Forms with real coefficients and differing degrees

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Systems of polynomial equations to be solved in integers. Ubiquitous in mathematics; relatively few general results.

Notation (density of solutions)

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_s]^R$ will be a system of R homogenous forms of degrees d_i in $s > \sum d_i$ variables with integer coefficients.
- We count solutions of $\vec{f} = \vec{0}$ in integers of size B, where B is big.

- $\vec{f}$ takes about $B^{\sum d_i}$ values; maybe it is zero about $\frac{1}{B^{\sum d_i}}$ of the time.
- That would mean about $B^{s-\sum d_i}$ solutions.
- Also need to consider the number of solutions modulo m for $m \in \mathbb{N}$.

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- We study $N_{\vec{f}}(B) = \#\{\vec{x} \in \mathbb{Z}^s \cap [-B, B]^s : \vec{f}(\vec{x}) = \vec{0}\}$.

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Heuristics

- Model $\vec{x}$ by a random real vector $\vec{X}$, and model $f_i(\vec{x})$ by $\lfloor f_i(\vec{X}) \rfloor$.
- That is, let $\vec{X}$ be a uniform random variable on $[-B, B]^s$. Maybe $N_{\vec{f}}(B) \sim (2B)^s \cdot \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R]$, which is typically $\asymp c_{\vec{f}} B^{s - \sum d_i}$.

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- But: if $f(\vec{x}) = x_1^2 + x_2^2 - 3x_3^2$, then $N_f(B) = 1$ as $\vec{x} = \vec{0}$ "mod 2^∞".
- Fix: let $\vec{X}_p$ be uniformly distributed on $\mathbb{Z}_p^s$. Predict

$$N_{\vec{f}}(B) - 1 = (1 + o(1))(2B)^s \mathbb{P}[\vec{f}(\vec{X}) \in [0,1)^R] \cdot \prod_p \lim_{N \to \infty} p^{NR} \mathbb{P}[p^N \mid \vec{f}(\vec{X}_p)].$$

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- We say $a = O(b)$ iff $a \ll b$ iff $|a| < Cb$ for some constant C. Also write $a \asymp b$ iff $a \ll b \ll a$. And put $a \sim b$ iff $a/b \to 1$ as $B \to \infty$. And put $a = o(b)$ iff $a/b \to 0$ as $B \to \infty$.

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- This is the *quantitative Hasse principle*; the *Manin-Peyre conjecture* is a more sophisticated version needed for $s \leq 2\sum d_i$ or $\vec{f}$ singular.

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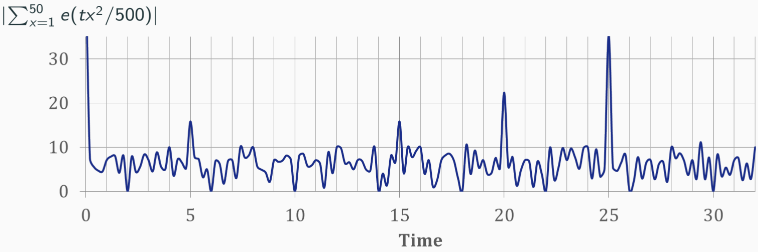
- Set $N_{\vec{f}}(B) := \#\{\vec{x} \in \mathbb{Z}^s : \vec{f}(\vec{x}) = \vec{0},\ \|\vec{x}\|_\infty \leq P\}$, $\vec{\alpha} \cdot \vec{f} = \sum^R \alpha_i f_i$.

$$N_{\vec{f}}(P) = \int_0^1 \cdots \int_0^1 \sum_{\substack{\vec{x} \in \mathbb{Z}^n \\ \|\vec{x}\|_\infty \leq B}} e^{2\pi i \vec{t} \cdot \vec{f}(\vec{x})}\, d\vec{t}$$



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Theorem (Birch 1962)

We have $N_{\vec{f}}(B) \sim cB^{s-dR}$ as above if $\vec{f}$ is smooth, $d_1 = \cdots = d_R = d$,

$$s \geq (d-1)2^{d-1}R(R+1) + R.$$

The Birch-Davenport circle method

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Hope for $s > 2dR$. Much work on the range for s if $R = 1$. For $R \geq 2$:

- $(d, R, s) = (2, 2, 11)$ by Munshi (2015) - $s = 10$, Li-RM-Vishe, soon!
- $d = 2, s \geq 9R$, RM (2018)
- $d = 3, s \geq 25R$, RM (2019)
- $(d, R, s) = (3, 2, 39)$ by Northey and Vishe (2024)

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Generalisations to unequal d_i.

- Browning–Dietmann–Heath-Brown (2014), $d_1 = 2, d_2 = 3$, $s \geq 29$.
- If $d_1, \ldots, d_R \leq d$, Browning–Heath-Brown (2017) handle around $d^3 R^2 2^D$ variables, improving on Schmidt (1985).

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$s \geq R + \sum_{i=1}^{R} d_i 2^{d_i} 3^{2d(d-d_i)}$.

Typically this is about $3^{2d^2}R$ variables; when $d_1 = \cdots = d_R = d$ it is $(1 + d2^d)R$ variables. Improve both Birch's result and, in big R, BHB.

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Key ideas: p-adic repulsion and a way to extract lower-order terms in exponential sums.

Notation

- $\vec{g}(\vec{x}) \in \mathbb{R}[\vec{x}]^R$ is a system of R forms in s variables of degrees $d_i \geq 2$.
- The function $M(B) := \#\{\vec{x} \in \mathbb{Z}^s : \|\vec{g}(\vec{x})\|_\infty \leq 1,\ \|\vec{x}\|_\infty \leq B\}$.
- $\vec{g}$ is nonsingular if the $R \times s$ Jacobian matrix $(\partial g_i(\vec{x})/\partial x_j)_{ij}$ has rank R at every nontrivial complex solution $\vec{x}$ to $\vec{g}(\vec{x}) = \vec{0}$.

- If $\vec{g}$ is nonsingular and the number of variables s is very large, can we estimate $M(B)$?
- In the case $R = 1$, $d = 2$ the breakthrough work of Margulis, which introduced ideas from ergodic theory, led to:

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Theorem (Eskin, Margulis and Mozes 1998)

Let g be a single quadratic form. Suppose g is nonsingular, and not a multiple of a form with rational coefficients.

If $s \geq 5$, then $M(B) \sim \nu B^{s-2}$ as $B \to \infty$ for some $\nu \geq 0$.

If $\vec{X}$ is a uniform RV on $[-B, B]^s$, then $\nu B^{s-2} \sim B^s \mathbb{P}[g(\vec{X}) \in [0, 1)]$.

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- $\vec{g}(\vec{x})$ is irrational if no $\vec{\alpha} \in \mathbb{R}^R \setminus \{\vec{0}\}$ satisfies $\vec{\alpha} \cdot \vec{g}(\vec{x}) \in \mathbb{Z}[\vec{x}]$.

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Theorem (Müller, 2008, slightly rephrased)

If $d_1 = \cdots = d_R = 2$, $\vec{g}$ is nonsingular and irrational, and $s \geq 9R$, then we have $M(B) \sim \nu B^{s-2}$ as $P \to \infty$ for some $\nu \geq 0$.

- Buterus-Götze-Hille-Margulis (2022): $R = 1$, $d = 2$, $s \geq 5$ by the circle method.

Diophantine inequalities: Higher degrees

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- Asymptotics for $d \geq 3$? Diagonal case: Freeman ('03), Wooley ('03)
- Schmidt (1980) gave a lower bound for $M(B)$ when s is (very) large.
- Chow (2014) showed that $\|g(\vec{x})\|_\infty \leq 1$ has a nontrivial integral solution when g is a cubic form and $n \geq 358\,823\,708$.

Theorem (RM 2024)

Let $d_i \leq d$. Suppose that $\vec{g}$ is nonsingular and irrational, and that

$$s \geq R + \sum_{i=1}^{R} d_i \max\{d_i - 2, 1\} 2^{d_i} 3^{2d(d-d_i)}.$$

Then $M(B) \sim \nu B^{n - \sum d_i}$ as $P \to \infty$ for $\nu = \lim B^{\sum d_i} \mathbb{P}[\vec{g}(\vec{X}) \in [0,1)^R]$.

Accessing the lower-degree part

Notation

- $e(t) = 2^{2\pi i t}$, $\Delta_{\vec{h}} f(\vec{x}) = f(\vec{x}+\vec{h}) - f(\vec{x})$, $\Delta_{\vec{h}_1,\dots,\vec{h}_r} = \Delta_{\vec{h}_1} \cdots \Delta_{\vec{h}_r}$
- $f^{[k]}$ is the degree k homogeneous part of a polynomial f
- NB $\Delta_{\vec{h}_1,\dots,\vec{h}_k} f^{[k]}$ is a multilinear form in the $\vec{h}_i$, independent of $\vec{x}$

Proposition

Given $w \in C_c^\infty(\mathbb{R}^s)$ there is $\tilde{w} \in C_c^\infty(\mathbb{R}^{ks})$ as follows. For any $f \in \mathbb{R}[\vec{x}]$ of degree $\leq d$, and any $1 < k < d$,

$$\left| \sum_{\vec{x}\in\mathbb{Z}^s} \frac{1}{B^s} w\left(\frac{\vec{x}}{B}\right) e\left(f(\vec{x})\right) \right|^{(2^{d-1}+1)\cdots(2^k+1)2^{k-1}3^{(k+1)(d-k-1)+1}}$$

$$\leq B^{-ks} \sum_{\vec{h}_1,\dots,\vec{h}_{k-1}\in\mathbb{Z}^s} \left| \sum_{\vec{h}_k\in\mathbb{Z}^s} \tilde{w}\left(\frac{\vec{h}_1}{B},\dots,\frac{\vec{h}_k}{B}\right) e\left(\Delta_{\vec{h}_1,\dots,\vec{h}_k} f^{[k]}\right) \right|$$

Lemma

Given $w \in C_c^\infty(\mathbb{R}^s)$ there is $\tilde{w} \in C_c^\infty(\mathbb{R}^{(d+1)s})$ as follows. For any $f \in \mathbb{R}[\vec{x}]$ of degree $\leq d$,

$$\left| \sum_{\vec{x} \in \mathbb{Z}^s} \frac{1}{B^s} w\left(\frac{\vec{x}}{B}\right) e\left(f(\vec{x})\right) \right|^{2^{d-1}+1} \leq \left| \sum_{\vec{x}_1, \vec{x}_d} \sum_{\vec{x}} \tilde{w}\left(\frac{\vec{x}_d}{B}, \dots, \frac{\vec{x}_1}{B}, \frac{\vec{x}}{B}\right) e\left(f^{[<d]} + F\left(\vec{x}; \vec{x}_1, \dots, \vec{x}_d\right)\right) \right|$$

where $F = f^{[d]} - \Delta_{\vec{x}+\vec{x}_1, \dots, \vec{x}+\vec{x}_d} f$ has degree $< d$ in $\vec{x}$ and ≤ 1 in each $\vec{x}_i$.

Lemma

For $L \in \mathrm{GL}_s(\mathbb{R})$, $A := \{\|\vec{x}\|_\infty \leq B : \|L\vec{x} - \vec{u}\|_\infty \leq 1/B \text{ some } \vec{u} \in \mathbb{Z}^s\}$. Given $w \in C_c^\infty(\mathbb{R}^{2s})$ there are $\tilde{w}_{\vec{c}}^{L,B} \in C_c^\infty(\mathbb{R}^s)$, whose values, support and derivatives are bounded in terms of w only, such that for all real g,

$$\left| \sum_{\vec{x}, \vec{y}} w\left(\frac{\vec{x}}{B}, \frac{\vec{y}}{B}\right) e\left(\vec{y} \cdot L\vec{x} + g(\vec{x})\right) \right|^3 \leq \sum_{\vec{c} \in A - A} \sum_{\vec{x}} \tilde{w}_{\vec{c}}^{L,B}\left(\frac{\vec{x}}{B}\right) e\left(\Delta_{\vec{c}} g(\vec{x})\right).$$