Bass-Serre Theory: Groups acting on trees

Geometric Group Theory without Boundaries

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Ref: my thesis, **Train Track maps on Graphs of Groups and Outer Automorphisms of Hyperbolic Groups**, Ch. 1

Lecture 2 7/21/20

Today:
- A taste of constructing the tree
- A helpful aside: graphs of spaces
- The fundamental group of a graph of groups
  - w/ examples
- Covering space theory, morphisms (maybe)

The Bass-Serre Tree: general nonsense

Recall Thm: Let $(\Gamma, \mathcal{G})$ be a graph of groups, and $p \in \Gamma$. Then there exists a group $G = \pi_1(\Gamma, \mathcal{G}, p)$, a tree T and an action of G on T s.t. the quotient graph of groups is $(\Gamma, \mathcal{G})$.

Note: If we assume G and T exist, then the orbit-stabilizer theorem says that vertices $\tilde{v}$ in T w/ $\pi(\tilde{v}) = v$ ...

$$\begin{array}{c} T \\ \Big\downarrow \pi \\ G \backslash T = \Gamma \end{array}$$

Similarly edges $\tilde{e}$ w/ $\pi(\tilde{e}) = e$

The Bass-Serre Tree: first observations

Fix $(\Gamma, \mathcal{G})$. Let v be a vertex of Γ.

Def. Write

$$st(v) = \{ \text{oriented edges } e : \tau(e) = v \}$$

Ex.

$$st(v) = \{ e, \bar{e} \}$$

If $G \curvearrowright T$ exists and $\tilde{v} \in T$ is a vertex in $\pi^{-1}(v)$

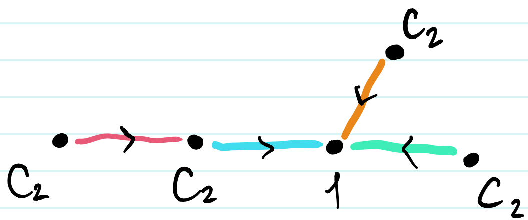
$$St(\tilde{v}) \cong$$

G_v-equivariant

The Bass-Serre Tree: construction using examples

First case: Γ is a tree, $G_e = 1$ $\forall$ edges e.

Ex.



We will build T from copies of Γ so that the observation holds

The Bass-Serre Tree

Second case: Γ is a tree.

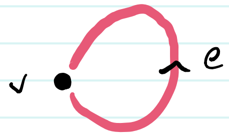
Ex.

$$\begin{tikzcd} C_4 \arrow[r, "C_2"] & C_6 & C_2 \arrow[l, "1"'] \end{tikzcd}$$

The Bass-Serre Tree

General case: construct the (ordinary) universal cover $\tilde{\Gamma}$ of Γ, for $\tilde{v} \in \tilde{\Gamma}$, set $\tilde{\mathcal{G}}_{\tilde{v}} = \mathcal{G}_v$ etc. Now build the tree for $(\tilde{\Gamma}, \tilde{\mathcal{G}})$

Ex.



$\mathcal{G}_v = \mathbb{Z} = \langle a \rangle \qquad \iota_e : b \longmapsto a^2$

$\mathcal{G}_e = \mathbb{Z} = \langle b \rangle \qquad \iota_{\bar{e}} : b \longmapsto a$

A Helpful Aside: Graphs of spaces

Given a graph of groups $(\Gamma, \mathcal{G})$,
choose, for each v, a space X_v w/ $\pi_1(X_v) \cong \mathcal{G}_v$,
for each e, X_e w/ $\pi_1(X_e) \cong \mathcal{G}_e$ and
maps $f_e, f_{\bar{e}} : X_e \longrightarrow X_{\tau(e)}, X_{\tau(\bar{e})}$
s.t. $f_{e*} : \mathcal{G}_e \longrightarrow \mathcal{G}_{\tau(e)}$ is ι_e.

Recall: Given $f: X \to Y$, the **mapping cylinder** of f is

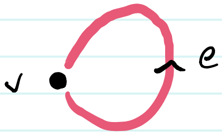
A Helpful Aside: Graphs of spaces

The **graph of spaces** $X = (\Gamma, \mathcal{G}, M_{f_e})$ is

$$X = \coprod_{e \in E(\Gamma)} M_{f_e} \Big/ \sim$$

where

Ex.



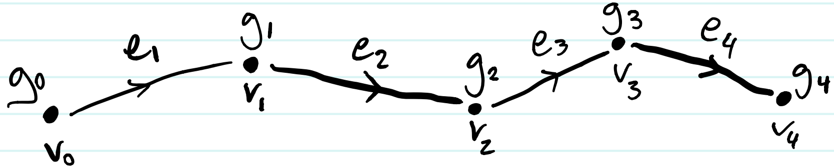
$G_v = \mathbb{Z} = \langle a \rangle$ $\qquad \iota_e : b \longmapsto a^2$

$G_e = \mathbb{Z} = \langle b \rangle$ $\qquad \iota_{\bar{e}} : b \longmapsto a$

Useful: The universal cover $\tilde{X}$ maps to T. Graphs of aspherical spaces are aspherical.

The Fundamental Group

A **path** in $(\Gamma, \mathcal{G})$ is a sequence



If $\gamma = g_0 e_1 \cdots e_n g_n$, write $\bar{\gamma} = g_n^{-1} \bar{e}_n \cdots \bar{e}_1 g_0^{-1}$.

Def. Two paths γ and γ' are **homotopic** (rel endpoints) if γ can be transformed into γ' by a finite sequence of moves

1.

2.

The Fundamental Group

A path $\gamma = g_0 e_1 \cdots e_n g_n$ is a **loop** if $\tau(\bar{e}_1) = \tau(e_n)$. it is **based** at $\tau(e_n)$.

Def. The **fundamental group** $\pi_1(\Gamma, \mathcal{G}, p)$ is the group of homotopy classes of loops based at p. The operation is concatenation.

$(\Gamma, \mathcal{G})$ is a **splitting** of $\pi_1(\Gamma, \mathcal{G}, p) = G$ if G is not equal to any $\mathcal{G}_v$.
G **splits** if it admits a splitting.

The Fundamental Group

Let $T_0 \subseteq \Gamma$ be a spanning tree

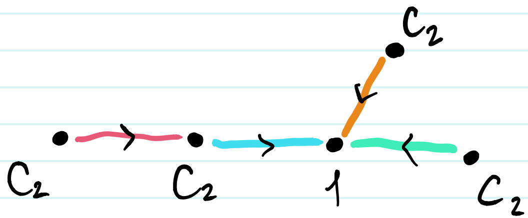
Thm $\pi_1(\Gamma, \mathcal{G}, p)$ is the quotient of

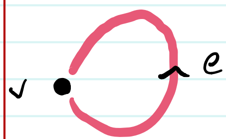
$$\mathop{\ast}_{v \in \Gamma} \mathcal{G}_v \ast F(E(\Gamma))$$

by the normal subgroup imposing the relations

1. $e = \bar{e}$,
2. $e = 1$ if $e \in T_0$
3. $\iota_{\bar{e}}(h)\, e = e\, \iota_e(h)$
   $h \in \mathcal{G}_e$

Examples:





$G_v = \mathbb{Z} = \langle a \rangle$ $\qquad \iota_e : b \longmapsto a^2$

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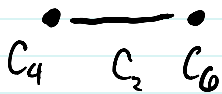
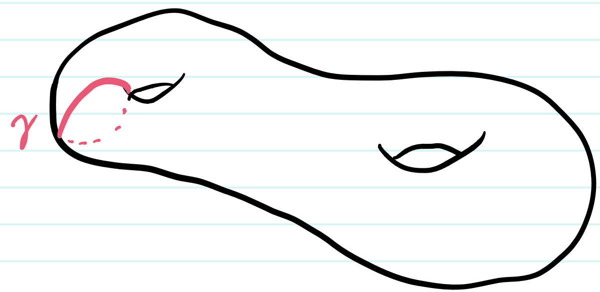
$SL_2(\mathbb{Z})$





A normal form

For each oriented edge e, choose a set S_e of representatives for $G_v / \iota_e(G_e)$ containing 1.

Each path γ is homotopic to a path of the form

$$g_0 e_1 \cdots e_n g_n$$

where $g_i \in S_{e_i}$ $1 \leq i \leq n$.

loops with nontrivial normal form are nontrivial in π_1

Morphisms *

Def. Since a graph of groups is a category Γ and a functor $\mathcal{G} : \Gamma \longrightarrow \text{Groups}^{\text{mono}}$, a **morphism** $f : (\Delta, \mathcal{L}) \longrightarrow (\Gamma, \mathcal{G})$ should be a **functor** $f : \Delta \longrightarrow \Gamma$ and a **morphism of functors** $f : \mathcal{L} \Longrightarrow \mathcal{G}f$.

It turns out the latter is a **pseudonatural transformation**

$$\begin{tikzcd}
\mathcal{L}_e \arrow[r, "f_e"] \arrow[d, "\iota_e"'] & \mathcal{G}_{f(e)} \arrow[d, "\mathrm{ad}(\delta_e)\cdot\iota_{f(e)}"] \\
\mathcal{L}_v \arrow[r, "f_v"'] & \mathcal{G}_{f(v)}
\end{tikzcd}$$

English:

The Local Map

Given $f:(\Delta,\mathcal{L}) \longrightarrow (\Gamma,\mathcal{G})$, for $v\in\Gamma$, if f does not collapse edges, define

$$f_{st(v)}: \coprod_{e\in st(v)} \mathcal{L}_v/\iota_e(\mathcal{L}_e)\times\{e\} \longrightarrow \coprod_{e\in st(f(v))} \mathcal{G}_{f(v)}/\iota_e(\mathcal{G}_e)\times\{e\}$$

$$(g\,\iota_e(\mathcal{L}_e), e) \longmapsto (f_v(g)\,\delta_e\,\iota_{f(e)}(\mathcal{G}_{f(e)}), f(e))$$

Notice: $f_v(g\,\iota_e(h))\,\delta_e = f_v(g)\,f_v\,\iota_e(h)\,\delta_e$

$= f_v(g)\,\delta_e\,\iota_{f(e)}\,f_e(h)\,\delta_e^{-1}\,\delta_e$

$\in f_v(g)\,\delta_e\,\iota_{f(e)}(\mathcal{G}_{f(e)})$.

Coverings

A morphism $f:(\Delta, \mathcal{Z}) \longrightarrow (\Gamma, \mathcal{G})$ is an **immersion** if it does not collapse edges and each $f_{st(v)}$ is injective.

If $f:\Delta \longrightarrow \Gamma$ is surjective and each $f_{st(v)}$ is a bijection, then $f:(\Delta, \mathcal{Z}) \longrightarrow (\Gamma, \mathcal{G})$ is a **covering map**.

Ex.