Plane partitions and MacMahon Theorem

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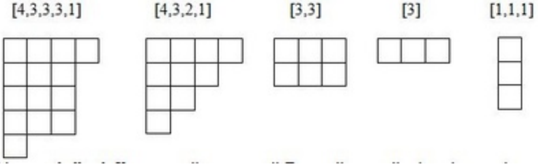
Partitions

Recall that an unordered partition of natural number can be represented as a Young diagram λ.



We denote the number of squares in λ by $|\lambda|$. One can consider the generating function for the number of partitions of integers

$$f_1(t) = \sum_{\lambda} t^{|\lambda|} = 1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 11t^6 + 15t^7 + 22t^8 + \dots$$

A simple theorem by Euler states the equality

$$f_1(t) = \prod_{i=1}^{\infty} \frac{1}{1 - t^i}$$

Generalizations to higher dimension

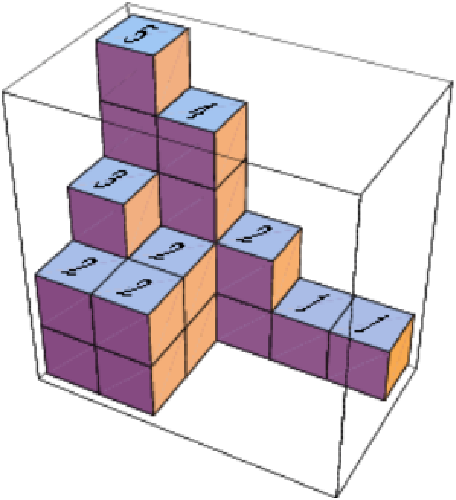
Note that there is a bijection between partitions and non-strictly decreasing sequences of natural numbers with finite support:
$a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0 \geq 0\ldots$
Similarly, one can consider plane partitions, which are sequences of natural numbers on a plane with finite support, that are decreasing along each axis.
Alternatively, one can think of plane partitions as of sequences of Young diagrams contained in each other:

$$\lambda_1 \supset \lambda_2 \supset \lambda_3 \supset \ldots$$

Their generating function starts as

$$f_2(t) = 1 + t + 3t^2 + 6t^3 + 13t^4 + 24t^5 + 48t^6 + 86t^7 + 160t^8 + \ldots$$

Example of a plane partition



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MacMahon Formula

Theorem

(Percy A. MacMahon, 19 century) The generating function $f_2(t)$ for plane partitions is given by

$$f_2(t) = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)^i}.$$

Remark

Judging by the cases with f_0, f_1, f_2 one can conjecture the general formula for n-dimensional partitions:

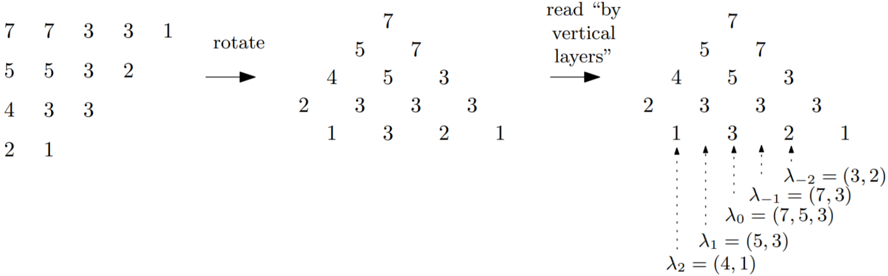
$$f_n(t) = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)^{\binom{n+i-2}{n-1}}}.$$

This is wrong for $n \geq 3$ and obtaining the exact function is one of the most important problems in Combinatorics.

Proof

The proof is *much* harder than the Euler formula.
We say that partition λ is interlacing with μ (and write $\lambda \succ \mu$) if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all i.
To each plane partition one associates a bi-infinite sequence of interlacing partitions:

$$\ldots \prec \lambda_{-3} \prec \lambda_{-2} \prec \lambda_{-1} \prec \lambda_0 \succ \lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \ldots$$



Proof

Let V be the vector space with formal orthonormal basis consisting of 1d partitions $V = \text{span}_{\mathbb{C}}(\lambda \vdash n, n \in \mathbb{N})$. The field will sometimes change to $\mathbb{C}((t, q, t'))$ and infinite combinations of vectors can be considered. Define *vertex operators* $\Gamma_+(t)$ and $\Gamma_-(t')$ on V as:

$$\Gamma_+(t)\lambda = \sum_{\mu \succ \lambda} t^{|\mu|-|\lambda|}\mu, \quad \Gamma_-(t')\lambda = \sum_{\lambda \succ \mu} {t'}^{|\lambda|-|\mu|}\mu.$$

So $\Gamma_+(t)$ interlaces the partition "upwards" in all possible ways, remembering how many boxes were added, while $\Gamma_-(t')$ does the same "downwards".

Proof

Lemma

Let $R_{\leq k}(q)$ be the generating function of plane partitions inside rectangle $k \times k$, equivalently it is generating for sequences

$$\emptyset = \lambda_k \prec \lambda_{k-1} \prec ... \prec \lambda_1 \prec \lambda_0 \succ \lambda_{-1} \succ ... \succ \lambda_{-k+1} \succ \lambda_{-k} = \emptyset$$

The $R_{\leq k}(q)$ is given by

$$(\emptyset, \Gamma_-(q^{-1})\Gamma_-(q^{-2})..\Gamma_-(q^{-k})\Gamma_+(q^{k+1})...\Gamma_+(q^{2k})\emptyset)$$

Proof.

This is equivalent to rephrasing that a 2d partition can be constructed by first interlacing up, and then interlacing down. The fact that the power of q will be correct is a simple computation that we will skip. □

Proof

Now, note that $\Gamma_-(q^{-k})\emptyset = \emptyset$ since only the empty partition interlaces itself down. So if the order of operators was fully reversed, the answer would be clear. This leads to an idea of computing the commutator.

Lemma

The following formula holds

$$\Gamma_-(t')\Gamma_+(t) = \frac{1}{1 - tt'}\Gamma_+(t)\Gamma_-(t').$$

Proof

Equivalent claim is that for any partition α we have

$$\Gamma_-(t')\Gamma_+(t)\alpha = \frac{1}{1-tt'}\Gamma_+(t)\Gamma_-(t')\alpha,$$

or that for any α, β we have

$$(\beta, \Gamma_-(t')\Gamma_+(t)\alpha) = (\beta, \frac{1}{1-tt'}\Gamma_+(t)\Gamma_-(t')\alpha).$$

Finally, this can be rewritten as

$$\sum_{\lambda \text{ s.t.} \alpha \prec \lambda \succ \beta} t'^{|\lambda|-|\beta|} t^{|\lambda|-|\alpha|} = \frac{1}{1-tt'} \sum_{\lambda \text{ s.t.} \alpha \succ \mu \prec \beta} t^{|\beta|-|\mu|} t'^{|\alpha|-|\lambda|}$$

Proof of the lemma

Let $\alpha \prec \lambda \succ \beta$ and $k \geq 0$.
Define $t_i, i \geq 0$ by $t_0 = k$ and $t_i = \min(\alpha_i, \beta_i) - \lambda_i$ for $i \geq 1$.
Then set $\mu_i = \max(\alpha_i, \beta_i) + t_i - 1$. It is easy to check that μ is a partition such that $\alpha \prec \mu \succ \lambda$, and that $|\mu| = |\alpha| + |\beta| - |\lambda| + k$. Indeed one has:

- We have $\mu_i \geq \alpha_i$ since $\mu_i - \alpha_i = (\max(\alpha_i, \beta_i) - \alpha_i) + (\min(\alpha_i - 1, \beta_i - 1) - \lambda_i - 1)$, and both terms are ≥ 0 (for the second term, this is because $\lambda \prec \alpha$ and $\lambda \prec \beta$, so $\lambda_i - 1 \leq \alpha_i - 1$ and same for β).
- We have $\mu_i + 1 \leq \alpha_i$ since $\mu_i + 1 - \alpha_i = (\max(\alpha_i + 1, \beta_i + 1) - \lambda_i) + (\min(\alpha_i, \beta_i) - \alpha_i)$, and both terms are ≤ 0 (for the first term, this is because $\lambda \prec \alpha$ and $\lambda \prec \beta$, so $\lambda_i \geq \alpha_i + 1$ and idem for β).

Moreover we have that
$|\mu| = \sum_{i\geq 1} \mu_i = k + \sum_{i\geq 1}(\min(\alpha_i, \beta_i) + \max(\alpha_i, \beta_i)) = k + |\alpha| + |\beta|$.
We leave the reciprocal bijection since it is similar.

Proof

Recall that

$$R_{\leq k}(q) = (\emptyset, \Gamma_-(q^{-1})\Gamma_-(q^{-2})..\Gamma_-(q^{-k})\Gamma_+(q^{k+1})...\Gamma_+(q^{2k})\emptyset)$$

Take the operator $\Gamma_+(q^{k+1})$ and send it to the left, commuting with the operators $\Gamma_-(q^{-k}), \Gamma_-(q^{-k+1})..\Gamma_-(q^{-1})$. The commutation relation will give us:

$$\frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot ... \cdot \frac{1}{1-q^k}.$$

Then send the operator $\Gamma_+(q^{k+2})$ through the same list $\Gamma_-(q^{-k}), \Gamma_-(q^{-k+1})..\Gamma_-(q^{-1})$, giving

$$\frac{1}{1-q^2} \cdot \frac{1}{1-q^3} \cdot ... \cdot \frac{1}{1-q^{k+1}}.$$

Proof

We see that $R_{\leq k}(q)$ starts with

$$\frac{1}{1-q} \cdot \frac{1}{(1-q^2)^2} \cdot \ldots \frac{1}{(1-q^k)^k} + \text{ terms of order greater than } q^k$$

by taking the limit $k \to \infty$ we conclude that MacMahon formula holds.