HOMEWORK 1: TOPOLOGY 2, FALL 2020
DUE SEPTEMBER 30

All answers should be given with proof. Proofs should be written in complete sentences and include justifications of each step. You are encouraged to use figures to support your arguments. The word *show* is synonymous with *prove*. This assignment has five problems and two pages.

Definition 1. An *abstract Δ-complex* (also known as a semisimplicial set or Δ-set) K is a collection of sets $\{K_n\}_{n\geq 0}$ equipped with functions $d_i : K_n \to K_{n-1}$ for each n and $0 \leq i \leq n$ such that

$$i \leq j \implies d_j \circ d_i = d_i \circ d_{j+1}.$$

The set K_n is called the set of *n-simplices* and d_i the ith *face map* of K.

The *geometric realization* $|K|$ of the abstract Δ-complex K is the quotient of the disjoint union $\bigsqcup_{n\geq 0} K_n \times \Delta^n$ by the equivalence relation $(\sigma, \eta_i(t)) \sim (d_i(\sigma), t)$ for all $\sigma \in K_{n+1}$ and $t = (t_0, \ldots, t_n) \in \Delta^n$.

A *Δ-complex structure* on a space X is a homeomorphism of X with the geometric realization of a Δ-complex.

Problem 1.
(1) Show that the definition of Δ-complex structure given above is equivalent to the definition given on p. 103 of Hatcher.
(2) Given a space X, construct an abstract Δ-complex $S(X)$ with $S(X)_n$ the set of singular n-simplices of X.
(3) Give an example of a space X not homeomorphic to $|S(X)|$ (hint: a very simple space suffices).
(4) Based on your answer to (3), how often do you think X is homeomorphic to $|S(X)|$?

Definition 2. Given an abstract Δ-complex K, the *homology of K*, denoted $H_*(K)$, is defined to be the homology of the chain complex

$$\cdots \xrightarrow{\partial_{n+1}} \mathbb{Z}\langle K_n \rangle \xrightarrow{\partial_n} \mathbb{Z}\langle K_{n-1} \rangle \xrightarrow{\partial_{n-1}} \cdots, \qquad \partial_n = \sum_{i=0}^{n} (-1)^i d_i.$$

The *simplicial homology* of a space with a Δ-complex structure is the homology of the corresponding abstract Δ-complex.

Problem 2.
(1) What is another name for $H_*(S(X))$?
(2) Work Hatcher 2.1.4. Before you compute the homology, write down the abstract Δ-complex structure explicitly.
(3) Do the same thing with Hatcher 2.1.5.

Definition 3. An *abstract simplicial complex* is a pair (S, Δ) such that

(1) S is a non-empty set,
(2) Δ is a non-empty set of non-empty subsets of S, and

(3) if $S' \in \Delta$ and $\varnothing \neq S'' \subseteq S'$, then $S'' \in \Delta$.

Problem 3. (1) Let (S, Δ) be an abstract simplicial complex. Given a total ordering of S, construct an abstract Δ-complex with set of n-simplices the elements of Δ with cardinality $n+1$.

We say that a Δ-complex structure on a space X is a *triangulation* if it arises from an abstract simplicial complex. Given a triangulation, we write V, E, and F for the number of 0-, 1-, and 2-simplices, respectively. In this problem, you may use the following fact, which was proven in Topology 1: in a triangulation of a surface, edges are identified *in pairs.*

(2) Show that $E \leq \binom{V}{2}$ for any triangulation of a space.
(3) Show that, in any triangulation of the orientable surface of genus g, we have
$$V \geq \frac{7+\sqrt{1+48g}}{2}.$$
You may assume the formula $2-2g = V - E + F$.
(4) Show that the sphere is the only orientable surface admitting a triangulation with $V \in \{4, 5, 6\}$.
(5) Exhibit a triangulation of the torus with $V = 7$. What are E and F?
(6) Explain the advantage of the extra generality of Δ-complexes in light of (5) and Example 2.3 in Hatcher.

Problem 4. Let $i : A \to X$ be the inclusion of a subspace.

(1) Is the induced homomorphism i_* on homology necessarily injective? Give a proof or a counterexample.
(2) Show that i_* is injective if A is a retract of X (Hatcher 2.1.11).

Problem 5. Recall that a sequence of homomorphisms is *exact* if the kernel of each homomorphism is equal to the image of the previous homomorphism.

(1) Suppose that the sequence $A \xrightarrow{\varphi} B \to C \to D \xrightarrow{\psi} E$ is exact. Show that $C = 0$ if and only if φ is surjective and ψ is injective.
(2) Classify (up to isomorphism) Abelian groups A fitting into an exact sequence
$$0 \to \mathbb{Z}/p^m\mathbb{Z} \to A \to \mathbb{Z}/p^n\mathbb{Z} \to 0.$$
(3) Classify (up to isomorphism) Abelian groups A fitting into an exact sequence
$$0 \to \mathbb{Z} \to A \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

These problems are Hatcher 2.1.14 and 2.1.15.