Homework 1 Solutions

Introduction to Quantum Information Science

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1. Stochastic and Unitary Matrices.

a) The solutions are listed below:

	A	B	C	D	E	F	G	H
stochastic		✓	✓					
unitary		✓		✓		✓	✓	

b) Suppose A is a $N \times N$ matrix that is both stochastic and unitary. Since A is stochastic, all elements of A $a_{ij} \in [0, 1]$. Additionally, since all $a_{ij} \in \mathbb{R}$, $A^H = A^T$. Since A is unitary, we can deduce that $A^H = A^T = A^{-1}$, or $AA^T = AA^{-1} = I$. Let a_i be the ith row vector of A. Then $a_i a_i^T = 1$ for all $i \in [1, N]$. Since A is stochastic, we also know that $1a_i^T = 1$, where the first 1 represents a $1 \times N$ row vector with all components set to 1. We can then subtract these two equations to get that $(1 - a_i)a_i^T = 0$, which implies that each element a_{ij} of a_i is either 0 or 1. Since $1a_i^T = 1$, exactly one component in a_i must be equal to 1, and the rest must be 0. Then A must be a permutation matrix.

c) The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ preserves the 4-norm of real vectors.

d) [Extra credit] From the lead up and also a bit of messing around with 2×2 matrices, it seems like only permutation matrices preserve 4-norms. This seems to make sense since there also aren't too many permutation matrices compared to unitary or stochastic matrices, and at least one of them (I) is trivial. This is far from rigorous though.

2. Tensor Products.

a) $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{2}{15} \\ \frac{8}{15} \\ \frac{1}{15} \\ \frac{4}{15} \end{pmatrix}$.

b) Only D cannot be factorized. The rest of the vector factorizations are below:

$$A = \begin{pmatrix} \frac{2}{9} \\ \frac{1}{9} \\ \frac{4}{9} \\ \frac{2}{9} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$E = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

3. Suppose there exists a real-valued matrix A such that $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$. Then

$$A^2 = \begin{pmatrix} a_1^2 + a_2 a_3 & a_2(a_1 + a_4) \\ a_3(a_1 + a_4) & a_4^2 + a_2 a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we have the system of equations

$$\begin{aligned} a_1^2 + a_2 a_3 &= 1 \\ a_2(a_1 + a_4) &= 0 \\ a_3(a_1 + a_4) &= 0 \\ a_4^2 + a_2 a_3 &= -1. \end{aligned}$$

Subtracting the first and last equations gives us $a_1^2 - a_4^2 = (a_1 + a_4)(a_1 - a_4) = 2$. Clearly, $a_1 + a_4 \neq 0$, thus $a_2 = a_3 = 0$. Then $a_4^2 = -1$, which implies that $a_4 \notin \mathbb{R}$. We conclude that no such matrix A can exist.

4. Dirac notation.

a) $\langle\psi|\phi\rangle = \frac{2i\langle 0|0\rangle + 3\langle 0|1\rangle + 4i\langle 1|0\rangle + 6\langle 1\rangle 1}{\sqrt{65}} = \frac{2i+6}{\sqrt{65}}$.

b) $A = \sqrt{\langle\phi|\phi\rangle} = \sqrt{(3i\,|1\rangle - 2i\,|0\rangle)(2i\,|0\rangle - 3i\,|1\rangle)} = \sqrt{13}$.

c) First, we show that the vectors are orthonormal. Two vectors are orthonormal if they are both unit vectors and their inner product is 0. $\langle i|i\rangle = \frac{1}{2}(|0\rangle - i\,|1\rangle)(|0\rangle + i\,|1\rangle) = 1$, $\langle -i|-i\rangle = \frac{1}{2}(|0\rangle + i\,|1\rangle)(|0\rangle - i\,|1\rangle) = 1$ and $\langle i|-i\rangle = \frac{1}{2}(|0\rangle - i\,|1\rangle)(|0\rangle - i\,|1\rangle) = 0$. Thus, the vectors are orthonormal. We also have to verify that $|i\rangle$ and $|-i\rangle$ are independent.

Suppose there exist scalars a, b such that $a|i\rangle + b|-i\rangle = \frac{1}{\sqrt{2}}((a+b)|0\rangle + i(a-b)|1\rangle) = 0$. This happens only if $a+b = a-b = 0$, or equivalently if $a = b = 0$. We conclude that $|i\rangle$ and $|-i\rangle$ are independent. Then $|i\rangle$ and $|-i\rangle$ form an orthonormal basis for $\mathbb{C}^2$.

d) $|\psi\rangle_{|i\rangle,|-i\rangle} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}|\psi\rangle = \frac{1}{\sqrt{26}}\begin{pmatrix} -i \\ -5 \end{pmatrix}$