HW5

Tuesday, November 3, 2020 - Due: Tuesday, November 10, 2020

1. Let $\Lambda = \mathbb{C}Q/I$ be the quotient algebra of the path algebra $\mathbb{C}Q$, where Q is the following quiver of type A_6 and the ideal I is generated by the following paths $\{abc, cd\}$:

$$1 \xleftarrow{a} 2 \xleftarrow{b} 3 \xleftarrow{c} 4 \xleftarrow{d} 5 \xleftarrow{e} 6$$

   (a) Describe all indecomposable projective representations.

   (b) Describe a projective resolution of the simple S_6.

   (c) Describe a projective resolution of the representation

$$0 \xleftarrow{0} 0 \xleftarrow{0} 0 \xleftarrow{0} 0 \xleftarrow{0} \mathbb{C} \xleftarrow{[7]} \mathbb{C}$$

2. Let $x \in Q_0$ be a sink and let $\mathcal{S}_x^+ : repQ \to repQ'$ be the reflection functor defined on representation $V = (\{V(x)\}_{x\in Q_0}, \{V(a)\}_{a\in Q_1})$ as:
$\mathcal{S}_x^+(V) = W$ where $W(y) = V(y)$ for all $y \neq x$, $y \in Q_0$ and $W(x)$ is the following kernel:

$$0 \to W(x) \xrightarrow{[W(a*)]} \bigoplus_{y \xrightarrow{a} x} V(y) \xrightarrow{[V(a)]} V(x)$$

   Let $x \in Q_0$ be a source and let $\mathcal{S}_x^- : repQ \to repQ'$ be the reflection functor defined on representation $V = (\{V(x)\}_{x\in Q_0}, \{V(a)\}_{a\in Q_1})$ as:
$\mathcal{S}_x^-(V) = U$ where $U(y) = V(y)$ for all $y \neq x$, $y \in Q_0$ and $U(x)$ is the following cokernel:

$$V(x) \xrightarrow{[V(a)]} \bigoplus_{x \xrightarrow{a} y} V(y) \xrightarrow{[U(a*)]} U(x) \to 0$$

   (a) Prove that there is a natural transformation between the functors:
$\mathcal{S}_1^- \mathcal{S}_1^+ \to Id_{repQ}$ for the quiver $1 \xleftarrow{a} 2$.

   (b) Prove that there is a natural transformation between the functors:
$\mathcal{S}_x^- \mathcal{S}_x^+ \to Id_{repQ}$ for a quiver Q, where $x \in Q_0$ is a **sink**.

   (c) Prove that there is no nonzero natural transformation between the functors:
$Id_{repQ} \to \mathcal{S}_1^- \mathcal{S}_1^+$ for the quiver $1 \xleftarrow{a} 2$.

3. Let Q be the quiver $1 \xleftarrow{a} 2$. Let V be the following representation of Q:

$$\mathbb{C}^2 \xleftarrow{\begin{bmatrix} 2 \\ 8 \end{bmatrix}} \mathbb{C}.$$

   Prove that the simple representation S_1,

$$\mathbb{C} \xleftarrow{0} 0$$

   is isomorphic to a direct summand of V.