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Optimization Models in Engineering

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Contents

1. Linear Algebra

1.a. Least-Squares Problem Statement

Definition 1.1 (Least Squares)
Assume matrix A and vectors $\vec{x}$ and $\vec{b}$. The problem defined by

$$\min_{\vec{x}} \|A\vec{x} - \vec{b}\|^2$$

is a Least Squares Problem (LSP).

Example 1.2
Assume we have two dimensional data set $\vec{x}$ and $\vec{y}$ and we want to formalize a LSP to find a linear correlation between x and y. We first formalize the goal linear correlation as

$$y = mx + c$$

where we want to find the optimal values for m and c to minimize the squared loss across all data points. Summarizing the above equation for all data points gives us

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Where

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} m \\ c \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

And therefore

$$\min_{\vec{x}} \|A\vec{x} - \vec{b}\|^2 = \min_{m,c} \sum_{i=1}^{n} (y_i - (mx_i + c))^2$$

Theorem 1.3 (Ordinary Least Squares)

Given the column space of the matrix A, for vector $\vec{b}$ not in the said column space, $A\vec{x} - \vec{b} = \vec{e}$ must be orthogonal to the columns of A. (Pythagora's theorem)

Therefore, the dot products of every column of A and $\vec{e}$ must be zero, i.e.

$$\begin{aligned} A^\top(A\vec{x} - \vec{b}) &= 0 \\ A^\top A\vec{x} - A^\top\vec{b} &= 0 \\ A^\top A\vec{x} &= A^\top\vec{b} \\ \vec{x} &= (A^\top A)^{-1}A^\top\vec{b} \end{aligned}$$

We conclude that the solution for Ordinary Least Squares (OLS) is

$$\vec{x}^* = \underset{\vec{x}}{\operatorname{argmin}} \|A\vec{x} - \vec{b}\|^2 = (A^\top A)^{-1}A^\top\vec{b}$$

1.b. Norm

Definition 1.4 (Norm)

A Norm is defined as

$$f := \mathbf{X} \to \mathbb{R}$$

For vector space $\mathbf{X}$.

The norm of x is denoted as $\|x\|$.

For any vector x and y, we have

- $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = \vec{0}$
- $\|x + y\| \leq \|x\| + \|y\|$
- $\|\alpha x\| = |\alpha| * \|x\|$

Definition 1.5 (l-p Norm)

Generally, l-p norm is defined as

$$\|\vec{x}\|_p := \left(\sum |x_i|^p\right)^{\frac{1}{p}}; \;\; 1 \leq p < \infty$$

Commonly used norms:

- $\|\vec{x}\|_1 := \sum |x_i|$
- $\|\vec{x}\|_2 := \sqrt{\sum |x_i|^2}$
- $\|\vec{x}\|_\infty := \max |x_i|$

Theorem 1.6 (Cauchy-Schwartz Inequality)

$$< \vec{x}, \vec{y} >= \vec{x}^\top \vec{y} = \|\vec{x}\|_2 \|\vec{y}\|_2 \cos\theta$$

Since $-1 \le \cos\theta \le 1$,

$$< \vec{x}, \vec{y} >= \vec{x}^\top \vec{y} \le \|\vec{x}\|_2 \|\vec{y}\|_2$$

Theorem 1.7 (Holder's Inequality)
For $p, q \ge 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$,

$$|\vec{x}^\top \vec{y}| \le \sum_{i=1}^{n} |x_i y_i| \le \|\vec{x}\|_p \|\vec{y}\|_p$$

i.e., Cauchy-Schwartz is a narrowed case of Holder's Inequality.

1.c. Gram-Schimdt

Theorem 1.8 (Gram-Schimdt/QR-decomposition)
Let X be a vector space with basis $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$, which is orthonormal. For any matrix A,

$$A = QR$$

$$\left[\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\right] = \left[\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\right] \begin{bmatrix} \vec{r}_{11} & \vec{r}_{12} & \cdots & \vec{r}_{1n} \\ 0 & \vec{r}_{22} & \cdots & \vec{r}_{2n} \\ 0 & 0 & \ddots & \vec{r}_{3n} \\ 0 & 0 & 0 & \vec{r}_{nn} \end{bmatrix}$$

Where Q is orthonormal andR is upper-triangular.

Theorem 1.9 (Fundamental Theorem of Linear Algebra)
For matrix $A \in \mathbb{R}^{m*n}$,

$$Null(A) \bigoplus Range(A^\top) = \mathbb{R}^n$$

Where $\oplus$ denotes "direct sum" and $Range(A^\top)$ is the column space of $A^\top$. With the said equation we can also conclude that

$$Range(A) \bigoplus Null(A^\top) = \mathbb{R}^m$$

Theorem 1.10 (orthogonal decomposition theorem)
X a vector space and S a subspace of X. Then for any $\vec{x}$ in X,

$$\vec{x} = \vec{s} + \vec{r}, \quad \vec{s} \in S, \quad \vec{r} \in S^{\perp}$$

Such that

$$S^{\perp} = \{\vec{r} \mid < \vec{r}, \vec{s} > = 0, \quad \forall \vec{s} \in S\}$$

Therefore,

$$\mathbf{X} = S \bigoplus S^{\perp}$$

Example 1.11 (Minimum Norm Problem)
We want to find

$$\min \|\vec{x}\|_2^2$$

subject to $A\vec{x} = \vec{b}$. From FTLA we know that

$$\vec{x} = \vec{y} + \vec{z} \;\; s.t. \;\; \vec{y} \in N(A; \;\; \vec{z} \in R(A^{\top}).$$

And

$$A(\vec{y} + \vec{z}) = 0 + A\vec{z} = \vec{b}$$

Since $\vec{y} \perp \vec{z}$,

$$\|\vec{x}\|_2^2 = \|y\|_2^2 + \|z\|_2^2$$

Consider $\vec{z} = A^{\top}\vec{w}$,

$$\begin{aligned} A\vec{z} &= \vec{b} \\ AA^{\top}\vec{w} &= \vec{b} \\ \vec{w} &= (AA^{\top})^{-1}\vec{b} \end{aligned}$$

Therefore

$$\vec{z} = \min \|\vec{x}\|_2^2 = A^{\top}(AA^{\top})^{-1}\vec{b}$$

1.d. Symmetric Matrices

Definition 1.12
Matrix A is symmetric if $A = A^{\top}$, i.e. $A_{ij} = A_{ji}$.
Set $\mathbb{S}^n$ means the set of symmetric matrices of dimension n.

Theorem 1.13 (Spectral Theorem)
If matrix A $\in \mathbb{S}^{\ltimes}$, then

- All eigenvalues of A are real numbers
- Eigenspaces are orthogonal
- $dim(N(\lambda_i I - A)) = \mu_i$ where μ_i is the algebraic multiplicity of λ_i

This means that A is always diagonalizable. i.e.:

$$A = U\Lambda U^\top$$

where U orthonormal and Λ diagonal. Orthonormal (or, unitary) means that the columns of U are orthogonal and all columns are normalized, i.e.

$$U^{-1} = U^\top$$

Theorem 1.14
For a diagonalizable n*n matrix A that has n linearly independent eigenvectors, A can be factorized as

$$A = U\Lambda U^\top$$

Where U orthonormal and Λ is a diagonal matrix consists of the eigenvalues of A such that

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_i \end{bmatrix}$$

Therefore it is also called an eigenvalue decomposition.

1.e. Principal Component Analysis

Definition 1.15
For $A \in \mathbb{S}$, its Rayleigh coefficient is defined as

$$R = \frac{\vec{x}^\top A\vec{x}}{\vec{x}^\top \vec{x}}$$

The Rayleigh coefficient can bound the eigenvalues of A such that,

$$\lambda_{min}(A) \le \frac{\vec{x}^\top A\vec{x}}{\vec{x}^\top \vec{x}} \le \lambda_{max}(A)$$

PCA is very similar to Singular Value Decomposition (SVD). SVD has more nice properties than PCA.

1.f. Singular Value Decomposition

Theorem 1.16 (SVD)
Let $A \in \mathbb{R}^{m*n}$, the SVD of A is given as

$$A = U\Sigma V^\top$$

Where

$$U \in \mathbb{R}^{m*m}, \ \ \Sigma \in \mathbb{R}^{m*n}, \ \ V \in \mathbb{R}^{n*n}$$

and Σ has real entries in its diagonal (the singular values) and zero's else where. If $Rank(A) = r$, we can rewrite A as

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^\top + \sigma_1 \vec{u}_1 \vec{v}_1^\top + \cdots + \sigma_r \vec{u}_r \vec{v}_r^\top$$

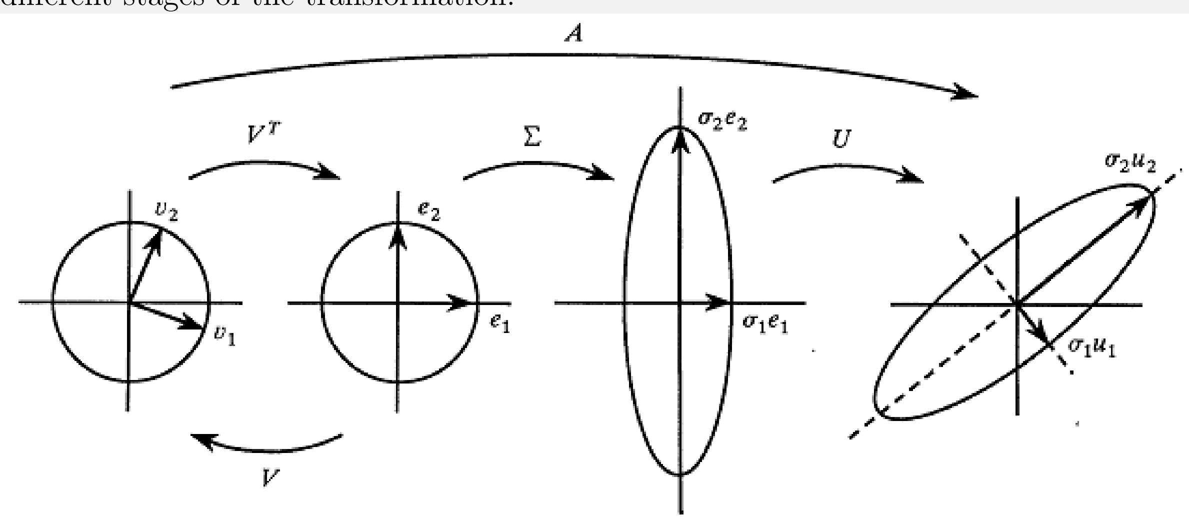
Remark 1.17 (geometric interpretation of SVD)
Consider linear transformation on vector $\vec{x}$ given by matrix A, s.t.

$$A\vec{x} = U\Sigma V^\top \vec{x}$$

SVD helps breaking the transformation into three smaller steps, i.e.

- orthonormal transformation (rotate/reflect) by V,
- scaling by Σ,
- orthonormal transformation by U.

The following illustration is an example of a 2D transformation $A\vec{x}$. It shows the decomposed linear transformation through the unit circles relative to the original unit circle at different stages of the transformation.



Theorem 1.18 (Proof of SVD)

For $A \in \mathbb{R}^{m*n}$, consider symmetric matrix $A^\top A$ that has eigenvalues $\lambda_1 \cdots \lambda_r > 0$ with corresponding eigenvectors $v_1 \cdots v_r$ and $\lambda_{r+1} \cdots \lambda_n = 0$. Then we know that

$$A^\top A\vec{v}_i = \lambda_i \vec{v}_i$$

Let

$$V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

Define $\sigma_i = \sqrt{\lambda_i}$, let

$$A\vec{v}_i = \sigma_i \vec{u}_i \quad i \leq r$$

for some vector $\vec{u}_i$.

Claim. $\vec{u}_i$ are orthonormal.

Proof.

$$\begin{aligned} \vec{u}_i^\top \vec{u}_j &= \frac{(A\vec{v}_i)^\top}{\sigma_i}\frac{(A\vec{v}_j)}{\sigma_j} \\ &= \frac{1}{\sigma_i \sigma_j}\vec{v}_i^\top A^\top A \vec{v}_j && A^\top A\vec{v}_j = \lambda_j \vec{v}_j \\ &= \frac{1}{\sigma_i \sigma_j}\vec{v}_i^\top \lambda_j \vec{v}_j \\ &= \frac{\lambda_j}{\sigma_i \sigma_j}\vec{v}_i^\top \vec{v}_j && \vec{v}_i \vec{v}_j \text{ orthonormal} \\ &= \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \end{aligned}$$

Therefore $\vec{u}_i$ are orthonormal. ■

Recall that A has rank r, we let

$$V_r = V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_r \\ | & & | \end{bmatrix}$$

Hence

$$AV_r = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_r \\ | & & | \end{bmatrix}\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} = U_r \Sigma_r$$
$$A = U\Sigma V^\top$$

Since V orthonormal and $V^{-1} = V^\top$

1.g. Low-Rank Approximation

Definition 1.19 (matrix norms)

There are two ways to interpret a matrix, either as an operator or as a block of data. Frobenius norm consider the matrix as a block of data.
Frobenius norm of matrix A is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{tr(A^\top A)}$$

Frobenius norm is invariant to orthonormal transformations, i.e. given U an orthonormal matrix,

$$\|UA\|_F = \|AU\|_F = \|A\|_F$$

Spectral norm, or l_2 norm, interpret the matrix as an operator and is defined as

$$\|A\|_2 = \max_{\|\vec{x}\|_2=1} \|A\vec{x}\|_2 = \max_{\|\vec{x}\|=1} \sqrt{\vec{x}^\top A^\top A\vec{x}} = \sqrt{\lambda_{max}(A^\top A)} = \sigma_{max}(A^\top A)$$

Intuitively, the spectral norm of a matrix A is the largest scaling that A can do (recall the Σ matrix that is used to scale the unit circle in the three steps of transformation after SVD).

Theorem 1.20 (Eckart-Young-Mirsky Theorem)

$A \in \mathbb{R}^{m*n}$. Do SVD gives us

$$A = U\Sigma V^\top = \sum_{i=1}^{n} \sigma_i \vec{u}_i \vec{v}_i^\top$$

Define

$$A_k = \sum_{i=1}^{k} \sigma_i \vec{u}_i \vec{v}_i^\top$$

We want to find the best k-rank (lower than r) approximation of A, i.e.

$$\underset{B\in\mathbb{R}^{m*n},\, Rank(B)=k}{\operatorname{argmin}} \|A-B\|_F$$

Suprisingly, Eckart-Young-Mirsky Theorem tells us that

$$\underset{B\in\mathbb{R}^{m*n},\, Rank(B)=k}{\operatorname{argmin}} \|A-B\|_F = A_k$$

Moreover,

$$\underset{B\in\mathbb{R}^{m*n},\, Rank(B)=k}{\operatorname{argmin}} \|A-B\|_2 = A_k$$

This theorem relates two completely different norms and is not obvious at all. It shows how fundamental SVD is, such that in any way of looking at a matrix, the decomposition shows up.

Remark 1.21

Eckart-Young-Mirsky Theorem can be used to **compress images**. For an image, the matrix that represents the pixels of the image can be reduced to a lower rank matrix, and hence a smaller set of data, while remains relatively high resolution. The A_k matrix **captures the key features of the image because it keeps k largest singular values and their corresponding vectors that contribute most to the dataset/transformation.**

Definition 1.22 (trace)

The trace of a matrix is defined as

$$trace := \mathbb{R}^{n*n} \to \mathbb{R}$$

$$trace(A) = \sum_{i=1}^{n} a_{ii}$$

Remark 1.23 (Orthonormal transformation invariance of Frobenius norm)

Proof that $\|UA\|_F = \|AU\|_F = \|A\|_F$

Proof. Recall that $\|A\|_F = \sqrt{tr(A^\top A)}$. By definition, for any matrices A and B, we have $tr(AB) = tr(BA)$ Then,

$$\begin{aligned}
\|AU\|_F &= \sqrt{tr((AU)^\top (AU))} \\
&= \sqrt{tr(U^\top A^\top AU)} \\
&= \sqrt{tr(UU^\top A^\top A)} \\
&= \sqrt{tr(A^\top A)} \\
&= \|A\|_F
\end{aligned}$$

■

Remark 1.24 (Frobenius norm is the sqrt of the sum of the squares of the singular values)

$$\begin{aligned}
\|A\|_F = \|U\Sigma V^\top\|_F &= \|\Sigma\|_F \\
&= \sqrt{\sum_{i=1}^{n} \sigma_i^2}
\end{aligned}$$

Remark 1.25 (Proof of Eckart-Young-Mirsky)
Goal: B: rank(k), $\|A - B\|_F \geq \|A - A_k\|_F$

Proof.

$$\|A - A_k\|_F = \|\sum_{i=k+1}^{n} \sigma_i \vec{u}_i \vec{v}_i\|_F = \sqrt{\sum_{i=k+1}^{n} \sigma_i^2}$$

Note that the goal is true iff

$$\sum_{i=1}^{n} \sigma_i^2(A - B) \geq \sum_{i=k+1}^{n} \sigma_i^2(A)$$

Further note that the previous statement is true iff:

$$\sigma_i^2(A - B) \geq \sigma_{k+i}^2(A)$$

Let $\sigma_{k+i}(A)$be the k+ith largest singular value of A. Hence

$$\sigma_{k+i}(A) = \sigma_{max}(A - A_k)$$

Denote A-B = C. Then

$$\sigma_i(A - B) = \sigma_i(C) = \|C - C_{i-1}\|_2$$

Since B has rank k,

$$\|B - B_k\|_2 = 0$$

Add it to the previous equation gives us

$$\begin{aligned}\sigma_i(A - B) &= \|C - C_{i-1}\|_2 + \|B - B_k\|_2 \\ &\geq \|C + B - C_{i-1} - B_k\|_2 \\ &\geq \|A - C_{i-1} - B_k\|_2\end{aligned}$$

Let $D = C_{i-1} + B_k$. Rank(D) $\leq$ i-1+k. Then

$$\sigma_i(A - B) \geq \|A - D\|_2$$

Consider the solution to the optimization problem

$$\underset{D,\, rank(D) \leq i+k-1}{\text{argmin}} \|A - D\|_2 = A_k + i - 1$$

$$\min_{rank(D) \leq i+k-1} \|A - D\|_2 = \sigma_{k+1}(A)$$

Finally, bring the above result back to the previous equation gives us

$$\sigma_i(A - B) \geq \sigma_{k+1}(A)$$

as desired. ■