LQR Controller for Reference Tracking Device

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***Abstract*—This paper describes a process of development and simulation of a Linear Quadratic Regulator (LQR) system for a rotational object tracker. The system is based on a device with two Direct Current (DC) motors installed and working in 'tank' mode, and controlled by the LQR controller. The document leads the reader through the system state space representation derivation, LQR controller design using MATLAB, and Simulink simulation. Suggestions on further work and improvements are covered at the end of the document.**

Keywords: Control Theory, Optimal Control, Linear Quadratic Regulator (LQR).

I. Introduction

A reference tracking device is a negative feedback system designed to continuously reduce error through application of a system actuator effort proportional to an input error. A system has three main components: plant to be controlled, controller with actuator, and sensing device to provide feedback.

In the present project, the plant is built of a circular platform where two DC motors are installed. The controller is an MCU that actuates the DC motors based on the input signal. Finally, the sensing device is an image sensor. More precisely, the MCU performs functions of image processing, error estimation, and controller operation. Figure 1 presents a scale drawing of the system with precise dimensions.

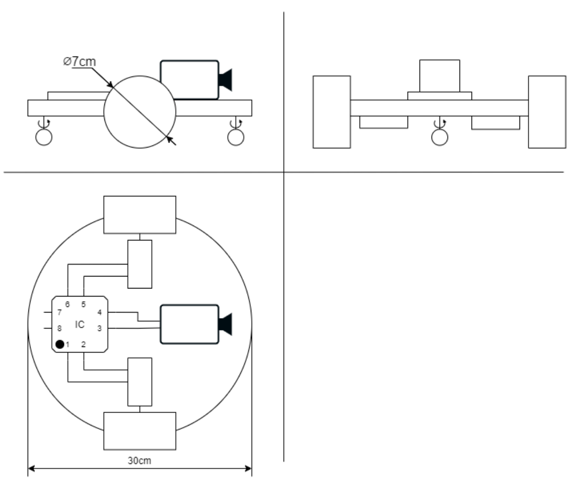

Fig. 1: Mechanical drawing of the reference tracker device.

The device operates in one dimension only. It rotates with respect to its center, changing an observation angle as an object moves within an observable space. Rotational movement is achieved through 'tank' mode DC motors operation, that is, rotating the wheels in opposite directions.

The tracking task outputs an object location error in rads. The error is estimated based on the wide angle of the camera sensor and detected object pixel location through the Equation 1.

$$Error_{rads} = 2\pi(O_{x,px} - \frac{W_{img}}{2})\frac{A_w}{W_{img}} \tag{1}$$

where $O_{v,px}$ is an object location on x axis, W_{img} is image width, and A_w is the wide angle of the camera sensor. Then this error is fed into the controller which operates the motors such as the error is reduced to zero.

DC motors is a fundamental part of the system. They transform energy from electrical to mechanical domain. The main components are a permanent magnet rotor, armature, and wire wound glsstator poles. Electrical energy is converted to mechanical energy by the rotated magnetic field forces induced between the stator poles and permanent magnet rotor [1]. Generally, it is a popular choice for low-power and motion-precision applications.

The project focuses mainly on designing and simulating the reference tracker device controller using Linear Quadratic Regulator (LQR) algorithm. The angular position of the device is controlled by the voltage supplied on the DC motors terminals. Supplying higher voltage results in reducing the angular error faster through faster rotation of the DC motors.

The rest of the document is organized as follows. Section II focuses on the derivation of the state space model of the object tracking device, as well as, description of the LQR optimization procedure. Section III describes computational part of the LQR feedback gain, the key point for the LQR control, and provides the simulation results using MATLAB and Simulink. The document ends with a conclusion and discussion on further model improvements and direction for the future work.

II. Theory

Before designing the LQR controller, it is necessary to deduce a state space representation of the system that will be controlled. The system dynamics can be described by the DC motors and device platform characteristics. For the sake of simplicity, some physical properties as solid body inertia for the platform and external friction coefficients for the wheels will be omitted.

A. State Space Modeling

To derive the state space representation of the whole system, first, the DC motor model will be described. Then, it will be augmented with the platform dynamics. Figure 2 presents a mechanical diagram of the system.

The DC motor is modeled as a linear transformation of current in the motor to induced torque [2]. Angular velocities of both motors are controlled by the input voltage U_a. The rotor resistor R_{a1} along with a back-electromotive force (EMF) creates a constant voltage drop. Motor current links both mechanical and electrical components while generating the torque. Motor inertia, viscous friction/damping structure, internal friction, and external load oppose the torque. Analyzing Figure 2 from electrical (Kirchhoff's Law) and mechanical (2^{nd} Law of Newton) viewpoints, the DC motor dynamics can be described by Equations 2 and 3.

$$u(t) = L\frac{di}{dt} + R_m i(t) + K_e \omega_r(t), \tag{2}$$

$$K_m i(t) = J\frac{d\omega_r}{dt} + K_d \omega_r(t) + \tau_l + \tau_f \tag{3}$$

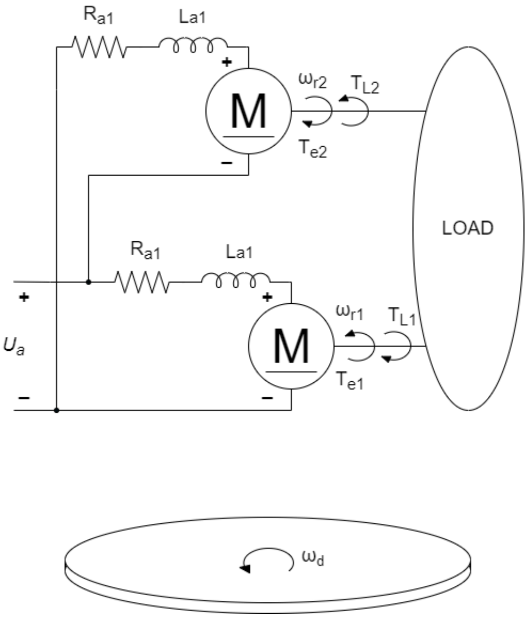

Fig. 2: Mechanical diagram of the reference tracker device. Two DC motors are on top and the platform is on the bottom.

where L and R_m are motor inductance and internal resistance, K_e is EMF, K_m is motor torque, K_d is viscous friction, J is motor inertia, and τ_l is the motor load and τ_f external friction. Rearranging terms in the Equations 2 and 3 and omitting load and friction terms

$$\frac{di}{dt} = -\frac{R_m}{L} i(t) - \frac{K_e}{L} \omega_r(t) + \frac{u(t)}{L}, \tag{4}$$
$$\frac{d\omega_r}{dt} = \frac{K_m}{J} i(t) - \frac{K_d}{J} \omega_r(t) \tag{5}$$

This form fits into a general form of state-space equations for linear systems [3, Chapter 7, Section 1], as in Equation block 6.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{6}$$

Introducing a state vector $x = [i, \omega_r]^T$, the state space representation for the DC motor is

$$\dot{x} = \begin{bmatrix} -\frac{R_m}{L} & -\frac{K_e}{L} \\ \frac{K_m}{J} & -\frac{K_d}{J} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u \tag{7}$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \tag{8}$$

The model described in the Equations 7 and 8 is suitable for the DC motor. Furthermore, to extrapolate this model for the device, Equation 9 can be used. It converts the DC motor angular velocity to the device angular velocity.

$$\omega_d = \omega_r \frac{C}{R_p}, \tag{9}$$
$$C = 2\pi R_w \tag{10}$$

where ω_d is device angular velocity in rad/s, ω_r is DC motor angular velocity, C is wheel circumference, R_p is radius of the device's platform, and R_w is wheel radius. Therefore, an updated state-space representation that models the device angular velocity, with state vector $x = [i, \omega_d]^T$, is presented in Equation 11.

$$\dot{x} = \begin{bmatrix} -\frac{R_m}{L} & -\frac{K_e}{L} \\ \frac{K_m}{J}\frac{C}{R_p} & -\frac{K_d}{J}\frac{C}{R_p} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u \tag{11}$$

Equations 11 and 8 model the system if it had only one DC motor. However, the tracking device has two DC motors. Since they work in 'tank' mode, that is, their dynamics are identical though absolutely opposed, their state transition matrices are identical. Their output matrices are different. Equations 12 and 13 describe the output matrices for both DC motors.

$$y_{m_1} = \begin{bmatrix} 0 & 1 \end{bmatrix} x \tag{12}$$
$$y_{m_2} = \begin{bmatrix} 0 & -1 \end{bmatrix} x \tag{13}$$

The complete model has eight independent parameters. Most of them are provided by the DC motor datasheet, empirical experimental results, and the device platform geometry.

B. LQR Controller

There are multiple ways of designing a state feedback controller: using pole placement technique [4] or using optimal control approaches such as LQR. For the former method, the designer knows and sets a vector of eigenvalues and calculates the gain values for the closed loop feedback. It is a non-intuitive method that requires advanced knowledge and system exploration. LQR approach uses a quadratic function of the state variable and the control inputs to achieve a compromise between minimizing the regulator error and the control effort [3, Chapter 10, Section 3]. The quadratic cost function to be minimized is presented in Equation 14.

$$J = \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t)) \tag{14}$$

where Q is a positive semi-definite symmetric matrix of regulator weights and R of effort weights, respectively. The optimal feedback gain is

$$K = R^{-1}B^TP \tag{15}$$

where P is a solution to the Riccati equation:

$$A^{-1}P + PA - PBR^{-1}B^TP + Q = 0 \tag{16}$$

Finally, the feedback control signal follows the Equation 17.

$$u = -Kx \tag{17}$$

The control structure with state-space representation is depicted on Figure 3.

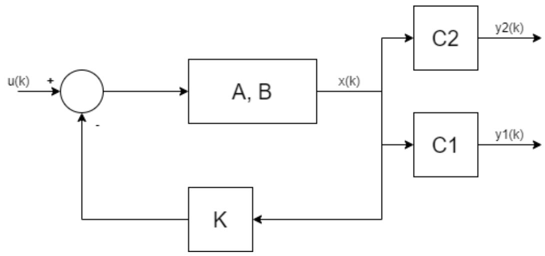

Fig. 3: Block diagram of the proposed control structure.

III. Simulation

For the simulation of the present project, MATLAB and Simulink were mainly used. On the other hand, a Crouzet brushless DC motor was used. Its datasheet can be found following this link, and important for the simulation parameters in Table I.

Parameter	Value
Phase-to-phase resistance, R_m	1.72Ω
Inductance, L	$3.8mH$
Torque constant, K_m	$74.5mNm/A$
Rotor inertia, J	$50gcm^2$
Viscous friction, K_d	$3.2mNms{-}1/rad$

TABLE I: DC motor parameters used for the project simulation.

Tracker device platform important geometry parameters are listed in Table II.

Parameter	Value
Platform radius, R_p	$15cm$
Wheel radius, R_w	$3.5cm$

TABLE II: Tracker device platform parameters used for the project simulation.

Before developing the state feedback control, open loop response was examined. The open loop response is presented in Figure 4.

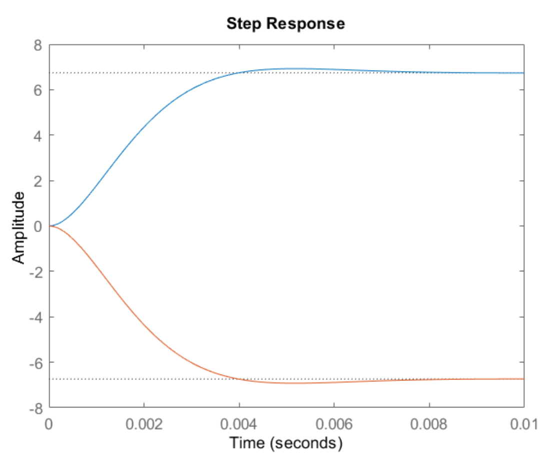

Fig. 4: Open loop step response.

Steady state (SS) error of more than 6 rad/s can be clearly observed. There is a serious need for a state feedback controller. Figure 5 presents step response with the LQR controller implemented.

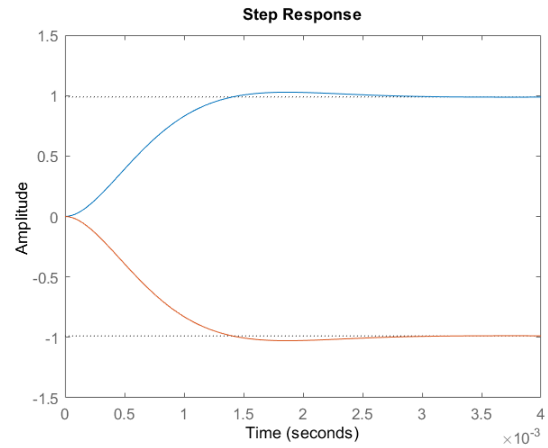

Fig. 5: Step response with the LQR controller.

The MATLAB code is presented in Listing 1.

Listing 1: Matlab simulation code.

```
clear all
close all

Rm = 1.72;    % Phase−to−phase resistance, Ohms
Lm = 3.8e−3;  % Inductance, H
Ka = 74.5e−3; % Torque constant, Nm/A
J  = 50e−7;   % Rotor intertia, kg*m^2
Ba = 3.2e−3;  % Torque/speed factor | sliding/viscous friction | damping ratio of intertia or rotor, N*m/(rad/s)

Rp = 15e−2;   % Tracker platfrom radius
Rw = 3.5e−2;  % Wheel radius
Cw = 2*pi*Rw; % Wheel circumference

% Since the device will act in a 'tank' mode, that is, both dc motors
%   will exercise 'mirrored' efforts, and, on the other hand,
%   both motors have 'identical' characteristics,
%   the state space and input matrices will be shared by both
%   motors. The output matrices will be different.

% State transition matrix
Am = [−Rm/Lm          −Ka/Lm;
      (Ka/J)*(Cw/Rp) −(Ba/J)*(Cw/Rp)];
```

```
% Input matrix
Bm = [1/Lm; 0];
% Output matrix
Cm1 = [0 1];
% Direct transmission matrix
Dm = 0;%0.1*Rp^2/4;

% Output matrix for the second motor. (Mirrored action)
Cm2 = [0 -1];


% Penalize bad performance
Q = [1 0;  % current consumption
     0 1]; % angular speed
% Penalize actuator effort
R = 1;     % voltage

[K, S, e] = lqr(Am, Bm, Q, R);
Acl = Am - Bm*K; % closed loop transition matrix

figure (1)
step(Am, Bm, Cm1, Dm)
hold
step(Am, Bm, Cm2, Dm)

figure (2)
step(Acl, Bm, Cm1, Dm)
hold
step(Acl, Bm, Cm2, Dm)
```

Simulink diagrams for the open loop and LQR-controlled loop are presented in Figures 6 and 7.

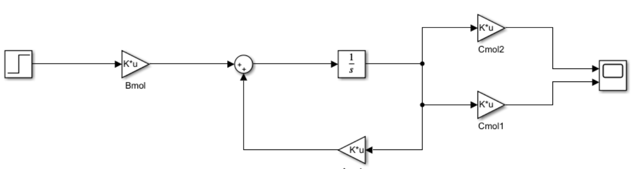

Fig. 6: Simulink diagram of the open loop system.

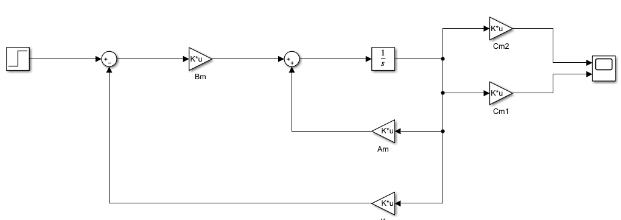

Fig. 7: Simulink diagram of the LQR-controlled system.

IV. Conclusion

A. Improvements and Future Work

There are many improvements that should be done to the project before its final prototyping. The following descriptions block highlights the most important improvements.

State Space Model Current state space model omits very important system parameters. Since it is heavily based on the DC motor state space representation, there is no room for taking into account platform dynamics. The device platform exercises considerable load function for the DC motors. It owns its inertia moment $I = \frac{mR^2}{2}$ and solid body rotational physics that must be considered.

Non-Linear Effects Current state space representation assumes system behavior to be linear. It is not always true, and negligence of considering it can result in significant disadvantages. Coulomb friction is one of the greatest sources of non-linearity [5]. To manage this behavior, an extension of non-linear nature, as suggest authors of [6], needs to be added to the current representation.

Feedforward Reference Gain Even though, the steady state (SS) error for the LQR-controlled system is zero, it is a good idea to assure the reference tracking zero error using feedforward gain approach described in [3, Chapter 10, Section 4.3].

Integral Action Feedforward reference gain is a good approach for assuring SS zero error, keeping the system as simple as possible. However, it can be very sensitive to the input noise and have poor ability to recover zero SS error after a disturbance occurred. Hence, the system needs to be augmented with the integral action. Using this approach, the system will always achieve zero SS error. One downside is that after the integral action implementation, it can be necessary to adjust the system poles that requires advanced knowledge of the system dynamics and control theory.

B. Meditation

Development and simulation of an LQR controller system was undertaken in the present project. It was a very interesting and intriguing journey to go through the process of application of theory to practice. Control theory turns extremely theoretical without practical projects. In this project, I faced many challenges related to the realm of practical applications, physics, and design. I am satisfied with achieved results, even though they can be improved.

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GLOSSARY

armature Winding (or set of windings) of an electric machine which carries alternating current.. 2

DC Direct Current. 2

EMF Moving component of an electromagnetic system in the electric motor, electric generator, or alternator.. 3

MCU Microcontroller is a small computer on a single metal-oxide-semiconductor (MOS) integrated circuit (IC) chip usually limited in computational and power resources and optimized for a specific application.. 2

rotor Moving component of an electromagnetic system in the electric motor, electric generator, or alternator.. 2

SS A condition of a physical system or device that does not change over time, or in which any one change is continually balanced by another, such as the stable condition of a system in equilibrium.. 4

torque Rotational equivalent of linear force.[1] It is also referred to as the moment, moment of force, rotational force or turning effect, depending on the field of study.. 2

wide angle The angle of view is the visible extent of the scene captured by the image sensor, stated as an angle.. 2