

# DSC 180A: Exercise Solutions

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## 1 Chapter 1 Exercises

EXERCISE 1 (1.6) Recall the protocol for the median of two lists discussed in the introduction. Consider the following variant of the median problem. Alice is given a set  $X \subseteq [n]$ , Bob is given a set  $Y \subseteq [n]$ , and their goal is to compute the median of the set  $X \cup Y$ . The difference here is that we take the union as sets, so there are no repetitions. Show that the communication complexity of this problem is at least  $\Omega(n/\log n)$ . *Hint: Compute the median of the union of sets, as well as the union of sets.*

*Proof.* This is a fooling set of  $\log n$  intervals, which cover all of  $n$ . Which blocks are in  $X \cup Y$  determines median, and there are  $2^{\frac{n}{\log n}}$  arrangements of blocks, so requires at least  $\frac{n}{\log n}$  to determine which arrangement and therefore which median is correct.  $\square$

## 2 Chapter 2 Exercises

EXERCISE 2 (2.1) Show that there is a matrix whose rank is 1, yet its communication complexity is 2. Conclude that +1 is necessary in "The communication complexity of  $M$  is at most  $\text{rank}(M) + 1$ .

*Proof.* Take  $M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ . It has rank 1 but clearly takes 2 bits of communication.  $\square$

EXERCISE 3 (2.6) For any symmetric matrix  $M \in \{0, 1\}^{n \times n}$ , with ones in diagonal entries, show that  $2^c \geq n^2/|M|$  where  $|M|$  is the number of ones in  $M$  and  $c$  is the deterministic communication complexity of  $M$ .

*Proof.* The protocol partitions the matrix into at most  $R \leq 2^c$  monochromatic rectangles. Consider only the 1-rectangles, labelling them  $R_1$  to  $R_t$  where  $t \leq R$ .

For each 1-rectangle  $R_r$ , let its row set be  $A_r$  and its column set  $B_r$ . The diagonal entries are all 1, so they are exactly the  $d_r := |A_r \cap B_r|$ . There are  $n$  diagonals so

$$\sum_{r=1}^t k_r = n.$$

If  $i, j \in A_r \cap B_r$ , then  $(i, j)$  is in  $R_r$  and evaluates to 1. There must be at least  $k_r^2$  of these entries. Hence  $|M|$  must be at least the sum of all these:

$$\sum_{r=1}^t k_r^2 \leq |M|.$$

We can apply Cauchy-Schwarz,  $(\sum k_r)^2 \leq t \sum k_r^2$ , to get

$$n^2 = \left( \sum_{r=1}^t k_r \right)^2 \leq t \sum_{r=1}^t k_r^2 \leq t |M|.$$

Since  $t \leq R \leq 2^c$  we have  $2^c \geq n^2/|M|$  as desired.  $\square$

### 3 Chapter 3 Exercises

**EXERCISE 4 (3.2)** Show that there is a protocol for computing greater-than with communication complexity  $\lceil \log(1/\epsilon) \rceil$  such that if the inputs are sampled uniformly and independently, then the average case error of the protocol is at most  $\epsilon$ .

*Proof.* We suppose that Alice and Bob each have some input  $x, y \in [k]$ . We then can divide this set into  $\frac{1}{\epsilon}$  intervals. Alice identifies which interval her input lands in, and communicates this to Bob, using

$$\log \frac{1}{\epsilon}$$

bits. If Bob's input is in a smaller interval, the function returns 1, and in all other cases the function returns 0. The function will have error in the worst case whenever Alice and Bob's inputs are in the same interval. The communication of the function is  $\log \frac{1}{\epsilon}$ , for Alice to tell Bob which interval her input is in, and the probability that the inputs are in the same interval is

$$\frac{1}{\frac{1}{\epsilon}} = \epsilon$$

as desired.  $\square$

## 4 Chapter 4 Exercises

**EXERCISE 5 (4.1)** Suppose there are  $k$  parties in the number-on-forehead model. The  $i$ th party has the bit  $X_i \in \{0, 1\}$  written on their forehead, and  $X_1, \dots, X_k$  are sampled independently and uniformly at random. Show that there is a protocol for each party to privately write down a guess for the bit on their own forehead, without *any* communication, in such a way that the probability that all parties guess correctly is  $1/2$ .

*Proof.* Let  $X_1, X_2, \dots, X_k \in \{0, 1\}$  be independent and uniformly random bits. In the number-on-forehead model, the  $i$ -th player sees all bits except  $X_i$  and must guess that value. We use the protocol below to guess the value of  $X_i$ :

$$g_i = \sum_{j \neq i}^k X_j \mod 2,$$

The condition for our guess to be correct is then,

$$X_i = \sum_{j \neq i}^k X_j \mod 2$$

We can determine whether or not the sum of all bits is zero by the following equation:

$$X_i + \sum_{j \neq i}^k X_j \mod 2 = 0$$

Which is the same as:

$$X_i = \sum_{j \neq i}^k X_j \mod 2$$

Therefore, we know that  $X_i = g_i$  only if the sum of all bits is even. Given that  $X_1, X_2, \dots, X_k \in \{0, 1\}$  are independent and uniformly random bits there are  $2^k$  unique bytes, assuming  $k \geq 1$ , we know that  $2^{k-1}$  of the unique bytes sum to an even number. This results in:

$$P(\text{Correct}) = \frac{\text{number of bytes with even sums}}{\text{total number of bytes}} = \frac{2^{k-1}}{2^k} = \frac{1}{2}$$

□