HW7

Tuesday, November 17, 2020 - Due: Tuesday, November 25, 2020

1. Let $\mathcal{C}$ be an additive $\mathbb{C}$-category. The radical of $\mathcal{C}$, denoted by $rad_{\mathcal{C}}$, is defined as:

$$rad_{\mathcal{C}}(X,Y) = \{h \in Hom_{\mathcal{C}}(X,Y) \mid 1_X - g \circ h \text{ is invertible } \forall g \in Hom_{\mathcal{C}}(Y,X)\}.$$

   Prove that:

$$rad_{\mathcal{C}}(X,Y) = \{h \in Hom_{\mathcal{C}}(X,Y) \mid 1_Y - h \circ g \text{ is invertible } \forall g \in Hom_{\mathcal{C}}(Y,X)\}.$$

2. (a) Prove $(f \in rad_C(X,Y)) \implies (f$ is neither split mono nor split epi$)$.
   (b) Show that the converse implication is not true, i.e. find a map f which is neither split mono nor split epi, but $f \notin rad_C(X,Y)$.

3. (a) Let $M \xrightarrow{f} N$ be a Λ-module homomorphism. Prove that the homomorphism $M \xrightarrow{\begin{bmatrix} f \\ f \end{bmatrix}} N \oplus N$ is not left minimal.
   (b) Prove that the map $M \oplus M \xrightarrow{\begin{bmatrix} 1_M & 1_M \end{bmatrix}} M$ is left minimal.

4. Let Q be the quiver $1 \leftarrow 2 \leftarrow 3$.
   (a) Prove that the map $P_1 \xrightarrow{incl} P_3$ is not right almost split map.
   (b) Prove that the map $P_1 \xrightarrow{incl} P_3$ is right minimal map.

5. Let Q be the following quiver:

$$\begin{tikzcd} & & 5 \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\ 1 & 2 & & 3 & 4 \end{tikzcd}$$

   (a) Prove that the following representation morphism must be equal to 0 morphism:

$$\begin{tikzcd} & & \mathbb{C} \arrow[dll, "{[1]}"'] \arrow[dl] \arrow[dr] \arrow[drr, "{[1]}"] & & & \xrightarrow{f} & & & \mathbb{C} \arrow[dll, "{[1]}"'] \arrow[dl, "{[1]}"] \arrow[dr, "{[1]}"'] \arrow[drr, "{[1]}"] & & \\ \mathbb{C} & 0 & & 0 & \mathbb{C} & & \mathbb{C} & \mathbb{C} & & \mathbb{C} & \mathbb{C}. \end{tikzcd}$$

   (b) Prove that $P_1 \oplus P_2 \oplus P_3 \xrightarrow{[i_1\ i_2\ i_3]} P_5$ is right minimal but is not right minimal almost split map. (Here the maps i_1 i_2 i_3 are the inclusions.)