From Diophantine equations to PDEs

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Outline

1. Analytic toolkit for Diophantine equations and inequalities
    1.1 Main example: the circle method for Manin's conjecture
    1.2 Comments on other work
2. Applying that toolkit to harmonic analysis and dispersive PDEs
    2.1 Main example: cluster estimates for the Laplace-Beltrami operator on flat tori
    2.2 Comments on other work
3. Vision and future plans

Diophantine equations

Systems of polynomial equations to be solved in integers. Ubiquitous in mathematics; relatively few general results.

Notation (density of solutions)

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_s]^R$ will be a system of R homogenous forms of degrees d_i in $s > \sum d_i$ variables with integer coefficients.
- We count solutions of $\vec{f} = \vec{0}$ in integers of size B, where B is big.

- $\vec{f}$ takes about $B^{\sum d_i}$ values; maybe it is zero about $\frac{1}{B^{\sum d_i}}$ of the time.
- That would mean about $B^{s - \sum d_i}$ solutions.
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Heuristics

- Model $\vec{x}$ by a random real vector $\vec{X}$, and model $f_i(\vec{x})$ by $\lfloor f_i(\vec{X}) \rfloor$.
- That is, let $\vec{X}$ be a uniform random variable on $[-B, B]^s$. Maybe $N_{\vec{f}}(B) \sim (2B)^s \cdot \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R]$, which is typically $\asymp c_{\vec{f}} B^{s - \sum d_i}$.

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- But: if $f(\vec{x}) = x_1^2 + x_2^2 - 3x_3^2$, then $N_f(B) = 1$ as $\vec{x} = \vec{0}$ "mod 2^∞".
- Fix: let $\vec{X}_p$ be uniformly distributed on $\mathbb{Z}_p^s$. Predict

$$N_{\vec{f}}(B) - 1 = (1 + o(1))(2B)^s \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R] \cdot \prod_p \lim_{N \to \infty} p^{NR} \mathbb{P}[p^N \mid \vec{f}(\vec{X}_p)].$$

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- We say $a = O(b)$ iff $a \ll b$ iff $|a| < Cb$ for some constant C. Also write $a \asymp b$ iff $a \ll b \ll a$. And put $a \sim b$ iff $a/b \to 1$ as $B \to \infty$. And put $a = o(b)$ iff $a/b \to 0$ as $B \to \infty$.

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The circle method

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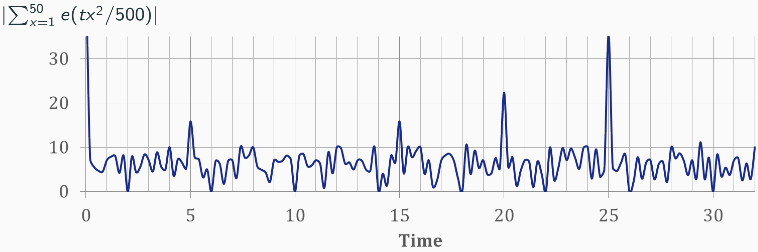
- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- Set $N_{\vec{f}}(B) := \#\{\vec{x} \in \mathbb{Z}^s : \vec{f}(\vec{x}) = \vec{0},\ \|\vec{x}\|_\infty \leq P\}$, $\vec{\alpha} \cdot \vec{f} = \sum^R \alpha_i f_i$.

$$N_{\vec{f}}(P) = \int_0^1 \cdots \int_0^1 \sum_{\substack{\vec{x} \in \mathbb{Z}^n \\ \|\vec{x}\|_\infty \leq B}} e^{2\pi i \vec{t} \cdot \vec{f}(\vec{x})}\, d\vec{t}$$



The Birch-Davenport circle method

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- $\vec{f}$ is smooth if $\vec{f} = 0$ defines a smooth $s - R$ dimensional complex manifold in $\mathbb{C}^s$ away from the origin.

Theorem (Birch 1962)

We have $N_{\vec{f}}(B) \sim cB^{s-dR}$ as above if $\vec{f}$ is smooth, $d_1 = \cdots = d_R = d$,

$$s \geq (d-1)2^{d-1}R(R+1) + R.$$

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Hope for $s > 2dR$. Much work on the range for s if $R = 1$. For $R \geq 2$:

- $(d, R, s) = (2, 2, 11)$ by Munshi (2015) - $s = 10$, Li-RM-Vishe, soon!
- $d = 2, s \geq 9R$, RM (2018)
- $d = 3, s \geq 25R$, RM (2019)
- $(d, R, s) = (3, 2, 39)$ by Northey and Vishe (2024)

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Few generalisations to unequal d_i.

- Browning–Dietmann–Heath-Brown (2014), $d_1 = 2, d_2 = 3$, $s \geq 29$.
- If $d_1, \ldots, d_R \leq d$, Browning–Heath-Brown (2017) handle around $d^3 R^2 2^D$ variables, improving on Schmidt (1985).

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Typically this is about $3^{2d^2} R$ variables; when $d_1 = \cdots = d_R = d$ it is $(1 + d2^d)R$ variables. Improve both Birch's result and, in big R, BHB.

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Key ideas: p-adic repulsion, multilinear repulsion, a way to extract lower-order terms in exponential sums.

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Further work

- I will give the first analogue of Birch's result for Diophantine inequalities $|f_i(\vec{x})| \leq 1$ with real coefficients and differing degrees.
- Counting equations that have solutions, within families of equations parametrized by solutions to quadratic or cubic equations (Serre's problem).
  - Quadratic forms with restricted variables: prime and almost-prime variables with Blomer-Grimmelt-Li;
  - future work with Naomi Bazlov (Warwick), Junxian Li (UC Davis), Alisa Sedunova (Purdue).
  - The elliptic sieve with Loughran, Nakahara, and Bhakta (2023) for rank 1 elliptic curves;
  - working on generalisations with Edison Au-Yeung (Warwick), Subham Bhakta (IISER), Shparlinski (UNSW).

Cluster estimates on manifolds

Notation

- M a compact Riemannian manifold, Δ_M Laplace-Beltrami operator.
- Let $\dim M = d$; analogous to $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^d$.
- Let $(\psi_j)_j$ be an orthonormal basis of eigenfunctions, $\Delta_M \psi_j = -\lambda_j^2 \psi_j$.
- $F(\Delta_M)\psi_j = F(\lambda_j)\psi_j$ (functional calculus).
- If $f = \sum_j \widehat{f}_j \psi_j$ put $\mathbf{1}_{(-1,1)}\left(\frac{\sqrt{-\Delta}-\lambda}{\delta}\right) f = \sum_{j:|\lambda_j - \lambda|<\delta} \widehat{f}_j \psi_j$.

- The universal estimate of Sogge (1988):

$$\|P_{\lambda,1}\|_{L^2 \to L^p} \lesssim \lambda^{\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} + \lambda^{\frac{d-1}{2}-\frac{d}{p}}.$$

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- For $p < \frac{2(d+1)}{d-1}$ the first term is bigger (*geodesic-focusing regime*).
- For fixed $p < \frac{2(d+1)}{d-1}$, the function $\lambda \mapsto \|P_{\lambda,\log^{-1}\lambda}\|_{L^2\to L^p}$ detects the curvature of the manifold (Huang-Sogge 2024)).
- We can really "hear the curvature of a drum"!

Cluster estimates: current work

Notation

- Let $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, $\Delta = \sum_1^3 \frac{\partial^2}{\partial x_i^2}$, $f_{\vec{k}}(\vec{x}) = e^{2\pi i \vec{k}\cdot\vec{x}}$, $\lambda_{\vec{k}} = \|\vec{k}\|_2$ ($\vec{k} \in \mathbb{Z}^3$).
- Define $P_{\lambda,\delta} = \mathbf{1}_{(-1,1)}\left(\frac{\sqrt{-\Delta}-\lambda}{\delta}\right)$.
- The Fourier transform on $\mathbb{T}^3$ is given by

$$\widehat{f}_{\vec{k}} = \int_{\mathbb{T}^3} f(\vec{x}) e^{-2\pi i \vec{k}\cdot\vec{x}}\, d\vec{x} \qquad (\vec{k} \in \mathbb{Z}^3).$$

- We study the $L^2 \to L^p$ operator norms of $P_{\lambda,\delta}$, with a focus on narrow spectral windows (small δ).
- Our goal is to establish bounds of the form, for suitable $C(p, \lambda, \delta)$,

$$\|P_{\lambda,\delta} f\|_p \leq C(p, \lambda, \delta)\|f\|_2.$$

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$$\|P_{\lambda,\delta} f\|_p = \left(\int_{\mathbb{T}^3} \Big(\sum_{\substack{\vec{k}\in\mathbb{Z}^3 \\ -\delta < \|\vec{k}\|_2 - \lambda < \delta}} \widehat{f}_{\vec{k}} e^{2\pi i \vec{k}\cdot\vec{x}} \Big)^p d\vec{x}\right)^{\frac{1}{p}} \leq C(p, \lambda, \delta)\|f\|_2.$$

- An exponential sum; a polynomial in integers $\|k\|_2^2 = k_1^2 + k_2^2 + k_3^2$.

Conjecture (Germain-Myerson 2022)

For $\lambda^{-1} \lesssim \delta \leq 1$ and $p \geq 2$, we conjecture that

$$\|P_{\lambda,\delta}\|_{L^2 \to L^p} \lesssim (\lambda\delta)^{\frac{1}{2}-\frac{1}{p}} + \lambda^{1-\frac{3}{p}}\delta^{\frac{1}{2}} + 1.$$

- Provided $\delta \geq \lambda^{-1}$, we expect

$$\|P_{\lambda,\delta}\|_{L^2 \to L^p} \lesssim \begin{cases} \lambda^{1-\frac{3}{p}}\delta^{\frac{1}{2}} & \text{if } p \geq 6, \text{ or } 4 \leq p \leq 6 \text{ and } \delta \geq \lambda^{2-\frac{p}{2}}, \\ (\lambda\delta)^{\frac{1}{2}-\frac{1}{p}} & \text{otherwise.} \end{cases}$$

- Joint with Pierre Germain (Imperial College London).
- Goal: prove the conjecture for $\delta \geq \lambda^{-1/2}$, possibly with a loss $\lambda^{o(1)}$.
- Builds on weaker result for $\mathbb{T}^d$ and relatively strong results for generic non-rectangular d-tori (Germain-Myerson, 2022).

Harmonic analysis to PDE: future work

- Spectral projector estimates in less traditional settings:
  - hyperbolic spaces with Vedran Sohinger (Warwick),
  - nilmanifolds with Hajer Bahouri (Sorbonne), Veronique Fischer (Bath),
  - Simple Lie groups with Damaris Schindler (Göttingen).
- Bigger picture: Strichartz estimates, applications to PDE.

$$-2\pi i \frac{\partial}{\partial t} f + \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) f = |f|^2 f \qquad \vec{x} \in (\mathbb{R}/L\mathbb{Z})^2$$

$$f(\vec{x}, t) = \sum_{\vec{u} \in \mathbb{Z}^2} a_{\vec{u}}(t) e(\vec{u} \cdot \vec{x}/L + |\vec{u}|^2 t/L^2) \qquad e(s) = e^{2\pi i s}$$

$$-2\pi i \partial_t a_{\vec{u}} = L^{-4} \sum_{\substack{\vec{p}, \vec{q}, \vec{r} \in \mathbb{Z}^2 \\ \vec{p} + \vec{q} = \vec{r} + \vec{u} \\ |\vec{p}|^2 + |\vec{q}|^2 = |\vec{r}|^2 + |\vec{u}|^2}} a_{\vec{p}} a_{\vec{q}} \bar{a}_{\vec{r}} \quad + \quad \text{oscillatory terms}$$

Tristan Buckmaster, Jalal Shatah, Hong Wang (NYU).

Vision

- Push the boundaries of analytic number theory, outward from Diophantine problems to harmonic analysis and applications to nonlinear dispersive PDE.
- I work to build these connections and show them to the world.
- Passionate about mentoring students through research projects and dissertations. Everyone can contribute to mathematical research.
- Collaborative and open approach: group meetings, inter-group idea exchange.

Motivation

- Wind driven waves on the ocean!

Motivation

- Wind driven waves on the ocean $[-L, L]^2$
- Zakharov–Filonenko: There is a simple theory in the big box limit $L \to \infty$.
- But it assumes waves don't have time to cross the box. Can't build a box big enough to test it experimentally.
- Let's do better!

Related problem

Nonlinear Schrödinger equation on a large torus

$$-2\pi i \frac{\partial}{\partial t} f + \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) f = |f|^2 f \qquad \vec{x} \in (\mathbb{R}/L\mathbb{Z})^2$$

Faou–Germain–Hani, Buckmaster–Germain–Hani–Shatah derived a simpler equation for L large, f and t small.

Related problem (Nonlinear Schrödinger equation)

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$$-2\pi i \partial_t a_{\vec{u}} = L^{-4} \int \delta_{\vec{p}+\vec{q}=\vec{r}+\vec{u}} \delta_{|\vec{p}|^2+|\vec{q}|^2=|\vec{r}|^2+|\vec{u}|^2} a_{\vec{p}} a_{\vec{q}} \bar{a}_{\vec{r}} \, d\vec{p} \, d\vec{q} \, d\vec{r}$$

Count weighted solutions to a quadratic equation; use the circle method

Water waves $\to$ some different counting problems. New techniques?
With Jalal Shatah, Tristan Buckmaster, Hong Wang (NYU).

Related problem (Nonlinear Schrödinger equation)

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