HOMEWORK 4: TOPOLOGY 2, FALL 2020
DUE DECEMBER 9

All answers should be given with proof. Proofs should be written in complete sentences and include justifications of each step. You are encouraged to use figures to support your arguments. The word *show* is synonymous with *prove*. This assignment has five problems and two pages.

Problem 1. Recall that the complex projective space $\mathbb{CP}^n$ is the space of (complex) 1-dimensional subspaces of $\mathbb{C}^{n+1}$, topologized as the quotient of $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ by the equivalence relation $v \sim \lambda v$, where λ is a complex number with $|\lambda| = 1$. Write $\pi_n : S^{2n+1} \to \mathbb{CP}^n$ for the quotient map.

(a) Show that the subspace $D_+ := \{(z_0, \ldots, z_n) \in S^{2n+1} : z_n \in \mathbb{R}_{\geq 0}\}$ is homeomorphic to D^{2n} (hint: consider the function $w \mapsto \sqrt{1 - |w|^2}$).

(b) Show that $\pi_n|_{D_+}$ is surjective and $\pi_n|_{\mathring{D}_+}$ is injective.

(c) Show that the following diagram commutes:

$$\begin{tikzcd} S^{2n-1} \arrow[r, "\cong"] \arrow[d, "\pi_{n-1}"'] & \partial D_+ \arrow[d, "\pi_n|_{\partial D_+}"] \\ \mathbb{CP}^{n-1} \arrow[r] & \mathbb{CP}^n. \end{tikzcd}$$

Conclude that $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a single $2n$-cell.

(d) Calculate $H_*(\mathbb{CP}^n)$ for $0 \leq n \leq \infty$ (don't work too hard).

Spoilers for this problem abound.

Problem 2. Suppose given the short exact sequence $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ of Abelian groups. Show that the following are equivalent.

(a) There is a homomorphism $\rho : B \to A$ such that $\rho \circ \varphi = \mathrm{id}_A$ (i.e., φ admits a retraction).

(b) There is a homomorphism $\sigma : C \to B$ such that $\psi \circ \sigma = \mathrm{id}_C$ (i.e., ψ admits a section).

(c) There is an isomorphism $B \cong A \oplus C$ making the following diagram commute

$$\begin{tikzcd} & B \arrow[dr, "\psi"] \arrow[dd, "\cong"] & \\ A \arrow[ur, "\varphi"] \arrow[dr, "\iota"'] & & C \\ & A \oplus C \arrow[ur, "\pi"'] & \end{tikzcd}$$

where ι is the inclusion and π the projection.

Hatcher calls this result the "Splitting Lemma." Hints and partial spoilers can be found there.

The remaining problems would be best approached after our discussion of the cup product.

Problem 3. Fix $k, \ell > 0$.

(a) Show that every map $S^{k+\ell} \to S^k \times S^\ell$ induces the trivial homomorphism on homology. This problem is Hatcher 3.2.11.
(b) Show that $S^k \times S^\ell$ and $S^k \vee S^\ell \vee S^{k+\ell}$ have isomorphic homology groups in every degree but are not homotopy equivalent.

Problem 4. Calculate the cohomology ring of Σ_g, the surface of genus g (a hint in different notation can be found in Hatcher 3.2.1).

Problem 5. In this problem, you may wish to use the relative cup product $H^k(X, A) \times H^\ell(X, B) \to H^{k+\ell}(X, A \cup B)$ as discussed on p. 209 of Hatcher.

(1) Show that, if $X = A \cup B$ with A and B acyclic, then the cup product of any pair of elements in $H^*(X)$ of positive degree vanishes.
(2) Show that there are no nontrivial products in the reduced cohomology of a suspension.
(3) Formulate and prove a generalization of (1) under the assumption that $X = \bigcup_{i=1}^n A_i$ with each A_i acyclic.

This problem is Hatcher 3.2.2.