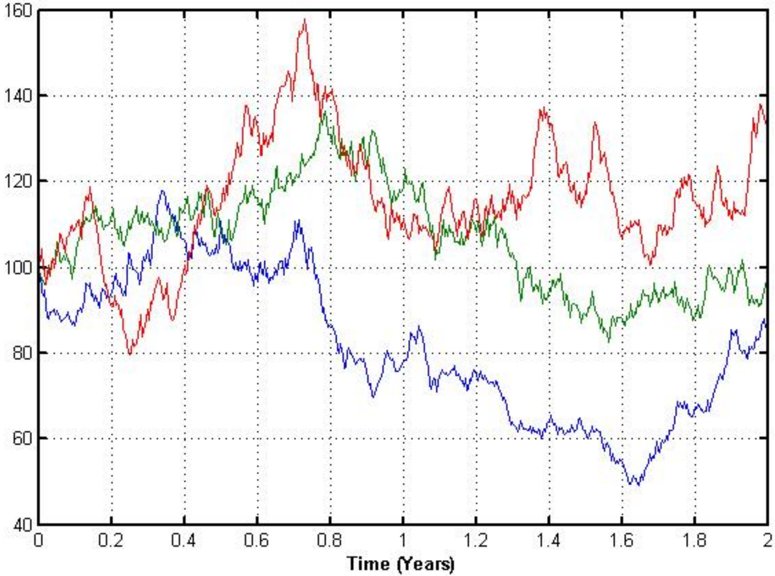
Black and Scholes assumed:

1. A continuously-compounded interest rate of r.
2. Geometric Brownian motion dynamics for the stock price, S_t, so that

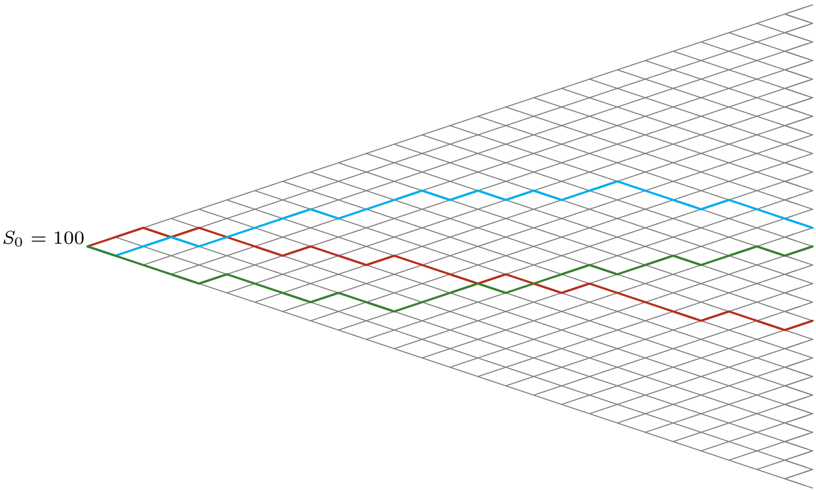
$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$$

   where W_t is a standard Brownian motion.
3. The stock pays a dividend yield of c.
4. Continuous trading with no transactions costs and short-selling allowed.

Sample Paths of Geometric Brownian Motion



Sample Paths of a Binomial Model



If number of periods $n \to \infty$ then binomial paths will looks like Brownian paths

The Black-Scholes Formula

- We know in the binomial model that the call option price is given by

$$C_0 = \mathsf{E}_0^{\mathbb{Q}}\left[e^{-rT}\max(S_T - K,\, 0)\right] \tag{4}$$

where $q_u = (R-d)/(u-d)$ and $q_d = 1 - q_u$.

- If we let number of periods $n \to \infty$ then we obtain the Black-Scholes GBM model where

$$S_t = S_0\, e^{(r-c-\sigma^2/2)t \,+\, \sigma W_t}$$

and where W_t is a Brownian motion under $\mathbb{Q}$.

- We can then evaluate (4) to obtain the Black-Scholes formula:

$$C_0 = S_0\, e^{-cT} N(d_1) \;-\; K e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\log(S_0/K) + (r - c + \sigma^2/2)\,T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and $N(d) = \mathsf{P}(N(0,1) \leq d)$.

The Black-Scholes Formula

- Note that μ does not appear in the Black-Scholes formula
  - just as p is not used in option pricing calculations for the binomial model.
- Black-Scholes obtained their formula using a similar replicating strategy argument to the one we used for the binomial model.
- European put option price, P_0, can be calculated from put-call parity

$$P_0 + S_0 e^{-cT} \;=\; C_0 + Ke^{-rT}.$$

- The Black-Scholes formula is arguably the most important and famous formula in all of finance and economics
  - it is used extensively in the financial industry
  - it has also led to an enormous amount of academic work since it's publication.
- We will see how it is used in practice
  - but will emphasize now that the GBM model is **not** a good approximation to security prices
    - and **everybody knows this!**

Financial Engineering & Risk Management

The Greeks: Delta and Gamma

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Review: The Black-Scholes Formula

The Black-Scholes formula for the price of a European call option with strike K and expiration T

$$C_0(S_0, \sigma, T) \; = \; S_0 e^{-cT} N(d_1) \; - \; K e^{-rT} N(d_2) \tag{5}$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + (r - c + \sigma^2/2)\,T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

and $N(d) = \mathsf{P}(N(0,1) \leq d)$.

Recall that r is the risk-free interest rate, c is the dividend yield and the stock price S_t satisfies

$$S_t \; = \; S_0 e^{(r-c-\sigma^2/2)t \; + \; \sigma W_t}$$

where W_t is a Brownian motion under $\mathbb{Q}$.

The "Greeks" refer to the partial (mathematical) derivatives of a (financial) derivative security price with respect to the model parameters

Delta

Definition. The delta of an option is the partial derivative of the option price with respect to the price of the underlying security.

The delta measures the sensitivity of the option price to the price of the underlying security.

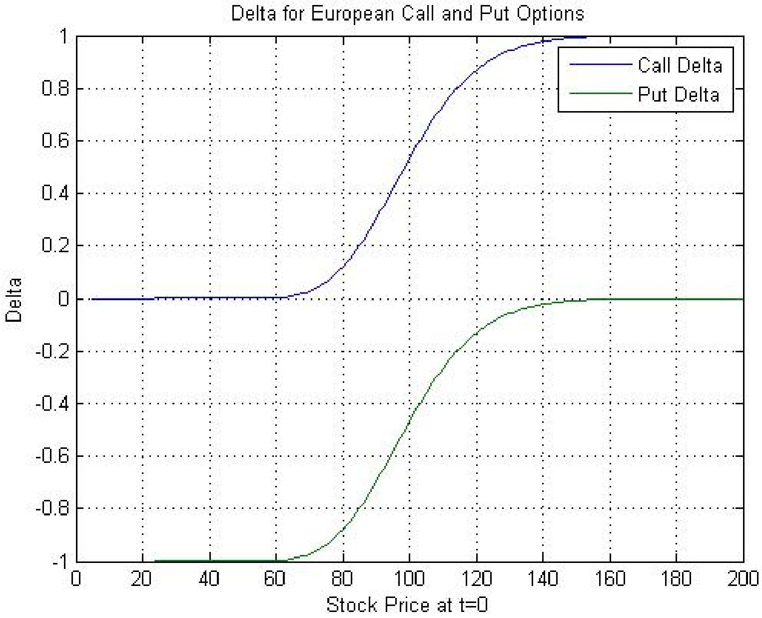
The delta of a European call option satisfies

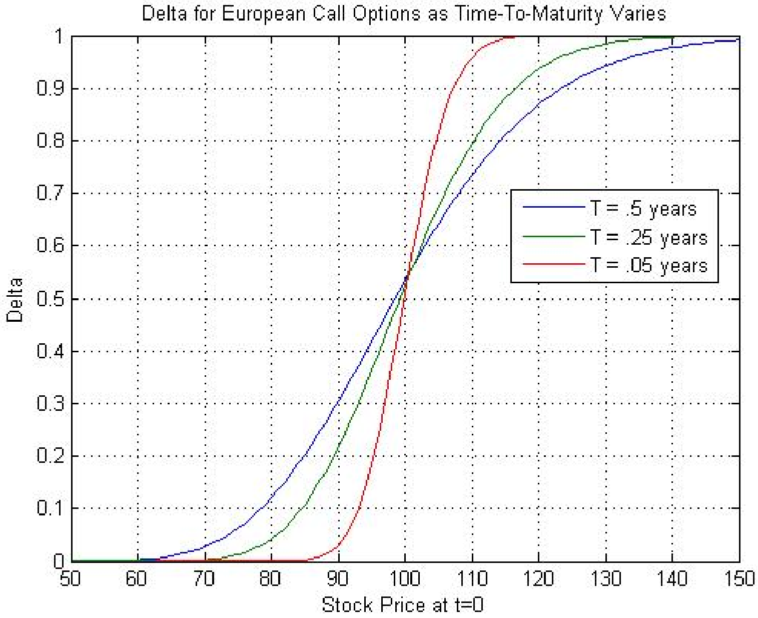
$$\text{delta} \; = \; \frac{\partial C}{\partial S} \; = \; e^{-cT} \; \mathsf{N}(d_1)$$

– follows from (5)

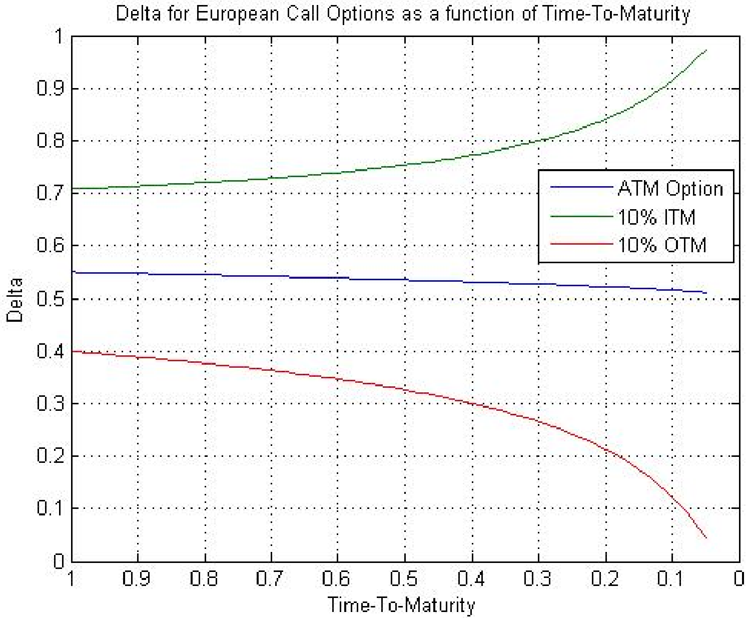
– although more tedious to calculate than it may appear to be!

The delta of a European put option is also easily calculated.


Delta for European Call and Put Options
Call Delta
Put Delta
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Delta
0
20
40
60
80
100
120
140
160
180
200
Stock Price at t=0



Delta for European Call Options as Time-To-Maturity Varies
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Delta
50
60
70
80
90
100
110
120
130
140
150
Stock Price at t=0
T = .5 years
T = .25 years
T = .05 years


Delta for European Call Options as a function of Time-To-Maturity
Delta
Time-To-Maturity
ATM Option
10% ITM
10% OTM
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

Gamma

Definition. The gamma of an option is the partial derivative of the option's delta with respect to the price of the underlying security.

Gamma measures the sensitivity of the option delta to the price of the underlying security.

The gamma of a call option satisfies

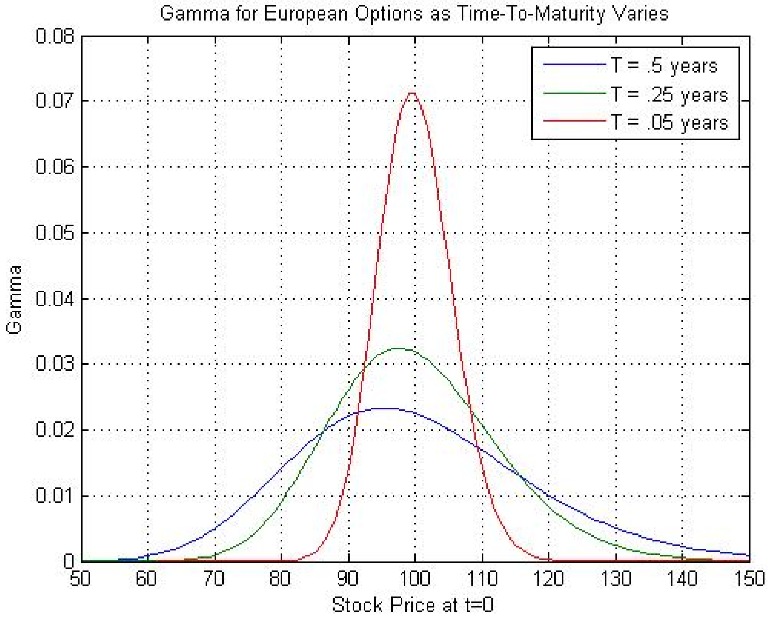
$$\text{gamma} \;=\; \frac{\partial^2 C}{\partial S^2} \;=\; e^{-cT}\,\frac{\phi(d_1)}{\sigma S\sqrt{T}}$$

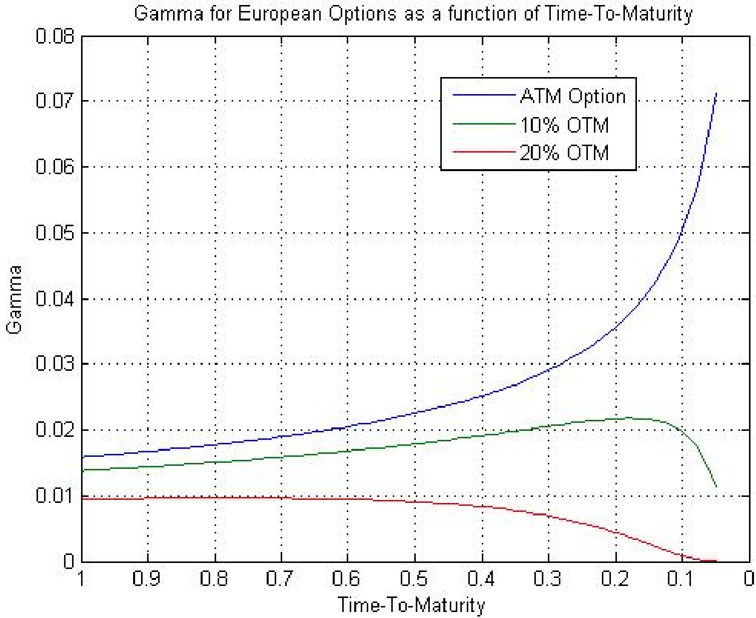
where $\phi(\cdot)$ is the standard normal PDF.

The gamma of a European put option is easily calculated (why?) from put-call parity

$$e^{-rT}\;K + C \;=\; e^{-cT}\;S + P. \tag{6}$$

Gamma for European options is always positive due to option convexity.


Gamma for European Options as Time-To-Maturity Varies
T = .5 years
T = .25 years
T = .05 years
Gamma
Stock Price at t=0
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
50
60
70
80
90
100
110
120
130
140
150


Gamma for European Options as a function of Time-To-Maturity
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Gamma
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Time-To-Maturity
ATM Option
10% OTM
20% OTM

Financial Engineering & Risk Management

The Greeks: Vega and Theta

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Review: The Black-Scholes Formula

The Black-Scholes formula for the price of a European call option with strike K and expiration T

$$C_0(S_0, \sigma, T) \;=\; S_0 e^{-cT} N(d_1) \;-\; K e^{-rT} N(d_2) \tag{7}$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + (r - c + \sigma^2/2)\,T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

and $N(d) = \mathsf{P}(N(0,1) \leq d)$.

Recall that r is the risk-free interest rate, c is the dividend yield and the stock price S_t satisfies

$$S_t \;=\; S_0 e^{(r-c-\sigma^2/2)t \,+\, \sigma W_t}$$

where W_t is a Brownian motion under $\mathbb{Q}$.

Vega

Definition. The vega of an option is the partial derivative of the option price with respect to the volatility parameter, σ.

Vega therefore measures the sensitivity of the option price to σ.

The vega of a call option satisfies

$$\text{vega} \;=\; \frac{\partial C}{\partial \sigma} \;=\; e^{-cT} S \sqrt{T}\, \phi(d_1)$$

where $\phi(\cdot)$ is the PDF of a standard normal random variable.

The vega of a European put option is easily calculated (why?) from put-call parity

$$e^{-rT}\, K + C \;=\; e^{-cT}\, S + P. \tag{8}$$

Question: Is the concept of vega inconsistent in any way with the Black-Scholes model?

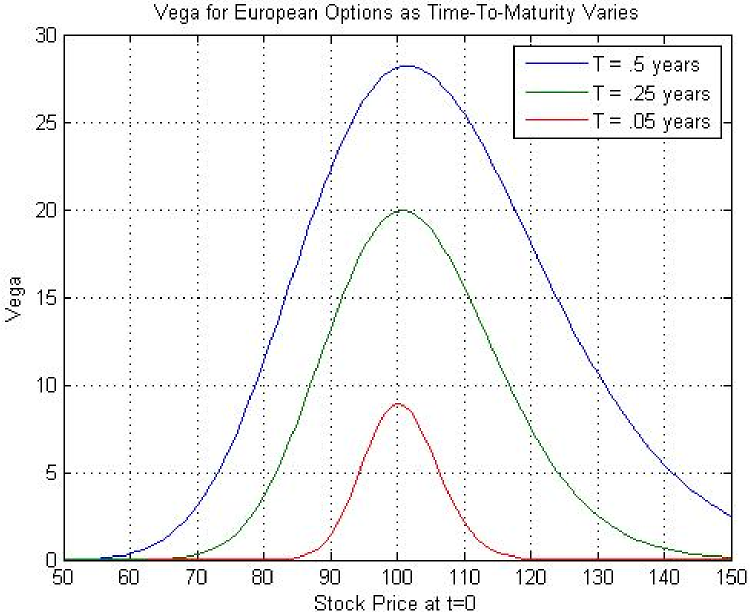

Figure : Vega as a Function of Stock Price

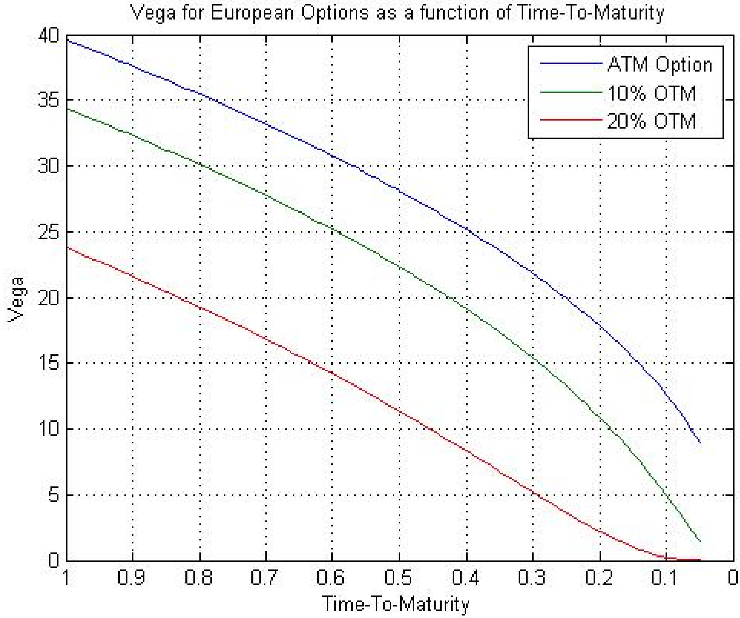

Figure : Vega as a Function of Time-to-Maturity

Theta

Definition. The theta of an option is the negative of the partial derivative of the option price with respect to time-to-maturity.

The theta of an option is therefore the sensitivity of the option price to a negative change in time-to-maturity. It satisfies

$$\begin{aligned}\text{theta} &= -\frac{\partial C}{\partial T} \\ &= -e^{-cT} S \phi(d_1) \frac{\sigma}{2\sqrt{T}} \; + \; c e^{-cT} S \mathsf{N}(d_1) \; - \; rKe^{-rT} \mathsf{N}(d_2)\end{aligned}$$

where $\phi(\cdot)$ is the standard normal PDF.

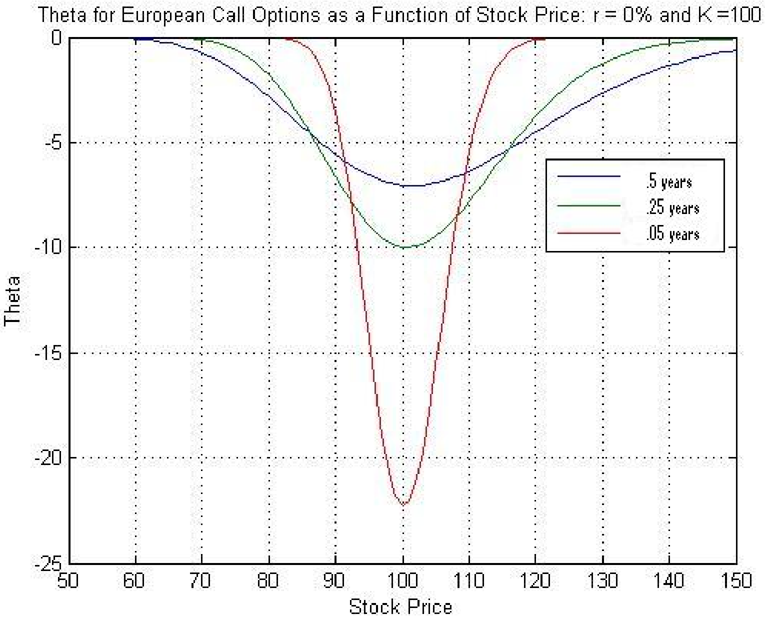

Figure : Theta as a Function of Stock Price

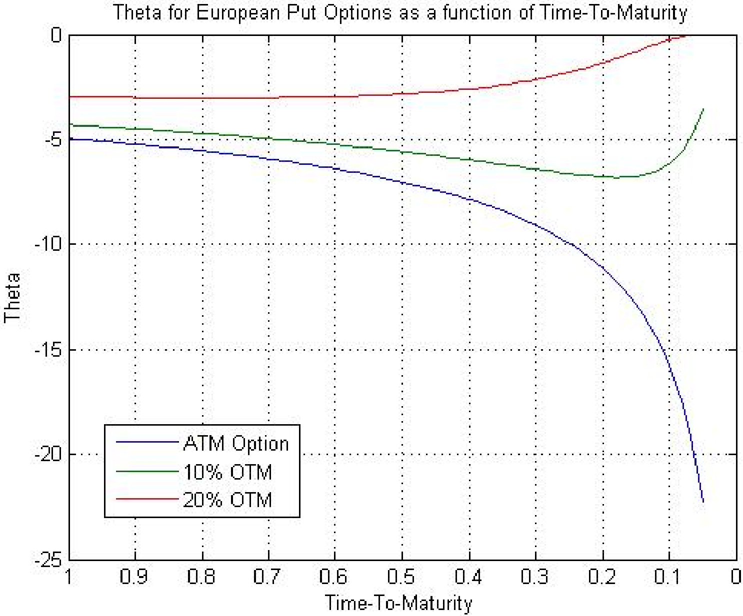

Figure : Theta as a Function of Time-to-Maturity

Financial Engineering & Risk Management

Risk-Management of Derivatives Portfolios

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Review: The Black-Scholes Formula

The Black-Scholes formula for the price of a European call option with strike K and expiration T

$$C_0(S_0, \sigma, T) \;=\; S_0 e^{-cT} N(d_1) \;-\; Ke^{-rT} N(d_2) \tag{9}$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + (r - c + \sigma^2/2)\,T}{\sigma\sqrt{T}}, \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

and $N(d) = \mathsf{P}(N(0,1) \leq d)$.

Recall that r is the risk-free interest rate, c is the dividend yield and the stock price S_t satisfies

$$S_t \;=\; S_0 e^{(r-c-\sigma^2/2)t \,+\, \sigma W_t}$$

where W_t is a Brownian motion under $\mathbb{Q}$.

Can easily calculate the Greeks from (9).

Delta-Gamma-Vega Approximations to Option Prices

Let us now view the option price as a function of S and σ only.

A simple application of Taylor's Theorem says

$$\begin{aligned} C(S+\Delta S, \sigma+\Delta\sigma) &\approx C(S,\sigma) + \Delta S \frac{\partial C}{\partial S} + \frac{1}{2}(\Delta S)^2 \frac{\partial^2 C}{\partial S^2} + \Delta\sigma \frac{\partial C}{\partial \sigma} \\ &= C(S,\sigma) \ + \ \Delta S\, \delta \ + \ \frac{1}{2}(\Delta S)^2\, \Gamma \ + \ \Delta\sigma \text{ vega.} \end{aligned}$$

We therefore obtain

$$\begin{aligned} \text{P\&L} &\approx \delta \Delta S \ + \ \frac{\Gamma}{2}(\Delta S)^2 \ + \ \text{vega } \Delta\sigma \\ &= \text{delta P\&L} \ + \ \text{gamma P\&L} \ + \ \text{vega P\&L} \ . \end{aligned}$$

When $\Delta\sigma = 0$, obtain the well-known delta-gamma approximation

– often used, for example, in historical Value-at-Risk (VaR) calculations.

Delta-Gamma-Vega Approximations to Option Prices

Can also write

$$\begin{aligned}
\text{P\&L} \quad &\approx \quad \delta S \left(\frac{\Delta S}{S}\right) + \frac{\Gamma S^2}{2}\left(\frac{\Delta S}{S}\right)^2 + \text{vega } \Delta\sigma \\
&= \quad \text{ESP} \times \text{Return} \; + \; \$ \text{ Gamma} \times \text{Return}^2 \; + \; \text{vega } \Delta\sigma \qquad (10)
\end{aligned}$$

where ESP denotes the equivalent stock position or "dollar" delta.

We could easily include a theta term in (10) and other "Greeks" if necessary.

Note that (10) also applies to portfolios of derivatives

- very useful for estimating P&L for relatively small moves in risk factors
  - and market participants use such approximations all the time
- but for very large moves in S or σ (10) breaks down
  - scenario analysis comes to the rescue then.

Scenario Analysis

Underlying	SPX Index

Sum of PnL	Vol Stress									
Underlying Stress	-10	-5	-2	-1	0	1	2	5	10	Grand Total
-20	(1,275)	(5,349)	(7,887)	(8,741)	(9,597)	(10,454)	(11,313)	(13,891)	(18,182)	(86,688)
-10	6,993	1,615	(1,509)	(2,536)	(3,555)	(4,568)	(5,575)	(8,562)	(13,439)	(31,138)
-5	9,571	3,902	639	(430)	(1,491)	(2,544)	(3,590)	(6,689)	(11,740)	(12,371)
-2	10,653	4,924	1,625	543	(531)	(1,597)	(2,656)	(5,795)	(10,915)	(3,750)
-1	10,940	5,210	1,905	821	(255)	(1,324)	(2,386)	(5,534)	(10,671)	(1,292)
0	11,194	5,469	2,163	1,078	0	(1,071)	(2,135)	(5,290)	(10,442)	966
1	11,413	5,703	2,398	1,312	234	(837)	(1,903)	(5,063)	(10,228)	3,029
2	11,599	5,911	2,611	1,526	448	(624)	(1,689)	(4,853)	(10,028)	4,901
5	11,972	6,391	3,123	2,045	972	(96)	(1,159)	(4,322)	(9,512)	9,415
10	12,031	6,748	3,585	2,533	1,482	434	(613)	(3,742)	(8,915)	13,542
20	10,499	6,082	3,260	2,300	1,333	359	(621)	(3,590)	(8,598)	11,024
Grand Total	105,589	46,605	11,913	453	(10,959)	(22,322)	(33,640)	(67,331)	(122,669)	(92,362)

Underlying and Volatility Stress Table

The pivot table shows the P&L for an options portfolio in various scenarios

- could also report the Greeks in each scenario
- can easily adapt this to portfolios with multiple underlying securities.

It's important to choose the risk factors and stress levels carefully

- can be very difficult for portfolios of complex derivatives

Financial Engineering & Risk Management

Delta-Hedging

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Delta-Hedging

Recall that the delta of a European call and put option, respectively, are given by

$$\text{Call-Delta} = e^{-cT}\mathsf{N}(d_1)$$

$$\text{Put-Delta} = e^{-cT}\mathsf{N}(d_1) - e^{-cT}$$

where

$$d_1 = \frac{\log(S_0/K) + (r - c + \sigma^2/2)\,T}{\sigma\sqrt{T}},$$

and $N(d) = \mathsf{P}(N(0,1) \leq d)$.

In the Black-Scholes model an option can be **replicated** exactly by following a **self-financing** trading strategy

– just as we did with the binomial model

– when we execute the s.f. trading strategy we say we are delta-hedging the option.

Delta-Hedging

But the Black-Scholes model assumes we can trade **continuously**

– of course this is not feasible in practice

So instead we "hedge" periodically

– feasible in practice but it means we can no longer exactly replicate the option payoff.

Let T be the option expiration and let S_i denote the value of the underlying security at time t_i where

$$0 = t_0 < t_1 < \cdots t_{n-1} < t_n = T.$$

If $V_0(S_0, \sigma_0)$ is the initial value of the option and $\Delta t := t_{i+1} - t_i$ for all i, then

$$V_{i+1} \;=\; V_i \;+\; \delta_i \left(S_{i+1} + S_i c \Delta t - S_i\right) \;+\; \left(V_i - \delta_i S_i\right)\left(e^{r\Delta t} - 1\right) \qquad (11)$$

defines (why?) a **self-financing** trading strategy where δ_i units of the security are held at each time t_i.

Delta-Hedging

If we let $\Delta t \to 0$ then this s.f. strategy will replicate the option payoff at time T

- otherwise it only replicates the payoff **approximately** so that

$$V_n \approx \text{Option Payoff at time } T$$

- assuming the σ parameter that we used to price the option is **"correct"**

i.e. if $\sigma_0 = \sigma$ where σ is the true volatility parameter in the price generating process so that

$$S_{i+1} = S_i \, e^{(\mu - \sigma^2/2)\Delta t \; + \; \sigma (W_{i+1} - W_i)}.$$

If we assume the wrong σ_0 then V_0 and all the δ_i's will be **"wrong"**

- and the s.f. trading strategy may not come close to replicating the option payoff.

Because we can't know the true σ in advance and because security prices don't even follow GBM's, the concept of dynamic replication is therefore only a theoretical concept

- so at best can only hope to replicate options **approximately.**

Financial Engineering & Risk Management

The Volatility Surface

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Problems with Black-Scholes

- Black-Scholes is an elegant model but for several reasons it does not perform very well in practice:
    - security prices often jump
        - this is not possible with GBM
    - security price returns tend to have fatter tails than those implied by the log-normal distribution
    - and returns are clearly not IID in practice.
- Market participants are well aware that the Black-Scholes model is a very poor approximation to reality
    - and have certainly known this since the Wall Street Crash of 1987.
- But the Black-Scholes model and the language of Black-Scholes model is still pervasive in finance
    - most derivatives markets still use aspects of Black-Scholes to quote prices and perform risk management.
- So even though Black-Scholes is clearly not a good approximation to market dynamics, it is still necessary to understand it.

The Volatility Surface

The incorrectness of Black-Scholes is most obviously manifested through the volatility surface

– a concept that is found throughout derivatives markets.

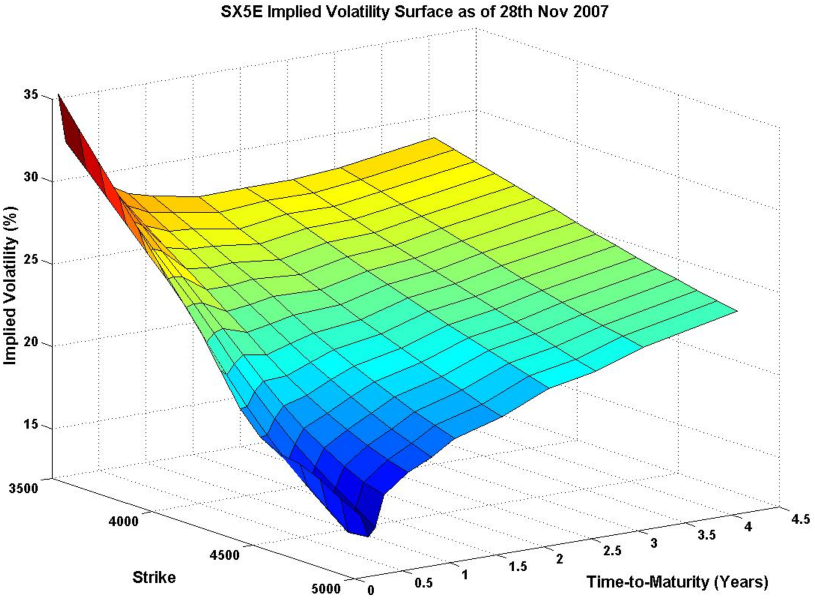
The volatility surface is constructed using **market prices** of European call and put options

– can also use American prices but it's a little trickier.

Definition. The volatility surface, $\sigma(K, T)$, is a function of the strike, K, and expiration, T. It is defined **implicitly** by

$$C_{\text{mkt}}(S, K, T) = C_{\text{BS}}\left(S, T, r, c, K, \sigma(K, T)\right) \tag{12}$$

where $C_{\text{mkt}}(S, K, T)$ denotes the market price of the call option with expiration, T, and strike, K, and $C_{\text{BS}}(\cdot)$ is the corresponding Black-Scholes formula for pricing this call option.


SX5E Implied Volatility Surface as of 28th Nov 2007
Implied Volatility (%)
35
30
25
20
15
3500
4000
4500
5000
Strike
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
Time-to-Maturity (Years)

The Volatility Surface

Question: Why will there will always be a unique solution, $\sigma(K, T)$, to (12)?

If BS model were correct then should have a flat volatility surface with $\sigma(K, T) = \sigma$ for all K and T

- and it would be **constant** through time.

In practice, however, volatility surfaces are not flat and they move about **randomly**.

Options with **lower** strikes tend to have **higher** implied volatilities

- for a given maturity, T, this feature is typically referred to as the volatility skew or smile.

The Volatility Surface

For a given strike, K, the implied volatility can be either increasing or decreasing with time-to-maturity

- in general $\sigma(K, T)$ converges to a constant as $T \rightarrow \infty$.
- for T small often observe an **inverted** volatility surface with short-term options having much higher volatilities than longer-term options
  - this is particularly true in times of **market stress**.

Single-stock options are generally American and in this case, call and put options typically give rise to different surfaces.

The fact that the volatility surface is not constant emphasizes just how wrong Black-Scholes is

- but every equity and FX derivatives trading desk computes the Black-Scholes implied volatility surface
- and the "Greeks" are computed (and used) using Black-Scholes
- use of the BS formula is often likened to "using the wrong number in the wrong formula to obtain the right price"!

Arbitrage Constraints on the Volatility Surface

The **shape** of the implied volatility surface is constrained by the absence of arbitrage. In particular:

1. Must have $\sigma(K, T) \geq 0$ for all strikes K and expirations T.
2. At any given maturity, T, the skew cannot be too steep
   – otherwise arbitrage opportunities, e.g. put-spread arbitrage, will exist.
3. Likewise the term structure of implied volatility cannot be too inverted
   – otherwise calendar spread arbitrages will exist.

In practice the implied volatility surface will not violate any of these restrictions
– otherwise there would be an arbitrage in the market.

These restrictions can be difficult to enforce, however, when we are stressing the volatility surface
– a task that is commonly performed for risk management purposes
– recall our earlier discussion on scenario analysis.