Homework 2 Solutions

Introduction to Quantum Information Science

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1. More fun with matrices.

a) The 2×2 unitary matrix is straightforward:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

b) A 4×4 unitary matrix satisfying the constraints is below:

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}.$$

c) Let $A = \begin{pmatrix} 0 & c & e \\ a & 0 & f \\ b & d & 0 \end{pmatrix}$ be a 3×3 unitary matrix with only diagonal 0 Since A is unitary, $A^H A = I$. We then have the equations

$$\begin{aligned} |a|^2 + |b|^2 &= 1 \qquad & \overline{a} f = a \overline{f} = 0 \\ |c|^2 + |d|^2 &= 1 \qquad & \overline{b} d = b \overline{d} = 0 \\ |e|^2 + |f|^2 &= 1 \qquad & \overline{c} e = c \overline{e} = 0. \end{aligned}$$

Then one of each pair $(a, f), (b, d), (c, e)$ must be 0. This implies that there exist non-diagonal entries in A that are 0, a contradiction. Thus, no such matrix A can exist.

2. Single Qubit Quantum Circuits.

a) $SHZH\,|0\rangle = SHZ\,|+\rangle = SH\,|-\rangle = S\,|1\rangle = i\,|1\rangle = |1\rangle$. If we measure in the $|0\rangle, |1\rangle$ basis, $P[|0\rangle] = 0$, $P[|1\rangle] = 1$.

b) $HYZR_{\pi/4}\,|0\rangle = HYZ\,|+\rangle = HY\,|-\rangle = H(iXZ\,|-\rangle) = H(iX\,|+\rangle) = H(i\,|+\rangle) = H\,|+\rangle = |0\rangle$. If we measure in the $|+\rangle, |-\rangle$ basis, $P[|+\rangle] = \frac{1}{2}$, $P[|-\rangle] = \frac{1}{2}$.

c) $HT\,|+\rangle = H(\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}} e^{i\pi/4}\,|1\rangle) = \frac{1}{\sqrt{2}}\,|+\rangle + \frac{1}{\sqrt{2}} e^{i\pi/4}\,|-\rangle$.

d) $TZT\,|+\rangle = TZ(\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}} e^{i\pi/4}\,|1\rangle) = T(\frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}} e^{i\pi/4}\,|1\rangle) = \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}} e^{i\pi/2}\,|1\rangle$.

3. Miscellaneous.

a) Intuitively, $|0\rangle + |+\rangle$ is the superposition of the $|0\rangle$ and $|+\rangle$ states. Let $|\phi\rangle = |0\rangle + |1\rangle$. Note that $\langle\phi|\phi\rangle = 2 + \sqrt{2}$. Then $|\psi\rangle = \frac{1}{\sqrt{2+\sqrt{2}}}|\phi\rangle$.

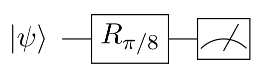
b) $H|\psi\rangle = \frac{1}{\sqrt{2+\sqrt{2}}}H(|0\rangle + |+\rangle) = \frac{1}{\sqrt{2+\sqrt{2}}}(|+\rangle + |0\rangle) = |\psi\rangle$. Then $|\psi\rangle$ is an eigenstate of the H gate with $\lambda = 1$.

c) There are 8 different states reachable from $|0\rangle$ with H and S:

$$\begin{aligned}
|0\rangle \qquad & H|0\rangle = |+\rangle \\
SH|0\rangle = |i\rangle \qquad & HSH|0\rangle = H|i\rangle = \frac{|+\rangle + i|-\rangle}{\sqrt{2}} \\
S^2H|0\rangle = |-\rangle \qquad & HS^2H|0\rangle = H|-\rangle = |1\rangle \\
s^3H|0\rangle = |-i\rangle \qquad & HS^3H|0\rangle = H|-i\rangle = \frac{|+\rangle - i|-\rangle}{\sqrt{2}}.
\end{aligned}$$

4. Distinguishability of states.

a) Consider the quantum circuit:



If $|\psi\rangle = |0\rangle$, $R_{\pi/8}|\psi\rangle = \cos(\frac{\pi}{8})|0\rangle + \sin(\frac{\pi}{8})|1\rangle$, so probability of failure is $\sin^2(\pi/8)$. If $|\psi\rangle = |+\rangle$, $R_{\pi/8}|\psi\rangle = \cos(\frac{3\pi}{8})|0\rangle + \sin(\frac{3\pi}{8})|1\rangle$, so probability of failure is $\cos^2(3\pi/8) = \sin^2(\pi/8)$. Then overall probability of failure is $\sin^2(\frac{\pi}{8})$.
Extra Credit: Suppose that we rotate by some angle $\epsilon \in [0, \frac{\pi}{4}]$. If $|\psi\rangle = |0\rangle$, $R_\epsilon|\psi\rangle = \cos\epsilon|0\rangle + \sin\epsilon|1\rangle$, and if $|\psi\rangle = |+\rangle, R_\epsilon|\psi\rangle = \cos(\frac{\pi}{4}+\epsilon)|0\rangle + \sin(\frac{\pi}{4}+\epsilon)|1\rangle$. Then the overall probability of failure $p_f = \sin^2(\epsilon) + \cos^2(\frac{\pi}{4}+\epsilon)$. The first derivative of this is $p_f' = 2\sin(\epsilon)\cos(\epsilon) - 2\sin(\frac{\pi}{4}+\epsilon)\cos(\frac{\pi}{4}+\epsilon) = 0$. Rearranging and converting the sin to cos, $\cos(\frac{\pi}{2}-\epsilon)\cos(\epsilon) = \cos(\frac{\pi}{4}-\epsilon)\cos(\frac{\pi}{4}+\epsilon)$. Then either

$$\frac{\pi}{2} - \epsilon = \frac{\pi}{4} - \epsilon, \quad \epsilon = \frac{\pi}{4} + \epsilon,$$

or

$$\frac{\pi}{2} - \epsilon = \frac{\pi}{4} + \epsilon, \quad \epsilon = \frac{\pi}{4} - \epsilon.$$

The first case is impossible, but in the second case we find that optimally $\epsilon = \frac{\pi}{8}$.

b) The probability of failure is equal to the probability of getting $|0\rangle$ and measuring 1 plus the probability of getting $|+\rangle$ and measuring 0. Then $P[\text{failure}] = \frac{1}{2}0 + \frac{1}{2}\frac{1}{2} = \frac{1}{4}$.