Evaluation of Cohen 1998's Biphasic Model

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1 Introduction

Taking Cohen's biphasic model and comparing Cohen's analytic solution with our numerical results [1]. As there is a discrepancy, take the case of $C_1 = C_2$ to prove which is correct.

2 Proof: $C_1 = C_2 \iff f(t) = C_1\varepsilon_{zz}(t)$

If $C_1 = C_2$ (happens if $\nu_{31} = 0.5$), then:

$$\frac{\widetilde{F(s)}}{\tilde{\varepsilon}_{zz}(s)} = \frac{C_1 I_0\left[\sqrt{s}\right] - C_2 C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}}{I_0\left[\sqrt{s}\right] - C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}} \tag{1}$$

$$= C_1 \frac{I_0\left[\sqrt{s}\right] - C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}}{I_0\left[\sqrt{s}\right] - C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}} \tag{2}$$

$$= C_1 \tag{3}$$

$$\textbf{if} \qquad \underbrace{s \neq 0}_{\text{Already known invalid point}} \qquad \textbf{and} \qquad \underbrace{I_0\left[\sqrt{s}\right] \neq C_0 \frac{I_1\left[\sqrt{s}\right]}{\sqrt{s}}}_{\text{Tricky condition, will return to it}} \tag{4}$$

If those conditions are satisfied, then:

$$f(t) = C_1 \tilde{\varepsilon}_{zz}(s) \tag{5}$$

$$= C_1 \dot{\varepsilon}_0 t_g \left(\frac{1 - \exp\left(-s\frac{t_0}{t_g}\right)}{s^2} \right) \tag{6}$$

$$= C_1 \varepsilon_{zz}(t) \tag{7}$$

It will be proven later that this is satisfied only with $\nu_{31} = 0.5$. See in Figure 1 (corresponding to 1) that the numerical solution is a perfect ramp as expected at this ν_{31} value, unlike Cohen's analytic solution.

Table 1: Parameters for lines plotted in Fig 1

	t_0/t_g	t_g	$\dot{\varepsilon}$	E_1	E_3	ν_{21}	ν_{31}	Δ_1	Δ_2	Δ_3	C_{11}	C_{12}	C_{13}	C_{33}	C_0	C_1	C_2
Line #1	0.246184	40.62	0.01	8.5	19	0.75	0.293	0.173188	0.549482	2.628	26.968415	22.111272	14.38	27.426	0.18	9.555	27.055
Line #2	0.246184	40.62	0.01	8.5	19	0.75	0.5	0.026316	0.507519	9.6428	163.928	159.071429	161.5	180.5	0.0296	7.823	7.823
Line #3	0.246184	40.62	0.01	8.5	19	0.75	0.707	-0.197368	0.443609	4e-16	-19.104762	-23.961905	-30.45	-24.067	-0.254	6.302	19.790

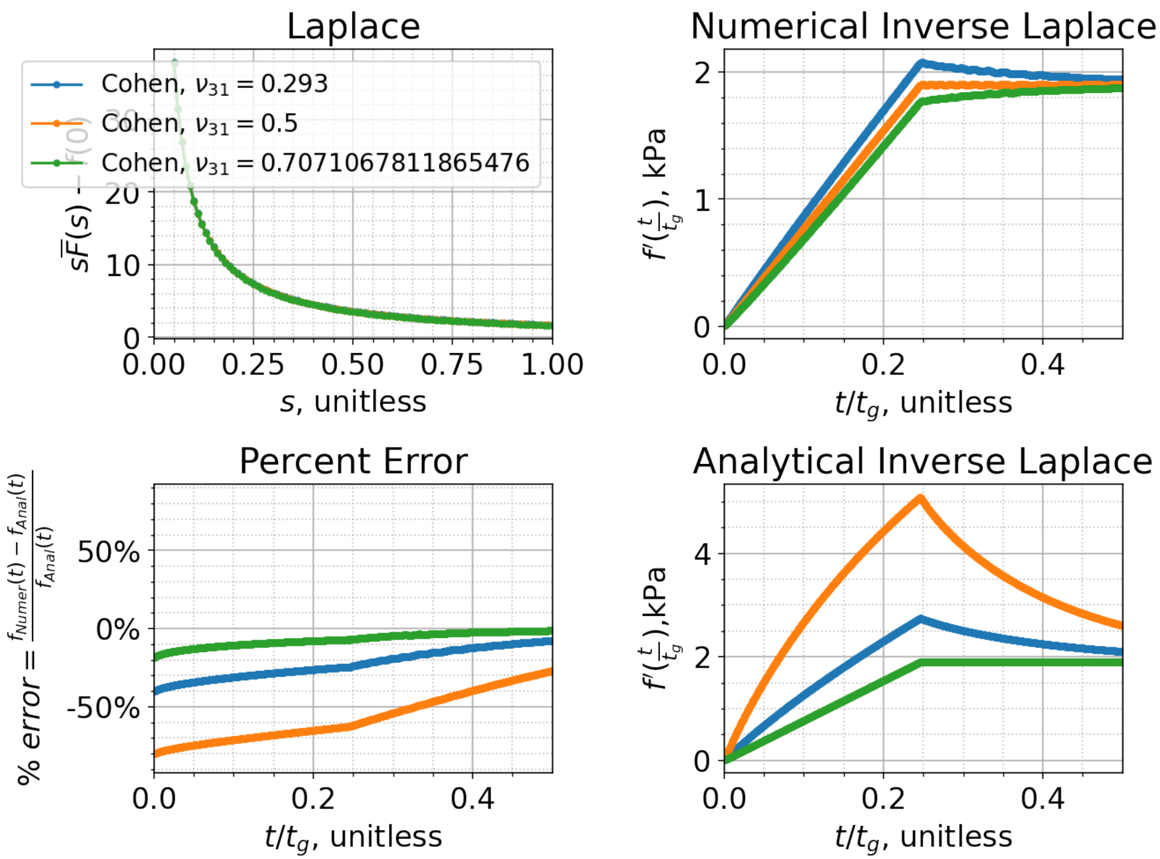

Figure 1: Note that the numerical solution is a perfect ramp at $\nu_{31} = 0.5$ as expected, unlike Cohen's proposed analytic solution.

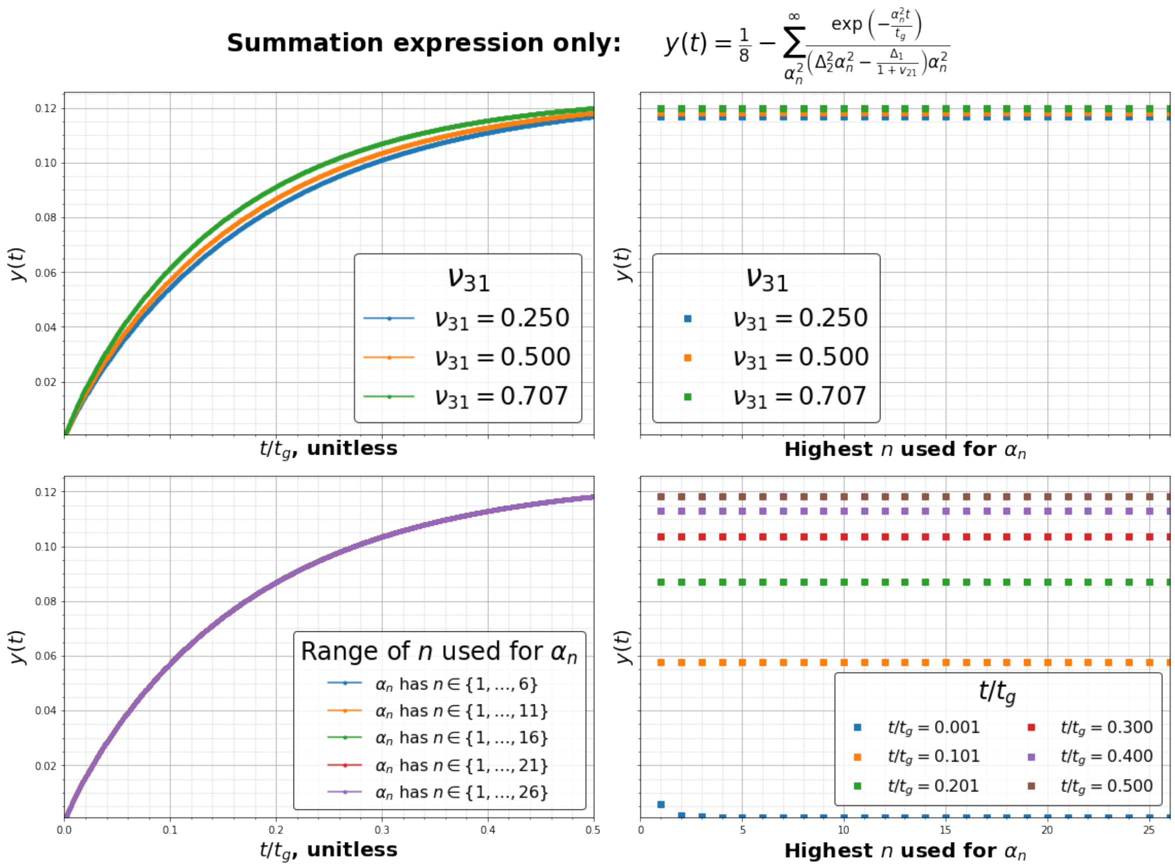

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Figure 2: Note that at $\nu_{31} = 0.5$, the summation expression is not uniformly 0, and thus $f(t)$ is not proportional to $varepsilon(t)$.

3 Proof #1 that $C_1 = C_2 \iff \nu_{31} = 0.5$

Set $C_1 = C_2$:

$$\overbrace{\frac{C_{11}+C_{12}-4C_{13}+2C_{33}}{C_{11}-C_{12}}}^{C_1} = 2\overbrace{\frac{C_{33}\left(C_{11}-C_{12}\right)+C_{11}\left(C_{11}+C_{12}-4C_{13}\right)+2C_{13}^2}{\left(C_{11}-C_{12}\right)^2}}^{C_2} \tag{8}$$

$$\left(C_{11}-C_{12}\right)\left(C_{11}+C_{12}-4C_{13}+2C_{33}\right) = 2C_{33}\left(C_{11}-C_{12}\right)+2C_{11}\left(C_{11}+C_{12}-4C_{13}\right)+4C_{13}^2 \tag{9}$$

$$\left(C_{11}-C_{12}\right)\left(C_{11}+C_{12}-4C_{13}\right) = 2C_{11}\left(C_{11}+C_{12}-4C_{13}\right)+4C_{13}^2 \tag{10}$$

$$-\left(C_{11}+C_{12}\right)\left(C_{11}+C_{12}-4C_{13}\right) = 4C_{13}^2 \tag{11}$$

$$-\frac{E_1}{\Delta_1}\cdot\frac{E_1\left(1-4\nu_{31}\right)}{\Delta_1} = 4\left(\frac{E_1\nu_{31}}{\Delta_1}\right)^2 \tag{12}$$

$$\left(\frac{E_1}{\Delta_1}\right)^2\left(4\nu_{31}-1\right) = 4\nu_{31}^2\left(\frac{E_1}{\Delta_1}\right)^2 \tag{13}$$

Knowing $\mathrm{E}_1 = 0$ is impossible as then $C_{11} = C_{12} = 0$
Thus, only one pair of solutions remains:

$$2\nu_{31} - 1 = 0 \tag{17}$$

$$\boxed{\nu_{31} = \frac{1}{2} \iff C_1 = C_2}$$

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4 Proof #2 that $C_1 = C_2 \iff \nu_{31} = 0.5$

Strategy: Get C_1 - C_2 directly in terms of constants (ν_{21}, ν_{31}, E_1, E_3), then set them equal each other.

4.1 Getting C_{11}, C_{12}, etc expressions directly in terms of parameters

$$C_{11} = \frac{E_1 \cdot \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right)}{(1+\nu_{21}) \cdot \Delta_1} \tag{18}$$

$$= \frac{E_1 \cdot \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right)}{(1+\nu_{21}) \left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)} \tag{19}$$

$$C_{12} = \frac{E_1 \cdot \left(\nu_{21} + \nu_{31}^2 \frac{E_1}{E_3}\right)}{(1+\nu_{21}) \cdot \Delta_1} \tag{20}$$

$$= \frac{E_1 \cdot \left(\nu_{21} + \nu_{31}^2 \frac{E_1}{E_3}\right)}{(1+\nu_{21}) \left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)} \tag{21}$$

$$C_{13} = \frac{E_1 \nu_{31}}{\Delta_1} \tag{22}$$

$$= \frac{E_1 \nu_{31}}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{23}$$

$$C_{33} = E_3 \cdot \left(1 + \frac{2\nu_{31}^2 \frac{E_1}{E_3}}{\Delta_1}\right) = \frac{\Delta_1 E_3 + 2\nu_{31}^2 E_1}{\Delta_1} \tag{24}$$

$$= E_3 \cdot \left(1 + \frac{2\nu_{31}^2 \frac{E_1}{E_3}}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}}\right) \tag{25}$$

$$= \frac{\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right) E_3 + 2\nu_{31}^2 E_1}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{26}$$

$$= \frac{E_3 \cdot (1 - \nu_{21})}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{27}$$

$$\tag{28}$$

4.2 Combinations of $\mathbf{C}_{11}, C_{12}, etcexpressionsg$

$$C_{11} - C_{12} = \frac{E_1}{1 + \nu_{21}} \tag{29}$$

$$\tag{30}$$

$$C_{11} + C_{12} = \frac{E_1}{\Delta_1} \tag{31}$$

$$= \frac{E_1}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{32}$$

$$\tag{33}$$

$$C_{13} = \frac{E_1 \nu_{31}}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{34}$$

$$\tag{35}$$

$$C_{33} = \frac{E_3 \cdot (1 - \nu_{21})}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{36}$$

$$\tag{37}$$

$$C_{11} + C_{12} - 4C_{13} = \frac{E_1 (1 - 4\nu_{31})}{\Delta_1} \tag{38}$$

$$= \frac{E_1 \cdot (1 - 4\nu_{31})}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{39}$$

$$\tag{40}$$

$$C_{11} + C_{12} - 4C_{13} + 2C_{33} = \frac{E_1 (1 - 4\nu_{31}) + 2E_3 (1 - \nu_{21})}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{41}$$

$$= \frac{E_1 + 2E_3 - 4E_1\nu_{31} - 2E_3\nu_{21}}{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}} \tag{42}$$

$$\tag{43}$$

4.3 Getting C_0, C_1, directly in terms of parameters

$$C_0 = \frac{C_{11} - C_{12}}{C_{11}} = \frac{\Delta_1}{1 - \nu_{31}^2 \frac{E_1}{E_3}} = \frac{1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}}{1 - \nu_{31}^2 \frac{E_1}{E_3}} \tag{44}$$

$$\tag{45}$$

$$C_1 = \frac{C_{11} + C_{12} - 4C_{13} + 2C_{33}}{C_{11} - C_{12}} \tag{46}$$

$$= \frac{(1 + \nu_{21}) \left(1 - 4\nu_{31} + 2\frac{E_3}{E_1}(1 - \nu_{21})\right)}{\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)} \tag{47}$$

$$\tag{48}$$

$$C_2 = 2\frac{C_{33}(C_{11} - C_{12}) + C_{11}(C_{11} + C_{12} - 4C_{13}) + 2C_{13}^2}{(C_{11} - C_{12})^2} \tag{49}$$

$$= 2\frac{\left(\frac{E_3 \cdot (1-\nu_{21})}{1-\nu_{21}-2\nu_{31}^2 \frac{E_1}{E_3}}\right)\left(\frac{E_1}{1+\nu_{21}}\right) + \left(\frac{E_1 \cdot \left(1-\nu_{31}^2 \frac{E_1}{E_3}\right)}{(1+\nu_{21})\left(1-\nu_{21}-2\nu_{31}^2 \frac{E_1}{E_3}\right)}\right)\left(\frac{E_1 \cdot (1-4\nu_{31})}{1-\nu_{21}-2\nu_{31}^2 \frac{E_1}{E_3}}\right) + 2\left(\frac{E_1 \nu_{31}}{1-\nu_{21}-2\nu_{31}^2 \frac{E_1}{E_3}}\right)^2}{\left(\frac{E_1}{1+\nu_{21}}\right)^2} \tag{50}$$

$$= 2\frac{E_3 E_1 (1 - \nu_{21})\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right) + E_1^2 \cdot \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right)(1 - 4\nu_{31}) + 2E_1^2 \nu_{31}^2 (1 + \nu_{21})}{\left(\frac{E_1}{1+\nu_{21}}\right)^2 \left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)^2 (1 + \nu_{21})} \tag{51}$$

$$= 2\frac{\frac{E_3}{E_1}(1 - \nu_{21})\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right) + \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right)(1 - 4\nu_{31}) + 2\nu_{31}^2 (1 + \nu_{21})}{\frac{\left(1-\nu_{21}-2\nu_{31}^2 \frac{E_1}{E_3}\right)^2}{1+\nu_{21}}} \tag{52}$$

$$= 2\frac{\frac{E_3}{E_1}\left(1 - \nu_{21}^2\right)\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right) + \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right)(1 - 4\nu_{31})(1 + \nu_{21}) + 2\nu_{31}^2 (1 + \nu_{21})^2}{\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)^2} \tag{53}$$

$$= 2(1 + \nu_{21})\,\frac{\frac{E_3}{E_1}(1 - \nu_{21})\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right) + \left(1 - \nu_{31}^2 \frac{E_1}{E_3}\right)(1 - 4\nu_{31}) + 2\nu_{31}^2 (1 + \nu_{21})}{\left(1 - \nu_{21} - 2\nu_{31}^2 \frac{E_1}{E_3}\right)^2} \tag{54}$$

$$\tag{55}$$

4.4 Solve for C_1=C_2

Knowing:

$$C_0 = \frac{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}}{1-\nu_{31}^2\frac{E_1}{E_3}} \tag{56}$$

$$\tag{57}$$

$$C_1 = \frac{(1+\nu_{21})\left(1-4\nu_{31}+2\frac{E_3}{E_1}(1-\nu_{21})\right)}{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}} \tag{58}$$

$$\tag{59}$$

$$C_2 = 2\frac{\frac{E_3}{E_1}\left(1-\nu_{21}^2\right)\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)+\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)(1-4\nu_{31})(1+\nu_{21})+2\nu_{31}^2(1+\nu_{21})^2}{\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)^2} \tag{60}$$

$$\tag{61}$$

We want to solve for C_1=C_2:

$$\overbrace{\frac{(1+\nu_{21})\left(1-4\nu_{31}+2\frac{E_3}{E_1}(1-\nu_{21})\right)}{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}}}^{C_1} = 2\overbrace{\frac{\frac{E_3}{E_1}\left(1-\nu_{21}^2\right)\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)+\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)(1-4\nu_{31})(1+\nu_{21})+2\nu_{31}^2(1+\nu_{21})^2}{\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)^2}}^{C_2} \tag{A}$$

$$\text{Multiply both sides by } \left(1-\nu_{21}-2\nu_{31}^2\tfrac{E_1}{E_3}\right)^2$$

$$(1+\nu_{21})\left(1-4\nu_{31}+2\frac{E_3}{E_1}(1-\nu_{21})\right)\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)=2\frac{E_3}{E_1}\left(1-\nu_{21}^2\right)\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)+2\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)(1-4\nu_{31})(1+\nu_{21})+4\nu_{31}^2(1+\nu_{21})^2$$

$$\text{Divide both sides by } 1+\nu_{21}$$

$$\left(1-4\nu_{31}+2\frac{E_3}{E_1}(1-\nu_{21})\right)\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)=\underbrace{2\frac{E_3}{E_1}(1-\nu_{21})\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)}_{\text{Move to lefthand side}}+2\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)(1-4\nu_{31})+4\nu_{31}^2(1+\nu_{21}) \tag{B}$$

$$(1-4\nu_{31})\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)=\underbrace{2\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)(1-4\nu_{31})}_{\text{Move to lefthand side}}+4\nu_{31}^2(1+\nu_{21}) \tag{C}$$

$$(1-4\nu_{31})\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}-2\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)\right)=4\nu_{31}^2(1+\nu_{21}) \tag{D}$$

$$(1-4\nu_{31})(-1-\nu_{21})=4\nu_{31}^2(1+\nu_{21}) \tag{E}$$

$$-(1-4\nu_{31})(1+\nu_{21})=4\nu_{31}^2(1+\nu_{21}) \tag{F}$$

$$-\left(1-4\nu_{31}+4\nu_{31}^2\right)(1+\nu_{21})=0 \tag{G}$$

$$-\left(4\nu_{31}^2-4\nu_{31}+1\right)(1+\nu_{21})=0 \tag{H}$$

$$-(2\nu_{31}-1)^2(1+\nu_{21})=0 \tag{I}$$

$$-(2\nu_{31}-1)^2(1+\nu_{21})=(C_1-C_2)\qquad \overbrace{\times\frac{\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)^2}{1+\nu_{21}}}^{\text{Multiplications/divisions done to get to this step}}$$

$$-\frac{(2\nu_{31}-1)^2(1+\nu_{21})^2}{\left(1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\right)^2}=C_1-C_2$$

$$\boxed{C_2-C_1=\left(\frac{(2\nu_{31}-1)(1+\nu_{21})}{1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}}\right)^2} \tag{J}$$

Since $\nu_{21}=-1 \implies C_{11}$ and C_{12} are undefined, we know $C_{11}=C_{12}$ has only one solution remaining:

$$\boxed{C_1=C_2 \iff \nu_{31}=0.5 \qquad \text{assuming } 1-\nu_{21}-2\nu_{31}^2\frac{E_1}{E_3}\neq 0}$$

References

[1] B Cohen, WM Lai, and VC Mow. "A transversely isotropic biphasic model for unconfined compression of growth plate and chondroepiphysis". In: (1998).