Electrical Engineering Mathematics

MZB221

Scott McCue

Queensland University of Technology

Week 1

Week 1: Infinite series

Prerecorded lecture

Sequences

A sequence is an **ordered list** of numbers

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

The sequence can be denoted by $\{a_n\}_{n=1}^{\infty}$, where a_n is the $n\text{th}$ term, or just $\{a_n\}$.

Example

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

Here $a_n = \dfrac{1}{n}$, for $n = 1, 2, \ldots$, or $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$.

Example

$$1, -1, 1, -1, 1, -1, \ldots$$

Here $a_n = (-1)^{n-1}$ or $\left\{(-1)^{n-1}\right\}$, for $n = 1, 2, \ldots$.

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Limits of Sequences

A sequence $\{a_n\}$ has the limit L if a_n approaches L as n approaches infinity. We write:

$$\lim_{n\to\infty} a_n = L, \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

If the limit exists we say that the sequence converges.
Otherwise, we say the sequence diverges.

Example

$a_n = \dfrac{1}{n}$ converges to 0.

Example

$a_n = (-1)^{n-1}$ oscillates, does not converge to a finite value as $n \to \infty$, so it diverges.

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$a_n = n$ diverges to infinity.

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The idea of the limit of sequence $\{a_n\}$ as $n \to \infty$ and the limit of a function $f(x)$ as $x \to \infty$ is essentially the same.

The only difference between is that n is an integer whereas x is any real number.

Importantly, if $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ where n is an integer, then

$$\lim_{n\to\infty} a_n = L.$$

This means that we can use our previous techniques for evaluating limits of functions to evaluate limits of sequences.

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Example

Determine the limit of the sequence $\left\{ \dfrac{4}{n^2} \right\}$

Example

Determine the limit of the sequences $\left\{\left(\frac{3}{4}\right)^n\right\}$ and $\left\{\left(\frac{4}{3}\right)^n\right\}$, provided they converge.

Series

Given a particular sequence $\{a_n\}$, we can formulate a sequence of sums,

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \ldots + a_n \end{aligned}$$

The new sequence $\{S_n\}$ is called the sequence of **partial sums**.

If $\{S_n\}$ converges, that is $\lim_{n\to\infty} S_n = L$, where L is a finite value, then we say that the **infinite series** $\sum_{i=1}^{\infty} a_i$ converges.

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Example

Consider the sequence $\{n\} = 1, 2, 3, 4, \ldots$.

The corresponding series is $\displaystyle\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 \ldots$

The sequence and series diverge to infinity.

Example

Consider the sequence $\left\{\frac{1}{10^n}\right\} = 0.1, 0.01, 0.001, \ldots$.

The corresponding series

$$\sum_{n=1}^{\infty} \frac{1}{10^n} = 0.1 + 0.01 + 0.001 + \ldots = 0.11111111$$

The sequence converges to 0 while the series converges to $\frac{1}{9}$.

Geometric Series

The geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and diverges if $|r| \geq 1$.

If $|r| < 1$, we have

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Harmonic series

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is a special case that you need to remember.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \dots$$

p-series

The harmonic series is a special case of the so-called p-**series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

which will pop up when we consider Fourier series.

Turns out that for $0 < p \leq 1$, the p-series diverges. Eg.,

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{2} + \frac{1}{\sqrt{5}} + \dots \quad \text{diverges.}$$

And for $p > 1$, the p-series converges. Eg.,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \frac{\pi^2}{6},$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625} + \dots = \frac{\pi^4}{90}.$$

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series and

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

(1) If $\rho < 1$, $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $\rho > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

(3) If $\rho = 1$, the ratio test is inconclusive.

A proof (not examinable) relies on the use of geometric series.

Example

Test the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n n}$$

for convergence.

What about $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}$?

Alternating Series

An alternating series has terms that have alternating signs $(+,-,+,-\ldots)$

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots$$

Alternating Series Test

If the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ satisfies

(1) $a_n > 0$ for all $n \geq 1$

(2) $a_{n+1} \leq a_n$ for all $n \geq 1$ $(\{a_n\}$ is decreasing)

(3) $\lim_{n\to\infty} a_n = 0$

then the series converges.

Example

Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converge or diverge?

Solution

Let $a_n = \dfrac{1}{n}$. Then

- $a_n > 0$ for all $n \geq 1$.
- a_n is decreasing since $\dfrac{1}{n+1} < \dfrac{1}{n}$ for all $n \geq 1$.
- $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$.

So $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by the alternating series test.

Summary and further comments on Week 1 material:

- A more rigorous treatment of limits of sequences is provided in MXB102 or equivalent unit.

- An infinite series is a sequence of partial sums.
  - If the terms in a series *do not* converge to zero, then the series itself cannot converge.
  - If the terms in a series *do* converge to zero, then the series may or may not converge, depending on how fast the terms approach zero.

- There are many tests for convergence of a series.
  In MZB221, you only need to know about the **harmonic series**, **geometric series**, **ratio test**, the **alternating series test**.

- Other tests you don't need to know include the integral test (but it's straightforward, so you could look it up), the comparison test (it's also straightforward), root tests, and so on.