

From Diophantine equations to PDEs

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1. **Analytic toolkit** for Diophantine equations and inequalities
 - 1.1 Main example: the **circle method** for Manin's conjecture
 - 1.2 Comments on other work
2. **Applying** that toolkit to harmonic analysis and dispersive PDEs
 - 2.1 Main example: **cluster estimates** for the Laplace-Beltrami operator on flat tori
 - 2.2 Comments on other work
3. **Vision** and **future plans**

Diophantine equations

Systems of polynomial equations to be solved in integers. Ubiquitous in mathematics; relatively few general results.

Notation (density of solutions)

- $\vec{f}(\vec{x}) \in \mathbb{Z}[x_1, \dots, x_s]^R$ will be a system of R homogenous forms of degrees d_i in $s > \sum d_i$ variables with integer coefficients.
 - We count solutions of $\vec{f} = \vec{0}$ in integers of size B , where B is big.
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- \vec{f} takes about B^{dR} values; maybe it is zero about $\frac{1}{B^{dR}}$ of the time.
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 - $\vec{\alpha} \cdot \vec{f} = \sum^R \alpha_i F_i$ is nonzero and indefinite for all $\vec{\alpha} \in \mathbb{R}^R \setminus \{\vec{0}\}$.
 - We study $N_{\vec{f}}(B) = \#\{\vec{x} \in \mathbb{Z}^s \cap [-B, B]^s : \vec{f}(\vec{x}) = \vec{0}\}$.
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Heuristics

- Model \vec{x} by a random real vector \vec{X} , and model $f_i(\vec{x})$ by $\lfloor f_i(\vec{X}) \rfloor$.
- That is, let \vec{X} be a **uniform random variable on $[-B, B]^s$** . Maybe $N_{\vec{f}}(B) \sim (2B)^s \cdot \mathbb{P}[\vec{f}(\vec{X}) \in [0, 1)^R]$, which is **typically $\sim c_{\vec{f}} B^{s - \sum d_i}$** .

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- But: if $f(\vec{x}) = x_1^2 + x_2^2 - 3x_3^2$, then $N_f(B) = 1$ as $\vec{x} = \vec{0} \pmod{2^\infty}$.
- Fix: let \vec{X}_p be uniformly distributed on \mathbb{Z}_p^s . Predict

$$N_{\vec{f}}(B) - 1 = (1 + o(1))(2B)^s \mathbb{P}[\vec{f}(\vec{X}) \in [0, 1]^R]$$

$$\cdot \prod_p \lim_{N \rightarrow \infty} p^{NR} \mathbb{P}[p^N \mid \vec{f}(\vec{X}_p)].$$

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- We say $a = O(b)$ iff $a \ll b$ iff $|a| < Cb$ for some constant C . Also write $a \sim b$ iff $a \ll b \ll a$. And put $a \sim b$ iff $a/b \rightarrow 1$ as $B \rightarrow \infty$. And put $a = o(b)$ iff $a/b \rightarrow 0$ as $B \rightarrow \infty$.

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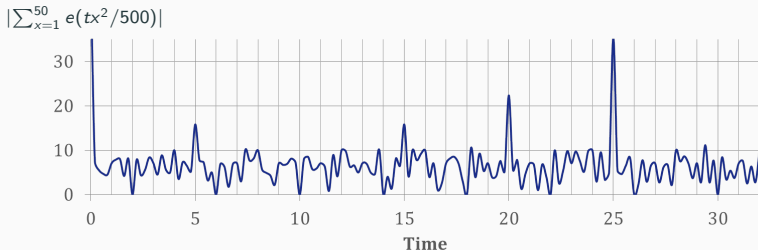
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The circle method

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- We set $N_{\vec{f}}(B) := \#\{\vec{x} \in \mathbb{Z}^s : \vec{f}(\vec{x}) = \vec{0}, \|\vec{x}\|_{\infty} \leq B\}$.

$$N_{\vec{f}}(P) = \int_0^1 \cdots \int_0^1 \sum_{\substack{\vec{x} \in \mathbb{Z}^n \\ \|\vec{x}\|_{\infty} \leq B}} e(\vec{\beta} \cdot \vec{f}(\vec{x})) d\vec{\beta}, \quad e(s) = e^{2\pi i s}$$



The Birch-Davenport circle method

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Theorem (Birch 1962)

We have $N_{\vec{f}}(B) \sim cB^{s-dR}$ as above if \vec{f} is **smooth**, $d_1 = \dots = d_R = d$,
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Hope for $s > 2dR$. Much work on the range for s if $R = 1$. For $R \geq 2$:

- $(d, R, s) = (2, 2, 11)$ by Munshi (2015) - $s = 10$, Li-RM-Vishe, soon!
- $d = 2, s \geq 9R$, RM (2018)
- $d = 3, s \geq 25R$, RM (2019)
- $(d, R, s) = (3, 2, 39)$ by Northey and Vishe (2024)

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Few generalisations to unequal d_i .

- Browning–Dietmann–Heath-Brown (2014), $d_1 = 2, d_2 = 3, s \geq 29$.
- If $d_1, \dots, d_R \leq d$, Browning–Heath-Brown (2017) handle around $d^3 R^2 2^D$ variables, improving on Schmidt (1985).

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- In the culmination of work announced at the 2021 Hausdorff circle method school, I achieve $s \geq R + \sum_{i=1}^R d_i 2^{d_i} 3^{2d(d-d_i)}$.

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Key ideas: p -adic repulsion, multilinear repulsion, a way to extract lower-order terms in exponential sums.

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Further work

- I will give the **first** analogue of Birch's result for Diophantine **inequalities** $|f_i(\vec{x})| \leq 1$ with real coefficients and differing degrees.
- Counting equations that have solutions, within families of equations parametrized by solutions to quadratic or cubic equations (**Serre's problem**).
 - Quadratic forms with restricted variables: prime and almost-prime variables with Blomer-Grimmelt-Li;
 - future work with Naomi Bazlov (Warwick), Junxian Li (UC Davis), Alisa Sedunova (Purdue).
 - The elliptic sieve with Loughran, Nakahara, and Bhakta (2023) for rank 1 elliptic curves;
 - working on generalisations with Edison Au-Yeung (Warwick), Subham Bhakta (IISER), Shparlinski (UNSW).

Cluster estimates on Riemannian manifolds

Notation

- M a **compact** Riemannian manifold, Δ_M Laplace-Beltrami operator.
- Let $\dim M = d$; analogous to $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ on \mathbb{R}^d .
- Let $(\psi_j)_j$ be an orthonormal basis of eigenfunctions, $\Delta_M \psi_j = -\lambda_j^2 \psi_j$.
- $F(\Delta_M) \psi_j = F(\lambda_j) \psi_j$ (functional calculus).
- If $f = \sum_j \hat{f}_j \psi_j$ put $\mathbf{1}_{(-1,1)} \left(\frac{\sqrt{-\Delta} - \lambda}{\delta} \right) f = \sum_{j: |\lambda_j - \lambda| < \delta} \hat{f}_j \psi_j$.

- The universal estimate of Sogge (1988):

$$\|P_{\lambda,1}\|_{L^2 \rightarrow L^p} \lesssim \lambda^{\frac{d-1}{2}(\frac{1}{2} - \frac{1}{p})} + \lambda^{\frac{d-1}{2} - \frac{d}{p}}.$$

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- For $p < \frac{2(d+1)}{d-1}$ the first term is bigger (*geodesic-focusing regime*).
- For fixed $p < \frac{2(d+1)}{d-1}$, the function $\lambda \mapsto \|P_{\lambda, \log^{-1} \lambda}\|_{L^2 \rightarrow L^p}$ detects the curvature of the manifold (Huang-Sogge 2024)).
- We can really “hear the curvature of a drum”!

Cluster estimates: current work

Notation

- Let $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, $\Delta = \sum_1^3 \frac{\partial^2}{\partial x_i^2}$, $f_{\vec{k}}(\vec{x}) = e^{2\pi i \vec{k} \cdot \vec{x}}$, $\lambda_{\vec{k}} = \|\vec{k}\|_2$ ($\vec{k} \in \mathbb{Z}^3$).
- Define $P_{\lambda, \delta} = \mathbf{1}_{(-1, 1)}\left(\frac{\sqrt{-\Delta} - \lambda}{\delta}\right)$.
- The Fourier transform on \mathbb{T}^3 is given by

$$\widehat{f}_{\vec{k}} = \int_{\mathbb{T}^3} f(\vec{x}) e^{-2\pi i \vec{k} \cdot \vec{x}} d\vec{x} \quad (\vec{k} \in \mathbb{Z}^3).$$

- We study the $L^2 \rightarrow L^p$ operator norms of $P_{\lambda, \delta}$, with a focus on narrow spectral windows (small δ).
- Our goal is to establish bounds of the form, for suitable $C(p, \lambda, \delta)$,

$$\|P_{\lambda, \delta} f\|_p \leq C(p, \lambda, \delta) \|f\|_2.$$

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- Our goal is to establish bounds of the form, for suitable $C(p, \lambda, \delta)$,

$$\|P_{\lambda, \delta} f\|_p = \left(\int_{\mathbb{T}^3} \left(\sum_{\substack{\vec{k} \in \mathbb{Z}^3 \\ -\delta < \|\vec{k}\|_2 - \lambda < \delta}} \widehat{f}_{\vec{k}} e^{2\pi i \vec{k} \cdot \vec{x}} \right)^p d\vec{x} \right)^{\frac{1}{p}} \leq C(p, \lambda, \delta) \|f\|_2.$$

- An **exponential sum**; a **polynomial in integers** $\|k\|_2^2 = k_1^2 + k_2^2 + k_3^2$.

Cluster estimates: current work

Conjecture (Germain-Myerson 2022)

For $\lambda^{-1} \lesssim \delta \leq 1$ and $p \geq 2$, we conjecture that

$$\|P_{\lambda,\delta}\|_{L^2 \rightarrow L^p} \lesssim (\lambda\delta)^{\frac{1}{2}-\frac{1}{p}} + \lambda^{1-\frac{3}{p}}\delta^{\frac{1}{2}} + 1.$$

- Provided $\delta \geq \lambda^{-1}$, we expect

$$\|P_{\lambda,\delta}\|_{L^2 \rightarrow L^p} \lesssim \begin{cases} \lambda^{1-\frac{3}{p}}\delta^{\frac{1}{2}} & \text{if } p \geq 6, \text{ or } 4 \leq p \leq 6 \text{ and } \delta \geq \lambda^{2-\frac{p}{2}}, \\ (\lambda\delta)^{\frac{1}{2}-\frac{1}{p}} & \text{otherwise.} \end{cases}$$

- Joint with Pierre Germain (Imperial College London).
- Goal: prove the conjecture for $\delta \geq \lambda^{-1/2}$, possibly with a loss $\lambda^{o(1)}$.
- Builds on weaker result for \mathbb{T}^d and relatively strong results for generic non-rectangular d -tori (Germain-Myerson, 2022).

Harmonic analysis: future work

- Spectral projector estimates in less traditional settings:
 - hyperbolic spaces with Vedran Sohinger (Warwick),
 - nilmanifolds with Hajer Bahouri (Sorbonne), Veronique Fischer (Bath),
 - Simple Lie groups with Damaris Schindler (Göttingen).
- Bigger picture: Strichartz estimates, applications to PDE.

- Push the boundaries of analytic number theory, outward from Diophantine problems to harmonic analysis and applications to nonlinear dispersive PDE.
- I work to build these connections and show them to the world.
- Passionate about mentoring students through research projects and dissertations. Everyone can contribute to mathematical research.
- Collaborative and open approach: group meetings, inter-group idea exchange.

Motivation

- Wind driven waves on the ocean!

Number theory and the sea

Motivation

- Wind driven waves on the ocean $[-L, L]^2$
- Zakharov–Filonenko: There is a simple theory in the big box limit $L \rightarrow \infty$.
- But it assumes waves don't have time to cross the box. Can't build a box big enough to test it experimentally.
- Let's do better!

Related problem

Nonlinear Schrödinger equation on a large torus

$$-2\pi i \frac{\partial}{\partial t} f + \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) f = |f|^2 f \quad \vec{x} \in (\mathbb{R}/L\mathbb{Z})^2$$

Faou–Germain–Hani, Buckmaster–Germain–Hani–Shatah derived a simpler equation for L large, f and t small.

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$$f(\vec{x}, t) = \sum_{\vec{u} \in \mathbb{Z}^2} a_{\vec{u}}(t) e(\vec{u} \cdot \vec{x}/L) \quad e(s) = e^{2\pi i s}$$

$$-2\pi i \partial_t a_{\vec{u}} = L^{-4} \sum_{\substack{\vec{p}, \vec{q}, \vec{r} \in \mathbb{Z}^2 \\ \vec{p} + \vec{q} = \vec{r} + \vec{u}}} a_{\vec{p}} a_{\vec{q}} \bar{a}_{\vec{r}} + \text{oscillatory terms}$$

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$$-2\pi i \partial_t a_{\vec{u}} = L^{-4} \int \delta_{\vec{p} + \vec{q} = \vec{r} + \vec{u}} \delta_{|\vec{p}|^2 + |\vec{q}|^2 = |\vec{r}|^2 + |\vec{u}|^2} a_{\vec{p}} a_{\vec{q}} \bar{a}_{\vec{r}} d\vec{p} d\vec{q} d\vec{r}$$

Count weighted solutions to a **quadratic equation**; use the circle method

Water waves → some different counting problems. New techniques?
With Jalal Shatah (NYU), Tristan Buckmaster (NYU).

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