CH2

- **What is Rank?**
  - Rank is the number of linearly independent variables in a matrix
  - Essentially the rank is the number of variables that are not accounted for by other variables
    - This means that the linearly independent variables need to be inputted into any sort of equation or formula
  - The fact that the rank is the number of linearly independent variables is very similar to the communication complexity problem, since the communication complexity is the number of variables/data that **needs** to be sent. The dependent variables do not need to be sent, only the independent variables, hence how Rank correlates to Communication Complexity

- **Properties of Rank**
  - Given a matrix A and matrix B:
    - If matrix A is a subset of matrix B: rank(A) <= rank(B)
    - rank([A C, 0 B]) >= rank(A) + rank(B)
    - |rank(A) - rank(B)| <= rank(A+B) <= rank(A) + rank(B)
  - The constants used to represent function g(i, j) do not change the rank
  - All monochromatic rectangles are submatricies with rank=1
  - *OTHERS*
    - The largest possible rank of an m x n matrix is the min{m,n}
    - rank(M) <= number of rank 1 matrices

- **Tensors**
  - The tensor product of two matrices is essentially the projection of one matrix onto another resulting in a n * n' dimensional space
  - The tensor product also has the property of multiplying rank, so given matrix A and B, and tensor product C, rank(C) = rank(A) * rank(B)
    - This intuitively makes sense because as we project onto different dimensions, the rank increases linearly

- **Lower Bounds Using Rank**
  - The logarithm of rank always gives a lower bound on communication complexity
    - Since given c bits, there are at most 2^c monochromatic rectangles
    - Since 2^c >= rank(M), we can rearrange such that c >= log2(M) giving us the lower

- **Non-Negative Rank**

  - ***Definition*.** The following are equivalent for M of size $m \times n$:
    - The smallest number of non-negative rank 1 matrices that sum to M

        - The smallest integer r such that $M = AB$ where A is a non-negative $m \times r$ matrix and B is a non-negative $r \times n$ matrix.
        - The smallest number of non-negative vectors sized $m \times 1$ such that every column of M is the non-negative coefficient linear combination of these vectors.

    - ***Example.*** The matrix $M = [1\,1\,0\,0;\ 1\,0\,1\,0;\ 0\,1\,0\,1;\ 0\,0\,1\,1]$ has $rank(M) = 3$ but $rank^{+}(M) = 4$.

    - ***Lemma.*** If a matrix is 3×3 or smaller, $rank(M) = rank^{+}(M)$.

    - ***Lemma (Fact 2.12).*** Notice that $rank(M) \leq rank_{+}(M) \leq min\{m, n\}$.

    - ***Remark.*** Non-negative rank always equals regular rank when $rank(M) \leq 2$. But as rank increases it is possible for the non-negative rank to be far greater than the regular rank.

    - ***Conjecture.*** Determining rank is $O(n^{3})$ using SVD or Gaussian Elimination. Determining non-negative rank is believed to be NP-hard, and checking if $rank(M) = rank^{+}(M)$ is also NP-hard.

- **Non-Negative Rank Bounds**

    - ***Remark.*** Despite the difficulty, non-negative rank can give us much better lower and upper bounds. Why? Because they can differ greatly.

    - ***Theorem (2.13).*** If M is not the all ones matrix, the CC of M is at least $log(rank^{+}(M) + 1)$. Follows from 2.2.

    - ***Lemma (2.15).*** If a boolean matrix M has a 1-cover of size 2^{c} then the CC of M is at most $O(c \cdot log(rank(M)))$.

    - ***Remark.*** Notice there is an exponential difference between the bounds.

    - ***Log-Rank Conjecture (Conjecture 2.16).*** There is a constant α such that the communication complexity of a non-constant matrix M is at most $log^{\alpha}(rank(M))$.

    - ***Theorem (2.14).*** The CC of a boolean matrix M is at most $O(log^{2}(rank^{+}(M)))$.

    - *Proof (2.14)*. Split $M = [R\ A;\ B\ C]$ where R is a 1-rectangle. WLOG, if Alice sees her input is in R and not B, she can send c bits to Bob to communicate this. Then they remove the need to search the other part of the matrix. Bob can do the same by symmetry.

Each iteration reduces $rank(M)$ by a constant factor. Thus, this can continue for only $O(log(rank(M)))$ steps by "binary search" depth.

- *Lemma (2.18)*. Any $m \times n$ matrix that has $0/1$ entries and rank $r \geq 0$ must contain a monochromatic submatrix of size at least $mn \cdot 2^{-O(\sqrt{r}\, log(r))}$.

- ***Theorem (2.17)***. If the rank of a matrix is $r > 1$, then its CC is at most $O(\sqrt{r}\, log^2(r))$.

- *Proof (2.17)*. Similar to 2.14. Binary chop the rectangles, which each reduces the matrix by a factor of $1 - 2^{-O(\sqrt{r}\, log(r))} = 1 - 2^{-k}$.

- *Remark*. Professor Lovett proved a tighter bound, $O(\sqrt{r}\, log(r))$, in 2014.

- **Properties of Boolean Matrices**

  - *Claim (2.19)*. If at least half of the entries in M are 0s, then there is a submatrix T of size at least $mn \cdot 2^{-O(\sqrt{r}\, log(r))}$ such that the fraction of 1s in T is at most $1/r^3$.

  - *Claim (2.20)*. If T is a 0/1 matrix of rank $r \geq 2$, where at most $1/r^3$ of the entries are 1s, then there is a submatrix consisting of at least half of the rows and half of the columns of T that only contains 0s.

  - ***Lemma (2.22)***. Any boolean matrix M of rank r can be expressed as $M = AB$ where A is an $m \times r$ matrix whose rows are vectors of length at most $\sqrt{r}$, and B is an $r \times n$ matrix whose columns are vectors of length at most 1.

  - *Theorem (2.21)*. Let $K \subseteq R^r$ be a symmetric convex body such that the unit ball is the most voluminous of all ellipsoids contained in K. Then every element of K has Euclidean length at most $\sqrt{r}$.

  - ***Proof Sketch (2.22).*** Notice that $M = A'B'$ is not unique; I can scale A, B by scalar multiples. Hand-waving, we can scale A such that it is a convex set of vectors that denote a convex polytope. By 2.21, the length of all rows of A is at most $\sqrt{r}$.

    Since M is boolean, we know that the dot product of any row of A with any column of B satisfies $0 \leq row(A) \cdot col(B) \leq 1$. Applying the 2-norm, and that $|row(A)|_2 \leq 1$ with equality is achieved when $r = 1$, we get by the triangle inequality that $|col(B)|_2 \leq 1$, as desired.

- Some interesting exercises
  - ***Ex 2.1.*** Show that there is a matrix whose rank is 1, yet its communication complexity is 2. Conclude that $+\ 1$ is necessary in "The communication complexity of M is at most $rank(M) + 1$.
  - *Solution*. Take $M = [0\ 0;\ 0\ 1]$.
  - *Ex 2.6*. For any symmetric matrix $M \in \{0, 1\}^{n\times n}$, with ones in diagonal entries, show that $2^c \geq n^2/|M|$ where $|M|$ is the number of ones in M and c is the deterministic communication complexity of M.
  - *Solution*. The protocol partitions the matrix into at most $R \leq 2^c$ monochromatic rectangles. Consider only the 1-rectangles, labelling them R_1 to R_t where $t \leq R$.

    For each 1-rectangle R_i, let its row set be A_i and column set B_r. The diagonal entries are all 1, so they are exactly the $d_r := |A_r \cap B_r|$. There are n diagonals so $\sum_{r=1}^{t} k_r = n$.

    If $i, j \in A_r \cap B_r$, then (i, j) is in R_r and evaluates to 1. There must be at least k_r^2 of these entries. Hence $|M|$ must be at least the sum of all these: $\sum_{r=1}^{t} k_r^2 \leq |M|$.

    We can apply Cauchy-Schwarz ($(\Sigma k_r)^2 \leq t\Sigma k_r^2$) to get

    $n^2 = (\sum_{r=1}^{t} k_r)^2 \leq t \sum_{r=1}^{t} k_r^2 \leq t|M|$.

    Since $t \leq R \leq 2^c$, we have $2^c \geq n^2/|M|$ as desired.