Orbifolds and Graphs of Groups as Étale Groupoids

an (ideally very gentle) introduction

Rylee Lyman

Rutgers University-Newark

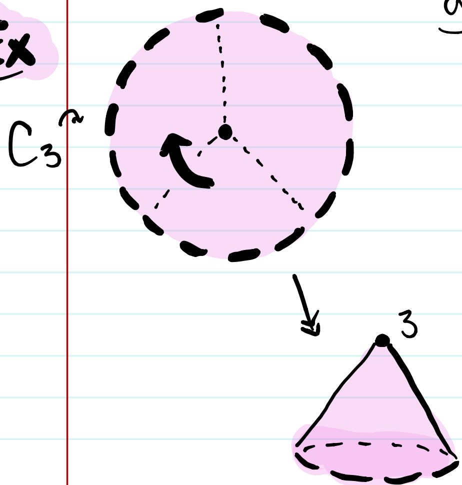
this is expository, but comes from joint work in progress with Tyrone Ghaswala

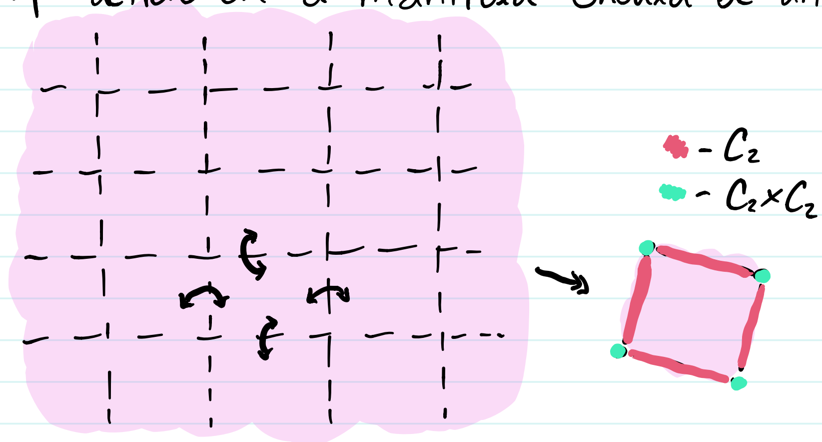
Orbifolds

Idea: If manifolds are locally modeled on $\mathbb{R}^n$, orbifolds are locally modeled on $\mathbb{R}^n$ modulo finite group actions.

The quotient of a properly discontinuous group action on a manifold should be an example

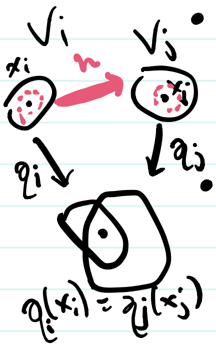
Ex





Orbifolds: The formal definition

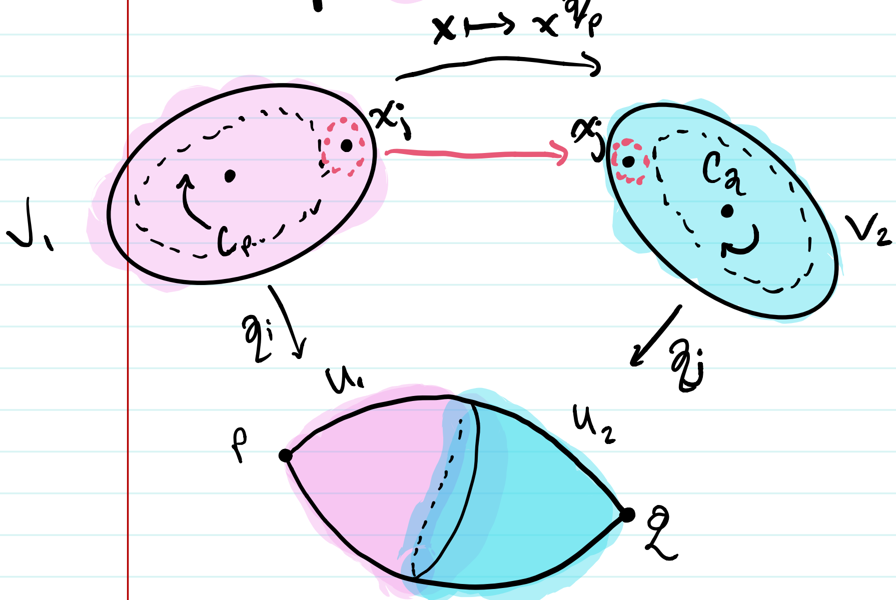
An n-dim'l orbifold structure on a Hausdorff space $\mathcal{O}$ is



- An open cover $\{U_i\}_{i \in I}$ of $\mathcal{O}$
- $\forall i$, a finite subgroup Γ_i acting smoothly (and effectively) on $V_i \subset \mathbb{R}^n$ open and a map $q_i : V_i \longrightarrow U_i$ s.t. q_i induces a homeomorphism $\Gamma_i \backslash V_i \xrightarrow{\sim} U_i$.
- $\forall x_i \in V_i$ and $x_j \in V_j$ s.t. $q_i(x_i) = q_j(x_j)$, $\exists$ a diffeomorphism h from a connected nbhd W of x_i to a nbhd of x_j s.t. $q_j \circ h = q_i|_W$

Compare the definition of a manifold!

Example: The (p,q)-football



Orbifolds

Neat! Ty and I are interested in the mapping class group of an orbifold

So, what's a map of an orbifold?

We-ell...

- Satake gives different definitions in different papers
- It's not clear whether some of Satake's maps can be composed
- Working with charts is awkward

So What to do?

The "modern" approach is to use **étale groupoids**

So what's a groupoid?

Idea: a groupoid is like a group with many "basepoints".

Ex The set of homotopy classes of paths in a space X is a groupoid:

- paths can be composed if their endpoints line up
- composition is associative
- every path has an inverse

Groupoids: The formal definition

A groupoid $\mathcal{G}$ is a set of objects $\mathcal{G}_0$, a set of arrows $\mathcal{G}_1$ and several structure maps

- two maps $\alpha, \omega : \mathcal{G}_1 \to \mathcal{G}_0$ sending an arrow $\underset{x}{\bullet} \xrightarrow{g} \underset{y}{\bullet}$ to its **source** x and its **target** y, respectively
- a map $\mathcal{G}_0 \to \mathcal{G}_1$ sending an object x to its **identity** arrow $1_x : x \to x$
- a **multiplication** map $\{(g,h) \in \mathcal{G}_1 \times \mathcal{G}_1 : \alpha(g) = \omega(h)\} \to \mathcal{G}_1$ sending $(g,h) \longmapsto gh$
- an **inversion** map $\mathcal{G}_1 \to \mathcal{G}_1$ sending $g : x \to y$ to $g^{-1} : y \to x$

Étale Groupoids

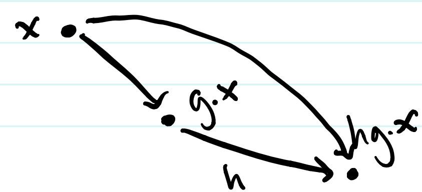
A **topological** groupoid is a groupoid $\mathcal{G}$ where $\mathcal{G}_0$ and $\mathcal{G}_1$ are spaces and the structure maps are continuous. It is **étale** if α and ω are étale maps, i.e. local homeomorphisms

Ex. Suppose G is a discrete group acting on a space X. The **action groupoid** $G \ltimes X$ is étale

$$(G \ltimes X)_0 = X, \quad (G \ltimes X)_1 = G \times X,$$

$$\alpha(g,x) = x, \quad \omega(g,x) = g \cdot x, \quad (h, g\cdot x)(g,x) = (hg, x)$$

a groupoid is **developable** if it's equivalent to an action groupoid



Orbifolds as Groupoids

Let $\{ q_i : V_i \longrightarrow U_i \}$ be our atlas of charts.

The space of objects will be $\coprod_i V_i$.

Given $x \in V_i$, $\exists$ an arrow $x \longrightarrow g \cdot x$ $\forall g \in \Gamma_i$

and for each change of charts h, an arrow $x \longrightarrow h(x)$

Topologizing the space of arrows uses "germs" of changes of charts

isotropy groups in groupoid are discrete

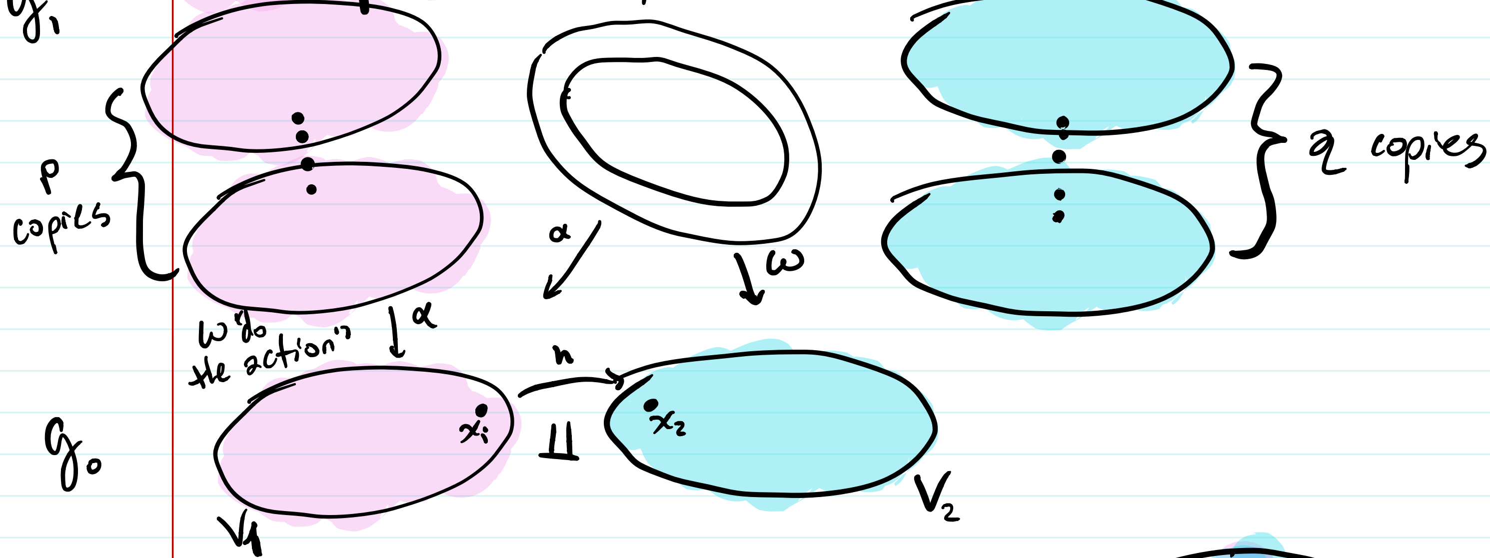
Fact: This groupoid $\mathcal{G}$ is étale, and $\alpha \times \omega : \mathcal{G}_1 \longrightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is proper.

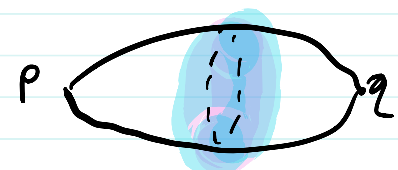
$\mathcal{G}_0$ and $\mathcal{G}_1$ are manifolds

orbifold groupoids are groupoids w/ these properties

isotropy groups are compact

Example: The (p,q)-football





Graphs of Groups

\mathscr{G}

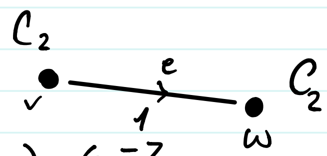
A graph of groups $(\Gamma, \mathscr{G})$ is a graph Γ, and groups $\mathscr{G}_v$ and $\mathscr{G}_e$ for each vertex v and edge e of Γ, together with injective homomorphisms

$$\iota_e : \mathscr{G}_e \to \mathscr{G}_{\tau(e)} \qquad \iota_{\bar{e}} : \mathscr{G}_e \to \mathscr{G}_{\tau(\bar{e})}$$

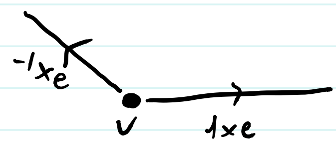
Let $st(v)$ be the set of oriented edges e w/ $\tau(e) = v$ and define

$$\widetilde{st}(v) = \coprod_{e \in st(v)} \mathscr{G}_v / \iota_e(\mathscr{G}_e) \times \{e \setminus \tau(\bar{e})\} \Big/ {\tau(e) \sim \tau(e')}$$

Ex.



$st(v) = \{\bar{e}\}$

$\widetilde{st}(v) =$ 

Graphs of Groups as Groupoids

Let $(\Gamma, \mathcal{G})$ be a graph of groups.

$\mathcal{G}_v \curvearrowright \tilde{st}(v)$ The space of objects will be $\coprod_{v \in \Gamma} \tilde{st}(v)$.

For $p \in \tilde{st}(v)$ and $g \in \mathcal{G}_v$, $\exists$ an arrow $p \longrightarrow g \cdot p$

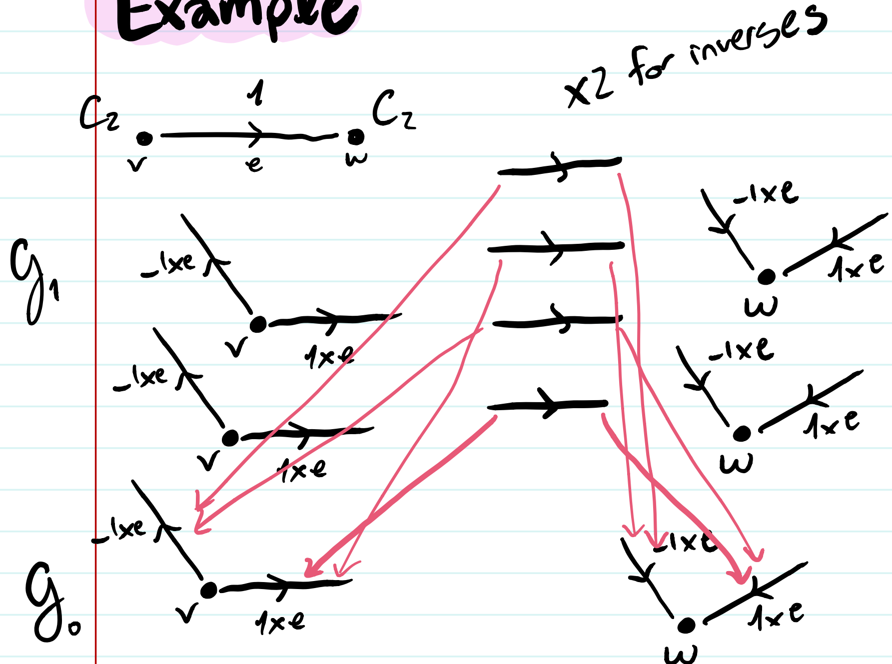
For p in the interior of an edge e, there is also an arrow

$$e_p : ([1], p) \in \tilde{st}(\tau(e)) \longrightarrow ([1], p) \in \tilde{st}(\tau(\bar{e}))$$

for $h \in \mathcal{G}_e$ $\qquad e_p \, \iota_e(h) = \iota_{\bar{e}}(h) \, e_p$

there are composites of $g \in \mathcal{G}_v$ w/ e_p

Example



C_2
1
C_2
v
e
w
x2 for inverses
$\mathcal{G}_1$
-1xe
v
1xe
-1xe
v
1xe
-1xe
$\mathcal{G}_0$
v
1xe
-1xe
w
1xe
-1xe
w
1xe
-1xe
w
1xe

Étale Groupoids

So, what are maps between étale groupoids?

We-ell...

A **functor** $f : \mathcal{G} \to \mathcal{H}$ is a pair of cts maps $f : \mathcal{G}_0 \to \mathcal{H}_0$ and $f : \mathcal{G}_1 \to \mathcal{H}_1$ that commute with the structure maps, e.g. $f(gh) = f(g)f(h)$ and $f(1_x) = 1_{f(x)}$.

A **natural transformation** η between functors $f, f' : \mathcal{G} \to \mathcal{H}$ is a cts map $\eta : \mathcal{G}_0 \to \mathcal{H}_1$ s.t. $\forall g : x \to y$ in $\mathcal{G}_1$,

$$f'(g)\eta(x) = \eta(y)f(g)$$

$$\begin{tikzcd} f(x) \arrow[r, "\eta(x)"] \arrow[d, "f(g)"'] & f'(x) \arrow[d, "f'(g)"] \\ f(y) \arrow[r, "\eta(y)"'] & f'(y) \end{tikzcd}$$

we'll identify functors related by a natural transformation

The Problem

A functor of (not topological) groupoids $f: \mathcal{G} \to \mathcal{H}$ is an **equivalence** if there exists $g: \mathcal{H} \to \mathcal{G}$ and natural transformations $\eta: gf \Rightarrow 1_{\mathcal{G}}$ and $\varepsilon: fg \Rightarrow 1_{\mathcal{H}}$

Thm Assuming the axiom of choice, $f: \mathcal{G} \to \mathcal{H}$ is an equivalence if and only if it is

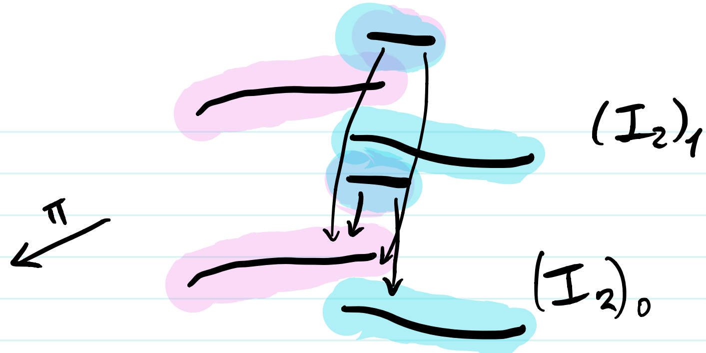
1) **fully faithful**: $\forall x, y \in \mathcal{G}_0$, f induces a bijection

$$\{g \in \mathcal{G}_1 : g: x \to y\} \longrightarrow \{g \in \mathcal{H}_1 : g: f(x) \to f(y)\}$$

2) **essentially surjective**: $\forall x \in \mathcal{H}_0 \ \exists \, g: f(y) \to x$

The problem is that theorem fails for continuous functors!

Example



I

π is fully faithful and essentially surjective but has no inverse!

It turns out that up to playing this open cover game, every equivalence is invertible

The Fix

We formally invert the equivalences

A **generalized map** of étale groupoids $\mathcal{G} \to \mathcal{H}$

$$\mathcal{G} \xleftarrow{\sim} \mathcal{G}' \longrightarrow \mathcal{H}$$

equivalence functor

it turns out you can always take $\mathcal{G}'$ to be obtained from $\mathcal{G}$ by the open cover construction

The mapping class group of an étale groupoid is (homotopy) equivalences up to homotopy/isotopy