Ryan D'Mello

Error Comparison of the *Euler's*, *Improved Euler's*, and *Runge-Kutta* Approximation Methods

Introduction

This paper investigates three methods for approximating the solutions of first-order differential equations: Euler's method, Improved Euler's method, and the Runge-Kutta method. The approximations are accomplished using the sequence and approach spelled out in *Exploration 7.5* in the textbook.

The specific formulas and calculations are implemented in an EXCEL spreadsheet. Although I also performed the implementation in Python as well (getting results identical to those from EXCEL), I have chosen to display the EXCEL-based results because they are more simple, intuitive, and far more convenient to visualize and tweak, and also because the exercise explicitly recommends "using a spreadsheet to make these lengthy calculations". Parts of the spreadsheet are displayed in this paper.

Section 1 explains how the EXCEL spreadsheet is set up and organized, and it addresses the computations associated with each of the three methods. ***Section 2*** addresses the first of the four exercises listed in *Exploration 7.5* (drawing a picture in the $tx - plane\ that\ illustrates\ the\ \ (t_k, x_k)\ to\ (t_{k+1}, x_{k+1})\ transition\ for\ each\ of\ the\ three\ methods$). ***Section 3*** shows the results of the calculations in EXCEL and computes the errors and error changes (thus addressing the second and third of the four exercises in *Exploration 7.5)*. ***Section 4*** is a summary and concluding discussion that addresses the last (fourth) exercise.

Section 1: EXCEL spreadsheet set up and organization

This section describes how the EXCEL spreadsheet is set up in order to perform the computations associated with the three methods, so it completely addresses the parts of Exploration 7.5 describing the computations in *1. Euler's method*, *2. Improved Euler's method*, and *3. (a), (b), (c), and (d) (Fourth-order) Runge-Kutta Method* (on page 154 and first half of page 155).

The EXCEL spreadsheet contains two tabs (see Figure 1 on the next page). The first tab is for the differential equation $x' = x$, where $x = x(t)$ is a function of t. The second tab is for the differential equation $x' = 2t(1 + x^2)$. Each tab has three sections, one for each of the three methods: Euler's, Improved Euler', and Runge-Kutta. In each of these methods, the value of Δt can be set (see cells shaded yellow in Row 7), and so can the values of the initial conditions (see gray cells in Row 10, $k = 0$). All other cells are computed using these initial values and Δt.

The first column of both spreadsheets (column A) contains the values of k, which provides the index for the subsequent sequential calculations (recurrence relations). This is followed by three columns for Euler's method (columns B, C, and D). Column B simply calculates t_{k+1} by adding Δt to t_k. Cell B10 is the initial value $\boldsymbol{t_0}$ **which we set to 0**. Column C computes x_{k+1} from x_k using the formula provided under Euler's method ($x_{k+1} = x_k + f(t_k, x_k)\,\Delta t$). The x_k values are the approximations to $x(t_k)$. Cell C10 is the value x_0 of x at the initial value t_0 , so $\boldsymbol{x_0} = \boldsymbol{x}(\boldsymbol{t_0}) = \boldsymbol{x}(\boldsymbol{0})$. Column D computes $f(t_k, x_k)$ based on which of the two cases we are dealing with: $x' = x$ or $x' = 2t(1 + x^2)$. (Figure 1 displays the tab for $\mathrm{x}' = \mathrm{x}$, with $\Delta\mathrm{t}$ set to 0.1.) Clearly, the solution of $x' = x$ is $x(t) = ke^t$, where k is a constant. If $x(0) = 1$ $(as\ stiputated\ in\ the\ assignment)$, then $k = 1$ and $\boldsymbol{x(t) = e^t}$. The solution of $x' = 2t(1 + x^2)$ can be determined as follows:

$$\int \frac{dx}{1 + x^2} = \int 2t\, dt$$

or $\tan^{-1} x = t^2 + k$, where k is constant, or $x(t) = \tan(t^2 + k)$. If we assume the initial condition $(t, x) = (0,0)$ $(or\ x(0) = 0)$, then $\boldsymbol{x(t) = \tan(t^2)}$. The values being approximated are $x(1)$, namely $e^1 = \boldsymbol{e}, and\ \tan(1^2) = \boldsymbol{\tan(1)}$.

FIGURE 1 ($x' = x$ and $\Delta t = 0.1$)

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This Excel tab is for the function f(t,x) = x' = x, with initial condition x(0) = 1. Hence, $f(t_k,x_k) = x_k$. Clearly, $x(t) = e^t$

BRIGHT YELLOW CELLS SHOULD BE EDITED TO REFLECT Δt — GRAY CELLS SHOULD BE EDITED TO REFLECT INITIAL VALUES

GREEN CELLS COLUMNS SHOULD REFLECT FUNCTION f (so $f(t_k,x_k) = x_k \sim x(t_k)$) — BLUE CELLS ARE ACTUAL FUNCTION VALUES

EULER's METHOD: $x_{k+1} = x_k + f(t_k,x_k)\Delta t$; Δt = 0.1

IMPROVED EULER's METHOD: $x_{k+1} = x_k + 0.5 * \{m_k + n_k\}\Delta t$ where $m_k = f(t_k,x_k)$; $n_k = f(t_{k+1}, x_k + f(t_k,x_k)\Delta t)$; Δt = 0.1

RUNGE-KUTTA METHOD: $x_{k+1} = x_k + (1/6) * (m_k + 2n_k + 2p_k + q_k)\Delta t$ where $m_k = f(t_k,x_k)$; $n_k = f(t_k + 0.5\Delta t, x_k + 0.5*m_k*\Delta t)$; $p_k = f(t_k + 0.5\Delta t, x_k + 0.5*n_k*\Delta t)$; $q_k = f(t_{k+1}, x_k + p_k*\Delta t)$; Δt = 0.1

k	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$ (Euler)	$f(t_k,x_k) = x_k$	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$ (Improved Euler)	$m_k = f(t_k,x_k)$	$n_k = f(t_{k+1}, x_k + f(t_k,x_k)\Delta t)$
0	0.0000000	1.0000000	1.0000000	0.0000000	1.0000000	1.0000000	1.1000000
1	0.1000000	1.1000000	1.1000000	0.1000000	1.1050000	1.1050000	1.2155000
2	0.2000000	1.2100000	1.2100000	0.2000000	1.2210250	1.2210250	1.3431275
3	0.3000000	1.3310000	1.3310000	0.3000000	1.3492326	1.3492326	1.4841559
4	0.4000000	1.4641000	1.4641000	0.4000000	1.4909021	1.4909021	1.6399923
5	0.5000000	1.6105100	1.6105100	0.5000000	1.6474468	1.6474468	1.8121914
6	0.6000000	1.7715610	1.7715610	0.6000000	1.8204287	1.8204287	2.0024715
7	0.7000000	1.9487171	1.9487171	0.7000000	2.0115737	2.0115737	2.2127311
8	0.8000000	2.1435888	2.1435888	0.8000000	2.2227889	2.2227889	2.4450678
9	0.9000000	2.3579477	2.3579477	0.9000000	2.4561818	2.4561818	2.7017999
10	1.0000000	**2.5937425**	2.5937425	1.0000000	**2.7140808**	2.7140808	2.9854889
11	1.1000000	2.8531167	2.8531167	1.1000000	2.9990593	2.9990593	3.2989653
12	1.2000000	3.1384284	3.1384284	1.2000000	3.3139606	3.3139606	3.6453566
13	1.3000000	3.4522712	3.4522712	1.3000000	3.6619264	3.6619264	4.0281191
14	1.4000000	3.7974983	3.7974983	1.4000000	4.0464287	4.0464287	4.4510716
15	1.5000000	4.1772482	4.1772482	1.5000000	4.4713037	4.4713037	4.9184341
16	1.6000000	4.5949730	4.5949730	1.6000000	4.9407906	4.9407906	5.4348697
17	1.7000000	5.0544703	5.0544703	1.7000000	5.4595736	5.4595736	6.0055310
18	1.8000000	5.5599173	5.5599173	1.8000000	6.0328288	6.0328288	6.6361117
19	1.9000000	6.1159090	6.1159090	1.9000000	6.6662759	6.6662759	7.3329035
20	2.0000000	**6.7274999**	6.7274999	2.0000000	**7.3662348**	7.3662348	8.1028583
21	2.1000000	7.4002499	7.4002499	2.1000000	8.1396895	8.1396895	8.9536585
22	2.2000000	8.1402749	8.1402749	2.2000000	8.9943569	8.9943569	9.8937926
23	2.3000000	8.9543024	8.9543024	2.3000000	9.9387644	9.9387644	10.9326408
24	2.4000000	9.8497327	9.8497327	2.4000000	10.9823346	10.9823346	12.0805681
25	2.5000000	10.8347059	10.8347059	2.5000000	12.1354798	12.1354798	13.3490277
26	2.6000000	11.9181765	11.9181765	2.6000000	13.4097051	13.4097051	14.7506757
27	2.7000000	13.1099942	13.1099942	2.7000000	14.8177242	14.8177242	16.2994966

k	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$ (Runge-Kutta)	$m_k = f(t_k,x_k)$	$n_k = f(t_k + 0.5\Delta t, x_k + 0.5*m_k*\Delta t)$	$p_k = f(t_k + 0.5\Delta t, x_k + 0.5*n_k*\Delta t)$	$q_k = f(t_{k+1}, x_k + p_k*\Delta t)$	ACTUAL VALUE of $x(t_k) = e^{t_k}$
0	0.0000000	1.0000000	1.0000000	1.0500000	1.0525000	1.1052500	1.0000000
1	0.1000000	1.1051708	1.1051708	1.1604294	1.1631923	1.2214901	1.1051709
2	0.2000000	1.2214026	1.2214026	1.2824727	1.2855262	1.3499552	1.2214028
3	0.3000000	1.3498585	1.3498585	1.4173514	1.4207261	1.4919311	1.3498588
4	0.4000000	1.4918242	1.4918242	1.5664155	1.5701450	1.6488387	1.4918247
5	0.5000000	1.6487206	1.6487206	1.7311567	1.7352785	1.8222485	1.6487213
6	0.6000000	1.8221180	1.8221180	1.9132239	1.9177792	2.0138959	1.8221188
7	0.7000000	2.0137516	2.0137516	2.1144392	2.1194736	2.2256990	2.0137527
8	0.8000000	2.2255396	2.2255396	2.3368165	2.3423804	2.4597776	2.2255409
9	0.9000000	2.4596014	2.4596014	2.5825815	2.5887305	2.7184745	2.4596031
10	1.0000000	**2.7182797**	2.7182797	2.8541937	2.8609894	3.0043787	2.7182818
11	1.1000000	3.0041635	3.0041635	3.1543717	3.1618821	3.3203517	3.0041660
12	1.2000000	3.3201139	3.3201139	3.4861196	3.4944198	3.6695559	3.3201169
13	1.3000000	3.6692930	3.6692930	3.8527577	3.8619309	4.0554861	3.6692967
14	1.4000000	4.0551956	4.0551956	4.2579554	4.2680934	4.4820050	4.0552000
15	1.5000000	4.4816839	4.4816839	4.7057681	4.7169723	4.9533811	4.4816891
16	1.6000000	4.9530263	4.9530263	5.2006777	5.2130602	5.4743324	4.9530324
17	1.7000000	5.4739403	5.4739403	5.7476373	5.7613221	6.0500725	5.4739474
18	1.8000000	6.0496391	6.0496391	6.3521211	6.3672452	6.6863636	6.0496475
19	1.9000000	6.6858847	6.6858847	7.0201789	7.0368936	7.3895741	6.6858944
20	2.0000000	**7.3890448**	7.3890448	7.7584970	7.7769696	8.1667417	7.3890561
21	2.1000000	8.1661568	8.1661568	8.5744646	8.5948800	9.0256448	8.1661699
22	2.2000000	9.0249983	9.0249983	9.4762482	9.4988107	9.9748793	9.0250135
23	2.3000000	9.9741649	9.9741649	10.4728731	10.4978085	11.0239457	9.9741825
24	2.4000000	11.0231561	##########	11.5743139	11.6018718	12.1833433	11.0231764
25	2.5000000	12.1824706	##########	12.7915941	12.8220503	13.4646756	12.1824940
26	2.6000000	13.4637112	##########	14.1368968	14.1705560	14.8807668	13.4637380
27	2.7000000	14.8797009	##########	15.6236860	15.6608852	16.4457894	14.8797317

x' = x   x' = 2t(1 + x^2)

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Figure 1 above illustrates what we have done. Note the two tabs (left bottom), one for $x' = x$ and the other for $x' = 2t(1 + x^2)$, the active tab here being for $x' = x$. The next four columns (E, F, G, & H) are for the Improved Euler's method, the only difference here being that we need an extra column (column H) for computing $n_k = f(t_{k+1}, x_k + f(t_k, x_k)\Delta t)$.

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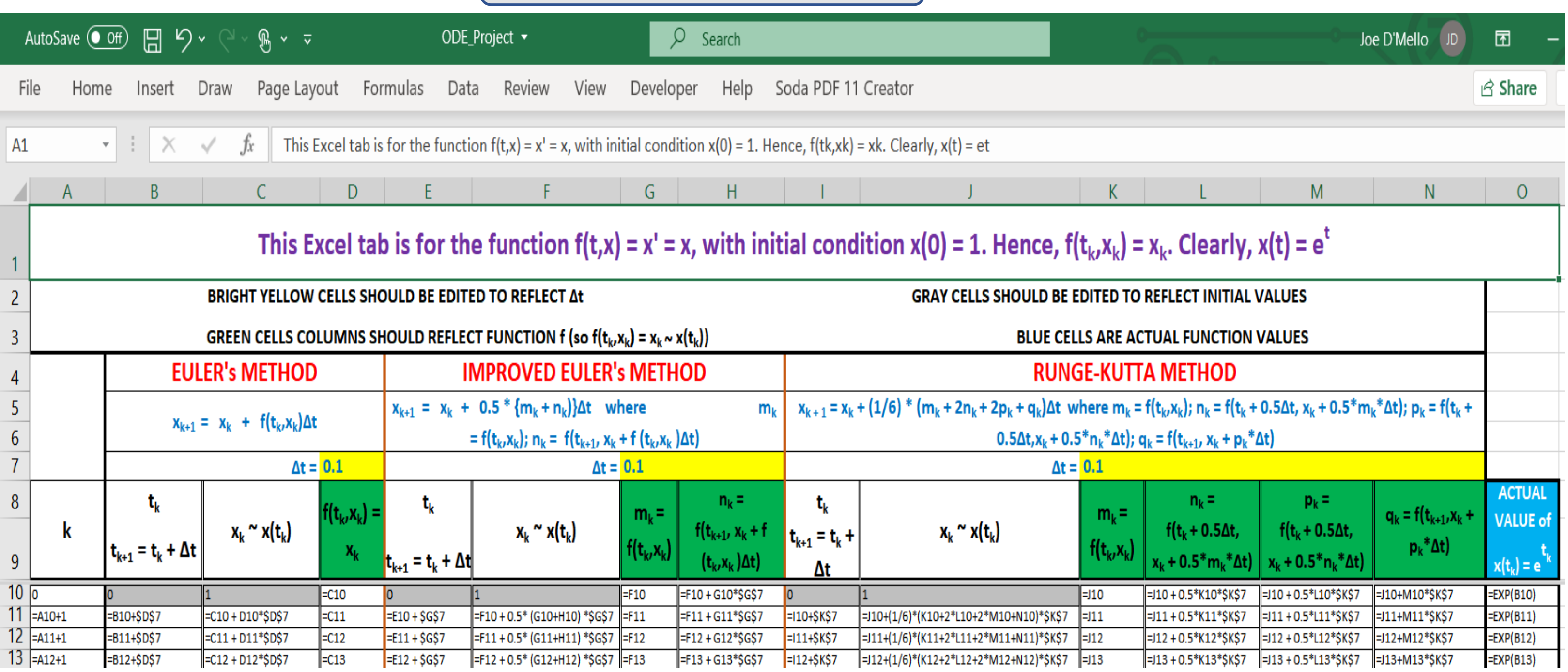
The next six columns (I, J, K, L, M, & N) are for the Runge-Kutta method, the two extra columns (compared to the four for Improved Euler's method) being for computing $p_k = f(t_k + \frac{\Delta t}{2}, x_k + \frac{n_k}{2}\Delta t)$ and $q_k = f(t_{k+1}, x_k + (\Delta t)p_k)$. Note that the formulas for calculating x_{k+1} from x_k for each of the three methods are shown in blue font (rows 5 and 6), and they employ the quantities $m_k, n_k,\ p_k, and\ q_k$. These formulas are coded into the cell computations (see Figure 2 and Figure 3 below) for calculating column x_k.

The last column (Column O) is the <u>actual</u> value of the function we are dealing with:

$\boldsymbol{x(t) = e^t}$ or $\boldsymbol{x(t) = \tan(t^2)}$

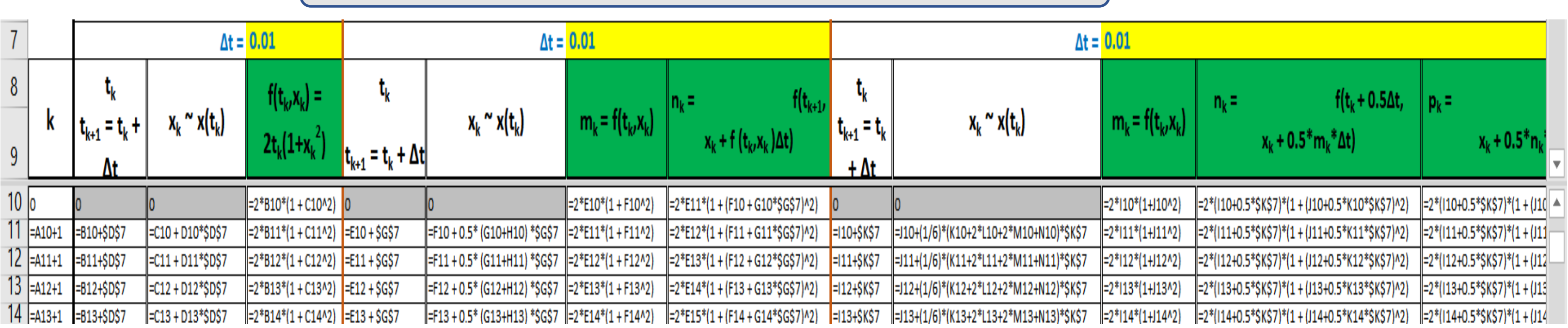
The recurrence formulas in the cells of the EXCEL spreadsheets essentially mirror those referred to above (and also specified in Exploration 7.5). I have included a screenshot of the formulas (for a few rows) in Figure 2 and Figure 3 below. (In Figure 3, not all columns are included because the entire view is too wide to fit on the page.)

FIGURE 2 (cell formulas for $x' = x$)



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8–9	k	tk, tk+1 = tk + Δt	xk ~ x(tk)	f(tk,xk) = xk	tk, tk+1 = tk + Δt	xk ~ x(tk)	mk = f(tk,xk)	nk = f(tk+1, xk + f (tk,xk)Δt)	tk, tk+1 = tk + Δt	xk ~ x(tk)	mk = f(tk,xk)	nk = f(tk + 0.5Δt, xk + 0.5*mk*Δt)	pk = f(tk + 0.5Δt, xk + 0.5*nk*Δt)	qk = f(tk+1,xk + pk*Δt)	ACTUAL VALUE of x(tk) = e^tk
10	0	0	1	`=C10`	0	1	`=F10`	`=F10 + G10*$G$7`	0	1	`=J10`	`=J10 + 0.5*K10*$K$7`	`=J10 + 0.5*L10*$K$7`	`=J10+M10*$K$7`	`=EXP(B10)`
11	`=A10+1`	`=B10+$D$7`	`=C10 + D10*$D$7`	`=C11`	`=E10 + $G$7`	`=F10 + 0.5* (G10+H10) *$G$7`	`=F11`	`=F11 + G11*$G$7`	`=I10+$K$7`	`=J10+(1/6)*(K10+2*L10+2*M10+N10)*$K$7`	`=J11`	`=J11 + 0.5*K11*$K$7`	`=J11 + 0.5*L11*$K$7`	`=J11+M11*$K$7`	`=EXP(B11)`
12	`=A11+1`	`=B11+$D$7`	`=C11 + D11*$D$7`	`=C12`	`=E11 + $G$7`	`=F11 + 0.5* (G11+H11) *$G$7`	`=F12`	`=F12 + G12*$G$7`	`=I11+$K$7`	`=J11+(1/6)*(K11+2*L11+2*M11+N11)*$K$7`	`=J12`	`=J12 + 0.5*K12*$K$7`	`=J12 + 0.5*L12*$K$7`	`=J12+M12*$K$7`	`=EXP(B12)`
13	`=A12+1`	`=B12+$D$7`	`=C12 + D12*$D$7`	`=C13`	`=E12 + $G$7`	`=F12 + 0.5* (G12+H12) *$G$7`	`=F13`	`=F13 + G13*$G$7`	`=I12+$K$7`	`=J12+(1/6)*(K12+2*L12+2*M12+N12)*$K$7`	`=J13`	`=J13 + 0.5*K13*$K$7`	`=J13 + 0.5*L13*$K$7`	`=J13+M13*$K$7`	`=EXP(B13)`

FIGURE 3 ($partial\ view$ of cell formulas for $x' = 2t(1 + x^2)$)

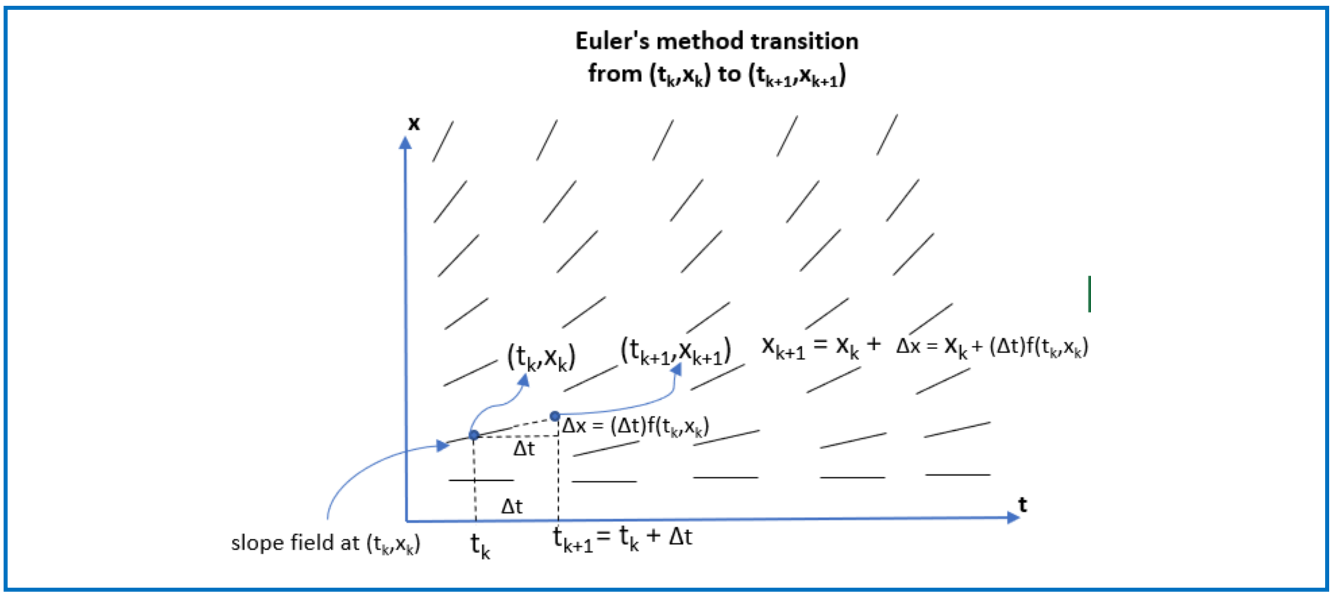


	k	tk, tk+1 = tk + Δt	xk ~ x(tk)	f(tk,xk) = 2tk(1+xk^2)	tk, tk+1 = tk + Δt	xk ~ x(tk)	mk = f(tk,xk)	nk = f(tk+1, xk + f (tk,xk)Δt)	tk, tk+1 = tk + Δt	xk ~ x(tk)	mk = f(tk,xk)	nk = f(tk + 0.5Δt, xk + 0.5*mk*Δt)	pk = … xk + 0.5*nk…
10	0	0	0	`=2*B10*(1 + C10^2)`	0	0	`=2*E10*(1 + F10^2)`	`=2*E11*(1 + (F10 + G10*$G$7)^2)`	0	0	`=2*I10*(1+J10^2)`	`=2*(I10+0.5*$K$7)*(1 + (J10+0.5*K10*$K$7)^2)`	`=2*(I10+0.5*$K$7)*(1 + (J10`
11	`=A10+1`	`=B10+$D$7`	`=C10 + D10*$D$7`	`=2*B11*(1 + C11^2)`	`=E10 + $G$7`	`=F10 + 0.5* (G10+H10) *$G$7`	`=2*E11*(1 + F11^2)`	`=2*E12*(1 + (F11 + G11*$G$7)^2)`	`=I10+$K$7`	`=J10+(1/6)*(K10+2*L10+2*M10+N10)*$K$7`	`=2*I11*(1+J11^2)`	`=2*(I11+0.5*$K$7)*(1 + (J11+0.5*K11*$K$7)^2)`	`=2*(I11+0.5*$K$7)*(1 + (J11`
12	`=A11+1`	`=B11+$D$7`	`=C11 + D11*$D$7`	`=2*B12*(1 + C12^2)`	`=E11 + $G$7`	`=F11 + 0.5* (G11+H11) *$G$7`	`=2*E12*(1 + F12^2)`	`=2*E13*(1 + (F12 + G12*$G$7)^2)`	`=I11+$K$7`	`=J11+(1/6)*(K11+2*L11+2*M11+N11)*$K$7`	`=2*I12*(1+J12^2)`	`=2*(I12+0.5*$K$7)*(1 + (J12+0.5*K12*$K$7)^2)`	`=2*(I12+0.5*$K$7)*(1 + (J12`
13	`=A12+1`	`=B12+$D$7`	`=C12 + D12*$D$7`	`=2*B13*(1 + C13^2)`	`=E12 + $G$7`	`=F12 + 0.5* (G12+H12) *$G$7`	`=2*E13*(1 + F13^2)`	`=2*E14*(1 + (F13 + G13*$G$7)^2)`	`=I12+$K$7`	`=J12+(1/6)*(K12+2*L12+2*M12+N12)*$K$7`	`=2*I13*(1+J13^2)`	`=2*(I13+0.5*$K$7)*(1 + (J13+0.5*K13*$K$7)^2)`	`=2*(I13+0.5*$K$7)*(1 + (J13`
14	`=A13+1`	`=B13+$D$7`	`=C13 + D13*$D$7`	`=2*B14*(1 + C14^2)`	`=E13 + $G$7`	`=F13 + 0.5* (G13+H13) *$G$7`	`=2*E14*(1 + F14^2)`	`=2*E15*(1 + (F14 + G14*$G$7)^2)`	`=I13+$K$7`	`=J13+(1/6)*(K13+2*L13+2*M13+N13)*$K$7`	`=2*I14*(1+J14^2)`	`=2*(I14+0.5*$K$7)*(1 + (J14+0.5*K14*$K$7)^2)`	`=2*(I14+0.5*$K$7)*(1 + (J14`

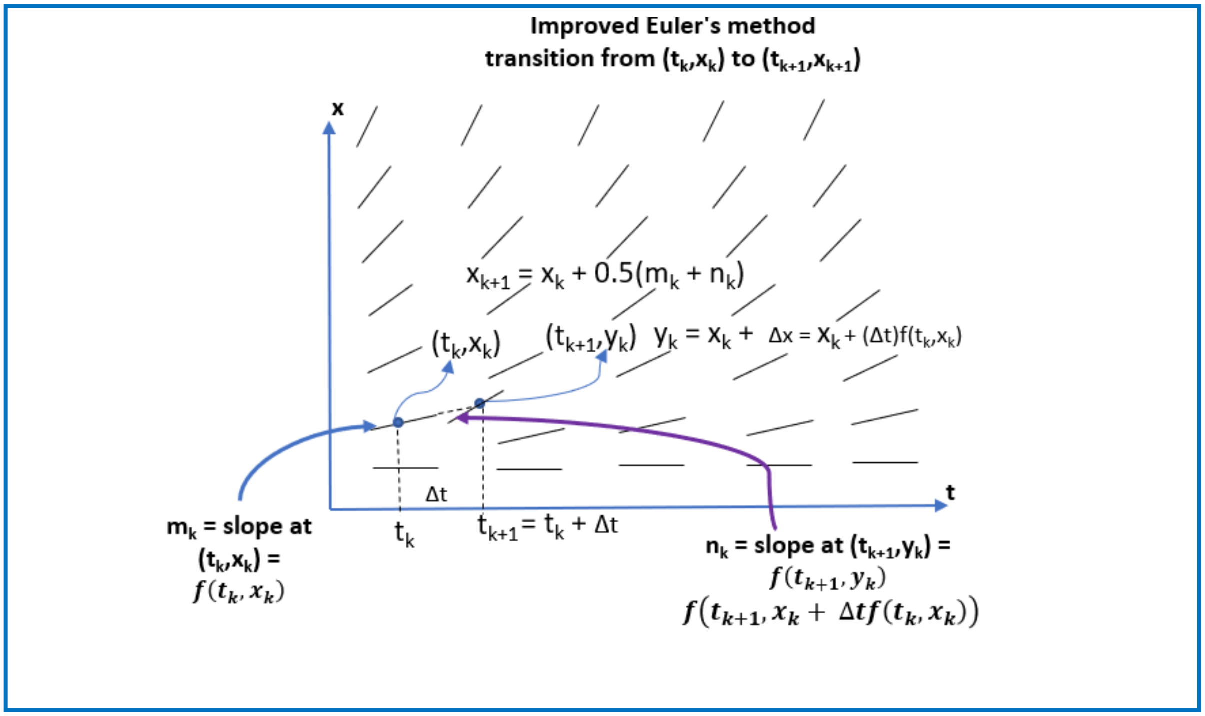
Section 2: The (t_k, x_k) *to* (t_{k+1}, x_{k+1}) *transition*

In this section, **we address the first of the four exercises** in *Exploration 7.5* by drawing pictures in the $tx - plane$ that illustrate and explain the process of moving from $(t_k, x_k)\ to\ (t_{k+1}, x_{k+1})$.

Euler's method: Referring to the picture below, in the case of Euler's method, we calculate the slope at (t_k, x_k), which is $f(t_k, x_k)$, then we draw the line through (t_k, x_k) whose slope is $f(t_k, x_k)$ until it intersects the vertical line $t = t_{k+1} = t_k + \Delta t$. The point of intersection of these two lines is the point (t_{k+1}, x_{k+1}), where obviously $x_{k+1} = x_k + \Delta x = x_k + \Delta t f(t_k, x_k)$, because $\frac{\Delta x}{\Delta t} = f(t_k, x_k)$, and therefore $\Delta x = (\Delta t) f(t_k, x_k)$.



Improved Euler's method: Referring to the picture below, in this case the increment Δx is calculated not by multiplying Δt by the slope at (t_k, x_k) (which is the calculation used in Euler's method) but by the average of the slopes at $(t_k, x_k)\ and\ \left(t_{k+1}, x_k + \Delta t f(t_k, x_k)\right)$. Since $x' = f(t, x)$, the slope at (t_k, x_k) is simply $m_k = f(t_k, x_k)$ and the slope at $\left(t_{k+1}, x_k + \Delta t f(t_k, x_k)\right)$ is simply $n_k = f\left(t_{k+1}, x_k + \Delta t f(t_k, x_k)\right)$. Thus $\Delta x = x_k + \Delta t(\frac{m_k + n_k}{2})$, so $x_{k+1} = x_k + \Delta x = x_k + \Delta t(\frac{m_k + n_k}{2})$. Using the average of the two slopes (for the slope field shown in the picture) enables us to move x_{k+1} a little higher, and therefore a little closer to $x(t_{k+1})$, thereby providing a better approximation of $x(t_{k+1})$ than would be provided by Euler's method.



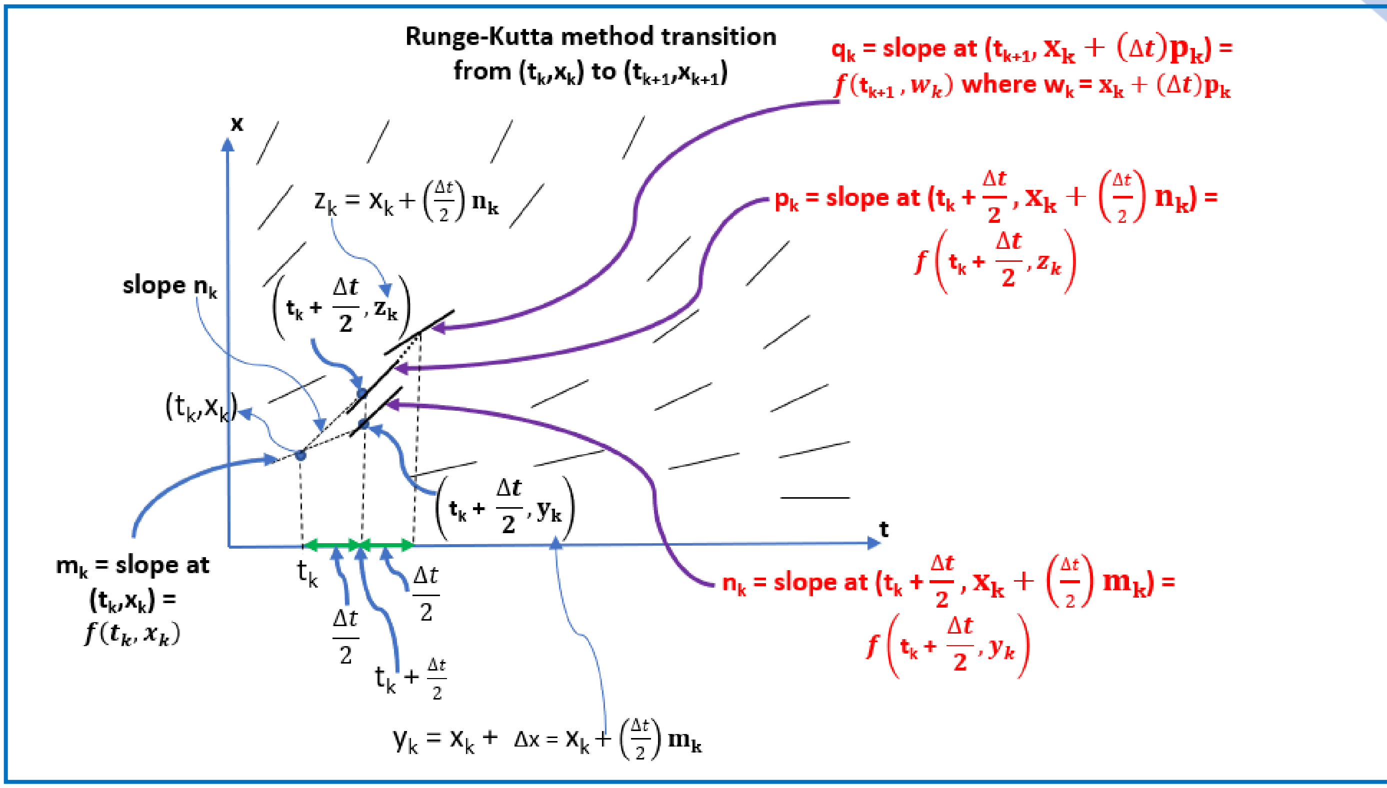
<u>Runge-Kutta method</u>: Referring to the picture below, in the Runge-Kutta method, we calculate four slopes $m_k, n_{k,}\, p_k, and\ q_k$. The slope $\boldsymbol{m_k} = f(t_k, x_k)$ is the usual slope calculated in the two methods above (Euler's and Improved Euler's). We then move a distance $\frac{\Delta t}{2}$ from x_k and calculate the slope $n_{k,}$ at the point $(t + \frac{\Delta t}{2}, x_k + \frac{\Delta t}{2}\, m_k)$, which of course is simply f$(t + \frac{\Delta t}{2}, x_k + \frac{\Delta t}{2}\, m_k)$. <u>It is easier to visualize this geometrically</u>: starting from (t_k, x_k) we move along the straight line through (t_k, x_k) **having slope $\boldsymbol{m_k}$** to a point that has a <u>horizontal</u> positive displacement (in the t direction) of $\frac{\Delta t}{2}$ from (t_k, x_k) and then set $\boldsymbol{n_k}$ to be the slope at that point.

We then repeat this traversal , the only difference being that this time again starting from (t_k, x_k) we move along the line through (t_k, x_k) **having slope $\boldsymbol{n_k}$**, to a point that has a <u>horizontal</u> positive displacement of $\frac{\Delta t}{2}$ from (t_k, x_k) and then set $\boldsymbol{p_k}$ to be the slope at that point.

We then repeat this same traversal starting from (t_k, x_k) for a third time, the difference here being that we move along the line through (t_k, x_k) **having slope $\boldsymbol{p_k}$**, to a point that has a <u>horizontal</u> positive displacement of $\boldsymbol{\Delta t}$ $(\boldsymbol{not}\ \frac{\boldsymbol{\Delta t}}{\boldsymbol{2}})$ from x_k and then set $\boldsymbol{q_k}$ to be the slope at that point.

To calculate x_{k+1} from x_k we use a "composite" slope that is a weighted average of the four slopes $m_k, n_{k,}\, p_k, and\ q_k$, ascribing to $n_k\ and\ p_k$ twice the weight we ascribe to $m_k\ and\ q_k$. That is, our composite slope is

$(\frac{m_k + 2n_k + 2p_k + q_k}{6})$ and therefore $\boldsymbol{x_{k+1}}$ **=** $\boldsymbol{x_k}$ **+** $(\frac{\boldsymbol{m_k + 2n_k + 2p_k + q_k}}{\boldsymbol{6}})\boldsymbol{\Delta t}$**.** Note that using the weighted average of the four slopes results in more upward displacement of x_{k+1} (than if we had used only one or two slopes) assuming, of course, the slope field depicted in the picture. Hence, we would expect Runge-Kutta would in general provide a better approximation of $x(t_{k+1})$ than would be provided by Euler's method or the Improved Euler's method.



This completes Exercise 1 of Exploration 7.5.

Section 3: Results of EXCEL calculations

In this section, **we address the second and third of the four exercises** in *Exploration 7.5* by showing the results of the EXCEL computations and analyzing the error and error changes, as Δt changes. Note that the cells shaded yellow in the spreadsheet are used to set the value of Δt. Exploration 7.5 requires that Δt take on three values: 0.1, 0.05, and 0.01.

Figure 1 above showed the result of the computations for the $x' = x$ case for $\Delta t = 0.1$ (Δt is set in the yellow cells in Row 7). When $\Delta t = 0.1$, t_k becomes 1 when $k = 10$ (see Row 20, $k = 10$ in Figure 1). The three approximations for $x(1) = e^1 = e$ are shown in the orange-shaded cells (in Row 20, $k = 10$), and the actual (exact) value is in the blue-shaded cell. (The orange shaded cells in Row 30, $k = 20$, approximate e^2, hence are not relevant to the exercise, so are not used below.) Only a few decimal places are used for the display, so it is not too wide to fit on the page, but higher precision is used when calculating the error and the error changes based on changing step size Δt.

Figure 4 below shows the spreadsheet results (Row 30, $k = 20$) for $x' = x$ and $\Delta t = 0.05$, and Figure 5 below shows the results (Row 110, $k = 100$) for $x' = x$ and $\Delta t = 0.01$.

Figure 6 below shows the spreadsheet results (Row 20, $k = 10$) for $x' = 2t(1 + x^2)$ and $\Delta t = 0.1$, Figure 7 below shows the results (Row 30, $k = 20$) for $x' = 2t(1 + x^2)$ and $\Delta t = 0.05$, and Figure 8 shows the results (Row 110, $k = 100$) for $x' = 2t(1 + x^2)$ and $\Delta t = 0.01$.

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FIGURE 4 ($x' = x$ and $\Delta t = 0.05$)

		$\Delta t =$ 0.05				$\Delta t =$ 0.05				$\Delta t =$ 0.05					
8 / 9	k	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$	$f(t_k,x_k) = x_k$	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$	$m_k = f(t_k,x_k)$	$n_k = f(t_{k+1}, x_k + f(t_k,x_k)\Delta t)$	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$	$m_k = f(t_k,x_k)$	$n_k = f(t_k + 0.5\Delta t, x_k + 0.5*m_k*\Delta t)$	$p_k = f(t_k + 0.5\Delta t, x_k + 0.5*n_k*\Delta t)$	$q_k = f(t_{k+1},x_k + p_k*\Delta t)$	ACTUAL VALUE of $x(t_k) = e^{t_k}$
27	17	0.8500000	2.2920183	2.2920183	0.8500000	2.3388488	2.3388488	2.4557913	0.8500000	2.3396468	2.3396468	2.3981379	2.3996002	2.4596268	2.3396469
28	18	0.9000000	2.4066192	2.4066192	0.9000000	2.4587148	2.4587148	2.5816506	0.9000000	2.4596030	2.4596030	2.5210931	2.5226303	2.5857345	2.4596031
29	19	0.9500000	2.5269502	2.5269502	0.9500000	2.5847240	2.5847240	2.7139601	0.9500000	2.5857095	2.5857095	2.6503523	2.6519683	2.7183080	2.5857097
30	20	1.0000000	**2.6532977**	2.6532977	1.0000000	**2.7171911**	2.7171911	2.8530506	1.0000000	**2.7182817**	2.7182817	2.7862387	2.7879377	2.8576786	2.7182818
31	21	1.0500000	2.7859626	2.7859626	1.0500000	2.8564471	2.8564471	2.9992695	1.0500000	2.8576510	2.8576510	2.9290922	2.9308783	3.0041949	2.8576511
32	22	1.1000000	2.9252607	2.9252607	1.1000000	3.0028400	3.0028400	3.1529820	1.1000000	3.0041659	3.0041659	3.0792700	3.0811476	3.1582232	3.0041660
33	23	1.1500000	3.0715238	3.0715238	1.1500000	3.1567356	3.1567356	3.3145723	1.1500000	3.1581927	3.1581927	3.2371475	3.2391214	3.3201488	3.1581929
34	24	1.2000000	3.2250999	3.2250999	1.2000000	3.3185183	3.3185183	3.4844442	1.2000000	3.3201167	3.3201167	3.4031196	3.4051947	3.4903765	3.3201169
35	25	1.2500000	3.3863549	3.3863549	1.2500000	3.4885923	3.4885923	3.6630219	1.2500000	3.4903427	3.4903427	3.5776013	3.5797828	3.6693319	3.4903430

FIGURE 5 ($x' = x$ and $\Delta t = 0.01$)

7		$\Delta t =$ 0.01				$\Delta t =$ 0.01				$\Delta t =$ 0.01					
8 / 9	k	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$	$f(t_k,x_k) = x_k$	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$	$m_k = f(t_k,x_k)$	$n_k = f(t_{k+1}, x_k + f(t_k,x_k)\Delta t)$	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$	$m_k = f(t_k,x_k)$	$n_k = f(t_k + 0.5\Delta t, x_k + 0.5*m_k*\Delta t)$	$p_k = f(t_k + 0.5\Delta t, x_k + 0.5*n_k*\Delta t)$	$q_k = f(t_{k+1},x_k + p_k*\Delta t)$	ACTUAL VALUE of $x(t_k) = e^{t_k}$
107	97	0.9700000	2.6252657	2.6252657	0.9700000	2.6379021	2.6379021	2.6642812	0.9700000	2.6379445	2.6379445	2.6511342	2.6512001	2.6644565	2.6379445
108	98	0.9800000	2.6515183	2.6515183	0.9800000	2.6644130	2.6644130	2.6910572	0.9800000	2.6644562	2.6644562	2.6777785	2.6778451	2.6912347	2.6644562
109	99	0.9900000	2.6780335	2.6780335	0.9900000	2.6911904	2.6911904	2.7181023	0.9900000	2.6912345	2.6912345	2.7046906	2.7047579	2.7182821	2.6912345
110	100	1.0000000	**2.7048138**	2.7048138	1.0000000	**2.7182369**	2.7182369	2.7454192	1.0000000	**2.7182818**	2.7182818	2.7318732	2.7319412	2.7456012	2.7182818
111	101	1.0100000	2.7318620	2.7318620	1.0100000	2.7455551	2.7455551	2.7730107	1.0100000	2.7456010	2.7456010	2.7593290	2.7593977	2.7731950	2.7456010
112	102	1.0200000	2.7591806	2.7591806	1.0200000	2.7731480	2.7731480	2.8008795	1.0200000	2.7731948	2.7731948	2.7870607	2.7871301	2.8010661	2.7731948
113	103	1.0300000	2.7867724	2.7867724	1.0300000	2.8010181	2.8010181	2.8290283	1.0300000	2.8010658	2.8010658	2.8150712	2.8151412	2.8292172	2.8010658
114	104	1.0400000	2.8146401	2.8146401	1.0400000	2.8291683	2.8291683	2.8574600	1.0400000	2.8292170	2.8292170	2.8433631	2.8434338	2.8576514	2.8292170

FIGURE 6 ($x' = 2t(1 + x^2)$ and $\Delta t = 0.1$)

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R5

This Excel tab is for the function $f(t,x) = x' = 2t(1 + x^2)$, with initial condition $x(0) = 0$. Hence, $f(t_k,x_k) = 2t_k(1 + x_k^2)$. Integrating $dx/(1 + x^2)$ and $(2t)dt$ yields $\arctan x = t^2$, or $x(t) = \tan(t^2)$

BRIGHT YELLOW CELLS SHOULD BE EDITED TO REFLECT Δt

GRAY CELLS SHOULD BE EDITED TO REFLECT INITIAL VALUES

GREEN CELLS COLUMNS SHOULD REFLECT FUNCTION f (so $f(t_k,x_k) = 2t_k(1 + x_k^2)$)

EULER's METHOD: $x_{k+1} = x_k + f(t_k,x_k)\Delta t$

IMPROVED EULER's METHOD: $x_{k+1} = x_k + 0.5 * \{m_k + n_k\}\Delta t$ where $m_k = f(t_k,x_k)$; $n_k = f(t_{k+1}, x_k + f(t_k,x_k)\Delta t)$

RUNGE-KUTTA METHOD: $x_{k+1} = x_k + (1/6) * (m_k + 2n_k + 2p_k + q_k)\Delta t$ where $m_k = f(t_k,x_k)$; $n_k = f(t_k + 0.5\Delta t, x_k + 0.5*m_k*\Delta t)$; $p_k = f(t_k + 0.5\Delta t, x_k + 0.5*n_k*\Delta t)$; $q_k = f(t_{k+1}, x_k + p_k*\Delta t)$

7		$\Delta t =$ 0.1				$\Delta t =$ 0.1				$\Delta t =$ 0.1					
8 / 9	k	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$	$f(t_k,x_k) = 2t_k(1+x_k^2)$	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$	$m_k = f(t_k,x_k)$	$n_k = f(t_{k+1}, x_k + f(t_k,x_k)\Delta t)$	t_k ($t_{k+1} = t_k + \Delta t$)	$x_k \sim x(t_k)$	$m_k = f(t_k,x_k)$	$n_k = f(t_k + 0.5\Delta t, x_k + 0.5*m_k*\Delta t)$	$p_k = f(t_k + 0.5\Delta t, x_k + 0.5*n_k*\Delta t)$	$q_k = f(t_{k+1}, x_k + p_k*\Delta t)$	ACTUAL VALUE of $x(t_k) = \tan(t_k^2)$
10	0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.2000000	0.0000000	0.0000000	0.0000000	0.1000000	0.1000025	0.2000200	0.0000000
11	1	0.1000000	0.0000000	0.2000000	0.1000000	0.0100000	0.2000200	0.4003600	0.1000000	0.0100004	0.2000200	0.3001200	0.3001876	0.4006406	0.0100003
12	2	0.2000000	0.0200000	0.4001600	0.2000000	0.0400190	0.4006406	0.6038480	0.2000000	0.0400217	0.4006407	0.5018032	0.5021198	0.6048853	0.0400213
13	3	0.3000000	0.0600160	0.6021612	0.3000000	0.0902434	0.6048863	0.8181761	0.3000000	0.0902445	0.6048864	0.7101623	0.7110696	0.8208274	0.0902438
14	4	0.4000000	0.1202321	0.8115646	0.4000000	0.1613966	0.8208391	1.0592827	0.4000000	0.1613808	0.8208350	0.9368774	0.9390218	1.0651694	0.1613795
15	5	0.5000000	0.2013886	1.0405574	0.5000000	0.2554026	1.0652305	1.3571883	0.5000000	0.2553442	1.0652007	1.2047602	1.2095514	1.3699214	0.2553419
16	6	0.6000000	0.3054443	1.3119555	0.6000000	0.3765236	1.3701240	1.7692069	0.6000000	0.3764066	1.3700184	1.5573256	1.5682731	1.7980738	0.3764029
17	7	0.7000000	0.4366399	1.6669161	0.7000000	0.5334901	1.7984564	2.4141567	0.7000000	0.5333948	1.7983140	2.0827740	2.1096734	2.4865200	0.5333881
18	8	0.8000000	0.6033315	2.1824142	0.8000000	0.7441208	2.4859452	3.5738706	0.8000000	0.7445569	2.4869841	2.9834964	3.0578860	3.7858064	0.7445438
19	9	0.9000000	0.8215729	3.0149676	0.9000000	1.0471116	3.7735968	6.0582367	0.9000000	1.0504829	3.7863257	4.8204937	5.0691844	6.8509977	1.0504551
20	10	1.0000000	**1.1230697**	4.5225709	1.0000000	**1.5387032**	6.7352153	12.9666646	1.0000000	**1.5574275**	6.8511610	9.6808849	10.8519747	17.5636272	1.5574077
21	11	1.1000000	1.5753267	7.6596395	1.1000000	2.5237972	16.2130155	43.6364255	1.1000000	2.6487693	17.6351536	30.9686281	42.8179361	117.6784863	2.6503246
22	12	1.2000000	2.3412907	15.5559411	1.2000000	5.5162693	75.4301444	446.0167182	1.2000000	7.3635488	132.5324422	491.8122050	2555.1707016	179678.7715393	7.6018261

FIGURE 7 ($x' = 2t(1 + x^2)$ and $\Delta t = 0.05$)

7		Δt = 0.05			Δt = 0.05				Δt = 0.05						
8 / 9	k	t_k $t_{k+1} = t_k + \Delta t$	$x_k \sim x(t_k)$	$f(t_k,x_k) = 2t_k(1+x_k^2)$	t_k $t_{k+1} = t_k + \Delta t$	$x_k \sim x(t_k)$	$m_k = f(t_k,x_k)$	$n_k = f(t_{k+1}, x_k + f(t_k,x_k)\Delta t)$	t_k $t_{k+1} = t_k + \Delta t$	$x_k \sim x(t_k)$	$m_k = f(t_k,x_k)$	$n_k = f(t_k + 0.5\Delta t, x_k + 0.5*m_k*\Delta t)$	$p_k = f(t_k + 0.5\Delta t, x_k + 0.5*n_k*\Delta t)$	$q_k = f(t_{k+1}, x_k + p_k*\Delta t)$	ACTUAL VALUE of $x(t_k) = \tan(t_k^2)$
27	17	0.8500000	0.7837949	2.7443685	0.8500000	0.8813028	3.0203810	3.7182393	0.8500000	0.8815023	3.0209787	3.3528253	3.3807346	3.7865380	0.8815007
28	18	0.9000000	0.9210133	3.3268780	0.9000000	1.0497683	3.7836245	4.8164925	0.9000000	1.0504576	3.7862300	4.2758764	4.3280184	4.9493679	1.0504551
29	19	0.9500000	1.0873572	4.1464568	0.9500000	1.2647713	4.9393281	6.5707016	0.9500000	1.2666525	4.9483762	5.7195570	5.8248233	6.8540652	1.2666487
30	20	1.0000000	1.2946801	5.3523929	1.0000000	1.5525220	6.8206492	9.6296519	1.0000000	1.5574125	6.8510674	8.1761509	8.4131930	10.3168158	1.5574077
31	21	1.0500000	1.5622997	7.2256387	1.0500000	1.9637795	10.1985032	15.6622729	1.0500000	1.9769673	10.3076390	12.8864494	13.5148824	17.6811308	1.9769704
32	22	1.1000000	1.9235816	10.3403659	1.1000000	2.6102989	17.1900533	29.9909034	1.1000000	2.6502292	17.6521726	23.7545538	25.9293143	38.1257217	2.6503246
33	23	1.1500000	2.4405999	16.0000145	1.1500000	3.7898229	35.3343418	76.5003270	1.1500000	3.9431095	38.0606580	58.6498025	71.1136233	137.3564658	3.9443382
34	24	1.2000000	3.2406007	27.6035823	1.2000000	6.5856896	106.4911374	357.1349263	1.2000000	7.5676426	139.8461142	302.3485451	563.0262979	3192.1098078	7.6018261

FIGURE 8 ($x' = 2t(1 + x^2)$ and $\Delta t = 0.01$)

7		Δt = 0.01			Δt = 0.01				Δt = 0.01						
8 / 9	k	t_k $t_{k+1} = t_k + \Delta t$	$x_k \sim x(t_k)$	$f(t_k,x_k) = 2t_k(1+x_k^2)$	t_k $t_{k+1} = t_k + \Delta t$	$x_k \sim x(t_k)$	$m_k = f(t_k,x_k)$	$n_k = f(t_{k+1}, x_k + f(t_k,x_k)\Delta t)$	t_k $t_{k+1} = t_k + \Delta t$	$x_k \sim x(t_k)$	$m_k = f(t_k,x_k)$	$n_k = f(t_k + 0.5\Delta t, x_k + 0.5*m_k*\Delta t)$	$p_k = f(t_k + 0.5\Delta t, x_k + 0.5*n_k*\Delta t)$	$q_k = f(t_{k+1}, x_k + p_k*\Delta t)$	ACTUAL VALUE of $x(t_k) = \tan(t_k^2)$
108	98	0.9800000	1.3757346	5.6695855	0.9800000	1.4294481	5.9649108	6.3704727	0.9800000	1.4295743	5.9656179	6.1658149	6.1715726	6.3834128	1.4295743
109	99	0.9900000	1.4324304	6.0426768	0.9900000	1.4911250	6.3824385	6.8357352	0.9900000	1.4912806	6.3833574	6.6070594	6.6138427	6.8511082	1.4912806
110	100	1.0000000	1.4928572	6.4572453	1.0000000	1.5572159	6.8498426	7.3587529	1.0000000	1.5574077	6.8510377	7.1021157	7.1101514	7.3771256	1.5574077
111	101	1.0100000	1.5574297	6.9196861	1.0100000	1.6282589	7.3754783	7.9495749	1.0100000	1.6284956	7.3770356	7.6601909	7.6697676	7.9716754	1.6284956
112	102	1.0200000	1.6266265	7.4376642	1.0200000	1.7048841	7.9695249	8.6205305	1.0200000	1.7051766	7.9715597	8.2925475	8.3040355	8.6473031	1.7051766

We can extract the relevant numbers from the orange and blue cells in the preceding Figures 1, 4, 5, 6, 7, & 8 (using EXCEL's increased precision) into the following summary table (Table 1 below) for the two differential equations: $x' = x$ and $x' = 2t(1 + x^2)$:

TABLE 1

x' = x	"Actual" (used) approximation of e	Δt = 0.1		Δt = 0.05		Δt = 0.01		Error Changes	
e = x(1)	2.71828182845235	Approximation	Error = $\rho_{0.1}$	Approximation	Error = $\rho_{0.05}$	Approximation	Error = $\rho_{0.01}$	$\rho_{0.1}/\rho_{0.05}$	$\rho_{0.5}/\rho_{0.01}$
Approximation of e (Euler's method)		2.59374246010000	-0.12453936835235	2.65329770514442	-0.06498412330793	2.70481382942153	-0.01346799903082	1.91645839	4.825076328
Approximation of e (Improved Euler's method)		2.71408084660822	-0.00420098184413	2.71719105435488	-0.00109077409747	2.71823686255996	-0.00004496589239	3.851376608	24.25781052
Approximation of e (Runge-Kutta)		2.71827974413517	-0.00000208431718	2.71828169265634	-0.00000013579601	2.71828182823440	-0.00000000021795	15.34888329	623.0689973

$x'=2t(1 + x^2)$	"Actual" (used) approximation of tan 1	Δt = 0.1		Δt = 0.05		Δt = 0.01		Error Changes	
tan 1 = x(1)	1.557407724654	Approximation	Error = $\rho_{0.1}$	Approximation	Error = $\rho_{0.05}$	Approximation	Error = $\rho_{0.01}$	$\rho_{0.1}/\rho_{0.05}$	$\rho_{0.5}/\rho_{0.01}$
Approximation of e (Euler's method)		1.123069650930	-0.434338073724	1.294680055958	-0.262727668696	1.492857209696	-0.064550514958	1.653187408	4.07010957
Approximation of e (Improved Euler's method)		1.538703241499	-0.018704483155	1.552522013674	-0.004885710980	1.557215875961	-0.000191848693	3.828405576	25.46648043
Approximation of e (Runge-Kutta)		1.557427530206	0.000019805552	1.557412497612	0.000004772958	1.557407739939	0.000000015285	4.149534345	312.2640723

We have at this point completed **Exercise 1** (Section 2), **Exercises 2(a), (b), (c), (d), (e), (f), (g), & (h)**, and **Exercise 3** in Exploration 7.5. Parts 2(g) & (h) are addressed in Table 1 above. The next section (Section 4) completes the fourth exercise.

Section 4: Summary and concluding discussion

In this section, **we address the last (fourth) of the four exercises** in *Exploration 7.5.* In Section 2 above (***The*** $\boldsymbol{(t_k, x_k)\ to\ (t_{k+1}, x_{k+1})\ transition}$), we explained and visualized the transition from $(t_k, x_k)\ to\ (t_{k+1}, x_{k+1})$, and we also provided an intuitive rationale for why we would generally expect the Improved Euler's method to provide a better approximation (of $x(t_{k+1})$) than Euler's method. Similarly, we provided an intuitive rationale for why the Runge-Kutta method would generally produce a better approximation of $x(t_{k+1})$ than the Improved Euler's method (and hence better than the one produced by Euler's method as well). Table 1 above (Section 3) bears out this intuition, clearly showing this to be the case for both $x' = x\ and\ x' = 2t(1 + x^2)$, and for all three values of Δt (0.1, 0.05, and 0.01), the This is obvious by inspecting the three error columns in the tables (for each of the two cases $x' = x\ and\ x' = 2t(1 + x^2)$).

Table 1 (Section 3) above also shows how the error changes as a result of shortening the step size. Obviously, the error decreases as the step size decreases (by inspecting the three error values in each row, for both cases). However, we can make another interesting observation by inspecting the values $a = \frac{\rho_{0.1}}{\rho_{0.05}}$ and $b = \frac{\rho_{0.05}}{\rho_{0.01}}$ individually; in both cases ($x' = x$ and $x' = 2t(1 + x^2)$), not only does the error decrease as we progress from the Euler's method to the Improved Euler's method to the Runge-Kutta method, but the error changes increase, meaning that the rate of decrease of the error (with respect to Δt) is higher as we progress through these methods. This does a double whammy on the error, thus a double benefit to a better approximation. Another equally interesting observation is that the rate of the rate of change of the error (with respect to Δt) also increases (because $\frac{b}{a}$ increases as well). In short, the leaps in accuracy as we progress from the Euler's method to the Improved Euler's method to the Runge-Kutta method are very pronounced and dramatic (as we shrink Δt).

Lastly, we will discuss why the Runge-Kutta method is called a fourth-order method. This is because it uses (a weighted average of) four slopes: $m_k, n_k, p_k, and\ q_k$. The equal weights of $n_k\ and\ p_k$ are each twice the equal weights of $m_k\ and\ q_k$. Euler's method could be considered a first-order method (uses one slope), and Improved Euler's method could be considered a second order method (uses average of two slopes). There are also third-order Runge-Kutta methods which use a weighted average of three slopes.