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Citation: Cohen, B., Lai, W. M., & Mow, V. C. (1998). A transversely isotropic biphasic model for unconfined compression of growth plate and chondroepiphysis.

Parameters

Choose the following parameters and compute the inversion

$$E_1 = 8.5\ kPa, \qquad E_3 = 19\ kPa, \qquad \nu_{21} = 0.75, \qquad \nu_{31} = 0.24,$$

$$t_g = 40.62\ sec, \qquad \dot{\varepsilon}_0 = 0.01\ sec^{-1}$$

$$t_0/t_g = 0.25$$

Setup Equations

$$\Delta_1 = 1 - \nu_{21} - 2\upsilon_{31}^2 \frac{E_1}{E_3}$$

$$\Delta_2 = \left(1 - \upsilon_{31}^2 \frac{E_1}{E_3}\right) / (1 + \nu_{21})$$

$$\Delta_3 = (1 - 2\upsilon_{31}^2) \frac{\Delta_2}{\Delta_1}$$

$$C_{11} = \frac{E_1 \cdot \left(1 - \upsilon_{31}^2 \frac{E_1}{E_3}\right)}{(1 + \upsilon_{21}) \cdot \Delta_1}$$

$$C_{12} = \frac{E_1 \cdot \left(\upsilon_{21} + \frac{\upsilon_{31}^2 E_1}{E_3}\right)}{(1 + \upsilon_{21}) \cdot \Delta_1}$$

$$C_{13} = \frac{E_1 \upsilon_{31}}{\Delta_1} = \frac{E_1 \upsilon_{31}}{1 - \nu_{21} - 2\upsilon_{31}^2 \frac{E_1}{E_3}}$$

$$C_{33} = E_3 \cdot \left(1 + \frac{2\upsilon_{31}^2 \frac{E_1}{E_3}}{\Delta_1}\right) = \frac{E_3 \left(\Delta_1 + 2\upsilon_{31}^2 \frac{E_1}{E_3}\right)}{\Delta_1} = \frac{\Delta_1 E_3 + 2\upsilon_{31}^2 E_1}{\Delta_1}$$

$$C_0 = \frac{C_{11} - C_{12}}{C_{11}}$$

$$C_1 = \frac{C_{11} + C_{12} - 4C_{13} + 2C_{33}}{C_{11} - C_{12}}$$

$$C_2 = 2 \cdot \frac{C_{33}\ (C_{11} - C_{12}) + C_{11}(C_{11} + C_{12} - 4C_{13}) + 2C_{13}^2}{(C_{11} - C_{12})^2}$$

Units of C_{11}, C_{12}, C_{13} are in kPa (the same dimension as $E_1$and E_3)

Units of C_0, C_1, C_{12} are non-dimensional

Units of $\Delta_0, \Delta_1, \Delta_2$ are non-dimensional

Laplace Solution

$$\tilde{\varepsilon}_{zz} = \dot{\varepsilon}_0 \cdot t_g \cdot \frac{1 - \exp\left(-s \cdot {}^{t_0}/_{t_g}\right)}{s^2}$$

Finally:

$$\widetilde{F(s)} = \tilde{\varepsilon}_{zz} \cdot \frac{C_1 I_0[\sqrt{s}] - C_2 \cdot C_0 \cdot \frac{I_1[\sqrt{s}]}{\sqrt{s}}}{I_0[\sqrt{s}] - C_0 \cdot \frac{I_1[\sqrt{s}]}{\sqrt{s}}}$$

Inversion (Time) Solution

$$f(t) = E_3 \dot{\varepsilon}_0 t + E_1 \dot{\varepsilon}_0 t_g \Delta_3 \left(\frac{1}{8} - \sum \frac{\exp\left(-\frac{\alpha_n^2 t}{t_g}\right)}{\alpha_n^2 \left[\Delta_2^2 \alpha_n^2 - \frac{\Delta_1}{1 + \nu_{21}}\right]} \right) \quad \text{if } t<t_0$$

$$f(t) = E_3 \dot{\varepsilon}_0 t_0 - E_1 \dot{\varepsilon}_0 t_g \Delta_3 \left(\sum_{\alpha_n^2} \frac{\exp\left(-\frac{\alpha_n^2 t}{t_g}\right) - \exp\left(-\frac{\alpha_n^2 (t - t_0)}{t_g}\right)}{\alpha_n^2 \left[\Delta_2^2 \alpha_n^2 - \frac{\Delta_1}{1 + \nu_{21}}\right]} \right) \quad \text{if } t \geq t_0$$

Limit in Time Case

$$\lim_{t\to\infty} f(t) = E_3 \dot{\varepsilon}_0 t_0 - E_1 \dot{\varepsilon}_0 t_g \Delta_3 \lim_{t\to\infty} \left(\sum_{\alpha_n^2} \frac{\exp\left(-\frac{\alpha_n^2 t}{t_g}\right) - \exp\left(-\frac{\alpha_n^2 (t - t_0)}{t_g}\right)}{\alpha_n^2 \left[\Delta_2^2 \alpha_n^2 - \frac{\Delta_1}{1 + \nu_{21}}\right]} \right)$$

$$= E_3 \dot{\varepsilon}_0 t_0 - E_1 \dot{\varepsilon}_0 t_g \Delta_3 \lim_{t\to\infty} \left(\sum_{\alpha_n^2} \frac{0 - 0}{\alpha_n^2 \left[\Delta_2^2 \alpha_n^2 - \frac{\Delta_1}{1 + \nu_{21}}\right]} \right) = E_3 \dot{\varepsilon}_0 t_0$$

Variable Manipulation to Help Laplace Limit Derivation

$$C_{11}-C_{12}=\frac{E_1\cdot\left(1-v_{21}-2v_{31}^2\frac{E_1}{E_3}\right)}{\left(1+v_{21}\right)\cdot\Delta_1}=\frac{E_1\Delta_1}{\left(1+v_{21}\right)\cdot\Delta_1}=\frac{E_1}{1+v_{21}}$$

$$C_{11}+C_{12}=\frac{E_1\cdot(1+v_{21})}{\left(1+v_{21}\right)\cdot\Delta_1}=\frac{E_1}{\Delta_1}$$

$$C_{11}+C_{12}-4C_{13}=\frac{E_1}{\Delta_1}-4\cdot\frac{E_1v_{31}}{\Delta_1}=\frac{E_1\cdot(1-4v_{31})}{\Delta_1}=\frac{E_1\cdot(1+v_{21})(1-4v_{31})}{\left(1+v_{21}\right)\cdot\Delta_1}$$

$$\begin{aligned}C_{11}+C_{12}-4C_{13}+2C_{33}&=\frac{E_1\cdot(1-4v_{31})}{\Delta_1}+2\cdot\frac{\Delta_1E_3+2v_{31}^2E_1}{\Delta_1}\\&=\frac{E_1\cdot(1-4v_{31})+2\Delta_1E_3+4v_{31}^2E_1}{\Delta_1}=\frac{E_1\cdot(1-4v_{31}+4v_{31}^2)+2\Delta_1E_3}{\Delta_1}\end{aligned}$$

$$\begin{aligned}C_0C_2&=\frac{C_{11}-C_{12}}{C_{11}}\cdot 2\frac{C_{33}\;(C_{11}-C_{12})+C_{11}(C_{11}+C_{12}-4C_{13})+2C_{13}^2}{(C_{11}-C_{12})^2}\\&=2\cdot\frac{C_{33}\;(C_{11}-C_{12})+C_{11}(C_{11}+C_{12}-4C_{13})+2C_{13}^2}{C_{11}(C_{11}-C_{12})}\\&=\frac{2C_{33}}{C_{11}}+\frac{2(C_{11}+C_{12}-4C_{13})}{C_{11}-C_{12}}+\frac{4C_{13}^2}{C_{11}(C_{11}-C_{12})}\end{aligned}$$

$$\begin{aligned}2C_1-C_0C_2&=2\frac{C_{11}+C_{12}-4C_{13}+2C_{33}}{C_{11}-C_{12}}-\left(\frac{2C_{33}}{C_{11}}+\frac{2(C_{11}+C_{12}-4C_{13})}{C_{11}-C_{12}}+\frac{4C_{13}^2}{C_{11}(C_{11}-C_{12})}\right)\\&=\frac{4C_{33}}{C_{11}-C_{12}}-\frac{2C_{33}}{C_{11}}-\frac{4C_{13}^2}{C_{11}(C_{11}-C_{12})}=\frac{4C_{33}C_{11}-2C_{33}(C_{11}-C_{12})-4C_{13}^2}{C_{11}(C_{11}-C_{12})}\\&=\frac{2C_{33}C_{11}+2C_{33}C_{12}-4C_{13}^2}{C_{11}(C_{11}-C_{12})}=\frac{2C_{33}(C_{11}+C_{12})-4C_{13}^2}{C_{11}(C_{11}-C_{12})}\end{aligned}$$

$$2-C_0=2-\frac{C_{11}-C_{12}}{C_{11}}=\frac{C_{11}+C_{12}}{C_{11}}=\frac{(C_{11}+C_{12})(C_{11}-C_{12})}{C_{11}(C_{11}-C_{12})}$$

Limit in Laplace Case

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s \cdot F(s) = \frac{C_1 - \frac{C_2 \cdot C_0}{2}}{1 - \frac{C_0}{2}} \cdot \lim_{s\to 0} s \cdot \tilde{\varepsilon}_{zz} = \frac{2 \cdot C_1 - C_2 \cdot C_0}{2 - C_0} \cdot \lim_{s\to 0} s \cdot \tilde{\varepsilon}_{zz}$$

$$\lim_{s\to 0} s \cdot \tilde{\varepsilon}_{zz} = \dot{\varepsilon}_0 \cdot t_g \cdot \lim_{s\to 0} \frac{1 - \exp\left(-s \cdot {t_0}/{t_g}\right)}{s} = \dot{\varepsilon}_0 t_g \cdot \lim_{s\to 0} \frac{\frac{t_0}{t_g} \exp\left(-s \cdot {t_0}/{t_g}\right)}{1} = \dot{\varepsilon}_0 t_g \cdot \frac{t_0}{t_g} = \dot{\varepsilon}_0 t_0$$

$$\begin{aligned}
\lim_{s\to 0} s \cdot F(s) &= \dot{\varepsilon}_0 t_0 \frac{2 \cdot C_1 - C_2 \cdot C_0}{2 - C_0} = \dot{\varepsilon}_0 t_0 \frac{2C_{33}(C_{11} + C_{12}) - 4C_{13}^2}{(C_{11} + C_{12})(C_{11} - C_{12})} \\
&= \dot{\varepsilon}_0 t_0 \frac{2\frac{\Delta_1 E_3 + 2v_{31}^2 E_1}{\Delta_1} \cdot \frac{E_1}{\Delta_1} - 4\left(\frac{E_1 v_{31}}{\Delta_1}\right)^2}{\frac{E_1}{\Delta_1} \cdot \frac{E_1}{1 + v_{21}}} \cdot \frac{\Delta_1^2 (1 + v_{21})}{\Delta_1^2 (1 + v_{21})} \\
&= \dot{\varepsilon}_0 t_0 (1 + v_{21}) \frac{2(\Delta_1 E_3 + 2v_{31}^2 E_1) \cdot E_1 - 4E_1^2 v_{31}^2}{E_1^2 \Delta_1} \\
&= 2\dot{\varepsilon}_0 t_0 (1 + v_{21}) \left(\Delta_1 \frac{E_3}{E_1} + 2v_{31}^2 - 2v_{31}^2\right) \div \Delta_1 = 2\dot{\varepsilon}_0 t_0 (1 + v_{21}) \frac{E_3}{E_1} \\
&= \dot{\varepsilon}_0 t_0 \cdot 2(1 + v_{21}) \frac{E_3}{E_1} = E_3 \dot{\varepsilon}_0 t_0 \cdot \frac{2(1 + v_{21})}{E_1}
\end{aligned}$$

Calculate the dimensional value as the solution provided was divided by (C11-C12)/2 to be nondimensional

$$\frac{C_{11} - C_{12}}{2} = \frac{E_1}{2(1 + v_{21})}$$

$$\lim_{s\to 0} s \cdot F(s) \cdot \frac{C_{11} - C_{12}}{2} = E_3 \dot{\varepsilon}_0 t_0 \cdot \frac{2(1 + v_{21})}{E_1} \cdot \frac{E_1}{2(1 + v_{21})} = E_3 \dot{\varepsilon}_0 t_0$$

Summary

$t \to \infty$ limit of Cohen's Laplace equation:

$$\lim_{s\to 0} s \cdot F(s) = \dot{\varepsilon}_0 t_0 \cdot 2(1 + v_{21}) \frac{E_3}{E_1}$$

This can be dimensionalized to:

$$\lim_{s\to 0} s \cdot F(s) \cdot \frac{C_{11} - C_{12}}{2} = E_3 \dot{\varepsilon}_0 t_0$$

$t \to \infty$ limit of Cohen's time-space equation:

$$\lim_{t \to \infty} f(t) = E_3 \dot{\varepsilon}_0 t_0$$

Therefore the inversion equation and the Laplace equation are not exactly equal as specified only because they were divided by different values to become nondimensional. Thus, their dimensional values are the same.