Notes on

Lie Groups and Lie Algebras[1]

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July 1, 2017
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[1] Modul Math Ma MMRM: Lie Groups and Lie Algebras

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10 Representation theory

Definition. A *representation of a Lie group* G on a vector space V is a homomorphism $\rho : G \to GL(V)$. A *representation of a Lie algebra* $\mathfrak{g}$ on V is a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

Definition. Given two representations V, W, the *homomorphism space* between V and W is

$$\mathrm{Hom}_G(V,W) = \{T : V \to W \text{ linear} \mid \rho_W(g) \circ T = T \circ \rho_V(g) \forall g \in G\}.$$

Similar for $\mathfrak{g}$-representations.

Theorem. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

1. *Every G-representation ρ defines a $\mathfrak{g}$-representation ρ_*.*
2. *If G is simply connected, then ρ and ρ_* are in a 1:1 correspondence.*

11 Interwining operators and Schur's lemma

Let V, W be two representations of G or $\mathfrak{g}$. We call $\mathrm{Hom}_{G \text{ or } \mathfrak{g}}(V, W)$ the *space of intertwining operators.*

Lemma (Schur). *Let V be an irreducible (complex) representation of G or $\mathfrak{g}$. Then:*

1. $\mathrm{Hom}_{G\ or\, \mathfrak{g}}(V, V) = \mathbb{C} \cdot \mathrm{id}_V$
2. *If W is an irreducible representation not isomorphic to V, then* $\mathrm{Hom}_{G\ or\, \mathfrak{g}}(V, W) = \{0\}$.

Proof. Let V, W be irreducible and $\phi \in \mathrm{Hom}_{G \text{ or } \mathfrak{g}}(V, W)$. Then:

- $\ker\phi \subseteq V$ is a subrepresentation since for $v \in \ker\phi$ we have $\phi \circ \rho_V(g)v = \rho_W(g) \circ \phi(v) = 0$. Thus $\rho_V(g)v \in \ker\phi$.
- $\mathfrak{I}\phi \subseteq W$ is a subrepresentation since for $w = \phi(v) \in \mathfrak{I}\phi$ we have $\rho_W(g)w = \phi \circ \rho_W(g)v \in \mathfrak{I}\phi$.

By irreducibility, $\ker\phi = 0$ or V and $\mathfrak{I}\phi = 0$ or W. So either $\phi = 0$ or ϕ is isomorphism. This proves the second part. Now observe that by the argument above, every non-zero $\phi \in \mathrm{Hom}_G(V, V)$ is invertible. Let $\lambda \in \mathbb{C}$ be an Eigenvalue of ϕ. Then $\phi - \lambda \cdot \mathrm{id}_V \in \mathrm{Hom}_G(V, V)$ is not invertible (not injective), therefore zero, i.e. $\phi = \lambda \cdot \mathrm{id}_V$. This proves the first part. $\square$

Example. Consider $GL(n, \mathbb{C})$ with its tautological representation. It is irreducible: every $U \subseteq \mathbb{C}^n$ can be mapped to every other $U' \subseteq \mathbb{C}^n$ of the same dimension through a matrix in $GL(n, \mathbb{C})$ (by completing the bases of U, U' to bases of $\mathbb{C}^n$ and using a basis transform matrix). Thus, $Z(GL(n, \mathbb{C})) = \{\lambda \cdot \mathrm{id}_n \mid \lambda \in \mathbb{C}^\times\}$; similarly, $\mathfrak{z}(\mathfrak{gl}(n, \mathbb{C})) = \{\lambda \cdot \mathrm{id} \mid \lambda \in \mathbb{C}\}$.

Since $\mathbb{C}^n$ is also irreducible as a representation of $SL(n, \mathbb{C})$, $U(n)$, $SU(n)$, $SO(n, \mathbb{C})$ (for ex. take orthogonal bases of U, U', then a basis transform matrix A will be orthogonal), similar argument yields:

$Z(SL(n, \mathbb{C})) = \{\lambda \cdot \mathrm{id} \mid \lambda^n = 1\}$	$\mathfrak{z}(\mathfrak{sl}(n, \mathbb{C})) = 0$
$Z(SU(n, \mathbb{C})) = \{\lambda \cdot \mathrm{id} \mid \lambda^n = 1\}$	$\mathfrak{z}(\mathfrak{su}(n, \mathbb{C})) = 0$
$Z(U(n, \mathbb{C})) = \{\lambda \cdot \mathrm{id} \mid \lvert\lambda\rvert = 1\}$	$\mathfrak{z}(\mathfrak{u}(n, \mathbb{C})) = \{\lambda \cdot \mathrm{id} \mid \lambda \in i\mathbb{R}\}$
$Z(SO(n, \mathbb{C})) = \{\pm 1\}, \{1\}$ if n odd	$\mathfrak{z}(\mathfrak{so}(n, \mathbb{C})) = 0$
$Z(SO(n, \mathbb{R})) = \{\pm 1\}, \{1\}$ if n odd	$\mathfrak{z}(\mathfrak{so}(n, \mathbb{R})) = 0$

Corollary. *If V is completely reducible representation, $V = \bigoplus_i n_i V_i$, V_i pairwise non-isomorphic irreducible representations, then*

$$\mathrm{Hom}_G(V, V) = \left\{ \bigoplus_i A_i \otimes \mathrm{id}_{V_i} \mid A_i \in \mathrm{End}(\mathbb{C}^{n_i}) \right\}.$$

Corollary. *Of G or $\mathfrak{g}$ is abelian, then every non-zero irreducible representation is one-dimensional.*

Proof. Wlog G is abelian, then $\rho(G) \subseteq \mathrm{Hom}_G(V, V) = \{\lambda \mathrm{id} \mid \lambda \in \mathbb{C}^\times\}$. Thus every subspace of V is invariant. $\square$

12 Unitary representations

Let V be a representation of G. Suppose that we have a scalar product (positive definite Hermitian form) $\langle\cdot,\cdot\rangle$ on V, which is G-*invariant*: $\langle\rho(g)v,\rho(g)w\rangle = \langle v,w\rangle$ for all $g\in G$. Equivalently: $\rho(g)\in U(V)$ for all $g\in G$, where $U(V) := \{f\in GL(V) \mid f \text{ unitary}\}$. In this case, if $U\subseteq V$ is $\rho(G)$-invariant, then $U^\perp$ is also invariant: $u\in U, v\in U^\perp$, then $\langle\rho(g)v,u\rangle = \langle v,\rho(g^{-1})u\rangle = 0$ since $\rho(g^{-1})u\in U$. So, if V is finite dimensional (or V is a Hilbert-space), then $V = U\oplus U^\perp$. Hence every subrepresentation has an invariant complement. Thus, if $\rho: G\to U(V,\langle\cdot,\cdot\rangle)$ is a finite-dimensional unitary representation, then ρ is completely reducible.

Question: Given a representation V, can we unitarise it, i.e. find a G-invariant scalar product on V?

Example. Let G be a finite group, $\rho: G\to GL(V)$ a representation, then ρ is unitarisable. Pick any scalar product $\langle\cdot,\cdot\rangle$ on V, consider the new scalar product

$$\langle v,w\rangle' := \frac{1}{|G|}\sum_{g\in G}\langle\rho(g)v,\rho(g)w\rangle.$$

Then $\langle\rho(h)v,\rho(h)w\rangle' := \frac{1}{|G|}\sum_{g\in G}\langle\rho(gh)v,\rho(gh)w\rangle \forall h\in G$.

Corollary. *Finite-dimensional representations of finite groups are completely reducible.*

Now a generalisation of this idea: for a finite group G, we had a probability measure $\#/|G|$. If there is a right-invariant probability measure μ on an arbitrary Lie group G, $\rho: G\to GL(V)$ a representation, $\langle\cdot,\cdot\rangle$ a scalar product on V, then

$$\langle v,w\rangle' := \int_G \langle\rho(g)v,\rho(g)w\rangle\, d\mu(g)$$

is G-invariant.

Definition. Let G be a topological group. A *(right) Haar measure* on G is a Borel measure μ s.t. $(R_h)_*\mu = \mu$ for all $h\in G$, where $R_h: G\to G, g\mapsto gh$.

Example. The Lebesgue measure λ is a (right) Haar measure on $\mathbb{R}$, it induces a Lebesgue measure on $\mathbb{R}/\mathbb{Z}$, which is a Haar measure on S^1.

Theorem (Haar). *Let G be a locally compact topological group. Then there existsa (right) Haar measure on G. Moreover, it is unique up to a positive constant.* □

Theorem. *Let G be a (real) Lie group. Then:*

1. *G is orientable (in a right invariant way) as a manifold, i.e. there exists a nowhere vanishing top differential form $\omega\in\Omega^n(G)$ ($n = \dim G$).*
2. *If G is compact, then for a fixed choice of a (right-invariant) orientation, there is a unique right-invariant $\omega\in\Omega^n(G)$ s.t. $\int_G \omega = 1$*

3. *For compact* G*,* ω *is also left invariant. Moreover,* $\omega(g^{-1}) = (-1)^{\dim G}\omega(g)$*.*

Proof. $\mathfrak{g} = T_1 G$, $\Lambda^n \mathfrak{g}^*$ is of dimension one. Take a non-zero element $\widetilde{\omega} \in \Lambda^n \mathfrak{g}^*$. Define $\omega(g) := (R_{g^{-1}})_* \widetilde{\omega} \in \Lambda^n (T_g G)^*$. Using $\Lambda^n (T_g G)^* \cong (\Lambda^n T_g G)^*$ we define $(R_{g^{-1}})^* \widetilde{\omega}(x) = \widetilde{\omega}((R_{g^{-1}})_* x)$. Then ω is non-vanishing, right invariant, which proves the first statement. □

Corollary. *On every compact Lie group, there exits a unique bi-invariant probability measure.*

This also works for general topological groups but needs a lot of functional analysis.

Theorem. *Every finite-dimensional representation of a compact Lie group* G *is completely reducible.*

Observation. A concrete example of "Weyl unitary trick": $SU(n)$ *is compact, hence every f.d. representation is completely reducible.*

$$\begin{tikzcd}
\text{\textit{Representations of} } \mathfrak{su}(n) \arrow[r, leftrightarrow, "1{:}1"] \arrow[d, leftrightarrow, "1{:}1"'] & \text{\textit{Representations of} } \mathfrak{su}(n)\otimes\mathbb{C} = \mathfrak{sl}(n,\mathbb{C}) \arrow[d, leftrightarrow, "1{:}1"] \\
\text{\textit{Representations of} } SU(n) & \text{\textit{Representations of} } SL(n,\mathbb{C})
\end{tikzcd}$$

Remark: $SL(n,\mathbb{C})$ is a very important Lie group.

Corollary. All finite-dimensional representations of $SL(n,\mathbb{C})$ are completely reducible

13 Characters, orthogonality relations, representations of compact Lie groups

Definition. Let V be a representation: $\rho : G \to GL(V)$, $v \in V$, $\alpha \in V^*$. The function $\rho^{v,\alpha} : G \to \mathbb{C}, g \mapsto \alpha(\rho(g)\cdot v)$ is called a *matrix coefficient* of ρ.

E.g. $\dim V < \infty$, $v_1, \dots, v_n$ a basis in V, $v_1^*, \dots, v_n^*$ dual basis in V^*. Then the matrix coefficient $\rho^{i,j}(g) = v_j^*(\rho(g)\cdot v_i)$ can be written as a matrix $(\rho(g))_{i,j}$. We can identify $GL(V) = GL(n,\mathbb{C})$ using the bases $v_1, \dots, v_n$ in V, $w_1, \dots, w_n$ in W. Then the matrix coefficients $\rho_V^{i,j}(\cdot)$, $\rho_W^{\alpha,\beta}(\cdot) : G \to \mathbb{C}$ are orthogonal:

Theorem (orthogonality relations). *1. Let* G *be a compact group,* V, W *non-isomorphic f.d. irreducible representation of* G*. Choose bases:* $v_1, \dots, v_n$*,* $w_1, \dots, w_n$

14 Structure theory

"Ideal" goal: classify Lie algebras.

Definition. Let $\mathfrak{g}$ be a Lie algebra. An *Ideal* $\mathfrak{h} \lhd \mathfrak{g}$ is a subspace $\mathfrak{h}$ s.t. $[\mathfrak{g},\mathfrak{h}] \subseteq \mathfrak{g}$.

Remark: $\mathfrak{h} \lhd \mathfrak{g} \iff \mathfrak{h}$ is a subrepresentation of $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. Observation: $\mathfrak{h}_1, \mathfrak{h}_2 \lhd \mathfrak{g} \implies \mathfrak{h}_1 \cap \mathfrak{h}_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2 \lhd \mathfrak{g}$.

Lemma. (1) $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ *a homomorphism* $\implies \ker f \lhd g_1$.
(2) *If* $\mathfrak{h} \lhd \mathfrak{g}_1$*, then* $\exists$ *Lie algebra* $\mathfrak{g}_2 = \mathfrak{g}_1/\mathfrak{h}$ *with a surjective map* $q : \mathfrak{g}_1 \to \mathfrak{g}_2$ *s.t.* $\ker q = \mathfrak{h}$.

Proof. Exercise. First part: Let $x \in \ker f$, $y \in \mathfrak{g}$. Then $f([x,y]) = [f(x), f(y)] = 0$. Second part: Let $q(x) := x + \mathfrak{h}$. Then $\ker q = \mathfrak{h}$ and $\mathfrak{g}_2$ with $[x+\mathfrak{h}, y+\mathfrak{h}] := [x,y]+\mathfrak{h}$ is a Lie algebra. □

Some canonical ideals:

- $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,y] = 0 \forall y \in \mathfrak{g}\}$.
- $\mathfrak{g}^{(1)} := [\mathfrak{g},\mathfrak{g}] = \mathrm{Span}\{[x,y] \mid x,y \in \mathfrak{g}\}$.

Lemma. *The quotient* $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ *is abelian. Moreover,* $[g,g]$ *is universal:* $\forall f : \mathfrak{g} \to \mathfrak{a}$ *hom.,* $\mathfrak{a}$ *an abelian Lie algebra,* $[\mathfrak{g},\mathfrak{g}] \subseteq \ker f$.

Proof. In $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$: $[x+[\mathfrak{g},\mathfrak{g}], y+[\mathfrak{g},\mathfrak{g}]] = [\mathfrak{g},\mathfrak{g}] = 0$. Let $f : \mathfrak{g} \to \mathfrak{a}$ as above. Then $0 = [f(x), f(y)] = f([x,y]) \implies [x,y] \in \ker f \forall x,y \in \mathfrak{g} \implies [\mathfrak{g},\mathfrak{g}] \subseteq \ker f$. □

Example. Is $[\mathfrak{gl}(n,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C})] = \mathfrak{gl}(n,\mathbb{C})$? Answer is no: $\mathrm{tr}[x,y] = \mathrm{tr}(xy - yx) = \mathrm{tr}(xy) - \mathrm{tr}(xy) = 0$. Thus $\mathfrak{gl}(n,\mathbb{C})^{(1)} \subseteq \mathfrak{sl}(n,\mathbb{C})$. Do we get equality? Basis of $\mathfrak{gl}(n,\mathbb{C})$: $\{E_{i,j}\}$, where $E_{i,j} := (\delta_{ij})_{i,j=1,\dots,n}$. Basis of $\mathfrak{sl}(n,\mathbb{C})$: $\{E_{i,j} \mid i \neq j\} \cup \{E_{1,1} - E_{2,2}, \dots, E_{n-1,n-1} - E_{n,n}\}$. Multiplication: $E_{i,j} E_{kl} = \delta_{j,k} E_{il}$ since

$$\begin{aligned}
(E_{i,j} E_{k,l})_{m,n} &= \sum_{r=1}^{n} (E_{i,j})_{m,r} \cdot (E_{k,l})_{r,n} \\
&= \sum_{r=1}^{n} \delta_{i,m} \delta_{j,r} \delta_{k,r} \delta_{l,n} \\
&= \delta_{i,m} \delta_{l,n} \delta_{j,k}
\end{aligned}$$

So $[E_{i,j}, E_{j,i}] = E_{i,i} - E_{j,j}$ and $[E_{i,i} - E_{j,j}, E_{i,j}] = E_{i,j} \implies \mathfrak{sl}(n,\mathbb{C})^{(1)} = \mathfrak{sl}(n,\mathbb{C})$.

Example. Let

$$\mathfrak{n} := \left\{ x \in \mathfrak{gl}(n,\mathbb{C}) \mid x = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}.$$

Then for $x, y \in \mathfrak{n}$:

$$[x,y] = xy - yx = \begin{pmatrix} 0 & 0 & & * \\ & \ddots & \ddots & \\ 0 & & 0 & 0 \end{pmatrix}.$$

So $[[\dots[[x_1, x_2], x_3], \dots], x_n] = 0$.

14.1 Solvable and nilpotent Lie algebras

Notation: Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra $\implies [\mathfrak{h}, \mathfrak{g}] := \operatorname{Span}\{[y, x] \mid x \in \mathfrak{h}, y \in \mathfrak{g}\}$.

Definition. Let $\mathfrak{g}$ be a Lie algebra, then $g' := [g, g]$ is called the *derived algebra.* Furthermore:

- the *derived series* of $\mathfrak{g}$ is defined inductively as follows: $g^{(0)} := g,\ g^{(k+1)} := [g^{(k)}, g^{(k)}] = (g^{(k)})'$,
- the *lower central series* of $\mathfrak{g}$ is also defined inductively as follows: $g_{(0)} := g,\ g_{(k+1)} := [g, g_{(k)}]$,
- $\mathfrak{g}$ is *solvable* iff $g^{(n)} = 0$ for some n,
- $\mathfrak{g}$ is *nilpotent* iff $g_{(n)} = 0$ for some n.

Example. $\mathfrak{g}$ abelian $\implies$ $\mathfrak{g}$ is solvable and nilpotent.

Proposition. (1) $\mathfrak{g}$ *is solvable*

iff $\exists n \in \mathbb{N} : [...[[x_1, x_2], [x_3, x_4]]...] = 0,\ x_i \in \mathfrak{g}$ *(2^n terms, n bracket levels).*
iff $\exists$ *sequence* $\mathfrak{h}^0 = \mathfrak{g} \supset \mathfrak{h}^1 \supset \mathfrak{h}^2 \supset ... \supset \mathfrak{h}^n = \{0\}$ *s.t.* $\mathfrak{h}^{i+1} \trianglelefteq \mathfrak{h}^i$ *and* $\mathfrak{h}^i/\mathfrak{h}^{i+1}$ *is abelian.*

(2) $\mathfrak{g}$ *is nilpotent*

iff $\exists n \in \mathbb{N} : [[...[[x_1, x_2], x_3], ...], x_n] = 0,\ x_i \in \mathfrak{g}$ *(n terms).*
iff $\exists$ *sequence of ideals* $\mathfrak{h}^0 = \mathfrak{g} \supset \mathfrak{h}^1 \supset \mathfrak{h}^2 \supset ... \supset \mathfrak{h}^n = \{0\}$ *s.t.* $\mathfrak{h}^i \trianglelefteq \mathfrak{g}$*,* $i > 0$ *and* $[\mathfrak{g}, \mathfrak{h}^i] \subseteq \mathfrak{h}^{i+1}$.

Proof. First part:

- $g^{(n+1)} = \operatorname{Span}\{[[...[[x_1, x_2], x_3], ...], x_n]\} \implies$ first equivalence. Since $\mathfrak{g}$ is solvable, $\mathfrak{h}^k := g^{(k)} \subset \mathfrak{g}$, $\mathfrak{g}^{(k+1)} \subset \mathfrak{g}^{(k)}$, $\mathfrak{g}^{(n)} = 0$. Furthermore $\mathfrak{g}^{(k+1)} = (\mathfrak{g}^{(k)})' \triangleleft \mathfrak{g}^{(k)}$ ideal, $\mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$ abelian by the property of derived subalgebras.
- Let $\mathfrak{g} \supset \mathfrak{h}^0 \supset \mathfrak{h}^1 \supset ... \supset \mathfrak{h}^n = 0$. Since $\mathfrak{g}/\mathfrak{h}^1$ abelian, $\mathfrak{h}^1 \supset [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}'$. Induction: suppose $\mathfrak{h}^k/\mathfrak{h}^{k+1}$ abelian, $\mathfrak{h}^k \supset \mathfrak{g}^{(k)} \implies \mathfrak{h}^{k+1} = \ker q \supset [\mathfrak{h}^k, \mathfrak{h}^k] \supset [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{g}^{(k+1)}$. So $\mathfrak{h}^k \supset g^{(k)} \forall k \implies \mathfrak{g}^{(k)} = 0 \implies \mathfrak{g}$ solvable.

Second part:

- $\mathfrak{g}_{(k)} = \operatorname{Span}\{[[...[[x_1, x_2], x_3], ...], x_n] \mid x_i \in \mathfrak{g}\}$.
- $\mathfrak{g}$ nilpotent $\implies \mathfrak{h}^k = \mathfrak{g}_{(k)} \triangleleft \mathfrak{g}$ by definition, $[\mathfrak{h}^k, \mathfrak{g}] = \mathfrak{g}_{(k+1)} \subseteq \mathfrak{h}^{k+1}$, $\mathfrak{h}^n = 0$. On the other hand, if $\mathfrak{h}^k = \mathfrak{g} \supset \mathfrak{h}^1 \supset ... \supset \mathfrak{h}^n = \{0\}$ is a sequence of ideals as in the statement, use induction to show $\mathfrak{h}^k \supset \mathfrak{g}_{(k)} : \mathfrak{h}^1 \triangleleft \mathfrak{g}, [\mathfrak{h}^0, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{h}^1$ as above. We have $h^{k+1} \supset [g, \mathfrak{h}^k] \supset [\mathfrak{g}, \mathfrak{g}_{(k)}] = \mathfrak{g}_{(k+1)} \implies$ if $\mathfrak{h}^n = 0$, then $\mathfrak{g}_{(n)} = 0 \implies \mathfrak{g}$ nilpotent. □

Example. Let

$$\mathfrak{b} := \left\{ x \in \mathfrak{gl}(n, \mathbb{C}) \mid x = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}.$$

Then for $x, y \in \mathfrak{b}$:

$$[x,y] = xy - yx = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}.$$

So $\mathfrak{b}' \subset \mathfrak{n}, \mathfrak{b}^{(k)} \subseteq \mathfrak{n}_{(k-1)} \implies \mathfrak{b}$ is solvable.

Definition. Let V be a vector space over $\mathbb{K}$. A *flag* in V is a chain of subspaces $\mathcal{F} = \{0 = V^0 \subsetneq V^1 \subsetneq V^2 \subsetneq ... \subsetneq V^k = V\}$. Denote:

$$\mathfrak{b}(\mathcal{F}) := \{x \in \operatorname{End}(V) \mid xV^i \subseteq V^i \forall i\},\ \mathfrak{n}(\mathcal{F}) := \{x \in \operatorname{End}(V) \mid xV^i \subseteq V^{i-1} \forall i > 0\}.$$

These are obviously Lie algebras.

Example. Let $V := \mathbb{K}^n$, $V^i := \operatorname{Span}\{e_1, ..., e_i\}$. Then we recover the Lie algebras $\mathfrak{b}$, $\mathfrak{n}$ from the two previous examples.

Lemma. $\mathfrak{n}(\mathcal{F})$ *is nilpotent.*

Proof. Let $\mathfrak{n}^l(\mathcal{F}) := \{x \in \operatorname{End}(V) \mid xV^i \subseteq V^{i-l} \forall i > 0\}$. Then if $x \in \mathfrak{n}^l, y \in \mathfrak{n}^{l'} \implies x \circ y \in \mathfrak{n}^{l+l'} \implies [x,y] \in \mathfrak{n}^{l+l'}$. We have $\mathfrak{n}(\mathcal{F})_{(1)} = \mathfrak{n}(\mathcal{F})' \subseteq \mathfrak{n}^2$. By induction: $\mathfrak{n}(\mathcal{F})_{(l)} \subset \mathfrak{n}^{l+1}$. Since $\mathfrak{n}^n = 0$, $\mathfrak{n}(\mathcal{F})_{(n)} = 0$ □

Theorem. (1) $\mathfrak{g}$ *is solvable resp. nilpotent iff* $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ *is solvable resp. nilpotent.*
(2) $\mathfrak{g}$ *solvable resp. nilpotent* $\implies$ *any subalgebra/quotient of* $\mathfrak{g}$ *is solvable/nilpotent.*
(3) $\mathfrak{g}$ *is nipotent* $\implies$ $\mathfrak{g}$ *is solvable.*
(4) *If* $\mathfrak{h} \lhd \mathfrak{g}$ *is an ideal s.t.* $\mathfrak{h}$ *is solvable and* $\mathfrak{g}/\mathfrak{h}$ *is solvable, then* $\mathfrak{g}$ *is solvable.*

Proof. (1) The property that iterated commutators vanish does not depend on the field.
(2) Vanishing properties of iterated commutators are inherited by subalgebras and quotients.
(3) $\mathfrak{g}^{(k)} \subseteq \mathfrak{g}_{(k)}$. Proof by induction: the statement is true for $k = 0$. Induction step: $g^{(k+1)} \subseteq [g^{(k)}, g^{(k)}] \subseteq [g_{(k)}, g_{(k)}] \subseteq [g, g_{(k)}] = g_{(k+1)}$.
(4) Let $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the canonical quotient map. Then $q(\mathfrak{g}^{(k)}) = (\mathfrak{g}/\mathfrak{h})^{(k)} = 0$ for $k > n_1$ as $\mathfrak{g}/\mathfrak{h}$ solvable $\implies \mathfrak{g}^{(k)} \subseteq \mathfrak{h}$ for $k > n_1$. Then $g^{(k+l)} \subset \mathfrak{h}^{(l)} = 0$ for $k > n_1, l > n_2$, so $\mathfrak{g}$ is solvable. □

Theorem (Lie). *Let V be a f.d. representation of a solvable Lie algebra $\mathfrak{g}$ (so we have $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$). Then $\exists$ a basis in V s.t.*

$$\rho(\mathfrak{g}) \subset \mathfrak{b} = \left\{ x \in \mathfrak{gl}(V) \mid x = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$$

wrt. that basis.

Remark: If $\mathfrak{g} = \mathbb{C}$, we know the proof from linear algebra, it goes by finding an eigenvector for $\rho(1)$, factoring it out a doing induction.

Proposition. *Let* $\mathfrak{g}$ *be solvable,* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ *a complex representation, then* $\exists v \in V$*, a common eigenvector for* $\rho(x) \forall x \in \mathfrak{g}$.

Proof. Induction on $\dim \mathfrak{g}$. As $\mathfrak{g}$ is solvable, $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \trianglelefteq \mathfrak{g}$. Let $\widetilde{\mathfrak{g}} \subset \mathfrak{g}$ be a subspace of codimension 1 s.t. $\widetilde{\mathfrak{g}} \supset [\mathfrak{g}, \mathfrak{g}]$. We have $\widetilde{\mathfrak{g}} \trianglelefteq \mathfrak{g}$, as $[\mathfrak{g}, \widetilde{\mathfrak{g}}] \subseteq [\mathfrak{g}, \mathfrak{g}] \subseteq \widetilde{\mathfrak{g}}$. By the induction assumption, $\exists v \in V$, a common eigenvector for $\rho(\widetilde{x}) \in \widetilde{\mathfrak{g}}$. Let $\mathfrak{g} = \widetilde{\mathfrak{g}} \oplus \mathbb{C}x$ as a vector space, where x is some missing vector. Let $W := \mathrm{Span}\{\rho(x)^k v \mid k \geq 0\}$ and $v^k := \rho(x)^k v$, $k \geq 0$ ($\rho(x)^0 := \mathrm{id}$). We claim that W is $\rho(\widetilde{\mathfrak{g}})$-invariant (hence $\rho(\mathfrak{g})$-invariant). Let $\widetilde{x} \in \widetilde{\mathfrak{g}}$ (so $[\widetilde{x}, x] \in \widetilde{\mathfrak{g}}$). Induction on k:

$$\rho(\widetilde{x})v^1 = \rho(x)\lambda(\widetilde{x})v^0 + \lambda([\widetilde{x}, x])v^0 = \lambda(\widetilde{x})v^1 + \text{terms from Span}\{v^0\},$$

where $\lambda : \widetilde{\mathfrak{g}} \to \mathbb{C}$ is the eigenvalue at v^0. Suppose that the statement is true for k and, then

$$\begin{aligned}\rho(\widetilde{x})v^{k+1} =& \rho(x)\rho(\widetilde{x})v^k + \rho([\widetilde{x}, x])v^k \\ =& \rho(x)\lambda(\widetilde{x})v^k + \lambda([\widetilde{x}, x])v^k + \text{terms from Span}\{v^0, ..., v^k\} \\ =& \lambda(\widetilde{x})v^{k+1} + \text{terms from Span}\{v^0, ..., v^k\}.\end{aligned}$$

Let $n := \min\{k \in \mathbb{N} \mid v^{k+1} \in \mathrm{Span}\{v^0, ..., v^k\}$, exists by finite dimension of V. Then $W = \mathrm{Span}\{v^0, ..., v^n\}$ and $\{v^0, ..., v^n\}$ is a basis in W. In this basis,

$$\rho(\widetilde{x}) = \begin{pmatrix} \lambda(\widetilde{x}) & & * \\ & \ddots & \\ 0 & & \lambda(\widetilde{x}) \end{pmatrix}.$$

Then $\mathrm{tr}|_W \rho(\widetilde{x}) = (n+1)\lambda(\widetilde{x}) \forall \widetilde{x} \in \mathfrak{g}$. This means $\lambda(\widetilde{x}) = 0$ for $\widetilde{x} \in \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ (as $\mathrm{tr}\,\rho(\mathfrak{g}') = 0$). Then, by the induction above, $\rho(\widetilde{x})v^k = \lambda(\widetilde{x})v^k \forall \widetilde{x} \in \widetilde{\mathfrak{g}}, k \in \mathbb{N}$. Now, let $w \in W$ be an eigenvector for $\rho(x)$ (exists since ρ is complex). Then w is an common eigenvector for $\rho(\mathfrak{g})$. □

Remark: If v is such a common eigenvector, then $\exists \lambda \in \mathfrak{g}^*$ s.t. $\rho(x)v = \lambda(x)v$, $x \in \mathfrak{g}$.

Proof (of the theorem). Let $v \in V$ be a common eigenvector for $\mathfrak{g} \implies \mathbb{C}v$ is an invariant subspace $\implies \exists \overline{\rho} : \mathfrak{g} \to \mathfrak{gl}(V/\mathbb{C}v)$, the quotient representation. Induction: for $\dim V = 0$, there is nothing to prove. Suppose the statement is true for $d - 1 = (\dim V) - 1$. Then by induction hypothesis, $\exists \overline{v}_1, ..., \overline{v}_{d-1} \in V/\mathbb{C}v$ s.t. $\overline{\rho}(x)$ is upper triangular wrt. $\overline{v}_1, ..., \overline{v}_{d-1}$. Then there exists a basis $v, v_1, ..., v_{d-1}$ in V s.t. $v_k + \mathbb{C}v = \overline{v}_k \forall k = 1, ..., d-1$ with

$$\rho(x) = \begin{pmatrix} \lambda(x) & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}.$$

□

Corollary. (1) $\mathfrak{g}$ *is solvable* $\implies$ *every irreducible representation is one-dimensional.*

(2) $\mathfrak{g}$ *is solvable* $\Longrightarrow \exists 0 \subset \mathfrak{h}^1 \subset ... \subset \mathfrak{h}^k = \mathfrak{g}$ *s.t.* $\mathfrak{h}^i \lhd \mathfrak{g}$ *and* $\mathfrak{h}^i/\mathfrak{h}^{i-1}$ *is one-dimensional.*
(3) $\mathfrak{g}$ *is solvable* $\Longleftrightarrow [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}'$ *is nilpotent.*

Proof. (1) Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be irreducible, then $\exists v \in V$, a common eigenvector for $\rho(x)$, $x \in \mathfrak{g} \Longrightarrow \mathbb{C}v$ is invariant $\Longrightarrow \mathbb{C}v = V$ by irreducibility.
(2) Consider the adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. Find a basis $x_1,...,x_d \in \mathfrak{g}$ s.t. $ad(x)$, $x \in \mathfrak{g}$ are upper triangular wrt. it. Then for $\mathfrak{h}^i = \text{Span}\{x_1,...,x_i\} \subseteq \mathfrak{g}$ we have $\mathfrak{h}^i \supseteq ad(\mathfrak{g})(\mathfrak{h}^i) = [\mathfrak{g},\mathfrak{h}^i]$, i.e. $\mathfrak{h}^i \lhd \mathfrak{g}$, $\mathfrak{h}^{i+1}/\mathfrak{h}^i$ are one-dimensional.
(3) Let $[\mathfrak{g},\mathfrak{g}]$ be nilpotent, then it is solvable. The quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian, hence solvable. Then g is also solvable. Let $\mathfrak{g}$ be solvable, then $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is solvable. We have $\ker ad = \mathfrak{z}(\mathfrak{g})$, so $ad\mathfrak{g} \cong \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, which is solvable. By Lie's Theorem, $ad\mathfrak{g} \subset \mathfrak{b}$. Then $[ad\mathfrak{g}, ad\mathfrak{g}] \subset [\mathfrak{b},\mathfrak{b}] \subset \mathfrak{n} \Longrightarrow [ad\mathfrak{g}, ad\mathfrak{g}]$ is nilpotent. □

Question: Is there an analogue of this for nilpotent $\mathfrak{g}$ with strictly upper triangular matrices? Answer: No in general. Take $\mathfrak{g} := \mathbb{C}$, $\rho(1) := \text{id}_{\mathbb{C}^2}$.

Definition. An endomorphism $T \in \text{End}(V)$ is called *nilpotent* if $\exists n \in \mathbb{N}$ s.t. $T^n = 0$.

Remark: $\mathfrak{n}$ is a nilpotent Lie algebra but it also consist of nilpotent operators.

Theorem. *Let* $\mathfrak{g}$ *be nilpotent,* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ *s.t.* $\{\rho(x) \mid x \in \mathfrak{g}\}$ *consists of nilpotent operators. Then* $\exists$ *basis in* V *s.t.* $\rho(x)$ *are strictly upper triangular.*

Proof. Exercise. $\mathfrak{g}$ is nilpotent, hence solvable. By Lie's Theorem, there exists a basis s.t. $\rho(x)$ is an upper triangular matrix wrt. it. The entries on the diagonal of $\rho(x)$, its eigenvalues, are zero since $0 = \rho^n(x)v = \lambda^n(x)v$ $(v \neq 0)$. □

Corollary (Engel's Theorem). *A Lie algebra* $\mathfrak{g}$ *is nilpotent* $\Longleftrightarrow ad(\mathfrak{g}) \subset \text{End}(\mathfrak{g})$ *consists of nilpotent operators.*

Proof. Let $\mathfrak{g}$ be nilpotent, then $\forall x \in \mathfrak{g}, y \in \mathfrak{g} : 0 = [x,[x,[x,...[x,y]...]]] = ad^n(x)(y) \Longrightarrow$ $ad(x)$ is nilpotent. If $ad(\mathfrak{g})$ consists of nilpotent operators $\Longrightarrow \exists$ basis $x_1,...,x_n$ in $\mathfrak{g}$ s.t. $ad(x) \in \mathfrak{n} \forall x \in \mathfrak{g}$. Let $\mathfrak{h}^i := \text{Span}\{x_1,...,x_i\} \lhd \mathfrak{g}$ since $h^i \supset ad(\mathfrak{g})(\mathfrak{h}^i) = [\mathfrak{g},\mathfrak{h}^i]$. By strict upper triangularity, $[\mathfrak{g},\mathfrak{h}^i] = ad(\mathfrak{g})(\mathfrak{h}^i) \subset \mathfrak{h}^{i-1} \Longrightarrow \mathfrak{g}$ is nilpotent. □

15 Radical, semisimple and reductive algebras

Definition. A Lie algebra is called

- *semisimple* if it does not contain any nonzero solvable ideal,
- *simple* if it does not contain any non-trivial ideals and is not abelian.

Remark: $\mathfrak{g}$ semisimple $\Longrightarrow \mathfrak{z}(\mathfrak{g}) = 0$.

Lemma. $\mathfrak{g}$ *simple* $\Longrightarrow$ $\mathfrak{g}$ *semisimple.*

Proof. Let $\mathfrak{h} \lhd \mathfrak{g}$ be a solvable ideal $\Longrightarrow \mathfrak{h} = \{0\}$ or $\mathfrak{h} = \mathfrak{g}$ by simplicity. If $\mathfrak{h} = 0$, there is nothing to prove. If $\mathfrak{h} = \mathfrak{g}$, then $\mathfrak{g}$ contains a non-trivial ideal or is abelian. □

Example. $\mathfrak{sl}(2,\mathbb{C})$ is simple.

Definition. Let $\mathfrak{g}$ be a Lie algebra. The *radical* of $\mathfrak{g}$, $\mathrm{rad}(\mathfrak{g})$, is a maximal solvable ideal in $\mathfrak{g}$ (i.e. a solvable ideal containing any other solvable ideal).

Proposition. $\mathrm{rad}(\mathfrak{g})$ *exists and is unique.*

Proof. If $\mathrm{rad}(\mathfrak{g})$ exists, it is necessarily unique, $\mathrm{rad}_1(\mathfrak{g}) \subseteq \mathrm{rad}_2(\mathfrak{g})$. Existence: Let $\mathfrak{h}_1 \lhd \mathfrak{g}$, $h_2 \lhd \mathfrak{g}$ solvable ideals $\implies \mathfrak{h}_1+\mathfrak{h}_2$ solvable, because $\mathfrak{h}_1 \lhd \mathfrak{h}_1+\mathfrak{h}_2, (\mathfrak{h}_1+\mathfrak{h}_2)/\mathfrak{h}_1 = \mathfrak{h}_2/(\mathfrak{h}_1\cap\mathfrak{h}_2)$ are solvable. So,

$$\mathrm{rad}(\mathfrak{g}) := \sum_{\substack{\mathfrak{h}\lhd\mathfrak{g}\\ \text{solvable}}} \mathfrak{h}$$

is solvable and maximal. □

Theorem. (1) $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ *is semisimple.*
(2) $\mathfrak{h} \lhd \mathfrak{g}$ *solvable s.t.* $\mathfrak{g}/\mathfrak{h}$ *semisimple, then* $\mathfrak{h} = \mathrm{rad}(\mathfrak{g})$.

Proof. (1) Let $\overline{\mathfrak{h}} \lhd \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ be solvable. If $q : \mathfrak{g} \to \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is the quotient map, then $\mathfrak{h} := q^{-1}(\overline{\mathfrak{h}}) \lhd \mathfrak{g}$ (since $q(x) \in \overline{\mathfrak{h}} \lhd \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$, $y \in \mathfrak{g} \implies q([x,y]) = [q(x),q(y)] \in \overline{\mathfrak{h}}$). For $q|_{\mathfrak{h}} : \mathfrak{h} \to \overline{\mathfrak{h}}$, we have $\ker q|_{\mathfrak{h}} = \mathfrak{h}\cap\mathrm{rad}(\mathfrak{g})$. So, $\mathfrak{h}\cap\mathrm{rad}(\mathfrak{g})$, $\overline{\mathfrak{h}}$ are solvable. Then $\mathfrak{h}$ is solvable $\implies \mathfrak{h} \subset \mathrm{rad}(\mathfrak{g}) \implies \overline{\mathfrak{h}} = 0$.
(2) Exercise. Let $\mathfrak{h} \lhd \mathfrak{g}$ solvable, $\mathfrak{g}/\mathfrak{h}$ semisimple. Then $\mathfrak{h} \subset \mathrm{rad}(\mathfrak{g})$ and the map $\varphi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is a well-defined homomorphism. Consequently $\mathrm{rad}(\mathfrak{g}) = \ker\varphi \lhd \mathfrak{g}/\mathfrak{h}$. If $\mathfrak{h} \subsetneq \mathrm{rad}(\mathfrak{g})$, then we get a contradiction to $\mathfrak{g}/\mathfrak{h}$ being semisimple ($\mathrm{rad}(\mathfrak{g})$ is a non-zero ideal in $\mathfrak{g}/\mathfrak{h}$). □

Observation: $0 \to \mathfrak{h}\cap\mathrm{rad}(\mathfrak{g}) \to \mathfrak{h} \to \overline{\mathfrak{h}} \to 0$ from (1) is exact.

Corollary. *For every Lie algebra* $\mathfrak{g}$ *exists a semisimple* $\mathfrak{g}_{ss}$ *and a short exact sequence* $0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g}/\mathrm{rad}(\mathfrak{g}) = \mathfrak{g}_{ss} \to 0$.

Example. Let $G := \mathrm{Iso}^+(\mathbb{R}^3) = \mathbb{R}^3 \ltimes SO(3) = \{x \mapsto Ax + b \mid A \in SO(3), b \in \mathbb{R}^3\}$ (the *Poincare Group*). Then $\mathfrak{g} = Lie(G) = \mathbb{R}^3 \oplus \mathfrak{so}(3)$ as a vector space, Lie-bracket is given by $[(b_1,A_1),(b_2,A_2)] := (A_1b_2 - A_2b_1, [A_1,A_2])$. There, $\mathrm{rad}(\mathfrak{g}) = \mathbb{R}^3 = \{(b,0) \mid b \in \mathbb{R}^3\} \subseteq \mathfrak{g}$, $\mathfrak{g}/\mathrm{rad}(\mathfrak{g}) = \mathfrak{so}(3)$.

Example. $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{z}(\mathfrak{gl}(n,\mathbb{C}))\oplus\mathfrak{sl}(n,\mathbb{C}) \implies \mathfrak{gl}(n,\mathbb{C})$ is not semisimple but it differs from a semisimple algebra just by the centre.

Definition. A Lie algebra is *reductive* if $\mathrm{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ ($\iff \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$).

16 Invariant bilinear forms

Definition. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, $B : \mathfrak{g}\times\mathfrak{g} \to \mathbb{K}$ be a bilinear form on $\mathfrak{g}$. Then B is called *invariant*, if $\forall x,y,z \in \mathfrak{g} : B(ad(x)y,z) + B(y,ad(x)z) = 0$ (i.e., $ad(x)$ is skew-adjoint w.r.t B).

Remark: If $\rho : G \to GL(V)$ is a representation, $B : V \times V \to \mathbb{C}$ is invariant if $B(gv, gw) = B(v, w) \forall g \in G, v, w \in V$. If B is invariant, $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$ the corresponding Lie algebra representation, then $B(\rho_*(x)v, w) + B(v, \rho_*(x)w) = 0 \forall x \in \mathfrak{g}, v, w \in V$ can be obtained by differentiation.

Lemma. *If $\mathfrak{h} \lhd \mathfrak{g}$ is an ideal, B an invariant bilinear form, then*

$$\mathfrak{h}^\perp := \{x \in \mathfrak{g} \mid B(x, y) = 0 \forall y \in \mathfrak{h}\}$$

is also an ideal.

Proof. $\mathfrak{h} \lhd \mathfrak{g}, y \in \mathfrak{h}^\perp \implies \forall z \in \mathfrak{h} \forall x \in \mathfrak{g}, 0 = B(y, ad(x)z) = -B(ad(x)y, z) \implies ad(x)y = [x, y] \in \mathfrak{h}^\perp$. □

Remark: $\mathfrak{h} \cap \mathfrak{h}^\perp \neq 0$ in general.

Example. $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$. Let $\rho(x) := \mathrm{tr}(xy)$.

- Symmetry: $B(x, y) = \mathrm{tr}(xy) = \mathrm{tr}(yx) = B(y, x)$.
- Invariance: $B(ad(x)y, z) = \mathrm{tr}([x, y]z) = \mathrm{tr}((xy - yx)z) = \mathrm{tr}(xyz - yxz) = \mathrm{tr}(yzx - yxz) = -\mathrm{tr}(y[x, z]) = -B(y, ad(x)z)$.

Proposition. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation over $\mathbb{K}$. Then the form $B_\rho(x, y) := \mathrm{tr}(\rho(x)\rho(y))$ is a symmetric bilinear form which is invariant.*

Proof. Last example. □

Theorem. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ s.t. $\exists \rho : \mathfrak{g} \to \mathfrak{gl}(V)$ with the form B_ρ non-degenerate. Then $\mathfrak{g}$ is reductive.*

Remark: We will prove this over $\mathbb{C}$, exercise: reduce $\mathbb{R}$ to $\mathbb{C}$.

Proposition. *Let $\mathfrak{g}$ be a Lie algebra, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a complex representation. If ρ is reducible, then all elements $h \in \mathrm{rad}(\mathfrak{g})$ act by scalar operators: $\rho(h) = \lambda(h) \cdot \mathrm{id}_V$ $\forall h \in \mathrm{rad}(\mathfrak{g})$. Moreover, $\lambda(h) = 0 \forall h \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$.*

Remark: This is an extension of Lie's Theorem about representation of solvable Lie algebras.

Proof. By Lie's Theorem, $\exists \lambda \in (\mathrm{rad}(\mathfrak{g}))^*$ and $v \in V$ s.t. $\rho(h)v = \lambda(h)v \forall h \in \mathrm{rad}(\mathfrak{g})$. Then $V_\lambda := \{w \in V \mid \rho(h)w = \lambda(h)w \forall h \in \mathrm{rad}(g)\} \neq 0$ and is invariant under $\rho(\mathrm{rad}(\mathfrak{g}))$. Furthermore, $\mathrm{tr}(\rho(h))|_{V_\lambda} = \lambda(h) \cdot \dim V_\lambda \implies \lambda([\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]) = 0$. Consequently $\forall x \in \mathfrak{g}, v \in V_\lambda, h \in \mathrm{rad}(\mathfrak{g})$:

$$\begin{aligned} \rho(h)\rho(x)v =& \rho(x)\rho(h)v + \rho([h, x])v \\ =& \rho(x)\lambda(h)v + \lambda([h, x])v \\ =& \lambda(h)\rho(x)v \end{aligned}$$

$\implies \rho(g)V_\lambda \subseteq V_\lambda \implies V_\lambda = V$ by irreducibility. □

Proof (of the theorem). It suffices to prove $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = 0$ (since $\mathrm{rad}(\mathfrak{g}) \supset \mathfrak{z}(\mathfrak{g})$). Let $x \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$. By previous proposition, $\rho_W(x) = 0$ for every irreducible representation $\rho_W : \mathfrak{g} \to \mathfrak{gl}(W) \implies B_{\rho_W}(x, y) = 0 \forall y \in \mathfrak{g}$.

Suppose now, that ρ_{W_1}, ρ_{W_2} are representations s.t. $B_{\rho_{W_1}}(x, y) = B_{\rho_{W_2}}(x, y) = 0 \forall y \in \mathfrak{g}$ and $\rho_{\widetilde{W}} : \mathfrak{g} \to \mathfrak{gl}(\widetilde{W})$ is a representation s.t. there is a short exact sequence $0 \to W_1 \to \widetilde{W} \to W_2 \to 0$ of $\mathfrak{g}$-representations. Claim: $B_{\rho_{\widetilde{W}}}(x, y) = 0 \forall y \in \mathfrak{g}$. Reason: $W_1 \cong V$ subrepresentation of $\widetilde{W}$ and $W_2 \cong \widetilde{W}/V$. Then $\widetilde{W} \cong V \oplus (\widetilde{W}/V)$ and $\rho_{V \oplus (\widetilde{W}/V)} = (\rho_V, \rho_{\widetilde{W}/V})$. So

$$\begin{aligned} B_{\rho_{\widetilde{W}}}(x, y) =& \mathrm{tr}(\rho_{\widetilde{W}}(x), \rho_{\widetilde{W}}(y) \\ =& \mathrm{tr}\left(\begin{pmatrix} \rho_{W_1}(x) & * \\ 0 & \rho_{W_2}(x) \end{pmatrix} \cdot \begin{pmatrix} \rho_{W_1}(y) & * \\ 0 & \rho_{W_2}(y) \end{pmatrix} \right) \\ =& B_{\rho_{W_1}}(x, y) + B_{\rho_{W_2}}(x, y) \end{aligned}$$

$\forall x, y \in \mathfrak{g}$. This implies $B_{\rho_W}(x, y) = 0 \forall y \in \mathfrak{g}, x \in [\mathfrak{g}, \mathrm{rad}\,\mathfrak{g}]$, since if W irreducible, we are done, otherwise we can do induction on dimension of W, where $\dim W_1, \dim W_2 < \dim W$. But B_{ρ_W} is non-degenerate $\implies x = 0$. □

Corollary. All classical Lie algebras $\mathfrak{gl}(n, \mathbb{K})$, $\mathfrak{sl}(n, \mathbb{K})$, $\mathfrak{so}(n, \mathbb{K})$, $\mathfrak{sp}(2n, \mathbb{K})$, $\mathfrak{u}(n)$, $\mathfrak{su}(n)$ are reductive.

Proof. Reason is common: The invariant bilinear form coming from the standard tautological representation is non-degenerate. Semisimplicity follows by $\mathfrak{z}(\mathfrak{g}) = 0$ for relevant $\mathfrak{g}$. The standard bilinear form descends from $\mathfrak{gl}(n, \mathbb{K})$:

$$B(x, y) = \mathrm{tr}(xy) = \sum_{i,j=1}^{n} x_{ij} y_{ji}$$

for $x, y \in \mathfrak{gl}(n, \mathbb{K})$. $\{E_{i,j}\}$ is a basis in $\mathfrak{gl}(n, \mathbb{K}) \implies \{E_{j,i}\}$ is a dual basis w.r.t. B. Restriction to classical Lie algebras:

- $\mathfrak{sl}(n, \mathbb{K})$: $\mathfrak{gl}(n, \mathbb{K}) = \mathbb{K} \cdot \mathrm{id} \oplus \mathfrak{sl}(n, \mathbb{K})$, an orthogonal decomposition w.r.t. B, so we only consider B nondegenerate on both separately $\implies B$ nondegenerate on $\mathfrak{sl}(n, \mathbb{K})$.
- $\mathfrak{so}(n, \mathbb{K})$: $y_{ji} = -y_{ij}$. Then $B(x, y) = -2\sum_{i>j} x_{ij} y_{ij}$ (is negative definite on $\mathbb{R}$). Since $\{E_{i,j} - E_{j,i} \mid i > j\}$ is a basis in $\mathfrak{so}(n, \mathbb{K}) \implies B$ nondegenerate there.
- $\mathfrak{u}(n) : B(x, y) = \sum_{i,j=1}^{n} x_{ij} y_{ji} = -\sum_{i,j=1}^{n} x_{ij} \overline{y_{ij}}$ (is negative definite sesquilinear, even on $\mathfrak{gl}(n, \mathbb{C}) \implies$ on all subspaces, $\mathfrak{u}(n)$ too. Hence $B(x, x) = 0$ only if $x = 0$.)
- $\mathfrak{sp}(n, \mathbb{K})$: Exercise. □

Definition. The *Killing-Cartan form* on a Lie algebra $\mathfrak{g}$ is the symmetric invariant bilinear form coming from the adjoint representation:

$$K^{\mathfrak{g}}(x, y) := \mathrm{tr}_{\mathfrak{g}}(ad(x) \cdot ad(y)).$$

Remark: If $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra, $K^{\mathfrak{h}} \neq K^{\mathfrak{g}}|_{\mathfrak{h}}$ in general. It does however if $\mathfrak{h} \lhd \mathfrak{g}$, see exercise 5.1.

Theorem (Cartan's criterion for solvability). *A Lie algebra* $\mathfrak{g}$ *is solvable iff* $K(\mathfrak{g}, \mathfrak{g}') = 0$.

Theorem (Cartan's criterion for semisimplicity). *A Lie algebra* $\mathfrak{g}$ *is semisimple iff* K *is nondegenerate.* $K(\mathfrak{g}, \mathfrak{g}') = 0$.

17 Appendix

17.1 Linear Algebra

Theorem. Let V be a finite-dimensional vector space over $\mathbb{K}$, $f \in \mathrm{End}(V)$. Then f is trigonalisable if the characteristic polynomial of f factorizes over $\mathbb{K}$.

Proof. Wlog $V = \mathbb{K}^n$. Induction on n: For $n = 1$, any $M \in \mathbb{K}^{1\times 1}$ is already upper triangular. Suppose that every $M \in \mathbb{K}^{(n-1)\times(n-1)}$ is upper triangular if the characteristic polynomial of M factorizes over $\mathbb{K}$. Let $M \in \mathbb{K}^{n\times n}$. By the assumption, M has at least one eigenvalue λ_1. Let v_1 be the associated eigenvector. Complete $\{v_1\}$ to a basis $\{v_1, ..., v_n\}$ of $\mathbb{K}^n$. □

Definition/Proposition. *Let V be a finite-dimensional vector space over $\mathbb{K}$, $v_1, ..., v_n$ a basis in V. Then for every $i \in \{1, ..., n\}$, there is exactly one linear map $v_i^* \in V^*$ such that $v_i^*(v_j) = \delta_{ij}$. The set $\{v_1^*, ..., v_n^*\}$ is called the* dual basis *and constitutes a basis in V^*.*

Definition. Let V be a vector space over $\mathbb{K}$, a *bilinear form* (a bilinear map $B : V \times V \to \mathbb{K}$) is

- *nondegenerate* if $B(v, w) = 0 \forall w \in V \implies v = 0$
- *skew-symmetric* if $B(v, w) = -B(w, v) \forall v, w \in V$
- *alternating* if $B(v, v) = 0 \forall v \in V$

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A symmetric bilinearform (hermitian sesquilinearform) B is

- *positive definite* if $B(v, v) > 0 \forall v \in V$
- *positive semidefinite* if $B(v, v) \geq 0 \forall v \in V$
- *negative definite* if $B(v, v) < 0 \forall v \in V$
- *negative semidefinite* if $B(v, v) \leq 0 \forall v \in V$

Proposition. *An alternating bilinear form $B : V \times V \to \mathbb{K}$ is skew-symmetric. If* $\mathrm{char}\,\mathbb{K} \neq 2$*, the converse is also true.*

Proof. By bilinearity, $0 = B(v + w, v + w) = B(v, v) + B(v, w) + B(w, v) + B(w, w) = B(v, w) + B(w, v)$. Converse: $B(v, v) = -B(v, v) \implies 2B(v, v) = 0 \implies B(v, v) = 0 \forall v$ if $\mathrm{char}\,\mathbb{K} \neq 2$. □

Proposition. *Let $B : V \times V \to \mathbb{R}$ be a nondegenerate bilinear form, then $\alpha(v)(w) := B(v, w)$ gives a isomorphism $V \to V^*$.*

Proof. By bilinearity of B, α is a linear map. It is injective: Let $\alpha(v) = 0 \implies B(v, w) = 0 \forall w \in V \implies v = 0$ since nondegenerate. Surjectivity follows from the Rank theorem: $\dim \mathrm{im}(\alpha) = n = \dim V^*$. □

17.2 Differential Forms

17.3 Haar Measure

17.4 Useful Formulas

$$\sum_k \delta_{i,k}\delta_{k,j} = \delta_{i,j} \tag{1}$$

17.5 Exercises

Exercise 5.3. Let $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$ be the subspace of block triangular matrices:

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid A \in \mathbb{C}^{k\times k}, B \in \mathbb{C}^{k\times(n-k)}, D \in \mathbb{C}^{(n-k)\times(n-k)} \right\}.$$

(1) Direct computation:

$$\begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 & A_1B_2 + B_1D_2 \\ 0 & D_1D_2 \end{pmatrix}.$$

Then

$$\left[\begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ 0 & D_2 \end{pmatrix}\right] = \begin{pmatrix} [A_1, A_2] & A_1B_2 + B_1D_2 - A_2B_1 - B_2D_1 \\ 0 & [D_1, D_2] \end{pmatrix}$$

(2) The $\mathfrak{h} \lhd \mathfrak{g}$ solvable $\iff$ $[\mathfrak{h},\mathfrak{h}]$ nilpotent. Since $[\mathfrak{h},\mathfrak{h}]$ has the form above, any A_1,A_2 and D_1, D_2 must commute, thus $A_i = \lambda_i E_k$, $D_i = \mu_i E_{n-k}$. The maximal solvable ideal, $\mathrm{rad}(\mathfrak{g})$ contains all such matrices. Furthermore,

$$\mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \cong \{A \in \mathbb{C}^{k\times k} \mid A \text{ upper triagonal}\} \oplus \{B \in \mathbb{C}^{(n-k)\times(n-k)} \mid B0 \text{ upper triagonal}\}$$