HOMEWORK 2: TOPOLOGY 2, FALL 2020
DUE OCTOBER 14

All answers should be given with proof. Proofs should be written in complete sentences and include justifications of each step. You are encouraged to use figures to support your arguments. The word *show* is synonymous with *prove*. This assignment has six problems and two pages.

Problem 1. Let $i : A \subseteq X$ be the inclusion of a subspace.

(a) Show that $H_0(X, A) = 0$ if and only if the intersection of A with every path component of X is nonempty.
(b) Show that $H_1(X, A) = 0$ if and only if $i_* : H_1(A) \to H_1(X)$ is surjective and every path component of X contains at most one path component of A.

This problem is Hatcher 2.1.16.

Problem 2. Compute the homology of $X = \big\{(x, y) \in [0, 1]^2 : x \in \mathbb{Q} \text{ or } y \in \{0, 1\}\big\}$. This problem is Hatcher 2.1.19.

Definition 1. The *cone* on a space X is the quotient $CX = X \times [0, 1]/X \times \{0\}$. The *suspension* of X is the further quotient $SX = CX/X \times \{1\}$.

Problem 3. (a) Draw schematic pictures of CX and SX.
(b) Show that CX is contractible for any X.
(c) Show that $\widetilde{H}_n(SX) \cong \widetilde{H}_{n-1}(X)$ for every $n \geq 0$ and X.
(d) Show that $SS^m \cong S^{m+1}$ for every $m \geq 0$ (hint: show that $CS^m \cong D^{m+1}$ and glue).
(e) Given a map $f : X \to Y$, give a reasonable definition of an induced map $Sf : SX \to SY$, and show that $\deg(Sf) = \deg(f)$ when $X = Y = S^m$.

This problem is partly Hatcher 2.1.20. Beware spoilers for some parts elsewhere in Hatcher.

Problem 4. Let $X = [0, 1]$ and $A = \{1/n\}_{n \geq 0} \cup \{0\}$. Show that $H_1(X, A) \ncong \widetilde{H}_1(X/A)$. This problem is Hatcher 2.1.26, and a hint can be found there.

Problem 5. (a) Suppose that m is even. Given a map $f : S^m \to S^m$, show that $f(x) = x$ or $f(x) = -x$ for some $x \in S^m$.
(b) Suppose that m is even. Show that any map $g : \mathbb{RP}^m \to \mathbb{RP}^m$ has a fixed point.
(c) For each odd m, construct a map $g : \mathbb{RP}^m \to \mathbb{RP}^m$ with no fixed point (hint: consider an invertible linear operator on $\mathbb{R}^{m+1}$ with no eigenvectors).

This problem is Hatcher 2.2.2.

Problem 6. Suppose that both rows are exact in the following commuting diagram of homomorphisms among Abelian groups:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'.
\end{tikzcd}$$

(a) Suppose that β and δ are surjective and that ϵ is injective. Show that γ is surjective.
(b) Suppose that β and δ are injective and that α is surjective. Show that γ is injective.

The five lemma is a standard result, and the proof appears in any number of textbooks. You are strongly encouraged not to look it up, but no one can stop you.