where we have used the fact that if $X \sim N(0, \sigma^2)$ then $E(X^4) = 3\sigma^4$.

We've seen that some stochastic integrals are normally distributed. However, not all stochastic integrals are normally distributed, as the previous example should show. This is almost immediate from the above example. It is apparent that (6.49) never goes below $-\frac{T}{2}$, which is a contradiction to the notion that $\int_0^T W_t \; dW_t$ is normally distributed because the normal distribution has support equal to the whole real line. In this particular case, we can say more. We know that if Z is standard normal, then Z^2 is chi-squared distributed. Now W_T has the same distribution as $\sqrt{T}Z$, sot that $\frac{1}{2}W_T^2$ has the same distribution as $\frac{T}{2}Z^2$, which is a gamma distribution because a constant times a chi-squared distribution is a gamma distribution. Finally, $\int_0^T W_t \, dW_t$ is a constant plus this, which has what's known as a generalized gamma distribution.

6.2 *Exercises*

Let $W = (W_t)_{t \geq 0}$ be a standard Brownian motion.

Exercise 6.1

As we've seen, the Ito integral is defined by sampling the integrand $b = (b_t)$ at the left endpoints of each subinterval,

$$\int_0^T b_t \, dW_t = \lim_{n\to\infty} \sum_{k=1}^{n} b_{(k-1)\Delta t} \left(W_{k\Delta t} - W_{(k-1)\Delta t} \right), \tag{6.54}$$

where $\Delta t = \Delta t(n) = \frac{T}{n}$. Using the left endpoints was deliberate and is essential in the application of the stochastic integral to finance in this course. However, ultimately it was a choice, and in this exercise we explore the implications of choosing other points, including the right endpoints and all points in between. We focus on perhaps the simplest non-trivial stochastic integral, $\int_0^T W_t \, dW_t$. To this end, define for $p \in [0,1]$ the ***p*-stochastic integral**

$$I_p \left(\int_0^T W_t \, dW_t \right) = \lim_{n\to\infty} \sum_{k=1}^{n} W_{(k-1+p)\Delta t} \left(W_{k\Delta t} - W_{(k-1)\Delta t} \right), \tag{6.55}$$

where $\Delta t = \Delta t(n) = \frac{T}{n}$. Observe that $p = 0$ corresponds to using the left endpoints, as we do in the Ito integral. In contrast, $p = 1$ corresponds to using the right endpoints, and $p = \frac{1}{2}$ corresponds to using the midpoints. We will prove the following formula,

$$I_p \left(\int_0^T W_t \, dW_t \right) = \frac{1}{2} W_T^2 - \frac{T}{2}(1 - 2p), \tag{6.56}$$

which demonstrates that the choice of p is critical to the calculation of the p-stochastic integral. Again, if the Riemann integral were applicable, then the choice of p would not matter in the result.

Much of the analysis will be similar to the calculation of $\int_0^T W_t \, dW_t$ as an Ito integral in the notes on stochastic integration.

1. Show that

$$\sum_{k=1}^{n} W_{(k-1+p)\Delta t} \left(W_{k\Delta t} - W_{(k-1)\Delta t} \right) = A_n + B_n + C_n, \tag{6.57}$$

where

$$A_n = \sum_{k=1}^{n} W_{(k-1)\Delta t} \left(W_{k\Delta t} - W_{(k-1)\Delta t} \right) \tag{6.58}$$

$$B_n = \sum_{k=1}^{n} \left(W_{(k-1+p)\Delta t} - W_{(k-1)\Delta t} \right) \left(W_{k\Delta t} - W_{(k-1+p)\Delta t} \right) \tag{6.59}$$

$$C_n = \sum_{k=1}^{n} \left(W_{(k-1+p)\Delta t} - W_{(k-1)\Delta t} \right)^2 \tag{6.60}$$

2. Observe that $A := \lim_{n\to\infty} A_n$ is just the Ito integral, which we calculated in the notes. Retrieve the value of A from the notes.

3. Use properties of the Brownian motion to show that

$$\left(W_{(k-1+p)\Delta t} - W_{(k-1)\Delta t} \right) \left(W_{k\Delta t} - W_{(k-1+p)\Delta t} \right) \stackrel{d}{=} \Delta t \sqrt{p(1-p)} Z_k Z_k^{\perp} \tag{6.61}$$

where Z_k and $Z_k^{\perp}$ each have a standard normal distribution and Z_k and $Z_k^{\perp}$ are independent random variables. (Note that the symbol $\stackrel{d}{=}$ means "has the same distribution as.") Use this and the law of large numbers to argue that $\lim_{n\to\infty} B_n = 0$.

4. Use properties of the Brownian motion to show that

$$W_{(k-1+p)\Delta t} - W_{(k-1)\Delta t} \stackrel{d}{=} \sqrt{p\Delta t} Z_k, \tag{6.62}$$

where Z_k has a standard normal distribution. Use this and the law of large numbers to argue that $\lim_{n\to\infty} C_n = pT$.

5. Combining the above, show the formula (6.56) holds.

6. For which value of p does follow the classical rules of calculus, i.e. $I_p(\int_0^T W_t \, dW_t) = \frac{1}{2} W_T^2$? This is the so-called Stratonovich integral, which has applications in physics.

Exercise 6.2

For fixed $T > 0$, the stochastic integral $\int_0^T b_t \, dW_t$ is a random variable. As such, we are naturally interested in computing its moments, such as its expected value and variance. In this exercise, we compute expectations for a certain class of stochastic integrals, namely

$$E\left[\int_0^T f(W_t)\,dW_t\right] \tag{6.63}$$

for an arbitrary but *bounded* function $f\colon \mathbb{R} \to \mathbb{R}$. We wonder if there's a pattern, and if so, whether it depends on f. We use the definition of the Ito integral to write

$$\int_0^T f(W_t)\,dW_t = \lim_{n\to\infty} F_n, \tag{6.64}$$

where

$$F_n = \sum_{k=1}^{n} f(W_{(k-1)\Delta t})\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right) \tag{6.65}$$

and $\Delta t = \Delta t(n) = \frac{T}{n}$.

As usual in analysis, we work with things that approximate our object, hoping that we can pass to the limit. The strategy is to show that $E[F_n] = 0$ for all n, and then to use the dominated convergence theorem (DCT) to commute the limit and the integral, like this:

$$E\left[\int_0^T f(W_t)\,dW_t\right] = E\left[\lim_{n\to\infty} F_n\right] \overset{DCT}{=} \lim_{n\to\infty} E[F_n] = 0 \tag{6.66}$$

Thus a perhaps startling result obtains in which the expectation of all stochastic integrals of the above form are zero, regardless of the function f. This can then be used to show that all stochastic integrals have expectation zero.

1. Argue that $E[F_n] = 0$ for each n. You may freely use the following fact about independent random variables: if X is independent of Y, then $f(X)$ is independent of $g(Y)$ where $f, g\colon \mathbb{R} \to \mathbb{R}$ are arbitrary functions.

2. Note: this part of the problem begins with an extended discussion, which you should absolutely read and understand – the **boldfaced** part below is what you ultimately need to show.

   Great, we've shown $E[F_n] = 0$. Normally we would now try to use the fact that f is bounded to show that $E|F_n|$ is finite for all n and $E|F_n|$ is bounded independently of n, which would let us use the dominated convergence theorem. Unfortunately, when I tried to do this I could show that $E|F_n|$ is finite for each n but the bound involved n ($\sqrt{n}$, in fact). So $E|F_n|$ is finite for all n, but not *uniformly* in n.

   So, what to do? This is a great opportunity to stretch our math muscles and learn about an important concept called "uniform integrability," which will eventually lead us to the *definitive* dominated convergence theorem.

We begin with the dominated convergence theorem, which implies that

$$\lim_{C\to\infty} E\left(|X|\, 1_{|X|\geq C}\right) = 0 \quad \Longleftrightarrow \quad E\left(|X|\right) < \infty \tag{6.67}$$

In words, this is saying that if the random variable is integrable (i.e., $E|X|$ is finite), then its far tails (the parts for which $|X| \geq C$) contribute vanishingly little to the expectation. Taking inspiration from this, a uniformly integrable sequence $\{X_n\}$ of random variables is a sequence in which the size of this contribution of the tails can be uniformly controlled over all elements X_n in the sequence. That is, a sequence $\{X_n\}$ of random variables is called *uniformly integrable* if

$$\lim_{C\to\infty} \left(\sup_n E\left(|X_n|\, 1_{|X|\geq C}\right)\right) = 0 \tag{6.68}$$

As usual in math, the definition precisely conveys the intuition of the concept but is itself perhaps hard to work with directly. We have several characterizations of uniform integrability that make it more useful in practice. We'll use one of these in a sec, but first let's discuss the use of the concept.

A central result is the definitive version of the dominated convergence theorem.

Theorem 6.2.1 *Let $\{X_n\}$ be a sequence of integrable random variables (that is, $E|X_n|$ is finite for each n) that converges to X in probability. Then*

$$E(X) = E\left(\lim_n X_n\right) = \lim_n E(X_n) \tag{6.69}$$

if and only if the sequence $\{X_n\}$ is uniformly integrable.

This is the definitive version of dominated convergence because the notions are equivalent, it's an "if and only if."

So, back to our exercise: we can apply dominated convergence if (and only if) we can show that our sequence is uniformly integrable. The key is the following useful characterization of uniform integrability:

Theorem 6.2.2 *Suppose that there exists some $\delta > 0$ such that*

$$\sup_n E|X_n|^{1+\delta} < \infty. \tag{6.70}$$

Then the sequence $\{X_n\}$ is uniformly integrable.

Thus, if we show that $\sup_n E|F_n|^2$ is finite, for example by showing that $E|F_n|^2$ is bounded by a constant *independent* of n, then the above result (with $\delta = 1$) tells us that $\{F_n\}$ is uniformly integrable, and we can use the definitive version of the dominated convergence theorem.

Now, using that f is bounded by assumption, i.e. $|f(x)| \leq M$ for all $x \in \mathbb{R}$ for some fixed $M \geq 0$, we find that

$$E(F_n^2) = E\left[\left(\sum_{k=1}^{n} f(W_{(k-1)\Delta t})\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right)\right)^2\right] \tag{6.71}$$

$$= E\left[\sum_{k=1}^{n} f(W_{(k-1)\Delta t})^2\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right)^2\right] \tag{6.72}$$

$$+ E\left[\sum_{k\neq j} f(W_{(k-1)\Delta t})\left(W_{k\Delta t} - W_{(k-1)\Delta t}\right) f(W_{(j-1)\Delta t})\left(W_{j\Delta t} - W_{(j-1)\Delta t}\right)\right] \tag{6.73}$$

$$\leq M^2 \sum_{k=1}^{n} E[(W_{k\Delta t} - W_{(k-1)\Delta t})^2] + M^2 \sum_{k\neq j} E[(W_{k\Delta t} - W_{(k-1)\Delta t})(W_{j\Delta t} - W_{(j-1)\Delta t})] \tag{6.74}$$

You can finish this: Use properties of the Brownian motion to show that the right hand side of the above is equal to M^2T.

Thus, $E[F_n^2] \leq M^2T$, a uniform bound independent of n, and therefore $\{F_n\}$ is uniformly integrable and we can use dominated convergence to finish the problem, as in (6.66). We conclude that

$$E\left[\int_0^T f(W_t)\, dW_t\right] = 0 \tag{6.75}$$

for every (!) bounded function $f\colon \mathbb{R} \to \mathbb{R}$.

Exercise 6.3

An important result in the theory of Ito integration is the Ito isometry, which can be stated as

$$E\left[\left(\int_0^T b_t\, dW_t\right)^2\right] = E\left(\int_0^T b_t^2\, dt\right). \tag{6.76}$$

The Ito isometry is crucial in proving the general existence of the Ito integral. It is also very useful for computing variances of stochastic integrals.

In this exercise, we will prove the Ito isometry for a very special class of integrands b. Let $0 = t_0 < t_1 < \cdots < t_n = T$ denote a (fixed)