HOMEWORK 3: TOPOLOGY 2, FALL 2020
DUE NOVEMBER 4

All answers should be given with proof. Proofs should be written in complete sentences and include justifications of each step. You are encouraged to use figures to support your arguments. The word *show* is synonymous with *prove*. This assignment has five problems and two pages.

Problem 1. The following facts about a CW complex X were used in class. Prove them from the definitions.

(a) A subcomplex $A \subseteq X$ is a CW complex with $A_n = A \cap X_n$.
(b) If $A_i \subseteq X$ is a finite subcomplex for $1 \leq i \leq r$, then $A := \bigcup_{i=1}^{r} A_i$ is so as well.
(c) The subset $\{x\} \subseteq X$ is closed for any $x \in X$.

You may find it helpful to prove that X carries the quotient topology from the disjoint union of its closed cells.

Problem 2. Compute the homology of the following spaces. You may find it useful to consider CW structures.

(a) The quotient of S^2 obtained by identifying the north and south poles.
(b) The quotient of S^2 obtained by identifying x with $-x$ for every x in the equatorial circle.
(c) The quotient of S^3 obtained by identifying x with $-x$ for every x in the equatorial sphere.
(d) The product $S^1 \times (S^1 \vee S^1)$.

This problem combines parts of Hatcher 2.2.9 and 2.2.10.

Definition 1. Let X and Y be CW complexes. A map $f : X \to Y$ is *cellular* if $f(X_n) \subseteq Y_n$.

Problem 3. Let $f : X \to Y$ be a cellular map.

(a) Give a reasonable definition of an induced homomorphism $f_*^{\mathrm{CW}} : H_*^{\mathrm{CW}}(X) \to H_*^{\mathrm{CW}}(Y)$ that does not appeal to the isomorphism with singular homology (this notation is not standard).
(b) Show that the isomorphism between cellular and singular homology is natural in the sense that the following diagram commutes:

$$\begin{tikzcd} H_*(X) \arrow[r, equal, "\sim"] \arrow[d, "f_*"'] & H_*^{\mathrm{CW}}(X) \arrow[d, "f_*^{\mathrm{CW}}"] \\ H_*(Y) \arrow[r, equal, "\sim"] & H_*^{\mathrm{CW}}(Y). \end{tikzcd}$$

This problem is Hatcher 2.2.17. Beware spoilers everywhere.

Problem 4. Let $q : S^1 \times S^1 \to S^2$ denote the quotient map collapsing the 1-skeleton of the torus to a point.

(a) Show that q induces an isomorphism on H_2. Conclude that q is not nullhomotopic.

(b) Show that every map $S^2 \to S^1 \times S^1$ is nullhomotopic (hint: covering spaces).

This problem is Hatcher 2.2.15.

Problem 5. Let X be a CW complex.

(a) Show that the intersection of two subcomplexes of X is also a subcomplex.

(b) Show that, if $X = A \cup B$ for finite subcomplexes A and B, then

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

(c) Let $p : \widetilde{X} \to X$ be a covering map. Explain how to endow $\widetilde{X}$ with a CW structure for which p is cellular. You need not give full details.

(d) In the situation of the previous problem, assume that X is finite as a CW complex and p is finite as a cover. Show that

$$\chi(\widetilde{X}) = n\chi(X),$$

where n is the degree of p.

Given a second CW complex Y, the product $X \times Y$ has a canonical CW structure. Think about how you would define this structure, and check your intuition on p. 523 of Hatcher.

(e) If X and Y are both finite, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

This problem combines Hatcher 2.2.20-22.