# Homework on Integrated Brownian Motion

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This exercise will explore the integrated Brownian motion, which is the random variable $I$ defined by[1]

$$I = \int_0^1 W_t \, dt. \tag{1}$$

Because paths of the Brownian motion are continuous functions of $t$, we can integrate them using normal Riemann integration. A valid expression for the integral for a given state $\omega$ in the state space $\Omega$ is therefore

$$I = \int_0^1 W_t \, dt = \lim_{n \to \infty} I_n \qquad \text{where} \qquad I_n = \frac{1}{n} \sum_{k=1}^{n} W_{\frac{k}{n}} \tag{2}$$

Note that this integral is defined path-by-path, or pathwise. This means that, for each $\omega \in \Omega$, one obtains a different continuous function (a sample path of the Brownian motion), which one then integrates. We see that the integrated Brownian motion is a random variable.

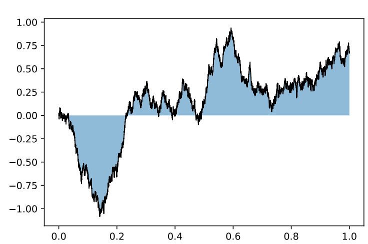


Figure 1: $I(\omega_1) > 0$

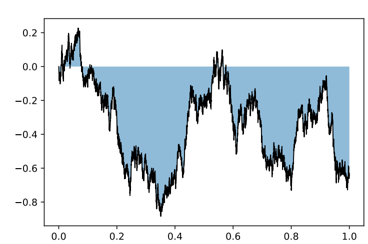


Figure 2: $I(\omega_2) < 0$

Figures 1 and 2 depict two sample paths on [0,1] of the Brownian motion. The value of the integrated Brownian motion for each path is the (signed) area, shaded in blue. It is even more apparent that the integrated Brownian motion

[1] Here we take the limits of the integral to be 0 and 1 but this is only for convenience and could be generalized to $a$ and $b$. One could also let the limits be 0 and $t$, where $t$ varies, which would produce a stochastic process.

is a random variable, and that its value in figure 1 is positive and its value in figure 2 is negative.

Our exercise will explore the integrated Brownian motion. It is divided into two parts. The first part will be programmatic and will numerically simulate the integrated Brownian motion to better understand its properties and form hypotheses about its distribution. The second part will test our hypotheses analytically.

*Part I: Numerical Simulation of the Integrated Brownian Motion*

Equation (2) gives us a recipe for approximating the integrated Brownian motion as the limit of Riemann sums. So to estimate the integrated Brownian motion we will, for large $n$ (where $n$ is as in (2)), simulate a Brownian path at $n$ points in $[0,1]$, add up these values, and then divide by $n$.

(a) In a python or R script, write a function of three variables $(t_0, t_n, n)$ called "areaBMpath" that simulates a path of the Brownian motion on $[t_0, t_n]$ using $n$ subintervals, adds them up, and then divides by $n$. The output is then an estimate of $\int_{t_0}^{t_n} W_t \, dt$.

(b) For $t_0 = 0, t_1 = 1$, and $n = 1,000$ simulate $\int_0^1 W_t \, dt$ by calling the function areaBMpath(0,1,1000). Repeat this over and over again, storing the resulting values in an array "arr," so that in the end you have 100,000 estimates stored in "arr."

(c) Plot a histogram of the 100,000 values in "arr." What do you notice about the likely distribution of $\int_0^1 W_t \, dt$?

(d) Compute the mean and standard deviation of "arr."

*Part II: Finding the Distribution Analytically*

Our numerical simulations strongly suggest that $\int_0^1 W_t \, dt$ is normally distributed with a certain mean and variance. We'll now *prove* that $\int_0^1 W_t \, dt$ is normally distributed and find its exact mean and variance.

Our plan is to show that each of the $I_n$ is normally distributed, and then hope by some magic that will be enough to conclude that the limit, i.e. $\int_0^1 W_t \, dt$, is normally distributed.[2]

(a) Let's first show that each of the $I_n$ is normally distributed with a certain mean and variance. Each $I_n$ is a sum of normal random variables, which we know is <u>not</u> necessarily normally distributed unless the summands are independent or, more generally, form a *multivariate* normal distribution

[2] This is a frequent mode of attack in mathematical analysis: If the object of interest is too complicated to work with directly, find a sequence of simpler objects that converges to it and work with these simpler objects before passing to the limit.

when collected into a random vector. We recall that *increments* of the Brownian motion are independent, so we follow that lead. Is there a trick? Can we make increments happen? Observe that

$$Y_n := \sum_{k=1}^{n} W_{\frac{k}{n}} = n\left(W_{\frac{1}{n}} - W_0\right) + (n-1)\left(W_{\frac{2}{n}} - W_{\frac{1}{n}}\right) + \cdots \tag{3}$$

$$+ 2\left(W_{\frac{n-1}{n}} - W_{\frac{n-2}{n}}\right) + \left(W_1 - W_{\frac{n-1}{n}}\right) \tag{4}$$

$$= \sum_{k=1}^{n} kX_k, \tag{5}$$

where

$$X_k := W_{\frac{n-k+1}{n}} - W_{\frac{n-k}{n}}. \tag{6}$$

Note that the $X_k$ are mutually disjoint increments of the Brownian motion and are therefore independent. Furthermore, each $X_k$ is normally distributed with zero mean and variance $1/n$ (Convince yourself that you know why...). Therefore, the sum $Y = \sum_{k=1}^{n} W_{\frac{k}{n}}$ is normally distributed with mean zero, which also shows that $I_n = \frac{1}{n}Y_n$ is normally distributed with mean zero. The more interesting computation is the variance $\sigma_n^2$ of $I_n$. Find $\sigma_n^2$. (*Hint*: You can freely use the identity $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.)

(b) Having shown that each $I_n$ is normally distributed with mean zero and variance $\sigma_n^2$, we now turn to showing that the sequence $\{I_n\}$ converges to a normal random variable. This is not guaranteed in general, e.g. if the variances $\sigma_n^2$ grow too fast. We will use the so-called continuity theorem, which is a famous result involving characteristic functions[3]. By that theorem we are done if, for all $t$, we can show that the characteristic functions $\varphi_{I_n}(t)$ of $I_n$ converge pointwise to the characteristic function $\varphi(t)$ of a normal random variable.

(i) Let $\sigma_n^2$ be the variance of $I_n$ that you found in part (a). The limit of $\sigma_n^2$ as $n$ goes to infinity exists. Find the value of this limit, $\sigma^2 = \lim_{n\to\infty} \sigma_n^2$.

(ii) Using the known formula for the characteristic function of a normal random variable (see previous homework or course slides or handouts), the characteristic function of $I_n$ is

$$\varphi_{I_n}(t) = e^{-\frac{1}{2}\sigma_n^2 t^2}. \tag{7}$$

Argue that

$$\lim_{n\to\infty} \varphi_{I_n}(t) = e^{-\frac{1}{2}\sigma^2 t^2}, \qquad \text{for all } t \in \mathbb{R} \tag{8}$$

Be sure to justify your steps.

[3]The curious can consult Wikipedia: `https://en.wikipedia.org/wiki/Levy's_continuity_theorem`

(iii) The right hand side of (8) is the characteristic function of a random variable $X$. We know that characteristic functions uniquely identify distributions. What is the distribution of $X$?

(iv) Tracing all the steps back, what is the distribution of $\int_0^1 W_t \, dt$? How does it compare to what we saw in our simulations?