DSC180A Notes: Rank, Non-Negative Rank, and Communication Complexity

1 What is Rank?

Definition. The rank of a matrix is the number of linearly independent variables (or columns/rows) in that matrix. Essentially, rank measures the number of variables not accounted for by other variables.

This idea connects closely to communication complexity: the communication complexity represents the number of independent pieces of data that must be sent. Dependent variables need not be transmitted—only the independent ones. Hence, rank correlates with communication complexity.

2 Properties of Rank

Given matrices A and B:

$$\text{If } A \subseteq B, \quad \text{rank}(A) \leq \text{rank}(B),$$

$$\text{rank} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \geq \text{rank}(A) + \text{rank}(B),$$

$$|\text{rank}(A) - \text{rank}(B)| \leq \text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

- Constants used in a function $g(i, j)$ do not change the rank.

- All monochromatic rectangles are submatrices with rank $= 1$.

- The largest possible rank of an $m \times n$ matrix is $\min(m, n)$.

- $\text{rank}(M) \leq$ (number of rank-1 matrices).

3   Tensors

The tensor product of two matrices is the projection of one matrix onto another, resulting in an $n \times n'$-dimensional space.

The tensor product multiplies ranks:

$$\text{If } C = A \otimes B, \quad \text{then rank}(C) = \text{rank}(A) \cdot \text{rank}(B).$$

This makes intuitive sense—projecting onto additional dimensions increases rank multiplicatively.

4   Lower Bounds Using Rank

The logarithm of the rank gives a lower bound on communication complexity:

$$2^c \geq \text{rank}(M) \quad \Rightarrow \quad c \geq \log_2 \text{rank}(M).$$

Since 2^c represents the maximum number of monochromatic rectangles that c bits can encode, $\log_2(\text{rank}(M))$ is a fundamental lower bound.

5   Non-Negative Rank

Definition. For a non-negative matrix M, the following are equivalent:

1. The smallest number of non-negative rank-1 matrices that sum to M.

2. The smallest integer r such that $M = UV$ where U and V are non-negative matrices of sizes $m \times r$ and $r \times n$, respectively.

3. The smallest number r such that every column of M is a non-negative linear combination of r non-negative vectors of size m.

Example. The matrix $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ has $\text{rank}(M) = 3$ but $\text{rank}_+(M) = 4$.

Lemma. If a matrix is 3×3 or smaller, $\text{rank}_+(M) = \text{rank}(M)$.

Lemma (Fact 2.12). $min\{m, n\} \geq \text{rank}_+(M) \geq \text{rank}(M)$.

Remark. Non-negative rank equals regular rank when $\text{rank}(M) \leq 2$, but may diverge for larger matrices.

Conjecture. Determining $\operatorname{rank}(M)$ is polynomial-time solvable, being $(O(n^3))$ (e.g., via SVD or Gaussian elimination), while determining $\operatorname{rank}_+(M)$ or checking whether $\operatorname{rank}_+(M) = \operatorname{rank}(M)$ is NP-hard.

6 Non-Negative Rank Bounds

Remark. Despite computational difficulty, non-negative rank often yields tighter lower or upper bounds on communication complexity.

[2.13] If M is not the all-ones matrix, the communication complexity of M is at least $\log_2 \operatorname{rank}_+(M)$. (Follows from Theorem 2.2.)

[2.15] If a Boolean matrix M has a 1-cover of size r, then the communication complexity of M is at most $\log_2 r$.

Remark. These bounds can differ exponentially.

7 Log-Rank Conjecture

[2.16] There exists a constant c such that for every non-constant matrix M,

$$\mathrm{CC}(M) \leq (\log \operatorname{rank}(M))^c.$$

8 Upper Bounds and Proof Sketches

[2.14] The communication complexity of a Boolean matrix M is at most $O(\operatorname{rank}(M) \log^2 \operatorname{rank}^+(M))$.

Proof. Partition M as $M = [A\ B]$, where A is a 1-rectangle. Without loss of generality, if Alice's input is in A, she sends 1 bit to Bob. Each iteration halves the matrix, akin to a binary search, so the process requires $O(\log n)$ rounds.

[2.18] Any $n \times n$ matrix with r ones and rank k contains a monochromatic submatrix of size at least $2^{-O(\sqrt{r}\log(r))}$.

[2.17] If $\operatorname{rank}(M) = r$, then $\mathrm{CC}(M) = O(\sqrt{r}\log^2(r))$.

Proof. Follows from iterative binary partitioning, reducing the matrix size by a constant factor at each step. A tighter bound of $O(\sqrt{\operatorname{rank}(M)}\log \operatorname{rank}(M))$ was proven by Lovett (2014).

9 Properties of Boolean Matrices

[2.19] If at least half of the entries in M are 0s, then there is a submatrix T of size at least

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$mn * 2^{-O(\sqrt{r}log(r))}$ such that the fraction of 1s in T is at most $\frac{1}{r^3}$.

[2.20] If M is a 0/1 matrix of rank $r \geq 2$ where at most $\frac{1}{r^3}$ entries are 1s, then there exists a submatrix with at least half the rows and half the columns of M that contains only 0s.

[2.22] Any Boolean matrix M of rank r can be expressed as $M = UV$, where U is an $m \times r$ matrix with vectors of lenght at most 1, and V an $r \times n$ matrix with each row vector of length at most $\sqrt{r}$.

[2.21] Let $K \subseteq R^T$ be a symmetric convex body such that the unit ball is the largest-volume ellipsoid contained in K. Then every element of K has Euclidean length at most $\sqrt{r}$.

Proof Sketch. Since scaling preserves convexity, we can assume K forms a convex polytope. By Theorem 2.21, all rows of U have norm $\leq \sqrt{r}$. Because M is Boolean, for each pair (u_i, v_j), the dot product satisfies $u_i^\top v_j \in \{0, 1\}$. Since $|row(A)|_2 \leq 1$ with equality achieved when r = 1, we get the triangle inequality that $|col(B)|_2 \leq 1$, as desired.