Evaluation of Cohen 1998's Biphasic Model

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1 Introduction

With regard to Cohen's biphasic model, we compare Cohen's analytic "solution" with our numerical results [1]. As there is a discrepancy between the two, we will consider the special case of $C_1 = C_2$ (pure ramped strain) and analytically derive the solution for this case in order to prove which of the two is incorrect.

1.1 Cohen's Laplace solution

$$\frac{\widetilde{F(s)}}{\tilde{\varepsilon}_{zz}(s)} = \frac{C_1 I_0\left[\sqrt{s}\right] - C_2 C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}}{I_0\left[\sqrt{s}\right] - C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}} \tag{1}$$

$$\tilde{\varepsilon}_{zz}(s) = \dot{\varepsilon}_0 \cdot t_g \cdot \frac{1 - \exp\left(-s\frac{t_0}{t_g}\right)}{s^2} \tag{2}$$

1.2 Cohen's proposed inversion (time) solution

1.2.1 Equation

$$f(t) = E_3\dot{\varepsilon}_0 t + E_1\dot{\varepsilon}_0 t_g \Delta_3 \left(\frac{1}{8} - \sum \frac{\exp\left(-\frac{\alpha_n^2 t}{t_g}\right)}{\alpha_n^2\left[\Delta_2^2\alpha_n^2 - \frac{\Delta_1}{1+v_{21}}\right]}\right) \text{ if t} < t_0 \tag{3}$$

$$f(t) = E_3\dot{\varepsilon}_0 t_0 - E_1\dot{\varepsilon}_0 t_g \Delta_3 \underbrace{\left(\sum_{\alpha_n^2} \frac{\exp\left(-\frac{\alpha_n^2 t}{t_g}\right) - \exp\left(-\frac{\alpha_n^2 (t-t_0)}{t_g}\right)}{\alpha_n^2\left[\Delta_2^2\alpha_n^2 - \frac{\Delta_1}{1+v_{21}}\right]}\right)}_{\text{Summation expression component}} \quad \text{if t} \geq t_0 \tag{4}$$

The above returns a **dimensionalized** value for $f(t)$ (kPa). To get a non-dimensionalized value (which is returned by the Laplace formula), divide by $\frac{C_{11}-C_{12}}{2}$
where

$$\Delta_1 = 1 - v_{21} - 2v_{31}^2\frac{E_1}{E_3} \tag{5}$$

$$\Delta_2 = \left(1 - v_{31}^2\frac{E_1}{E_3}\right) / \left(1 + v_{21}\right) \tag{6}$$

$$\Delta_3 = \left(1 - 2v_{31}^2\right)\frac{\Delta_2}{\Delta_1} \tag{7}$$

$$C_{11} = \frac{E_1 \cdot \left(1 - v_{31}^2\frac{E_1}{E_3}\right)}{(1+v_{21})\cdot\Delta_1} \tag{8}$$

$$C_{12} = \frac{E_1 \cdot \left(v_{21} + \frac{v_{31}^2 E_1}{E_3}\right)}{(1+v_{21})\cdot\Delta_1} \tag{9}$$

$$C_{13} = \frac{E_1 v_{31}}{\Delta_1} = \frac{E_1 v_{31}}{1 - v_{21} - 2v_{31}^2\frac{E_1}{E_3}} \tag{10}$$

$$C_{33} = E_3 \cdot \left(1 + \frac{2v_{31}^2\frac{E_1}{E_3}}{\Delta_1}\right) = \frac{E_3\left(\Delta_1 + 2v_{31}^2\frac{E_1}{E_3}\right)}{\Delta_1} = \frac{\Delta_1 E_3 + 2v_{31}^2 E_1}{\Delta_1} \tag{11}$$

$$\tag{12}$$

Blue color indicates a part of the equation that was derived by the other equations that were given.

$$C_0 = \frac{C_{11} - C_{12}}{C_{11}} \tag{13}$$

$$C_1 = \frac{C_{11} + C_{12} - 4C_{13} + 2C_{33}}{C_{11} - C_{12}} \tag{14}$$

$$C_2 = 2 \cdot \frac{C_{33}\left(C_{11} - C_{12}\right) + C_{11}\left(C_{11} + C_{12} - 4C_{13}\right) + 2C_{13}^2}{\left(C_{11} - C_{12}\right)^2} \tag{15}$$

1.2.2 Example parameters

$$
\begin{gathered}
E_1 = 8.5\text{kPa}, \quad E_3 = 19\text{kPa} \\
v_{21} = 0.75, \quad v_{31} = 0.24 \\
t_g = 40.62\text{sec}, \quad t_0/t_g = 0.25, \quad \dot{\varepsilon}_0 = 0.01\text{sec}^{-1}
\end{gathered}
$$

1.2.3 Units

C_{11}, C_{12}, C_{13}, C_{33} are in kPa (the same dimension as E_1 and E_3)
C_0, C_1, C_{12} are in non-dimensional units
$\Delta_0, \Delta_1, \Delta_2$ are in non-dimensional units

2 Using limit as $t \to \infty$

2.1 Motivation

This didn't disprove either solution. However, it helped me realize that you have to keep track of whether the resulting f(t) is dimensional or not i.e. whether or not is multipled by $\frac{C_{11}-C_{12}}{2}$.

2.2 Limit of Cohen's proposed time solution

$$
\lim_{t\to\infty} f(t) = E_3\dot{\varepsilon}_0 t_0 - E_1\dot{\varepsilon}_0 t_g \Delta_3 \lim_{t\to\infty}\left(\sum_{\alpha_n^2} \frac{\exp\left(-\frac{\alpha_n^2 t}{t_g}\right) - \exp\left(-\frac{\alpha_n^2 (t-t_0)}{t_g}\right)}{\alpha_n^2\left[\Delta_2^2\alpha_n^2 - \frac{\Delta_1}{1+v_{21}}\right]}\right) \tag{16}
$$

$$
= E_3\dot{\varepsilon}_0 t_0 - E_1\dot{\varepsilon}_0 t_g \Delta_3 \lim_{t\to\infty}\left(\sum_{\alpha_n^2} \frac{0-0}{\alpha_n^2\left[\Delta_2^2\alpha_n^2 - \frac{\Delta_1}{1+v_{21}}\right]}\right) \tag{17}
$$

$$
= E_3\dot{\varepsilon}_0 t_0 \tag{18}
$$

2.3 Manipulation of variables to help the Laplace limit derivation

$$C_{11} - C_{12} = \frac{E_1 \cdot \left(1 - \upsilon_{21} - 2\upsilon_{31}^2 \frac{E_1}{E_3}\right)}{(1+\upsilon_{21}) \cdot \Delta_1} = \frac{E_1 \Delta_1}{(1+\upsilon_{21}) \cdot \Delta_1} \tag{19}$$

$$= \frac{E_1}{1+\upsilon_{21}} \tag{20}$$

$$C_{11} + C_{12} = \frac{E_1 \cdot (1+\upsilon_{21})}{(1+\upsilon_{21}) \cdot \Delta_1} = \frac{E_1}{\Delta_1} \tag{21}$$

$$C_{11} + C_{12} - 4C_{13} = \frac{E_1}{\Delta_1} - 4 \cdot \frac{E_1 \upsilon_{31}}{\Delta_1} = \frac{E_1 \cdot (1 - 4\upsilon_{31})}{\Delta_1} = \frac{E_1 \cdot (1+\upsilon_{21})(1-4\upsilon_{31})}{(1+\upsilon_{21}) \cdot \Delta_1} \tag{22}$$

$$C_{11} + C_{12} - 4C_{13} + 2C_{33} = \frac{E_1 \cdot (1 - 4\upsilon_{31})}{\Delta_1} + 2 \cdot \frac{\Delta_1 E_3 + 2\upsilon_{31}^2 E_1}{\Delta_1} \tag{23}$$

$$= \frac{E_1 \cdot (1 - 4\upsilon_{31}) + 2\Delta_1 E_3 + 4\upsilon_{31}^2 E_1}{\Delta_1} = \frac{E_1 \cdot \left(1 - 4\upsilon_{31} + 4\upsilon_{31}^2\right) + 2\Delta_1 E_3}{\Delta_1} \tag{24}$$

$$C_0 C_2 = \frac{C_{11} - C_{12}}{C_{11}} \cdot 2 \frac{C_{33} \quad (C_{11} - C_{12}) + C_{11}(C_{11} + C_{12} - 4C_{13}) + 2C_{13}^2}{(C_{11} - C_{12})^2} \tag{25}$$

$$= 2 \cdot \frac{C_{33} \quad (C_{11} - C_{12}) + C_{11}(C_{11} + C_{12} - 4C_{13}) + 2C_{13}^2}{C_{11}(C_{11} - C_{12})} \tag{26}$$

$$= \frac{2C_{33}}{C_{11}} + \frac{2(C_{11} + C_{12} - 4C_{13})}{C_{11} - C_{12}} + \frac{4C_{13}^2}{C_{11}(C_{11} - C_{12})} \tag{27}$$

$$2C_1 - C_0 C_2 = 2\frac{C_{11} + C_{12} - 4C_{13} + 2C_{33}}{C_{11} - C_{12}} - \left(\frac{2C_{33}}{C_{11}} + \frac{2(C_{11} + C_{12} - 4C_{13})}{C_{11} - C_{12}} + \frac{4C_{13}^2}{C_{11}(C_{11} - C_{12})}\right) \tag{28}$$

$$= \frac{4C_{33}}{C_{11} - C_{12}} - \frac{2C_{33}}{C_{11}} - \frac{4C_{13}^2}{C_{11}(C_{11} - C_{12})} \tag{29}$$

$$= \frac{4C_{33}C_{11} - 2C_{33}(C_{11} - C_{12}) - 4C_{13}^2}{C_{11}(C_{11} - C_{12})} \tag{30}$$

$$= \frac{2C_{33}C_{11} + 2C_{33}C_{12} - 4C_{13}^2}{C_{11}(C_{11} - C_{12})} = \frac{2C_{33}(C_{11} + C_{12}) - 4C_{13}^2}{C_{11}(C_{11} - C_{12})} \tag{31}$$

$$2 - C_0 = 2 - \frac{C_{11} - C_{12}}{C_{11}} = \frac{C_{11} + C_{12}}{C_{11}} = \frac{(C_{11} + C_{12})(C_{11} - C_{12})}{C_{11}(C_{11} - C_{12})} \tag{32}$$

2.4 Limit of Laplace solution

$$\begin{aligned}
\lim_{t\to\infty} f(t) &= \lim_{s\to 0} s\cdot F(s) && (33)\\
&= \frac{C_1 - \frac{C_2\cdot C_0}{2}}{1-\frac{C_0}{2}} \cdot \lim_{s\to 0} s\cdot \tilde{\varepsilon}_{zz} && (34)\\
&= \frac{2C_1 - C_2\cdot C_0}{2-C_0} \cdot \lim_{s\to 0} s\cdot \tilde{\varepsilon}_{zz} && (35)\\
\lim_{s\to 0} s\cdot \tilde{\varepsilon}_{zz} &= \dot{\varepsilon}_0 \cdot t_g \cdot \lim_{s\to 0} \frac{1-\exp\left(-s\cdot t_0/t_g\right)}{s} && (36)\\
&= \dot{\varepsilon}_0 t_g \cdot \lim_{s\to 0} \frac{\frac{t_0}{t_g}\exp\left(-s\cdot t_0/t_g\right)}{1} && (37)\\
&= \dot{\varepsilon}_0 t_g \cdot \frac{t_0}{t_g} = \dot{\varepsilon}_0 t_0 && (38)\\
\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\cdot F(s) &= \frac{2C_1 - C_2\cdot C_0}{2-C_0} \cdot \dot{\varepsilon}_0 t_0 && (39)\\
&= \dot{\varepsilon}_0 t_0 \frac{2C_{33}\left(C_{11}+C_{12}\right) - 4C_{13}^2}{\left(C_{11}+C_{12}\right)\left(C_{11}-C_{12}\right)} && (40)\\
&= \dot{\varepsilon}_0 t_0 \frac{2\frac{\Delta_1 E_3 + 2v_{31}^2 E_1}{\Delta_1}\cdot \frac{E_1}{\Delta_1} - 4\left(\frac{E_1 v_{31}}{\Delta_1}\right)^2}{\frac{E_1}{\Delta_1}\cdot\frac{E_1}{1+v_{21}}\left(1+v_{21}\right)} && (41)\\
&= \dot{\varepsilon}_0 t_0 \left(1+v_{21}\right) \frac{2\left(\Delta_1 E_3 + 2v_{31}^2 E_1\right)\cdot E_1 - 4E_1^2 v_{31}^2}{E_1^2 \Delta_1} && (42)\\
&= 2\dot{\varepsilon}_0 t_0 \left(1+v_{21}\right)\left(\Delta_1 \frac{E_3}{E_1} + 2v_{31}^2 - 2v_{31}^2\right) \div \Delta_1 = 2\dot{\varepsilon}_0 t_0 \left(1+v_{21}\right)\frac{E_3}{E_1} && (43)\\
&= \dot{\varepsilon}_0 t_0 \cdot 2\left(1+v_{21}\right)\frac{E_3}{E_1} = E_3 \dot{\varepsilon}_0 t_0 \cdot \frac{2\left(1+v_{21}\right)}{E_1} && (44)
\end{aligned}$$

Calculate the dimensional value as the solution provided was divided by (C11-C12)/2 to be nondimensional

$$\begin{aligned}
\frac{C_{11}-C_{12}}{2} &= \frac{E_1}{2\left(1+v_{21}\right)} && (45)\\
\lim_{s\to 0} s\cdot F(s)\cdot \frac{C_{11}-C_{12}}{2} &= E_3\dot{\varepsilon}_0 t_0 \cdot \frac{2\left(1+v_{21}\right)}{E_1}\cdot \frac{E_1}{2\left(1+v_{21}\right)} && (46)\\
&= E_3 \dot{\varepsilon}_0 t_0 && (47)
\end{aligned}$$

2.5 Summary

The summary is as follows:
$t \to \infty$ limit of Cohen's Laplace equation:

$$\lim_{s\to 0} s\cdot F(s) = \dot{\varepsilon}_0 t_0 \cdot 2\left(1+v_{21}\right)\frac{E_3}{E_1}$$

This can be dimensionalized to:

$$\lim_{s\to 0} s\cdot F(s)\cdot \frac{C_{11}-C_{12}}{2} = E_3\dot{\varepsilon}_0 t_0$$
$$\lim_{t\to\infty} f(t) = E_3\dot{\varepsilon}_0 t_0$$

Therefore the inversion equation and the Laplace equation are not exactly equal as specified only because they were divided by different values to become nondimensional. Thus, their dimensional values are the same.

3 Special case: pure ramped strain

3.1 Proof: $C_1 = C_2 \iff f(t) = C_1\varepsilon_{zz}(t)$

When $C_1 = C_2$, then:

$$\frac{\widetilde{F(s)}}{\tilde{\varepsilon}_{zz}(s)} = \frac{C_1 I_0\left[\sqrt{s}\right] - C_2 C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}}{I_0\left[\sqrt{s}\right] - C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}} \tag{48}$$

$$= C_1 \frac{I_0\left[\sqrt{s}\right] - C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}}{I_0\left[\sqrt{s}\right] - C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}} \tag{49}$$

$$= C_1 \tag{50}$$

$$\textbf{if} \qquad \underbrace{s \neq 0}_{\text{Already known invalid point}} \qquad \textbf{and} \qquad \underbrace{I_0\left[\sqrt{s}\right] \neq C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}}_{\text{Tricky condition, will return to it}} \tag{51}$$

If those conditions are satisfied, then:

$$f(t) = \mathscr{L}^{-1}\left\{C_1 \tilde{\varepsilon}_{zz}(s)\right\} \tag{52}$$

$$= C_1 \mathscr{L}^{-1}\left\{\dot{\varepsilon}_0 t_g \left(\frac{1 - \exp\left(-s\frac{t_0}{t_g}\right)}{s^2}\right)\right\} \tag{53}$$

$$= C_1 \varepsilon_{zz}(t) \tag{54}$$

It will be proven in later sections the special condition of $C_1 = C_2$ is satisfied only when $\nu_{31} = 0.5$. See in Figure 1 (corresponding to Table 1) that the numerical solution is a perfect ramp as expected at this ν_{31} value, unlike Cohen's analytic solution.

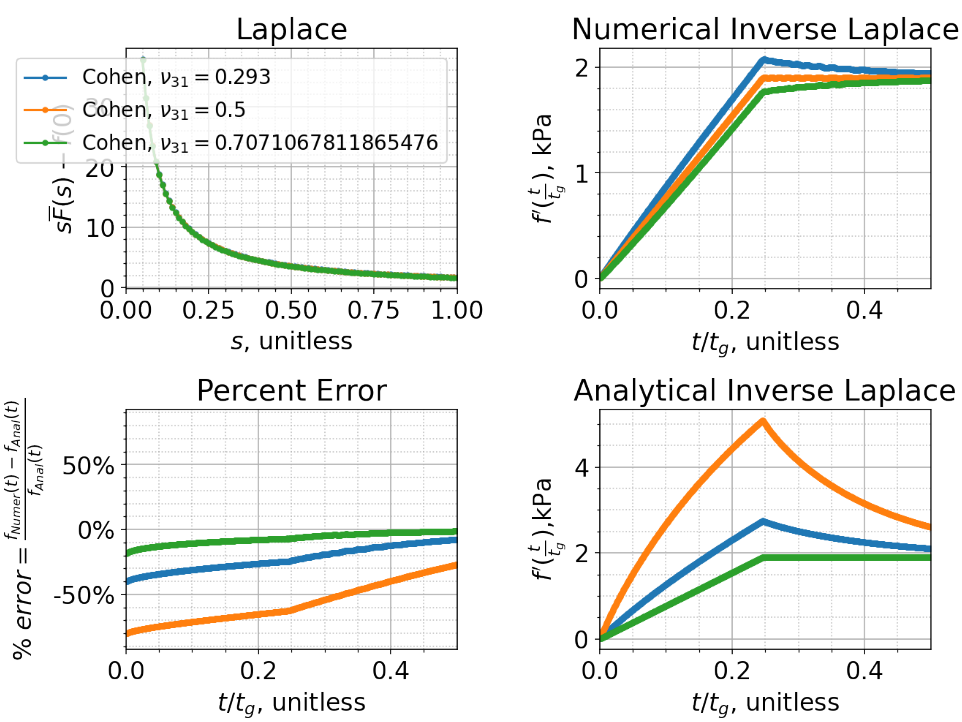

Figure 1: This is our numerical solution. Note that the numerical solution is a perfect ramp at $\nu_{31} = 0.5$ as expected, unlike Cohen's proposed analytic solution.

Table 1: Parameters for lines plotted in Figure 1

	t_0/t_g	t_g	$\dot{\varepsilon}$	E_1	E_3	ν_{21}	ν_{31}	Δ_1	Δ_2	Δ_3	C_{11}	C_{12}	C_{13}	C_{33}	C_0	C_1	C_2
Line #1	0.246184	40.62	0.01	8.5	19	0.75	0.293	0.173188	0.549482	2.628	26.968415	22.111272	14.38	27.426	0.18	9.555	27.055
Line #2	0.246184	40.62	0.01	8.5	19	0.75	0.500	0.026316	0.507519	9.6428	163.928	159.071429	161.5	180.5	0.0296	7.823	7.823
Line #3	0.246184	40.62	0.01	8.5	19	0.75	0.707	-0.197368	0.443609	4e-16	-19.104762	-23.961905	-30.45	-24.067	-0.254	6.302	19.790

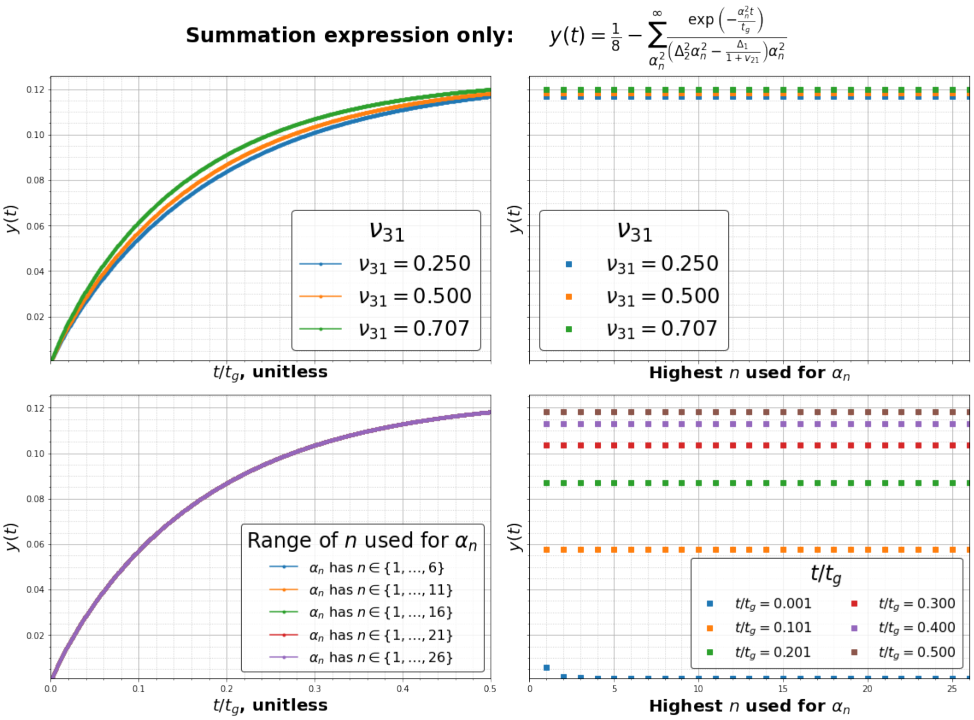

Figure 2: This is the summation expression component of Cohen's analytic solution. Note that at $\nu_{31} = 0.5$, the value is not uniformly 0, and thus $f(t)$ is not proportional to $\varepsilon(t)$.

3.2 Proof #1 that $C_1 = C_2 \iff \nu_{31} = 0.5$

Set $C_1 = C_2$:

$$\overbrace{\frac{C_{11} + C_{12} - 4C_{13} + 2C_{33}}{C_{11} - C_{12}}}^{C_1} = \overbrace{2\frac{C_{33}\left(C_{11} - C_{12}\right) + C_{11}\left(C_{11} + C_{12} - 4C_{13}\right) + 2C_{13}^2}{\left(C_{11} - C_{12}\right)^2}}^{C_2} \tag{55}$$

$$\left(C_{11} - C_{12}\right)\left(C_{11} + C_{12} - 4C_{13} + 2C_{33}\right) = 2C_{33}\left(C_{11} - C_{12}\right) + 2C_{11}\left(C_{11} + C_{12} - 4C_{13}\right) + 4C_{13}^2 \tag{56}$$

$$\left(C_{11} - C_{12}\right)\left(C_{11} + C_{12} - 4C_{13}\right) = 2C_{11}\left(C_{11} + C_{12} - 4C_{13}\right) + 4C_{13}^2 \tag{57}$$

$$-\left(C_{11} + C_{12}\right)\left(C_{11} + C_{12} - 4C_{13}\right) = 4C_{13}^2 \tag{58}$$

$$-\frac{E_1}{\Delta_1} \cdot \frac{E_1\left(1 - 4\nu_{31}\right)}{\Delta_1} = 4\left(\frac{E_1\nu_{31}}{\Delta_1}\right)^2 \tag{59}$$

$$\left(\frac{E_1}{\Delta_1}\right)^2\left(4\nu_{31} - 1\right) = 4\nu_{31}^2\left(\frac{E_1}{\Delta_1}\right)^2 \tag{60}$$

$$\left(4\nu_{31}^2 - 4\nu_{31} + 1\right)\left(\frac{E_1}{\Delta_1}\right)^2 = 0 \tag{61}$$

$$\left(2\nu_{31} - 1\right)^2\left(\frac{E_1}{\Delta_1}\right)^2 = 0 \tag{62}$$

$$\tag{63}$$

We know $E_1 = 0$ is impossible because then $C_{11} = C_{12} = 0$. Thus, only solution remains: $2\nu_{31} - 1 = 0$. Therefore:

$$\boxed{\nu_{31} = \frac{1}{2} \iff C_1 = C_2}$$

3.3 Proof #2 that $C_1 = C_2 \iff \nu_{31} = 0.5$

This is an alternative proof to Proof #1.
Strategy: Derive C_1 - C_2 directly in terms of constants (ν_{21}, ν_{31}, E_1, E_3), then set them equal each other.

3.3.1 Derive C_{11}, C_{12}, C_{13}, and C_{33} expressions directly in terms of ν_x and E_x parameters

$$C_{11} = \frac{E_1 \cdot \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right)}{(1+\nu_{21}) \cdot \Delta_1} = \frac{E_1 \cdot \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right)}{(1+\nu_{21}) \left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)} \tag{64}$$

$$C_{12} = \frac{E_1 \cdot \left(\nu_{21} + \nu_{31}^2 \frac{E_1}{E_3}\right)}{(1+\nu_{21}) \cdot \Delta_1} = \frac{E_1 \cdot \left(\nu_{21} + \nu_{31}^2 \frac{E_1}{E_3}\right)}{(1+\nu_{21}) \left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)} \tag{65}$$

$$C_{13} = \frac{E_1 \nu_{31}}{\Delta_1} = \frac{E_1 \nu_{31}}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{66}$$

$$C_{33} = E_3 \cdot \left(1 + \frac{2\nu_{31}^2 \frac{E_1}{E_3}}{\Delta_1}\right) = \frac{\Delta_1 E_3 + 2\nu_{31}^2 E_1}{\Delta_1} \tag{67}$$

$$= E_3 \cdot \left(1 + \frac{2\nu_{31}^2 \frac{E_1}{E_3}}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}}\right) = \frac{\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right) E_3 + 2\nu_{31}^2 E_1}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{68}$$

$$= \frac{E_3 \cdot (1 - \nu_{21})}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{69}$$

$$\tag{70}$$

3.3.2 Derive combinations of C_{11}, C_{12}, etc expressions

Motivation: This will make future steps (e.g. deriving C_0 in terms of v_x and E_x parameters) easier.

$$C_{11} - C_{12} = \frac{E_1}{1+\nu_{21}} \tag{71}$$

$$\tag{72}$$

$$C_{11} + C_{12} = \frac{E_1}{\Delta_1} = \frac{E_1}{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}} \tag{73}$$

$$\tag{74}$$

$$C_{13} = \frac{E_1\nu_{31}}{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}} \tag{75}$$

$$\tag{76}$$

$$C_{33} = \frac{E_3\cdot(1-\nu_{21})}{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}} \tag{77}$$

$$\tag{78}$$

$$\begin{aligned} C_{11}+C_{12}-4C_{13} &= \frac{E_1\,(1-4\nu_{31})}{\Delta_1} && (79)\\ &= \frac{E_1\cdot(1-4\nu_{31})}{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}} && (80) \end{aligned}$$

$$\tag{81}$$

$$\begin{aligned} C_{11}+C_{12}-4C_{13}+2C_{33} &= \frac{E_1\,(1-4\nu_{31})+2E_3\,(1-\nu_{21})}{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}} && (82)\\ &= \frac{E_1+2E_3-4E_1\nu_{31}-2E_3\nu_{21}}{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}} && (83) \end{aligned}$$

$$\tag{84}$$

3.3.3 Getting C_0, C_1, and C_2 directly in terms of ν_x and E_x parameters

$$C_0 = \frac{C_{11} - C_{12}}{C_{11}} = \frac{\Delta_1}{1 - \nu_{31}^2 \frac{E_1}{E_3}} = \frac{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}}{1 - \nu_{31}^2 \frac{E_1}{E_3}} \tag{85}$$

$$\tag{86}$$

$$C_1 = \frac{C_{11} + C_{12} - 4C_{13} + 2C_{33}}{C_{11} - C_{12}} \tag{87}$$

$$= \frac{(1 + \nu_{21}) \left(1 - 4\nu_{31} + 2\frac{E_3}{E_1} (1 - \nu_{21})\right)}{\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)} \tag{88}$$

$$\tag{89}$$

$$C_2 = 2\frac{C_{33} (C_{11} - C_{12}) + C_{11} (C_{11} + C_{12} - 4C_{13}) + 2C_{13}^2}{(C_{11} - C_{12})^2} \tag{90}$$

$$= 2\frac{\left(\frac{E_3 \cdot (1-\nu_{21})}{1-\nu_{21}-2\nu_{31}^2 \frac{E_1}{E_3}}\right) \left(\frac{E_1}{1+\nu_{21}}\right) + \left(\frac{E_1 \cdot \left(1-\nu_{31}^2 \frac{E_1}{E_3}\right)}{(1+\nu_{21})\left(1-\nu_{21}-2\nu_{31}^2 \frac{E_1}{E_3}\right)}\right) \left(\frac{E_1 \cdot (1-4\nu_{31})}{1-\nu_{21}-2\nu_{31}^2 \frac{E_1}{E_3}}\right) + 2\left(\frac{E_1 \nu_{31}}{1-\nu_{21}-2\nu_{31}^2 \frac{E_1}{E_3}}\right)^2}{\left(\frac{E_1}{1+\nu_{21}}\right)^2} \tag{91}$$

$$= 2\frac{E_3 E_1 (1 - \nu_{21}) \left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right) + E_1^2 \cdot \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right) (1 - 4\nu_{31}) + 2E_1^2 \nu_{31}^2 (1 + \nu_{21})}{\left(\frac{E_1}{1+\nu_{21}}\right)^2 \left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)^2 (1 + \nu_{21})} \tag{92}$$

$$= 2\frac{\frac{E_3}{E_1} (1 - \nu_{21}) \left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right) + \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right) (1 - 4\nu_{31}) + 2\nu_{31}^2 (1 + \nu_{21})}{\frac{\left(1-\nu_{21}-2\nu_{31}^2 \frac{E_1}{E_3}\right)^2}{1+\nu_{21}}} \tag{93}$$

$$= 2\frac{\frac{E_3}{E_1} \left(1 - \nu_{21}^2\right) \left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right) + \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right) (1 - 4\nu_{31}) (1 + \nu_{21}) + 2\nu_{31}^2 (1 + \nu_{21})^2}{\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)^2} \tag{94}$$

$$= 2 (1 + \nu_{21}) \frac{\frac{E_3}{E_1} (1 - \nu_{21}) \left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right) + \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right) (1 - 4\nu_{31}) + 2\nu_{31}^2 (1 + \nu_{21})}{\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)^2} \tag{95}$$

$$\tag{96}$$

3.3.4 Solve for C_1=C_2

Knowing:

$$C_0 = \frac{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}}{1 - \nu_{31}^2 \frac{E_1}{E_3}} \tag{97}$$

$$\tag{98}$$

$$C_1 = \frac{(1 + \nu_{21}) \left(1 - 4\nu_{31} + 2\frac{E_3}{E_1} (1 - \nu_{21})\right)}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{99}$$

$$\tag{100}$$

$$C_2 = 2\frac{\frac{E_3}{E_1} \left(1 - \nu_{21}^2\right) \left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right) + \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right) (1 - 4\nu_{31}) (1 + \nu_{21}) + 2\nu_{31}^2 (1 + \nu_{21})^2}{\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)^2} \tag{101}$$

$$\tag{102}$$

We want to solve for C_1=C_2:

$$\overbrace{\frac{(1+\nu_{21})\left(1-4\nu_{31}+2\frac{E_3}{E_1}(1-\nu_{21})\right)}{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}}}^{C_1} = \overbrace{2\frac{\frac{E_3}{E_1}(1-\nu_{21}^2)\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)+\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)(1-4\nu_{31})(1+\nu_{21})+2\nu_{31}^2(1+\nu_{21})^2}{\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)^2}}^{C_2} \tag{A}$$

Multiply both sides by $\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)^2$

$$\begin{aligned}(1+\nu_{21})\left(1-4\nu_{31}+2\frac{E_3}{E_1}(1-\nu_{21})\right)\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right) =&2\frac{E_3}{E_1}\left(1-\nu_{21}^2\right)\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)\\ &+2\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)(1-4\nu_{31})(1+\nu_{21})\\ &+4\nu_{31}^2(1+\nu_{21})^2\end{aligned}$$

Divide both sides by $1+\nu_{21}$

$$\begin{aligned}\left(1-4\nu_{31}+2\frac{E_3}{E_1}(1-\nu_{21})\right)\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right) =&2\frac{E_3}{E_1}(1-\nu_{21})\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)\\ &+2\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)(1-4\nu_{31})+4\nu_{31}^2(1+\nu_{21})\end{aligned} \tag{B}$$

$$(1-4\nu_{31})\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right) = \underbrace{2\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)(1-4\nu_{31})}_{\text{Move to lefthand side}}+4\nu_{31}^2(1+\nu_{21}) \tag{C}$$

$$(1-4\nu_{31})\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}-2\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)\right) =4\nu_{31}^2(1+\nu_{21}) \tag{D}$$

$$(1-4\nu_{31})(-1-\nu_{21}) =4\nu_{31}^2(1+\nu_{21}) \tag{E}$$

$$-(1-4\nu_{31})(1+\nu_{21}) = 4\nu_{31}^2(1+\nu_{21}) \tag{F}$$

$$-\left(1-4\nu_{31}+4\nu_{31}^2\right)(1+\nu_{21}) =0 \tag{G}$$

$$-\left(4\nu_{31}^2-4\nu_{31}+1\right)(1+\nu_{21}) =0 \tag{H}$$

$$-(2\nu_{31}-1)^2(1+\nu_{21}) =0 \tag{I}$$

$$-(2\nu_{31}-1)^2(1+\nu_{21}) = (C_1-C_2) \qquad \times\overbrace{\frac{\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)^2}{1+\nu_{21}}}^{\text{Multiplications/divisions done to get to this step}}$$

$$-\frac{(2\nu_{31}-1)^2(1+\nu_{21})^2}{\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)^2} =C_1-C_2$$

$$\boxed{C_2-C_1 = \left(\frac{(2\nu_{31}-1)(1+\nu_{21})}{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}}\right)^2} \tag{J}$$

Since $\nu_{21}=-1 \implies C_{11}$ and C_{12} are undefined, we know $C_{11}=C_{12}$ has only one solution remaining:

$$\boxed{C_1=C_2 \iff \nu_{31}=0.5 \qquad \text{assuming } 1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\neq 0}$$

References

[1] B Cohen, WM Lai, and VC Mow. “A transversely isotropic biphasic model for unconfined compression of growth plate and chondroepiphysis”. In: (1998).