CTs for free products

Rylee Alanza Lyman

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Abstract

The fundamental group of a finite graph of groups with trivial edge groups is a free product. We are interested in those outer automorphisms of such a free product that permute the conjugacy classes of the vertex groups. We show that in particular cases of interest, such as where vertex groups are themselves finite free products of finite and cyclic groups, given such an outer automorphism, after passing to a positive power, the outer automorphism is represented by a particularly nice kind of relative train track map called a CT. CTs were first introduced by Feighn and Handel for outer automorphisms of free groups. We develop the theory of attracting laminations for and principal automorphisms of free products. We prove that outer automorphisms of free products satisfy an index inequality reminiscent of a result of Gaboriau, Jaeger, Levitt and Lustig and sharpening a result of Martino. Finally, we prove a result reminiscent of a theorem of Culler on the fixed subgroup of an automorphism of a free product whose outer class has finite order.

A homotopy equivalence $f\colon G \to G$ of a connected graph G is a *train track map* when it maps vertices to vertices and f^k restricts, on each edge of G, to an immersion for all $k \geq 1$. Train track maps were introduced by Bestvina–Handel in [BH92], where they showed that an appropriate generalization, called a *relative train track map,* exists representing each $\varphi \in \mathrm{Out}(F_n)$, the outer automorphism group of a free group of finite rank n.

There are by now many proofs of the existence of relative train track maps for free products: the original proof by Collins–Turner [CT94] used graphs of spaces à la Scott and Wall [SW79]; Francaviglia and Martino [FM15] used the Lipschitz metric on Outer Space and viewed relative train track maps as twisted equivariant maps of trees; and the present author [Lym21] worked directly in the graph of groups.

In the intervening time, the theory of relative train track maps for $\mathrm{Out}(F_n)$ has progressed, first with *improved relative train track maps* in [BFH00], and more recently with *CTs* in [FH11]. Free products have lagged behind. The purpose of this paper is to rectify this situation.

Let F be a group that decomposes as a free product of the form

$$F = A_1 * \cdots * A_n * F_k,$$

where the A_i are countable groups and F_k is a free group of rank k. Let $[[A_i]]$ denote the conjugacy class of A_i, and write $\mathcal{A} = \{[[A_1]], \ldots, [[A_n]]\}$. Write $\mathrm{Out}(F, \mathcal{A})$ for the subgroup of $\mathrm{Out}(F)$ consisting of outer automorphisms that permute the conjugacy classes in $\mathcal{A}$.

Theorem A. *Each rotationless $\varphi \in \mathrm{Out}(F, \mathcal{A})$ is represented by a CT.*

See Section 6 for the definition of a CT, and see Section 5 for the definition of rotationless. The question of whether a given $\varphi \in \mathrm{Out}(F, \mathcal{A})$ has a rotationless power seems to require some restrictions on groups in $\mathcal{A}$: our sufficient condition in Proposition 5.7 requires finite-order elements of $A \in \mathcal{A}$ to have bounded order and requires each automorphism $\Phi\colon A \to A$ to have a bound on the period of Φ-periodic elements. It would be interesting to know to what extent this sufficient condition is necessary. We prove in Corollary 5.8 together with

Proposition 5.19 that if F is a finite free product of finite and cyclic groups (so that each group in $\mathcal{A}$ is again of this form) then each $\varphi \in \mathrm{Out}(F, \mathcal{A})$ has a rotationless power.

We present one application of Theorem A. The original application of relative train track maps in [BH92] was to prove the *Scott conjecture,* which predicts that for an automorphism $\Phi\colon F_n \to F_n$, the rank of the fixed subgroup $\mathrm{Fix}(\Phi)$ satisfies

$$\mathrm{rank}(\mathrm{Fix}(\Phi)) \leq n.$$

Collins and Turner [CT94] generalized this result to free products; here the *(Kurosh) rank* of a free product $F = A_1 * \cdots * A_n * F_k$ is the quantity $n + k$. A subgroup H of F inherits a free product decomposition from $\mathcal{A}$ and thus there is a notion of the *(Kurosh subgroup) rank* of H, which a priori may be infinite. Their results imply that if $\Phi\colon (F, \mathcal{A}) \to (F, \mathcal{A})$ is an automorphism of F preserving $\mathcal{A}$, then

$$\mathrm{rank}(\mathrm{Fix}(\Phi)) \leq n + k.$$

In actual fact, they state their results only for the Grushko decomposition of a finitely generated group, but their proofs do not use this assumption in any essential way.

Back in the setting of free groups, Gaboriau, Jaeger, Levitt and Lustig [GJLL98] define an *index* $i(\varphi)$ for an outer automorphism $\varphi \in \mathrm{Out}(F_n)$; it is a sum over automorphisms $\Phi\colon F_n \to F_n$ representing φ up to an equivalence relation called *isogredience* (see Section 5 for a definition) of the quantity

$$\max\left\{0, \mathrm{rank}(\mathrm{Fix}(\Phi)) + \frac{1}{2}\alpha(\Phi) - 1\right\},$$

where $\alpha(\Phi)$ is the number of $\mathrm{Fix}(\Phi)$-orbits of *attracting fixed points* (see Section 4) for the action of Φ on the Gromov boundary ∂F_n. They prove that

$$i(\varphi) \leq n - 1.$$

Martino [Mar99] established the analogous result for free products.

Most recently, Feighn and Handel define a new index $j(\varphi)$ that gives weight 1 rather than $\frac{1}{2}$ to certain classes of attracting fixed points called *eigenrays*, see Section 7 for details. Using CTs in [FH18], they show that their index satisfies

$$j(\varphi) \leq n - 1.$$

Following [GH19], we say a *Grushko* $(F, \mathcal{A})$*-tree* is a simplicial tree T equipped with an F-action with trivial edge stabilizers such that $\mathcal{A}$ is the set of conjugacy classes of nontrivial vertex stabilizers. Rather than consider the Gromov boundary of some (and hence any) Grushko $(F, \mathcal{A})$-tree T, we, like Guirardel–Horbez, consider the *Bowditch boundary,* which has the advantage of being compact (when the groups in $\mathcal{A}$ are countable) and which is obtained from the Gromov boundary by adding the vertices in T of infinite valence, suitably topologized. An automorphism may fix one of these points of infinite valence without fixing any elements of its stabilizer group. In our version of the index theorem (see Theorem 7.1 and Corollary 7.16) we count the number of $\mathrm{Fix}(\Phi)$-orbits of such infinite valence points as well, and we obtain the following theorem.

Theorem B. *Suppose that each group in $\mathcal{A}$ is finitely generated. If $\varphi \in \mathrm{Out}(F, \mathcal{A})$ has a rotationless power, the index $j(\varphi)$ satisfies*

$$j(\varphi) \leq n + k - 1.$$

Because our proof uses CTs, the rotationless power assumption is necessary. A key step in the proof uses "Nielsen realization" for free products, due to Hensel–Kielak [HK18], who require groups in $\mathcal{A}$ to be finitely generated. It would be interesting to know whether a

Nielsen realization theorem holds without the finite generation assumption, perhaps following Hensel–Kielak's proof using Grushko $(F, \mathcal{A})$-trees rather than particular Cayley graphs for F.

Along the way to proving Theorem B, we prove a version of a theorem of Culler [Cul84, Theorem 3.1], which may be of independent interest. See Proposition 7.8 for a precise statement. The realization statement in the theorem relies on [HK18]. A subgroup of F is non-peripheral if it is not conjugate into any $A \in \mathcal{A}$.

Theorem C. *Suppose $\varphi \in \mathrm{Out}(F, \mathcal{A})$ has finite order, that each $A \in \mathcal{A}$ is finitely generated, and that Φ represents φ. There exists an automorphism of graphs of groups $f : \mathcal{G} \to \mathcal{G}$ representing φ. Either* $\mathrm{Fix}(\Phi)$ *is cyclic and non-peripheral or* $\mathrm{Fix}(\Phi)$ *is conjugate to the fundamental group of a graph of groups $\mathcal{H}$ that has an injective-on-edges immersion into a component of* $\mathrm{Fix}(f)$.

Culler's theorem says that if Φ represents $\varphi \in \mathrm{Out}(F_n)$ and $\mathrm{Fix}(\Phi)$ is not cyclic, then it is a free factor that is a component of $\mathrm{Fix}(f)$. In our theorem, the injective-on-edges assertion is meant to replace the free factor conclusion, which is too strong in general. Likewise, because of the presence of almost fixed but not fixed edges, one should not expect an entire component of $\mathrm{Fix}(f)$ to appear in the statement.

The broad-strokes strategy of this paper is to find the correct equivariant perspective so that arguments scattered across [BH92], [BFH04], [BFH04], [FH11], [FH18], and other papers can be adapted without too much extra effort. We have made some effort to synthesize these results, providing a self-contained exposition with the exception of the proof of the basic existence of relative train track maps, for which the reader is referred to [CT94], [FM15] or [Lym21].

Despite the similarities with the case of free groups, there are also some interesting differences. We describe a few for the expert reader. Perhaps the first difference is the presence of the word *almost* in constructions like *almost periodic, almost Nielsen path,* and *almost linear.* Here *almost* refers to the presence of vertex group elements.

A reader familiar with relative train track theory for $\mathrm{Out}(F_n)$ might like to imagine beginning with a relative train track map of a graph $f : G \to G$, and then collapsing each component of some filtration element G_{r-1} to obtain a map of graphs of groups $f' : \mathcal{G} \to \mathcal{G}$. We show in Section 1 that f' is again a relative train track map, but to define the map of the collapsed graph of groups $f' : \mathcal{G} \to \mathcal{G}$, one needs to make some choices (see Section 1 for details). The failure of f to preserve these choices results in the presence of vertex group elements at the ends of images of edges with endpoints in G_{r-1}.

Thus a non-exponentially growing edge E of H_r in G becomes an *almost periodic edge* of H'_r in $\mathcal{G}$; here "almost" because in general we have $f'(E) = Eg$ for some vertex group element g. By making different choices, we may transform for instance almost fixed edges into genuinely fixed edges, but it is typically not possible to do this simultaneously for each edge incident to a vertex with nontrivial vertex group. An *almost Nielsen path* differs from a Nielsen path in that $f(\sigma)$ is homotopic to $g\sigma h$ for vertex group elements g and h. An *almost linear edge* has a suffix which is an almost Nielsen path.

A second difference, observed by Martino in his thesis [Mar98], has to do with almost linear edges that are not genuinely linear. Suppose $f : G \to G$ is a relative train track map and that E is an edge in an exponentially growing stratum H_r such that $f(E) = Eu$ for some path u. If $\tilde{E}$, $\tilde{u}$ and $\tilde{f} : \Gamma \to \Gamma$ are lifts to the universal cover such that the path $\tilde{f}(\tilde{E}) = \tilde{E}\tilde{u}$, then

$$\tilde{E} \subset \tilde{f}_\sharp(\tilde{E}) \subset \tilde{f}^2_\sharp(\tilde{E}) \subset \cdots$$

is an increasing sequence of tight paths in Γ whose union is a ray $\tilde{R}_{\tilde{E}}$ converging to a fixed point $P \in \partial\Gamma \cong \partial F_n$. The number of H_r-edges of the path $f^k_\sharp(u)$ grows exponentially so [GJLL98, Proposition 1.1] implies that P is an attracting fixed point. In fact, one does not need exponential growth of u to show that P is an ("algebraic") attracting fixed point in this way, merely that it grows at all. If it does not grow, then since there are finitely many paths of a given length in a graph, some $f^k_\sharp(u)$ is a periodic Nielsen path, and P is instead in the boundary of the fixed subgroup of the automorphism corresponding to the lift $\tilde{f}$. For free products, it is no longer true in general that there are finitely many graph-of-groups paths of a given length, and in fact if $f^k_\sharp(u)$ is a concatenation of almost Nielsen paths but not a Nielsen path, then one cannot expect P to be in the boundary of the fixed subgroup. Fortunately, Martino shows that in this situation although P is not an "algebraic" attracting fixed point, it is still an attracting fixed point; see Proposition 3.4 and Example 7.15.

Since we work in the Bowditch boundary $\partial(F, \mathcal{A})$, rather than the Gromov boundary, there are also fixed points coming from vertices of infinite valence. It appears possible for these points to be neither in the boundary of the fixed subgroup nor attractors. Fortunately the vertices of infinite valence determine a well-defined subspace of $\partial(F, \mathcal{A})$, so the distinction is less important. As in [FH11, Section 3], we discuss *principal automorphisms, principal periodic points* and *principal lifts.* A first guess might be that since vertices with nontrivial vertex group are permuted by homotopy equivalences of graphs of groups, these periodic points might be principal. In fact this is true with one exception, the case of the cyclic group of order 2. Vertices with C_2 vertex group need not be principal from the perspective of the Bowditch boundary, and in fact often are not principal, see Example 7.13.

An extreme example of this behavior is the case of *dihedral pairs;* subgraphs of groups of $\mathcal{G}$ isomorphic to the quotient of $\mathbb{R}$ by the standard action of $C_2 * C_2$ which are f-periodic with no outward-pointing periodic directions. Although dihedral pairs are preserved by rotationless f, since C_2 points are not principal, there is no guarantee that f induces the trivial outer automorphism of $C_2 * C_2$, and indeed there are examples where f does not, see Example 7.14. In Feighn–Handel's construction of CTs, one key step is to get rid of periodic circles with no outward-pointing periodic directions by "untwisting" them. Imagine this taking place on a graph equipped with a C_2 action, so that the quotient is a graph of groups with C_2 vertex groups. The problem is that this "untwisting" cannot be done C_2-equivariantly. Rather than get rid of dihedral pairs, we are forced to build them into the theory. Thus a useful key feature of CTs for free groups: namely that every filtration element contains a principal point, fails for free products in general.

Here is the organization of the paper. We give a quick review of Bass–Serre theory and relative train track maps in Section 1 by way of setting the scene. We expect many readers to be interested in the case of $(F, \mathcal{A})$ where F is a free group, and we describe collapsing relative train track maps from graphs to graphs of groups. Section 2 is given to proving the analogue of [FH11, Theorem 2.19], which provides better relative train track maps for all outer automorphisms $\varphi \in \mathrm{Out}(F, \mathcal{A})$. In Section 3, we study the action of automorphisms of $(F, \mathcal{A})$ on the Bowditch boundary. We give an account of a result from Martino's thesis, an analogue of [GJLL98, Proposition 1.1] for free products. We develop the theory of *attracting laminations* for outer automorphisms of free products in Section 4. Attracting laminations are very useful in the study of $\mathrm{Out}(F_n)$, and we have provided a bit more of the theory than we actually have need of in the paper, for example in Proposition 4.13 the existence of a homomorphism from the stabilizer of an attracting lamination to $\mathbb{Z}$. In Section 5, we study principal automorphisms of $(F, \mathcal{A})$ and define rotationless outer automorphisms and relative train track maps and prove their equivalence under the technical assumption of the existence of a rotationless iterate of a relative train track map. The construction of CTs for rotationless outer automorphisms of free products is accomplished in Section 6. *Geometric strata* play an important role in the definiton of an improved relative train track map in [BFH00] but do not figure into the definition of a CT. We have not attempted to develop a theory of geometric strata for free products, but such a project would surely be interesting. Theorem B and Theorem C are proven in Section 7. One useful consequence of Feighn–Handel's index inequality is the following. They prove that if $\varphi \in \mathrm{Out}(F_n)$ is in the kernel of the action of $\mathrm{Out}(F_n)$ on the homology of F_n with $\mathbb{Z}/3\mathbb{Z}$-coefficients and induces the trivial permutation on a certain finite set whose size is bounded by their index theorem, then φ is rotationless. Such a result for $\mathrm{Out}(F, \mathcal{A})$, say for F a free product of finite and cyclic groups, would certainly be interesting. Certainly such $\mathrm{Out}(F, \mathcal{A})$ are virtually torsion free, and our index theorem provides a bound on the relevant finite set, but we are unaware of a torsion-free subgroup of finite index with the special properties $\mathrm{IA}_n(\mathbb{Z}/3\mathbb{Z})$ enjoys.

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1 Relative train track maps on graphs of groups

The purpose of this section is to set the scene for the later sections. We define graphs of groups with trivial edge groups, as well as maps and homotopies thereof. We recall the definition of a relative train track map and prove a proposition about relative train track maps realizing a given nested sequence of φ-invariant free factor systems. We describe how to collapse relative train track maps on graphs to obtain relative train track maps on graphs of groups and conversely, how to blow up relative train track maps on graphs of free groups to relative train track maps on graphs given relative train track maps representing the maps on vertex groups.

Graphs of groups with trivial edge groups. A *graph of groups with trivial edge groups* $\mathcal{G}$ is a graph G together with an assignment of groups $\mathcal{G}_v$ to the vertices of G. Throughout the paper, unless otherwise specified, our graphs of groups will be assumed to be connected and finite.

Associated to a graph of groups $\mathcal{G}$ with trivial edge groups with underlying graph G, we can build a *graph of spaces* $X_\mathcal{G}$. See [SW79] for more details. For each vertex v of G, take a connected CW complex X_v that is a $K(\mathcal{G}_v, 1)$ with one vertex $\star_v$ and fix an identification $\pi_1(X_v, \star_v) = \mathcal{G}_v$. If V is the set of vertices of G and E is the set of edges (which we think of as coming with a choice of orientation), the graph of spaces $X_\mathcal{G}$ is the quotient of the disjoint union

$$\coprod_{v \in V} X_v \amalg E \times [0,1]$$

by the equivalence relation identifying the point $(e,1)$ with $\star_{\tau(e)}$ and $(e,0)$ with $\star_{\iota(e)}$, where $\tau(e)$ denotes the terminal vertex of the edge e and $\iota(e)$ denotes the initial vertex. There is a natural "retraction" $X_\mathcal{G} \to G$ that collapses each X_v to the point $\star_v$.

The *fundamental group of the graph of groups* $\pi_1(\mathcal{G})$ is the fundamental group of the graph of spaces $X_\mathcal{G}$. For convenience, choose a basepoint $p \in X_\mathcal{G}$ in the image of the retraction $X_\mathcal{G} \to G$. Each loop in $\pi_1(X_\mathcal{G}, p)$ may be represented by an *edge path* γ of the form

$$\gamma = e_1' g_1 e_2 g_2 \ldots e_k g_k e_{k+1}'.$$

Here $e_2, \ldots, e_k$ are edges of G, e_1' and e_{k+1}' are terminal and initial segments of edges, respectively, and the concatenation $e_1' e_2 \ldots e_k e_{k+1}'$ is a path in G. The g_i for $1 \le i \le k$ are elements of $\pi_1(X_{v_i}, \star_{v_i}) = \mathcal{G}_{v_i}$, where $v_i = \tau(e_i) = \iota(e_{i+1})$. We allow γ to begin or end at vertices, in which case the initial or terminal segments should be dropped from the notation. A path is *nontrivial* if it contains a (segment of) an edge e_i.

Notice that the notion of an edge path makes sense without reference to the space $X_\mathcal{G}$. Homotopy rel endpoints of paths in $X_\mathcal{G}$ yields a corresponding notion of homotopy for edge paths in $\mathcal{G}$; it is generated by adding or removing segments of the form $e\bar{e}$ for e an oriented edge of $\mathcal{G}$ and $\bar{e}$ the edge e in the opposite orientation. An edge path is *tight* if it cannot be shortened by a homotopy.

The Bass–Serre tree. The fundamental group of a finite graph of groups with trivial edge groups is a free product of the form

$$F = A_1 * \cdots * A_n * F_k,$$

where the A_i are the nontrivial vertex groups of $\mathcal{G}$ and F_k, a free group of rank k, is the ordinary fundamental group of G. Recall from the introduction our notation $\mathcal{A}$ for the set of conjugacy classes of the A_i. We will call such a graph of groups a *Grushko* $(F,\mathcal{A})$*-graph of groups.* We underline that although we use the term Grushko, this splitting is *not* assumed to be the Grushko splitting of a finitely generated group. For example F may be a free group, and the A_i may be free factors of F.

A *Grushko* $(F,\mathcal{A})$*-tree* is a simplicial tree T equipped with a minimal F-action with trivial edge stabilizers and vertex stabilizers trivial or conjugate to some A_i. The fundamental theorem of Bass–Serre theory asserts that associated to each Grushko $(F,\mathcal{A})$-tree, there is a quotient Grushko $(F,\mathcal{A})$-graph of groups (which is finite if the tree is *minimal,* i.e. there is no proper F-invariant subtree) and conversely associated to any (not necessarily finite) Grushko $(F,\mathcal{A})$-graph of groups $\mathcal{G}$, there is a Grushko $(F,\mathcal{A})$-tree Γ called the *Bass–Serre tree* for $\mathcal{G}$. For the constructions of the quotient graph of groups and Bass–Serre tree, the reader is referred to [Ser03], [Bas93], or [Lym21, Section 1]. Let us remark that to identify a Grushko $(F,\mathcal{A})$-tree Γ as the Bass–Serre tree of a Grushko $(F,\mathcal{A})$-graph of groups $\mathcal{G}$, there is a choice not only of a basepoint p in $\mathcal{G}$ and a lift $\tilde{p}$ to Γ, but also a choice of fundamental domain for the action of F on Γ containing $\tilde{p}$. However, beginning with the graph of groups and constructing the Bass–Serre tree requires merely the choice of a basepoint, as in the case of topological spaces.

Given a vertex v of G, write $[p,v]$ for the set of homotopy classes rel endpoints of paths in $\mathcal{G}$ from p to v. There is a natural right action of $\mathcal{G}_v$ on $[p,v]$: an element g sends the homotopy class of a path γ to the homotopy class of the composite path γg. The vertex set of Γ is the set

$$\coprod [p,v]/\mathcal{G}_v$$

where the disjoint union is over the set of vertices of G. One can extend the above definition, defining points of Γ to be (equivalence classes of) homotopy classes of paths beginning at p. Alternatively one may say that two vertices $[\gamma]\mathcal{G}_v$ and $[\gamma']\mathcal{G}_w$ are connected by an edge if $\bar{\gamma}'\gamma$ is homotopic to a path of length one, i.e. of the form geg' for vertex group elements g and g' and e an edge of G.

Maps of graphs of groups. A *map* of graphs of groups (with trivial edge groups) $f\colon \mathcal{G} \to \mathcal{G}'$ is a pair of maps $f\colon G \to G'$ and $f_X\colon X_{\mathcal{G}} \to X_{\mathcal{G}'}$ such that the following diagram commutes

$$\begin{array}{ccc} X_{\mathcal{G}} & \xrightarrow{f_X} & X_{\mathcal{G}'} \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r'} \\ G & \xrightarrow{f} & G', \end{array}$$

where r and r' are the retractions. A *homotopy* between maps of graphs of groups is a commutative diagram of homotopies. A map $f\colon \mathcal{G} \to \mathcal{G}'$ is a *homotopy equivalence* if (as one might expect) there is a map $g\colon \mathcal{G}' \to \mathcal{G}$ such that gf and fg are homotopic to the respective identity maps. It is not too hard to see that each map $f\colon \mathcal{G} \to \mathcal{G}'$ is homotopic to a map that sends the vertices $\star_v$ of $X_{\mathcal{G}}$ to vertices $\star_{v'}$ of $X_{\mathcal{G}'}$ and thus sends edges of $G \subset X_{\mathcal{G}}$ to (possibly trivial) edge paths in $\mathcal{G}'$. Throughout the rest of the paper, we will understand a *map* of graphs of groups to be a map of this kind, following the convention established in [Lym21, Section 1].

Like edge paths, a map of graphs of groups $f\colon \mathcal{G} \to \mathcal{G}$ that is a homotopy equivalence has a combinatorial shadow in $\mathcal{G}$ that does not depend on $X_{\mathcal{G}}$: The data of f is a continuous map $f\colon G \to G$ taking vertices to vertices along with, for each edge e of G, an edge path for which we write $f(e)$ in $\mathcal{G}$ (such that the underlying path in G agrees with the image of e under the continuous map $f\colon G \to G$), and for each vertex v of G with nontrivial vertex group, an isomorphism $f_v\colon \mathcal{G}_v \to \mathcal{G}_{f(v)}$. Although such data is sufficient to determine a map $f\colon \mathcal{G} \to \mathcal{G}$, let us remark that not every such map is a homotopy equivalence; one can check whether one has a homotopy equivalence by performing *folds* as in [BF91] or [Dun98]. Homotopy for f has a combinatorial shadow in $\mathcal{G}$ generated by the following operations.

1. For each edge e of G, alter the edge path $f(e)$ (thought of as an edge path in $\mathcal{G}$) by a homotopy.
2. For a vertex v of G with trivial vertex group and an edge path γ with endpoints at vertices and initial vertex $f(v)$, replace $f(v)$ with the terminal vertex of γ and append γ to the edge path $f(e)$ for each oriented edge e with $\tau(e) = v$.
3. For a vertex v of G with nontrivial vertex group and a group element $g \in \mathcal{G}_{f(v)}$, replace $f_v\colon \mathcal{G}_v \to \mathcal{G}_{f(v)}$ with the map $x \mapsto gf_v(x)g^{-1}$, and append g^{-1} to the edge path $f(e)$ for each oriented edge e with $\tau(e) = v$.

A map $f\colon \mathcal{G} \to \mathcal{G}$ acts on edge paths by the rule that f sends e to the edge path $f(e)$ in $\mathcal{G}$ and sends the vertex group element $g \in \mathcal{G}_v$ to $f_v(g) \in \mathcal{G}_{f(v)}$. For initial and terminal segments of edges, there is a small amount of indeterminacy: suppose we divide the edge e into $e'e''$ at the preimage of a vertex v with nontrivial vertex group such that the image $f(e)$ contains the vertex group element $g \in \mathcal{G}_v$. Then we may freely factor $g = g'g''$ and say that $f(e')$ ends with g' and $f(e'')$ begins with g''. We will be careful about choosing factorizations but note that in practice, there is usually a convenient choice of factorization, and different choices yield the same result.

Lifting maps to the Bass–Serre tree. Given a path $\gamma \in \mathcal{G}$, let $[\gamma]$ denote its homotopy class rel endpoints. Given a basepoint $p \in G$ and a path σ from p to $f(p)$, the homotopy equivalence f induces an automorphism $\Phi\colon \pi_1(\mathcal{G}, p) \to \pi_1(\mathcal{G}, p)$ defined as

$$\Phi([\gamma]) = [\sigma f(\gamma)\bar{\sigma}].$$

Differing choices of path σ change Φ within its outer class $\varphi \in \mathrm{Out}(\pi_1(\mathcal{G}))$. We say f *represents* φ. If in addition to sending vertices to vertices, the map f sends edges to nontrivial tight edge paths, we say that f is a *topological representative* of φ.

Let Γ be the Bass–Serre tree of $\mathcal{G}$ with basepoint $\tilde{p}$ lifting p. The choice of path σ also defines a lift $\tilde{f}\colon \Gamma \to \Gamma$ defined as follows. If $\tilde{v}$ is the vertex $[\gamma]\mathcal{G}_v$ of Γ, The point $\tilde{f}(\tilde{x})$ is the vertex $[\sigma f(\gamma)]\mathcal{G}_{f(v)}$. Similarly if $\tilde{x}$ is a point $[\gamma]$ in the interior of an edge, then $\tilde{f}(\tilde{x})$ is the point $[\sigma f(\gamma)]$. One notes that this is well-defined. The lifted map $\tilde{f}$ is Φ-*twisted equivariant* in the sense that

$$\tilde{f}(g.\tilde{x}) = \Phi(g).\tilde{f}(\tilde{x})$$

for each point $\tilde{x}$ in Γ and each element g of $\pi_1(\mathcal{G}, p)$. We say that the lifted map $\tilde{f}$ *corresponds to* or *is determined by* Φ and vice versa.

Directions, the map Df. Let v be a vertex of G, and let $\mathrm{st}(v)$ denote the set of oriented edges e of G with initial vertex $\iota(e) = v$. A *direction* at v is an element of the set

$$\coprod_{e \in \mathrm{st}(v)} \mathcal{G}_v \times \{e\}.$$

An identification of Γ with the Bass–Serre tree of $\mathcal{G}$ determines, in particular, a bijection between the set of directions at v and those of any lift $\tilde{v}$ in Γ. To wit, if $\tilde{e}$ is an edge of Γ with initial vertex $\tilde{v} = [\gamma]\mathcal{G}_v$, where we suppose that γ ends with a trivial vertex group element, then this bijection sends the edge $\tilde{e}$ with initial vertex $\tilde{v}$ and terminal vertex $\tilde{w}$ to the pair (g, e) if $\tilde{w} = [\gamma g e]\mathcal{G}_w$. The map $g \mapsto [\gamma g \bar{\gamma}]$ defines an isomorphism from $\mathcal{G}_v$ to the stabilizer of $\tilde{v}$ in Γ making the bijection above $\mathcal{G}_v$-equivariant.

A topological representative $f\colon \mathcal{G} \to \mathcal{G}$ determines a map Df from the set of directions based at v to the set of directions based at $f(v)$ in the following way. Let e be an edge in $\mathrm{st}(v)$, and suppose the edge path $f(e)$ begins with $g_0 e_1$. We define

$$Df(g, e) = (f_v(g)g_0, e_1).$$

Note that $f(ge)$ begins $f_v(g)g_0e_1$, so another way of describing this map is as the first vertex group element and edge of the edge path $f(ge)$. A direction (g, e) is *almost periodic* if $Df^k(g, e) = (g', e)$ for some $k \geq 1$, and *almost fixed* if $k = 1$. If we can (possibly by increasing k) take $g' = g$, then the direction is *periodic* or *fixed,* respectively.

A lift $\tilde{f}$ of f is determined by the image of the point $\tilde{p}$ together with (if $\tilde{p}$ is a vertex with nontrivial vertex group) the direction $D\tilde{f}(\tilde{e})$ for an edge $\tilde{e}$ with $\iota(\tilde{e}) = \tilde{p}$. Alternatively, if $\tilde{p}$ is a vertex with nontrivial vertex group and $\tilde{x} \neq \tilde{p}$ is a second point, then $\tilde{f}$ is determined by the image of $\tilde{p}$ and the image of $\tilde{x}$. It follows that the correspondence between lifts and automorphisms defined in the previous titled paragraph is a bijection (for topological representatives, or more generally, maps that do not collapse edges to vertices).

Suppose v is a vertex with nontrivial vertex group $\mathcal{G}_v$, and that $\tilde{v}$ is the point $[\gamma]\mathcal{G}_v$ of Γ. Suppose further that $f\colon \mathcal{G} \to \mathcal{G}$ fixes v. Then if $\tilde{f}$ is the lift of f determined by the path σ from p to $f(p)$, we see that $\tilde{f}(\tilde{v}) = \tilde{v}$ if and only if $\sigma = \gamma h f(\bar{\gamma})$ for some element $h \in \mathcal{G}_v$. In this situation, if $\tilde{e}$ is the edge connecting the vertex $\tilde{v}$ to the vertex $\tilde{w} = [\gamma g e]\mathcal{G}_w$, we have that

$$\tilde{f}(\tilde{w}) = [\sigma f(\gamma) f_v(g) f(e)]\mathcal{G}_{f(w)} = [\gamma h f_v(g) f(e)]\mathcal{G}_{f(w)},$$

and we see that in the notation of the previous paragraph and under the $\mathcal{G}_v$-equivariant bijection of directions at $\tilde{v}$ with directions at v, where $\tilde{e}$ corresponds to (g, e), we have that $D\tilde{f}(\tilde{e})$ corresponds to $(hf_v(g)g_0, e_1)$.

Markings, filtrations, transition matrices. Given a group F that splits as a free product of the form

$$F = A_1 * \cdots * A_n * F_k,$$

where as usual the A_i are groups and F_k is a free group of rank k, consider the following Grushko $(F, \mathcal{A})$-graph of groups $\mathbb{G}$, the *thistle with n prickles and k petals.* There are $n + 1$ vertices; one of which, $\star$, has trivial vertex group, and the others each have vertex group isomorphic to some A_i. There are $n + k$ edges, the first n of which connect a vertex with nontrivial vertex group to $\star$, and the remaining k of which form loops based at $\star$. A *marked graph of groups* is a Grushko $(F, \mathcal{A})$-groups $\mathcal{G}$ together with a homotopy equivalence, the *marking,* $m\colon \mathbb{G} \to \mathcal{G}$. (Note that in this paper, a marked graph of groups always has trivial edge groups.) It is mildly convenient to allow the marking m to *not* map the vertex $\star$ to a vertex of $\mathcal{G}$. Fix once and for all an identification $F = \pi_1(\mathbb{G}, \star)$. Every automorphism $\Phi\colon (F, \mathcal{A}) \to (F, \mathcal{A})$ permuting the conjugacy classes in $\mathcal{A}$ admits a (pointed) topological representative $f\colon (\mathbb{G}, \star) \to (\mathbb{G}, \star)$. The marking on a marked graph of groups $(\mathcal{G}, m)$ provides an identification of $\pi_1(\mathcal{G}, m(\star))$ with F. Thus it makes sense to say that a topological representative $f\colon \mathcal{G} \to \mathcal{G}$ represents $\varphi \in \mathrm{Out}(F, \mathcal{A})$.

A *filtration* on a marked graph of groups $\mathcal{G}$ with respect to a topological representative $f\colon \mathcal{G} \to \mathcal{G}$ is a (strictly) increasing sequence

$$\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$$

of f-invariant subgraphs. We assume that G_1 is *nontrivial* in the sense that it contains an edge of G. (Some authors would begin with the vertices of G with nontrivial vertex group as G_0. This makes no difference for our purposes.) The subgraphs in the filtration are not required to be connected.

The rth *stratum* of $\mathcal{G}$ is the subgraph H_r containing those edges of G_r not contained in G_{r-1}. When we consider G_r and H_r as graphs of groups in their own right, the vertex groups are equal to what they are in $\mathcal{G}$; in the language of [Bas93], we work primarily with subgraphs of groups, not subgraphs of subgroups. An edge path has *height* r if it is contained in G_r and meets the interior of H_r.

A *turn* based at a vertex v of G is an unordered pair of directions in $\mathcal{G}$ based at v. The map of directions Df determines a self-map of the set of turns in $\mathcal{G}$ also called Df. A turn is *degenerate* if it is of the form $\{(g,e),(g,e)\}$, and *illegal* if some iterate of Df maps it to a degenerate turn. A turn is *legal* if it is not illegal. A path $\gamma = \dots e_k g_k e_{k+1} \dots$ is said to *cross* or *contain* the turn $\{(1,\bar{e}_k),(g_k,e_{k+1})\}$. There is a diagonal action of $\mathcal{G}_v$ on the set of turns based at v; what is primarily important is the $\mathcal{G}_v$-orbit of the turn. If a path has height r and contains no illegal turns in H_r, then we say that it is *r-legal.*

Given a stratum H_r, the *transition (sub)matrix* M_r is the square matrix whose columns are the edges of H_r and whose (i,j)-entry is the number of times the f-image of the jth edge crosses the ith edge in either direction. A filtration is *maximal* when each M_r is either irreducible or the zero matrix. If M_r is irreducible, we say that H_r is an *irreducible stratum* and a *zero stratum* otherwise. All filtrations in this paper will be assumed to be maximal unless otherwise stated; we will think of a maximal filtration as part of the data of a topological representative $f\colon \mathcal{G} \to \mathcal{G}$.

Associated to each irreducible stratum H_r, the matrix M_r has a *Perron–Frobenius eigenvalue* $\lambda_r \geq 1$. If $\lambda_r > 1$, we say that H_r is an *exponentially growing stratum.* Otherwise we have $\lambda_r = 1$, M_r is a transitive permutation matrix and we say that H_r is a *non-exponentially growing stratum.* In the literature, "exponentially growing" and "non-exponentially growing" are often shortened to "EG" and "NEG," respectively.

Relative train track maps. Let $f\colon \mathcal{G} \to \mathcal{G}$ be a topological representative with associated filtration $\varnothing = G_0 \subset \cdots \subset G_m = G$. Given a path σ in $\mathcal{G}$, let $f_\sharp(\sigma)$ denote the (unique) tight path homotopic rel endpoints to $f(\sigma)$. The map f is a *relative train track map* if for every exponentially growing stratum H_r, we have

(EG-i) Directions in H_r are mapped to directions in H_r by Df; every turn with one edge in H_r and the other in G_{r-1} is legal.

(EG-ii) If $\sigma \subset G_{r-1}$ is a homotopically nontrivial path with endpoints in $H_r \cap G_{r-1}$, then $f_\sharp(\sigma)$ is as well.

(EG-iii) If $\sigma \subset G_r$ is a tight r-legal path, then $f(\sigma)$ is an r-legal path.

We have the following theorem, for which the reader is referred to [CT94], [FM15] or [Lym21, Theorem 3.2].

Theorem 1.1. *Every $\varphi \in \mathrm{Out}(F,\mathcal{A})$ is represented by a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ on a marked graph of groups $\mathcal{G}$.*

We extend this theorem by bringing in free factor systems.

Free factor systems. A $(F,\mathcal{A})$*-free splitting* is a simplicial tree T equipped with an action of F with trivial edge stabilizers in which subgroups in $\mathcal{A}$ are elliptic (i.e. fix points of T). We assume that some element of F is hyperbolic (i.e. not elliptic) in T. A *proper free factor of F relative to $\mathcal{A}$* is a vertex stabilizer in some $(F,\mathcal{A})$-free splitting. It has *positive complexity* in the sense of [Lym21, Section 2] if it is not contained in $\mathcal{A}$. If F^i is a free factor of F relative to $\mathcal{A}$, let $[[F^i]]$ denote its conjugacy class. If $F^1,\dots,F^m$ are free factors of F relative to $\mathcal{A}$ with positive complexity and $F^1 * \cdots * F^m$ is a free factor of F (in the sense that there is a subgroup F^{m+1} such that $F = (F^1 * \cdots * F^m) * F^{m+1}$), the collection $\{[[F^1]],\dots,[[F^m]]\}$ is a *free factor system.* We stress that we only allow free factors of positive complexity to be part of a free factor system.

Example 1.2. Suppose $f\colon \mathcal{G} \to \mathcal{G}$ is a topological representative and $\mathcal{G}$ is a marked graph of groups and $G' \subset \mathcal{G}$ is an f-invariant subgraph with noncontractible (hence nontrivial) connected components $C_1, \ldots, C_k$. Here we say C_1 is *noncontractible* if it contains at least two vertices with nontrivial vertex group or has nontrivial ordinary fundamental group; otherwise we say C is *contractible*. The conjugacy classes $[[\pi_1(\mathcal{G}|_{C_i})]]$ of the fundamental groups of the C_i are well-defined. We define

$$\mathcal{F}(G') = \{[[\pi_1(\mathcal{G}|_{C_1})]], \ldots, [[\pi_1(\mathcal{G}|_{C_k})]]\}.$$

Notice that each $\pi_1(\mathcal{G}|_{C_i})$ has positive complexity. We say that G' *realizes* $\mathcal{F}(G')$.

The group $\mathrm{Out}(F, \mathcal{A})$ acts on the set of conjugacy classes of free factors of F relative to $\mathcal{A}$. If $\varphi \in \mathrm{Out}(F, \mathcal{A})$ is an outer automorphism and F' a free factor of F then we say $[[F']]$ is *φ-invariant* if $\varphi([[F']]) = [[F']]$. In this case there is some automorphism $\Phi\colon (F, \mathcal{A}) \to (F, \mathcal{A})$ representing φ such that $\Phi(F') = F'$. The restriction $\Phi|_{F'}$ is well-defined up to an inner automorphism of F', so it induces an outer automorphism $\varphi|_{F'} \in \mathrm{Out}(F')$, which we will call the *restriction* of φ to F'. More generally one can restrict φ to any subgroup of F which is its own normalizer. A free factor system is φ-invariant if each conjugacy class of each free factor contained in it is φ-invariant.

There is a partial order $\sqsubset$ on free factor systems: we say that $[[F^1]] \sqsubset [[F^2]]$ if F^1 is conjugate to a subgroup of F^2, and we say that $\mathcal{F}_1 \sqsubset \mathcal{F}_2$ for free factor systems $\mathcal{F}_1$ and $\mathcal{F}_2$ if for each $[[F^i]] \in \mathcal{F}_1$, there exists $[[F^j]] \in \mathcal{F}_2$ such that $[[F^i]] \sqsubset [[F^j]]$.

Proposition 1.3. *Let $\varphi \in \mathrm{Out}(F, \mathcal{A})$ be an outer automorphism. If $\mathcal{F}_1 \sqsubset \cdots \sqsubset \mathcal{F}_d$ is a nested sequence of φ-invariant free factor systems relative to $\mathcal{A}$, then there is a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ representing φ such that each free factor system is realized by some element of the filtration $\varnothing = G_0 \subset \cdots \subset G_m = G$.*

Proof. The first step is to construct a topological representative $f\colon \mathcal{G} \to \mathcal{G}$ and an associated filtration such that each $\mathcal{F}_i$ is realized by a filtration element. We proceed by induction on d, the length of the nested sequence of φ-invariant free factor systems. The case $d = 0$ is vacuous. Let $\mathcal{F}_d = \{[[F^1]], \ldots, [[F^\ell]]\}$, and choose automorphisms $\Phi_i\colon (F, \mathcal{A}) \to (F, \mathcal{A})$ representing φ such that $\Phi_i(F^i) = F^i$. For each j with $1 \le j \le d-1$ and each i satisfying $1 \le i \le \ell$, write $\mathcal{F}_j^i$ for the set of conjugacy classes of free factors in $\mathcal{F}_j$ that are conjugate into F^i. Then for each i,

$$\mathcal{F}_1^i \sqsubset \cdots \sqsubset \mathcal{F}_{d-1}^i$$

is a nested sequence of φ-invariant free factor systems. By induction, there are graphs of groups $\mathcal{G}^i$ and topological representatives $f_i\colon \mathcal{G}^i \to \mathcal{G}^i$ representing the restriction of φ to F^i together with associated (not necessarily maximal!) filtrations $G_1^i \subset \cdots \subset G_{d-1}^i$ such that G_j^i realizes $\mathcal{F}_j^i$. Inductively, we may assume that f_i fixes some vertex v_i of $\mathcal{G}^i$, that $\mathcal{G}^i$ has no valence-one or valence-two vertices with trivial vertex group, and that the marking on $\mathcal{G}^i$ identifies F^i with $\pi_1(\mathcal{G}^i, v_i)$ and Φ_i with $(f_i)_\sharp\colon \pi_1(\mathcal{G}^i, v_i) \to \pi_1(\mathcal{G}^i, v_i)$.

Take a complementary free factor $F^{\ell+1}$ so that $F = F^1 * \cdots F^\ell * F^{\ell+1}$ and an associated thistle $\mathcal{G}^{\ell+1}$. (If $F^{\ell+1}$ is trivial, $\mathcal{G}^{\ell+1}$ is a vertex $\star$.) The graph of groups $\mathcal{G}$ is constructed as follows: begin with the disjoint union of the $\mathcal{G}^i$. Glue $\mathcal{G}^{\ell+1}$ to $\mathcal{G}^1$ by identifying the vertex $\star$ of $\mathcal{G}^{\ell+1}$ with trivial vertex group with the fixed point v_1. Then attach an edge E_i connecting v_1 to v_i for $2 \le i \le \ell$. The resulting graph of groups has no valence-one or valence-two vertices with trivial vertex group. Collapsing each E_i to v_1 gives a homotopy equivalence of $\mathcal{G}$ onto a graph whose fundamental group is naturally identified with $F^1 * \cdots F^\ell * F^{\ell+1} = F$. This provides the (essential data of) a marking on $\mathcal{G}$.

Define $f\colon \mathcal{G} \to \mathcal{G}$ from the f_i as follows. For $2 \le i \le \ell$, there is $c_i \in F$ such that $\Phi_1(x) = c_i \Phi_i(x) c_i^{-1}$ for all $x \in F$. Let γ_i be loops based at v_1 that are identified under the marking with c_i. On each $\mathcal{G}^i$ for $2 \le i \le \ell$, set $f = f_i$. Define $f(E_i) = \gamma_i E_i$, and define f on $\mathcal{G}^{\ell+1}$ according to Φ_1. Then $f_\sharp\colon \pi_1(\mathcal{G}, v_1) \to \pi_1(\mathcal{G}, v_1)$ induces $\Phi_1\colon (F, \mathcal{A}) \to (F, \mathcal{A})$. For $1 \le j \le d-1$, define $G_j = \bigcup_{j=1}^{\ell} G_j^i$, and define $G_d = \bigcup_{i=1}^{\ell} \mathcal{G}^i$. Then

$$\varnothing = G_0 \subset \cdots \subset G_d \subset G_{d+1} = G$$

is an f-invariant filtration, and each $\mathcal{F}_i$ for i satisfying $1 \leq i \leq d$ is realized by G_i. Complete this filtration to a maximal filtration (by adding in intermediate f-invariant subgraphs). This completes the first step.

The next step is to promote our topological representative to a relative train track map. Note (cf. the proof of [BFH00, Lemma 2.6.7]) that the moves described in [Lym21, Section 2 and 3] all preserve the property of realizing free factors. More precisely, suppose C_1 and C_2 are disjoint noncontractible components of some filtration element G_r of $\mathcal{G}$ and that $f'\colon \mathcal{G}' \to \mathcal{G}'$ is obtained from $f\colon \mathcal{G} \to \mathcal{G}$ by collapsing a pretrivial forest, folding, subdivision, invariant core subdivision, valence-one homotopy or valence-two homotopy. If $p\colon \mathcal{G} \to \mathcal{G}'$ is the identifying homotopy equivalence, then $p(C_1)$ and $p(C_2)$ are disjoint, noncontractible subgraphs of $\mathcal{G}'$.

To see this in the case of collapsing a pretrivial forest X, note that because C_1 and C_2 are noncontractible, there exists $k \geq 1$ such that $f^k(C_1) \subset C_1$ and $f^k(C_2) \subset C_2$. If some component of X intersected C_1 and C_2, then the same must be true of $f^k(X)$, thus since C_1 and C_2 are disjoint, this contradicts the fact that X is pretrivial.

Thus we may work freely and apply the proof of [Lym21, Theorem 3.2] to produce a relative train track map. ☐

The proof of Theorem 1.1 in [Lym21] has the following useful corollary. Let $f\colon \mathcal{G} \to \mathcal{G}$ be a topological representative. Let $\mathrm{PF}(f)$ be the set of Perron–Frobenius eigenvalues of the exponentially growing strata of f in nonincreasing order. The set of $\mathrm{PF}(f)$ for f a topological representative of $\varphi \in \mathrm{Out}(F, \mathcal{A})$ satisfying a certain condition called *boundedness* ordered lexicographically has a minimum, $\mathrm{PF}_{\min}$ (see [Lym21, Section 3] and note that boundedness depends solely on $\mathrm{PF}(f)$). The proof of Theorem 1.1 shows that if f is a bounded topological representative satisfying (EG-i) but not (EG-iii), then $\mathrm{PF}(f)$ can be decreased. Therefore we have the following corollary.

Corollary 1.4 (Corollary 3.7 of [Lym21]). *If $f\colon \mathcal{G} \to \mathcal{G}$ is a topological representative satisfying (EG-i) and with $\mathrm{PF}(f) = \mathrm{PF}_{\min}$, then the exponentially growing strata of f satisfy (EG-iii).*

We also have the following useful characterization of (EG-ii).

Lemma 1.5. *Let $f\colon \mathcal{G} \to \mathcal{G}$ be a topological representative and H_r an exponentially growing stratum that satisfies (EG-i). Property (EG-ii) for H_r is a finite property for any contractible component C of G_{r-1} which does not contain a vertex with nontrivial vertex group. In the contrary case there exists $k \geq 1$ such that $f^k(C) \subset C$ and (EG-ii) for connecting paths in C is equivalent to the condition that each vertex in $H_r \cap C$ is periodic; i.e. f^k permutes the elements of the set $H_r \cap C$.*

Proof. The first statement, where C is contractible and contains no vertex with nontrivial vertex group, is clear, since there are only finitely many tight paths with endpoints at vertices in C.

So suppose that there exists $k \geq 1$ such that $f^k(C) \subset C$. Then since H_r satisfies (EG-i), f^k maps the finite set of vertices $H_r \cap C$ into itself. If f^k does not permute the elements of $H_r \cap C$, then there are a pair of vertices v and w whose f^k-image is equal, say to u. Then there is a path σ connecting v to w whose image under $f^k_\sharp$ is trivial (consider what a homotopy inverse to f^k does to u) and (EG-ii) fails.

So suppose that f^k permutes the elements of $H_r \cap C$. If α is a tight connecting path in C with distinct endpoints in $H_r \cap C$, then $f^k(\alpha)$ also has distinct endpoints and therefore $f_\sharp(\alpha)$ is nontrivial. If instead α is a tight connecting path in C with identical endpoints, say v, then it determines a nontrivial loop in $\pi_1(\mathcal{G}, v)$. If $f_\sharp(\alpha)$ fails to be nontrivial then α must have the form $\sigma g \bar{\sigma}$ for $g \in \mathcal{G}_w$ an element of a vertex group and $f(v) = f(w)$. Since C is a component of G_{r-1}, vertices with nontrivial vertex group are permuted, and $f^k(w) = f^k(v) \in H_r \cap C$, we conclude that in fact $w \in H_r \cap C$ and therefore $v = w$. Since α was tight, we conclude σ is a nontrivial loop in $\pi_1(\mathcal{G}, v)$, and $f_\sharp(\sigma)$ (and therefore $f_\sharp(\alpha)$)

is nontrivial provided σ is not itself a path of the form $\rho g'\bar{\rho}$ for $g' \in \mathcal{G}_v$. On the other hand, if it *is* of that form, then we can ask the question of ρ and so on until at last we come to a path μ which is too short to be of the above form. We have that $f_\sharp(\mu)$ is nontrivial. Recall that nondegenerate turns of the form $\{(g_1, e), (g_2, e)\}$ where g_1 and g_2 are distinct are legal, therefore $f_\sharp(\mu g''\bar{\mu}) = f_\sharp(\mu)f_v(g'')f_\sharp(\bar{\mu})$ is a tight concatenation of tight paths, and we conclude that $f_\sharp(\alpha)$ is nontrivial. Therefore (EG-ii) holds for connecting paths in C in this case. ☐

Collapsing relative train track maps. Suppose $f\colon G \to G$ and $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ are a relative train track map and filtration in the sense of [BH92] (i.e. in our sense with all vertex groups trivial) and fix r satisfying $2 \le r \le m$. To aid the reader interested in the case $F = F_n$, we describe how to collapse the filtration element G_{r-1} of f to obtain a new relative train track map $f'\colon \mathcal{G}' \to \mathcal{G}'$ on a Grushko $(F, \mathcal{F}(G_{r-1}))$-graph of groups and a collapse map $p\colon G \to \mathcal{G}'$.

Begin by defining a subgraph H of G inductively as follows: begin with $H = G_{r-1}$ and then repeatedly add to H all those edges whose f-image is entirely contained in H. Each edge of $H \setminus G_{r-1}$ belongs to a zero stratum. The subgraph H is f-invariant; let G' be the graph obtained by collapsing each component of H to a point, and let $p\colon G \to G'$ be the collapse map. For each component C of H, let T_C be a maximal tree in C, and $c \in C$ a vertex. For v a vertex of C, let η_v be the unique tight path in T_C from c to v.

Each component C of H corresponds to a vertex v_C of G'; set $\mathcal{G}'_{v_C} = \pi_1(C, c)$. Each vertex v of G not contained in H corresponds to a vertex, also called v, of G'; let $\mathcal{G}'_v$ be the trivial group. As a map of graphs of groups, p is the "identity" on edges not contained in H and sends an edge path γ in H with initial vertex v and terminal vertex w to the vertex group element $[\eta_v\gamma\bar{\eta}_w]$.

The map $f\colon G \to G$ descends to a map $f'\colon G' \to G'$. We would like the following diagram to commute for paths in G with endpoints either in the complement of H or at the vertices $c \in C$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"] & G \arrow[d, "p"] \\ \mathcal{G}' \arrow[r, "f'"] & \mathcal{G}'. \end{tikzcd}$$

Suppose that $f'(v_C) = v_{C'}$. Choose a path σ_C from c' to $f(c)$. The rule $[\gamma] \mapsto [\sigma_C f(\gamma)\bar{\sigma}_C]$ defines a homomorphism $f'_{v_C}\colon \mathcal{G}_{v_C} \to \mathcal{G}_{f'(v_C)}$. One checks that if $\mathcal{G}_{v_C}$ is nontrivial, then this map is an isomorphism. A naïve definition of the map $f'\colon \mathcal{G}' \to \mathcal{G}'$ on edges would be to define $f'(E)$ as $pf(E)$. In fact this is correct except at the endpoints, where we need to multiply by some vertex group element. To see why, consider a pair of edges E and E' with initial endpoints v and v', respectively, in a component C of H. Let γ be the unique tight path in T_C connecting v to v'. Then the path $\bar{E}\gamma E'$ in $\mathcal{G}$ projects to the path $\bar{E}E'$ in $\mathcal{G}'$. The equation $f'p = pf$ implies that $f'(\bar{E}E') = pf(\bar{E}\gamma E')$: it may happen that $pf(\gamma)$ is contained in $T_{f(C)}$, but in general this will not be the case. Define the first vertex group element of $f(E)$ to be $p(\sigma_C f(\eta_v)) = [\sigma_C f(\eta_v)\bar{\eta}_{f(v)}]$. Then check that the vertex group element $pf(\gamma)$ satisfies

$$pf(\gamma) = pf(\bar{\eta}_v\eta_{v'}) = p(f(\bar{\eta}_v)\bar{\sigma}_C\sigma_C f(\eta_{v'})) = [\eta_{f(v)}f(\bar{\eta}_v)\bar{\sigma}_C\sigma_C f(\eta_{v'})\bar{\sigma}_{f(v')}]$$

as required.

Lemma 1.6. *The map $f'\colon \mathcal{G}' \to \mathcal{G}'$ is a relative train track map.*

Proof. First we show that f' is a topological representative if f was. Indeed, if $f(E)$ is a tight path, then $pf(E)$ is a tight path, for if $f(E)$ contains a subpath of the form $E'b\bar{E}'$, where b is in H and E' is not, then b determines a nontrivial loop, so $p(b)$ is a nontrivial vertex group element. Put another way, $f'_\sharp(E)$ is obtained from $f_\sharp(E)$ by possibly adding vertex group elements at the ends and replacing maximal subpaths in H with the corresponding vertex group elements. Thus we see that if H_i is an irreducible stratum for $i > r-1$, then no edge of H_i belongs to H, and the edges of H_i determine an irreducible stratum H'_j in $\mathcal{G}'$ of the same kind. Likewise if H_i is a zero stratum but not every edge of H_i belongs to H, then H_i determines a zero stratum H'_j in $\mathcal{G}'$.

Property (EG-i) is clearly satisfied for each exponentially growing stratum H'_j. If H'_j is exponentially growing and D is a component of G_{i-1} such that $f^k(D) \subset D$ for some $k \geq 1$, then each vertex v' of $H'_j \cap D$ corresponds either to a vertex v of G or a component C of H. In either case, the preimage of v' contains an f-periodic vertex, so the vertex v' is f'-periodic. If D is not eventually mapped into itself, then D is contractible and contains no vertices with nontrivial vertex group. Each homotopically nontrivial path γ' in D with endpoints in $H'_j \cap D$ has lifts to a homotopically nontrivial path γ in $H_i \cap p^{-1}(D)$. The path γ' may have many lifts, but each lift satisfies that $f_\sharp(\gamma)$ is homotopically nontrivial and not entirely contained in H, so $f'_\sharp(\gamma')$ is homotopically nontrivial; this proves (EG-ii) by Lemma 1.5. Property (EG-iii) for H'_j follows from (EG-ii) and (EG-iii) for H_i. Namely, suppose E and E' are edges of H_i with terminal endpoint v and initial endpoint w respectively, that b is a maximal subpath of the component C of H, and that EbE' is a subpath of the f^k-image of some edge. Then the vertex group element at the center of $f'(Ep(b)E')$ is

$$[\eta_{f(v)} f(\bar{\eta}_v)\bar{\sigma}_C\sigma_C f(\eta_v) f(b) f(\bar{\eta}_w)\bar{\sigma}_C\sigma_C f(\eta_w)\bar{\eta}_{f(w)}] = pf(b).$$

If $f'(\bar{E})$ and $f'(E)$ begin with the same edge of H'_j, then since $f(b)$ is homotopically nontrivial, it must form a loop and therefore the turn crossed by $p(EbE')$ is not mapped to a degenerate turn by Df. Since the subpath and k were arbitrary, we conclude that (EG-iii) holds. $\square$

Example 1.7. Consider the graph G in Figure 1, and the relative train track map

$$f \begin{cases} \alpha_1 \mapsto \alpha_2 \mapsto \alpha_3 \mapsto \alpha_4 \mapsto \beta_1\gamma_1\alpha_1 \\ \beta_1 \mapsto \beta_2 \mapsto \beta_3 \mapsto \beta_4 \mapsto \alpha_1\beta_1 \\ \gamma_1 \mapsto \gamma_2 \mapsto \gamma_3 \mapsto \gamma_4 \mapsto \gamma_1\alpha_1 \\ x_1 \mapsto x_2 \mapsto x_3 \mapsto x_4 \mapsto x_1\beta_1 \\ A \mapsto BE\bar{A}x_1\alpha_1\bar{x}_1A \\ B \mapsto C \\ C \mapsto D\bar{E} \\ D \mapsto A \\ E \mapsto \bar{B}x_2\beta_2\gamma_2\bar{x}_2BE. \end{cases}$$

There are three strata: H_1 consists of the edges with Greek letters and is exponentially growing, $H_2 = \{x_1, x_2, x_3, x_4\}$ is non-exponentially growing, and $H_3 = \{A, B, C, D, E\}$ is exponentially growing. The illegal turns in the interior of exponentially growing strata are drawn sharp in the figure.

We will collapse G_2. Since H_3 is exponentially growing, no edge in H_3 maps entirely into G_2, so the subgraph H is G_2. There are four components, C_1, C_2, C_3 and C_4 containing those edges with matching subscripts. Choose the maximal trees consisting of the edges x_i and γ_i, and let the basepoints c_i be the terminal endpoints of the x_i. Take the paths σ_{C_i} to be the trivial paths at c_i. The vertex groups are all isomorphic to F_2, with generators a_i and b_i corresponding to the loops α_i and $\beta_i\gamma_i$, respectively. The isomorphisms of vertex groups are generated by the rule

$$\begin{cases} a_1 \mapsto a_2 \mapsto a_3 \mapsto a_4 \mapsto b_1a_1 \\ b_1 \mapsto b_2 \mapsto b_3 \mapsto b_4 \mapsto a_1b_1a_1. \end{cases}$$

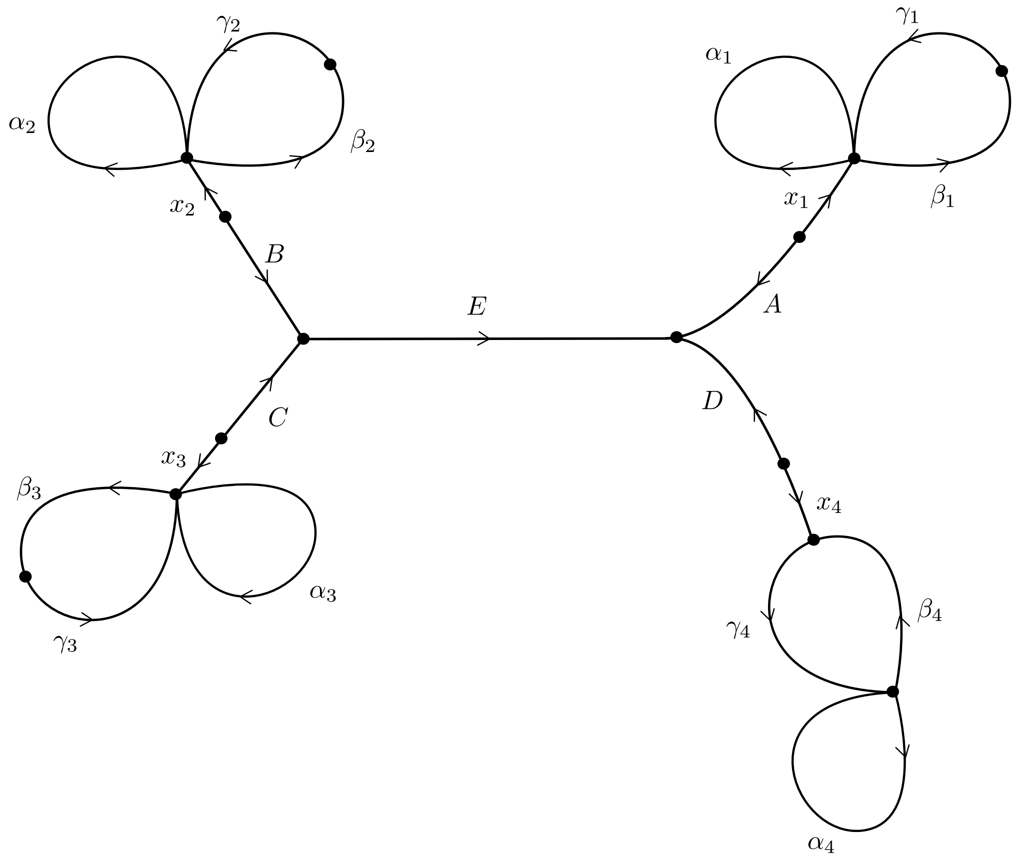

Figure 1: The relative train track map $f\colon G \to G$.

The relative train track map $f'\colon \mathcal{G}' \to \mathcal{G}'$ is defined on edges as

$$f' \begin{cases} A \mapsto BE\bar{A}a_1A \\ B \mapsto C \\ C \mapsto D\bar{E} \\ D \mapsto a_1b_1A \\ E \mapsto \bar{B}b_2BE. \end{cases}$$

See Figure 2.

There is a dual process of "blowing up" a relative train track map representing $\varphi \in \operatorname{Out}(F, \mathcal{A})$ to a "finer" free product decomposition $(F, \mathcal{B})$ where $\mathcal{B} \sqsubset \mathcal{A}$. In general it is not quite true that relative train track maps blow up to relative train track maps: one needs to restore properties (EG-i) and (EG-ii) by performing the "(invariant) core subdivision" and "collapsing inessential connecting paths" moves of [BH92] or [Lym21]. Blowing up plays no role in what follows, so we do not pursue the idea further.

Bounded cancellation. To finish out this section, we prove the following bounded cancellation lemma for free products (cf. [Coo87] or [BFH00, Lemma 2.3.1] for free groups), which is surely well-known to experts, but for which we were unable to find a published complete, general proof.

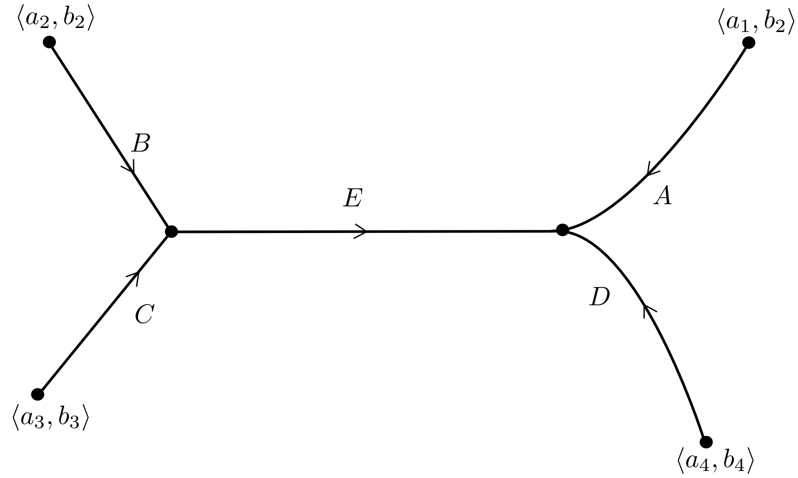

Figure 2: The relative train track map $f' \colon \mathcal{G}' \to \mathcal{G}'$.

Lemma 1.8. *Let $\mathcal{G}$ and $\mathcal{G}'$ be marked graphs of groups with no valence-one vertices with trivial vertex group, and let $h \colon \mathcal{G} \to \mathcal{G}'$ be a homotopy equivalence taking vertices to vertices. There is a constant C with the following properties.*

1. *If $\rho = \alpha\beta$ is a tight concatenation of tight paths in $\mathcal{G}$, then $h_\sharp(\rho)$ is obtained from $h_\sharp(\alpha)$ and $h_\sharp(\beta)$ by concatenating and cancelling $c \le C$ edges (and vertex group elements) from the terminal end of $h_\sharp(\alpha)$ with c edges (and vertex group elements) from the initial end of $h_\sharp(\beta)$.*

2. *If $\tilde h \colon \Gamma \to \Gamma'$ is a lift of h to the Bass–Serre trees, $\tilde\alpha$ is a line in Γ (a proper, linear embedding $\mathbb{R} \to \Gamma$) and $\tilde x \in \tilde\alpha$, then $\tilde h(\tilde x)$ can be connected to the line $\tilde h_\sharp(\tilde\alpha)$ by a path with $c \le C$ edges.*

3. *Suppose that $\tilde h \colon \Gamma \to \Gamma'$ is a lift to the Bass–Serre trees and that $\tilde\alpha \subset \Gamma$ is a (finite) tight path. Define $\tilde\beta \subset \Gamma'$ by removing C initial and C terminal edges from $\tilde h_\sharp(\tilde\alpha)$. If $\tilde\gamma$ is a line in Γ that contains $\tilde\alpha$ as a subpath, then $\tilde h_\sharp(\tilde\gamma)$ contains $\tilde\beta$ as a subpath.*

Proof. Note that if h_1 and h_2 are homotopy equivalences that additionally map edges to edges or collapse edges to vertices and satisfy the conclusions of this lemma with constants C_1 and C_2, then their composition h_2h_1 satisfies the conclusions of this lemma with constant $C_1 + C_2$. This is because $(h_2h_1)_\sharp = (h_2)_\sharp(h_1)_\sharp$.

If $H \subset \mathcal{G}$ is a (contractible) forest, then collapsing each component of H to a point yields a homotopy equivalence $h \colon \mathcal{G} \to \mathcal{G}'$. It is not hard to see that h satisfies the conclusions of the lemma with $C = 0$.

Since the trivial group is, in particular, finitely generated, we may use [Dun98, Theorem 2.1], which says that after collapsing a contractible forest, we may decompose h (or more properly speaking, its subdivision into a map sending edges to edges) into a finite product of folds, each of which is a homotopy equivalence.

We claim that a homotopy equivalence which is a single fold $f \colon \mathcal{G} \to \mathcal{G}'$ of a pair of edges g_1e_1 and g_2e_2 with $\iota(e_1) = \iota(e_2)$ satisfies the conclusions of the lemma with constant $C = 1$, from which the lemma follows. Note that because edge groups of marked graphs of groups are trivial, we may assume that e_1 and e_2 are distinct edges of $\mathcal{G}$ (in order for f to be a fold factor of h), and thus we have $\tau(e_1) \ne \tau(e_2)$ for otherwise the fold would fail to be a homotopy equivalence. For the same reason, at most one of $\tau(e_1)$ and $\tau(e_2)$ has nontrivial vertex group.

We prove that the first conclusion holds; the arguments for the others are similar. Suppose first that $\alpha = \alpha'\bar{e}_1 g_1^{-1}$ is a tight concatenation, where α' is some tight, possibly trivial path in $\mathcal{G}$ and that similarly $\beta = g_2 e_2 \beta'$ for some tight path β'. By assumption, the initial edge (if any) of $\bar{\alpha}'$ is distinct from the initial edge of β' and at least one of these edges (if any) is not e_1 or e_2. Therefore the concatenation $f_\sharp(\alpha')f_\sharp(\beta')$ is tight and equal to $f_\sharp(\alpha\beta)$. In all other cases, $f_\sharp(\alpha\beta) = f_\sharp(\alpha)f_\sharp(\beta)$, so we see that f satisfies the first conclusion of this lemma with constant $C = 1$. □

2 Improving Relative Train Track Maps

As a step towards the existence of CTs, Feighn and Handel construct in [FH11, Theorem 2.19] relative train track maps that satisfy a number of extra properties. The goal of this section is to prove the existence of such relative train track maps for outer automorphisms of free products. To state the theorem we require some more terminology. We begin by introducing *splittings* of edge paths in order to introduce a family of paths that cannot be split, the *indivisible periodic almost Nielsen paths.*

Splittings A decomposition of a path, line or circuit σ into subpaths $\sigma = \dots\sigma_1\sigma_2\dots$ is a *splitting,* written with raised dots as $\sigma = \dots\cdot\sigma_1\cdot\sigma_2\cdot\dots$ if $f^k_\sharp(\sigma) = \dots f^k_\sharp(\sigma_1)f^k_\sharp(\sigma_2)\dots$. That is, $f^k_\sharp(\sigma)$ is obtained from $f^k(\sigma)$ by tightening each $f^k(\sigma_i)$ and then concatenating. For our purposes, merely performing multiplication in a vertex group does not count as tightening.

The main application of the relative train track properties is the following lemma. Note that the definition of a relative train track map makes sense in the event that $f\colon \mathcal{G} \to \mathcal{G}$ is merely a map of graphs of groups that sends vertices to vertices and edges to nontrivial edge paths. We work mainly with relative train track maps that are topological representatives (so f maps edges to *tight* edge paths), but this more expansive definition allows for an iterate of such a relative train track map to be a relative train track map.

Lemma 2.1 (cf. Lemma 2.9(2) of [FH11])**.** *Suppose $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map with exponentially growing stratum H_r, and that σ is a path with endpoints at vertices of H_r which is r-legal. Then the decomposition of σ into single edges of H_r and maximal subpaths in G_{r-1} is a splitting.*

In particular if f is also a topological representative then if $\sigma = \alpha_1\beta_1\dots\alpha_n\beta_n$ is a decomposition into subpaths where $\alpha_i \subset H_r$ and $\beta_i \subset G_{r-1}$ (allow α_1 and β_n to be trivial, but assume all others are nontrivial), then

$$f_\sharp(\sigma) = f(\alpha_1)f_\sharp(\beta_1)\dots f(\alpha_n)f_\sharp(\beta_n)$$

is a tight concatenation of tight paths.

Note that strictly speaking the filtration for f ought to be enlarged to create the filtration for f^k, but that (EG-i), (EG-ii) and (EG-iii) are still satisfied by f^k for any stratum H_r which is an exponentially growing stratum for f.

Proof. Let $\sigma = \dots\alpha_1\alpha_2\dots$ be a decomposition of the r-legal path σ into single edges of H_r and maximal subpaths in G_{r-1}, and fix $k \geq 1$. Each $f^k_\sharp(\sigma_i)$ is nontrivial, either because σ_i is an edge of H_r and thus $f^k_\sharp(\sigma_i)$ is r-legal by (EG-iii), or by (EG-ii). Either by assumption or by (EG-i), the turn at the common endpoint of $\bar{\sigma}_i$ and σ_{i+1} is legal, so no cancellation occurs at the common endpoint of $f^k_\sharp(\sigma_i)$ and $f^k_\sharp(\bar{\sigma}_{i+1})$.

In particular, if f is a topological representative with decomposition $\sigma = \alpha_1\beta_1\dots\alpha_n\beta_n$ as in the statement, then $f(\alpha_1)$ is already a tight path by (EG-iii) because α_i is legal. The path $f_\sharp(\beta_i)$ is nontrivial (provided β_i is) by (EG-ii), and the turn at the common endpoint of $f(\alpha_i)$ and $f_\sharp(\bar{\beta}_i)$ is nondegenerate by (EG-i), so

$$f_\sharp(\sigma) = f(\alpha_1)f_\sharp(\beta_1)\dots f(\alpha_n)f_\sharp(\beta_n)$$

is a tight concatenation of tight paths. $\square$

An application of the above lemma is the following result. The proof is identical to the original, using only Lemma 2.1 (which is [BH92, Lemma 5.8]) and the topology of the underlying graph of $\mathcal{G}$, so we omit it.

Lemma 2.2 (Lemma 5.10 of [BH92])**.** *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map and that H_r is an exponentially growing stratum. There is a length function $L_r(\sigma)$ for paths σ in G_r with the property that $L_r(f(\sigma)) = L_r(f_\sharp(\sigma)) = \lambda_r L_r(\sigma)$ for any r-legal path σ in G_r. If we assume that f acts linearly with respect to some metric on the underlying graph of $\mathcal{G}$, then if σ contains an initial or terminal segment of an edge in H_r, then $L_r(\sigma) > 0$.*

We have an application to splittings of paths that are not necessarily legal. Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map, that H_r is an exponentially growing stratum, that σ is a tight path in $\mathcal{G}$, and that α is a subpath of σ in G_r with endpoints at vertices. If there are k edges of H_r to the left and to the right of α in σ, define $W_k(\alpha)$ to be the subpath of σ that begins with the kth edge of H_r to the left of α and ends with the kth edge of H_r to the right of α. We say that α is *k-protected in σ* if its first and last edges are in H_r, if $W_k(\alpha)$ is in G_r and if $W_k(\alpha)$ is r-legal.

Lemma 2.3 (cf. Lemma 4.2.2 of [BFH00])**.** *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map and that H_r is an exponentially growing stratum. There is a constant K such that if σ is a tight path in $\mathcal{G}$ and if α in G_r is a K-protected subpath of σ, then σ can be split at the endpoints of α.*

Proof. We follow the proof in [BFH00, Lemma 4.2.2]. Choose ℓ so that the f^ℓ-image of an edge in H_r contains at least two edges in H_r. Let $K = 2\ell C$, where C is a bounded cancellation constant for f as in Lemma 1.8.

We show that if $\sigma = \sigma_1\alpha\sigma_2$, then $f^i_\sharp(\sigma) = f^i_\sharp(\sigma_1)f^i_\sharp(\alpha)f^i_\sharp(\sigma_2)$ for $1 \le i \le \ell$ and that $f^\ell_\sharp(\alpha)$ is K-protected in $f^\ell_\sharp(\sigma)$, so the result follows by induction.

So take i satisfying $1 \le i \le \ell$. Let $W_K(\alpha) = \tau_1\alpha\tau_2$. Since α begins and ends with edges of H_r and $W_K(\alpha)$ is r-legal, Lemma 2.1 implies that $f^i_\sharp(W_K(\alpha)) = f^i_\sharp(\tau_1)f^i_\sharp(\alpha)f^i_\sharp(\tau_2)$. Write $\sigma_1 = \beta_1\tau_1$ and $\sigma_2 = \tau_2\beta_2$. By Lemma 1.8 (applied iteratively), when $f^i_\sharp(\beta_1)f^i_\sharp(\tau_1)$ and $f^i_\sharp(\tau_2)f^i_\sharp(\beta_2)$ are tightened to compute $f^i_\sharp(\sigma_1)$ and $f^i_\sharp(\sigma_2)$, at most iC edges are canceled. But τ_i contains at least $2\ell C$ edges, so we conclude that $f^i_\sharp(\sigma_1)f^i_\sharp(\alpha)f^i_\sharp(\sigma_2)$ is a tight concatenation of tight paths. In fact, the same argument applies with α replaced by $W_{\ell C}(\alpha)$. By the choice of ℓ, we have that $f^\ell_\sharp(W_{\ell C}(\alpha))$ contains $W_K(f^\ell_\sharp(\alpha))$, so $f^\ell_\sharp(\alpha)$ is K-protected in $f^\ell_\sharp(\tau)$. $\square$

Almost Nielsen paths. A path σ is a *periodic almost Nielsen path* with respect to a topological representative $f\colon \mathcal{G} \to \mathcal{G}$ if σ is nontrivial and $f^k_\sharp(\sigma) = g\sigma g'$ for some $k \ge 1$ and vertex group elements g and g'. The minimal such k is the *period* of σ, and σ is an *almost Nielsen path* if it has period 1. If g and g' are trivial, then we have *periodic Nielsen paths* and *Nielsen paths.* A periodic almost Nielsen path is *indivisible* if it cannot be written as a concatenation of nontrivial periodic almost Nielsen paths. (Note that we do not allow interior vertex group elements to change.) We consider an equivalence relation on periodic almost Nielsen paths, where σ is is equivalent to ρ if $\rho = g\sigma g'$, for vertex group elements g and g'

Lemma 2.4 (cf. Lemma 5.11 [BH92])**.** *Suppose $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map and that H_r is an exponentially growing stratum. There are only finitely many equivalence classes of indivisible periodic almost Nielsen paths σ in G_r that meet the interior of H_r. Each such σ has exactly one illegal turn in H_r, thus $\sigma = \alpha\beta$, where α and β are r-legal paths and the turn in H_r at the common vertex of $\bar\alpha$ and β is illegal.*

An indivisible periodic almost Nielsen path of height r has period 1 if and only if the directions determined by α and $\bar\beta$ are almost fixed.

Proof. The proof follows [BH92, Lemma 5.11] and [BFH00, Lemma 4.2.5]. Let σ be an indivisible periodic almost Nielsen path that meets the interior of H_r. By Lemma 2.2, σ cannot be r-legal. So let $\sigma = \alpha\beta\ldots$ be a decomposition of σ into maximal r-legal subpaths. Let k be the period of σ. For $m \geq 1$, let a_m be the smallest initial segment of α that satisfies $f_\sharp^{km}(a_m) = g''\alpha$ for some vertex group element g''. Let b_m be the complementary segment of α, so $\alpha = a_m b_m$. There is a largest initial segment c_m of β such that $f_\sharp^{km}(b_m c_m)$ is trivial. Let d_m be the complementary segment of β, so $\beta = c_m d_m$. It is not hard to see that we must have $f_\sharp^k(a_{m+1}) = g a_m$ and $f_\sharp^k(b_{m+1}c_{m+1}) = b_m c_m$. Therefore $f_\sharp^k(\bigcap_{m=1}^\infty a_m) = \bigcap_{m=1}^\infty a_m$ and $f_\sharp^k(\bigcup_{m=1}^\infty b_m c_m) = \bigcup_{m=1}^\infty b_m c_m$. Since σ is indivisible, we must have that $\bigcap_{m=1}^\infty a_m$ is a point and that $\bigcup_{m=1}^\infty b_m c_m$ is the interior of σ.

We have shown that σ satisfies the following conditions:

1. The path $\sigma = \alpha\beta$ has exactly one illegal turn in H_r.
2. The initial and terminal (partial) edges of σ are in H_r.
3. The number of H_r-edges in $f_\sharp^\ell(\sigma)$ is bounded independently of ℓ.

We will show that there are only finitely many paths σ satisfying these three conditions up to the equivalence relation above.

Let σ be a path satisfying these conditions, and as above write $\sigma = \alpha\beta$, where α and β are r-legal. When $f^\ell(\alpha)$ and $f^\ell(\beta)$ are tightened to $f_\sharp^\ell(\alpha)$ and $f_\sharp^\ell(\beta)$, no edges of H_r are canceled. When $f_\sharp^\ell(\alpha)f_\sharp^\ell(\beta)$ is tightened to $f_\sharp^\ell(\sigma)$, an initial segment of $f_\sharp^\ell(\bar\alpha)$ is canceled with an initial segment of $f_\sharp^\ell(\beta)$. Since $f_\sharp^\ell(\sigma)$ has an illegal turn in H_r, (it must have at least one by Lemma 2.2 and cannot have more because α and β are r-legal) the first edges which are not canceled are contained in H_r. Since nondegenerate turns with the same underlying edge are legal, the first edges which are not canceled are distinct edges. Therefore the cancellation between $f_\sharp^\ell(\alpha)$ and $f_\sharp^\ell(\bar\beta)$ is determined by the ordered list of oriented edges of $f_\sharp^\ell(\alpha)$ and $f_\sharp^\ell(\bar\beta)$ in H_r: the two paths cancel until the first distinct edges of H_r are reached. Call these lists $f_\sharp^\ell(\alpha) \cap H_r$ and $f_\sharp^\ell(\bar\beta) \cap H_r$, and note that they are determined by $\alpha \cap H_r$ and $\bar\beta \cap H_r$, respectively.

We claim that $\sigma \cap H_r$ takes on only finitely many values as σ varies over paths satisfying the three conditions above. Note that if σ has a splitting, then one of the pieces of the splitting is r-legal and meets the interior of H_r. Then the number of H_r edges of σ would grow without bound by Lemma 2.1 in contradiction to our assumption. But Lemma 2.3 then implies that the number of H_r edges of σ is bounded independently of σ. Therefore it follows that $\sigma \cap H_r$ takes on only finitely many values except for the possibility that differing amounts of the first and last edges α_0 and β_0 may occur. That is, suppose σ' is another path satisfying conditions 1 through 3 above, and that $\sigma \cap H$ and $\sigma' \cap H_r$ are identical except that the lengths of α_0 and α_0' and the lengths of β_0 and β_0' differ. For concreteness, suppose α_0' is a proper subset of α_0, and that A is what is left over. Since A is r-legal, the number of H_r-edges in $f_\sharp^\ell(A)$ grows without bound. This implies that all of $f_\sharp^\ell(\bar\alpha') \cap H_r$ is canceled with a proper initial segment X of $f_\sharp^\ell(\beta) \cap H_r$ for sufficiently large ℓ. But X, like all initial segments of $f_\sharp^\ell(\beta)$, either contains or is contained in $f_\sharp^\ell(\beta') \cap H_r$. If "contains," then all of $f_\sharp^\ell(\bar\beta') \cap H_r$ is canceled with part of $f_\sharp^\ell(\alpha') \cap H_r$. If "contained in," then all of $f_\sharp^\ell(\alpha') \cap H_r$ is canceled with part of $f_\sharp^\ell(\bar\beta') \cap H_r$. In either case $f_\sharp^\ell(\sigma')$ is legal, a contradiction.

Property 2, the fact that the number of H_r edges of α and β is bounded independently of σ, and the fact that α_0 and β_0 take on only finitely many values (up to multiplication by elements of $\mathcal{G}_{\iota(\alpha_0)}$ and $\mathcal{G}_{\tau(\beta_0)}$ if α_0 and β_0 are whole edges) implies that there exists $\ell > 0$ independent of σ such that $f_\sharp^\ell(\sigma)$ is obtained from $f_\sharp^\ell(\alpha_0)$ and $f_\sharp^\ell(\beta_0)$ by concatenating and canceling at the juncture. This implies that $f_\sharp^\ell(\sigma)$ itself takes on only finitely many values up to multiplication by vertex group elements at the ends, and therefore so does σ.

We now turn to the second paragraph of the lemma. Suppose σ is an indivisible periodic almost Nielsen path of height r with period p, and write $\sigma = \alpha\beta$. The proof follows [FH11, Lemma 2.11]. After subdividing at the endpoints of $f_\sharp^k(\sigma)$ for $0 \leq k \leq p-1$, we may assume the endpoints of σ are vertices. Since α and β are r-legal, the relative train track property implies that Df maps the initial directions of α and $\bar\beta$ to the initial directions of $f_\sharp(\alpha)$ and $f_\sharp(\beta)$ respectively. If σ has period 1, then we see that these directions are almost fixed, since $f_\sharp(\sigma) = g\sigma g'$ is obtained from $f_\sharp(\alpha)$ and $f_\sharp(\bar\beta)$ by cancelling their maximal common terminal segment.

Now suppose that the initial directions of α and $\bar\beta$ are almost fixed and write $f_\sharp(\sigma) = \alpha_1\beta_1$. The first edge E of α is also the first edge of α_1, and the argument above shows that α and α_1 are initial segments of $f_\sharp^{Np}(E)$ for large N. Thus either α is an initial segment of $g\alpha_1$ for some vertex group element g or vice versa. Assume α is an initial segment of $g\alpha_1$.

Suppose toward a contradiction that $\alpha_1 = g\alpha\gamma$ for some nontrivial path γ. The path $\alpha_2 = \bar\beta\gamma$ is a subpath of $f_\sharp^{Np}(\bar\beta)$ for large N and is thus r-legal. The path $\alpha_2\beta_1 \simeq \beta\bar\alpha\alpha_1\beta_1$ is a nontrivial periodic almost Nielsen path with exactly one illegal turn in H_r. It is therefore indivisible. Notice that $\bar\beta$ and $g'^{-1}\bar\beta_1$ share a common initial subpath in H_r for some vertex group element g' by assumption, therefore the same is true of α_2 and $g'^{-1}\bar\beta_1$. In fact, the argument above shows that α_2 and $g'^{-1}\bar\beta_1$ are initial subpaths of a common path δ. They cannot be equal, so one is a proper initial subpath of the other. But note that the difference between the number of H_r edges in $f_\sharp^{Np}(\alpha_2)$ and $f_\sharp^{Np}(\bar\beta_1)$ must grow exponentially in N, contradicting the fact that $\alpha_2\beta_1$ is a periodic almost Nielsen path. This contradiction shows that $\alpha_1 = g\alpha$, and a symmetric argument shows that $\beta_1 = \beta g'$, so $p = 1$. ☐

Let P_r be the set of equivalence classes of paths satisfying items 1 through 3 in the proof of Lemma 2.4; this is a finite set. The following lemma will be used in Section 6.

Lemma 2.5 (cf. Lemma 4.2.6 of [BFH00])**.** *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map and that H_r is an exponentially growing stratum. If σ is a path in G_r with the property that each path $f_\sharp^k(\sigma)$ has the same number of illegal turns in H_r, then σ can be split into subpaths that are either r-legal or elements of P_r.*

Proof. The proof is identical to [BFH00, Lemma 4.2.6]. We prove the statement by induction on m, the number of illegal turns in H_r that σ has. If $m = 0$, then we are done. Suppose that $m = 1$ and that σ cannot be split; we will show that σ belongs to P_r. The hypothesis that σ has exactly one illegal turn in H_r is satisfied. If the number of H_r-edges in $f_\sharp^\ell(\sigma)$ is not bounded independent of ℓ, then Lemma 2.3 implies that σ may be split. Similarly if the first and last (possibly partial) edges of σ are not contained in H_r, then Lemma 2.1 implies that σ may be split at the initial vertex of the first edge of H_r in σ. Therefore if σ cannot be split, then σ belongs to P_r.

So suppose $m > 1$. Choose lifts $\tilde f\colon \Gamma \to \Gamma$ and $\tilde\sigma$ in Γ and decompose $\tilde\sigma = \tilde\sigma_1 \dots \tilde\sigma_{m+1}$ so that each juncture projects to an illegal turn in H_r and each $\tilde\sigma_i$ projects to an r-legal path. By hypothesis, each $\tilde f_\sharp^k(\tilde\sigma)$ has a decomposition $\tilde f_\sharp^k(\tilde\sigma) = \tilde\tau_1^k \dots \tilde\tau_{m+1}^k$ into maximal r-legal subpaths. The set $\tilde S_k^2 = \{\tilde x \in \tilde\sigma_2 : \tilde f^k(\tilde x) \in \tilde f_\sharp^k(\tilde\sigma)\}$ is closed, and one can argue by induction that f^N maps $\bigcap_{k=1}^N(\tilde S_N^2)$ onto $\tilde\tau_2^N$ for all $N \geq 1$. Therefore $\tilde S^2 = \bigcap_{k=1}^\infty \tilde S_k^2$ is nonempty. It is not hard to see that σ can be split at a point x if and only if $\tilde f^k(\tilde x)$ belongs to $\tilde f_\sharp^k(\tilde\sigma)$ for all $k \geq 0$. Therefore σ can be split at (the projection of) any point in $\tilde S^2$. This splits $\tilde\sigma$ into subpaths that have fewer than m illegal turns in H_r, and thus induction completes the proof. ☐

Non-exponentially growing strata. A non-exponentially growing stratum H_r is *almost periodic* if the edges $E_1, \dots, E_k$ of H_r satisfy $f(E_i) = g_iE_{i+1}h_i$ for vertex group elements g_i and h_i and with indices taken mod k. If $k = 1$, we say H_r is an *almost fixed* stratum and E_1 is an *almost fixed* edge.

If H_r is a non-exponentially growing but not periodic stratum for a topological representative $f\colon \mathcal{G} \to \mathcal{G}$, each edge e has a subinterval which is eventually mapped back over e, so the subinterval contains a periodic point. After declaring all of these periodic points to be vertices, reordering, reorienting, and possibly replacing H_r with two non-exponentially growing strata, we may assume the edges $E_1, \dots, E_k$ of H_r satisfy $f(E_i) = g_i E_{i+1} u_i$, where indices are taken mod k, g_i is a vertex group element and u_i is a path in G_{r-1}. Henceforth we always adopt this convention.

Dihedral Pairs. A *dihedral pair* is a pair of (oriented) edges E_i and E_j with the following properties.

1. E_i and E_j have a common initial vertex v with trivial vertex group. We refer to v as the *center vertex of the dihedral pair E_i and E_j.*
2. E_i and E_j have terminal vertices each with valence one in G and C_2 vertex group.
3. The edges E_i and E_j are either fixed or swapped by f. (Up to homotopy, we may assume that $f(E_i) = E_j$ and $f(E_j) = E_i$, rather than for example $E_j g$, where g is the nontrivial element of C_2.)

Dihedral pairs will play an exceptional role in Section 6. If the edges E_i and E_j are fixed by f, they determine fixed strata H_i and H_j that are forests. By rearranging strata, we may assume that $H_j = H_{i+1}$. In this situation, we say that H_i is the *bottom half of the dihedral pair.*

For a topological representative $f\colon \mathcal{G} \to \mathcal{G}$, let $\mathrm{Per}(f)$ denote the set of f-periodic points in G (as a map of topological spaces). The subset of points with period 1 is $\mathrm{Fix}(f)$. A subgraph $C \subset G$ is *wandering* if $f^k(C) \subset \overline{G \setminus C}$ for all $k \geq 1$ and is *non-wandering* otherwise.

The *core* of a subgraph $C \subset G$ is the minimal subgraph (of groups) K of C such that the inclusion is a homotopy equivalence.

Enveloped zero strata. Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a topological representative, that $u < r$ and that the following hold.

1. The stratum H_u is irreducible.
2. The stratum H_r is exponentially growing. Each component of G_r is noncontractible.
3. For each i satisfying $u < i < r$, the stratum H_i is a zero stratum that is a component of G_{r-1}, and each vertex of H_i has valence at least two in G_r.

Then we say that each H_i is *enveloped by H_r*, and write $H_r^z = \bigcup_{k=u+1}^{r} H_k$.

The following theorem is the main result of this section; its proof occupies the remainder of the section.

Theorem 2.6 ([FH11] Theorem 2.19)**.** *Given an outer automorphism $\varphi \in \mathrm{Out}(F, \mathcal{A})$ there is a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ on a marked Grushko $(F, \mathcal{A})$-graph of groups and filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ representing φ satisfying the following properties:*

- *(V) The endpoints of all indivisible periodic almost Nielsen paths are vertices.*
- *(P) If a periodic or almost periodic stratum H_m is a forest, then either there exists a filtration element G_j such that $\mathcal{F}(G_j) \neq \mathcal{F}(G_\ell \cup H_m)$ for any filtration element G_ℓ or H_m is the bottom half of a dihedral pair.*
- *(Z) Each zero stratum H_i is enveloped by an exponentially growing stratum H_r. Each vertex of H_i is contained in H_r and meets only edges in $H_i \cup H_r$.*
- *(NEG) The terminal endpoint of an edge in a non-exponentially growing stratum H_i is periodic, and if the stratum is not almost periodic, the terminal endpoint is contained in a filtration element G_j with $j < i$ that is its own core.*

(F) *The core of a filtration element G_r is a filtration element, unless H_r is the bottom half of a dihedral pair, in which case G_{r-1} and G_{r+1} are their own core.*

Moreover, if $\mathcal{F}_1 \sqsubset \cdots \sqsubset \mathcal{F}_d$ is a nested sequence of φ-invariant free factor systems, we may assume that each free factor system is realized by some filtration element.

Before we turn to the proof, we prove a lemma and a simple consequence of Theorem 2.6 that we will need later.

Lemma 2.7 ([FH11] Lemma 2.10)**.** *Let H_r be an exponentially growing stratum of a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ and v a vertex of H_r. Then there is a legal turn in G_r based at v.*

Proof. The proof is identical to [FH11, Lemma 2.10]. There is a point w in the interior of an edge E of H_r and $j > 0$ such that $f^j(w) = v$. By (EG-iii), the point v is in the interior of an r-legal path. By (EG-i) and (EG-iii), the turn based at v crossed by this path is legal. □

In particular, the lemma implies that if v has valence one in G_r, then v has nontrivial vertex group.

Lemma 2.8. *Suppose $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map satisfying properties (Z) and (NEG). If v is a vertex of a filtration element G_k that is its own core, then there is a legal turn based at v in G_k.*

Proof. The proof is by induction on k; the statement is vacuously true for $k = 0$. If v has nontrivial vertex group, then the lemma holds, since nondegenerate turns of the form $\{(g_1, e), (g_2, e)\}$ with the same underlying edge e are legal. So suppose v has trivial vertex group. Since G_k is its own core, v has valence at least two in G_k, and we may assume that there is an illegal turn based at v. By Lemma 2.7 and property (Z) we may assume that v is not incident to any edges of exponentially growing or zero strata. The directions determined by periodic edges and the initial endpoints of non-periodic non-exponentially growing edges are periodic, so since we assume there is an illegal turn, we conclude that v is the terminal endpoint of a non-periodic non-exponentially growing edge, and thus by (NEG) v is contained in a lower stratum that is its own core; we conclude that there is a legal turn based at v by induction. □

Proof of Theorem 2.6. We adapt the proof of [FH11, Theorem 2.19, pp. 56–62]. Begin with a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ which has no valence-one vertices with trivial vertex group. By Proposition 1.3, we may assume that the filtration for f realizes our nested sequence of φ-invariant free factor systems. There is no loss in assuming f to be a topological representative, so we shall.

Property (V). To prove (V), we need to collect more information about almost Nielsen paths. We saw in Lemma 2.4 that if $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map and H_r is an exponentially growing stratum, then there are only finitely many equivalence classes of indivisible periodic almost Nielsen paths of height r.

If H_r is a zero stratum, then there are no indivisible periodic almost Nielsen paths of height r. If H_r is a periodic or an almost periodic stratum, then there are no indivisible periodic almost Nielsen paths of height r. (Note that if E is an almost periodic edge, then it is a periodic almost Nielsen path, but it is not indivisible.)

If H_r is a non-exponentially growing stratum which is not almost periodic, then assuming that our relative train track map f acts linearly with respect to some metric on G, there are no periodic points in the interior of each edge in H_r. Thus the endpoints of periodic almost Nielsen paths of height r, if there are any, are vertices.

All together, this implies that (V) can be accomplished by declaring a finite number of periodic points in the interior of exponentially growing strata to be vertices. The resulting map is still a relative train track map.

Let us remark that for the remainder of our construction, if $f\colon \mathcal{G} \to \mathcal{G}$ is our relative train track map satisfying (V) and $f'\colon \mathcal{G}' \to \mathcal{G}'$ is the final relative train track map, then there is a bijection $H_r \to H'_s$ from the exponentially growing strata for f to those for f' such that

1. H_r and H'_s have the same number of edges, and
2. The number of equivalence classes of indivisible almost Nielsen paths for f of height r (which is finite by Lemma 2.4) is equal to the number of equivalence classes of indivisible almost Nielsen paths for f' of height s.

For all of the moves we will use, the first item is obvious. For some of the moves we will use, namely valence-two homotopies not involving exponentially growing strata and reordering strata, both properties are obvious. For *sliding,* which is defined below, the second item is part of Lemma 2.10. For the remaining moves, namely *tree replacements,* which are defined below, and collapsing forests away from exponentially growing strata we have the following lemma, which demonstrates that the second item holds.

Lemma 2.9 (cf. Lemma 2.16 of [FH11]). *Suppose that the topological representatives $f\colon \mathcal{G} \to \mathcal{G}$ and $f'\colon \mathcal{G}' \to \mathcal{G}'$ are relative train track maps with exponentially growing strata H_r and H'_s respectively such that all indivisible periodic almost Nielsen paths of height r or s respectively have endpoints at vertices. Suppose further that $p\colon \mathcal{G} \to \mathcal{G}'$ is a homotopy equivalence such that the following hold.*

1. *The map p is such that $p(G_r) = G'_s$, $p(G_{r-1}) = G'_{s-1}$ and p induces a bijection between the set of edges of H_r and the set of edges of H'_s.*
2. *We have $p_\sharp f_\sharp(\sigma) = f'_\sharp p_\sharp(\sigma)$ for all tight paths σ in G_r with endpoints at vertices.*

Then $p_\sharp$ induces a period-preserving bijection between the indivisible periodic almost Nielsen paths of height r for f and those of height s for f'.

Proof. The proof is identical to [FH11, Lemma 2.16]. Let σ be a tight path of height r with endpoints at vertices, and let $\sigma' = p_\sharp(\sigma)$. It has height s.

Observe that no edges in H'_s are cancelled when $p(\sigma)$ is tightened to $p_\sharp(\sigma)$, for if there were, then σ has a subpath of the form $\sigma_0 = E\tau\bar{E}$, where E is an edge of H_r and $p_\sharp(\sigma_0)$ is trivial. But the closed path σ_0 determines a nontrivial element of the fundamental group of $\mathcal{G}$ by assumption, so this is a contradiction, from which it follows by item 1 that

3. The number of edges of H_r in σ is equal to the number of H'_s edges of $\sigma' = p_\sharp(\sigma)$.

We claim that σ is r-legal if and only if σ' is s-legal. If E is an edge of H_r, then by item 2 and the fact that f and f' are topological representatives, we have $f'(p(E)) = f'_\sharp(p_\sharp(E)) = p_\sharp(f_\sharp(E)) = p_\sharp(f(E))$. By item 3, this path has the same number of edges in H'_s as $f(E)$ has edges in H_r. It follows that before tightening, $f(\sigma)$ has as many edges in H_r as $f'(\sigma')$ has in H'_s. After tightening, item 3 implies that $p_\sharp f_\sharp(\sigma) = f'_\sharp(\sigma')$ has as many edges in H'_s as $f_\sharp(\sigma)$ has in H_r, so we see that σ is r-legal if and only if σ' is r-legal. What's more, the number of illegal turns of σ in H_r equals the number of illegal turns of σ' in H'_s.

Assume that σ is an indivisible periodic almost Nielsen path with height r and period k. By item 2, we have

$$(f')^k_\sharp(\sigma') = (f')^k_\sharp(p_\sharp(\sigma)) = p_\sharp f^k_\sharp(\sigma) = p_\sharp(g\sigma g') = g''\sigma' g'''$$

where g, g', g'' and g''' are vertex group elements. Therefore σ' is a periodic almost Nielsen path. If σ is indivisible, then the first and last edges of σ belong to H_r and σ has exactly one illegal turn in H_r. By the argument above, σ' has these properties for H'_s and so is indivisible. Let E_i and E_j be the first and last edges of σ. Then by Lemma 2.4, k is the minimum positive integer such that E_i and E_j are the first and last edges of $f^k_\sharp(\sigma)$. By item 2, it follows that the period of σ' equals the period of σ. To prove that if σ' is an indivisible periodic almost Nielsen path of period k then σ is as well, observe that if σ' is a tight path in G'_s whose first edge is $p(E_i)$ and whose last edge is $p(E_j)$, then there is a unique tight path σ whose first edge is E_i, whose last edge is E_j and satisfies $p_\sharp(\sigma) = \sigma'$. If σ' is a periodic almost Nielsen path of period k, then the uniqueness of σ implies that σ is a periodic almost Nielsen path of period k. As above, if σ' is indivisible, then it has exactly one illegal turn in H'_s, and σ has exactly one illegal turn in H_r, so σ is indivisible. □

Sliding. The move *sliding* was introduced in [BFH00, Section 5.4, p. 579]. Suppose H_i is a non-periodic, non-exponentially growing stratum that satisfies our convention: the edges $E_1, \ldots, E_k$ of H_i satisfy $f(E_j) = g_j E_{j+1} u_j$ where indices are taken mod k, g_j is a vertex group element and u_j is a path in G_{i-1}. We will call the edge of H_i we focus on E_1. Let α be a path in G_{i-1} from the terminal vertex of E_1 to some vertex of G_{i-1}. Define a new graph of groups $\mathcal{G}'$ by removing E_1 from $\mathcal{G}$ and gluing in a new edge E'_1 with initial vertex equal to the initial vertex of E_1 and terminal vertex the terminal vertex of the path α. See Figure 3. Define homotopy equivalences $p\colon \mathcal{G} \to \mathcal{G}'$ and $p'\colon \mathcal{G}' \to \mathcal{G}$ by sending each edge other than E_1 and E'_1 to itself, and defining $p(E_1) = E'_1\bar{\alpha}$ and $p'(E'_1) = E_1\alpha$. Define $f'\colon \mathcal{G}' \to \mathcal{G}'$ by tightening $pfp'\colon \mathcal{G}' \to \mathcal{G}'$. If G_r is a filtration element of $\mathcal{G}$, define $G'_r = p(G_r)$. The G'_r form the filtration for $f'\colon \mathcal{G}' \to \mathcal{G}'$.

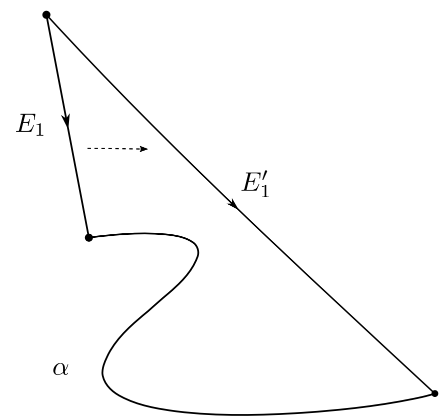

Figure 3: Sliding E_1 along α

Lemma 2.10 ([FH11] Lemma 2.17)**.** *Suppose $f'\colon \mathcal{G}' \to \mathcal{G}'$ is obtained from a topological representative $f\colon \mathcal{G} \to \mathcal{G}$ by* sliding *E_1 along α as described above. Let H_i be the non-exponentially growing stratum of $\mathcal{G}$ containing E_1, and let k be the number of edges of H_i.*

1. *$f'\colon \mathcal{G}' \to \mathcal{G}'$ is a relative train track map if $f\colon \mathcal{G} \to \mathcal{G}$ was.*
2. *$f'|_{G'_{i-1}} = f|_{G_{i-1}}$.*
3. *If $k = 1$, then $f'(E'_1) = g_1 E'_1[\bar{\alpha} u_1 f(\alpha)]$, where $[\gamma]$ denotes the path obtained from γ by tightening.*
4. *If $k \neq 1$, then $f'(E_k) = g_k E'_1[\bar{\alpha} u_k]$, $f'(E'_1) = g_1 E_2[u_1 f(\alpha)]$ and $f'(E_j) = g_j E_{j+1} u_j$ for $2 \le j \le k-1$.*
5. *For each exponentially growing stratum H_r, the map $p_\sharp$ defines a bijection between the set of indivisible periodic Nielsen paths in $\mathcal{G}$ of height r and the indivisible Nielsen paths in $\mathcal{G}'$ of height r.*

Proof. The proof is a straightforward adaptation of [BFH00, Lemma 5.4.1]. We include a proof for completeness. Suppose first that $k = 1$. Then

$$f'(E_1') = (pfp')_\sharp(E_1') = (pf)_\sharp(E_1\alpha) = p_\sharp(g_1E_1f(\alpha)) = g_1E_1'[\bar{\alpha}u_1f(\alpha)].$$

Next suppose that $k \neq 1$. Then

$$f'(E_k) = (pfp')_\sharp(E_k) = p_\sharp(g_kE_1[u_k]) = g_kE_1'[\bar{\alpha}u_k],$$

and

$$f'(E_1') = (pfp')_\sharp(E_1') = (pf)_\sharp(E_1\alpha) = g_1E_2[u_1f(\alpha)].$$

This proves items 3 and 4.

Item 2 is clear from the definition of sliding. To prove item 1, it suffices to consider each stratum H_j with $j > i$. So suppose $f\colon \mathcal{G} \to \mathcal{G}$ was a relative train track map. If H_j is a zero stratum and E is an edge of H_j, then $f(E) \subset G_{j-1}$, so $f'(E) = (pf)_\sharp(E) \subset G'_{j-1}$, and H'_j is still a zero stratum. Similarly if H_j is a non-exponentially growing stratum, then H'_j is still non-exponentially growing. For any nontrivial tight paths $\beta \subset \mathcal{G}$ and $\gamma' \subset \mathcal{G}'$ with endpoints at vertices, we have $(p'p)_\sharp(\beta) = \beta$ and $(pp')_\sharp(\gamma') = \gamma'$, so $p_\sharp(\beta)$ and $p'_\sharp(\gamma')$ are nontrivial.

Suppose H_r is an exponentially growing stratum and E is an edge of H_r. Property (EG-i) implies that

$$f(E) = a_1b_1a_2b_2\ldots a_\ell b_\ell a_{\ell+1},$$

where $a_i \subset H_r$ and $b_i \subset G_{r-1}$ are nontrivial tight paths. Then

$$f'(E) = (pf)_\sharp(E) = a_1p_\sharp(b_1)a_2p_\sharp(b_2)\ldots a_\ell p_\sharp(b_\ell)a_{\ell+1}.$$

Thus H'_r is exponentially growing, its transition matrix is equal to the transition matrix of H_r, and (EG-i) is satisfied. If γ' in G'_{r-1} is a nontrivial path with endpoints in $H'_r \cap G'_{r-1}$, then $p'_\sharp(\gamma')$ is nontrivial, and so are $(fp')_\sharp(\gamma')$ and $(pfp')_\sharp(\gamma') = f'_\sharp(\gamma')$. Therefore H'_r satisfies (EG-ii). In fact, (EG-iii) is also satisfied: the map Df of turns in H_r is unchanged, so the legality or illegality of turns in H'_r is identical to the situation in H_r. Thus $f'\colon \mathcal{G}' \to \mathcal{G}'$ is a relative train track map.

If σ' in $\mathcal{G}'$ satisfies $f'^k_\sharp(\sigma') = g\sigma'g'$ for vertex group elements g and g', then $p'_\sharp(\sigma')$ satisfies

$$f^k_\sharp(p'_\sharp(\sigma')) = (p'p)_\sharp f^k_\sharp p'_\sharp(\sigma') = p'_\sharp f'^k_\sharp(\sigma') = g''p'_\sharp(\sigma')g''',$$

where for example $g'' = p'_\sharp(g)$ is a vertex group element. Similarly, if $\sigma \in \mathcal{G}$ satisfies $f^k_\sharp(\sigma) = g\sigma g'$, then $p_\sharp(\sigma)$ satisfies

$$f'^k_\sharp(p_\sharp(\sigma)) = p_\sharp f^k_\sharp(p'p)_\sharp(\sigma) = p_\sharp f^k_\sharp(\sigma) = g''p_\sharp(\sigma)g''',$$

where for example $g'' = p_\sharp(\sigma)$ is a vertex group element. If $\sigma' \subset G'_r$, then $p'_\sharp(\sigma') \subset G_r$, and if $\sigma \subset G_r$, then $p_\sharp(\sigma) \subset G'_r$. Thus $p_\sharp$ induces a period-preserving, height-preserving bijection from the set of periodic almost Nielsen paths for $f\colon \mathcal{G} \to \mathcal{G}$ to the set of periodic almost Nielsen paths for $f'\colon \mathcal{G}' \to \mathcal{G}'$. ☐

Property (NEG) part one. We will first show that the terminal vertex of an edge in a non-exponentially growing stratum H_i is either periodic or has valence at least three. This is automatic if H_i is almost periodic, so assume that H_i is not almost periodic. Let $E_1, \ldots, E_k$ be the edges of H_i. As usual, assume $f(E_j) = g_jE_{j+1}u_j$, where indices are taken mod k, g_k is a vertex group element and u_j is a path in G_{i-1}. Suppose the terminal vertex v_1 of E_1 is not periodic and has valence two. Then v_1 has trivial vertex group. If E is the other edge incident to v_1, then E does not belong to an exponentially growing stratum by Lemma 2.7.

We perform a valence-two homotopy of $\bar{E}_1$ over E as in [Lym21, Lemma 2.5] or [BH92, Lemma 1.13]. If v is a vertex of an exponentially growing stratum, $f(v) \neq v_1$, so before collapsing the pretrivial forest, properties (EG-i) through (EG-iii) are preserved. The pretrivial forest is inductively constructed as follows: any edge which was mapped to E is added, then any edge which is mapped into the pretrivial forest is added. Thus the argument above shows that no vertex of an exponentially growing stratum is incident to any edge in the pretrivial forest, so the property of being a relative train track map is preserved. After repeating this process finitely many times, the terminal vertex of E_1 is either periodic or has valence at least three in $\mathcal{G}$.

Finally, we arrange that v_1 is periodic: the component of G_{i-1} containing v_1 is non-wandering (because $f^{k-1}(u_1)$ is contained in it), so contains a periodic vertex w_1. Choose a path α in G_{i-1} from v_1 to w_1 and slide E_1 along α. No valence-one vertices are created because v_1 was assumed to have valence at least three. Repeating this process for each edge in a non-almost periodic, non-exponentially growing stratum, we establish the first part of (NEG), namely the following.

(NEG*) The terminal vertex of each edge in a non-almost periodic, non-exponentially growing stratum is periodic.

Property (Z) part one. Property (Z) has several parts. Let H_i be a zero stratum. For H_i to be enveloped, let H_u the first irreducible stratum below H_i and H_r the first irreducible stratum above. One condition we need is that no component of G_r is contractible. Following [FH11, p. 58], we postpone that step and merely show here that each component is non-wandering.

First we arrange that if a filtration element G_i has a wandering component, then H_i is a wandering component. Suppose that G_i has wandering components. Call their union W and their complement N. If N is not precisely equal to a union of strata, the difference is that N contains part but not all of a zero stratum, so we may divide this zero stratum to arrange so that N is a union of strata. Thus W is a union of zero strata. Since N is f-invariant, we may push all strata in W higher than all strata in N. We define a new filtration. Strata in N and higher than G_i remain unchanged. The strata that make up W will be the components of W. If C and C' are such components, C' will be higher than C if $C' \cap f^k(C) = \varnothing$ for all $k \geq 0$. We complete this to an ordering on the components of W, yielding the new filtration.

Now we work toward showing that zero strata are enveloped by exponentially growing strata. Suppose that K is a component of the union of all zero strata in $\mathcal{G}$, that H_i is the highest stratum that contains an edge of K and that H_u is the highest irreducible stratum below H_i. We aim to show that $K \cap G_u = \varnothing$. So assume $K \cap G_u \neq \varnothing$. By the previous paragraph, because H_u is irreducible, each component of G_u is non-wandering, so K meets G_u in a unique component C of G_u. If each vertex of K has valence at least two in $C \cup K$, then each edge of K belongs to a tight path in K with endpoints in C, and we may close this path up in C to form a tight loop in $K \cup C$. But some iterate of $f \colon \mathcal{G} \to \mathcal{G}$ maps $K \cup C$ into C, so this is a contradiction. Therefore some vertex $v \in K$ has valence one in $K \cup C$. Because some iterate of $f \colon \mathcal{G} \to \mathcal{G}$ maps $K \cup C$ into C, if we can show this vertex has valence one in G, it must have trivial vertex group.

This valence-one vertex v is not periodic, so by (NEG*), v is not an endpoint of an edge in a non-exponentially growing stratum. We saw in Lemma 1.5 that (EG-ii) for an exponentially growing stratum H_r is equivalent to the condition that vertices of $H_r \cap G_{r-1}$ contained in non-wandering components of G_{r-1} are periodic. Since $K \cup C$ is non-wandering, v is not the endpoint of an edge in an exponentially growing stratum above H_i. By construction, v is also not the endpoint of an edge of another zero stratum. Thus v has valence one in $\mathcal{G}$, but we have not produced any valence-one vertices with trivial vertex group so far. This contradiction shows that $G_u \cap K = \varnothing$.

The lowest edges in K are mapped either to another zero stratum or into G_u. In any case, by connectivity, K is wandering, so we can reorganize zero strata so that $K = H_i$. Repeating this for each component of the union of all zero strata, we have arranged that if

H_i is a zero stratum and H_r is the first irreducible stratum above H_i, then H_i is a component of G_{r-1}.

Let H_r be the first irreducible stratum above H_i. Because H_r is irreducible, no component of G_r is wandering, so the component that contains H_i intersects H_r. No vertex of H_i is periodic, so (NEG*) implies H_r is exponentially growing, and the argument above shows that vertices of H_i meet only edges of H_i and H_r and every vertex of H_i has valence at least two in G_r. This satisfies every part of the definition of the zero stratum H_i being enveloped by H_r, except that we have not shown that all components of G_r are non-contractible, only that they are non-wandering.

Note also that H_i is contained in the core of G_r: one obtains the core of G_r by repeatedly removing from G_r any edge incident to a valence-one vertex with trivial vertex group. Each vertex of H_i has valence at least two in G_r, and Lemma 2.7 says that each valence-one vertex of H_r has nontrivial vertex group.

Tree replacement. The final part of property (Z) we will establish now is that every vertex of H_i is contained in H_r. We do so by Feighn and Handel's method of *tree replacement.* Replace H_i with a tree H'_i whose vertex set is exactly $H_i \cap H_r$. We may do so for every zero stratum at once, (with *a priori* different exponentially growing strata H_r, of course) and call the resulting graph of groups $\mathcal{G}'$. Let X denote the union of all irreducible strata. There is a homotopy equivalence $p'\colon \mathcal{G}' \to \mathcal{G}$ that is the identity on edges in X and sends each edge in a zero stratum H'_i to the unique path in H_i with the same endpoints. Choose a homotopy inverse $p\colon \mathcal{G} \to \mathcal{G}'$ that also restricts to the identity on edges in X and maps each zero stratum H_i to the corresponding tree H'_i. Define $f'\colon \mathcal{G}' \to \mathcal{G}'$ by tightening $pfp'\colon \mathcal{G}' \to \mathcal{G}'$. The map f' still satisfies (EG-i). Because $p_\sharp$ and $p'_\sharp$ send nontrivial paths with endpoints in X to nontrivial paths with endpoints in X, (EG-ii) is preserved as well. Because $\mathrm{PF}(f) = \mathrm{PF}(f')$, Corollary 1.4 implies f' satisfies (EG-iii) as well. Nothing we have done so far changes the realization of free factor systems by filtration elements as well. We replace $f\colon \mathcal{G} \to \mathcal{G}$ by $f'\colon \mathcal{G}' \to \mathcal{G}'$ in what follows.

Property (P). Let $\mathcal{F}_1 \sqsubset \cdots \sqsubset \mathcal{F}_d$ be our chosen nested sequence of φ-invariant free factor systems. If none was specified, instead define this sequence to be the sequence determined by $\mathcal{F}(G_r)$ as G_r varies among the filtration elements of $\mathcal{G}$. We will show that if H_m is an almost periodic forest that is not the bottom half of a dihedral pair, then there is some $\mathcal{F}_i$ that is not realized by $H_m \cup G_\ell$ for any filtration element G_ℓ. Assume that this does not hold for some almost periodic forest H_m. Then for all i satisfying $1 \le i \le d$, there is some filtration element G_ℓ such that $\mathcal{F}(H_m \cup G_\ell) = \mathcal{F}_i$. In this case we will collapse an invariant forest containing H_m, reducing the number of non-exponentially growing strata. Iterating this process establishes (P).

Let Y be the set of all edges in $G \setminus H_m$ eventually mapped into H_m by some iterate of f. Each edge of Y is thus contained in a zero stratum. We want to arrange that if α is a tight path in a zero stratum with endpoints at vertices that is not contained in Y, then $f_\sharp(\alpha)$ is not contained in $Y \cup H_m$. If there *is* such a path α such that $f_\sharp(\alpha)$ is contained in $Y \cup H_m$, let E_i be an edge crossed by α and not contained in Y. Perform a tree replacement as above, removing E_i and adding in an edge with endpoints at the endpoints of α. By our preliminary form of property (Z), if a vertex incident to E_i has valence two, then it is an endpoint of α, so this process does not create valence-one vertices with trivial vertex group. The image of the new edge is contained in $Y \cup H_m$, so we add it to Y. Because there are only finitely many paths in zero strata with endpoints at vertices, we need only repeat this process finitely many times if necessary.

Let $\mathcal{G}'$ be the graph of groups obtained by collapsing each component of $H_m \cup Y$ to a point, and let $p\colon \mathcal{G} \to \mathcal{G}'$ be the quotient map. For each component C of $H_m \cup Y$, choose a vertex $c \in C$. If C contains a vertex with nontrivial vertex group, choose that vertex as c. If an oriented edge E has initial vertex in C, let γ_C be the unique tight path containing no vertex group elements from c to $\iota(E)$; otherwise let γ_E be the trivial path. Identify the edges of $G \setminus (Y \cup H_m)$ with the edges of G' and define $f \colon \mathcal{G}' \to \mathcal{G}'$ on each edge E of the complement by tightening $pf(\gamma_E E \gamma_{\bar{E}})$. By construction, $f' \colon \mathcal{G}' \to \mathcal{G}'$ is a topological representative of φ, and $f'(E)$ is obtained from $f(E)$ by removing all occurrences of edges in $Y \cup H_m$ and possibly multiplying by vertex group elements at the ends. If $p(H_r)$ is not collapsed to a point, the stratum H_r and $p(H_r)$ are thus of the same type, and we see that f' has one less non-exponentially growing stratum and possibly fewer zero strata. The previous properties, (NEG*) and our preliminary form of (Z) are still satisfied.

Let H_r be an exponentially growing stratum. We will verify (EG-ii). Suppose first that C is a wandering component of $p(G_{r-1})$. Then there are only finitely many tight paths with endpoints in $C \cap p(H_r)$. For each such path α', there is a unique tight path α in G_{r-1} such that $p(\alpha) = \alpha'$. By assumption, α is not contained in $Y \cup H_m$, so $f_\sharp(\alpha)$ is not contained in $Y \cup H_m$ by construction, and thus we see that $f'_\sharp(\alpha') = (pf)_\sharp(\alpha)$ is nontrivial. By Lemma 1.5, for each non-wandering component C of $p(G_{r-1})$, (EG-ii) is equivalent to checking that each vertex of $p(H_r) \cap C$ is periodic. Let v' be a vertex of $p(H_r) \cap C$. By assumption, there is a vertex $v \in H_r$ such that $p(v) = v'$. If v is periodic, we are done. Thus we may assume that v is not in H_m (which is almost periodic). If $v \in Y$, then the component of G_{r-1} containing v is a zero stratum (and thus wandering), contradicting our assumption that C is non-wandering. Therefore v is the only preimage of v'. By the same reasoning, the component of G_{r-1} containing v must be non-wandering, so we conclude that v and thus v' is periodic. This verifies (EG-ii). It is easy to see that (EG-i) is still satisfied, and that $\mathrm{PF}(f) = \mathrm{PF}(f')$, so Corollary 1.4 implies (EG-iii) is still satisfied, so $f' \colon \mathcal{G}' \to \mathcal{G}'$ is a relative train track map.

It remains to check that our family of free factor systems is still realized. Let $\mathcal{F}_j$ be such a free factor system. By assumption on H_m, there is G_ℓ such that $\mathcal{F}(G_\ell \cup H_m) = \mathcal{F}_j$. Each non-contractible component of $G_\ell \cup H_m$ is mapped into itself by some iterate of f. Since Y is eventually mapped into H_m, some iterate of f induces a bijection between the non-contractible components of $G_\ell \cup H_m \cup Y$ and those of $G_\ell \cup H_m$, so $p(G_\ell)$ realizes $\mathcal{F}_j$. Repeating this process decreases the number of non-exponentially growing strata, so eventually property (P) is established.

Here is the main consequence of (P) that we use.

Lemma 2.11. *Suppose that $f \colon \mathcal{G} \to \mathcal{G}$ satisfies (P), that H_m is an almost periodic forest that is not the bottom half of a dihedral pair, and that a vertex v has trivial vertex group and valence one in H_m. Then $v \in G_j$ for some $j < m$.*

Proof. Suppose $f \colon \mathcal{G} \to \mathcal{G}$ and H_m are as in the statement. The restriction of f to H_m acts transitively on the components of H_m, and either each component is a single edge, or f transitively permutes the valence-one vertices of H_m (necessarily with trivial vertex group in this contrary case, since H_m is a forest). Property (P) says that there exists a filtration element G_j such that $\mathcal{F}(G_j) \neq \mathcal{F}(G_\ell \cup H_m)$ for any filtration element G_ℓ. In particular, $\mathcal{F}(G_j) \neq \mathcal{F}(G_j \cup H_m)$, so we see that $j < m$. Since H_m is a forest, if it is disjoint from G_j, it contributes nothing to $\mathcal{F}(G_j \cup H_m)$ (recall that we define free factor systems to be made up of free factors with *positive complexity,* ruling out the fundamental group of each component of H_m). Therefore we conclude that some and hence every edge of H_m is incident to a vertex of G_j. In fact, some and hence every valence one vertex of H_m with trivial vertex group and valence one in H_m must be a vertex of G_j. □

Property (Z). To complete the proof of property (Z), we must show that if H_r is an exponentially growing stratum, then all non-wandering components of G_r are in fact non-contractible. If C is a non-wandering component, the lowest stratum H_i containing an edge of C is either exponentially growing or almost periodic. If H_i is exponentially growing, Lemma 2.7 shows that each vertex of H_i has valence at least two in C (and thus H_i) or has nontrivial vertex group, showing that C is noncontractible. If H_i is the bottom half of a dihedral pair, then $r > i + 1$ and C contains the full dihedral pair and is thus non-contractible. If H_i is almost periodic but not the bottom half of a dihedral pair, then

Lemma 2.11 shows that H_i is not a forest so C is non-contractible. Thus property (Z) follows from the form of (Z) we have already established.

Property (NEG). Let E be an edge in a non-exponentially growing stratum H_i which is not almost periodic. Let C be the component of G_{i-1} containing the terminal vertex v of E; it is non-wandering by our work proving (NEG*). By the argument in the previous paragraph, if H_ℓ is the lowest stratum containing an edge of C, then H_ℓ is either almost periodic or exponentially growing, and the argument in the previous paragraph shows that H_ℓ is either non-contractible or the bottom half of a dihedral pair. In fact, in the former case H_ℓ is its own core, while in the latter $H_\ell \cup H_{\ell+1}$ is its own core. In the exponentially growing case this follows since Lemma 2.7 shows that every valence-one vertex of H_ℓ has nontrivial vertex group. To see this in the almost periodic case, observe that if some edge of H_ℓ is incident to a vertex with trivial vertex group and valence one in H_ℓ, then every edge has this property and H_ℓ is a forest, in contradiction to (P) and Lemma 2.11. Write D for H_ℓ if H_ℓ is its own core, or for the dihedral pair $H_\ell \cup H_{\ell+1}$ otherwise.

Choose a periodic vertex w in D and a path γ from v to w. If D is a dihedral pair, let w be the center vertex of the dihedral pair. Slide E along γ. The result is a relative train track map which still realizes our sequence of free factor systems and still satisfies (Z). Working up through the filtration repeating this process establishes (NEG). This time sliding may have introduced valence-one vertices with trivial vertex group, but (NEG), (Z) and Lemma 2.7 imply that only valence-one vertices are mapped to the valence-one vertices created. We perform valence-one homotopies to remove each of these vertices. If property (P) is not satisfied, restore it using the process above. Since the number of non-exponentially growing strata decreases, this process terminates.

Property (F). We want to show that the core of each filtration element is a filtration element. If H_ℓ is a zero stratum, then $\mathcal{F}(G_\ell) = \mathcal{F}(G_{\ell-1})$, so assume that H_ℓ is irreducible, and thus G_ℓ has no contractible components unless H_ℓ is the bottom half of a dihedral pair. So assume H_ℓ is not the bottom half of a dihedral pair. If a vertex v with trivial vertex group has valence one in G_ℓ, then Lemma 2.7 and (Z) imply the incident edge E belongs to a non-exponentially growing stratum H_i. If H_i were almost periodic, it would be a forest, because every edge would be incident to a vertex of valence one in G_ℓ. This contradicts Lemma 2.11. This exhausts the possibilities: v must be the initial endpoint of a non-exponentially growing edge which is not almost periodic. All edges in such a stratum have initial vertex a valence-one vertex of G_ℓ, and no vertex of valence at least two in G_ℓ maps to them. Thus we may push all such non-exponentially growing strata H_i above $G_\ell \setminus H_i$. After repeating this process finitely many times, $\mathcal{F}(G_\ell)$ is realized by a filtration element that is its own core. Working upwards through the strata, we have arranged that each $\mathcal{F}(G_\ell)$ is realized by a filtration element that is its own core. To complete the proof, rearrange strata by pushing the dihedral pair down the filtration so that if H_ℓ is the bottom half of a dihedral pair, then $G_{\ell-1}$ is its own core. Then $G_{\ell+1}$ is also its own core and (F) is satisfied. □

3 Fixed points at infinity

The purpose of this section is to study the action of automorphisms of $(F, \mathcal{A})$ on the *Bowditch boundary* (to be defined presently) of some and hence any Bass–Serre tree associated to a marked graph of groups. This boundary was previously considered by Guirardel and Horbez in [GH19]. We define *attractors* and *repellers* for the action of an automorphism Φ on the boundary $\partial(F, \mathcal{A})$. The main result is a proposition from Martino's Thesis [Mar98], which is an analogue of [GJLL98, Proposition 1.1] for free products.

The boundary $\partial(F,\mathcal{A})$. We begin this section by collecting some basic information about our boundary of the free product F and the action of F and automorphisms of F on it.

Let $\mathbb{G}$ be the thistle with n prickles and k petals associated to our standing free product decomposition

$$F = A_1 * \cdots * A_n * F_k,$$

and let $\star$ be the vertex of $\mathbb{G}$ with trivial vertex group. Let T be the Bass–Serre tree of $\mathbb{G}$, equipped with a basepoint $\tilde\star$ and fundamental domain, thus defining an action of F on T.

Recall that the Gromov boundary $\partial_\infty T$ of the tree T may be identified with the set of singly infinite tight paths beginning at $\tilde x$. Given a point $\tilde x \in T$, a *half-tree* based at $\tilde x$ is a component of $T \setminus \{\tilde x\}$ together with those boundary points $\xi \in \partial_\infty T$ such that the intersection of the tight path ξ with the component contains infinitely many edges.

The *observer's topology* on $T \cup \partial_\infty T$ is generated by the set of half-trees, which form a sub-basis for the topology. For more on the observer's topology, see [CHL07, GH19, Kno19]. In the case where the groups A_i are countable, the observer's topology on $T \cup \partial_\infty T$ is second countable and compact. It is easy to see that the observer's topology is Hausdorff. A sequence of points $\{\tilde x_n\}$ in $T \cup \partial_\infty T$ converges to a point $\tilde x$ in the observer's topology if for every $\tilde y \neq \tilde x$ in T, the points $\tilde x_n$ eventually belong to the same half-tree based at $\tilde y$ as $\tilde x$.

The *Bowditch boundary* ∂T of T is, as a set, the union of the Gromov boundary $\partial_\infty T$ of T together with the set $V_\infty(T)$ of vertices of T with infinite stabilizer. If $\tilde x$ is a vertex with finite valence or a point in the interior of an edge of T, it is easy to find a basic open neighborhood of $\tilde x$ (a finite intersection of half-trees) containing only vertices of finite valence or points in the interior of an edge. Thus ∂T is closed in $T \cup \partial_\infty T$, so it is compact. It is not hard to argue that it is a Cantor set. The subspace $\partial_\infty T$ inherits the usual (visual) topology; if we fix the basepoint $\tilde\star$, a sequence $\{\tilde x_n\}$ in $T \cup \partial_\infty T$ converges to $\xi \in \partial_\infty T$ if and only if for every finite subpath $\tilde\gamma$ of the ray $\tilde R_{\tilde\star,\xi}$ determining ξ, the tight path or ray $\tilde R_{\tilde\star,\tilde x_n}$ contains $\tilde\gamma$ for all large n.

Let $(\mathcal{G}, m)$ be a marked graph of groups, let $p = m(\star)$, and let Γ be Bass–Serre tree for $\mathcal{G}$. We always assume a basepoint $\tilde p \in \Gamma$ has been chosen, thus defining an action of F on Γ. As in Section 1 or [Lym21, Proposition 1.2], there is a lift $\tilde m\colon (T,\tilde\star) \to (\Gamma,\tilde p)$ to the Bass–Serre trees. This lift is an F-equivariant quasi-isometry, and Guirardel and Horbez prove in [GH19, Lemma 2.2] that it has a unique continuous extension to a homeomorphism $\hat m\colon \partial T \to \partial\Gamma$. They prove that the homeomorphism $\hat m$ does not depend on the F-equivariant map $\tilde m$ and furthermore that the map $\hat m$ sends $\partial_\infty T$ to $\partial_\infty\Gamma$ and $V_\infty(T)$ to $V_\infty(\Gamma)$. This allows us to identify the Bowditch boundaries of any marked graph of groups with ∂T. We shall let ∂T play the role for free products that the Gromov boundary of the free group plays for free groups, denoting it $\partial(F,\mathcal{A})$. Let us denote the subspace $\partial_\infty T$ as $\partial_\infty(F,\mathcal{A})$ and the subspace $V_\infty(T)$ as $V_\infty(F,\mathcal{A})$.

There is also a convenient algebraic interpretation of $\partial(F,\mathcal{A})$ analogous to that in [Coo87] and [Mar99]. To wit, the orbit map $g \mapsto g.\tilde\star$ yields a well-defined compact topology on $F \cup \partial(F,\mathcal{A})$ that we now describe. A choice of fundamental domain for T containing $\tilde\star$ corresponds to a choice of orientation for each edge in $\mathbb{G}$ that forms a loop based at $\star$. Choose the A_i in their conjugacy classes so that elements of each A_i fix a vertex in our fundamental domain, and choose the free basis S for F_k whose elements correspond to loops in $\mathbb{G}$ that traverse a single loop edge once in the positive orientation. A *letter* is an element of the set $\bigcup_{i=1}^n (A_i \setminus \{1\}) \cup S \cup S^{-1}$. A *word* is a finite or infinite sequence of letters of the form

$$w = x_1x_2x_3\cdots.$$

A word is *reduced* if adjacent letters neither cancel nor coalesce into a single letter. Each element g of F can be written uniquely as a reduced word w. Corresponding to each finite reduced word w there is a tight path in T from $\tilde\star$ to another lift of $\star$; the path projects to the element of $F = \pi_1(\mathbb{G},\star)$ corresponding to w. (This correspondence is perfect because $\star$ has trivial vertex group. In general there would be more group elements than paths.) Points in $\partial_\infty(F,\mathcal{A})$ correspond to infinite reduced words, while points in $V_\infty(F,\mathcal{A})$ correspond to cosets gA, where $A \in \{A_1, \dots, A_n\}$ is infinite. There is a unique reduced word w representing a representative of the coset gA such that the last letter of w does not belong to A.

A *prefix* of a reduced word ξ is a finite word w such that there exists ξ' so that $\xi = w\xi'$ is reduced as written. A sequence of points in $F \cup \partial(F, \mathcal{A})$ converges to a point $\xi \in \partial_\infty(F, \mathcal{A})$ if and only if for each prefix of ξ the associated sequence of words (which may be finite or infinite) eventually shares that prefix.

A sequence of points in $F \cup \partial(F, \mathcal{A})$ converges to a point $wA \in V_\infty(F, \mathcal{A})$ if and only if each associated word eventually has a prefix of the form wg for g a letter of A, but for each such g, the prefix wg appears only finitely many times.

Each $c \in F$ acts on Γ as an automorphism T_c of the natural projection $\Gamma \to \mathcal{G}$ and induces a homeomorphism $\hat{T}_c \colon \partial(F, \mathcal{A}) \to \partial(F, \mathcal{A})$. If c is *peripheral* (i.e. conjugate into some A_i) but nontrivial, then $\hat{T}_c$ acts with a single fixed point on $\partial(F, \mathcal{A})$ if it is conjugate into an infinite A_i and without fixed point if it is conjugate into a finite A_i. If c is nonperipheral, then $\hat{T}_c$ has two fixed points in $\partial(F, \mathcal{A})$, a sink $\hat{T}_c^+$ and a source $\hat{T}_c^-$. The line (proper, linear embedding of $\mathbb{R}$) in Γ whose endpoints in $\partial_\infty(F, \mathcal{A})$ are $\hat{T}_c^+$ and $\hat{T}_c^-$ is the *axis* of T_c; denote it by A_c. The image of A_c in $\mathcal{G}$ is a *circuit* (a tight path in $\mathcal{G}$ that forms a loop and remains tight however it is cut into a path) that corresponds under the marking to the conjugacy class of c (or of a root of c).

Recall from Section 1 that given a map $f \colon \mathcal{G} \to \mathcal{G}$ representing an outer automorphism $\varphi \in \mathrm{Out}(F, \mathcal{A})$, choosing a path σ in $\mathcal{G}$ from the basepoint p to $f(p)$ defines both an automorphism $\Phi \colon (F, \mathcal{A}) \to (F, \mathcal{A})$ and a lift $\tilde{f}$ of f to the Bass–Serre tree Γ which is Φ-twisted equivariant. This defines a bijection between the set of lifts of $f \colon \mathcal{G} \to \mathcal{G}$ to Γ and the set of automorphisms $\Phi \colon (F, \mathcal{A}) \to (F, \mathcal{A})$ representing φ. Following Feighn–Handel [FH11], we say that $\tilde{f}$ *corresponds to* or *is determined by* Φ and vice versa.

Under the identification of $\partial\Gamma$ with $\partial(F, \mathcal{A})$, a lift $\tilde{f}$ also determines a homeomorphism $\hat{f}$ of $\partial(F, \mathcal{A})$. An automorphism $\Phi \colon (F, \mathcal{A}) \to (F, \mathcal{A})$ determines a bijection of the set of infinite words and cosets of infinite A_i; it is not hard to see that this bijection is open, and thus Φ determines a homeomorphism $\hat{\Phi} \colon \partial(F, \mathcal{A}) \to \partial(F, \mathcal{A})$.

Lemma 3.1. *Assume notation as in the previous paragraph. We have $\hat{f} = \hat{\Phi}$ if and only if $\tilde{f}$ corresponds to Φ.*

Proof. Let $m' \colon \mathcal{G} \to \mathbb{G}$ be a homotopy inverse for m, and let $\tilde{m}' \colon \Gamma \to T$ be the lift such that $\tilde{m}'\tilde{m}$ is equivariantly homotopic to the identity; thus $\hat{m}'\hat{m}$ is the identity of $\partial(F, \mathcal{A})$. The action of $\hat{f}$ on $\partial(F, \mathcal{A})$ is the extension of $\tilde{m}'\tilde{f}\tilde{m} \colon T \to T$ to the Bowditch boundary of T. Up to equivariant homotopy, we may assume that $\tilde{m}'\tilde{f}\tilde{m}$ fixes $\tilde{\star}$. If $\tilde{f}$ corresponds to Φ, then $\tilde{m}'\tilde{f}\tilde{m}(g.\tilde{\star}) = \Phi(g).\tilde{\star}$, so it follows that $\hat{f}(\xi) = \hat{\Phi}(\xi)$ for all $\xi \in \partial(F, \mathcal{A})$. The same argument shows that if instead $\tilde{f}$ corresponds to $\Phi' \neq \Phi$, then $\hat{f} \neq \hat{\Phi}$. $\square$

Lemma 3.2. *Assume that $\tilde{f} \colon \Gamma \to \Gamma$ corresponds to $\Phi \colon (F, \mathcal{A}) \to (F, \mathcal{A})$. The following are equivalent.*

1. *$c \in \mathrm{Fix}(\Phi)$.*
2. *T_c commutes with $\tilde{f}$.*
3. *$\hat{T}_c$ commutes with $\hat{f}$.*

The above also imply the following for all $c \in F$ and any automorphism $\Phi \colon (F, \mathcal{A}) \to (F, \mathcal{A})$. If c is not peripheral and $\mathrm{Fix}(\hat{f})$ is nonempty, then the following are also equivalent to the above.

4. *$\mathrm{Fix}(\hat{T}_c) \subset \mathrm{Fix}(\hat{f})$.*
5. *$\mathrm{Fix}(\hat{f})$ is $\hat{T}_c$-invariant.*

Proof. Assume item 1. For all $\tilde{x} \in \Gamma$, by Φ-twisted equivariance, we have

$$\tilde{f} T_c(\tilde{x}) = T_{\Phi(c)} \tilde{f}(\tilde{x}) = T_c \tilde{f}(\tilde{x}),$$

which proves item 2. Item 3 follows from item 2 by extending the equal maps $\tilde{f} T_c$ and $T_c \tilde{f}$ to $\partial(F, \mathcal{A})$. Assume item 3. We have that $\hat{f} \hat{T}_c$ and $\hat{T}_c \hat{f}$ are the extensions of $\tilde{f} T_c$ and $T_c \tilde{f}$ to $\partial(F, \mathcal{A})$. We have that Φ-twisted equivariance and the effectiveness of the action of F on $\partial(F, \mathcal{A})$ imply $\Phi(c) = c$, proving item 1.

Assume item 3, and let $\xi \in \mathrm{Fix}(\hat{f})$. We have

$$\hat{f}(\hat{T}_c(\xi)) = \hat{T}_c \hat{f}(\xi) = \hat{T}_c(\xi),$$

which proves item 5. Item 4 is trivially true if c is peripheral and conjugate into a finite A_i. If c is conjugate into an infinite A_i, let ξ be the fixed point of $\hat{T}_c$. We have

$$\hat{f}(\xi) = \hat{f}(\hat{T}_c(\xi)) = \hat{T}_c(\hat{f}(\xi)),$$

so by the uniqueness of the fixed point ξ, we conclude $f(\xi) = \xi$.

Assume item 3 and suppose c is nonperipheral. We have

$$\hat{T}_c \hat{f}(\hat{T}_c^+) = \hat{f} \hat{T}_c(\hat{T}_c^+) = \hat{f}(\hat{T}_c^+)$$

and similarly for $\hat{T}_c^-$. Moreover, the above equation implies that $\hat{f}(\hat{T}_c^+)$ is a sink for $\hat{T}_c$, so we conclude that $\mathrm{Fix}(\hat{T}_c) \subset \mathrm{Fix}(\hat{f})$. Assuming item 5, note that any closed nonempty $\hat{T}_c$-invariant set contains both of the fixed points of $\hat{T}_c$. If we assume $\mathrm{Fix}(\hat{f})$ is nonempty, item 5 implies item 4. So assume item 4. Ghen we have

$$\hat{T}_{\Phi(c)}(\hat{T}_c^+) = \hat{T}_{\Phi(c)} \hat{f}(\hat{T}_c^+) = \hat{f}(\hat{T}_c(\hat{T}_c^+)) = \hat{f}(\hat{T}_c^+) = \hat{T}_c^+$$

and similarly $\hat{T}_{\Phi(c)}(\hat{T}_c^-) = \hat{T}_c^-$, so $\Phi(c)$ and c share an axis and thus a common root. That is, $c = a^k$ and $\Phi(c) = a^\ell$ for positive k and ℓ and some nonperipheral, root-free $a \in F$. Then $\Phi(c) = \Phi(a)^k = a^\ell$. Since nonperipheral elements have unique roots, this implies that $\Phi(a) = a^j$ for some positive j. But then $\Phi^{-1}(a)^j = a$, so since a is root-free, we conclude $\Phi(a) = a$ and thus $k = \ell$ and $c \in \mathrm{Fix}(\Phi)$, so item 4 implies item 1. □

Given a subgroup H of F, the action of H on T induces a decomposition of H as a free product. We say that H has *finite (Kurosh subgroup) rank* if this free product decomposition is of the form $H = B_1 * \cdots * B_p * F_\ell$, or equivalently if the action of H on its minimal subtree T_H in T is cocompact. In this situation we define the *(Kurosh subgroup) rank* of H to be $p + \ell$.

Lemma 3.3. *The inclusion $T_H \to T$ defines a closed embedding of ∂T_H into ∂T that is well-defined independent of T, so we denote ∂T_H as $\partial(H, \mathcal{A}|_H)$.*

Proof. The H-minimal subtree T_H of T is convex in T, so there is a well-defined inclusion of Gromov boundaries $\partial_\infty T_H \to \partial_\infty T$. If a vertex of T_H has infinite stabilizer in H, it has infinite stabilizer in F, so there is also a well-defined inclusion $V_\infty(T_H) \to V_\infty(T)$. The argument in the proof of [GH19, Lemma 2.1] and [GH19, Lemma 2.2] implies that the injective set map $\partial T_H \to \partial T$ is continuous. Since ∂T_H is compact, the map is an embedding. To show it is closed, it therefore suffices to show that ∂T_H is closed in ∂T.

Suppose that $\{\xi_n\}$ is a sequence in ∂T converging to some point $\xi \notin T_H$. There is a unique closest point $\tilde{x}$ of T_H closest to ξ in the sense that $\tilde{x}$ is the only point of T_H on the ray $\tilde{R}_{\tilde{x},\xi}$. Assume first that $\tilde{x} \neq \xi$. Then because ξ_n is eventually in the same half-tree based at $\tilde{x}$ as ξ, we see that $\xi_n \notin \partial T_H$ for n large. If $\tilde{x} = \xi$, then $\tilde{x}$ has finite valence in T_H but infinite valence in T. Let $\tilde{e}_1, \ldots, \tilde{e}_m$ be the edges T_H with initial vertex $\tilde{x}$ and let $\tilde{v}_1, \ldots, \tilde{v}_m$ be the corresponding terminal vertices. Again we see that because ξ_n must be in the same half-tree based at $\tilde{v}_i$ as ξ for n large, we see that $\xi_n \notin \partial T_H$ for n large.

Bounded cancellation implies that if $(\mathcal{G}, m)$ is a marked graph of groups, then $\tilde{m}(T_H)$ is in a bounded neighborhood of Γ_H, the H-minimal subtree of Γ. Thus the inclusion of Gromov boundaries is well-defined independent of T; it is clear that the same is true for the inclusion of infinite-stabilizer points, so the embedding is independent of T. $\square$

A point $\xi \in \partial_\infty(F, \mathcal{A})$ is an *attractor* for $\hat{\Phi}$ if the set U of all points ζ such that the sequence $\{\Phi^n(\zeta)\}$ converges to ξ contains an open neighborhood of ξ. If ζ is an attractor for $\hat{\Phi}^{-1}$, then we say that ζ is a *repeller* for $\hat{\Phi}$.

Proposition 3.4 (cf. Lemma 2.3 of [FH11], Proposition 1.1 of [GJLL98], Proposition 5.1.14 of [Mar98]). *Assume that $\tilde{f}\colon \Gamma \to \Gamma$ corresponds to $\Phi\colon (F, \mathcal{A}) \to (F, \mathcal{A})$ and that $\mathrm{Fix}(\hat{\Phi})$ contains at least three points in $\partial_\infty(F, \mathcal{A})$. Denote $\mathrm{Fix}(\Phi) = \mathbb{F}$ and $\mathbb{T} = \{T_c : c \in \mathbb{F}\}$. Then*

1. *If $\partial(\mathbb{F}, \mathcal{A}|_{\mathbb{F}}) \cap \partial_\infty(F, \mathcal{A})$ is nonempty, it is naturally identified with the closure of*
$$\{\hat{T}_c^{\pm} : T_c \in \mathbb{T}\}$$
in $\partial(F, \mathcal{A})$. None of these points is isolated in $\mathrm{Fix}(\hat{\Phi})$.
2. *Every point in $(\mathrm{Fix}(\hat{\Phi}) \setminus \partial(\mathbb{F}, \mathcal{A}|_{\mathbb{F}})) \cap \partial_\infty(F, \mathcal{A})$ is isolated and is either an attractor or a repeller for $\hat{\Phi}$.*

We note that in fact item 2 in the proposition holds without the assumption on the number of fixed points of $\hat{\Phi}$.

The proof of this proposition is admittedly more complicated than the corresponding statement for automorphisms of free groups. The reason is that for automorphisms of free groups, if a boundary point ξ is an attractor for an automorphism Φ, then it is *superlinearly* attracting, in the sense that if ξ_i is the ith prefix of ξ and $k(i)$ is the part of the length of $\Phi(\xi_i)$ which is common with ξ, then $k(i) - i$ goes to infinity with i. Superlinear growth dominates bounded cancellation, which allows one to more easily argue that one has an attractor. For automorphisms of free products, there is no such guarantee of superlinear growth, and in fact there exist attractors for which $k(i) - i$ remains bounded. Arguing that one has these kinds of attractors or repellers when the given fixed point is not in the boundary of the fixed subgroup is more delicate.

Here is an example, due to Martino: let A be a countably infinite group and $\Theta\colon A \to A$ an automorphism of infinite order. Assume further that Θ does not act periodically on any nontrivial element of A. Let $h\colon A \to A'$ be an isomorphism and $F = A * A' * F_1$; let t be a generator for F_1. Consider the automorphism $\Phi\colon F \to F$ defined as $\Phi(a) = h(\Theta(a))$ for $a \in A$, $\Phi(a') = h^{-1}(a')$ for $a' \in A'$, and finally as $\Phi(t) = tg$ for some fixed nontrivial element $g \in A$. Then $\mathrm{Fix}(\Phi)$ is trivial and
$$\xi = tgh(\Theta(g))\Theta(g)h(\Theta^2(g))\Theta^2(g)\ldots$$
is a linear attractor, in the sense that $k(i) = i + 1$.

Proof. If $\partial(\mathbb{F}, \mathcal{A}|_{\mathbb{F}}) \cap \partial_\infty(F, \mathcal{A})$ contains at least three points then no point of $\partial(\mathbb{F}, \mathcal{A}|_{\mathbb{F}})$ is isolated in itself, so it is not isolated in $\mathrm{Fix}(\hat{\Phi})$. If instead $\partial(\mathbb{F}, \mathcal{A}|_{\mathbb{F}}) \cap \partial_\infty(F, \mathcal{A})$ consists of two points (i.e. $\mathbb{F}$ is F_1 or $C_2 * C_2$), then $\partial(\mathbb{F}, \mathcal{A}|_{\mathbb{F}}) = \{T_c^{\pm}\}$ for some $c \in \mathrm{Fix}(\Phi)$. There is a point $\xi \in \mathrm{Fix}(\hat{\Phi}) = \mathrm{Fix}(\hat{f})$ contained in $\partial_\infty(F, \mathcal{A})$ different from $\hat{T}_c^{\pm}$. Since $\hat{T}_c$ acts with source–sink dynamics on $\partial(F, \mathcal{A})$ and $\mathrm{Fix}(\hat{f})$ is $\hat{T}_c$-invariant by Lemma 3.2, we see that $\lim_{n\to\infty} \hat{T}_c^n(\xi) = \hat{T}_c^+$ and similarly $\lim_{n\to\infty} \hat{T}_c^{-n}(\xi) = \hat{T}_c^-$, so neither of these points are isolated in $\mathrm{Fix}(\hat{\Phi})$.

Now consider any fixed point $\xi \in \partial_\infty(F, \mathcal{A})$. We follow the outline of [GJLL98, Proposition 1.1] and [Mar98, Proposition 5.1.14]. Think of ξ as an infinite word
$$\xi = x_1x_2x_3\cdots$$
and write $\xi_i = x_1 \cdots x_i$. Consider the words $w_i = \xi_i^{-1}\Phi(\xi_i)$.

A few observations are in order. First, since $\Phi(\xi) = \xi$, we have $\lim_{i\to\infty} \Phi(\xi_i) = \xi$, so we can write each $\Phi(\xi_i)$ as $\xi_{k(i)}z_i$, where $k(i) \to \infty$ with i. By bounded cancellation, which for words says that if uv is reduced as written, then at most $2B$ letters of $\Phi(u)\Phi(v)$ cancel to form the reduced word for $\Phi(uv)$, the length of the word z_i is bounded by B independently of i.

Superlinear attractors and repellers. Suppose that the length of the words w_i grows without bound. Then the equation $\Phi(\xi_i) = \xi_{k(i)}z_i$ and our bound on the length of z_i implies that $|k(i) - i|$ goes to infinity and in fact either $k(i) - i$ goes to infinity or $i - k(i)$ goes to infinity.

Suppose first that $k(i) - i$ goes to infinity. Then there exists j such that if $i \leq j$, then $k(i) > i + B$. Consider the basic open neighborhood U_i of ξ given by

$$U_i = \{\zeta : \zeta = \xi_i\zeta' \text{ is reduced as written}\}.$$

(We do include boundary points of the form wA in U_i.) Since $k(j) > j + B$, bounded cancellation implies that $\Phi(U_j) \subset U_{k(j)-B} \subset U_j$. Since $k(k(j)) > k(j) + B$, we have some sequence k_n such that $k_n \to \infty$ with n such that

$$\bigcap_{n=1}^{\infty} \Phi^n(U_j) \subset \lim_{n\to\infty} U_{k_n} = \{\xi\}.$$

This shows that ξ is an attractor for the action Φ.

Suppose instead that $k(i) - i$ goes to $-\infty$. Since ξ is fixed by Φ, it is fixed by Φ^{-1} and as above we may write $\Phi^{-1}(\xi_i) = \xi_{\bar{k}(i)}\bar{z}_i$. The equation $\xi_i = \Phi^{-1}(\xi_{k(i)})\Phi^{-1}(z_i)$ implies that $\bar{k}(k(i)) - i$ is bounded independent of i. We have

$$\bar{k}(k(i)) - k(i) = (\bar{k}(k(i)) - i) - (k(i) - i)$$

and the latter goes to infinity. Since $k(i)$ goes to infinity and $k(i+1)-k(i)$ and $\bar{k}(i+1)-\bar{k}(i)$ are both bounded, this implies $\bar{k}(i) - i$ goes to infinity. The argument in the previous paragraph applies to show that in this case ξ is a repeller for the action of Φ.

In the boundary of the fixed subgroup. So suppose that the length of w_i is uniformly bounded, and observe that if $w_i = w_p$, then $\xi_p\xi_i^{-1}$ is fixed by Φ. If we write $v_i = \xi_i^{-1}\Phi^{-1}(\xi_i)$, note that the same argument applies: if $v_i = v_p$, then $\xi_p\xi_i^{-1}$ is fixed by Φ. If infinitely many of the words w_i or infinitely many of the words v_i are drawn from a set of only finitely many letters, then v_i or w_i takes on some value infinitely often. If this happens, note that for fixed i, we have

$$\lim_{p\to\infty} \xi_p\xi_i^{-1} = \xi,$$

so $\xi \in \partial(\mathbb{F}, \mathcal{A}|_{\mathbb{F}})$.

We claim that this latter event, infinitely many words drawn from finitely many letters, actually occurs if thre exists j such that for all $i \geq j$, we have $k(i) < i$ and $\bar{k}(i) < i$. Since $k(i)$ goes to infinity, we may increase j so that if $i \geq j$, then additionally $\bar{k}(k(i)) < k(i)$.

Claim 3.5 (cf. Lemma 5.1.13 of [Mar98])**.** *There is a finite set of letters L with the property that if $k(i-1) < k(i) < i$ and $\bar{k}(k(i)) < k(i)$, then either every letter of $v_{k(i)}$ belongs to L or every letter of w_i belongs to L.*

Proof of Claim 3.5. Since $\Phi\colon (F, \mathcal{A}) \to (F, \mathcal{A})$ permutes the conjugacy classes in $\mathcal{A}$, we have that Φ sends each A_i to a conjugate of some A_j, say $g_iA_jg_i^{-1}$. Let L' be the union of the set of letters occuring in the normal form for each g_i and occuring in each $\Phi(s_i^{\pm1})$ for s_i in our fixed free basis S for F_k together with their inverses. (It follows that each $s_i^{\pm1}$ occurs in L'.)

We have

$$v_{k(i)} = \xi_{k(i)}^{-1}\Phi^{-1}(\xi_{k(i)}) = \xi_{k(i)}^{-1}\xi_{\bar{k}(k(i))}\bar{z}_{k(i)} = x_{k(i)}^{-1}\cdots x_{\bar{k}(k(i))+1}^{-1}\bar{z}_{k(i)}$$

since $\bar{k}(k(i)) < k(i)$. Since $\xi_{k(i)}v_{k(i)} = \Phi^{-1}(\xi_{k(i)})$, we must have that the first letter of $v_{k(i)}$ belongs to the same free factor as $x_{k(i)}$, therefore $x_i^{-1}\cdots x_{k(i)+i}^{-1}v_{k(i)}$ is reduced as written. Note that

$$\xi_i = \Phi^{-1}(\Phi(\xi_i)) = \Phi^{-1}(\xi_{k(i)})\Phi^{-1}(z_i)$$

so

$$\Phi^{-1}(z_i^{-1}) = \xi_i^{-1}\Phi^{-1}(\xi_{k(i)}) = x_i^{-1}\cdots x_{k(i)+1}^{-1}v_{k(i)}.$$

Notice that $\Phi(x_i) = z_{i-1}^{-1}x_{k(i-1)+1}\cdots x_{k(i)}z_i$. Since we assume that $k(i-1) < k(i)$, and since by assumption the first letter of z_{i-1}^{-1} cannot be $x_{k(i-1)+1}$, we see that z_i is a terminal subword of $\Phi(x_i)$. Since x_i is a single letter, at most one letter of z_i and thus z_i^{-1} does not belong to L'.

Let L'' be the union of the set of letters constructed for Φ^{-1} the way L' was constructed for Φ. Let L''' be the union of L' and L''. Thus every letter of $\Phi^{-1}(z_i^{-1})$ is a product of at most three letters of L''' except at most one (consider what happens when a product of elements of L''' is reduced). Let $L^{(4)}$ be the set of nonidentity products of at most three elements of L'''. Therefore either every letter of $v_{k(i)}$ belongs to $L^{(4)}$ or each of $x_{k(i)+1},\dots,x_i$ belongs to $L^{(4)}$. Now

$$w_i = \xi_i^{-1}\Phi(\xi_i) = \xi_i^{-1}\xi_{k(i)}z_i = x_i^{-1}\cdots x_{k(i)+1}^{-1}z_i$$

and we see that each letter of w_i is a product of at most two elements of $L^{(4)}$, so enlarging once more to a set L proves the claim (notice that the construction of L is independent of i). $\square$

Given Claim 3.5, notice that since $k(i)$ goes to infinity with i, we have that $k(i-1) < k(i)$ occurs infinitely often, so either infinitely many $v_{k(i)}$ or infinitely many w_i are written with a finite set of letters L. In either case, we see that $\xi \in \partial(\mathbb{F}, \mathcal{A}|_{\mathbb{F}})$.

Linear attractors So we shall suppose to the contrary: the word lengths of w_i and v_i remain uniformly bounded, but given any finite set of letters A, there exists j_0 so large that if $i > j_0$, then w_i and v_i each contain a letter not in A. Claim 3.5 shows that we have that either $k(i) \geq i$ or $\bar{k}(i) \geq i$ occurs infinitely often.

We will show that if $k(i) \geq i$ occurs infinitely often, then ξ is an attractor for Φ. If instead $\bar{k}(i) \geq i$ occurs infinitely often, then the same argument will show that ξ is a repeller for Φ. (Of course only one can occur.) We do this by considering a subsequence i_r where $k(i_r)$ is strictly increasing and showing that if $k(i_r) \geq i_r$ for some r, then $k(i_{r+1}) \geq i_{r+1}$. We show that $i_{r+1} - i_r$ is bounded by some constant C, and in fact that $\Phi^{2C+2}(U_{i_r}) \subset U_{i_{r+1}}$. It follows that the open neighborhood U^r of ξ given by

$$U^r = U_{i_r} \cup \Phi(U_{i_r}) \cup \cdots \cup \Phi^{2C+1}(U_{i_r})$$

satisfies $\Phi(U^r) \subset U^r$. A boundary point ζ is in $\bigcap_{n=1}^{\infty}\Phi^n(U^r)$ if for all n, there exists a nonnegative $c_n \leq 2C+1$ such that $\zeta \in \Phi^{n+c_n}(U_{i_r})$. Write $n + c_n = 2mC + c_n'$ for some nonnegative $c_n' \leq 2C+1$. Then $\zeta \in \Phi^{c_n'}(U_{i_{r+m}})$. Thus

$$\bigcap_{n=1}^{\infty}\Phi^n(U^r) = \lim_{m\to\infty} U^{r+m} = \{\xi\}$$

and we have shown that ξ is an attractor for Φ.

Here is the definition of the subsequence i_r. Define i_1 to be the first i such that $k(i) > 0$ and choose i_r inductively such that i_r is the least integer satisfying $k(i_r) > k(i_{r-1})$.

We claim that $i_{r+1} - i_r$ is bounded by a constant C chosen as follows. Let C be such that if the length of a reduced word w is at least C, then the length of $\Phi(w)$ is greater than $3B$, where B is the bounded cancellation constant. To see that C exists, note that the length of $\Phi^{-1}(w)$ is bounded by some constant multiple of the length of w, namely the maximum of the length of the image of a single letter under Φ^{-1}.

Suppose $i_{r+1} - i_r > C$. Then $\Phi(\xi_{i_{r+1}-1}) = \Phi(\xi_{i_r})\Phi(x_{i_r+1} \cdots x_{i_{r+1}-1})$ has length greater than $k(i_r) - 2B + 3B = k(i_r) + B$ by bounded cancellation. But by assumption $k(i_{r+1} - 1) < k(i_r)$, and hence the length of $z_{i_{r+1}-1}$ is greater than B, a contradiction.

Therefore to complete the proof, we show that if $k(i_r) \geq i_r$ for sufficiently large r, then $k(i_{r+1}) \geq i_{r+1}$ and $\Phi^{2C+2}(U_{i_r}) \subset U_{i_{r+1}}$. To do this, we need to choose a finite set of letters to avoid. We now turn to defining that finite set of letters.

We begin by recalling a result of Collins and Turner restated (allowing infinite words) by Martino.

Lemma 3.6 (Proposition 2.4 of [CT88], Lemma 5.1.3 of [Mar98])**.** *There exists a constant ℓ with the following property. Let L' be the finite set of letters from the beginning of Claim 3.5. Given one of our free factors A_i, we may write $\Phi(A_i) = g_i A_j g_i^{-1}$. Given a possibly infinite word Y, suppose $\Phi(Y) = g_i x Y'$ is reduced as written, where $x \in A_j$. Then either*

(a) Y begins with a letter from A_i, or

(b) x is the product of at most ℓ letters from L'.

Before continuing with the construction of our finite set, we need a little notation. If x is a letter in some A_i, we may write $\Phi(x)$ uniquely as $\Phi(x) = \mu_x x^\Phi \mu_x^{-1}$, where μ_x is a reduced word, x^Φ is a letter, and this product is reduced as written. The map $x \mapsto x^\Phi$ is a bijection of the set of letters that belong to $\bigcup_{i=1}^n (A_i \setminus \{1\})$. Given a finite set of letters S and a positive integer m, we write S^m for the finite set of letters that may be written as the product of at most m letters in S. Thus $S \subset S^m$ and if S is closed under taking inverses, so is S^m.

We say that a letter x has *Martino's property P* if it satisfies the following three conditions:

1. $x \notin L'$. (Recall that this implies that x is in some A_i.)
2. x^Φ is not the product of at most 3ℓ letters in L', i.e. $x^\Phi \notin (L')^{3\ell}$.
3. If y and $y' \in (L')^\ell$ are in the same factor A_j as x^Φ, then $(yx^\Phi y')^\Phi \notin (L')^{3\ell}$.

It is clear that there exists a finite set of letters M such that if $x \notin M$, then x has Martino's property P. Since $x \mapsto x^\Phi$ is a bijection, there is also a finite set of letters M' such that if $x^\Phi \notin M'$, then x has Martino's property P. We may suppose this set M is closed under taking inverses. As it happens, we need the following stronger notion.

Say that a letter z is a *descendent* of a letter x if there exist y and y' in $(L')^\ell \cup \{1\}$ such that $z = yx^\Phi y'$. More generally, say that z is a *k-descendent* of x for some positive integer k if there are letters $x = z_0, z_1, \ldots, z_k = z$ such that z_i is a descendent of z_{i-1} for i satisfying $1 \leq i \leq k$.

Claim 3.7 (Corollary 5.1.10 of [Mar98])**.** *There is a finite set of letters N such that if $x^\Phi \notin N$, then x and all k-descendents of x have Martino's property P for k satisfying $1 \leq k \leq C$.*

Proof of Claim 3.7. The proof is a corrected version of the discussion preceding [Mar98, Corollary 5.1.10]. Suppose first that some descendent of x fails to have Martino's property P. Then there exist y and y' in $(L')^\ell \cup \{1\}$ such that $yx^\Phi y' \in M$. Since $x \mapsto x^\Phi$ is a bijection and y and y' are drawn from finitely many possibilities, it is clear that there is a finite set N_1 which we may assume to be closed under taking inverses such that if $x \notin N_1$, then all descendents of x have Martino's property P.

Now suppose that there is a finite set N_i of letters such that if $x \notin N_i$, then all i-descendents of x have Martino's property P. If some $(i+1)$-descendent of x fails to have property P, then x has a descendent $yx^{\Phi}y' \in N_i$. The argument above shows that x belongs to a finite set N_{i+1}. Therefore the finite set $N = M' \cup \bigcup_{k=1}^{C} N_i$ satisfies the conclusion of the claim. We assume that N is closed under taking inverses. $\square$

By assumption, there is j_0 so large that if $i \geq j_0$, then w_i contains a letter not in N^{C+1}. The final step is the following claim.

Claim 3.8. *Suppose $i \geq j_0$, that $k(i) \geq i$, and that $k(i) > k(i-1)$. Then there exist integers $i = m_0 < m_1 < \cdots < m_{C+1}$ such that*

$$\Phi(U_{m_k}) \subset U_{m_{k+1}-1}$$

and

$$\Phi^2(U_{m_k}) \subset U_{m_{k+1}}$$

for k satisfying $0 \leq k \leq C$.

Proof of Claim 3.8. Consider $w_i = \xi_i^{-1}\Phi(\xi_i) = x_{i+1} \cdots x_{k(i)} z_i$. This is reduced as written, but there may be no x terms if $i = k(i)$. Since $i \geq j_0$, some letter of w_i does not belong to N^{C+1}. We claim that in fact there is some $j \leq i$ such that the following conditions are satisfied.

1. $x_j^{\Phi} \notin N$.

2. We may write $\Phi(\xi_{j-1})\mu_{x_j} = \xi_i z$ for some possibly trivial word z such that this product is reduced as written if z is nontrivial.

3. We may write $\Phi(\xi_{j-1})\mu_{x_j} x_j^{\Phi} = \xi_i z'$ for some possibly trivial word z' such that this product is reduced as written if z' is nontrivial.

Recall that since $k(i) > k(i-1)$, we have

$$\Phi(x_i) = z_{i-1}^{-1} x_{k(i-1)+1} \cdots x_{k(i)} z_i.$$

This product is not necessarily reduced as written, but $x_{k(i-1)+1}$ is not entirely canceled. First note that if the letter of w_i not belonging to N^{C+1} is in z_i, then in fact that letter is $x_i^{\Phi} \notin N$ and items 2 and 3 follow.

So suppose the letter is x_k for $i+1 \leq k \leq k(i)$. We follow and correct [Mar98, Lemma 5.1.7]. Let s be the least integer such that $k(i_s) \geq k$. Note that $i_s \leq i$. We have $k(i_s - 1) \leq k(i_{s-1}) < k \leq k(i_s)$. If

$$\Phi(x_{i_s}) = z_{i_s-1}^{-1} x_{k(i_s-1)+1} \cdots x_{k(i_s)} z_{i_s}$$

is reduced as written or if $k > k(i_s - 1) + 1$, then x_k occurs in the image of x_{i_s}, and in fact we have $x_k = x_{i_s}^{\Phi}$ and items 2 and 3 follow with $j = i_s$. If neither is the case, then we may write $z_{i_s-1} = xz'$ reduced as written where x is in the same factor as $x_k = x_{k(i_s-1)+1}$ but not equal to it. We have that $x_{i_s}^{\Phi} = x^{-1}x_k$. Now, either

(a) $x^{-1}x_k \notin N^C$, or

(b) $x^{-1}x_k \in N^C$, in which case $x^{-1} \notin N^C$, since $x_k \notin N^{C+1}$.

If item (a) holds, then items 1 through 3 hold with $j = i_s$. To see this, note that

$$\Phi(\xi_{i_s-1})\mu_{x_{i_s}} = \xi_{k(i_s-1)} z_{i_s-1} z'^{-1} = \xi_{k-1} x$$

and

$$\Phi(\xi_{i_s-1})\mu_{x_{i_s}} x_{i_s}^{\Phi} = \xi_{k(i_s-1)} z_{i_s-1} z'^{-1} x^{-1} x_k = \xi_k.$$

(Recall that $k \geq i+1$.)

Suppose item (b) holds. Notice that if $k-1 = k(i_s - 1) > k(i_s - 2)$, then

$$\Phi(x_{i_s-1}) = z_{i_s-2}^{-1} x_{k(i_s-2)+1} \cdots x_{k-1} x z',$$

$x_{k(i_s-2)+1}$ is not entirely canceled, and we see that x occurs in the image of x_{i_s-1}, so we conclude that $x = x_{i_s-1}^{\Phi}$ and items 1 through 3 hold with $j = i_s - 1$.

So suppose $k(i_s - 1) \leq k(i_s - 2)$. Because $k(i_{s-1}) > k(i_{s-1} - 1)$, we conclude that $i_s - 1 > i_{s-1}$. Notice that

$$k-1 = k(i_s - 1) \leq k(i_s - 2) \leq k(i_{s-1}) \leq k-1,$$

so we conclude $k(i_s - 1) = k(i_s - 2) = k(i_{s-1}) = k - 1$.

The argument proceeds as above. If x occurs in the image of x_{i_s-1}, we are done. If not, then $z_{i_s-2} = x'z''$, where this product is reduced as written and x and x' are in the same factor. If $x = x'$, then $x' \notin N^{C-1}$, and we may proceed considering x_{i_s-2}. If $x \neq x'$, then $x'^{-1}x$ occurs in the image of x_{i_s-1} since $\Phi(x_{i_s-1}) = z''^{-1}x'^{-1}xz'$. If $x'^{-1}x \notin N^{C-1}$, then we conclude with $j = i_s - 1$. If not, then $x'^{-1} \notin N^{C-1}$.

We may repeat this argument, reducing our candidate for j by one and our index q of N^q. We always reach a positive conclusion if $k(j) > k(j-1)$, but we know that $k(i_{s-1}) > k(i_{s-1} - 1)$, so we do reach a positive conclusion in at most C steps.

We now follow [Mar98, Lemma 5.1.8]. Notice that since $x_j^{\Phi} \notin N$, the letter x_j has Martino's property P. Suppose we have $\zeta \in U_j$, so $\zeta = \xi_j \zeta'$ is reduced as written. Then

$$\Phi(\zeta) = \Phi(\xi_{j-1}) \mu_{x_j} x_j^{\Phi} \mu_{x_j}^{-1} \Phi(\zeta')$$

The Collins–Turner result Lemma 3.6 quoted above implies that since $x_j^{\Phi} \notin (L')^{3\ell}$ by Martino's property P, the letter x_j^{Φ} is not entirely canceled in the above product, so there exist y and $y' \in (L')^{\ell}$ (possibly trivial) so that

$$\Phi(\xi_j \zeta') = w y x_j^{\Phi} y' \zeta''$$

for some words w and ζ'' such that this product is reduced as written if we count $yx_j^{\Phi}y'$ as a single letter. In fact, since x_j^{Φ} does not entirely cancel we may write $w = \xi_i w'$ for some possibly trivial word w' such that the product is reduced as written if w' is nontrivial. In fact, since $j \leq i$, we may write $w = \xi_j w''$ for some possibly trivial word w'' such that the product is reduced as written if w'' is nontrivial. To sum up, we have

$$\Phi(\xi_{j-1}) \mu_{x_j} x_j^{\Phi} = \xi_j w'' y x_j^{\Phi},$$

where this product is reduced as written. Now if we apply Φ again, we have

$$\begin{aligned}
\Phi^2(\xi_j) = \Phi(\xi_j w'' y x_j^{\Phi} \mu_{x_j}^{-1}) &= \Phi(\xi_{j-1}) \mu_{x_j} x_j^{\Phi} \mu_{x_j}^{-1} \Phi(w'') \Phi(y x_j^{\Phi}) \Phi(\mu_{x_j}^{-1}) \\
&= \xi_j w'' (y x_j^{\Phi}) \mu_{x_j}^{-1} \Phi(w'') \Phi(y x_j^{\Phi}) \Phi(\mu_{x_j}^{-1})
\end{aligned}$$

Since w'' does not begin with a letter in the same factor as x_j, the Collins–Turner result Lemma 3.6 implies we may find $y'' \in (L')^{\ell} \cup \{1\}$ such that

$$\xi_j w'' (y x_j^{\Phi}) \mu_{x_j}^{-1} \Phi(w'') = \xi_j w'' (y x_j^{\Phi} y'') v$$

for some possibly trivial word v such that this product is reduced as written if we count $(yx_j^{\Phi}y'')$ as a single letter and if we discard v if it is trivial. Thus we have

$$\Phi^2(\xi_j) = \xi_j w'' (y x_j^{\Phi} y'') v \Phi(y x_j^{\Phi}) \Phi(\mu_{x_j}^{-1})$$

Notice that if $v = \mu^{-1}_{yx_j^\Phi}$ and the letters $(yx_j^\Phi y'')$ and $(yx_j^\Phi)^\Phi$ are in the same factor, then Lemma 3.6 implies $yx_j^\Phi y'' \in (L')^\ell$, which contradicts the second item in Martino's property P. Therefore applying Lemma 3.6 once more to $\Phi(\mu^{-1}_{x_j})$, we may write

$$\Phi^2(\xi_j) = \xi_j w''(yx_j^\Phi y''')v'[z(yx_j^\Phi)^\Phi z']v''$$

where either $v' \neq 1$ or x_j^Φ and $(x_j^\Phi)^\Phi$ are in different factors, where $z' \in (L')^\ell \cup \{1\}$, and where v'' is a possibly trivial word.

Since $\Phi(\xi_i\zeta') = \xi_j w''(yx_j^\Phi y')\zeta''$ and $\mu_{yx_j^\Phi y'} = \mu_{yx_j^\Phi}$, we have

$$\Phi^2(\xi_j\zeta') = \xi_j w''(yx_j^\Phi y''')v'[z(yx_j^\Phi y')^\Phi z'']\zeta'''$$

where $z'' \in (L')^\ell \cup \{1\}$ and thus this product is reduced upon counting $(yx_j^\Phi y''')$ and $[z(yx_j^\Phi y')^\Phi z'']$ as single letters. By Martino's property P, both of these letters are non-trivial.

We claim that $(yx_j^\Phi y''') = x_{m_1}$ for some $m_1 > i$. Since $\xi_j w''(yx_j^\Phi y''')$ is always a subword of $\Phi(\xi_j\zeta')$, it is a subword of $\Phi^2(\xi) = \xi$, so we do have that $yx_j^\Phi y''' = x_{m_1}$ for some m_1. Since we saw that $\xi_j w'' = \xi_i w'$, we have $m_1 > i$.

Note that setting $i = m_0$, this proves that

$$\Phi(U_{m_0}) \subset U_{m_1 - 1}$$

and

$$\Phi^2(U_{m_0}) \subset U_{m_1}.$$

We would like to repeat this argument with m_1 (and later m_s) playing the role of both j and i to find m_s for $2 \leq s \leq C+1$. To do this, we need to know that x_{m_s} has property P provided that $x_{m_{s-1}}$ did and that we may write

1. $\Phi(\xi_{m_s - 1})\mu_{x_{m_s}} = \xi_{m_s} z$ and
2. $\Phi(\xi_{m_s - 1})\mu_{x_{m_s}} x^\Phi_{m_s} = \xi_{m_s} z'$

for possibly trivial words z and z' such that the respective products are reduced as written. First, observe that x_{m_s} is an s-descendent of x_j, so it has Martino's property P if $s \leq C$. The other property follows inductively once we note that

$$\Phi(\xi_{m_1 - 1})\mu_{x_{m_1}} = \Phi(\xi_j w'')\mu_{x_{m_1}} = \xi_j w'' x_{m_1} v' z$$

where either $v' \neq 1$ or x_{m_1} and z are in different factors and is otherwise reduced as written and

$$\Phi(\xi_{m_1 - 1})\mu_{x_{m_1}} x^\Phi_{m_1} = \xi_j w'' x_{m_1} v'(z x^\Phi_{m_1}). \qquad \square$$

Note that if $i_r \geq j_0$ and $k(i_r) \geq i_r$, then i_r satisfies the hypotheses of Claim 3.8. Since $i_{r+1} - i_r \leq C$, there is s with $m_s \leq i_{r+1} < m_{s+1}$, so $k(i_{r+1}) \geq m_{s+1} - 1 \geq i_{r+1}$, and we see that $\Phi^{2C+2}(U_{i_r}) \subset U_{m_{C+1}} \subset U_{i_{r+1}}$. $\square$

The following result is implied by the main result of [Mar99].

Lemma 3.9. *There are only finitely many $\mathbb{T}$-orbits of points in $(\mathrm{Fix}(\hat{\Phi}) \cap \partial_\infty(F, \mathcal{A})) \setminus \partial(\mathbb{F}, \mathcal{A}|_{\mathbb{F}})$.*

We will give an estimate for the number of $\mathbb{T}$-orbits of points in $\mathrm{Fix}(\hat{\Phi}) \setminus \partial(\mathbb{F}, \mathcal{A}|_{\mathbb{F}})$ in Section 7.

4 Attracting Laminations

We now turn to attracting laminations for free products, prove their existence and develop some of their properties. We end up with a little more of the theory than is actually needed for this paper, (For example, the existence of a homomorphism $\mathrm{PF}_{\Lambda^+}\colon \mathrm{Stab}(\Lambda^+) \to \mathbb{Z}$ from the stabilizer of an attracting lamination Λ^+ to $\mathbb{Z}$) but we hope that it will prove useful for future work.

The space $\tilde{\mathcal{B}}$. In [GH19], Guirardel and Horbez consider *algebraic laminations* for free products. To wit, let $\tilde{\mathcal{B}}$ be the space

$$\tilde{\mathcal{B}} = (\partial(F,\mathcal{A}) \times \partial(F,\mathcal{A}) \setminus \Delta)/\mathbb{Z}/2\mathbb{Z}$$

where Δ is the diagonal and $\mathbb{Z}/2\mathbb{Z}$ acts by permuting the factors. The diagonal action of F on $\partial(F,\mathcal{A}) \times \partial(F,\mathcal{A})$ descends to an action on $\tilde{\mathcal{B}}$. An element of $\tilde{\mathcal{B}}$ is an *algebraic line,* an unordered pair (α,ω) of distinct elements of $\partial(F,\mathcal{A})$. Let $\mathcal{B}$ be the quotient of $\tilde{\mathcal{B}}$ by the action of F. An *algebraic lamination* is a closed, F-invariant subset of $\tilde{\mathcal{B}}$ or equivalently a closed set in $\mathcal{B}$, and the lines it comprises are called *leaves.*

Lines in Γ and in $\mathcal{G}$. Let $\mathcal{G}$ be a marked graph of groups with Bass–Serre tree Γ. Between any two boundary points ξ and ζ, there is a unique tight path in Γ called the *line* $\tilde{L}_{\xi,\zeta}$ from ξ to ζ. Thus to a point (α,ω) in $\tilde{\mathcal{B}}$ we may associate the unoriented image of the line $\tilde{L}_{\alpha,\omega}$, thus yielding a *space of lines in* Γ*,* which we will denote $\tilde{\mathcal{B}}(\Gamma)$. A line may be finite, singly infinite, or bi-infinite, according to whether its endpoints lie in $V_\infty(F,\mathcal{A})$ or $\partial_\infty(F,\mathcal{A})$. In [BFH00], a *space of lines in* Γ*,* denoted $\tilde{\mathcal{B}}(\Gamma)$ is considered. It is equipped with a "compact–open" topology: if $\tilde{\gamma} \subset \Gamma$ is a finite tight edge path, (even with endpoints at vertices, say), define $N(\tilde{\gamma}) \subset \tilde{\mathcal{B}}(\Gamma)$ to be the set of lines that contain $\tilde{\gamma}$ as a subpath. In the free group setting, the sets $N(\tilde{\gamma})$ form a basis for the topology on $\tilde{\mathcal{B}}(\Gamma)$. For our purposes, the sets $N(\tilde{\gamma})$ will only form a basis for the *bi-infinite* lines.

To make the map $\tilde{\mathcal{B}} \to \tilde{\mathcal{B}}(\Gamma)$ a homeomorphism, a basic open neighborhood of a line $\tilde{\lambda}$ in Γ must be given by a pair of disjoint basic open neighborhoods of the endpoints α and ω of $\tilde{\lambda}$. In other words, a sequence of lines $\{\lambda_n\}$ in Γ converges to λ if for each $\tilde{y} \in \Gamma$ distinct from the endpoints (α,ω) of λ, there is a choice of order for the endpoints (α_n,ω_n) of λ_n such that α_n belongs to the same half-tree based at $\tilde{y}$ as α and ω_n belongs to the same half-tree based at $\tilde{y}$ as ω for n large.

Each line in Γ may be projected to a *line in* $\mathcal{G}$*.* See [Lym21, Section 1] for more details on projecting from Γ to $\mathcal{G}$. The *space of lines in* $\mathcal{G}$ is denoted $\mathcal{B}(\mathcal{G})$. Projection defines a natural projection map from $\tilde{\mathcal{B}}(\Gamma)$ to $\mathcal{B}(\mathcal{G})$, and we give $\mathcal{B}(\mathcal{G})$ the quotient topology. Given a tight edge path γ in $\mathcal{G}$, define $N(\gamma)$ to be the set of those lines in $\mathcal{G}$ that contain γ as a subpath. The sets $N(\gamma)$ form a basis for any *bi-infinite* line in $\mathcal{B}(\mathcal{G})$.

If $f\colon \mathcal{G} \to \mathcal{G}'$ is a homotopy equivalence, the homeomorphism $\hat{f}\colon \partial\Gamma \to \partial\Gamma'$ yields a homeomorphism $\tilde{f}_\sharp\colon \tilde{\mathcal{B}}(\Gamma) \to \tilde{\mathcal{B}}(\Gamma')$ and a homeomorphism $f_\sharp\colon \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{G}')$. If $\beta \in \mathcal{B}$ corresponds to λ in $\mathcal{B}(\mathcal{G})$, we say that λ *realizes* β in $\mathcal{G}$.

Laminations. Recall that a *lamination* is a closed set of lines in $\mathcal{G}$ or a closed, F-invariant set of lines in Γ. The lines it comprises are the *leaves* of the lamination. We are interested in *attracting laminations,* for which we need a little more terminology.

Recall that an element of F is *peripheral* if it is conjugate into a vertex group of some (and hence any) marked graph of groups. The conjugacy class of a nonperipheral element of F determines a periodic bi-infinite line in $\mathcal{B}(\mathcal{G})$ that runs over the tight circuit determined by the conjugacy class. A line β in $\mathcal{B}$ is *carried* by the conjugacy class of a free factor $[[F^i]]$ if it is in the closure of the periodic lines in $\mathcal{B}$ determined by the conjugacy classes of nonperipheral elements of F^i. If $\mathcal{G}$ is a marked graph of groups and $K \subset \mathcal{G}$ is a connected subgraph such that $[[\pi_1(\mathcal{G}|_K)]] = [[F^i]]$, then β is carried by $[[F^i]]$ if and only if the realization of β in $\mathcal{G}$ is contained in K.

Given $\varphi \in \mathrm{Out}(F, \mathcal{A})$, we say that $\beta' \in \mathcal{B}$ is *weakly attracted to* $\beta \in \mathcal{B}$ *under the action of* φ if $\varphi^k_\sharp(\beta') \to \beta$ (note that $\mathcal{B}$ is not Hausdorff). A subset $U \subset \mathcal{B}$ is an *attracting neighborhood of* $\beta \in \mathcal{B}$ *for the action of* φ if $\varphi_\sharp(U) \subset U$ and if $\{\varphi^k_\sharp(U) : k \geq 0\}$ is a neighborhood basis for $\beta \in \mathcal{B}$. Finally, a line σ in $\mathcal{G}$ is *birecurrent* if it is bi-infinite and every subpath of σ occurs infinitely often as a subpath of each end of σ. Our first lemma on attracting laminations says that birecurrence is a property of the abstract line.

Lemma 4.1 ([BFH00] Lemma 3.1.4)**.** *If some realization of $\beta \in \mathcal{B}$ in a marked graph of groups is birecurrent, then every realization of β in a marked graph of groups is birecurrent. If β is birecurrent, then $\varphi_\sharp(\beta)$ is birecurrent for every $\varphi \in \mathrm{Out}(F, \mathcal{A})$.*

Proof. The proof is identical to [BFH00, Lemma 3.1.4]. Suppose that σ and σ' are realizations of β in marked graphs of groups $\mathcal{G}$ and $\mathcal{G}'$ respectively. Since the subspaces $V_\infty(F, \mathcal{A})$ and $\partial_\infty(F, \mathcal{A})$ are well-defined, σ is bi-infinite if and only if σ' is. Suppose that σ is birecurrent, and let $h\colon \mathcal{G} \to \mathcal{G}$ be a homotopy equivalence that respects the markings. Let C be the bounded cancellation constant for h. Choose lifts $\tilde\sigma \subset \Gamma$, $\tilde\sigma' \subset \Gamma'$ and $\tilde h\colon \Gamma \to \Gamma'$ such that $\tilde h_\sharp(\tilde\sigma) = \tilde\sigma'$. Let $\tilde\sigma'_0$ be a finite subpath of $\tilde\sigma'$. Extend it to a finite subpath $\tilde\tau'$ of $\tilde\sigma'$ by adding C initial and terminal edges. Choose a finite subpath $\tilde\tau$ of $\tilde\sigma$ such that $\tilde h_\sharp(\tilde\tau)$ contains $\tilde\tau'$. Since σ is birecurrent, each end of $\tilde\sigma$ contains infinitely many copies $\tilde\tau_i$ of $\tilde\tau$. Define $\tilde\mu'_i$ by removing C initial and terminal edges of $\tilde h_\sharp(\tilde\tau_i)$. Bounded cancellation implies that $\tilde\mu'_i$ is a subpath of $\tilde\sigma'$. By construction, each $\tilde\mu'_i$ contains a copy of $\tilde\sigma'_0$, so we have shown that σ' is birecurrent.

As in the original, replacing h with a topological representative $f\colon \mathcal{G} \to \mathcal{G}$ of $\varphi \in \mathrm{Out}(F, \mathcal{A})$ in the argument above shows that $\varphi_\sharp(\beta)$ is birecurrent for all $\varphi \in \mathrm{Out}(F, \mathcal{A})$ provided that β is birecurrent. ☐

A closed subset $\Lambda^+ \subset \mathcal{B}$ is an *attracting lamination for* φ if it is the closure of a single line β such that:

1. The line β is birecurrent.
2. The line β has an attracting neighborhood for the action of some iterate of φ.
3. The line β is not carried by any φ-periodic F_1 or $C_2 * C_2$ free factor.

The line β is said to be *generic* for Λ^+. The set of attracting laminations for φ is denoted $\mathcal{L}(\varphi)$.

Lemma 4.2 ([BFH00] Lemma 3.1.6)**.** *$\mathcal{L}(\varphi)$ is φ-invariant.*

Proof. Again the proof is essentially identical to [BFH00, Lemma 3.1.6]. Suppose that β is generic for $\Lambda^+ \in \mathcal{L}(\varphi)$. Lemma 4.1 shows that $\varphi_\sharp(\beta)$ is birecurrent. If V is an attracting neighborhood for β under the action of φ^s for some $s \geq 1$, then $U = \varphi_\sharp(V)$ is an attracting neighborhood for $\varphi_\sharp(\beta)$ under the action of φ^s. If $[[B]]$ is a φ-periodic free factor of the form F_1 or $C_2 * C_2$ that carries $\varphi_\sharp(\beta)$, and if $\Phi^{-1}\colon (F, \mathcal{A}) \to (F, \mathcal{A})$ represents φ^{-1}, then $[[\Phi^{-1}(B)]]$ is a φ-periodic free factor of the same form that carries β. Therefore we have shown that $\varphi_\sharp(\beta)$ is generic with respect to some lamination $\varphi_\sharp(\Lambda^+) \in \mathcal{L}(\varphi)$. ☐

A nonnegative integral matrix M is *aperiodic* if there is some positive power k such that M^k has all positive entries. Aperiodic matrices are irreducible. Any irreducible matrix with a nonzero diagonal entry is aperiodic. Suppose $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map with filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ and that H_r is an exponentially growing stratum. We say that H_r is *aperiodic* if its transition matrix is aperiodic, and that $f\colon \mathcal{G} \to \mathcal{G}$ is *eg-aperiodic* if each exponentially growing stratum is aperiodic.

Lemma 4.3 ([BFH00] Lemma 3.1.9)**.** *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map and that H_r is an aperiodic exponentially growing stratum. There is an attracting lamination Λ^+ with generic leaf β such that H_r is the highest stratum crossed by the realization λ of β in $\mathcal{G}$.*

For $f\colon \mathcal{G} \to \mathcal{G}$ and H_r and $k \geq 0$, define a *k-tile* to be a path of the form $f_\sharp^k(E)$, for E an edge of H_r, in either orientation. A path in $\mathcal{G}$ is a *tile* if it is a k-tile for some k. A *k-tiling* of a path in G_r is a decomposition of the path into subpaths that are either k-tiles, vertex group elements, or contained in G_{r-1}. A bi-infinite path λ has an *exhaustion by tiles* if each of its subpaths occurs as a subpath of a tile in λ. If λ has an exhaustion by tiles, then (EG-iii) implies that λ is r-legal.

Proof. We follow [BFH00, Lemma 3.1.9]. Choose an edge E of H_r and $m > 0$ such that $f_\sharp^m(E) = \alpha E \beta$ for some nontrivial tight paths α and β in G_r. Write $h = f^m$ and choose lifts $\tilde{E}$, $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{h}\colon \Gamma \to \Gamma$ so that $\tilde{h}(\tilde{E}) = \tilde{\alpha}\tilde{E}\tilde{\beta}$. Define $\tilde{\tau}_j = \tilde{h}_\sharp^j(\tilde{E})$; it is the lift of a jm-tile. We have $\tilde{\tau}_0 = \tilde{E}$, $\tilde{\tau}_1 = \tilde{\alpha}\tilde{\tau}_0\tilde{\beta}$ and more generally $\tilde{\tau}_{j+1} = \tilde{\alpha}_j\tilde{\tau}_j\tilde{\beta}_j$ for nontrivial paths $\tilde{\alpha}_j$ and $\tilde{\beta}_j$. The $\tilde{\tau}_j$ are an increasing sequence of lifts of tiles whose union is a bi-infinite path $\tilde{\lambda}$ that is fixed by $\tilde{h}_\sharp$.

We claim that the projection $\lambda \subset \mathcal{G}$ realizes an element $\beta \in \mathcal{B}$ that is generic with respect to some element of $\mathcal{L}(\varphi)$. After replacing m by a multiple if necessary, we may assume that the $h_\sharp$-image of any edge in H_r contains at least two edges in H_r. Define $\tilde{\lambda}_k$ to be the subpath of $\tilde{\lambda}$ that begins with the kth lift of an edge of H_r to the left of $\tilde{E}$ and ends with the kth lift of an edge of H_r to the right of $\tilde{E}$. Project to $\lambda_k \subset G_r$ and define $V_k = N(\lambda_k)$. Lemma 2.1 implies that $\tilde{h}_\sharp(\tilde{\lambda}_k) \supset \tilde{\lambda}_{2k}$. Bounded cancellation in the form of Lemma 3.2 part 3 implies $h_\sharp(V_k) \subset V_{k+1}$ for all sufficiently large k. The V_k are a neighborhood basis for λ, and so for sufficiently large k, V_k is an attracting neighborhood of λ for the action of φ^m.

Since the difference between the number of edges in $\tilde{h}_\sharp(\tilde{\lambda}_k)$ and the number of edges in $\tilde{\lambda}_k$ grows without bound, the $\tilde{\lambda}_k$ cannot be subpaths of a single $\tilde{h}_\sharp$-invariant axis. In other words, λ is not a circuit, and so cannot be carried by any F_1 or $C_2 * C_2$ free factor.

Note that by construction λ has an exhaustion by tiles. We will show that λ has a k-tiling for all $k \geq 1$. Notice that each 1-tile has a 0-tiling, and inductively each $(k+1)$-tile has a k-tiling and thus an ℓ-tiling for $\ell \leq k+1$. A k-tiling of $\tilde{\tau}_i$ determines and is determined by a finite set of vertices $\tilde{V}_i$ in $\tilde{\tau}_i \subset \tilde{\lambda}$. Given a vertex $\tilde{v} \in \tilde{\lambda}$, we may, after passing to a subsequence, assume that either $\tilde{v} \in \tilde{V}_i$ for all large i or $\tilde{v} \notin \tilde{V}_i$ for all large i. We then may consider (with this subsequence) another vertex of $\tilde{\lambda}$. The set of vertices that satisfy $\tilde{v} \in \tilde{V}_i$ for all large i determine a k-tiling of $\tilde{\lambda}$ and thus of λ.

We have that (EG-i) implies that the first and last edges of any tile are contained in H_r. Thus each end of λ must contain infinitely many edges in H_r. Bounded cancellation and the existence of k-tilings for all k imply that each tile occurs infinitely often in each end of λ. Since every finite subpath of λ is contained in a tile, λ is birecurrent. This shows that λ is generic for some element of $\mathcal{L}(\varphi)$. ☐

Lemma 4.4 ([BFH00] Lemma 3.1.10)**.** *Assume that $\beta \in \mathcal{B}$ is a generic leaf of some $\Lambda^+ \in \mathcal{L}(\varphi)$, that $f\colon \mathcal{G} \to \mathcal{G}$ and $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ are a relative train track map and filtration representing φ and that λ is the realization of β in $\mathcal{G}$.*

1. *The highest stratum H_r crossed by λ is exponentially growing.*

2. *The bi-infinite path λ is r-legal.*

Define tiles *for H_r as before the proof of Lemma 4.3.*

3. *The bi-infinite path λ has a k-tiling for all $k \geq 1$.*

4. *The bi-infinite path λ has an exhaustion by tiles.*

Proof. We argue by induction on the structure of F. The base cases of $F = A_1$, F_1 and $C_2 * C_2$ are trivial, so we assume the result holds for outer automorphisms of proper free factors of F. The rest of the proof follows as in [BFH00, Lemma 3.1.10].

Suppose first that λ is contained in G_{m-1}. Let s be the smallest positive integer so that the component C of G_{m-1} containing the image of λ is f^s-invariant. The inductive hypothesis applied to the restriction of φ to $\pi_1(C)$ completes the proof of item 1.

Now suppose that λ contains edges of H_m. Since λ is birecurrent, it crosses some edge of H_m infinitely many times. Now choose $s \geq 1$ so that λ has an attracting neighborhood for the action of φ^s, and let $\alpha \neq \lambda$ be a periodic line that is weakly attracted to λ under the action of $\varphi^s_\sharp$. (Let us remark that it is here that we use the third point in the definition of an attracting lamination to ensure such a periodic line exists.) Since $f^{s\ell}_\sharp(\alpha) \to \lambda$ as $\lambda \to \infty$, $f^\ell_\sharp$ cannot act with period other than 1 on α. Since $\lambda \neq \alpha$, the map $f^s_\sharp$ cannot fix α. Since α is periodic, it corresponds to a circuit in $\mathcal{G}$.

Suppose the number of edges in the circuit $f^{s\ell}_\sharp(\alpha)$ does not grow without bound but that $f^{s\ell}_\sharp(\alpha)$ is not periodic. Since there are only finitely many underlying paths in G of a given length, it follows that there exists $M \geq 1$ such that $f^{sM\ell}_\sharp(\alpha)$ has the same underlying path as α but necessarily the vertex group elements differ. Consider a lift $\tilde\alpha$ of α to Γ, and a lift $\tilde h = \tilde f^{sM}$ such that $\tilde f^{sM}_\sharp(\tilde\alpha)$ and $\tilde\alpha$ share a subpath $\tilde\sigma$ with endpoints at points $\tilde v$ and $\tilde w$ of infinite valence in Γ with the property that the corresponding vertex group elements in the circuit α (which may be the same vertex group element) are not periodic. We claim that the sequence $\tilde h^\ell_\sharp(\tilde\alpha)$ converges to $\tilde\sigma$ in $\tilde{\mathcal{B}}(\Gamma)$. Indeed, consider a point $\tilde y$ distinct from $\tilde v$ and $\tilde w$ in Γ. We will show that the initial endpoint of $\tilde h^\ell_\sharp(\tilde\sigma)$ belongs to the same half-tree at $\tilde y$ as $\tilde v$ for large ℓ. The same argument shows that the terminal endpoint of $\tilde h^\ell_\sharp(\tilde\sigma)$ belongs to the same half-tree at $\tilde y$ as $\tilde w$ for large ℓ. Consider the direction at $\tilde v$ determined by the path from $\tilde v$ to $\tilde y$. A sufficient condition for the initial endpoint of $\tilde h^\ell_\sharp(\tilde\sigma)$ to be contained in the same half-tree at $\tilde y$ as $\tilde v$ is for this direction to be distinct from the direction at $\tilde v$ determined by the ray from $\tilde v$ to the initial endpoint of $h^\ell_\sharp(\tilde\sigma)$. Since by assumption this latter direction is not periodic, this is indeed the case for large ℓ. This shows that $\tilde h^\ell_\sharp(\tilde\alpha)$ converges to $\tilde\sigma$. This is a contradiction, as there are neighborhoods of λ that do not contain σ (indeed, take some subpath of λ longer than σ).

Therefore the number of edges of $f^{s\ell}_\sharp(\alpha)$ grows without bound. Since $f^{s\ell}_\sharp(\alpha)$ converges to λ, which has infinitely many edges of H_m, the number of H_m-edges of $f^{s\ell}_\sharp(\alpha)$ grows without bound. This implies that H_m is exponentially growing.

Now suppose that H_r is the highest exponentially growing stratum crossed by λ. To prove that λ is r-legal, let j be the number of illegal turns of the circuit α in H_r. Since $f_\sharp$ does not create new illegal turns in H_r, the number of illegal turns of $f^{s\ell}_\sharp(\alpha)$ in H_r is bounded by j. Take a finite subpath λ_0 of λ. By the definition of weak convergence, we have that λ_0 is a subpath of $f^{s\ell}_\sharp(\alpha)$ for all ℓ sufficiently large. Since the length of $f^{s\ell}_\sharp(\alpha)$ increases without bound, for ℓ sufficiently large, the subpath λ_0 is covered by two fundamental domains of the circuit $f^{s\ell}_\sharp(\alpha)$, so the number of illegal turns of λ_0 in H_r is at most $2j$. But since λ is birecurrent and we may choose λ_0 arbitrarily, this uniform bound actually implies that λ is r-legal.

Now consider item 3. Fix $k \geq 1$ and let $\tilde\lambda \subset \Gamma$ be a lift of λ. Recall from the proof of Lemma 4.3 that a k-tiling of λ corresponds to a subdivision of $\tilde\lambda$ and thus the vertices of $\tilde\lambda$ that are the endpoints of the subdivision pieces. Let q be the number of edges in α. Given a finite subpath $\lambda_0 \subset \lambda$, let $\lambda_1 \subset \lambda$ be a finite subpath that contains $2q+1$ copies of λ_0. (The path λ_1 exists by birecurrence.) As in the argument for item 2, if ℓ is sufficiently large, then λ_1 occurs as a subpath of the periodic line determined by $f^{s\ell}_\sharp(\alpha)$ that is covered by two fundamental domains. In particular, at least one copy of λ_0 occurs as a subpath of $f^{s\ell}_\sharp(E)$ for some edge E of G_r. We conclude that λ is an increasing union of finite subpaths that have k-tilings. The k-tilings of these subpaths correspond to finite sets $\tilde V_i$ of vertices of $\tilde\lambda$. After passing to subsequences as in the proof of Lemma 4.3, we obtain a k-tiling of λ.

To complete the proof of item 4, we need only show that each finite subpath $\lambda_0 \subset \lambda$ is actually a subpath of a tile. Choose a finite subpath $\lambda_0 \subset \lambda$. Birecurrence implies that there is a finite subpath $\lambda_1 \subset \lambda$ that contains two disjoint copies of λ_0. Assume further that λ_0 contains an edge of H_r. By item 3, λ has a k-tiling, where k is chosen so large that each k-tile is longer than λ_1. In any k-tiling of λ there are at most two k-tiles that intersect λ_1; one of these must contain a copy of λ_0, completing the proof of item 4. $\square$

Corollary 4.5 ([BFH00] Corollary 3.1.11)**.** *Assume that $f\colon \mathcal{G} \to \mathcal{G}$ and $\varnothing \subset G_0 \subset G_1 \subset \cdots \subset G_m = G$ are a relative train track map and filtration representing φ, that H_r is an aperiodic exponentially growing stratum and that* tiles *(see the paragraph before the proof of Lemma 4.3) are defined with respect to H_r. Assume further that $\beta \in \mathcal{B}$ is Λ^+-generic for $\Lambda^+ \in \mathcal{L}(\varphi)$ and that H_r is the highest stratum crossed by the realization of $\beta \in \mathcal{G}$. Then $\{N(\tau) : \tau \text{ is a tile}\}$ is a neighborhood basis in $\mathcal{B}$ for β. It follows that all such β have the same closure.*

Proof. The proof is identical to [BFH00, Corollary 3.1.11] Let $\lambda \subset \mathcal{G}$ be the realization of β. By Lemma 4.4 item 3, λ has a k-tiling for all k. Suppose k_0 is such that M^{k_0} is positive. Then the $f_\sharp^{k_0}$-image of an edge of H_r contains every edge of H_r, and thus each $(k_0 + \ell)$-tile contains every ℓ-tile. More generally if $k - \ell \geq k_0$, each k-tile contains every ℓ-tile. It follows that λ contains every tile. Conversely, item 4 of Lemma 4.4 implies that every subpath of λ is contained in a tile in λ. $\square$

The previous three results imply that for any relative train track map representing φ and any aperiodic exponentially growing stratum H_r, there is a *unique* attracting lamination $\Lambda^+ \in \mathcal{L}(\varphi)$ with the property that H_r is the highest stratum crossed by the realization $\lambda \subset \mathcal{G}$ of a Λ^+-generic leaf. We will say that H_r is *the stratum determined by* Λ^+ and that Λ^+ is *the attracting lamination associated to* H_r.

Lemma 4.6 ([BFH00] Lemma 3.1.13)**.** *The set $\mathcal{L}(\varphi)$ is finite.*

Proof. The proof is identical to [BFH00, Lemma 3.1.13]. Choose a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ and filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ representing φ. If $f\colon \mathcal{G} \to \mathcal{G}$ is eg-aperiodic, then the paragraph before the lemma implies a one-to-one correspondence between the exponentially growing strata of f and the attracting laminations in $\mathcal{L}(\varphi)$, so the lemma holds.

Suppose on the other hand that some H_r is not aperiodic. Then by general theory (see [Sen81]) there is a partition of the edges of H_r into $s > 1$ sets $P_1, \ldots, P_s$ such that for each edge E in P_i, the edge path $f_\sharp(E)$ only crosses edges in G_{r-1} and P_{i+1}, where indices are taken mod s. The sth power of the transition matrix M_r^s is not irreducible, so the filtration for f must be enlarged to obtain a filtration for f^s. Replacing f by f^s has the effect of replacing H_r by s exponentially growing strata. If we choose s maximal, the transition matrix of each of these s exponentially growing strata is aperiodic. Therefore some iterate of φ is represented by an eg-aperiodic relative train track map. Since $\mathcal{L}(\varphi^p) = \mathcal{L}(\varphi)$ for all $p \geq 1$, we are reduced to the previous case. $\square$

Lemma 4.7 ([BFH00] Lemma 3.1.14)**.** *The following are equivalent.*

1. *Each element of $\mathcal{L}(\varphi)$ is φ-invariant.*
2. *Each element of $\mathcal{L}(\varphi)$ has an attracting neighborhood for $\varphi_\sharp$.*
3. *Every relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ representing φ is eg-aperiodic.*
4. *Some relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ representing φ is eg-aperiodic.*

Proof. The proof is identical to [BFH00, Lemma 3.1.14]. It is clear that item 3 implies item 4.

Suppose that $f: \mathcal{G} \to \mathcal{G}$ is an eg-aperiodic relative train track map for φ, that $\Lambda^+ \in \mathcal{L}(\varphi)$ is an attracting lamination and that H_r is the exponentially growing stratum associated to Λ^+. If λ in G_r is Λ^+-generic, then $f_\sharp(\lambda)$ is $\varphi(\Lambda^+)$-generic. Since H_r is the highest stratum crossed by $f_\sharp(\lambda)$, Corollary 4.5 implies that $\varphi(\Lambda^+) = \Lambda^+$, so item 4 implies item 1.

Suppose that $f: \mathcal{G} \to \mathcal{G}$ is a relative train track map representing φ and that H_r is an exponentially growing stratum which is not aperiodic. As in the proof of the previous lemma, there is a partition of the edges of H_r into $s > 1$ sets $P_1, \ldots, P_s$ such that for each edge E of P_i, the edge path $f_\sharp(E)$ crosses only edges in G_{r-1} and P_{i+1} with indices taken mod s. When we replace f by f^s, the exponentially growing stratum H_r divides into s exponentially growing strata, one for each P_i. By Lemma 4.3, these contribute elements to $\mathcal{L}(\varphi)$ that do not have attracting neighborhoods for the action of φ. Therefore item 2 implies item 3.

Finally suppose that Λ^+ is φ-invariant, that β is a Λ^+-generic leaf and that V is an attracting neighborhood for β with respect to the action of φ^s. Each $\varphi^i_\sharp(\beta) \in \varphi^i_\sharp(V)$ is generic with respect to Λ^+. Corollary 4.5 implies that $\beta \in \varphi^i_\sharp(V)$. Therefore

$$U = V \cap \varphi_\sharp(V) \cap \cdots \cap \varphi_\sharp^{s-1}(V)$$

is a neighborhood of β that satisfies $\varphi_\sharp(U) \subset U$ and $\varphi^s_\sharp(U) \subset V$. Therefore U is an attracting neighborhood for β and we see that item 1 implies item 2. ☐

Lemma 4.8 ([BFH00] Lemma 3.1.15)**.** *Assume that H_r is an aperiodic exponentially growing stratum for a relative train track map $f: \mathcal{G} \to \mathcal{G}$, that $\Lambda^+ \in \mathcal{L}(\varphi)$ is associated to H_r and that σ is a leaf of Λ^+ that is not entirely contained in G_{r-1}. Then the closure of δ is all of Λ^+. If σ is birecurrent, then it is Λ^+-generic.*

Proof. The proof is identical to [BFH00, Lemma 3.1.15]. Fix $k \geq 1$. By Lemma 4.4 item 3, each Λ^+-generic line has a k-tiling. Since σ is a weak limit of Λ^+-generic leaves, σ is an increasing union of finite subpaths that have k-tilings. (This proves that σ is not a finite path between vertices with infinite vertex group) The argument in the proof of item 3 of Lemma 4.4 shows that σ has a k-tiling. If $\sigma \not\subset G_{r-1}$, then σ must contain at least one k-tile. Since k is arbitrary, Corollary 4.5 and the fact that each k-tile contains every ℓ-tile for $k - \ell$ sufficiently large (see again the proof of Corollary 4.5) implies that the closure of σ contains each Λ^+-generic line and thus contains Λ^+. It follows that σ is not carried by any F_1 or $C_2 * C_2$ free factor. The fact that σ has an attracting neighborhood for the action of some iterate of φ follows from the fact that every neighborhood of a generic leaf is also a neighborhood of σ. If σ is birecurrent, then all the items in the definition of an attracting lamination are satisfied and we conclude that σ is Λ^+-generic. ☐

Lemma 4.9 (Lemma 3.1.16 of [BFH00])**.** *A generic leaf of $\Lambda^+ \in \mathcal{L}(\varphi)$ is never a circuit.*

Proof. The proof is identical to [BFH00, Lemma 3.1.16]. A set in $\mathcal{B}$ consisting of a single circuit is closed. Therefore if a Λ^+-generic leaf β is a circuit, then $\Lambda^+ = \{\beta\}$. Choose a relative train track map $f: \mathcal{G} \to \mathcal{G}$ representing φ. Since $\mathcal{L}(\varphi)$ is finite and φ-invariant, the realization of λ in $\mathcal{G}$ for β is invariant for the action of some iterate of $f_\sharp$. But λ is contained in G_r for some exponentially growing stratum H_r, is r-legal, and crosses edges in H_r. Therefore the length of λ (as a circuit) must both be bounded and grow without bound, a contradiction. ☐

Given a point $\xi \in \partial_\infty(F, \mathcal{A})$, Guirardel and Horbez [GH19, Definition 4.9] define the *limit set* of ξ to be the lamination $\Lambda(\xi)$ defined as follows. (Here we think of $\Lambda(\xi)$ as an F-invariant closed subset of $\tilde{\mathcal{B}}$.) Given a line $\tilde\beta$ with endpoints α and ω, we have that $\tilde\beta \in \Lambda(\xi)$ if (up to swapping α and ω), there exists a sequence $\{g_n\}$ of elements of F converging to α such that $g_n.\xi$ converges to ω. They remark [GH19, Remark 4.10] that if we fix a marked graph of groups $\mathcal{G}$ with Bass–Serre tree Γ and a point $\tilde{x} \in \Gamma$, then if $\tilde\lambda$ realizes $\tilde\beta$ in Γ we have that $\tilde{\beta} \in \Lambda(\xi)$ if and only if $\tilde{\lambda}$ is a limit of translates of the ray $\tilde{R}_{\tilde{x},\xi}$. Bounded cancellation implies that $\varphi_\sharp(\Lambda(\xi)) = \lambda(\hat{\Phi}(\xi))$. If ξ belongs to $\mathrm{Fix}(\hat{\Phi})$, then $\Lambda(\xi)$ is $\varphi_\sharp$-invariant.

We would like to relate the construction of attracting laminations to the idea of attracting *fixed points* at infinity for the action of some lift $\tilde{f}\colon \Gamma \to \Gamma$. This is accomplished in the following lemma.

Lemma 4.10 (cf. Lemma 2.13 of [FH11])**.** *Suppose that H_r is an exponentially growing stratum of a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$, that $\tilde{f}\colon \Gamma \to \Gamma$ is a lift and that $\tilde{v} \in \mathrm{Fix}(\tilde{f})$.*

1. *If E is an oriented edge in H_r and $\tilde{E}$ is a lift that determines a fixed direction at $\tilde{v}$, then there is a unique ray $\tilde{R} \subset \Gamma$ that begins with $\tilde{E}$, intersects $\mathrm{Fix}(\tilde{f})$ only in $\tilde{v}$, converges to an attractor $\xi \in \mathrm{Fix}(\hat{f})$ and whose image in $\mathcal{G}$ is r-legal and has height r. The limit set of ξ is the unique attracting lamination associated to H_r.*
2. *Suppose that E' is another oriented edge in H_r, that $\tilde{E}' \neq \tilde{E}$ determines a fixed direction at $\tilde{v}$ and that $\tilde{R}'$ is the ray associated to $\tilde{E}'$ as in item 1. Suppose further that the projection of the turn $(\bar{\tilde{E}}, \tilde{E}')$ is contained in the path $f^k_\sharp(E'')$ for some $k \geq 1$ and some edge E'' of H_r. Then the line $\bar{\tilde{R}}\tilde{R}'$ projects to a generic leaf of the attracting lamination Λ^+ associated to H_r.*

Proof. We follow [FH11, Lemma 2.13]. Lemma 2.1 and (EG-i) imply that $\tilde{f}(\tilde{E}) = \tilde{E} \cdot \tilde{\mu}_1$ for some non-trivial r-legal path μ_1 of height r that ends with an edge of H_r. If we apply Lemma 2.1 again, we have $\tilde{f}^2_\sharp(\tilde{E}) = \tilde{E} \cdot \tilde{\mu}_1 \cdot \tilde{\mu}_2$ where μ_2 is an r-legal path of height r that ends with an edge of H_r. Iterating this produces a nested increasing sequence of paths $\tilde{E} \subset \tilde{f}(\tilde{E}) \subset \tilde{f}_\sharp(\tilde{E}) \subset \cdots$ whose union is a ray $\tilde{R}$ that converges to a point $\xi \in \mathrm{Fix}(\hat{f})$. This point is an attractor: bounded cancellation shows that if we think of ξ as an infinite word, we are in the "superlinear attractor" case of Proposition 3.4.

If $\tilde{R}'$ is another ray whose image in $\mathcal{G}$ is r-legal and has height r that begins with $\tilde{E}$ and converges to some point $\xi' \in \mathrm{Fix}(\hat{f})$, then $\tilde{R}'$ has a splitting into terms that project to either edges in H_r or maximal subpaths in G_{r-1}. The edge $\tilde{E}$ is a term of this splitting, and since $\tilde{f}_\sharp(\tilde{R}') = \tilde{R}'$, one can argue by induction that $\tilde{f}^k_\sharp(\tilde{E})$ is an initial segment of $\tilde{R}'$ for all k, which implies that $\tilde{R}' = \tilde{R}$. This proves the uniqueness in item 1.

Since $f_\sharp(E)$ contains E, the stratum H_r is aperiodic, so there is a unique attracting lamination Λ^+ associated to H_r. Recall the definition of *tiles* from before the proof of Lemma 4.3. Suppose $\tilde{\lambda}$ is the realization in Γ of a generic leaf of Λ^+. By Corollary 4.5, $\tilde{\lambda}$ has a neighborhood basis that consists of lifts of tiles. Let $\tilde{\tau} \subset \tilde{\lambda}$ be such a tile. By construction, every tile occurs infinitely often in $\tilde{R}$, so some translate of $\tilde{R}$ belongs to $N(\tilde{\gamma})$. It follows that $\lambda \in \Lambda(\xi)$, so $\Lambda^+ \subset \Lambda(\xi)$. We will show that ξ is an endpoint of a leaf $\tilde{\beta}$ such that $\beta \in \Lambda^+$. By Lemma 4.8, since β is not entirely contained in G_{r-1}, the closure of β is all of Λ^+. By [GH19, Lemma 4.12], we have that $\Lambda(\xi) \subset \Lambda^+$, completing the proof of item 1.

Arguing as in the proof of Lemma 2.7, one can show that there is a legal turn based at $\tilde{v}$ projecting into G_r and in the image of $D\tilde{f}$, one of whose directions is $\tilde{E}$. Say the other direction is $\tilde{E}'$. If the direction $\tilde{E}'$ is periodic, we may iterate until it is fixed, determining a ray $\tilde{R}'$. If the original edge $\tilde{E}'$ belonged to H_r, this new edge, still called $\tilde{E}'$ belongs to H_r, and we are in the case of item 2.

By construction, every subpath of $\bar{\tilde{R}}\tilde{R}'$ is contained in a lift of a tile except possibly those subpaths crossing the turn $(\bar{\tilde{E}}, \tilde{E}')$. Since we assume, as in item 2, that the projection of the turn $(\bar{\tilde{E}}, \tilde{E}')$ is contained in the path $f^k_\sharp(E'')$ for some $k \geq 1$ and some edge E'' of H_r, then the line $\bar{\tilde{R}}\tilde{R}'$ is contained in Λ^+. In this situation, the line is birecurrent by construction. Lemma 4.8 implies that it is Λ^+-generic, completing the proof of item 2.

If $\tilde{E}'$ instead belongs to G_{r-1}, then the argument in the previous paragraph applies, except the line $\bar{\tilde{R}}\tilde{R}'$ is not birecurrent. Nonetheless it is contained in Λ^+.

If finally the direction $\tilde{E}'$ is not periodic, then $\tilde{v}$ has infinite valence, and we may take $\tilde{R}$ to be the line in question; again it is contained in Λ^+, but it is not birecurrent because it is not bi-infinite. $\square$

For later use, we record the following lemma.

Lemma 4.11. *There is a unique free factor F^i of minimal rank whose conjugacy class $[[F^i]]$ carries every line in Λ^+.*

Proof. We are grateful to Lee Mosher for suggesting the following proof. Since an attracting lamination Λ^+ is the closure of a single line β, any free factor that carries β carries every line in Λ^+. Since $[[F]]$ carries β, there is at least one free factor of minimal rank that carries β. Suppose that β is carried by both $[[F^1]]$ and $[[F^2]]$. For $i = 1, 2$ choose a marked thistle $\mathcal{G}_i$ with vertex $\star_i$ and a subgraph K_i so that the marking identifies $\pi_1(K_i, \star_i)$ with F^i. Then in Γ_i, the Bass–Serre tree for $\mathcal{G}_i$, the F^i-minimal subtree is disjoint from all its translates by elements of $F \setminus F^i$. It follows under the identification of $\partial\Gamma_i$ with $\partial(F, \mathcal{A})$ that $\partial(F^i, \mathcal{A}|_{F^i}) \subset \partial(F, \mathcal{A})$ is disjoint from all its translates by elements of $F \setminus F^i$. Each translate $c.\partial(F^i, \mathcal{A}|_{F^i})$ is stabilized by the corresponding conjugate cF^ic^{-1}. Let $\tilde{\beta}$ be a line in $\tilde{\mathcal{B}}$ that lifts β. Since β is carried by F^i, there is a unique translate $c_i.\partial(F^i, \mathcal{A}|_{F^i})$ that contains the endpoints of $\tilde{\beta}$. Let $F^3 = c_1F^1c_1^{-1} \cap c_2F^2c_2^{-1}$. The F^3-minimal subtree contains the realization of $\tilde{\beta}$ in T. It follows that F^3 is a free factor (of positive complexity) whose conjugacy class $[[F^3]]$ carries β. This contradicts minimality, so we conclude that $[[F^1]] = [[F^2]]$. $\square$

If $\mathcal{F}$ is a free factor system $\mathcal{F} = \{[[F^1]], \ldots, [[F^\ell]]\}$, define the *complexity* of $\mathcal{F}$ to be zero if $\mathcal{F}$ is trivial, and to be the complexities of the F^i rearranged in non-increasing order if $\mathcal{F}$ is nontrivial. Order the complexities of free factor systems of F lexicographically.

Corollary 4.12. *For any subset $B \subset \mathcal{B}$, there is a unique free factor system $\mathcal{F}(B)$ of minimal complexity that carries every element of B.*

Proof. Since $[[F]]$ carries every element of B, there is at least one free factor system $\mathcal{F}_1$ of minimal complexity that carries every element of B. Suppose $\mathcal{F}_2$ is another free factor system of minimal complexity that carries every element of B. Define $\mathcal{F}_1 \wedge \mathcal{F}_2$ to be the set of free factors of positive complexity in the set $\{[[F^i \cap (F^j)^c]] : [[F^i]] \in \mathcal{F}_1,\ [[F^j]] \in \mathcal{F}_2,\ c \in F\}$. The Kurosh subgroup theorem implies that $\mathcal{F}_1 \wedge \mathcal{F}_2$ is a possibly empty free factor system of F. The argument in the previous lemma implies that $\mathcal{F}_1 \wedge \mathcal{F}_2$ carries every element of B.

Notice that if $\mathcal{F}_1 \wedge \mathcal{F}_2 \neq \mathcal{F}_1$, then its complexity is strictly smaller: each free factor $F^i \cap (F^j)^c$ of positive complexity is a free factor of F^i, and thus it either equals F^i or has strictly smaller complexity. The minimality assumption therefore shows that $\mathcal{F}_1 = \mathcal{F}_2$, proving uniqueness. $\square$

Our final goal in this section is the following proposition. Suppose that Λ^+ is an attracting lamination for some element of $\mathrm{Out}(F, \mathcal{A})$. Define the *stabilizer* of Λ^+ to be

$$\mathrm{Stab}(\Lambda^+) = \{\psi \in \mathrm{Out}(F, \mathcal{A}) : \psi_\sharp(\Lambda^+) = \Lambda^+\}.$$

Proposition 4.13 (cf. Corollary 3.3.1 of [BFH00])**.** *There is a homomorphism*

$$\mathrm{PF}_{\Lambda^+} \colon \mathrm{Stab}(\Lambda^+) \to \mathbb{Z}$$

such that we have $\psi \in \ker(\mathrm{PF}_{\Lambda^+})$ if and only if $\Lambda^+ \notin \mathcal{L}(\psi)$ and $\Lambda^+ \notin \mathcal{L}(\psi^{-1})$.

Assume that $f \colon \mathcal{G} \to \mathcal{G}$ and $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ are a relative train track map and filtration for an element of $\mathrm{Stab}(\Lambda^+)$ and that Λ^+ is the attracting lamination associated to the (necessarily aperiodic) exponentially growing stratum H_r. For any path $\sigma \subset \mathcal{G}$, define $\mathrm{EL}_r(\sigma)$ to be the edge length of σ counting only the edges of H_r that are contained in σ. We say that $\psi \in \mathrm{Stab}(\Lambda^+)$ *asymptotically expands Λ^+ by the factor μ* if for every choice of $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ and $f\colon \mathcal{G} \to \mathcal{G}$ as above, every topological representative $g\colon \mathcal{G} \to \mathcal{G}$ of ψ and for all $\eta > 0$, we have

$$\mu - \eta < \frac{\mathrm{EL}_r(g_\sharp(\sigma))}{\mathrm{EL}_r(\sigma)} < \mu + \eta$$

whenever σ is contained in a Λ^+-generic leaf and $\mathrm{EL}_r(\sigma)$ is sufficiently large.

For the remainder of this section, we fix the relative train track map $f\colon \mathcal{G} \to \mathcal{G}$, the filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$, the stratum H_r and the attracting lamination Λ^+. The following proposition has the useful corollary that the Perron–Frobenius eigenvalue associated to an aperiodic exponentially growing stratum of a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ depends only on φ and the element of $\mathcal{L}(\varphi)$ that is associated to the stratum.

Proposition 4.14 (cf. Proposition 3.3.3 of [BFH00])**.** *We have the following.*

1. *Every $\psi \in \mathrm{Stab}(\Lambda^+)$ asymptotically expands Λ^+ by some factor $\mu = \mu(\psi)$.*
2. *$\mu(\psi\psi') = \mu(\psi)\mu(\psi')$.*
3. *$\mu(\psi) > 1$ if and only if $\Lambda^+ \in \mathcal{L}(\psi)$.*
4. *If $\Lambda^+ \in \mathcal{L}(\psi)$ and $f'\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map for ψ, then $\mu(\psi) = \mu'_s$ is the Perron–Frobenius eigenvalue for the transition matrix M'_s of the exponentially growing stratum H'_s associated to Λ^+.*

Proof of Proposition 4.13. The proof is identical to the proof of [BFH00, Corollary 3.3.1]. Define $\mathrm{PF}_{\Lambda^+}(\psi) = \log \mu(\psi)$. Proposition 4.14 and the observation that

$$\mathrm{PF}_{\Lambda^+}(\psi^{-1}) = -\,\mathrm{PF}_{\Lambda^+}(\psi)$$

implies that each $\mu(\psi)$ (or its multiplicative inverse) other than 1 occurs as the Perron–Frobenius eigenvalue for an irreducible matrix of uniformly bounded size. It follows ([BH92, p. 37] or the argument in the proof of [Lym21, Theorem 2.2]) that the image of PF_{Λ^+} is an infinite discrete subset of $\mathbb{R}$ and thus is isomorphic to $\mathbb{Z}$. Abusing notation, identify the image with $\mathbb{Z}$ and call the resulting homomorphism PF_{Λ^+}. It is clear from Proposition 4.14 that $\psi \in \ker(\mathrm{PF}_{\Lambda^+})$ if and only if $\Lambda^+ \notin \mathcal{L}(\psi)$ and $\Lambda^+ \notin \mathcal{L}(\psi^{-1})$. $\square$

Let E_i be an edge of H_r and let μ_r be the Perron–Frobenius eigenvalue for M_r, the transition matrix associated to H_r. The Perron–Frobenius theorem [Sen81] implies that the matrices $\mu_r^{-n}M_r^n$ converge to a matrix $M^\star$ with the property that all of its columns are multiples of each other. Let $A = (a_i)$ be the vector obtained from a column of $M^\star$ by multiplying so that the sum $\sum a_i$ is 1; call it a *frequency vector.* Let τ_i^k be the k-tile $f_\sharp^k(E_i)$.

A 0-tiling of a Λ^+-generic leaf λ is a decomposition into edges of H_r, vertex group elements, and maximal subpaths in G_{r-1}; the leaf λ has a unique 0-tiling. This yields a 1-tiling of $f_\sharp(\lambda)$ and inductively a k-tiling of $f_\sharp^k(\lambda)$ called the *standard k-tiling* of $f_\sharp^k(\lambda)$. Notice that $\lambda = f_\sharp^k(\gamma_k)$ for some Λ^+-generic leaf γ_k. It follows that every Λ^+-generic leaf has a standard k-tiling for $k \geq 0$.

Suppose σ is a finite subpath of a Λ^+-generic leaf λ. Let $\alpha_{ik}(\sigma)$ be the proportion, among those k-tiles in the standard k-tiling of λ that are entirely contained within σ, of the tiles that are equal to τ_i^k.

Lemma 4.15 (cf. Lemma 3.3.5 of [BFH00])**.** *Given $\epsilon > 0$ and $k \geq 0$ and a Λ^+-generic leaf λ, if σ is a finite subpath of λ with $\mathrm{EL}_r(\sigma)$ sufficiently large, then $a_i - \epsilon < \alpha_{ik}(\sigma) < a_i + \epsilon$.*

Proof. We follow the proof of [BFH00, Lemma 3.3.5]. Suppose first that σ is the ℓ-tile τ_j^ℓ and that $k = 0$. Observe that by Lemma 2.1 (cf. [BFH00, Lemma 3.1.8(4)]), the (i,j)-entry of M_r^ℓ is the number of times τ_j^ℓ crosses the ith edge in either direction, and that $\alpha_{i0}(\tau_j^\ell)$ is the (i,j)-entry of M_r^ℓ divided by the sum of the entries in the jth column of M_r^ℓ. By the Perron–Frobenius theorem, we have that $\alpha_{i0}(\tau_j^\ell)$ converges to a_i as ℓ increases. Since $\mathrm{EL_r}(\tau_j^\ell)$ increases with ℓ, we see that the lemma holds in this case. Observe further that $\alpha_{ik}(\tau_j^{\ell+k}) = \alpha_{i0}(\tau_j^\ell)$ by Lemma 2.1, so for ℓ sufficiently large, the lemma holds for ℓ-tiles with arbitrary $k \geq 0$. An easy calculation shows that if the results of the lemma hold with fixed k and ϵ for each ℓ-tile, then the lemma holds when σ has an ℓ-tiling induced by the standard ℓ-tiling of λ. (That is, σ is a union of ℓ-tiles, vertex group elements and paths in G_{r-1}.) Finally for general σ, in the standard ℓ-tiling of λ, at most two ℓ-tiles intersect σ but are not entirely contained in σ. If $\mathrm{EL}_r(\sigma)$ is sufficiently large (compared with ℓ) the contribution of these two tiles is negligible, so the lemma follows. ☐

Lemma 4.16 (cf. Lemma 3.3.6 of [BFH00])**.** *Suppose that $g\colon \mathcal{G} \to \mathcal{G}$ is a topological representative of $\psi \in \mathrm{Out}(F,\mathcal{A})$ and that Λ^+ is ψ-invariant. Then there exists a constant C_1 depending only on g satisfying $\mathrm{EL}_r(g_\sharp(\delta)) < C_1$ for any subpath $\delta \subset G_{r-1}$ of a Λ^+-generic leaf.*

Proof. We follow the proof of [BFH00, Lemma 3.3.6]. Suppose that the lemma fails. Then there exist Λ^+-generic leafs λ_j and finite subpaths $\delta_j \subset G_{r-1}$ of λ_j such that $g_\sharp(\delta_j)$ contains at least one edge in H_r, even after the first and last j edges have been removed. If we pass to a subsequence, we may assume that the paths δ_j form an increasing sequence whose union is a line $\delta^\star \subset G_{r-1}$ with the property that $g_\sharp(\delta^\star) \not\subset G_{r-1}$. But note that $\delta^\star$ is a leaf of Λ^+ whose closure is not all of Λ^+ (since it contains no edges of H_r), so $g_\sharp(\delta^\star)$ is also a line in Λ^+ whose closure is not all of Λ^+. But this contradicts Lemma 4.8 since $g_\sharp(\delta^\star)$ is not entirely contained in G_{r-1}. ☐

Proof of Proposition 4.14. We follow the proof of [BFH00, Proposition 3.3.3]. Let us recall our standing assumption: $f\colon \mathcal{G} \to \mathcal{G}$ and $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ are a relative train track map and associated filtration, H_r is an aperiodic exponentially growing stratum with associated lamination Λ^+, and $g\colon \mathcal{G} \to \mathcal{G}$ is a topological representative of $\psi \in \mathrm{Stab}(\Lambda^+)$. Let a_i be the frequency vector coefficient corresponding to the edge E_i of H_r, and let τ_i^k be the k-tile $f_\sharp^k(E_i)$. Suppose σ is a finite subpath of a Λ^+-generic leaf. Define

$$\mu_k = \frac{\sum_i a_i \,\mathrm{EL}_r(g_\sharp(\tau_i^k))}{\sum_i a_i \,\mathrm{EL}_r(\tau_i^k)}.$$

Fixing $\epsilon > 0$, we will show that if k is sufficiently large (depending on ϵ) and $\mathrm{EL}_r(\sigma)$ is sufficiently large (depending on ϵ and k), then

$$(1-\epsilon)\mu_k \leq \frac{\mathrm{EL}_r(g_\sharp(\sigma))}{\mathrm{EL}_r(\sigma)} \leq (1+\epsilon)\mu_k.$$

It follows that the μ_k form a Cauchy sequence and thus converge to a limit $\mu = \lim_{k\to\infty}\mu_k$. Recall that in order to show that ψ asymptotically expands Λ^+ by the factor μ, we must show that for every choice of relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ and filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ and every topological representative $g\colon \mathcal{G} \to \mathcal{G}$ of ψ, we have for all $\eta > 0$,

$$\mu - \eta < \frac{\mathrm{EL}_r(g_\sharp(\sigma))}{\mathrm{EL}_r(\sigma)} < \mu + \eta$$

whenever σ is contained in a Λ^+-generic leaf and $\mathrm{EL}_r(\sigma)$ is sufficiently large. The first equation shows that the second holds for $\sigma \subset \lambda$ sufficiently long relative to η and our particular choice of f and g.

Let $\mathrm{bcc}(g)$ be a bounded cancellation constant for $g\colon \mathcal{G} \to \mathcal{G}$ and let C_1 be the constant from Lemma 4.16. Given $\epsilon > 0$, write $x \sim_\epsilon y$ to mean that $|x-y|$ is small relative to ϵ. We may choose k so large that for all i, we have

1. $\dfrac{C_1}{\mathrm{EL}_r(\tau_i^k)} \sim_\epsilon 0$ and

2. $\dfrac{\operatorname{bcc}(g)}{\mathrm{EL}_r(\tau_i^k)} \sim_\epsilon 0$.

We may furthermore choose $\sigma \subset \lambda$ so long that

3. $\alpha_{ik}(\sigma) \sim_\epsilon a_i$ and

4. $\dfrac{\mathrm{EL}_r(\tau_i^k)}{\mathrm{EL}_r(\sigma)} \sim_\epsilon 0$.

As in [BFH00], we make simplifying assumptions in the calculation that introduce small errors. First, we assume that σ begins and ends with a complete k-tile from the standard k-tiling of λ; that is, σ is a concatenation of k-tiles γ_j, vertex group elements and maximal subpaths $\delta_\ell \subset G_{r-1}$. The error in making this assumption is at most twice the number of H_r-edges in a k-tile, so is controlled by item 4. The second assumption is that the approximation in item 3 is exact. The third assumption is that $\mathrm{EL}_r(g_\sharp(\delta_\ell)) = 0$ for each maximal subpath $\delta_\ell \subset G_{r-1}$. Here the error is controlled by item 1. (The true upper bound is C_1, and by assuming that C_1 is much smaller than $\mathrm{EL}_r(\tau_i^k)$, we are in effect able to ignore each $\mathrm{EL}_r(g_\sharp(\delta_\ell))$.) The final assumption is that $g_\sharp(\sigma) \subset g_\sharp(\lambda)$ is a tight concatenation of the paths $g_\sharp(\gamma_j)$, vertex group elements and the paths $g_\sharp(\delta_\ell)$ with no cancellation beyond multiplication in vertex groups. This produces an error bounded by $2\operatorname{bcc}(g)N(\sigma)$, where $N(\sigma)$ is the number of k-tiles σ contains. This error is controlled by item 2.

Given these assumptions, we have

$$\mathrm{EL}_r(g_\sharp(\sigma)) = \sum_i N(\sigma) a_i \,\mathrm{EL}_r(g_\sharp(\tau_i^k)).$$

If we let g be the identity, we see that our assumptions calculate

$$\mathrm{EL}_r(\sigma) = \sum_i N(\sigma) a_i \,\mathrm{EL}_r(\tau_i^k),$$

so we see that

$$\frac{\mathrm{EL}_r(g_\sharp(\sigma))}{\mathrm{EL}_r(\sigma)} = \mu_k,$$

verifying the equation above.

If $g^\star\colon \mathcal{G} \to \mathcal{G}$ is another topological representative of ψ, then there are lifts $\tilde g\colon \Gamma \to \Gamma$ and $\tilde g^\star\colon \Gamma \to \Gamma$ such that the distance between $\tilde g(\tilde x)$ and $\tilde g^\star(\tilde x)$ is bounded independently of $\tilde x$, from which it follows that $\mathrm{EL}_r(g_\sharp(\sigma)) - \mathrm{EL}_r(g^\star_\sharp(\sigma))$ is bounded independently of σ, so hence μ does not depend on the choice of g.

If instead $f^\star\colon \mathcal{G}^\star \to \mathcal{G}^\star$ and $\varnothing = G_0^\star \subset G_1^\star \subset \cdots \subset G_M^\star = G^\star$ is another relative train track map and filtration representing φ, Λ^+ is the attracting lamination associated to $H_s^\star$ and $\mathrm{EL}_s^\star$ is the edge length function that counts edges of $H_s^\star$ in $\mathcal{G}^\star$, we will argue that the following holds.

Choose a homotopy equivalence $h\colon \mathcal{G} \to \mathcal{G}^\star$ that respects the markings and maps edges to tight edge paths. One argues exactly as above that there exists a positive constant ν such that for all $\epsilon > 0$, we have

$$(1-\epsilon)\nu < \frac{\mathrm{EL}_s^\star(h_\sharp(\sigma))}{\mathrm{EL}_r(\sigma)} < (1+\epsilon)\nu$$

whenever σ is contained in a Λ^+-generic leaf and $\mathrm{EL}_r(\sigma)$ is sufficiently large. We leave the details to the reader.

Suppose that $\hat g\colon \mathcal{G}^\star \to \mathcal{G}^\star$ represents ψ. Given any finite path $\sigma \subset G_r$, we have that $h_\sharp g_\sharp(\sigma)$ and $\hat g_\sharp h_\sharp(\sigma)$ differ by initial and terminal segments of uniformly bounded size. This, together with the equation above, shows that

$$\frac{\mathrm{EL}_s^\star(\hat g_\sharp h_\sharp(\sigma))}{\mathrm{EL}_s^\star(h_\sharp(\sigma))} \sim \frac{\mathrm{EL}_s^\star(h_\sharp g_\sharp(\sigma))}{\mathrm{EL}_s^\star(h_\sharp(\sigma))} \sim \frac{\mathrm{EL}_r(g_\sharp(\sigma))}{\mathrm{EL}_r(\sigma)},$$

where the error of approximation in each $\sim$ goes to 0 as $\mathrm{EL}_r(\sigma)$ or equivalently $\mathrm{EL}^{\star}_s(h_\sharp(\sigma))$ grows. We conclude that μ is independent of the choice of relative train track map and filtration and thus that ψ asymptotically expands Λ^+ by the factor μ, proving part 1 of Proposition 4.14.

Now suppose that $g\colon \mathcal{G} \to \mathcal{G}$ and $g'\colon \mathcal{G} \to \mathcal{G}$ are topological representatives for ψ and ψ' respectively. If $\sigma \subset \mathcal{G}$ is contained in a Λ^+-generic leaf, then (by bounded cancellation) there is a subpath σ' of $g_\sharp(\sigma)$ that is contained in a Λ^+-generic leaf and that differs from $g_\sharp(\sigma)$ only in an initial and terminal segment of uniformly bounded length. Similarly $g'_\sharp g_\sharp(\sigma)$ differs from $g'_\sharp(\sigma')$ only in an initial and terminal segment of uniformly bounded length. Therefore $\mu(\psi\psi') = \mu(\psi)\mu(\psi')$, proving part 2.

To prove part 3, we will show that $\mu = \mu(\psi) > 1$ holds if and only if we have $\Lambda^+ \in \mathcal{L}(\psi)$. Suppose first that $\mu > 1$, and that σ_0 is a subpath of a Λ^+-generic leaf. Let σ_1 be the finite subpath obtained from $g_\sharp(\sigma_0)$ by removing initial and terminal segments of length $\mathrm{bcc}(g)$. Bounded cancellation implies that $g_\sharp(N(\sigma_0)) \subset N(\sigma_1)$. Since $\mu > 1$, if $\mathrm{EL}_r(\sigma_0)$ is sufficiently large, we have $\mathrm{EL}_r(\sigma_1) > \mathrm{EL}_r(\sigma_0)$. Inductively we may produce subpaths σ_k such that $\mathrm{EL}_r(\sigma_k)$ is increasing and such that $g_\sharp(N(\sigma_{k-1})) \subset N(\sigma_k)$, from which it follows that $g^k_\sharp(N(\sigma_0)) \subset N(\sigma_k)$. Since $g^k_\sharp(\lambda)$ is Λ^+-generic, Corollary 4.5 implies that $\lambda \in N(\sigma_k)$ for all k.

Since λ has an exhaustion by tiles by Lemma 4.4, σ_0 is contained in an ℓ-tile in λ for some ℓ. If k is sufficiently large, since $\mathrm{EL}_r(\sigma_k)$ grows with k, by Lemma 4.15, σ_k contains every ℓ-tile at least once. Therefore in particular $N(\sigma_k) \subset N(\sigma_0)$. Recall from Corollary 4.5 that the tiles form a neighborhood basis for λ. The same argument shows that given ℓ', there is k' sufficiently large so that $\sigma_{k'}$ contains every ℓ'-tile, so the neighborhoods $N(\sigma_j)$ as j varies form a neighborhood basis for λ. It follows that $N(\sigma_0)$ is an attracting neighborhood for the action of $\psi^k_\sharp$. This shows that $\Lambda^+ \in \mathcal{L}(\psi)$.

Finally, if $\Lambda^+ \in \mathcal{L}(\psi)$, we may by the independence of choices in item 1 assume that the relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ used to compute $\mu = \mu(\psi)$ represents ψ, that $g = f$, and that γ is a k-tile τ^k_i for some large k. Under these assumptions, we have that $\mathrm{EL}_r(g_\sharp(\sigma))$ is the sum of the ith column of M^{k+1}_r, the $(k+1)$-st power of the transition matrix. Meanwhile $\mathrm{EL}_(\sigma)$ is the ith column sum of M^k_r. The Perron–Frobenius theorem implies that

$$\frac{\mathrm{EL}_r(g_\sharp(\sigma))}{\mathrm{EL}_r(\sigma)} \to \mu_r,$$

the Perron–Frobenius eigenvalue of M_r, as $k \to \infty$. Thus $\mu = \mu_r > 1$, completing the proof of item 3 and also proving item 4. $\square$

5 Principal automorphisms

The purpose of this section is to develop the so-called "Nielsen theory" for outer automorphisms of free products. We define principal automorphisms and rotationless outer automorphisms, and show that in cases of particular interest every outer automorphism has a rotationless power. (Rotationless outer automorphisms were originally called "forward rotationless" in [FH11], but see [FH18].)

Principal automorphisms. Given an outer automorphism $\varphi \in \mathrm{Out}(F,\mathcal{A})$ and an automorphism $\Phi\colon (F,\mathcal{A}) \to (F,\mathcal{A})$ representing it, let $\mathrm{Fix}_N(\hat\Phi) \subset \mathrm{Fix}(\hat\Phi)$ denote the subset of the set of fixed points of $\hat\Phi$ containing all fixed points in $V_\infty(F,\mathcal{A})$ along with those fixed points in $\partial_\infty(F,\mathcal{A})$ that are not repellers. We say that Φ is a *principal automorphism* if either of the following conditions holds.

1. $\mathrm{Fix}_N(\hat\Phi)$ contains at least three points.

2. $\mathrm{Fix}_N(\hat\Phi)$ is a two-point set that is neither the endpoints of an axis nor the endpoints of a lift of a generic leaf β of an attracting lamination $\Lambda^+ \in \mathcal{L}(\varphi)$.

If $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map representing φ, then we call the lift $\tilde{f}\colon \Gamma \to \Gamma$ corresponding to a principal automorphism Φ a *principal lift.*

Nielsen almost equivalence and isogredience. In [FH11], Feighn and Handel prove a kind of dictionary between certain equivalence classes in $\mathrm{Fix}(f)$ called *Nielsen classes* and *isogredience classes* of principal automorphisms. A pair of automorphisms Φ_1 and Φ_2 are *isogredient* if there exists $c \in F$ such that $\Phi_2 = i_c\Phi_1 i_c^{-1}$, where i_c is the inner automorphism $x \mapsto cxc^{-1}$. In the language of lifts, we say that $\tilde{f}_1$ is isogredient to $\tilde{f}_2$ if $\tilde{f}_2 = T_c\tilde{f}_1T_c^{-1}$. The idea is that if $\tilde{f}_1$ and $\tilde{f}_2$ are lifts of f that have nonempty fixed sets, then they are isogredient if and only if those fixed sets project to the same Nielsen class in $\mathrm{Fix}(f)$.

For free products, the situation is complicated by the following facts. Firstly, a path in the Bass–Serre tree Γ with endpoints in $\mathrm{Fix}(\tilde{f}_1)$ need not project to an honest Nielsen path for f, merely an almost Nielsen path. Second, if $\tilde{v}$ is a vertex of Γ that projects to a vertex v with nontrivial vertex group, then two lifts $\tilde{f}_1$ and $\tilde{f}_2$ may fix $\tilde{v}$ without being isogredient. What's more, we will show that if $\mathrm{Fix}(\tilde{f}) = \{\tilde{v}\}$ and $\tilde{f}$ is principal, then $D\tilde{f}$ must fix a direction at $\tilde{v}$, so not all lifts of f fixing $\tilde{v}$ are principal. To get the correct division of $\mathrm{Fix}(f)$, we have to make the following somewhat artificial-seeming definition.

Let $f\colon \mathcal{G} \to \mathcal{G}$ be a relative train track map. Two points x and y in $\mathrm{Fix}(f)$ are *Nielsen almost equivalent* or belong to the same *almost Nielsen class* if they are the endpoints of an almost Nielsen path for f. We must warn the reader: Nielsen almost equivalence is *not* an equivalence relation, since it fails to be transitive in general. A vertex may belong to *multiple* almost Nielsen classes.

Recall that a direction (g,e) based at a vertex v of $\mathcal{G}$ is *almost fixed* by f if $Df(g,e) = (g',e)$. It is *almost periodic* if it is almost fixed by some iterate of Df. If we have $g = g'$ in the above equation, of course, the direction is *fixed* or *periodic.*

To account for principal lifts satisfying $\mathrm{Fix}(\tilde{f}) = \{\tilde{v}\}$, we will add yet more almost Nielsen classes associated to a vertex v with nontrivial vertex group: let $e_1,\ldots,e_m$ be the oriented edges beginning at v determining almost fixed directions for Df. That is, $Df(1,e_i) = (g_i,e_i)$ for some vertex group element $g_i \in \mathcal{G}_v$. Define an equivalence relation on the set $\{e_1,\ldots,e_m\}$ where $e_i \sim e_j$ if there exists g_i and g_j in $\mathcal{G}_v$ such that $Df(1,e_i) = (g_i,e_i)$ and $g_i^{-1}Df(g_j,e_j) = (g_j,e_j)$. The vertex v should belong to an almost Nielsen class for each equivalence class $[e_i]$. For some of these e_i, there is an almost Nielsen path based at v that begins $e_i\cdots$; for these e_i we do not need to an add an almost Nielsen class; for the remainder of these e_i, the almost Nielsen class is $\{v\}$. For technical reasons, we also need to add an *exceptional* almost Nielsen class $\{v\}$ that does not correspond to any $[e_i]$ above. Thus a *non-exceptional* almost Nielsen class either corresponds to a non-trivial almost Nielsen path, is a single point w (possibly in the interior of an edge) that has trivial vertex group if w is a vertex, or is determined by a vertex v with nontrivial vertex group and at least one almost fixed direction at v.

Lemma 5.1. *There are finitely many almost Nielsen classes.*

Proof. Since G is a finite graph, $\mathrm{Fix}(f)$ has finitely many connected components. Each component of $\mathrm{Fix}(f)$ that contains an edge splits into finitely many almost Nielsen classes, since points in the interior of the same almost fixed edge are Nielsen almost equivalent. Each vertex belongs to finitely many almost Nielsen classes, since there are finitely many almost Nielsen paths for f and finitely many oriented edges incident to that vertex. An isolated point in $\mathrm{Fix}(f)$ in the interior of an edge belongs to a single almost Nielsen class. $\square$

Suppose $\tilde{f}\colon \Gamma \to \Gamma$ is a lift of $f\colon \mathcal{G} \to \mathcal{G}$. Any tight path $\tilde{\alpha}$ in Γ with endpoints in $\mathrm{Fix}(\tilde{f})$ projects to an almost Nielsen path α for f. Conversely, if α is an almost Nielsen path for f and $\tilde{f}$ fixes an endpoint of $\tilde{\alpha}$ with trivial stabilizer, then $\tilde{f}$ also fixes the other endpoint. If $\mathrm{Fix}(\tilde{f})$ is a single vertex $\tilde{v}$ that projects to a vertex v with nontrivial vertex group, we say that $\mathrm{Fix}(\tilde{f})$ projects to the exceptional almost Nielsen class $\{v\}$ if there is no fixed direction for $\tilde{f}$ at $\tilde{v}$. Otherwise, suppose $\tilde{e}$ determines a fixed direction for $D\tilde{f}$ based at $\tilde{v}$ corresponding to the direction (g, e) based at v. Then (g, e) is almost fixed for Df; say $Df(1, e) = (h, e)$, and $D\tilde{f}(x, e) = gh^{-1} f_v(g^{-1}) Df(x, e)$ for $x \in \mathcal{G}_v$. If $e \sim e'$, let h' be such that $h^{-1} Df(h', e) = (h', e)$. Then

$$D\tilde{f}(gh', e') = gh^{-1} f_v(g^{-1}) Df(gh', e') = gh^{-1} Df(h', e') = (gh', e').$$

Thus $\mathrm{Fix}(\tilde{f})$ is either empty or projects to a single almost Nielsen class in $\mathrm{Fix}(f)$.

Lemma 5.2 (cf. Lemma 3.8 of [FH11])**.** *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ represents $\varphi \in \mathrm{Out}(F, \mathcal{A})$ and that $\tilde{f}_1$ and $\tilde{f}_2$ are lifts of f with nonempty fixed point sets that project to non-exceptional almost Nielsen classes. Then $\tilde{f}_1$ and $\tilde{f}_2$ belong to the same isogredience class if and only if $\mathrm{Fix}(\tilde{f}_1)$ and $\mathrm{Fix}(\tilde{f}_2)$ project to the same almost Nielsen class in $\mathrm{Fix}(f)$.*

Proof. If $\tilde{f}_2 = T_c \tilde{f}_1 T_c^{-1}$, then $\mathrm{Fix}(\tilde{f}_2) = T_c \,\mathrm{Fix}(\tilde{f}_1)$. Furthermore if $\tilde{E}$ determines a fixed direction for $D\tilde{f}_1$ at $\tilde{v} \in \mathrm{Fix}(\tilde{f}_1)$, then $T_c(\tilde{E})$ determines a fixed direction for $D\tilde{f}_2$ at $T_c(\tilde{v}) \in \mathrm{Fix}(\tilde{f}_2)$. Therefore $\mathrm{Fix}(\tilde{f}_2)$ and $\mathrm{Fix}(\tilde{f}_1)$ project to the same almost Nielsen class in $\mathrm{Fix}(f)$. We remark that this holds without assumption on $\mathrm{Fix}(\tilde{f}_1)$ and $\mathrm{Fix}(\tilde{f}_2)$.

Conversely if $\mathrm{Fix}(\tilde{f}_2)$ and $\mathrm{Fix}(\tilde{f}_1)$ have the same non-exceptional almost Nielsen class as a projection, then there exists $\tilde{x} \in \mathrm{Fix}(\tilde{f}_2)$ and T_c such that $T_c^{-1}(\tilde{x}) \in \mathrm{Fix}(\tilde{f}_1)$. If $\tilde{x}$ projects to a point with trivial associated group, then we are done, as $\tilde{f}_2$ and $T_c \tilde{f}_1 T_c^{-1}$ are lifts of f that agree on a point and are thus equal.

If instead $\tilde{x}$ projects to a vertex with nontrivial vertex group, then by the assumption that $\mathrm{Fix}(\tilde{f}_2)$ projects to a non-exceptional almost Nielsen class, there is either a second point $\tilde{y}$ in $\mathrm{Fix}(\tilde{f}_2)$ or there is an edge $\tilde{e}$ incident to $\tilde{x}$ determining a fixed direction for $\tilde{f}_2$. In the former case, we may choose c such that $T_c^{-1}(\tilde{y}) \in \mathrm{Fix}(\tilde{f}_1)$ and we are done, as $\tilde{f}_2$ and $T_c \tilde{f}_1 T_c^{-1}$ agree on two points and are thus equal. In the contrary case, since $\mathrm{Fix}(\tilde{f}_1)$ projects to the same almost Nielsen class as $\mathrm{Fix}(\tilde{f}_2)$, we may choose c such that $T_c^{-1}(\tilde{e})$ determines a fixed direction for $\tilde{f}_1$. Then $T_c \tilde{f}_1 T_c^{-1}$ and $\tilde{f}_2$ agree on a point and a direction at that point and are thus equal. □

We show below that principal lifts $\tilde{f}$ have nonempty fixed sets which project to non-exceptional almost Nielsen classes. Since there are finitely many non-exceptional almost Nielsen classes, it follows that there are finitely many isogredience classes of principal automorphisms Φ representing each $\varphi \in \mathrm{Out}(F, \mathcal{A})$.

Rotationless automorphisms. Let $P(\varphi)$ denote the set of principal automorphisms $\Phi\colon (F, \mathcal{A}) \to (F, \mathcal{A})$ representing $\varphi \in \mathrm{Out}(F, \mathcal{A})$. Note that if $\Phi \in P(\varphi)$ is principal and $k \geq 1$, then Φ^k is also principal for φ^k. Let $\mathrm{Per}_N(\hat{\Phi})$ denote the set of non-repelling periodic points for the action of $\hat{\Phi}$ on $\partial(F, \mathcal{A})$. (That is, the set of periodic points that are not repellers in $\partial_\infty(F, \mathcal{A})$ for some positive iterate of $\hat{\Phi}$.)

An outer automorphism φ is *rotationless* if for all $\Phi \in P(\varphi)$, we have $\mathrm{Per}_N(\hat{\Phi}) = \mathrm{Fix}_N(\hat{\Phi})$ and for all $k \geq 1$, the map $\Phi \mapsto \Phi^k$ defines a bijection $P(\varphi) \to P(\varphi^k)$.

We assume throughout the remainder of the paper that $F \neq F_1$ or $C_2 * C_2$, and that the free product decomposition of F has positive complexity, so $F \neq A_1$. The Bowditch boundary of each of the former free products is two points which are the endpoints of an axis, so there are no principal automorphisms of F_1 or $C_2 * C_2$. Nevertheless for notational convenience, we will say that the identity outer automorphism and either outer automorphism of these groups are rotationless respectively.

We now turn to showing that principal lifts have nonempty fixed point sets that project to non-exceptional almost Nielsen classes. Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ represents $\varphi \in \mathrm{Out}(F, \mathcal{A})$ and that $\tilde{f}\colon \Gamma \to \Gamma$ is a lift to the Bass–Serre tree. We say that $\tilde{f}$ *moves* $\tilde{z} \in \Gamma$ *towards* $P \in \mathrm{Fix}(\tilde{f})$ if the tight ray from $\tilde{f}(\tilde{z})$ to P does not contain $\tilde{z}$. Similarly we say that $\tilde{f}$ *moves $\tilde{y}_1$ and $\tilde{y}_2$ away from each other* if the tight path in Γ connecting $\tilde{f}(\tilde{y}_1)$ to $\tilde{f}(\tilde{y}_2)$ contains $\tilde{y}_1$ and $\tilde{y}_2$ and if $\tilde{f}(\tilde{y}_1) < \tilde{y}_1 < \tilde{y}_2 < \tilde{f}(\tilde{y}_2)$ in the order induced by the orientation on that path.

Recall that $\partial(F, \mathcal{A})$ is identified with the Bowditch boundary of Γ. Thus it makes sense to say that points in Γ are close to $P \in \partial(F, \mathcal{A})$ or that P is the limit of points in Γ.

Lemma 5.3 (cf. Lemma 3.15 of [FH11])**.** *Suppose that $P \in \partial_\infty(F, \mathcal{A})$ is a point in $\mathrm{Fix}(\hat{f})$ and that there does not exist $c \in \Gamma$ such that $\mathrm{Fix}(\hat{f}) = \{\hat{T}_c^{\pm}\}$.*

1. *If P is an attractor for $\hat{f}$ then $\tilde{z}$ moves toward P under the action of $\tilde{f}$ for all $\tilde{z} \in \Gamma$ that are sufficiently close to P.*
2. *If P is the endpoint of an axis A_c or if P is the limit of points in Γ that are either fixed by $\tilde{f}$ or move toward P under the action of $\tilde{f}$, then $P \in \mathrm{Fix}_N(\hat{f})$.*

Proof. The proof follows [FH11, Lemma 3.15]. If P is not the endpoint of an axis A_c, then the statement follows from the proof of Proposition 3.4 and bounded cancellation.

If P is the endpoint of an axis A_c, then $\mathrm{Fix}(\hat{f})$ contains $\hat{T}_c^{\pm}$ and at least one other point. It follows from Lemma 3.2 that $P \in \partial(\mathbb{F}, \mathcal{A}|_{\mathbb{F}})$ and from Proposition 3.4 that this point is not isolated in $\mathrm{Fix}(\hat{f})$; thus it is neither an attractor nor a repeller, so $P \in \mathrm{Fix}_N(\hat{f})$. $\square$

Lemma 5.4 (cf. Lemma 3.16 of [FH11])**.** *If $\tilde{f}$ moves $\tilde{y}_1$ and $\tilde{y}_2$ away from each other, then $\tilde{f}$ fixes a point in the interval bounded by $\tilde{y}_1$ and $\tilde{y}_2$.*

Proof. We follow the proof of [FH11, Lemma 3.16]. Let α_0 denote the oriented tight path connecting $\tilde{y}_1$ to $\tilde{y}_2$ and let α_1 denote the oriented tight path connecting $\tilde{f}(\tilde{y}_1)$ to $\tilde{f}(\tilde{y}_2)$. Let $r\colon \Gamma \to \alpha_1$ be the retraction onto the nearest point in α_1 and let $\tilde{g} = r\tilde{f}\colon \alpha_0 \to \alpha_1$. Since $\tilde{f}$ moves $\tilde{y}_1$ and $\tilde{y}_2$ away from each other, $\tilde{\alpha}_0$ is a proper subpath of $\tilde{\alpha}_1$ and $\tilde{g}$ is a surjection. If $\tilde{y}$ is the first point in $\tilde{\alpha}_0$ such that $\tilde{g}(\tilde{y}) = \tilde{y}$, then $\tilde{y}_1 < \tilde{y} < \tilde{y}_2$ and $\tilde{g}(\tilde{z}) < \tilde{g}(\tilde{y})$ for $\tilde{y}_1 < \tilde{z} < \tilde{y}$. It follows that $\tilde{f}(\tilde{y})$ belongs to $\tilde{\alpha}_1$, so $\tilde{y}$ is fixed by $\tilde{f}$. $\square$

Corollary 5.5 (cf. Corollary 3.17 of [FH11])**.** *If $\tilde{f}$ is a principal lift, then $\mathrm{Fix}(\tilde{f})$ is nonempty and projects to a non-exceptional almost Nielsen class in $\mathrm{Fix}(f)$.*

Proof. The proof follows [FH11, Corollary3.17]. Suppose first that there is a nonperipheral element c such that A_c has its endpoints in $\mathrm{Fix}_N(\hat{f})$ and so T_c commutes with $\tilde{f}$. Either some point of the axis A_c is fixed by $\tilde{f}$ or not. If there is a fixed point $\tilde{x}$, then note that

$$\tilde{f}T_c(\tilde{x}) = T_c\tilde{f}(\tilde{x}) = T_c(\tilde{x}),$$

so $\mathrm{Fix}(\tilde{f})$ projects to a non-exceptional almost Nielsen class in $\mathrm{Fix}(f)$.

If there is no fixed point on A_c, then there is a point in A_c that moves toward one of the endpoints of A_c, say to P, for if we assume the contrary, then $\tilde{f}$ moves any two points of A_c away from each other, contradicting our assumed lack of fixed point. Since $\tilde{f}$ commutes with T_c, there are points in Γ arbitrarily close to P that move toward P. The same property holds for an attractor $P \in \mathrm{Fix}(\hat{f})$ by Lemma 5.3.

Therefore if $\mathrm{Fix}_N(\hat{f})$ contains $\{\hat{T}_c^{\pm}\}$ and at least one other point in $\partial_\infty(F, \mathcal{A})$, but we assume that there is no fixed point on each axis A_c satisfying $\{\hat{T}_c^{\pm}\} \subset \mathrm{Fix}_N(\hat{f})$, then there are distinct P_1 and P_2 in $\mathrm{Fix}_N(\hat{f})$ and $\tilde{x}_1$ and $\tilde{x}_2$ in Γ such that $\tilde{x}_i$ is close to and moves toward P_i. It follows that $\tilde{f}$ moves $\tilde{x}_1$ and $\tilde{x}_2$ away from each other. Lemma 5.4 produces a fixed point $\tilde{y}$.

If $\mathrm{Fix}_N(\hat{f})$ contains $\{\hat{T}_c^{\pm}\}$ and at least one other point $\tilde{y}$ in $V_\infty(F, \mathcal{A})$, then $\tilde{y} \in \mathrm{Fix}(\tilde{f})$. The argument in the first paragraph shows that $T_c(\tilde{y})$ is also fixed, so $\mathrm{Fix}(\tilde{f})$ projects to a non-exceptional almost Nielsen class in $\mathrm{Fix}(f)$.

If $\tilde{f}$ is a principal lift corresponding to Φ but $\mathrm{Fix}_N(\hat{f})$ contains only one point in $\partial_\infty(F, \mathcal{A})$ then there is a conjugate of some infinite vertex group A_i sent to itself by Φ. This conjugate fixes a unique point $\tilde{x}$ in Γ, which must (by Φ-twisted equivariance) be fixed by $\tilde{f}$. Let $P \in \partial_\infty(F, \mathcal{A})$ be in $\mathrm{Fix}_N(\hat{f})$. It is an attractor, so there exist points on the ray from $\tilde{x}$ to P close to P that move toward P under $\tilde{f}$. Either the direction determined by the tight ray from $\tilde{x}$ to P is fixed by $\tilde{f}$, or there are points on the tight ray from $\tilde{x}$ to P moved away from each other by $\tilde{f}$. In either case, $\mathrm{Fix}(\tilde{f})$ projects to a non-exceptional almost Nielsen class in $\mathrm{Fix}(f)$.

Finally if $\tilde{f}$ is a principal lift corresponding to Φ but $\mathrm{Fix}(\hat{f})$ contains no points in $\partial_\infty(F, \mathcal{A})$ then there are two conjugates of (not necessarily distinct) A_i sent to themselves by Φ. These conjugates fix unique points $\tilde{x}$ and $\tilde{y}$ in Γ, which must be fixed by $\tilde{f}$. The tight path from $\tilde{x}$ to $\tilde{y}$ projects to an almost Nielsen path, so $\mathrm{Fix}(\tilde{f})$ projects to a non-exceptional almost Nielsen class in $\mathrm{Fix}(f)$. □

There are two cases in which a lift $\tilde{f}$ whose fixed point set projects to a non-exceptional almost Nielsen class is not a principal lift. The first is when $\mathrm{Fix}(\hat{f})$ is the endpoints of a generic leaf of an attracting lamination $\Lambda^+ \in \mathcal{L}(\varphi)$. In this case $\mathrm{Fix}(\tilde{f})$ is a single point $\tilde{x}$ and there are exactly two periodic directions at $\tilde{x}$. The second case is when $\mathrm{Fix}(\hat{f})$ is the endpoints of an axis A_c. As it happens, this latter case occurs when $\mathrm{Fix}(\tilde{f}) = A_c$. In this latter case, the axis A_c projects to either a topological circle in $\mathrm{Fix}(f)$ or to the quotient of $\mathbb{R}$ by the standard action of $C_2 * C_2$, i.e. the quotient is a subgraph of subgroups of $\mathcal{G}$ that is topologically an interval with C_2 vertex groups at the endpoints. In both cases, the lift becomes principal after passing to an iterate unless there are exactly two periodic directions at each point of this component of $\mathrm{Fix}(f)$. In the case where $\mathrm{Fix}(\tilde{f})$ projects to the quotient of $\mathbb{R}$ by the standard action of $C_2 * C_2$, notice that if a vertex group $\mathcal{G}_v$ in question is finite but not equal to C_2, then there are more than two periodic directions at any lift $\tilde{v}$ in $\mathrm{Fix}(\tilde{f})$. If $\mathcal{G}_v$ is infinite, then $\mathrm{Fix}_N(\hat{f})$ is infinite, so $\tilde{f}$ is principal.

Points in $\mathrm{Per}(f)$ are *Nielsen almost equivalent* if they are Nielsen almost equivalent as fixed points for some iterate of f. Thus two periodic points x and y are Nielsen almost equivalent if they are the endpoints of a periodic almost Nielsen path. We say that a point $x \in \mathrm{Per}(f)$ is *principal* if neither of the following conditions are satisfied.

1. The point x has finite vertex group if it is a vertex, is not the endpoint of a periodic almost Nielsen path and there are exactly two periodic directions at x, both of which are contained in the same exponentially growing stratum.

2. The point x is contained in a component C of $\mathrm{Per}(f)$ which is either a topological circle or isomorphic as a graph of groups to the quotient of $\mathbb{R}$ by the standard action of $C_2 * C_2$ and each point in C has exactly two periodic directions.

If $x \in \mathrm{Per}(f)$ is a vertex with nontrivial vertex group, then it may be principal in multiple ways, according to the ways the edges incident to x determine different non-exceptional almost Nielsen classes. Note that a vertex with nontrivial vertex group is always principal unless that vertex group is C_2.

(Note that we follow Definition 3.5 in [FH18], which is a corrected version of [FH11, Definition 3.18].) Lifts of principal periodic points in $\mathcal{G}$ to Γ are *principal*.

We say that $f\colon \mathcal{G} \to \mathcal{G}$ is *rotationless* if it satisfies the following conditions.

1. Each principal vertex is fixed.

2. Each almost periodic direction at a principal vertex is almost fixed.

3. Suppose v is a principal vertex with nontrivial vertex group and that $(1, e)$ is an almost fixed direction at v, say $Df(1, e) = (g, e)$. If a direction d is periodic for the map $d \mapsto g^{-1}.Df(d)$, it is fixed.

In practice we apply these definitions to relative train track maps $f\colon \mathcal{G} \to \mathcal{G}$ that satisfy the conclusions of Theorem 2.6. Principal periodic points are thus either contained in almost periodic edges or are vertices. We have that any endpoint of an indivisible periodic almost Nielsen path (containing an edge, of course) is principal, as is the initial endpoint of any non-exponentially growing edge that is not almost periodic. This latter property implies that for a rotationless relative train track map each non-exponentially growing stratum that is not periodic is a single edge.

Lemma 5.6. *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a rotationless relative train track map, that $v \in G$ is a principal vertex with nontrivial vertex group and that $g = f^k$ for some $k \geq 1$. Let $\tilde{v}$ be a lift of v fixed by a lift $\tilde{g}$ of g. Suppose that $\tilde{g}$ fixes a direction at $\tilde{v}$. There is a lift $\tilde{f}$ of f fixing $\tilde{v}$ and all the same directions at $\tilde{v}$ as $\tilde{g}$.*

Proof. Using the $\mathcal{G}_v$-equivariant bijection from Section 1, we identify the set of directions at $\tilde{v}$ with

$$\coprod_{e \in \mathrm{st}(v)} \mathcal{G}_v \times \{e\}.$$

Since f is rotationless and v is principal, each Df-almost periodic direction $(1, e_i)$ at v is almost fixed, say $Df(1, e_i) = (g_i, e_i)$. By the discussion in Section 1, the map $D\tilde{f}$ takes the form $D\tilde{f}(x, e_i) = (hf_v(x)g_i, e_i)$ for some element $h = h(\tilde{f}) \in \mathcal{G}_v$. Suppose (x, e_i) is fixed by $D\tilde{g}$. Take $h = xg_i^{-1}f_v(x^{-1})$, and observe that

$$D\tilde{f}(x, e_i) = (xg_i^{-1}f_v(x^{-1})f_v(x)g_i, e_i) = (x, e_i).$$

Furthermore, by definition there exists $h' = h'(\tilde{g}) \in \mathcal{G}_v$ such that

$$D\tilde{g}(x, e_i) = (h'f_v^k(x)f_v^{k-1}(g_i)f_v^{k-2}(g_i)\cdots f_v(g_i)g_i, e_i),$$

and the condition that $D\tilde{g}(x, e_i) = (x, e_i)$ says that

$$h' = xg_i^{-1}f_v(g_i^{-1})\cdots f_v^{k-2}(g_i^{-1})f_v^{k-1}(g_i^{-1})f_v^k(x^{-1}) = hf_v(h)\cdots f_v^{k-1}(h).$$

In other words, $D\tilde{g} = D\tilde{f}^k$.

Therefore to complete the proof, we need only show that if a direction has $D\tilde{f}$-period k, it is fixed. Observe that if a direction (y, e_j) has period k under $D\tilde{f}$, then $(x^{-1}y, e_j)$ has period k under $D(T_{x^{-1}}\tilde{f}T_x)$. If the direction $(1, e_j)$ satisfies $Df(1, e_j) = (g_j, e_j)$, then $D(T_{x^{-1}}\tilde{f}T_x)(y, e_j) = (g_i^{-1}f_v(y)g_j, e_j)$. By the definition of rotationless, if a direction is periodic for $D(T_{x^{-1}}\tilde{f}T_x)$, it is fixed, so the same is true for $D\tilde{f}$. □

The following proposition provides a criterion for showing that a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ has a rotationless iterate.

Proposition 5.7. *Let $f\colon \mathcal{G} \to \mathcal{G}$ be a relative train track map. Suppose that each infinite vertex group $\mathcal{G}_v$ for $v \in G$ has a bound on the order of finite order elements. If $(1, e_i)$ is an almost periodic direction at v satisfying say $Df(1, e_i) = (g_i, e_j)$, suppose additionally that there exists $k > 0$ such that the automorphism $f_{v,i}\colon \mathcal{G}_v \to \mathcal{G}_v$ defined as $f_{v,i}(x) = g_i^{-1}f_v(x)g_i$ has the property that if an element x is $f_{v,i}^k$-periodic, it is fixed. If the automorphism f_v also has this property, then $f\colon \mathcal{G} \to \mathcal{G}$ has a rotationless iterate.*

Proof. Since there are finitely many principal vertices and finitely many directions at principal vertices with finite vertex group, and finitely many almost periodic directions of the form $(1, e_i)$ at vertices with infinite vertex group, there is $N > 0$ such that each principal vertex for $f^N\colon \mathcal{G} \to \mathcal{G}$ is fixed, each almost periodic (and hence periodic) direction at a vertex with finite vertex group is fixed, and each almost periodic direction at a vertex with infinite vertex group is almost fixed. By increasing N we may additionally assume that the condition in the proposition holds for each almost periodic direction at each vertex with infinite vertex group.

Write $g = f^N$. Suppose that v is a vertex with infinite vertex group and that a direction (x, e_i) is Dg-periodic. Then $(1, e_i)$ is Dg-almost periodic, hence $Dg(1, e_i) = (g_i, e_i)$ for some $g_i \in \mathcal{G}_v$. Suppose $Dg^k(x, e_i) = (x, e_i)$. By definition, we have

$$Dg^k(x, e_i) = (g_v^k(x)g_v^{k-1}(g_i)g_v^{k-2}(g_i)\cdots g_v(g_i)g_i, e_i),$$

so writing $h = g_v(x)g_ix^{-1}$, we see that

$$g_v^k(x)g_v^{k-1}(g_i)g_v^{k-2}(g_i)\cdots g_v(g_i)g_ix^{-1} = g_v^{k-1}(h)g_v^{k-2}(h)\cdots g_v(h)h = 1.$$

Again by definition we have

$$Dg^{k+1}(x,e_i) = (g_v^k(h)g_v^{k-1}(h)g_v^{k-2}(h)\cdots g_v(h)hx, e_i) = (hx, e_i),$$

so it follows that h is g_v-periodic and hence fixed, since we have

$$h = g_v^k(h)g_v^{k-1}(h)\cdots g_v(h)h = g_v^k(h).$$

But then $Dg^k(x,e_i) = (h^k x, e_i) = (x, e_i)$, so h has finite order. In fact, the period of (x,e_i) is the order of h.

Now suppose the direction d is periodic for the map $d \mapsto g_i^{-1}.Dg(d)$. Write $d = (x, e_j)$ and suppose $(g_i^{-1}.Dg)^k(d) = d$. Then the direction $(1, e_j)$ is almost periodic, so satisfies $Dg(1,e_j) = (g_j, e_j)$ for some $g_j \in \mathcal{G}_v$. By definition we have

$$g_i^{-1}.Dg(x,e_j) = (g_i^{-1}g_v(x)g_j, e_j) = (g_i^{-1}g_j g_{v,j}(x), e_j),$$

where $f_{v,j}\colon \mathcal{G}_v \to \mathcal{G}_v$ is, as in the statement, the automorphism $x \mapsto g_j^{-1}g_v(x)g_j$. Write $h' = x^{-1}g_i^{-1}g_j g_{v,j}(x)$. For $\ell \geq 1$, we have

$$(g_i^{-1}.Dg)^\ell(x,e_j) = (xhg_{j,v}(h')\cdots g_{j,v}^{\ell-1}(h'), e_j).$$

The condition that $(g_i^{-1}.Dg)^k(x,e_j) = (x,e_j)$ says that $h'g_{v,j}(h')\cdots g_{v,j}^{k-1}(h') = 1$, and the argument above shows that h' is $g_{v,j}$-periodic and hence fixed, and thus that it must have finite order. In fact, the period of (x,e_j) is the order of h'.

By assumption, there exists $M \geq 1$ such that if x is an element of an infinite vertex group of finite order, then $x^M = 1$. The arguments in the previous paragraphs show that g^{NM} is rotationless. ☐

Corollary 5.8. *If each infinite vertex group of $\mathcal{G}$ is a free product of the form $B_1 * \cdots * B_m * F_\ell$, where the B_i are finite groups and F_ℓ is free of finite rank ℓ, every relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ has a rotationless iterate.*

In principal, it appears the statement should hold for vertex groups which are virtually finite-rank free.

Proof. Suppose $\mathcal{G}_v$ is a free product of the form in the statement. Any finite order element x of $\mathcal{G}_v$ is conjugate into some B_i, so there is a bound on the order of x. Therefore by Proposition 5.7 it suffices to show that there exists $k > 0$ such that for each automorphism Φ of $\mathcal{G}_v$, if an element of $\mathcal{G}_v$ is Φ^k-periodic, it is fixed. Let $\Pi(\Phi)$ denote the subgroup of $\mathcal{G}_v$ consisting of all Φ-periodic elements. Then $\Pi(\Phi)$ is again a free product of finite and cyclic groups and the restriction $\Phi|_{\Pi(\Phi)}$ is a periodic automorphism. Suppose toward a contradiction that $x_0,\ldots,x_{m+\ell}$ are $m+\ell+1$ elements of distinct free factors for $\Pi(\Phi)$. There exists $N \geq 1$ such that $x_0,\ldots,x_n$ are all Φ-periodic of period dividing N. Thus they belong to the fixed subgroup of Φ^N, which has rank bounded by $m+\ell$ by the main theorem of [CT94] (the Scott conjecture), a contradiction. Therefore $\Pi(\Phi)$ has a finite free product decomposition with factors that are subgroups of the B_i. It follows that the free product $\Pi(\Phi)$ takes on finitely many values as Φ varies, and that the minimum index of a finite-index free subgroup of $\Pi(\Phi)$ and the rank of this minimum-index free subgroup are both bounded independent of Φ. We see that some uniform power of Φ restricts to a periodic automorphism of a free group of finite rank. The order of this restriction is bounded depending only on the rank, and we claim that except in the case where $\Pi(\Phi) = C_2 * C_2$, an automorphism of $\Pi(\Phi)$ fixing this finite-index free subgroup fixes the whole group. Since periodic automorphisms of $C_2 * C_2$ clearly have bounded order, this will complete the proof.

We are grateful to Sami Douba for explaining the following argument. Write $G = \Pi(\Phi)$. The statement is true for F_1, so assume that $G \neq F_1$. Since $G \neq C_2 * C_2$ nor F_1 and G is a free product of finite and cyclic groups, it is a hyperbolic group that acts effectively on its Gromov boundary ∂G. Let $H < G$ be a subgroup of finite index. We claim that if

$\Psi: G \to G$ is an automorphism such that $\Psi|_H$ is the identity, then Ψ is the identity. There exists $M > 0$ such that if $g \in G$ has infinite order, then $g^M \in H$. Each such g has a unique attracting fixed point $\hat{T}_g^+$ in ∂G, and Ψ induces a Ψ-twisted equivariant homeomorphism $\hat{\Psi}: \partial G \to \partial G$. We have

$$T_g^+ = T_{g^M}^+ = T_{\Psi(g^M)}^+ = T_{\Psi(g)^M}^+ = T_{\Psi(g)}^+ = \hat{\Psi}(T_g^+).$$

Since the set of attracting fixed points of elements of infinite order is dense in ∂G, we have that $\hat{\Psi}$ is the identity. Therefore for any element $g \in G$ and $\xi \in \partial G$, we have

$$g.\xi = \hat{\Psi}(g.\xi) = \Psi(g).\hat{\Psi}(\xi) = \Psi(g).\xi.$$

Because the action of G on ∂G is effective, we conclude that $g = \Psi(g)$. $\square$

Lemma 5.9 (cf. Lemma 3.19 of [FH11]). *Suppose that $f: \mathcal{G} \to \mathcal{G}$ is a relative train track map satisfying the conclusions of Theorem 2.6. For every exponentially growing stratum H_r, there is a principal vertex whose link contains a periodic or almost periodic direction in H_r.*

Proof. If v has nontrivial vertex group not equal to C_2, we are done, so suppose all vertices of H_r have vertex group trivial or C_2. If some vertex $v \in H_r$ belongs to a non-contractible component of G_{r-1} we know that v is periodic by Lemma 1.5. By (EG-i), some iterate of Df induces a self-map of the directions based at v in H_r and a self-map of the directions based at v in G_{r-1}. Therefore there are at least two periodic directions based at v, at least one in H_r and one out of H_r, so v is principal.

So suppose no vertex of v belongs to a non-contractible component of G_{r-1}. Then H_r^z is a union of components of G_r, and in fact $H_r^z = H_r$. We assume as above that all nontrivial vertex groups are C_2. We may also suppose that each periodic point of H_r is not the endpoint of a periodic almost Nielsen path—that is, H_r contains no periodic almost Nielsen paths.

The argument is similar to [BFH04, Lemma 5.2]. Let τ_1 and τ_2 be legal paths in H_r so that the initial ends of $\bar{\tau}_1$ and τ_2 determine an illegal but nondegenerate turn. By Lemma 2.3 and the proof of Lemma 2.4, if τ_1 and τ_2 are sufficiently long then for some $k > 0$, we have that $f_\sharp^k(\tau_1\tau_2)$ is legal and not all of $f_\sharp^k(\bar{\tau}_1)$ and $f_\sharp^k(\tau_2)$ is canceled when $f_\sharp^k(\tau_1)f_\sharp^k(\tau_2)$ is tightened to $f_\sharp^k(\tau_1\tau_2)$. We may write $f_\sharp^k(\tau_1) = \mu_1\eta$ and $f_\sharp^k(\tau_2) = \bar{\eta}\mu_2$ such that $f_\sharp^k(\tau_1\tau_2) = \mu_1\mu_2$ for legal paths μ_1, μ_2 and η. By assumption, each of the three turns determined by these paths is a legal turn. By assumption there are finitely many directions in H_r, so we may iterate until each of the directions determined by $\bar{\mu}_1$, $\bar{\mu}_2$ and η are fixed; they remain distinct. Each direction determines an attractor for $\hat{f}$ in a way that we now describe. This shows that the common vertex of each of these fixed directions is principal.

Suppose that an edge E of H_r determines a fixed direction, that $\tilde{E}$ is a lift of E and that $\tilde{f}: \Gamma \to \Gamma$ is a lift of f that fixes the direction determined by $\tilde{E}$. Then $f(E) = Eu$. Since $f|_{H_r}$ is an irreducible train track map, for all $k \geq 1$, we have $[f^k(E)] = Euf(u)f^2(u)\dots f^{k-1}(u)$. Let $\tilde{u}$ denote the lift of u that begins at the terminal endpoint of $\tilde{E}$ and let

$$\tilde{\gamma} = \tilde{E}\tilde{u}\tilde{f}(\tilde{u})\tilde{f}^2(\tilde{u})\dots.$$

Again by irreducibility of $f|_{H_r}$, the length of $f^k(u)$ goes to infinity as k goes to infinity, so by Proposition 3.4, the endpoint ξ of $\tilde{\gamma}$ is a (superlinear) attractor for $\tilde{f}$. $\square$

Notice that Lemma 5.9 implies that the transition matrix of an exponentially growing stratum of a rotationless relative train track map $f: \mathcal{G} \to \mathcal{G}$ representing $\varphi \in \mathrm{Out}(F, \mathcal{A})$ and satisfying the conclusions of Theorem 2.6 has at least one nonzero diagonal entry and so is aperiodic. This defines a bijection between $\mathcal{L}(\varphi)$ and the set of exponentially growing strata of f.

Our next goal is to relate principal points in Γ to principal automorphisms, with the eventual goal of showing that a rotationless relative train track map represents a rotationless outer automorphism.

Lemma 5.10 (cf. Lemma 3.21 of [FH11])**.** *Suppose that $\tilde f\colon \Gamma \to \Gamma$ is a principal lift of a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$.*

1. *For each attractor $P \in \mathrm{Fix}_N(\hat f) \cap \partial_\infty(F,\mathcal{A})$ there is a (not necessarily unique) fixed point $\tilde x \in \mathrm{Fix}(\tilde f)$ such that the interior of the ray $\tilde R_{\tilde x,P}$ that starts at $\tilde x$ and ends at P is fixed point free.*
2. *If $P \in \mathrm{Fix}_N(\hat f) \cap \partial_\infty(F,\mathcal{A})$ is an attractor, $\tilde x$ is a fixed point and if the interior of the ray $\tilde R_{\tilde x,P}$ is fixed point free, then no point in the interior of $\tilde R_{\tilde x,P}$ is mapped to $\tilde x$ by $\tilde f$. It follows that the initial direction determined by $\tilde R_{\tilde x,P}$ is fixed.*
3. *If P and Q are distinct attractors in $\mathrm{Fix}_N(\hat f)$, if $\tilde x$ is a fixed point and if the interiors of both rays $\tilde R_{\tilde x,P}$ and $\tilde R_{\tilde x,Q}$ are fixed point free then the directions determined by $\tilde R_{\tilde x,P}$ and $\tilde R_{\tilde x,Q}$ are distinct.*

Proof. To find $\tilde x \in \mathrm{Fix}(\tilde f)$ and $\tilde R_{\tilde x,P}$ as in item 1, begin with any ray $\tilde R'$ that begins in $\mathrm{Fix}(\tilde f)$ and converges to P. Since P is not in the boundary of the fixed subgroup of $f_\sharp$, there is a last point $\tilde x$ of $\mathrm{Fix}(\tilde f)$ in $\tilde R'$. Item 3 follows from Lemma 5.4; the line from P to Q in Γ contains a fixed point. That fixed point must be $\tilde x$, thus it follows that $\tilde R_{\tilde x,P}$ and $\tilde R_{\tilde x,Q}$ determine distinct directions at $\tilde x$. Similarly if some point $\tilde y$ in the interior of $\tilde R_{\tilde x,P}$ maps to $\tilde x$ by $\tilde f$, then there is a fixed point in the interior of $\tilde R_{\tilde x,P}$, so item 2 follows. □

Corollary 5.11 (cf. Corollary 3.22 of [FH11])**.** *Assume that $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map satisfying the conclusions of Theorem 2.6. If $\tilde f$ is a principal lift of f, then each element of $\mathrm{Fix}(\tilde f)$ is principal.*

Proof. Let Φ be the automorphism corresponding to $\tilde f$. If $\mathrm{Fix}(\Phi)$ is not F_1, $C_2 * C_2$ or some subgroup of an A_i, then $\mathrm{Fix}(\tilde f)$ is neither a single point nor a single axis and we are done. If $\mathrm{Fix}(\Phi)$ is either F_1 or $C_2 * C_2$, then $\mathrm{Fix}(\tilde f)$ is infinite and $\mathrm{Fix}_N(\hat f)$ contains an attractor or a point in $V_\infty(F,\mathcal{A})$. In either case (with the former being a consequence of Lemma 5.10) some $\tilde x \in \mathrm{Fix}(\tilde f)$ has a fixed direction that does not come from a fixed edge or an almost Nielsen path with one endpoint in $V_\infty(F,\mathcal{A})$ and we are done. In the remaining case, $\mathrm{Fix}(\Phi)$ is a possibly trivial subgroup of some A_i and $\mathrm{Fix}(\hat f) \cap \partial_\infty(F,\mathcal{A})$ is a set of attractors and does not contain the endpoints of any axis. Clearly $\mathrm{Fix}(\tilde f)$ is not an axis. If $\mathrm{Fix}(\tilde f)$ is a single vertex $\tilde x$ with infinite stabilizer in $\mathbb{T}$, Lemma 5.10 implies that $\tilde x$ has a fixed direction and we are done. So suppose $\tilde x$ has finite (possibly trivial) stabilizer. The only case where $\tilde x$ is *not* principal is if there are only two periodic directions at $\tilde x$ and these two directions are determined by lifts of oriented edges $\tilde E_1$ and $\tilde E_2$ that belong to the same exponentially growing stratum. Then Lemma 4.10 together with Lemma 5.10 implies that $\mathrm{Fix}_N(\hat f)$ is the endpoint set of a lift of a generic leaf of an attracting lamination, contradicting the assumption that $\tilde f$ is principal. Therefore we conclude that $\tilde x$ is principal. □

Lemma 5.12 (cf. Lemma 3.23 of [FH11])**.** *Suppose that $\mathrm{Fix}(\tilde f)$ is empty. Then there is a ray $\tilde R \subset \Gamma$ converging to an element $P \in \mathrm{Fix}(\hat f)$ and there are points in $\tilde R$ arbitrarily close to P that move toward P.*

Proof. The proof is identical to [FH11, Lemma 3.23]. Given a vertex $\tilde y$ of Γ, say that the initial edge of the tight path from $\tilde y$ to $\tilde f(\tilde y)$ is *preferred* by $\tilde y$. Begin with some vertex $\tilde y_0$ and inductively define $\tilde y_{i+1}$ to be the other endpoint of the edge preferred by $\tilde y_i$. If some edge $\tilde E$ is preferred by *both* of its endpoints, then $\tilde f$ maps some subinterval of $\tilde E$ over all of $\tilde E$, reversing orientation. It follows that $\mathrm{Fix}(\tilde f) \neq \varnothing$. Therefore we conclude that the vertices $\tilde y_i$ are contained in a ray that converges to some $P \in \mathrm{Fix}(\hat f)$, and that $\tilde y_i$ moves toward P. □

The main consequence of Lemma 5.12 is the following result, for which we set notation.

Restricting to G_{r-1} for non-exponentially growing strata. Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map satisfying the conclusions of Theorem 2.6, that H_r is a single edge E_r, and that $f(E_r) = E_r u$ for some nontrivial path $u \subset G_{r-1}$. Fix a lift $\tilde{E}_r$ of E_r and let $\tilde{f}\colon \mathcal{G} \to \mathcal{G}$ be the lift of f that fixes the initial endpoint of $\tilde{E}_r$. By property (NEG), the component C of G_{r-1} containing the terminal endpoint of E_r is non-contractible. Denote the copy of the Bass–Serre tree for C that contains the terminal endpoint of $\tilde{E}_r$ by Γ_{r-1} and the restriction of $\tilde{f}$ to Γ_{r-1} by $h\colon \Gamma_{r-1} \to \Gamma_{r-1}$.

The elements of F (thought of as automorphisms of the projection $\Gamma \to \mathcal{G}$) that preserve Γ_{r-1} define a free factor $F(C)$ of positive complexity such that $[[F(C)]] = [[\pi_1(C)]]$. The closure in $\partial(F,\mathcal{A})$ of $\{\hat{T}_c^{\pm} : c \in F(C)\}$ is naturally identified with $\partial(F(C),\mathcal{A}|_{F(C)})$ and with the Bowditch boundary of Γ_{r-1}. We have $\hat{h} = \hat{f}|_{\partial\Gamma_{r-1}} \colon \partial(F(C),\mathcal{A}|_{F(C)}) \to \partial(F(C),\mathcal{A}|_{F(C)})$.

The following lemma is an immediate application of Lemma 5.12.

Lemma 5.13 (cf. Lemma 3.25 of [FH11]). *Assuming notation as in the previous paragraphs, if $\mathrm{Fix}(h) = \varnothing$ then there is a ray $\tilde{R} \subset \Gamma_{r-1}$ converging to an element $P \in \mathrm{Fix}(\hat{h})$ and points in $\tilde{R}$ arbitrarily close to P that move toward P.*

The following lemma is the key step in proving the correspondence between principal points in Γ and principal automorphisms.

Lemma 5.14 (cf. Lemma 3.26 of [FH11]). *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a rotationless relative train track map satisfying the conclusions of Theorem 2.6, that $\tilde{f}\colon \Gamma \to \Gamma$ is a lift of f, that $\tilde{v}$ is a fixed point for $\tilde{f}$ and that $D\tilde{f}$ fixes the direction at $\tilde{v}$ determined by an edge $\tilde{E}$ which is a lift of an edge $E \subset H_r$. Suppose that H_r is not the bottom half of a dihedral pair. There is a fixed point $P \in \mathrm{Fix}(\hat{f})$ such that the ray $\tilde{R}$ from the initial endpoint of $\tilde{E}$ to P contains $\tilde{E}$, projects into G_r and has the following properties provided that $P \in \partial_\infty(F,\mathcal{A})$.*

1. *There are points in $\tilde{R}$ arbitrarily close to P that either move toward P or are fixed by $\tilde{f}$. If $\mathrm{Fix}(\hat{f}) \neq \{\hat{T}_c^{\pm}\}$ for any nonperipheral $c \in F$, then P belongs to $\mathrm{Fix}_N(\hat{f})$.*
2. *If H_r is an exponentially growing stratum, then P is an attractor whose limit set is the unique attracting lamination of height r, the interior of $\tilde{R}$ is fixed point free and $\tilde{R}$ projects to an r-legal ray in G_r.*
3. *If H_r is non-exponentially growing but not almost fixed then $\tilde{R} \setminus \tilde{E}$ projects into G_{r-1}.*
4. *No point in the interior of $\tilde{R}$ is mapped to $\tilde{v}$ by any iterate of $\tilde{f}$. This item holds without the assumption that $P \in \partial_\infty(F,\mathcal{A})$.*

Proof. We follow the argument in [FH11, Lemma 3.26]. The second statement in item 1 follows from the first and Lemma 5.3.

We proceed by induction on r, beginning with $r = 1$. If G_1 is not almost fixed, then it is exponentially growing. Lemma 4.10 and Lemma 5.10 imply the existence of P satisfying the conclusions of the lemma.

If G_1 is almost fixed, then since we assume that H_1 is not the bottom half of a dihedral pair, it consists of a single edge, and by property (F), we have that G_1 is its own core. Write $\mathbb{F} = \mathrm{Fix}(\Phi|_{\pi_1(G_1,v)})$. If the splitting induced by G_1 of $\mathbb{F}$ is nontrivial, then we may choose P to be the endpoint of any ray $\tilde{R}$ that begins with $\tilde{E}$ and is contained in $\mathrm{Fix}(\tilde{f})$. If the splitting of $\mathbb{F}$ is trivial, then since f is rotationless, at least one vertex group of G_1 is infinite, and we may take P to be a fixed point for $\tilde{f}$ with infinite valence connected to $\tilde{v}$ by a path of length at most two. This completes the $r = 1$ case, so assume the lemma holds for edges with height less than r.

If H_r is exponentially growing, then again the existence of P follows from Lemma 4.10 and Lemma 5.10. Therefore we may assume that H_r is non-exponentially growing. Suppose first that H_r is not almost fixed. Let $h\colon \Gamma_{r-1} \to \Gamma_{r-1}$ be as in the paragraph "Restricting to G_{r-1} for non-exponentially growing strata." If $\mathrm{Fix}(h)$ is nonempty, then the initial endpoint of $\tilde{E}$ and some $\tilde{x} \in \mathrm{Fix}(h)$ cobound an indivisible almost Nielsen path. If $\tilde{x}$ has infinite valence then it is the endpoint P. If $\tilde{x}$ has finite valence, then $\tilde{x}$ is principal, there is a fixed direction in Γ_{r-1} based at $\tilde{x}$, and the existence of P follows from the inductive hypothesis. The case where $\mathrm{Fix}(h) = \varnothing$ follows from Lemma 5.13.

If H_r is almost fixed but not a forest, then the argument for the base case applies. So suppose H_r is an almost fixed forest, that is not the bottom half of a dihedral pair, i.e. a single edge E with distinct endpoints v and w and at least one incident vertex group trivial. If w has infinite vertex group, the endpoint $\tilde{w}$ of $\tilde{E}$ is P and the ray $\tilde{R}$ is the edge $\tilde{E}$. So suppose w has finite, possibly trivial vertex group. By (F) and Lemma 2.11, each vertex with trivial vertex group is contained in a non-contractible component C of G_{r-1} and is thus principal for f. Since f is rotationless, $\tilde{v}$ is connected to a lift of this principal vertex by a path of fixed edges for $\tilde{f}$ beginning with $\tilde{E}$ of length at most two. The existence of $\tilde{R}$ then follows from the inductive hypothesis. □

Suppose $f\colon \mathcal{G} \to \mathcal{G}$ is a topological representative with non-exponentially growing stratum H_i which is not almost periodic and consists of a single edge E_i. A *basic path of height* i is a path of the form $E_i\gamma$ or $E_i\gamma\bar{E}_i$, where $\gamma \subset G_{i-1}$ is a nontrivial tight path.

Lemma 5.15 (cf. Lemma 4.1.4 of [BFH00])**.** *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ and E_i are as above, and that $\sigma \subset G_i$ is a tight path that meets the interior of H_i and whose endpoints are not contained in the interior of E_i. The path σ has a splitting whose pieces are basic paths of height i or are contained in G_{i-1}.*

Proof. The proof is identical to [BFH00, Lemma 4.1.4]. Choose a lift $\tilde{f}\colon \Gamma \to \Gamma$ and a lift $\tilde{\sigma} \subset \Gamma$. A splitting of σ is determined by a set of points in $\tilde{\sigma}$. Fix $k > 0$. Suppose that $\tilde{x}$ is a point of $\tilde{\sigma}$ decomposing $\tilde{\sigma}$ into $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ and that $f^k(\tilde{x})$ belongs to $\tilde{f}^k_\sharp(\tilde{\sigma})$. Then $\tilde{f}^k_\sharp(\tilde{\sigma}) = \tilde{f}^k_\sharp(\tilde{\sigma}_1)\tilde{f}^k_\sharp(\tilde{\sigma}_2)$. Observe further that the set of points $\tilde{x}$ satisfying $\tilde{f}^k(\tilde{x}) \in \tilde{f}^k_\sharp(\tilde{\sigma})$ is closed.

Notice that there is an initial segment E_i^k of E_i such that $f^k(E_i) = E_i$, and that no other points of G_i are mapped into the interior of E_i by f^k. Suppose a copy of E_i cancels with a copy of $\bar{E}_i$ when $f^k(\sigma)$ is tightened to $f^k_\sharp(\sigma)$. Then there is a subpath μ of σ connecting a copy of E_i^k to a copy of $\bar{E}_i^k$ such that $f^k_\sharp(\mu)$ is trivial (with trivial vertex group element). But since μ is a closed path and f is a homotopy equivalence, this is impossible. Therefore no such cancellation occurs. The argument in the previous paragraph shows that if $\tilde{x}$ is a point in the interior of a lift of E_i^k or $\bar{E}_i^k$ decomposing $\tilde{\sigma}$ into $\tilde{\sigma}_1\tilde{\sigma}_2$, then $\tilde{f}^k_\sharp(\tilde{\sigma}) = \tilde{f}^k_\sharp(\tilde{\sigma}_1)\tilde{f}^k_\sharp(\tilde{\sigma}_2)$. In fact, the same holds true if $\tilde{x}$ is the initial vertex of any lift of $\tilde{E}_i$ or the terminal vertex of any lift of $\bar{E}_i$. Since k was arbitrary, $\tilde{\sigma}$ can be split at these points. It follows from the definition of a splitting that $\tilde{\sigma}$ can be split at all of these points simultaneously, and the result follows. □

Lemma 5.16. *Suppose $f\colon \mathcal{G} \to \mathcal{G}$ is rotationless and that γ is an indivisible almost Nielsen path. Then at least one of the initial and terminal directions of γ is fixed.*

Proof. Suppose r is the highest stratum such that γ meets the interior of H_r. H_r cannot be a zero stratum, nor can it be almost fixed since γ is assumed to be indivisible. If H_r is exponentially growing, the lemma follows from Lemma 2.4. If H_r is non-exponentially growing, then it is a single edge E_r and the lemma follows from Lemma 5.15: up to reversing the orientation of γ and multiplying by vertex group elements at the ends, we have $\gamma = E_r\sigma$ or $\gamma = E_r\sigma\bar{E}_r$ for a nontrivial path $\sigma \subset G_{r-1}$. In the former case the initial direction of γ is fixed; in the latter both the initial and terminal directions are fixed. □

Now, under the assumption that $f\colon \mathcal{G} \to \mathcal{G}$ is rotationless, we can prove the converse of Corollary 5.11.

Corollary 5.17 (cf. Corollary 3.27 of [FH11]). *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map satisfying the conclusions of Theorem 2.6 and is rotationless. If some, and hence every, $\tilde{x} \in \mathrm{Fix}(\tilde{f})$ is principal and $\mathrm{Fix}(\tilde{f})$ projects to a non-exceptional almost Nielsen class, then $\tilde{f}$ is principal.*

Proof. We follow the proof in [FH11, Corollary 3.27]. Assume that each point in $\mathrm{Fix}(\tilde{f})$ is principal. If $\mathrm{Fix}(\tilde{f})$ contains two points of infinite valence, then $\tilde{f}$ is principal.

Assume that $\mathrm{Fix}(\tilde{f})$ contains a vertex $\tilde{v}$ of infinite valence. If there is a fixed direction based at $\tilde{v}$, Lemma 5.14 produces a second point in $\mathrm{Fix}_N(\hat{f})$, and we see that $\tilde{f}$ is principal. If there is no fixed direction at $\tilde{v}$, by the assumption that $\mathrm{Fix}(\hat{f})$ projects to a non-exceptional almost Nielsen class, there is a second point $\tilde{x}$ in $\mathrm{Fix}(\tilde{f})$, necessarily of finite valence. The tight path $\tilde{\gamma}$ from $\tilde{v}$ to $\tilde{x}$ projects to an almost Nielsen path γ for f. Since we assume that there is no fixed direction at $\tilde{v}$, by re-choosing $\tilde{x}$ we may assume that γ is indivisible, and therefore by Lemma 5.16 that the terminal direction of $\tilde{\gamma}$ is fixed by $D\tilde{f}$.

Let x be the vertex of G that $\tilde{x}$ projects to. Since $f\colon \mathcal{G} \to \mathcal{G}$ is rotationless, the restriction of Df to the set of almost periodic directions based at x is the identity. We saw in Section 1 that the map $D\tilde{f}$ is Df followed by left-multiplication by some element of $\mathcal{G}_x$; in other words, if the edge $\tilde{e}$ corresponds to an almost periodic direction (x, e) based at x, the direction $D\tilde{f}(\tilde{e})$ corresponds to (hx, e) for some element $h \in \mathcal{G}_x$. Since there is a $D\tilde{f}$-fixed direction based at $\tilde{x}$, we must have $h = 1$. It follows by Lemma 2.8 that there is a second periodic and hence fixed direction based at $\tilde{x}$, producing by Lemma 5.14 a second point in $\mathrm{Fix}_N(\hat{f})$, and we are done.

Suppose every fixed point $\tilde{x}$ has finite valence. The argument in the previous paragraphs produces a fixed point $\tilde{x}$ with a fixed direction. By Lemma 2.8, there is a legal turn based at $\tilde{x}$, so there are at least two periodic and hence fixed directions at $\tilde{x}$. As in [FH11], observe that if some $\tilde{x} \in \mathrm{Fix}(\tilde{f})$ has three fixed directions, then Lemma 5.14 produces at least three points in $\mathrm{Fix}_N(\hat{f})$, and we are done. So suppose that there are either zero or exactly two fixed directions at each $\tilde{x} \in \mathrm{Fix}(\tilde{f})$. By the assumption that each $\tilde{x}$ is principal, if $\mathrm{Fix}(\tilde{f})$ contains an edge, then $\mathrm{Fix}(\tilde{f})$ may not be a line, so some edge of $\mathrm{Fix}(\tilde{f})$ is incident to a valence-one vertex $\tilde{w}$ of $\mathrm{Fix}(\tilde{f})$. By Lemma 2.11, $\tilde{w}$ projects to a valence-one vertex w of some fixed stratum, so it is contained in a lower filtration element; by property (F), in proving Lemma 2.11 there is no loss in assuming that this filtration element G_k is its own core. But then Lemma 2.8 implies that there is a legal turn based at w in G_k, and thus $\tilde{w}$ has at least three fixed directions, since we assume $\tilde{w}$ has finite valence. Thus by this contradiction we may assume that $\mathrm{Fix}(\tilde{f})$ contains no edges.

Choose an edge E in H_r and a lift $\tilde{E}$ whose initial direction is fixed and based at some $\tilde{x} \in \mathrm{Fix}(\tilde{f})$. Let $\tilde{R}$ be the ray that begins with $\tilde{E}$ and ends at some fixed point $P \in \mathrm{Fix}(\hat{f})$ as in Lemma 5.14. Since we assume every fixed point in $\mathrm{Fix}(\tilde{f})$ has finite valence, we have that $P \in \partial_\infty(F, \mathcal{A})$. If H_r is exponentially growing, then the limit set of P is an attracting lamination, which implies by Lemma 4.9 that P is not the endpoint of an axis. If H_r is non-exponentially growing, then the ray $\tilde{R} \setminus \tilde{E}$ is contained in G_{r-1}, so P is not the endpoint of an axis that contains $\tilde{E}$. It follows that the line composed of $\tilde{R}$ and the ray determined by the second fixed direction at $\tilde{x}$ is not an axis. It follows that $\mathrm{Fix}(\hat{f})$ is not equal to $\{\hat{T}_c^{\pm}\}$ for any nonperipheral $c \in F$ and that every point in $\mathrm{Fix}(\hat{f})$ produced by Lemma 5.14 is contained in $\mathrm{Fix}_N(\hat{f})$. We have shown that $\mathrm{Fix}_N(\hat{f})$ contains at least two points and is not the endpoint set of an axis.

So assume arguing toward a contradiction that $\mathrm{Fix}_N(\hat{f})$ is the endpoint set of a lift $\tilde{\lambda}$ of a generic leaf of an attracting lamination. Since λ is birecurrent and contains E, the stratum H_r is exponentially growing and the second fixed direction based at $\tilde{x}$ comes from an edge of H_r. Item 2 of Lemma 5.14 implies that λ is r-legal and hence does not contain any indivisible almost Nielsen paths of height r. It follows that $\tilde{x}$ is the only fixed point in $\tilde{\lambda}$. Since $\mathrm{Fix}(\tilde{f})$ is principal it must contain a point other than $\tilde{x}$; that point would have a fixed direction that does not come from the initial edge of a ray converging to an endpoint of $\tilde{\lambda}$. This contradiction completes the proof. ☐

We now turn to proving that rotationless relative train track maps satisfying the conclusions of Theorem 2.6 represent rotationless outer automorphisms and vice versa.

Lemma 5.18 (cf. Lemma 3.28 of [FH11])**.** *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a rotationless relative train track map satisfying the conclusions of Theorem 2.6. Every periodic almost Nielsen path with principal endpoints is an almost Nielsen path (i.e. has period one).*

Proof. The proof follows [FH11, Lemma 3.28]. We may assume that our periodic almost Nielsen path σ is either a single (almost periodic) edge or an indivisible periodic almost Nielsen path. In the former case, σ is an almost periodic edge with a principal endpoint, so is almost fixed. Therefore we may assume that σ is indivisible.

The proof is by induction on the height r of σ. The base case of $r = 0$ is vacuous. The case that H_r is exponentially growing follows from the second paragraph of Lemma 2.4.

Therefore we may assume that H_r is a single non-exponentially growing edge E_r which is not almost fixed. Lemma 5.15 implies that up to reversing the orientation of σ and multiplying by vertex group elements at the ends we have $\sigma = E_r\mu$ or $E_r\mu\bar{E}_r$ for some path $\mu \subset G_{r-1}$. Let $\tilde{E}_r$ be a lift of E_r with initial endpoint $\tilde{v}$ and $\tilde{f}\colon \Gamma \to \Gamma$ the lift that fixes $\tilde{v}$ and the initial direction of $\tilde{E}_r$. Let $h\colon \Gamma_{r-1} \to \Gamma_{r-1}$ be as in the paragraph "Restricting to G_{r-1}". By Corollary 5.17, $\tilde{f}$ is principal. Let $\tilde{w}$ be the terminal endpoint of the lift of $\tilde{\sigma} = \tilde{E}_r\tilde{\mu}$ that begins with $\tilde{E}_r$.

If $\sigma = E_r\mu$, then $\tilde{w}$ belongs to Γ_{r-1}. If the period p of σ is not 1, then the tight path $\tilde{\tau} = h_\sharp(\tilde{\mu})$ connecting $\tilde{w}$ to $h(\tilde{w})$ projects to a nontrivial periodic almost Nielsen path $\tau \subset G_{r-1}$ that is closed because $\tilde{w}$ projects to $w \in \mathrm{Fix}(f)$. Since $\tilde{w}$ is principal, the inductive hypothesis implies that τ has period one. The projection of the closed path $\tilde{\tau}h(\tilde{\tau})\dots h^{p-1}(\tilde{\tau})$ is therefore homotopic to τ^p. But τ and therefore τ^p determine non-peripheral conjugacy classes in F, while $\tilde{\sigma}\tilde{\tau}h(\tilde{\tau})\dots h^{p-1}(\tilde{\tau})$ is homotopic to $\tilde{\sigma}$, so this is a contradiction; we conclude that $p = 1$ in the case that $\sigma = E_r\mu$.

Suppose now that $\sigma = E_r\mu\bar{E}_r$. If $\mathrm{Fix}(h^p)$ is nonempty, then the tight paths $\tilde{\sigma}_1$ connecting $\tilde{v}$ to $\tilde{x} \in \mathrm{Fix}(h^p)$ and $\tilde{\sigma}_2$ connecting $\tilde{x}$ to $\tilde{w}$ are periodic almost Nielsen paths whose concatenation is homotopic to $\tilde{\sigma}$. By the preceding case σ_1 and σ_2 and hence σ have period one. Therefore assume that $\mathrm{Fix}(h^p) = \varnothing$.

Let $T_c\colon \Gamma \to \Gamma$ be the automorphism of the natural projection satisfying $T_c(\tilde{v}) = \tilde{w}$ and taking the initial direction of $\tilde{\sigma}$ to the terminal direction of $\tilde{\sigma}$. Then T_c is a nonperipheral element of F, it commutes with $\tilde{f}^p$ and its axis A_c is contained in Γ_{r-1}. Lemma 3.2 implies that $\hat{T}_c^\pm \in \mathrm{Fix}(\hat{h}^p)$. If Φ is the principal automorphism corresponding to $\tilde{f}$, then we have that $\hat{T}_{\Phi(c)} = \hat{f}\hat{T}_c\hat{f}^{-1}$, which implies that $\hat{T}^\pm_{\Phi(c)} = \hat{h}(\hat{T}_c^\pm) \in \mathrm{Fix}(\hat{h}^p)$ and that $A_{\Phi(c)}$ is contained in Γ_{r-1}. If $\{\hat{T}_c^\pm\}$ is not $\hat{h}$-invariant, then $\mathrm{Fix}_N(\hat{h}^p)$ contains the four points $\{\hat{T}_c^\pm\} \cup \{\hat{h}(\hat{T}_c^\pm)\}$, and h^p is a principal lift of $f|_C$, where C is the component of G_{r-1} that contains the terminal endpoint of E_r. But $\mathrm{Fix}(h) = \varnothing$, which contradicts Corollary 5.5, so we conclude $\hat{T}_c^\pm \in \mathrm{Fix}(\hat{h})$. It follows that $\tilde{f}$ commutes with T_c, and hence that $\tilde{w} \in \mathrm{Fix}(\tilde{f})$, so $p = 1$. This completes the inductive step. □

The following is the main result of this section.

Proposition 5.19 (cf. Proposition 3.29 of [FH11])**.** *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a relative train track map representing $\varphi \in \mathrm{Out}(F, \mathcal{A})$ and satisfying the conclusions of Theorem 2.6. Then if $f\colon \mathcal{G} \to \mathcal{G}$ is rotationless, so is φ. If we assume additionally that each such f has a rotationless iterate, then if φ is rotationless, so is $f\colon \mathcal{G} \to \mathcal{G}$.*

Proof. We follow the proof of [FH11, Proposition 3.29]. Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is rotationless, write $g = f^k$ for some $k \geq 1$, and suppose that $\tilde{g}\colon \Gamma \to \Gamma$ is a principal lift of g. Corollary 5.5 and Corollary 5.11 imply that $\mathrm{Fix}(\tilde{g})$ is a nonempty set of principal fixed points projecting to a non-exceptional almost Nielsen class. Since f is rotationless, for each $\tilde{v} \in \mathrm{Fix}(\tilde{g})$, by Lemma 5.6 there is a lift $\tilde{f}\colon \Gamma \to \Gamma$ that fixes $\tilde{v}$ and all the same directions at $\tilde{v}$ that $\tilde{g}$ does.

Observe that if Φ_1 and Φ_2 are distinct representatives of φ, then $\text{Fix}_N(\hat{\Phi}_1) \cap \text{Fix}_N(\hat{\Phi}_2)$ is contained in $\text{Fix}(\hat{\Phi}_1^{-1}\hat{\Phi}_2) = \text{Fix}(\hat{T}_c)$, which is either empty, a single point in $V_\infty(F, \mathcal{A})$ or the endpoints $\{\hat{T}_c^{\pm}\}$ of an axis. Therefore, if Φ_1 and Φ_2 are distinct and principal, then $\text{Fix}_N(\hat{\Phi}_1) \neq \text{Fix}_N(\hat{\Phi}_2)$.

It therefore suffices to show that $\text{Fix}_N(\hat{f}) = \text{Fix}_N(\hat{g})$, since the argument in the previous paragraph shows that this implies $\tilde{f}^k = \tilde{g}$, so the map $\Phi \mapsto \Phi^k$ will be a surjection $P(\varphi) \to P(\varphi^k)$, and thus (again by the previous paragraph) a bijection.

Suppose $\tilde{\sigma}$ is the tight path connecting $\tilde{v}$ to another point in $\text{Fix}(\tilde{g})$. This path projects to an almost Nielsen path for $g = f^k$, and hence by Lemma 5.18 to an almost Nielsen path for f. Thus we see that $\text{Fix}(\tilde{f}) = \text{Fix}(\tilde{g})$, and thus $\hat{f}$ and $\hat{g}$ have the same fixed points in $V_\infty(F, \mathcal{A})$. It also follows that $\tilde{g}$ and $\tilde{f}$ commute with the same nonperipheral automorphisms of the natural projection, and Proposition 3.4 implies that $\text{Fix}_N(\hat{f})$ and $\text{Fix}_N(\hat{g})$ have the same non-isolated fixed points in $\partial_\infty(F, \mathcal{A})$.

Each isolated point $P \in \text{Fix}_N(\hat{g}) \cap \partial_\infty(F, \mathcal{A})$ is an attractor for $\hat{g}$. It suffices to show that $P \in \text{Fix}_N(\hat{f})$. By Lemma 5.10, there is a ray $\tilde{R}$ that terminates at P that intersects $\text{Fix}(\tilde{g})$ only in its initial endpoint and whose initial direction is fixed by $D\tilde{g}$ and thus by $D\tilde{f}$. Let us assume that the height r of the initial edge $\tilde{E}$ of $\tilde{R}$ is minimal among all choices of $\tilde{R}$ converging to P. By Lemma 5.14, there is a ray $\tilde{R}'$ that converges to some $P' \in \text{Fix}(\hat{f})$. It suffices to show that $P = P'$, since a repeller for $\hat{f}$ cannot be an attractor for $\hat{g}$. If H_r is exponentially growing, then item 2 of Lemma 5.14 implies that P' is an attractor for $\hat{f}$ (and hence for $\hat{g}$) and item 3 of Lemma 5.10 applied to $\hat{g}$ implies that $P = P'$. We may therefore assume that H_r is non-exponentially growing. If $\tilde{E}$ is fixed, or if there is a $\tilde{g}$-fixed point $\tilde{x}$ in $\tilde{R}' \setminus \tilde{E}$, then the ray connecting $\tilde{x}$ to P is in G_{r-1}, in contradiction to our choice of $\tilde{R}$. Therefore we may assume that $\text{Fix}(\tilde{g})$ intersects the interior of $\tilde{R}'$ trivially. Item 1 of Lemma 5.14 implies there exists $\tilde{x} \in \tilde{R}'$ that is moved towards P' by f. Then Lemma 5.4 implies that $P = P'$. Therefore if $f \colon \mathcal{G} \to \mathcal{G}$ is rotationless, then φ is rotationless.

Now suppose φ is rotationless and that $f \colon \mathcal{G} \to \mathcal{G}$ is a relative train track map satisfying the conclusions of Theorem 2.6 and representing φ. By assumption there is a rotationless iterate g satisfying $g = f^k$ for some $k > 0$. Given a principal vertex $v \in \text{Fix}(g)$, there is a lift $\tilde{v}$ of v and a principal lift $\tilde{g}$ of g that fixes $\tilde{v}$. Since φ is rotationless, there is a (principal) lift $\tilde{f}$ of f such that $\tilde{f}^k = \tilde{g}$ satisfying $\text{Fix}_N(\hat{f}) = \text{Fix}_N(\hat{g})$. To complete the proof we must show that $\tilde{v}$ is fixed by $\tilde{f}$ and that each direction fixed by $D\tilde{g}$ is fixed by $D\tilde{f}$. Note that the assumption that $\text{Fix}_N(\hat{g}) = \text{Fix}_N(\hat{f})$ implies that if $\tilde{x}$ is a vertex with infinite stabilizer fixed by $\tilde{g}$, then it is fixed by $\tilde{f}$.

Suppose that there is at most a single $D\tilde{g}$-fixed direction at $\tilde{v}$. Then $\tilde{v}$ has infinite stabilizer and is thus fixed by $\tilde{f}$. There is at most one $D\tilde{f}$-periodic direction at $\tilde{v}$. Suppose there is a $D\tilde{g}$-fixed direction $\tilde{d}$ at $\tilde{v}$. The edge determined by $\tilde{d}$ extends to a ray $\tilde{R}$ converging to a fixed point $P \in \text{Fix}_N(\hat{g}) = \text{Fix}_N(\hat{f})$. The ray $\tilde{R}$ satisfies $f_\sharp(\tilde{R}) = \tilde{R}$. If the direction $\tilde{d}$ is not fixed by $D\tilde{f}$, then there is a point $\tilde{x}$ in the interior of $\tilde{R}$ satisfying $\tilde{f}(\tilde{x}) = \tilde{v}$, contradicting item 4 of Lemma 5.14 applied to $\tilde{g} = \tilde{f}^k$, so we conclude the direction $\tilde{d}$ is fixed by $D\tilde{f}$.

Now suppose there are at least two $D\tilde{g}$-fixed directions $\tilde{d}_1$ and $\tilde{d}_2$ at $\tilde{v}$. (We no longer assume that $\tilde{v}$ has infinite stabilizer.) The edges determining these directions extend to rays $\tilde{R}_1$ and $\tilde{R}_2$ that converge to P_1 and P_2 in $\text{Fix}_N(\hat{g}) = \text{Fix}_N(\hat{f})$; denote the line connecting P_1 to P_2 by $\tilde{\gamma}$. We have $\tilde{f}_\sharp(\tilde{\gamma}) = \tilde{\gamma}$ and the turn $(\tilde{d}_1, \tilde{d}_2)$ is legal for $\tilde{g}$ and hence for $\tilde{f}$. If it is not the case that $\tilde{f}(\tilde{v})$ belongs to $\tilde{\gamma}$, then there exists $\tilde{y} \in \tilde{\gamma}$ such that $\tilde{f}(\tilde{v}) = \tilde{f}(\tilde{y})$. But then we have $\tilde{f}^k(\tilde{y}) = \tilde{f}^k(\tilde{v}) = \tilde{v}$, which contradicts item 4 of Lemma 5.14 applied to $\tilde{g}$. Therefore $\tilde{f}(\tilde{v})$ belongs to $\tilde{\gamma}$. Now suppose that $\tilde{f}(\tilde{v}) \neq \tilde{v}$. Write $\tilde{v}_0 = \tilde{v}$ and orient $\tilde{\gamma}$ so that $\tilde{v} < \tilde{f}(\tilde{v})$ in the order induced from the orientation. There exist $\tilde{v}_i \in \tilde{\gamma}$ for $1 \leq i \leq k$ such that $\tilde{v}_i < \tilde{v}_{i-1}$ and $\tilde{f}(\tilde{v}_i) = \tilde{v}_{i-1}$. But then $\tilde{f}^k(\tilde{v}_k) = \tilde{v}$, again contradicting item 4 of Lemma 5.14. Therefore $\tilde{f}(\tilde{v}) = \tilde{v}$. A final application of item 4 of Lemma 5.14 implies that the directions $\tilde{d}_i$ are fixed by $D\tilde{f}$. □

Proposition 5.20 (cf. Lemma 3.30 of [FH11])**.** *Suppose $\varphi \in \text{Out}(F, \mathcal{A})$ is rotationless.*

1. *Each periodic non-peripheral conjugacy class is fixed and each representative of that conjugacy class is fixed by some principal automorphism representing φ.*
2. *Each attracting lamination Λ^+ in $\mathcal{L}(\varphi)$ is φ-invariant.*
3. *A free factor of positive complexity that is invariant under an iterate of φ is φ-invariant.*

Proof. We follow in outline the argument in [FH11, Lemma 3.30].

Item 2 follows from Lemma 5.9, Proposition 5.19 and Lemma 4.7.

For item 1, suppose c is a non-peripheral element whose conjugacy class is φ^k-invariant for some $k \geq 1$. We will show that there is a principal automorphism $\Phi_k \in P(\varphi^k)$ that fixes c. By Lemma 3.2, this is equivalent to the condition that $\mathrm{Fix}_N(\hat{\Phi}_k)$ contains $\hat{T}_c^{\pm}$. Since φ is rotationless, we may assume that $k = 1$, completing the proof of item 1.

Let $f\colon \mathcal{G} \to \mathcal{G}$ be a relative train track map satisfying the conclusions of Theorem 2.6 and representing φ^k, and let $\tilde{f}\colon \Gamma \to \Gamma$ be a lift commuting with T_c. Since φ is rotationless, so is φ^k, and Proposition 5.19 implies f is rotationless. We show that we may assume $\mathrm{Fix}(\tilde{f})$ is nonempty, following [BFH00, Lemma 4.1.2]. We have $\tilde{f}_\sharp(A_c) = A_c$. The set $\tilde{S}_\ell = \{\tilde{x} \in A_c : \tilde{f}^\ell(\tilde{x}) \in A_c\}$ is closed. An induction argument reveals that $\tilde{f}^N$ maps $\bigcap_{\ell=1}^N \tilde{S}_\ell$ onto A_c for all $N \geq 1$. Since $\bigcap_{\ell=1}^N \tilde{S}_\ell$ is T_c-invariant and nonempty, it intersects each fundamental domain of A_c. It follows that the T_c-invariant set $\bigcap_{\ell=1}^\infty \tilde{S}_\ell$ is nonempty, and any T_c-orbit in $\bigcap_{\ell=1}^\infty \tilde{S}_\ell$ determines a splitting of the circuit σ (a splitting into a path) that A_c projects to in $\mathcal{G}$. This path is an almost Nielsen path for f, and we conclude there is a lift $\tilde{f}$ commuting with T_c with $\mathrm{Fix}(\tilde{f})$ nonempty. In particular by property (V), $\tilde{f}$ fixes a vertex of Γ.

If some fixed vertex $\tilde{v}$ has infinite valence, we are done, as $\tilde{f}$ is principal. So assume all fixed vertices $\tilde{v}$ have finite valence. We argue as in [BFH04, Lemma 5.2]. By Lemma 2.8, there are at least two fixed directions based at $\tilde{v}$. If some edge $\tilde{E}$ based at $\tilde{v}$ determines a fixed direction but not a fixed edge, then Lemma 5.14 produces a fixed point $P \in \mathrm{Fix}(\hat{f})$ that is not the endpoint of A_c. Suppose then that every fixed direction based at $\tilde{v}$ is determined by a fixed edge.

Thus there is no loss in supposing that $\mathrm{Fix}(\tilde{f}) = A_c$. The image of A_c in $\mathcal{G}$ is a subgraph of groups that is a component of $\mathrm{Fix}(f)$ that is either a topological circle or the quotient of $\mathbb{R}$ by the standard action of $C_2 * C_2$. After reordering the filtration, we may assume that this subgraph of groups is a filtration element G_i. By our standing assumption that $F \neq F_1$ or $C_2 * C_2$, there is a lowest stratum H_j above G_i that contains an edge incident to a vertex v of G_i. Because by assumption the edges of H_j do not determine fixed directions at v, it follows that H_j is an edge E_j and that $f(E_j) = E_j\sigma^m$ for some nonzero $m \neq 0$ by (NEG). Let $\tilde{E}_j$ be a lift of E_j with terminal endpoint in A_c and replace $\tilde{f}$ with the lift that fixes the initial endpoint of $\tilde{E}_j$. For this lift, A_c is setwise but not pointwise fixed, and either the initial vertex of $\tilde{E}_j$ has infinite valence, or by Lemma 2.8 and Lemma 5.14 there is a fixed point $P \in \mathrm{Fix}(\hat{f})$ that is not an endpoint of A_c. This completes the proof of the claim.

We now turn to the proof of item 3, following the argument in [FH11, Lemma 3.30]. Suppose that a free factor B of positive complexity is φ^k-invariant for some $k \geq 1$. If $B = F_1$, then it is φ-invariant by the first item of this lemma. If $B = C_2 * C_2$, let c be a generator of the index-two F_1 subgroup of B. By item 1, the conjugacy class of c is φ-invariant, and it follows that the conjugacy class of B is φ-invariant. So assume B is neither F_1 nor $C_2 * C_2$. Let $\mathcal{C}$ be the set of lines γ in $\mathcal{B}$ that are carried by B and for which there exists a principal lift Φ of an iterate of φ and a lift $\tilde{\gamma}$ of γ whose endpoints are contained in $\mathrm{Fix}_N(\hat{\Phi})$. Since φ is rotationless, each γ is $\varphi_\sharp$-invariant, so $\mathcal{C}$ is φ-invariant. It is clearly carried by B. It follows by uniqueness of $\mathcal{F}(\mathcal{C})$ as in Corollary 4.12 that $\mathcal{F}(\mathcal{C})$ is φ-invariant. So to complete the proof, we need to show that no proper φ-invariant free factor system of B carries $\mathcal{C}$.

Suppose to the contrary that such a free factor system $\mathcal{F}$ exists. By Theorem 2.6 there is a relative train track map $g\colon \mathcal{G}' \to \mathcal{G}'$ representing $\varphi^k|_B$ in which $\mathcal{F}$ is represented by a proper filtration element $G'_r \subset G'$. After replacing $\varphi|_B$ and g by iterates, we may assume they are rotationless. We claim that there is a principal vertex $v \in G'$ whose link contains an edge E of $G' \setminus G'_r$ that determines an almost fixed direction. If $G' \setminus G'_r$ contains an exponentially growing stratum, this follows from Lemma 5.9. If not, then the highest stratum is non-exponentially growing, thus it consists of a single edge whose initial vertex is principal. We claim that there is a principal lift $\tilde{g}\colon \Gamma' \to \Gamma'$ and a line $\tilde{\gamma}$ whose endpoints are contained in $\mathrm{Fix}_N(\hat{g})$ whose projected image γ crosses E and so is not carried by G'_r. This follows from Lemma 5.14: one endpoint of $\tilde{\gamma}$ is $\tilde{v}$ if $\mathcal{G}'_v$ is infinite, otherwise there are at least two periodic and hence fixed directions at v, one of which is determined by E. The automorphism $\Phi' \in P(\varphi^k|_B)$ determined by $\tilde{g}$ extends to a principal automorphism $\Phi \in P(\varphi^k)$ with $\mathrm{Fix}_N(\hat{\Phi}') \subset \mathrm{Fix}_N(\hat{\Phi})$. Thus we conclude $\gamma \in \mathcal{C}$ in contradiction to our choice of $\mathcal{F}$. $\square$

Corollary 5.21. *If φ is rotationless and B is a φ-invariant free factor of positive complexity, then $\theta = \varphi|_B$ is rotationless.*

Proof. The proof is identical to [FH11, Corollary 3.31]. Item 1 of the previous lemma handles the case $B = F_1$ and $B = C_2 * C_2$. Let $f\colon \mathcal{G} \to \mathcal{G}$ be a relative train track map satisfying the conclusions of Theorem 2.6 with associated filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ such that the conjugacy class of B is realized by G_i for some i. (We may assume that G_i is its own core by (F).) Proposition 5.19 implies that $f\colon \mathcal{G} \to \mathcal{G}$ is rotationless. The restriction of f to G_i is a rotationless relative train track map representing θ and satisfying the conclusions of Theorem 2.6, so by Proposition 5.19, we conclude that θ is rotationless. $\square$

6 CTs for free products

In this section we construct CTs for free products. We begin by recalling the necessary definitions from [FH11, Section 4].

Almost linear edges Given a non-peripheral element $c \in F$, let $[c]_u$ be the *unoriented conjugacy class determined by* c. That is, $[c]_u = [d]_u$ if d is conjugate to c or c^{-1}. If σ is a closed path, write $[\sigma]_u$ be the *unoriented conjugacy class determined by* σ thought of as a circuit.

Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a rotationless relative train track map satisfying the conclusions of Theorem 2.6. Each non-almost periodic non-exponentially growing stratum H_i is a single edge E_i satisfying $f(E_i) = g_iE_iu_i$ for a vertex group element g_i and for a nontrivial path $u_i \subset G_{i-1}$. If we assume additionally that the terminal vertex of E_i is fixed, (this will be part of the definition of a CT) then the path u_i is closed. Feighn and Handel note that u_i is sometimes called the *suffix* for E_i. If u_i is an almost Nielsen path, then we say that E_i is an *almost linear edge,* and a *linear edge* if it is a Nielsen path (that is, $f_\sharp(u_i) = u_i$ on the nose). In the situation of a linear edge, we define the *axis* for E_i to be $[w_i]_u$, where w_i is root free and $u_i = w_i^{d_i}$ for some nonzero integer d_i.

Suppose that E_i is an almost linear edge with suffix u_i satisfying $f_\sharp(u_i) = gu_ih$ for vertex group elements g and h in $\mathcal{G}_v$. Suppose that $f(E_i) = x_iE_i \cdot u_i$ is a splitting and that that E_i is also a linear edge for $f^k_\sharp$, i.e. that $f^k_\sharp(u_i \cdot f_\sharp(u_i) \cdots f^{k-1}_\sharp(u_i)) = u_i \cdot f_\sharp(u_i) \cdots f^{k-1}_\sharp(u_i)$. Let $g_0 = h_0 = 1$ and inductively define $g_\ell = f_v(g_{\ell-1})g$ and $h_\ell = hf_v(h_{\ell-1})$. Then we have

$$u_1 \cdot f_\sharp(u_i) \cdots f^{k-1}_\sharp(u_i) = g_0u_ih_0g_1u_i \dots g_{k-1}u_ih_{k-1}.$$

The condition that E_i is a linear edge for $f^k_\sharp$ implies that $g_k = 1$. An easy calculation shows that $g_{k+\ell} = f^\ell_v(g_k)g_\ell = g_\ell$. Since we have $h_{k+\ell}g_{k+\ell+1} = h_\ell g_{\ell+1}$ for $0 \le \ell \le k-2$, this implies in particular that $h_k = h_0$ and hence that $h_{k+\ell} = h_\ell$ for all $\ell \ge 0$. The argument in the proof of Proposition 5.7 implies that if f is rotationless, the condition that $g_k = h_k = 1$ implies that actually $g = h = 1$, so E_i was a linear edge to begin with.

If the terminal vertex of E_i is the center vertex of a non-fixed dihedral pair and u_i is contained in that dihedral pair, then u_i is a periodic Nielsen path of period two, as is $u_i f_\sharp(u_i)$. In this situation, we say that E_i is a *dihedral linear edge,* and we define the *axis* for E_i to be $[w_i]_u$ where w_i is root free and $u_i f_\sharp(u_i) = w_i^{d_i}$ for some nonzero integer d_i. In this case, if E and E' are the edges of the dihedral pair, we may (after reversing the orientation of w_i) write $w_i = \sigma\tau$, where $\sigma = Eg\bar{E}$, $\tau = E'g'\bar{E}'$ and g and g' denote the nontrivial elements of the respective copies of C_2. We have $f_\sharp(\sigma) = \tau$ and vice versa, and u_i is equal to some alternating product of σ and τ, say $u_i = (\sigma\tau)^{d_i}$ (in which case we say u_i is *even*) or $(\sigma\tau)^{d_i}\sigma$ (in which case we say u_i is *odd*), for some integer d_i, which must be nonzero in the even case.

If E_i and E_j are linear edges such that there exists nonzero integers d_i and d_j and a closed root-free almost Nielsen path w such that the suffixes u_i and u_j satisfy $u_i = w^{d_i}$ and $u_j = w^{d_j}$, where d_i and d_j have the same sign, then a path of the form $gE_i w^p \bar{E}_j h$ for integer p and vertex group elements g and h is called an *exceptional path.* Notice that if $E_i w^p \bar{E}_j$ is an exceptional path, then for $k \geq 0$ we have

$$f_\sharp^k(E_i w^p \bar{E}_j) = g_k E_i w^{p+k(d_i-d_j)} \bar{E}_j h_k$$

for vertex group elements g_k and h_k.

If E_i and E_j are dihedral linear edges with the same axis $w = \sigma\tau$ so that the suffixes u_i and u_j are alternating products of σ and τ, then a path of the form $gE_i\alpha\bar{E}_j h$, where α is an alternating product of σ and τ, and for vertex group elements g and h is sometimes exceptional and sometimes not according to the parities of u_i, u_j and α. We have the following cases. We always assume that d_i and d_j have the same sign. To ease notation, we assume that $f_\sharp(E_i) = E_i u_i$ and $f_\sharp(E_j) = E_j u_j$; the general calculation is essentially identical.

1. If u_i and u_j are even, i.e. $u_i = (\sigma\tau)^{d_i}$ and $u_j = (\sigma\tau)^{d_j}$, then $E_i(\sigma\tau)^p\bar{E}_j$ is an exceptional path, $E_i(\sigma\tau)^p\sigma\bar{E}_j$ is not and we have

$$f_\sharp(E_i(\sigma\tau)^p\bar{E}_j) = E_i(\tau\sigma)^{p-d_i+d_j}\bar{E}_j.$$

2. If u_i is even and u_j is odd, i.e. $u_j = (\sigma\tau)^{d_j}\sigma$ then both $E_i(\sigma\tau)^p\bar{E}_j$ and $E_i(\sigma\tau)^p\sigma\bar{E}_j$ are exceptional and we have

$$f_\sharp(E_i(\sigma\tau)^p\bar{E}_j) = E_i(\tau\sigma)^{p-d_i-d_j-1}\tau\bar{E}_j \text{ and } f_\sharp(E_i(\sigma\tau)^p\sigma\bar{E}_j) = E_i(\tau\sigma)^{p-d_i+d_j+1}\bar{E}_j.$$

3. If u_i and u_j are both odd, then $E_i(\sigma\tau)^p\bar{E}_j$ is an exceptional path, $E_i(\sigma\tau)^p\sigma\bar{E}_j$ is not and we have

$$f_\sharp(E_i(\sigma\tau)^p\bar{E}_j) = E_i(\sigma\tau)^{p+d_i-d_j}\bar{E}_j.$$

Therefore we have that $f_\sharp$ induces a height-preserving bijection on the set of exceptional paths, that $E_i w^p \bar{E}_j$ is a (periodic) almost Nielsen path if and only if (u_i and u_j have the same parity and) $d_i = d_j$ and that the interior of $E_i w^p \bar{E}_j$ is an increasing union of pre-trivial paths. In fact, if we assume that $f\colon \mathcal{G} \to \mathcal{G}$ satisfies the conclusions of Theorem 2.6 and is rotationless, it follows by Lemma 5.18 that all dihedral linear edges have odd suffix (and that there is no version of an odd-suffix dihedral linear edge in the case that the dihedral pair is fixed).

Reduced filtration A filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ is *reduced with respect to* $\varphi \in \mathrm{Out}(F, \mathcal{A})$ if whenever a free factor system $\mathcal{F}'$ is φ^k-invariant for some $k > 0$ and $\mathcal{F}(G_{r-1}) \sqsubset \mathcal{F}' \sqsubset \mathcal{F}(G_r)$, then either $\mathcal{F}' = \mathcal{F}(G_{r-1})$ or $\mathcal{F}' = \mathcal{F}(G_r)$.

Taken paths, complete splittings If E is an edge in an irreducible stratum H_r and $k > 0$, then a maximal subpath σ of $f_\sharp^k(E)$ in a zero stratum H_i is said to be *r-taken* (or simply *taken*). If the zero stratum in question is enveloped by an exponentially growing stratum, then σ is a connecting path. Recall that if $f\colon \mathcal{G} \to \mathcal{G}$ satisfies the conclusions of Theorem 2.6, then each zero stratum H_i of $f\colon \mathcal{G} \to \mathcal{G}$ is a wandering component of G_i and hence not incident to vertices with nontrivial vertex group.

A path σ is *completely split* if it has a splitting, called a *complete splitting,* into subpaths each of which is either a single edge in an irreducible stratum (possibly with vertex group elements on either end), an indivisible almost Nielsen path, an exceptional path, or a connecting path in a zero stratum that is both maximal and taken.

A relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ is *completely split* if

1. the path $f(E)$ is completely split for each edge E in an irreducible stratum.

2. If σ is a taken connecting path in a zero stratum, then $f_\sharp(\sigma)$ is completely split.

Lemma 6.1 (cf. Lemma 4.6 of [FH11])**.** *If $f\colon \mathcal{G} \to \mathcal{G}$ is a completely split relative train track map for which each zero stratum H_i is a wandering component of G_i and σ is a completely split path, then $f_\sharp(\sigma)$ is completely split. If $\sigma = \sigma_1 \cdots \sigma_k$ is a complete splitting, then $f_\sharp(\sigma)$ has a complete splitting which refines $f_\sharp(\sigma) = f_\sharp(\sigma_1) \cdots f_\sharp(\sigma_k)$.*

The assumption that zero strata are wandering allows us to use the definition of *taken* above. This is not a serious assumption, since we shall work with relative train track maps satisfying the conclusions of Theorem 2.6.

Proof. Suppose σ_i is a term in a complete splitting of σ. In all cases, $f_\sharp(\sigma_i)$ is completely split, since $f_\sharp$ carries indivisible almost Nielsen paths to indivisible almost Nielsen paths and exceptional paths to exceptional paths. If σ_i is gE_ih, where E_i is an edge in an irreducible stratum and g and h are vertex group elements with h nontrivial, the final term in the complete splitting of E_i is not a connecting path in a zero stratum. The path $f_\sharp(\sigma)$ therefore has a complete splitting that refines $f_\sharp(\sigma) = f_\sharp(\sigma_1) \cdots f_\sharp(\sigma_k)$, since each maximal subpath of $f_\sharp(\sigma)$ in a zero stratum is contained in a single $f_\sharp(\sigma_i)$. $\square$

Folding the indivisible almost Nielsen path. The final definition and lemma we need before defining CTs have to do with indivisible almost Nielsen paths in exponentially growing strata. Suppose that H_r is an exponentially growing stratum of a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ and that ρ is an indivisible almost Nielsen path of height r. Suppose further that the map f satisfies the following properties.

1. The topological representative f represents $\varphi \in \mathrm{Out}(F, \mathcal{A})$.

2. The filtration on $f\colon \mathcal{G} \to \mathcal{G}$ realizes a nested sequence $\mathcal{C}$ of φ-invariant free factor systems.

3. Each contractible component of a filtration element is a union of zero strata.

4. The endpoints of all indivisible almost Nielsen paths of exponentially growing height are vertices.

We will define an operation called *folding ρ* that produces a new relative train track map $f'\colon \mathcal{G}' \to \mathcal{G}'$ and show that this new relative train track map satisfies the above properties.

Write $\rho = \alpha\beta$ as a concatenation of maximal r-legal subpaths as in Lemma 2.4 and let g_1E_1 and g_2E_2 in H_r be the initial edges and vertex group elements of $\bar\alpha$ and β respectively. If one of the edge paths $f(g_iE_i)$ for $i = 1$ or 2 is an initial subpath of the other, then we say that *the fold at the illegal turn of ρ is a full fold* and that it is a *partial fold* otherwise. In the case of a full fold, if $f(g_1E_1) \neq f(g_2E_2)$, then the full fold is *proper,* otherwise it is *improper.*

Suppose that $f(g_1E_1)$ is a proper initial subpath of $f(g_2E_2)$, so that we are in the case of a proper full fold. Write $\bar{\alpha} = g_1E_1bE_3$ where b is a possibly trivial subpath of $\bar{\alpha}$ in G_{r-1} or a single vertex group element and E_3 is an edge of H_r. The initial edge of $f(E_3)$ and the first edge of $f(\beta)$ that is not canceled with $f(g_1E_1b)$ when $f(\alpha)f(\beta)$ is tightened to $g\alpha\beta h$ belong to H_r. We may decompose the edge E_2 into subpaths $E_2 = E_2''E_2'$ such that $f(g_2E_2'') = f(E_1b)$ and such that the first edge in $f(E_2')$ is in H_r. Form a new graph of groups $\mathcal{G}'$ by identifying E_2'' with E_1b. The quotient map $F\colon \mathcal{G} \to \mathcal{G}'$ is called the *extended fold determined by* ρ.

We may think of $G \setminus E_2$ as a subgraph of groups of $\mathcal{G}'$ on which the map F is the identity. By construction, we have that $F(E_2) = E_1bE_2'$. The strata of the filtration on $\mathcal{G}'$ is defined so that $H_i' = H_i$ if $i \neq r$ and $H_r' = (H_r \setminus E_2) \cup E_2'$. The map f factors as gF for some map $g\colon \mathcal{G}' \to \mathcal{G}$. The map $g\colon \mathcal{G}' \to \mathcal{G}$ is called *the map induced by the extended fold.* Define $f'\colon \mathcal{G}' \to \mathcal{G}'$ from $Fg\colon \mathcal{G}' \to \mathcal{G}'$ by tightening the images of edges. We say that $f'\colon \mathcal{G}' \to \mathcal{G}'$ is obtained from $f\colon \mathcal{G} \to \mathcal{G}$ by *folding* ρ and that $\rho' = F_\sharp(\rho)$ is *the indivisible almost Nielsen path determined by* ρ. If the fold at the illegal turn of ρ' is itself proper, then we may iterate this process, which is referred to as *iteratively folding* ρ.

Lemma 6.2 (cf. Lemma 2.22 of [FH11])**.** *Given notation as above, the map $f'\colon \mathcal{G}' \to \mathcal{G}'$ is a relative train track map that satisfies properties 1 through 4 above.*

Proof. We follow the proof of [FH11, Lemma 2.22], assuming explicitly that $f\colon \mathcal{G} \to \mathcal{G}$ is a topological representative. By construction, we have $f'|_{G_{r-1}'} = f|_{G_{r-1}}$. If E is an edge in H_r, then $f(E)$ does not cross the illegal turn in ρ, and if $E \neq E_2$, then the path $f'(E)$ is obtained from $f(E)$ replacing each occurrence of E_2 with E_1bE_2' (and by tightening, which is not necessary if we assume f is a topological representative). We also have that $f'(E_2')$ is obtained from $f(E_2')$ by replacing each occurrence of E_2 with E_1bE_2'. Therefore H_r' satisfies properties (EG-i) through (EG-iii).

So suppose H_i is a stratum above H_r. If H_i is non-exponentially growing, then H_i' is non-exponentially growing. If H_i is a zero stratum, then for each edge E_i of H_i, we have $f(E_i) = f_\sharp(E_i)$ is nontrivial because we assume that f is a topological representative. Observe that F does not identify points that are not identified by f. Since $F|_{E_i}$ is the identity, we have $(Fg)_\sharp(E_i) = (Ff)_\sharp(E_i)$, and this path is nontrivial. This shows that no edges are collapsed when Fg is tightened to f' and that if $\sigma \subset H_i$ is a path with endpoints at vertices, then $f_\sharp'(\sigma)$ is nontrivial.

So suppose that H_i is exponentially growing, that E_i is an edge of H_i, and that $f(E_i) = \sigma_1\mu_1\sigma_2 \ldots \mu_\ell\sigma_{\ell+1}$ is the decomposition of $f(E_i)$ into subpaths σ_j in H_m and μ_k in G_{i-1}. Then we have

$$f(E_i) = (Fg)_\sharp(E_i) = (Ff)_\sharp(E_i) = \sigma_1\mu_1'\sigma_2 \ldots \mu_\ell'\sigma_{\ell+1}$$

where $\mu_j' = F_\sharp(\mu_j)$ is nontrivial because $f_\sharp(\mu_j)$ is nontrivial. We conclude that H_i satisfies (EG-i) and (EG-iii).

Suppose σ' is a connecting path for H_i'. If σ' is contained in a contractible component of G_{i-1}', it is contained in a zero stratum H_k', then it is disjoint from G_r and thus identified by F with a connecting path σ in H_k, and we see that $f_\sharp'(\sigma')$ is nontrivial because $f_\sharp(\sigma)$ is. If σ' is contained in a noncontractible component of G_{i-1}', by property (Z) for $f\colon \mathcal{G} \to \mathcal{G}$ it is contained in a non-contractible component C of G_{i-1}'. Vertices in $H_i \cap C$ are periodic for f by Lemma 1.5. Since F does not identify points that are not identified by f, we see that these vertices are periodic for f', proving (EG-ii). Therefore $f'\colon \mathcal{G}' \to \mathcal{G}'$ is a relative train track map. The additional properties listed above follow from the corresponding properties for f. ☐

CTs. A relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ and filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ is a *CT* (for *completely split improved relative train track map*) if it satisfies the following properties. Compare [FH11, Definition 4.7].

1. (**Rotationless**) The map $f\colon \mathcal{G} \to \mathcal{G}$ is rotationless.

2. (**Completely Split**) The map $f\colon \mathcal{G} \to \mathcal{G}$ is completely split.

3. (**Filtration**) The filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ is reduced. The core of a filtration element G_r is a filtration element unless H_r is the bottom half of a dihedral pair, in which case G_{r-1} and G_{r+1} are their own core.

4. (**Vertices**) The endpoints of all indivisible periodic (necessarily fixed) almost Nielsen paths are (necessarily principal) vertices. The terminal endpoint of each non-almost fixed non-exponentially growing edge is fixed and is either principal or the center vertex of a dihedral pair. In the latter case, the suffix of the non-exponentially growing edge is contained in the dihedral pair.

5. (**Almost Periodic Edges**) Each almost periodic edge is either almost fixed or belongs to a dihedral pair. If an almost fixed edge E_r is not part of a dihedral pair, each endpoint of E_r is principal, and if an endpoint of E_r has trivial vertex group and E_r does not form a loop, then G_{r-1} is its own core and that endpoint of E_r is contained in G_{r-1}.

6. (**Zero Strata**) If H_i is a zero stratum, then H_i is enveloped by an exponentially growing stratum H_r, each edge in H_i is r-taken and each vertex in H_i is contained in H_r and has link contained in $H_i \cup H_r$.

7. (**Linear Edges**) For each linear edge E_i there is a closed, root-free Nielsen path w_i such that $f(E_i) = g_i E_i w_i^{d_i}$ for some nonzero integer d_i. If E_i and E_j are distinct linear edges with the same axis, then $w_i = w_j$ and $d_i \neq d_j$. For each dihedral linear edge E_i with axis $w_i = \sigma\tau$, we have $f(E_i) = g_i E_i (\sigma\tau)^{d_i}\sigma$ for some integer d_i. If E_i and E_j are distinct dihedral linear edges with the same axis, then $d_i \neq d_j$.

8. (**NEG Almost Nielsen Paths**) If the highest edges in an indivisible almost Nielsen path σ belong to a non-exponentially growing stratum, then there is a linear or dihedral linear edge E_i with w_i as in (Linear Edges) and there exists $k \neq 0$ such that $\sigma = gE_i w_i^k \bar{E}_i h$ for vertex group elements g and h.

9. (**EG Almost Nielsen Paths**) If H_r is exponentially growing and ρ is an indivisible almost Nielsen path of height r, then $f|_{G_r} = \theta \circ f_{r-1} \circ f_r$ where

   (a) $f_r\colon G_r \to \mathcal{G}^1$ is a composition of proper extended folds defined by iteratively folding ρ,

   (b) $f_{r-1}\colon \mathcal{G}^1 \to \mathcal{G}^2$ is a composition of folds involving edges in G_{r-1}, and

   (c) $\theta\colon \mathcal{G}^2 \to G_r$ is an isomorphism of graphs of groups.

Observe that if $f\colon \mathcal{G} \to \mathcal{G}$ is a CT, (Vertices) and Lemma 1.5 imply that a vertex whose link contains edges in more than one irreducible stratum is principal unless it is the center vertex of a dihedral pair.

The remainder of the section is given to studying properties of CTs with an eye towards their construction in the proof of Theorem 6.15. Our first lemma says, in particular, that the complete splitting of a path is unique up to "shuffling" vertex group elements between the terms.

Lemma 6.3 (cf. Lemma 4.11 of [FH11])**.** *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a CT, that σ is a circuit or a path and that $\sigma = \sigma_1 \dots \sigma_\ell$ is a decomposition into subpaths, each of which is either a single edge in an irreducible stratum (possibly with vertex group elements on either end), an indivisible almost Nielsen path, an exceptional path, or a connecting path in a zero stratum that is maximal and taken. Suppose additionally that each turn $(\bar{\sigma}_i, \sigma_{i+1})$ is legal. Then the following hold.*

1. *The decomposition $\sigma = \sigma_1 \dots \sigma_\ell$ is the unique complete splitting of σ up to the equivalence relation where $\sigma_1 \dots \sigma_\ell$ is equivalent to $\sigma'_1 \dots \sigma'_{\ell'}$ if $\ell = \ell'$ and $\sigma_i = g_i \sigma'_i h_i$ for (possibly trivial) vertex group elements g_i and h_i.*

2. *Each pretrivial subpath τ of σ is contained in a single σ_i.*

3. *A subpath of σ that has the same height as σ and is either an almost fixed edge or an indivisible almost Nielsen path is equal to $g_i \sigma_i h_i$ for some i and vertex group elements g_i and h_i.*

Proof. We follow [FH11, Lemma 4.11]. There is no loss in assuming that f is a topological representative, so as in the original, we shall. Let $\tilde\sigma = \tilde\sigma_1 \dots \tilde\sigma_\ell$ be a lift of σ and let $\tilde f \colon \Gamma \to \Gamma$ be a lift of $f \colon \mathcal{G} \to \mathcal{G}$. We first establish the following property.

4. If σ_i is not a maximal taken connecting path in a zero stratum, then for each $k \geq 0$, there exist nontrivial initial and terminal subpaths $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ of $\tilde\sigma_i$ such that $\tilde f^{-k}(\tilde f^k(\tilde x)) \cap \tilde\sigma = \{\tilde x\}$ for each point $\tilde x$ in the union of the interiors of $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$.

We prove item 4 by induction, first on k and then on ℓ. Recall that we assume $\tilde f$ acts linearly with respect to some metric on Γ. It is clear that item 4 holds for $k = 0$, so assume that item 4 holds for any iterate of $\tilde f$ (and all ℓ) less than k.

Now assume that $\ell = 1$, i.e. that $\sigma = \sigma_1$. If σ_1 is an exceptional path or an indivisible almost Nielsen path of non-exponentially growing height, then by (NEG Almost Nielsen Paths), the first and last edges of σ_1 are non-exponentially growing, and the existence of the initial and terminal segments $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ mapping over theses edges follows as in the proof of Lemma 5.15. If σ_1 is an indivisible almost Nielsen path of exponentially growing height, the statement follows from the proof of Lemma 2.4. The final possibility is that σ_1 is an edge E in an irreducible stratum. By (Zero Strata), the first and last terms in any complete splitting of $f(E)$ are not connecting paths in zero strata. By the inductive hypothesis, there exist initial and terminal subpaths $\tilde\alpha'$ and $\tilde\beta'$ of $\tilde f(\tilde E)$ such that

$$\tilde f^{-(k-1)}(\tilde f^{k-1}(\tilde x)) \cap \tilde f(\tilde E) = \{\tilde x\}$$

for all $\tilde x$ in the interior of $\tilde\alpha'$ or $\tilde\beta'$. Since we assume that f is a topological representative, $\tilde f|_{\tilde E}$ is an embedding, so we may pull back $\tilde\alpha'$ and $\tilde\beta'$ to initial and terminal subpaths $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ of $\tilde E$ that satisfy item 4. This completes the case $\ell = 1$.

Now suppose that item 4 holds for k and σ if the decomposition of σ in the statement has fewer than $\ell \geq 2$ terms. There are three cases, depending on whether σ_1 or σ_2 are taken maximal connecting paths in zero strata. Suppose first that σ_1 is such a path in a zero stratum H_p. By (Zero Strata), σ_2 is an edge in an exponentially growing stratum H_r with $r > p$. Define $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ using $\tilde\sigma_2 \dots \tilde\sigma_\ell$ in place of $\tilde\sigma$. Observe that the intersection of $\tilde f^k(\tilde\sigma_1)$ with the interior of $\tilde f^k(\tilde\alpha_{2,k})$ is empty, because $\tilde f^k(\tilde\sigma_1)$ projects to a path of height less than r and $\tilde f^k(\tilde\alpha_{2,k})$ is an embedded path whose initial direction projects to a direction of height r. Furthermore, the interior of $\tilde f^k(\tilde\alpha_{2,k})$ separates $\tilde f^k(\tilde\sigma_1)$ from each $\tilde f^k(\tilde\beta_{i,k})$ and from each $\tilde f^k(\tilde\alpha_{i,k})$ if $i > 2$, so it follows that $\tilde f^k(\tilde\sigma_1)$ is disjoint from these sets, proving that $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ satisfy item 4 with respect to σ.

Suppose next that σ_2 is a taken maximal connecting path in a zero stratum H_p. Again by (Zero Strata), we have that σ_1 and σ_3 (if $\ell \geq 3$) are edges in an exponentially growing stratum H_r with $r > p$. Define $\tilde\alpha_{1,k}$ and $\tilde\beta_{1,k}$ using $\tilde\sigma_1$ in place of $\tilde\sigma$. For $i > 2$, define $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ using $\tilde\sigma_2 \dots \tilde\sigma_\ell$ in place of $\tilde\sigma$. As in the previous case, we have that $\tilde f^k(\tilde\sigma_2)$ is disjoint from the interior of $\tilde f^k(\tilde\beta_{1,k})$ and the interior of $\tilde f^k(\tilde\alpha_{3,k})$. Also as in the previous case, this implies that $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ satisfy item 4 with respect to σ.

As a final case, suppose that neither σ_1 or σ_2 are taken maximal connecting paths in zero strata. Define $\tilde\alpha_{1,k}$ and $\tilde\beta_{1,k}$ using $\tilde\sigma_1$ in place of $\tilde\sigma$. For $i \geq 2$, define $\tilde\alpha_{i,k}$ and $\tilde\beta_{i,k}$ using $\tilde\sigma_2 \dots \tilde\sigma_\ell$ in place of $\tilde\sigma$. Because the turn $(\bar\sigma_1, \sigma_2)$ is legal, the interiors of $\tilde f^k(\tilde\alpha_{1,k})$ and

$\tilde{f}^k(\tilde{\beta}_{2,k})$ are disjoint. The proof concludes as in the previous two cases. This completes the inductive step and with it the proof of item 4.

If τ is a pretrivial path, choose $k > 0$ so that $f^k_\sharp(\tau)$ is trivial. For each point $\tilde{x} \in \tilde{\tau}$ there exists $\tilde{y} \neq \tilde{x}$ such that $\tilde{f}^k(\tilde{x}) = \tilde{f}^k(\tilde{y})$. If σ_i is not a taken maximal connecting path in a zero stratum and if τ intersects the interior of σ_i, then we must have that τ is contained in the interior of σ_i by item 4. No two consecutive σ_i are taken maximal connecting paths in zero strata, so this proves item 2. It also follows that $\sigma = \sigma_1 \ldots \sigma_m$ is a splitting, hence a complete splitting, of σ.

Suppose that $\sigma = \sigma'_1 \ldots \sigma'_1$ is a complete splitting as well. If σ'_i is an exceptional path or an indivisible almost Nielsen path then observe that the interior of σ'_i is an increasing union of pre-trivial subpaths. (This was remarked after the definition of exceptional paths, of which indivisible almost Nielsen paths of non-exponentially growing height are examples by (NEG Almost Nielsen Paths), and follows from the proof of Lemma 2.4 for indivisible almost Nielsen paths of exponentially growing height.) Item 2 implies that σ'_i is contained in some σ_j. Since σ_j is not a single edge and is not contained in a zero stratum, it must be an indivisible almost Nielsen path or an exceptional path. By symmetry we conclude that $\sigma'_i = g\sigma_j h$ for vertex group elements g and h. The terms that are taken maximal connecting paths in zero strata are the maximal subpaths of σ in the complement of the indivisible almost Nielsen paths and exceptional paths, contained in zero strata, so these subpaths are the same in each decomposition. The remaining edges determine terms in the complete splitting. This proves that complete splittings are unique up to the equivalence relation in item 1.

Finally, an almost fixed edge of maximal height in σ is clearly not contained in a zero stratum, is not an indivisible almost Nielsen path nor an exceptional path in σ and so must determine a term in the complete splitting of σ. By item 2, an indivisible almost Nielsen path in σ must be contained in a single σ_i by item 2. If it has maximal height, then again we conclude that it must be all of σ_i. This proves item 3. ☐

Corollary 6.4 (cf. Corollary 4.12 of [FH11])**.** *Let $f\colon \mathcal{G} \to \mathcal{G}$ be a CT and σ a completely split path with complete splitting $\sigma = \sigma_1 \ldots \sigma_s$. If τ is an initial segment of σ with terminal endpoint in σ_j, then $\tau = \sigma_1 \ldots \sigma_{j-1} \cdot \mu_j$ is a splitting, where μ_j is the initial segment of σ_j that is contained in τ. If τ is a nontrivial almost Nielsen path (or more generally, if $f_\sharp(\tau)$ has the same underlying path in G as τ) then σ_i is an almost Nielsen path for $i \leq j$ and if σ_j is not a single almost fixed edge, then $\mu_j = \sigma_j$.*

Proof. The proof is identical to [FH11, Lemma 4.12]; the main statement follows immediately from item 2 of Lemma 6.3. The statement about almost Nielsen paths follows from the fact that a proper initial segment of σ_j is an almost Nielsen path only when σ_j is an almost fixed edge. ☐

Lemma 6.5 (cf. Lemma 4.36 of [FH11])**.** *Suppose $f\colon \mathcal{G} \to \mathcal{G}$ is a CT and that $\tilde{f}\colon \Gamma \to \Gamma$ is a principal lift. The following hold.*

1. *If the vertex $\tilde{v}$ belongs to* $\mathrm{Fix}(\tilde{f})$ *and $\tilde{E}$ is a non-fixed edge determining a fixed direction at $\tilde{v}$, then $\tilde{E} \subset \tilde{f}_\sharp(\tilde{E}) \subset \tilde{f}^2_\sharp(\tilde{E}) \subset \cdots$ is an increasing sequence of paths whose union is a ray $\tilde{R}$ that converges to some fixed point $P \in \mathrm{Fix}_N(\hat{f}) \cap \partial_\infty(F, \mathcal{A})$. The interior of $\tilde{R}$ is fixed point free.*

2. *For every isolated point $P \in \mathrm{Fix}_N(\hat{f}) \cap \partial_\infty(F, \mathcal{A})$, there exists $\tilde{E}$ and $\tilde{R}$ as in Item 1 that converges P. The edge E is non-linear.*

Proof. The proof is identical to [FH11, Lemma 4.36]. Given $\tilde{E}$ as Item 1 and for $m > 0$, Lemma 6.1 implies that $\tilde{E} \subset \tilde{f}_\sharp(\tilde{E}) \subset \cdots \subset \tilde{f}^m_\sharp(\tilde{E})$ is a nested sequence of completely split paths which define a ray $\tilde{R}$ that converges to a fixed point P. Let $\tilde{w}$ be the terminal endpoint of $\tilde{E}$. Since the sequence of points $\tilde{f}^m(\tilde{w})$ limits to P and each point in the sequence moves toward P under the action of $\tilde{f}$, we have $P \in \mathrm{Fix}_N(\hat{f})$ by Lemma 5.3. By Corollary 6.4, the ray $\tilde{R}$ intersects $\mathrm{Fix}(\tilde{f})$ only in its initial endpoint, proving item 1.

If $P \in \mathrm{Fix}_N(\hat{f}) \cap \partial_\infty(F, \mathcal{A})$ is isolated, then Proposition 3.4 implies that $\tilde{f}$ moves points that are sufficiently close to P toward P. Therefore we may choose a ray $\tilde{R}$ that converges to P and intersects $\mathrm{Fix}(\tilde{f})$ only in its initial endpoint. The initial edge $\tilde{E}$ of $\tilde{R}$ determines a fixed direction by Lemma 5.4, so by item 1 extends to a fixed-point-free ray $\tilde{R}'$ converging to a point $Q \in \mathrm{Fix}_N(\hat{f})$. Lemma 5.4 implies that in fact $P = Q$. Lemma 3.2 and item 1 of Proposition 3.4 imply that P is not an endpoint of the axis of T_c for some non-peripheral $c \in F$, so it follows by Lemma 6.7 below that E is not a linear edge. ☐

Lemma 6.6 (cf. Lemma 4.21 of [FH11]). *Suppose $f\colon \mathcal{G} \to \mathcal{G}$ is a CT and H_i is a non-exponentially growing stratum that is not a dihedral pair. Then H_i is a single edge E_i. If E_i is not almost periodic, then there is a nontrivial closed path $u_i \subset G_{i-1}$ such that $f(E_i) = E_i \cdot u_i$. The closed path u_i is contained in a filtration element G_j with $j < i$ that is its own core.*

Let us remark that unlike in [FH11], the path u_i need not form a circuit. One way this can fail is the case of a dihedral linear edge, where $u_i = (\sigma\tau)^{d_i}\sigma$ is its own homotopy inverse.

Proof. The proof is identical to [FH11, Lemma 4.21]. If H_i consists of almost periodic edges, then the lemma follows from (Almost Periodic Edges). Otherwise (Rotationless), (Completely Split) and (Vertices) imply that H_i is a single edge E_i and that there is a nontrivial closed path u_i such that $f(E_i) = E_i u_i$ is completely split. To show that $f(E_i) = E_i \cdot u_i$ is a splitting, we need to show that the first term in the complete splitting of $E_i u_i$ is the single edge E_i. Clearly the first term is not contained in a zero stratum. It cannot be an indivisible almost Nielsen path by (NEG Almost Nielsen Paths). The final possibility we must eliminate is that the first term is an exceptional path. It cannot be an exceptional path if E_i is not linear, so suppose E_i is linear or dihedral linear. By (Linear Edges), we have $f(E_i) = E_i w_i^{d_i}$ in the linear case, and $f(E_i) = E_i(\sigma\tau)^{d_i}\sigma$ in the latter. If E_j is another linear edge with the same axis, then assuming notation as in (Linear Edges) A path of the form $E_i w^p \bar{E}_j$ cannot be a subpath of $f(E_i)$, so we conclude that the first term in the complete splitting of $f(E_i)$ is not an exceptional path. Therefore $f(E_i) = E_i \cdot u_i$.

By (Filtration), to prove the final claim, it suffices to show that if u_i is contained in G_j, it is contained in the core of G_j. If the terminal vertex of E_i is the center vertex of a dihedral pair, then by (Vertices), u_i is contained in the dihedral pair and the claim follows. So suppose the terminal vertex of E_i is principal and has finite vertex group. The turn $(Df^{k-1}(\bar{u}_i), Df^k(u_i))$ is the image under Df^k of the legal turn $(\bar{E}_i, u_i)$ and is therefore legal for $k \geq 1$. Since f is rotationless, v is principal and there are finitely many directions based at v, we have that $Df^k(d)$ is independent of k for all directions d based at v for sufficiently large k. Therefore for sufficiently large k, we have that $(Df^k(\bar{u}_i), Df^k(u_i))$ is legal, and we conclude that $(u_i, \bar{u}_i)$ is legal and hence nondegenerate, so u_i forms a circuit. It follows that if the terminal vertex of E_i has finite vertex group and u_i is contained in G_j, then it is contained in the core of G_j. If the turn $(u_i, \bar{u}_i)$ is degenerate, then the argument above shows that the terminal vertex of E_i has infinite vertex group, and in particular there is a circuit of the form $u_i g_i$ for vertex group element g_i, and we conclude that if u_i is contained in G_j, then it is contained in the core of G_j. ☐

If $f\colon \mathcal{G} \to \mathcal{G}$ is a CT, the lemma implies that $f\colon \mathcal{G} \to \mathcal{G}$ satisfies the conclusions of Theorem 2.6.

Lemma 6.7 (cf. Lemma 4.22 of [FH11]). *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a rotationless relative train track map and that the stratum H_i is a single edge E_i such that $f(E_i) = E_i \cdot u_i$ for some nontrivial closed path u_i in G_{i-1}. Suppose that $f|_{G_{i-1}}$ satisfies (Vertices) and (Almost Periodic Edges). Suppose either that there are no almost Nielsen paths of height i or that E_i is a linear or dihedral linear edge and all almost Nielsen paths of height i have the form $E_i w_i^k \bar{E}_i$ for $k \neq 0$ and where w_i is the axis of E_i. If $h\colon \Gamma_{i-1} \to \Gamma_{i-1}$ is the lift of $f|_{G_{i-1}}$ as* *in the paragraph "Restricting to G_{i-1} for non-exponentially growing strata," then we have the following.*

1. $\mathrm{Fix}(h) = \varnothing$.
2. *The edge E_i is linear or dihedral linear if and only if there is some nonperipheral $c \in F$ such that T_c preserves Γ_{i-1} and the restriction $T_c|_{\Gamma_{i-1}} : \Gamma_{i-1} \to \Gamma_{i-1}$ commutes with h and whose axis covers u_i if E_i is linear and $u_i f_\sharp(u_i)$ if E_i is dihedral linear.*

Proof. We follow the proof of [FH11, Lemma 4.22]. Write $\tilde{f} : \Gamma \to \Gamma$ and $\tilde{E}_i$ as in the paragraph "Restricting to G_{i-1} for non-exponentially growing strata." We have $\tilde{f}(\tilde{E}_i) = \tilde{E}_i \cdot \tilde{u}_i$, where $\tilde{u}_i \subset \Gamma_{i-1}$ is a lift of u_i and h maps the terminal vertex $\tilde{x}_1$ of $\tilde{E}_i$ to the terminal endpoint $\tilde{x}_2$ of $\tilde{u}_i$. If $\tilde{v}$ belongs to $\mathrm{Fix}(h)$ and $\tilde{\gamma}$ is the unique tight path from $\tilde{x}_1$ to $\tilde{v}$, then $\tilde{E}_i\tilde{\gamma}$ projects to an almost Nielsen path for f that is not of the form $E_i w_i^k \bar{E}_i$, so we conclude $\mathrm{Fix}(h) = \varnothing$.

If E_i is a (non-dihedral) linear edge, the assumption that $f(E_i) = E_i \cdot u_i$ is a splitting implies that the closed path u_i represents a nonperipheral conjugacy class in F. There is a choice of representative c such that T_c preserves Γ_{i-1} and $T_c(\tilde{x}_1) = \tilde{x}_2$. We have $T_c h(\tilde{x}_1) = hT_c(\tilde{x}_1)$ is the terminal endpoint of the lift of u_i that begins at $\tilde{x}_2$, and both maps agree on the initial direction of $\tilde{u}_i$. Thus T_c commutes with h. If E_i is dihedral linear, the closed path $u_i f_\sharp(u_i)$ represents a nonperipheral conjugacy class in F. There is a choice of representative c such that T_c preserves Γ_{i-1} and $T_c(\tilde{x}_1)$ is the terminal endpoint $\tilde{x}_3$ of $\tilde{u}_i h(\tilde{u}_i)$. We have that $T_c h(\tilde{x}_1) = hT_c(\tilde{x}_1)$ is the terminal endpoint of the lift of u_i that begins at $\tilde{x}_3$. Since the central vertex of a dihedral pair has trivial vertex group, T_c commutes with h.

For the converse, suppose that there is some nonperipheral element $c \in F$ such that T_c preserves Γ_{i-1} and commutes with h. Corollary 5.5 implies that since $\mathrm{Fix}(h) = \varnothing$, the map h is not a principal lift of $f|_{G_{i-1}}$, so the endpoints of the axis of T_c are the only fixed points in $\partial\Gamma_{i-1}$. But the ray $\tilde{u}_i \cdot h_\sharp(\tilde{u}_i) \cdot h_\sharp^2(\tilde{u}_i) \cdots$ converges to a fixed point in $\partial\Gamma_{i-1}$, so the endpoint of this ray is contained in the axis of T_c. Therefore u_i is a periodic Nielsen path. Write $u_i = \sigma_1 \cdots \sigma_m$, where each σ_j is either an almost periodic edge or an indivisible periodic almost Nielsen path. If some endpoint of some σ_j is principal, then all endpoints of all σ_j are principal and each σ_j and hence u_i has period one as a periodic almost Nielsen path. The argument after the definition of almost linear edges implies that u_i has period one as a periodic Nielsen path, and thus E_i is a linear edge. If no endpoint is principal, then by (Vertices) and (Almost Periodic Edges), each σ_j is an edge in a dihedral pair, and in fact it follows that u_i is contained in a single dihedral pair. The assumption on almost Nielsen paths of height i implies that E_i is a linear edge if the dihedral pair is fixed and a dihedral linear edge otherwise. It also follows that the axis of T_c covers u_i or $u_i f_\sharp(u_i)$ as appropriate. ☐

The next several lemmas will be used in verifying (EG Almost Nielsen Paths).

Lemma 6.8 (cf. Lemma 4.16 of [FH11])**.** *Suppose that H_r is an aperiodic exponentially growing stratum of a relative train track map $f : \mathcal{G} \to \mathcal{G}$, that ρ is an indivisible almost Nielsen path of height r and that ρ and H_r satisfy the conclusions of (EG Almost Nielsen Paths). Then the fold at the illegal turn of each indivisible almost Nielsen path obtained by iteratively folding ρ is proper.*

Proof. The proof is identical to [FH11, Lemma 4.16]. We may assume that $G = G_r$.

Define the *data set S for f and ρ* to be the ordered sequences of edges in H_r in ρ and in $f(E)$ for each edge E of H_r. As in the proof of Lemma 2.4, we see that S determines the type of fold (partial, proper or improper) of the fold at the illegal turn of ρ: if we decompose ρ into a concatenation of r-legal paths $\alpha\beta$, then edges from $f(\bar{\alpha})$ cancel with edges of $f(\beta)$ until the first distinct H_r edges are reached. Assuming that the fold is proper so that the extended fold is defined, S also determines the data set for the relative train track map and indivisible almost Nielsen path obtained by folding ρ. So define S_k to be the data set for the relative train track map and indivisible almost Nielsen path obtained by folding ρ k times, assuming that the folds are defined.

We adopt the notation of (EG Almost Nielsen Paths). We have that $f_r\colon \mathcal{G} \to \mathcal{G}^1$ is the composition of finitely many, say K, proper extended folds defined by iteratively folding ρ. Therefore S_K is defined. Since $f|_{G_r} = \theta \circ f_{r-1} \circ f_2$, the isomorphism θ determines a bijection between the edges of the top stratum of $\mathcal{G}^2$ (and thus $\mathcal{G}^1$, since f_{r-1} does not affect edges in the top stratum) and the edges of H_r. Furthermore this isomorphism takes S_K to S_0. Therefore the sequence of S_k for $k \geq 0$ is periodic with period K so the fold at the illegal turn of each indivisible almost Nielsen path obtained by iteratively folding ρ is always proper. □

Proposition 6.9 (cf. Lemma 4.17 of [FH11] and Theorem 5.15 of [BH92])**.** *Suppose $f\colon \mathcal{G} \to \mathcal{G}$ is an eg-aperiodic relative train track map and that for each exponentially growing stratum H_r such that there exists an indivisible almost Nielsen path ρ of height r, the fold at the illegal turn of each indivisible almost Nielsen path obtained by iteratively folding ρ is proper. The following hold.*

1. *For each exponentially growing stratum H_r there is, up to equivalence and reversal of orientation, at most one indivisible almost Nielsen path ρ of height r.*

Supposing ρ exists, we have the following.

2. *Supposing the illegal turn of ρ in H_r is based at the vertex v, the $\mathcal{G}_v$-orbit of this turn are the only nondegenerate illegal turns in H_r.*

3. *The path ρ crosses every edge in H_r.*

4. *The length $L_r(\rho)$ (see Lemma 2.2) satisfies $L_r(\rho) = 2\sum L_r(E)$, where the sum is taken over the edges of H_r.*

Proof. We follow the argument in [BH92, Lemma 3.9]. Fix r and assume that an indivisible almost Nielsen path ρ of height r exists. By assumption, the fold at the illegal turn of each indivisible almost Nielsen path obtained by iteratively folding ρ is always proper, so we may always continue folding.

If $f\colon \mathcal{G}_1 \to \mathcal{G}_1$ is obtained from $f\colon \mathcal{G} \to \mathcal{G}$ by folding ρ, the length function L_r in Lemma 2.2 descends to a length function L^1_r on $\mathcal{G}_1$. Since $\mathcal{G}_1$ is obtained from $\mathcal{G}$ by identifying a pair of intervals of some equal L_r-length x, we have $\sum L^1_r(E) = \sum L_r(E) - x$, where in each case the sum is taken over the edges of H_r. The indivisible almost Nielsen path ρ_1 of height r in $\mathcal{G}_1$ determined by ρ satisfies $L^1_r(\rho_1) = L_r(\rho) - 2x$. If $L_r(\rho) \neq 2\sum L_r(E)$, then

$$\left|\frac{L^1_r(\rho_1)}{\sum L^1_r(E)} - 2\right| = \left|\frac{L_r(\rho) - 2x}{\sum L_r(E) - x} - 2\right| = \left|\frac{L_r(\rho) - 2\sum L_r(E)}{\sum L_r(E) - x}\right| > \left|\frac{L_r(\rho) - 2\sum L_r(E)}{\sum L_r(E)}\right|,$$

and the last expression is equal to $|L_r(\rho)/\sum L_r(E) - 2|$. If we iteratively fold ρ, obtaining relative train track maps $f_i\colon \mathcal{G}_i \to \mathcal{G}_i$ and indivisible almost Nielsen paths ρ_i, we see that the ratio $L^i_r(\rho_i)/\sum L^i_r(E)$ takes on infinitely many values. But on the other hand, as in the proof of [CT94, Theorem 2.11, Step 3], notice that the ratio $L^i_r(\rho_i)/\sum L^i_r(E)$ is determined by the underlying graph map of $f\colon \mathcal{G} \to \mathcal{G}$, since the location of indivisible almost Nielsen paths are determined by fixed points of f and the metric is determined by the transition matrix of H_r. Therefore the argument in [BH92, Lemma 3.7] applies to show that we have a contradiction; we recount it for the reader's convenience. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic as graphs of groups and if after identifying we find that the associated relative train track maps $f_1\colon \mathcal{G}_1 \to \mathcal{G}_1$ and $f_2\colon \mathcal{G}_2 \to \mathcal{G}_2$ have the property that for each edge E of the identified graph, the underlying paths of $f_1(E)$ and $f_2(E)$ are the same, then it follows that the associated ratios above are equal. Since the number of edges in $\mathcal{G}$ remains constant under proper folds and there are only finitely many irreducible matrices of a given Perron–Frobenius eigenvalue, it follows that there are only finitely many possibilities for the maps $f_i\colon \mathcal{G}_i \to \mathcal{G}_i$, at least up to this coarse notion of equivalence. This contradiction implies that $L_r(\rho) = 2\sum L_r(E)$.

Now suppose that T is an illegal turn in $\mathcal{G}$ that is distinct from the illegal turn in ρ. Take ℓ to be the smallest positive integer such that $Df^\ell(T)$ is degenerate. Suppose first that $\ell = 1$. We may fold the edges determining T (after possibly subdividing) to obtain a topological representative which may not be a relative train track map because (EG-i) and (EG-ii) may fail. Restore these properties by the moves "(invariant) core subdivision" and "collapsing inessential connecting paths" as in [Lym21, Lemmas 3.4 and 3.5] or [BH92, Lemmas 5.13 and 5.14]. By Corollary 1.4, (EG-iii) is satisfied, resulting in a relative train track map $\hat{f}\colon \hat{\mathcal{G}} \to \hat{\mathcal{G}}$ and indivisible almost Nielsen path $\hat{\rho}$ determined by ρ. The map $\hat{f}$ is still eg-aperiodic and the fold at $\hat{\rho}$ is still proper. Invariant core subdivision and collapsing inessential connecting paths do not affect the resulting length function $\hat{L}_r$, so we have that $\sum \hat{L}_r(E) < \sum L_r(E)$ and $\hat{L}_r(\hat{\rho}) = L_r(\rho)$, so that $\hat{L}_r(\hat{\rho})/(\sum \hat{L}_r(E)) > 2$. This contradicts our previous arguments. Therefore we must have that $\ell \neq 1$ and $Df^{\ell-1}(T)$ is in the $\mathcal{G}_v$-orbit of the illegal turn in ρ. Replacing the turn T with $Df^{\ell-2}(T)$, we may assume $\ell = 2$. Fold ρ to obtain a relative train track map $f_1\colon \mathcal{G}_1 \to \mathcal{G}_1$, let ρ_1 be the indivisible almost Nielsen path in $\mathcal{G}_1$ determined by ρ and let T_1 be the turn in $\mathcal{G}_1$ determined by T. Then $Df_1(T_1)$ is degenerate (since the previously illegal turn at ρ is now degenerate), and we reach a contradiction as in the previous case.

Inductively define $f_k\colon \mathcal{G}_k \to \mathcal{G}_k$ and ρ by folding ρ_{k-1} in $\mathcal{G}_{k-1}$, starting with $f_0 = f\colon \mathcal{G} \to \mathcal{G}$ and $\rho_0 = \rho$. For $k \geq 0$, we have $\sum L_r^{k+1}(E) = \sum L_r^k(E) = x_k$. By the existence of θ conjugating L_r^K to a multiple of L_r^0, we see that $x_k/(\sum L_r^k(E))$ takes on finitely many values, so it is uniformly bounded below. It follows that $\sum L_r^k(E)$ goes to zero with k, so each $L_r^k(E)$ does as well, proving that ρ crosses every edge of H_r.

Now suppose $\rho' = \alpha'\beta'$ is another indivisible almost Nielsen path of height r. We must show that ρ is equivalent to ρ' (after possibly reversing the orientation). Since there is one $\mathcal{G}_v$-orbit of illegal turns in H_r, ρ and ρ' have the same illegal turn. (Recall from Section 1 our convention on which turns a path crosses.) After reorienting ρ' if necessary, we may assume that the initial edge of $\bar{\alpha}$ equals the initial edge of $\bar{\alpha}'$ and the initial edge of β equals the initial edge of β'. We factor the vertex group element at the illegal turn so that it is equal for α and α' and thus for β and β'. Let ρ'_k be the indivisible almost Nielsen path in $\mathcal{G}_k$ determined by ρ'. Write $\rho_k = \alpha_k\beta_k$ and $\rho'_k = \alpha'_k\beta'_k$ as in Lemma 2.4.

Suppose that $\alpha \neq g\alpha'$ for any vertex group element g. Then if the initial endpoints of α and α' are distinct, the same is true of α_k and α'_k. If the endpoints are equal, then $\alpha\bar{\alpha}'$ is a homotopically nontrivial loop, and the same is true of $\alpha_k\bar{\alpha}'_k$. We see that $\alpha_k \neq \alpha'_k$ for all k. For each $k \geq 0$, write ν_k for the maximum common terminal subinterval of α_k and α'_k and let μ_k and μ'_k be the complementary initial segments of $\alpha_k = \mu_k\nu_k$ and $\alpha'_k = \mu'_k\nu_k$. Suppose that the turn $\{\bar{\mu}_k, \mu'_k\}$ is not in the same $\mathcal{G}_v$-orbit as $\{\bar{\alpha}_k, \beta_k\}$. Then we have $L_r^{k+1}(\mu_{k+1}) = L_r^k(\mu_k)$. But the fact that the (orbits of) turns above are distinct implies that $\{\bar{\mu}_{k+1}, \mu'_{k+1}\}$ is not in the $\mathcal{G}_v$-orbit of the turn $\{\bar{\alpha}_{k+1}, \beta_{k+1}\}$, and we conclude that $L_r^{k+i}(\mu_{k+i}) = L_r^k(\mu_k)$ for $i > 0$, contradicting the fact that $L_r^k(\rho_k) = 2\sum L_r^k(E)$ goes to zero as k increases.

Therefore for $k \geq 0$ we may assume that the turns $\{\bar{\alpha}_k, \beta_k\}$ and $\{\bar{\mu}_k, \mu'_k\}$ are in the same $\mathcal{G}_v$-orbit. Then ν_{k+1} is obtained from ν_k by deleting a terminal subinterval of length x_k and adding an initial interval of length x_k, so we have $L_r^{k+1}(\nu_{k+1}) = L_r^k(\nu_k)$, again contradicting the fact that $L_r^k(\rho_k)$ goes to zero. This proves that ρ is equivalent to ρ'. ☐

Lemma 6.10 (cf. Lemma 4.24 of [FH11]). *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a rotationless relative train track map satisfying the conclusions of Theorem 2.6, that H_r is an exponentially growing stratum satisfying (EG Almost Nielsen Paths), and that ρ is an indivisible almost Nielsen path of height r. Then the following hold.*

1. $H_r^z = H_r$.

2. *If $\rho = a_1 b_1 \dots b_\ell a_{\ell+1}$ is the decomposition of ρ into subpaths a_i of height r and maximal subpaths b_i in G_{r-1}, then each b_i is an almost Nielsen path.*

3. *If E is an edge of H_r, then each maximal subpath of $f(E)$ in G_{r-1} is a path b_i from item 2. In particular $f(E)$ splits into edges in H_r (possibly with vertex group elements at either end) and almost Nielsen paths in G_{r-1}.*

Proof. We adopt the notation of (EG Almost Nielsen Paths); the proof is identical to [FH11, Lemma 4.24]. The maps f_r, f_{r-1} and θ induce bijections on the set of components in the filtration element of height $r-1$. It follows that $f|_{G_r} = \theta f_{r-1} f_r$ induces a bijection on the set of components of G_{r-1} and hence that each such component is non-wandering. By (Z) we conclude that item 1 holds.

For item 2, let $(f_r)_\sharp(\rho) = a'_1 b'_1 \dots a'_m b'_m a'_{m+1}$ be the decomposition into subpaths a'_j of height r and maximal subpaths b'_j in G_{r-1}. Since proper folds do not create new paths b'_j, we see that the set of b_j is contained in the set of b_i. Now let $(\theta f_{r-1})_\sharp(a'_1 b'_1 \dots a'_m b'_{m+1}) = c_1 d_1 \dots d_p c_{p+1}$ be the decomposition into subpaths c_k of height r and maximal subpaths d_k in G_{r-1}. For each k there exists j such that d_k differs from $(\theta f_{r-1})_\sharp(b'_j)$ by multiplying at the ends by vertex group elements. Since $a_1 b_1 \dots b_\ell a_{\ell+1} = c_1 d_1 \dots d_p c_{p+1}$, we conclude that up to multiplying at the ends by vertex group elements, $f_\sharp$ permutes the paths b_i. Since f is rotationless and the b_i cannot be contained in a dihedral pair, each b_i is an almost Nielsen path.

If E is an edge of H_r then by construction, each maximal subpath of $f_r(E)$ in G_{r-1} is one of the paths b_i. By item 2, each b_i is an almost Nielsen path for f and hence for θf_{r-1}, proving item 3. ☐

Changing the marking on G_j. To construct CTs, we require a move that plays the role of sliding for exponentially growing and zero strata.

Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a rotationless relative train track map satisfying the conclusions of Theorem 2.6 with respect to the filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$, that j is such that $1 \le j \le m$, every component of G_j is non-contractible and that f fixes every vertex in G_j whose link is not contained in G_j. Define a homotopy equivalence $g\colon \mathcal{G} \to \mathcal{G}$ by setting $g|_{G_j} = f|_{G_j}$ and setting $g|_{(G \setminus G_j)}$ to be the identity. Suppose $\tau\colon \mathbb{G} \to \mathcal{G}$ is the original marked graph of groups. Let $\mathcal{G}'$ be the marked graph of groups $g\tau\colon \mathbb{G} \to \mathcal{G}$. Therefore $\mathcal{G}$ and $\mathcal{G}'$ have the same underlying graph of groups, and there is a natural identification of $\mathcal{G}$ with $\mathcal{G}'$; we will use this identification when discussing edges and strata.

Define $f'\colon \mathcal{G}' \to \mathcal{G}'$ by setting $f'|_{G'_j} = f|_{G_j}$ and $f'(E) = (gf)_\sharp(E)$ for all edges E in H_i with $i > j$. We say that $f'\colon \mathcal{G}' \to \mathcal{G}'$ is obtained from $f\colon \mathcal{G} \to \mathcal{G}$ by *changing the marking on G_j via f*. We have the following lemma.

Lemma 6.11 (cf. Lemma 4.27 of [FH11])**.** *Suppose that $f'\colon \mathcal{G}' \to \mathcal{G}'$ is obtained from $f\colon \mathcal{G} \to \mathcal{G}$ by changing the marking on G_j via f. Then the following hold.*

1. *We have $f'|_{G_j} = f|_{G_j}$.*

2. *For every path σ with endpoints at vertices and $k > 0$, we have $g_\sharp f_\sharp^k(\sigma) = (f')_\sharp^k g_\sharp(\sigma)$.*

3. *If $f\colon \mathcal{G} \to \mathcal{G}$ represents $\varphi \in \mathrm{Out}(F, \mathcal{A})$, then $f'\colon \mathcal{G}' \to \mathcal{G}'$ is a homotopy equivalence representing $\varphi \in \mathrm{Out}(F, \mathcal{A})$.*

4. *There is a one-to-one correspondence between almost Nielsen paths for f and almost Nielsen paths for f'.*

5. *The map $f'\colon \mathcal{G}' \to \mathcal{G}'$ is a rotationless relative train track map satisfying the conclusions of Theorem 2.6 with respect to the original filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$.*

Proof. The proof is identical to [FH11, Lemma 4.27]. Item 1 is immediate from the definitions, as is the fact that f' preserves the original filtration. Observe that the following statements also hold.

6. If a point $x \in G$ satisfies $f(x) \neq f'(x)$, then $x \notin G_j$ and $f(x)$ and $f'(x)$ belong to G_j. In particular, $\mathrm{Fix}(f) = \mathrm{Fix}(f') \subset \mathrm{Fix}(g)$; furthermore $\mathrm{Per}(f) = \mathrm{Per}(f') \subset \mathrm{Per}(g)$. Under the identification of $\mathcal{G}$ with $\mathcal{G}'$, we see that Df and Df' have the same fixed and periodic directions.

7. Suppose that E is an edge of H_i for $i > j$ and that $f(E) = \mu_1\nu_1\mu_2 \dots \nu_{k-1}\mu_k$, where the ν_ℓ are maximal subpaths in G_j. Allow the case where μ_1 and μ_k are trivial. Then we have $f'(E) = \mu_1 f_\sharp(\nu_1)\mu_2 \dots f_\sharp(\nu_{k-1})\mu_k$. Since f fixes the endpoints of each ν_ℓ, it follows that the $f\sharp(\nu_\ell)$ are nontrivial.

8. Therefore each stratum H_i has the same type for f as for f'.

To prove item 2, it suffices to consider the case where $k = 1$ and σ is a single edge E. If $E \subset G_j$, then $g_\sharp f_\sharp(E) = f_\sharp(f_\sharp(E)) = f'_\sharp g_\sharp(E)$. If $E \subset G_i$ for $i > j$, then $g_\sharp f_\sharp(E) = f'_\sharp(E) = f'_\sharp g_\sharp(E)$. Item 2 implies item 3.

If ρ' is a tight path in $\mathcal{G}$ with endpoints in $\mathrm{Fix}(f') = \mathrm{Fix}(f)$, then there is a unique tight path ρ with the same endpoints such that $g_\sharp(\rho) = \rho'$. Condition 2 implies that ρ' is an almost Nielsen path for f' if and only if it is an almost Nielsen path for f, proving item 4.

By items 1, 6 and 7, to show that $f' \colon \mathcal{G}' \to \mathcal{G}'$ is a relative train track map, it suffices to show that each exponentially growing stratum H_i for $i > j$ satisfies (EG-ii). Suppose first that σ is a connecting path contained in a non-contractible component C of G_{i-1}. By Lemma 1.5, each vertex of $H_i \cap C$ is periodic; has valence at least two and is principal for f and hence fixed by f', so we conclude by Lemma 1.5 that $f'_\sharp(\sigma)$ is nontrivial. If σ is instead contained in a contractible component of G_{i-1}, then it is contained in a zero stratum that has height greater than j because it is contained in H_i^z. We have $(f')_\sharp(\sigma) = g_\sharp f_\sharp(\sigma)$. If $f_\sharp(\sigma)$ is contained in a non-contractible component of G_{i-1}, then $g_\sharp f_\sharp(\sigma)$ is nontrivial by the previous argument. If not, then $g_\sharp f_\sharp(\sigma) = f_\sharp(\sigma)$ and we are done. The remaining properties of Theorem 2.6, and thus item 5, follow from the definitions and items 4 and 6. □

Lemma 6.12 (cf. Lemma 4.25 of [FH11])**.** *If $f \colon \mathcal{G} \to \mathcal{G}$ is a CT and σ is a path in G_r with endpoints at vertices, then $f^k_\sharp(\sigma)$ is completely split for all sufficiently large k.*

Proof. The proof is identical to [FH11, Lemma 4.25]; we proceed by induction on r, the height of σ. There are no paths of height $r = 0$, so suppose that σ has height $r \geq 1$ and that the lemma holds for paths of height less than r. By Lemma 6.1, Lemma 6.3 and the inductive hypothesis, it suffices to show that some path $f^k_\sharp(\sigma)$ has a splitting into subpaths that are either themselves completely split or contained in G_{r-1}. This is immediate if H_r is a zero stratum or an almost periodic stratum. If H_r is non-exponentially growing but not almost periodic, then by Lemma 5.15, σ has a splitting into basic paths of height r and subpaths in G_{r-1}, and the desired splitting of $f^k_\sharp(\sigma)$ follows by Lemma 6.6. If H_r is exponentially growing, Lemma 2.5 implies that some $f^k_\sharp(\sigma)$ splits into pieces, each of which is either r-legal or part of the finite set P_r of equivalence classes of paths satisfying the conditions in the proof of Lemma 2.4. After iteration, the elements in P_r get mapped to indivisible periodic almost Nielsen paths, which are almost Nielsen paths by (Vertices). Lemma 2.1 implies that the r-legal paths in G_r split into single edges in H_r and subpaths in G_{r-1}. This completes the inductive step. □

Sliding revisited. The last tool we need before turning to Theorem A is the following proposition.

Proposition 6.13 (cf. Proposition 4.35 of [FH11])**.** *Suppose that $f \colon \mathcal{G} \to \mathcal{G}$ is a relative train track map that satisfies (EG Almost Nielsen Paths), that $f|_{G_{s-1}}$ is a CT, and that H_s*

is a non-exponentially growing stratum with a single edge E_s and that there does not exist a path μ in G_{s-1} such that $E_s\mu$ is an almost Nielsen path.

There exists a path τ in G_{s-1} with initial endpoint equal to the terminal vertex of E_s such that after sliding along τ, the following conditions are satisfied.

1. *$f(E_s) = g_s E_s \cdot u_s$ is a nontrivial splitting.*
2. *If σ is a path or circuit with endpoints at vertices and height s, then there exists $k \geq 0$ such that $f^k_\sharp(\sigma)$ splits into subpaths of the following type.*
   (a) *E_s or $\bar{E}_s$.*
   (b) *An exceptional path of height s.*
   (c) *A subpath of G_{s-1}.*
3. *u_s is completely split and its initial vertex is either principal or the center vertex of a dihedral pair; in the latter case u_s is contained in the dihedral pair and E_s is a linear or dihedral linear edge.*
4. *$f|_{G_s}$ satisfies (Linear Edges).*

Proof. We follow [BFH00, Proposition 5.4.3]. We adopt the notation of "Restricting to G_{s-1} for non-exponentially growing strata" from Section 5. In particular we have the map $h\colon \Gamma_{s-1} \to \Gamma_{s-1}$. If h had a fixed point, the path in Γ beginning at $\tilde{E}_s$ and ending at this fixed point would project to an almost Nielsen path of the form $gE_s\mu$. The assumption that there is no such path therefore implies that h is fixed-point free.

By Lemma 5.12, given any starting vertex $\tilde{v}$ in Γ_{s-1}, there is a ray $\tilde{R}$ beginning at $\tilde{v}$ and converging to a fixed point $P \in \partial_\infty \Gamma_{s-1}$. The ray has the property that it shares a nontrivial initial segment with the tight path from $\tilde{v}$ to $h(\tilde{v})$. There are points along this ray arbitrarily close to P that move towards P. In fact, the point P is independent of the choice of starting vertex $\tilde{v}$, for if the fixed point produced by choosing $\tilde{v}'$ were different, then $\tilde{f}$ would move a pair of points away from each other, contradicting Lemma 5.4.

Therefore we have, for any vertex $\tilde{v}$ in Γ_{s-1}, a ray $\tilde{R}_{\tilde{v}}$ beginning at $\tilde{v}$ and converging to P with the property that $\tilde{R}_{\tilde{v}}$ shares a nontrivial initial segment with the tight path from $\tilde{v}$ to $h(\tilde{v})$. Given $\tilde{v}$ and $\tilde{w}$, the rays $\tilde{R}_{\tilde{v}}$ and $\tilde{R}_{\tilde{w}}$ share a common terminal subray. Fix a vertex $\tilde{v}_0$ and inductively define $\tilde{v}_{i+1}$ to be the first vertex of the common subray of $\tilde{R}_{\tilde{v}_i}$ and $\tilde{R}_{h(\tilde{v}_i)}$. Given two points $\tilde{v}$ and $\tilde{w}$ in Γ_{r-1}, write $[\tilde{v},\tilde{w}]$ for the unique tight path from $\tilde{v}$ to $\tilde{w}$. We have

$$\tilde{R}_{\tilde{v}_0} = [\tilde{v}_0,\tilde{v}_1][\tilde{v}_1,\tilde{v}_2][\tilde{v}_2,\tilde{v}_3]\ldots$$

and we have that $h_\sharp([\tilde{v}_i,\tilde{v}_{i+1}]) = [h(\tilde{v}_i), h(\tilde{v}_{i+1})]$ contains the path $[\tilde{v}_{i+1},\tilde{v}_{i+2}]$. Let $\tilde{Y}_m$ be the set of points $\tilde{y}$ in $[\tilde{v}_0,\tilde{v}_1]$ such that $h^i(\tilde{y})$ belongs to $[\tilde{v}_i,\tilde{v}_{i+1}]$ for i satisfying $0 \leq i \leq m$. The map h^m sends $\tilde{Y}_m$ over all of $[\tilde{v}_m,\tilde{v}_{m+1}]$, so $\tilde{Y}_m$ is nonempty, and thus the intersection $\bigcap_{m=0}^\infty \tilde{Y}_m$ is nonempty. Let $\tilde{p}$ be a point in the intersection and notice that $\tilde{p}$ has the property that $\{\tilde{p}, h(\tilde{p}), h^2(\tilde{p}),\ldots\}$ is an ordered subset of a ray converging to P. Let $\tilde{X}$ be the set of such points $\tilde{p}$. Observe that $h(\tilde{X}) \subset \tilde{X}$. Observe as well that given $\tilde{x} \in \tilde{X}$ and a point $\tilde{y}$ in $[\tilde{x}, h(\tilde{x})]$ such that the decomposition $[\tilde{x},\tilde{y}]\cdot[\tilde{y},h(\tilde{x})]$ is a splitting (for f and hence h), then $\tilde{y}$ belongs to $\tilde{X}$.

We would like to find a vertex in $\tilde{X}$. Let ℓ be the smallest positive integer such that there exists $\tilde{x} \in \tilde{X}$ such that the projection of $[\tilde{x}, h(\tilde{x})]$ is contained in G_ℓ and choose such a point $\tilde{x}$. Notice that H_ℓ cannot be a zero stratum. Suppose first that H_ℓ is almost periodic, so H_ℓ consists of at most two edges, E_ℓ and E'_ℓ. Since h is a topological representative, we have that the path $[\tilde{x}, h(\tilde{x})]$ is not contained in a single edge and thus contains a vertex $\tilde{v}$ that projects to the initial or terminal vertex of some edge of H_ℓ, say E_ℓ. Lemma 5.15 implies that $[\tilde{x}, h(\tilde{x})]$ can be split at $\tilde{v}$ and thus that $\tilde{v} \in \tilde{X}$. (Let us remark that the hypotheses of that lemma require H_ℓ to be a single edge, but the proof also works for a dihedral pair.)

Suppose that H_ℓ is non-exponentially growing but not almost periodic, so H_ℓ consists of a single edge E_ℓ and $f(E_\ell) = g_\ell E_\ell u_\ell$ for some path u_ℓ in $G_{\ell-1}$. After replacing $\tilde{x}$ by $h^k(\tilde{x})$ for some $k \geq 0$ if necessary, we may assume that the path $[\tilde{x}, h(\tilde{x})]$ contains an entire edge $\tilde{e}$ projecting to either E_ℓ or $\bar{E}_\ell$ and that the projection of $h(\tilde{x})$ is not contained in the interior of E_ℓ. If $\tilde{e}$ projects to E_ℓ, let $\tilde{v}$ be the initial vertex of $\tilde{e}$; otherwise let it be the terminal vertex of $\tilde{e}$. Lemma 5.15 implies that $[\tilde{x}, h(\tilde{x})]$ can be split at $\tilde{v}$, so $\tilde{v} \in \tilde{X}$.

Finally suppose H_ℓ is exponentially growing. After replacing $\tilde{x}$ by $h^k(\tilde{x})$ for some $k \geq 0$ if necessary, we may assume that each path $[h^i(\tilde{x}), h^{i+1}(\tilde{x})]$ projects to a path with the same number of illegal turns in H_ℓ. Lemma 2.5 produces a splitting of $[\tilde{x}, h(\tilde{x})]$. If one of the resulting pieces is not ℓ-legal, it is a lift $\tilde{\rho}$ of one of finitely many equivalence classes of paths $\rho \in P_\ell$. Let $\tilde{v}$ be the initial endpoint of $\tilde{\rho}$. Replacing $\tilde{v}$ by $h^k(\tilde{v})$ if necessary, we may assume that the point $\tilde{v}$ projects to a periodic point in the interior of H_ℓ. In fact, after iterating we may assume that $\tilde{\rho}$ projects to an indivisible periodic (hence fixed by (Vertices)) almost Nielsen path of height ℓ, so we may assume that $\tilde{v}$ is a vertex. If there are no $\tilde{\rho}$ pieces, then $[\tilde{x}, h(\tilde{x})]$ projects to an ℓ-legal path. After replacing $\tilde{x}$ by $h^k(\tilde{x})$ we may assume that the path $[\tilde{x}, h(\tilde{x})]$ contains an entire edge that projects into H_ℓ. Lemma 2.1 implies that $[\tilde{x}, h(\tilde{x})]$ may be split at any endpoint $\tilde{v}$ of this edge. Replacing $\tilde{v}$ by $h^k(\tilde{v})$ if necessary, we may assume that $\tilde{v}$ projects to a periodic vertex.

The path in Γ_{s-1} from the terminal vertex of $\tilde{E}_s$ to $\tilde{v}$ projects to a path τ in G_{s-1}. It is immediate from the definition of sliding and Lemma 2.10 that after sliding along τ, we have $f(E_s) = g_s E_s u_s$, where u_s is the projected image of the path $[\tilde{v}, h(\tilde{v})]$. Thus by replacing $\tilde{v}$ with $h^k(\tilde{v})$ for some $k \geq 0$, we may replace u_s with $f^k_\sharp(u_s)$. Since we assume that $f|_{G_{s-1}}$ satisfies (Completely Split), we may assume by Lemma 6.12 that u_s is completely split. By replacing $\tilde{v}$ with another vertex in $\tilde{X}$, we may also alter u_s as follows: if $u_s = \alpha \cdot \beta$ is a coarsening of the complete splitting of u_s, we may alter u_s so that the terminal vertex of E_s is the initial vertex of β.

Therefore to arrange item 3, it suffices to prove that either some term of the complete splitting of u_s has a principal endpoint or that u_s is entirely contained in a dihedral pair. So suppose u_s is not entirely contained in a dihedral pair. Then since all other non-principal vertices are contained in exponentially-growing strata, the only way the complete splitting of u_s could fail to contain a principal vertex is if ℓ is exponentially growing and each height-ℓ term of the complete splitting of u_s is a single edge. But then by replacing u_s by $f^k_\sharp(u_s)$ for sufficiently large k, we may assume that u_s has a long ℓ-legal segment so that every edge in H_ℓ appears as a term in the complete splitting of u_s. Lemma 5.9 implies the existence of a principal vertex.

If E_s is a non-dihedral linear edge, choose a root-free almost Nielsen path w_s and $d_s \neq 0$ so that $u_s = w_s^{d_s}$. If $E_t \subset G_{s-1}$ is a linear edge with the same axis, then after reversing the orientation on w_s and multiplying d_s by -1, we may assume that w_s and w_t agree as oriented circuits. After sliding to change the order of the edges in w_s we may further assume $w_s = w_t$. Since $E_s \bar{E}_t$ is not an almost Nielsen path by assumption, we conclude that $d_s \neq d_t$, proving item 4, (Linear Edges) in this case.

If instead E_s is dihedral linear, write the axis for E_s as $\sigma\tau$ so that $u_s = (\sigma\tau)^{d_s}$ or u_s is homotopic to $(\sigma\tau)^{d_s}\sigma$ for some integer d_s (with $d_s \neq 0$ in the first case). If E_t is a linear (hence dihedral linear) edge in G_{s-1} with the same axis, then u_t also has the above form for some integer d_t. Since by assumption $E_s \bar{E}_t$ is not an almost Nielsen path by assumption, we conclude that if u_s and u_t are both of the former type or both of the latter type, then $d_s \neq d_t$, again proving item 4.

Following [BFH00, Proposition 5.4.3], we will show that if w_s is a nontrivial initial segment of u_s, then $E_s \cdot w_s$ is a splitting. This is equivalent to the claim that $[\tilde{v} h^i(\tilde{v})]$ is contained in $[\tilde{v}, h^i(\tilde{y})]$ for all points $\tilde{y}$ in the path $[\tilde{v}, h(\tilde{v})]$.

Since $\tilde{v} \in \tilde{X}$, we have that $f(E_s) = E_s \cdot u_s$ is a nontrivial splitting. In order to prove item 2, we will prove the stronger statement

5. If $\tilde{w}_s$ is an initial segment of u_s, then $E_s \cdot w_s$ is a splitting.

This is equivalent to showing that for all $\tilde{y} \in [\tilde{v}, h(\tilde{v})]$, we have that $[\tilde{v}, h^i(\tilde{v})]$ is contained in $[\tilde{v}, h^i(\tilde{y})]$ for all $i \geq 0$. Write $\tilde{u}_s = [\tilde{v}, h(\tilde{v})]$. There is a splitting $\tilde{u}_s = \tilde{\sigma}_1 \cdot \tilde{\sigma}_2 \cdots \tilde{\sigma}_n$ provided by Lemma 5.15 if H_ℓ is non-exponentially growing and by Lemma 2.5 if H_ℓ is exponentially growing. We have $\tilde{R}_{\tilde{v}} = \tilde{u}_s \cdot h_\sharp(\tilde{u}_s) \cdot h_\sharp^2(\tilde{u}_s) \cdots$ which yields a splitting $\tilde{R}_{\tilde{v}} = \tilde{\sigma}_1 \cdot \tilde{\sigma}_2 \cdots$, where $\tilde{\sigma}_{in+j} = h_\sharp^i(\tilde{\sigma}_j)$. To show that the stated claim holds, it suffices to show that $h^i(\tilde{u}_s)$ intersects $h_\sharp^{i-1}(\tilde{u}_s)$ in a point. We will prove the stronger statement that $h^i(\tilde{\sigma}_j)$, which tightens to $\tilde{\sigma}_{in+j}$, intersects $\tilde{\sigma}_{in+j-1}$ in a point.

The proof breaks into cases, depending on whether H_ℓ is exponentially growing or non-exponentially growing. Suppose first that it is non-exponentially growing. If the initial edge of $\tilde{\sigma}_j$ is a lift of E_ℓ, then Lemma 5.15 implies that $h^i(\tilde{\sigma}_j)$ is a lift of E_ℓ possibly followed by a sequence of edges lifting edges in $G_{\ell-1}$ and possibly terminating in a lift of $\bar{E}_\ell$. If $h^i(\tilde{\sigma}_j)$ *does* terminate in a lift of $\bar{E}_\ell$, then the sequence of edges lifting edges in $G_{\ell-1}$ tightens to a nontrivial path. Observe then that the initial lift of E_ℓ is disjoint from the rest of $h^i(\tilde{\sigma}_j)$, so prevents $h^i(\tilde{\sigma}_j)$ from intersecting $\tilde{\sigma}_{in+j-1}$ in more than one point. On the other hand, if $\tilde{\sigma}_j$ does not begin with a lift of E_ℓ, then the terminal end of $\tilde{\sigma}_{in+j-1}$ is a lift of $\bar{E}_\ell$ and $h^i(\tilde{\sigma}_j)$ is a sequence of edges lifting edges in $G_{\ell-1}$ possibly followed by a lift of $\bar{E}_\ell$. Again, if the last edge of $h^i(\tilde{\sigma}_j)$ is a lift of $\bar{E}_\ell$, then the sequence of edges lifting edges in $G_{\ell-1}$ tightens to a nontrivial path. We see that edges lifting edges in $G_{\ell-1}$ cannot cross the last edge of $\tilde{\sigma}_{in+j-1}$ and the final lift of $\bar{E}_\ell$ cannot because it is separated by the homotopically nontrivial path the edges lifting edges in $G_{\ell-1}$ tighten to form. This completes the analysis in the case that H_ℓ is non-exponentially growing.

Suppose on the other hand that H_ℓ is exponentially growing. If $\tilde{\sigma}_j = \tilde{\rho}$ for some path $\rho \in P_\ell$, write $\rho = \alpha\beta$ for a decomposition of ρ into ℓ-legal subpaths. Although the terminal end of $h^i(\tilde{\alpha})$ and the initial end of $h^i(\beta)$ agree up to a point, there is an initial subpath of $h^i(\tilde{\alpha})$ that is disjoint from the rest of $h^i(\tilde{\sigma}_j)$; as above this initial subpath prevents $h^i(\tilde{\sigma}_j)$ from intersecting $\tilde{\sigma}_{in+j-1}$ in more than a point. If $\tilde{\sigma}_j$ is not a lift of some $\rho \in P_\ell$, then σ_j is ℓ-legal. If the initial edge of σ_j is in H_ℓ, then Lemma 2.1 implies that the initial edge of $h^i(\tilde{\sigma}_j)$ is disjoint from the rest of $h^i(\tilde{\sigma}_j)$, and thus prevents $h^i(\tilde{\sigma}_j)$ from intersecting $\tilde{\sigma}_{in+j-1}$ in more than a point. If instead the initial edge of σ_j is in $G_{\ell-1}$, then the terminal edge of $\tilde{\sigma}_{in+j-1}$ projects to H_ℓ and prevents $h^i(\tilde{\sigma}_j)$ from intersecting $\tilde{\sigma}_{in+j-1}$ in more than a point: this is because edges that project to $G_{\ell-1}$ cannot cross into H_ℓ and edges that project into H_ℓ are part of $h_\sharp^i(\tilde{\sigma}_j)$. This completes the analysis in the exponentially growing case and with it the proof of item 5.

Before proving item 2, we establish the following.

6. The path u_s is a periodic Nielsen path if and only if there is a non-peripheral element $c \in F$ such that T_c preserves Γ_{i-1} and h and $T_c|_{\Gamma_{i-1}}$ commute. In this case the infinite ray $\tilde{R}_{\tilde{v}} = \tilde{u}_s h_\sharp(\tilde{u}_s) h_\sharp^2(\tilde{u}_s) \ldots$ is contained in the axis of T_c.

To see this, notice that first if $f_\sharp^k(u_s) = u_s$ for some $k > 0$, the fact that $f(E_s) = E_s \cdot u_s$ is a splitting implies that the element c of $\pi_1(G_{s-1}, v)$ determined by the loop $u_s f_\sharp(u_s) \ldots f_\sharp^{k-1}(u_s)$ is nonperipheral, and the ray $\tilde{u}_s h_\sharp(\tilde{u}_s) h_\sharp^2(\tilde{u}_s) \ldots$ is contained in the axis of T_c. Since $h_\sharp$ preserves the axis of T_c, h commutes with T.

Conversely, suppose that there is a nonperipheral element $c \in F$ such that T_c preserves Γ_{s-1} and $T_c|_{\Gamma_{s-1}}$ commutes with h. We have $T_c([\tilde{x}, h(\tilde{x})]) = [T_c(\tilde{x}), h(T_c(\tilde{x}))]$ for all $\tilde{x}$, so in particular $\tilde{R}_{T_c(\tilde{v})} = T_c(\tilde{R}_{\tilde{v}})$. Thus $\tilde{R}_{\tilde{v}}$ and $T_c(\tilde{R}_{\tilde{v}})$ have an infinite end in common, and therefore $\tilde{R}_{\tilde{v}}$ and the axis of T_c have an infinite end in common. It follows that $h^k(\tilde{v})$ is contained in the axis of T_c for all sufficiently large k. This implies that there is a uniform bound to the length of $[h^k(\tilde{v}), h^{k+1}(\tilde{v})]$ and thus that $f_\sharp^k(u_s)$ takes on only finitely many values up to multiplying by vertex group elements at the ends. These vertex group elements themselves take on only finitely many values, so by replacing $\tilde{v}$ with some $h^k(\tilde{v})$ if necessary, we may assume that u_s is a periodic Nielsen path and that $\tilde{R}_{\tilde{v}}$ is contained in the axis of T_c. This verifies item 6.

We now turn to the proof of item 2, for which we follow the argument in [BFH00, Lemma

5.5.1]. Note first that by Lemma 5.15, if σ is a path or circuit with endpoints at vertices and height s, then σ has a splitting into subpaths in G_{s-1} and paths of the form $E_s\gamma$, $E_s\gamma\bar{E}_s$ or $\gamma\bar{E}_s$, where γ is a path in G_{s-1}. Therefore to complete the proof of item 2, we will show that the following hold, assuming γ is a nontrivial tight path in G_{s-1}.

(i) If $E_s\gamma$ (respectively $E_s\gamma\bar{E}_s$) can be split at a point in the interior of the copy of E_s, then $f_\sharp^m(E_s\gamma) = E_s \cdot \gamma_1$ (respectively $f_\sharp^m(E_s\gamma\bar{E}_s) = E_s \cdot \gamma_1\bar{E}_s$) for some $m \geq 0$ and tight path γ_1 in G_{s-1}.

(ii) If $E_s\gamma$ has no splittings, then $f_\sharp^m(E_s\gamma)$ is an exceptional path of height s for some $m \geq 0$.

(iii) If $E_s\gamma\bar{E}_s$ has no splittings, then $E_s\gamma\bar{E}_s$ is an exceptional path of height s.

Item 5 implies that for any nontrivial initial segment σ_1 of E_s, some $f_\sharp^m(\sigma_1) = E_s \cdot \gamma'$, where γ' is contained in G_{s-1}. Thus if a path σ of height s splits as $\sigma = \sigma_1 \cdot \sigma_2$ where σ_1 is a nontrivial initial segment of E_i, then some $f_\sharp^m(\sigma)$ has a splitting of the form $E_i \cdot \sigma'$. Item (i) follows.

Therefore write σ for a path of the form $E_s\gamma$ or $E_s\gamma\bar{E}_s$ and suppose that σ has no splittings.

Step 1: Cancelling large middle segments. Suppose that $\sigma = \sigma_1'\sigma_2'\sigma_3'$ is a decomposition of σ into nontrivial subpaths. We will show that there exists $M > 0$ and a decomposition $\sigma = \sigma_1\sigma_2\sigma_3$ where σ_1 is an initial subpath of σ_1' and σ_3 is a terminal subpath of σ_3' such that $f_\sharp^M(\sigma) = f_\sharp^M(\sigma_1)f_\sharp^M(\sigma_3)$ and the indicated juncture is a vertex. By making σ_1' and σ_3' smaller, we may assume that they are contained in the initial and terminal edges of σ respectively.

Choose lifts $\tilde{f}$ and $\tilde{\sigma} = \tilde{\sigma}_1'\tilde{\sigma}_2'\tilde{\sigma}_3'$. Observe that the set

$$\tilde{S}_k = \{\tilde{x} \in \tilde{\sigma} : \tilde{f}^k(\tilde{x}) \in \tilde{f}_\sharp^k(\tilde{\sigma})\}$$

is closed and that $\tilde{\sigma}$ can be split at any point of $\bigcap_{k=1}^\infty \tilde{S}_k$. Since there are no splittings, this infinite intersection contains only the endpoints of $\tilde{\sigma}$, and thus there exists $M > 0$ such that $\bigcap_{k=1}^M \tilde{S}_k \subset \tilde{\sigma}_1' \cup \tilde{\sigma}_3'$. One can argue by induction that $\tilde{f}^N$ maps $\bigcap_{k=1}^N \tilde{S}_k$ onto $\tilde{f}_\sharp^N(\tilde{\sigma})$ for all $N \geq 1$. Since the lift of $\tilde{E}_s$ that is the initial edge of $\tilde{f}^k(\tilde{\sigma})$ is not canceled when $\tilde{f}^k(\tilde{\sigma})$ is tightened, each $\tilde{S}_k$ and thus $\bigcap_{k=1}^M \tilde{S}_k$ contains an initial segment of a lift of E_s. Therefore choose a point $\tilde{x}$ in the intersection of $\bigcap_{k=1}^M \tilde{S}_k$ with $\tilde{\sigma}_1'$ with the property that $\tilde{f}^M(\tilde{x})$ is as close to the terminal end of $\tilde{f}_\sharp^M(\sigma)$ as possible. This condition guarantees that $\tilde{f}^M(\tilde{x})$ is a vertex; let $\tilde{\sigma}_1$ be the initial segment of the lift of E_s terminating at $\tilde{x}$. We have that $\tilde{f}_\sharp^M(\tilde{\sigma}_1)$ is a proper initial subinterval of $\tilde{f}_\sharp^M(\tilde{\sigma})$, for if not, then $f_\sharp^M(\sigma) = f^i\sharp(E_s w_s)$ for some i and some initial segment w_s of u_s. This contradicts item 5 above and the assumption that σ has no splittings. Finally observe that there are points of $\bigcap_{k=1}^M \tilde{S}_k$ in $\tilde{\sigma}_3'$ that map arbitrarily close to $\tilde{f}^M(\tilde{x})$, and since $\bigcap_{k=1}^M \tilde{S}_k$ is closed there exists $\tilde{y}$ in the intersection of $\bigcap_{k=1}^M \tilde{S}_k$ and $\tilde{\sigma}_3'$ such that $\tilde{f}^M(\tilde{y}) = \tilde{f}^M(\tilde{x})$. The subdivision of $\tilde{\sigma}$ at $\tilde{x}$ and $\tilde{y}$ provides the decomposition $\sigma = \sigma_1\sigma_2\sigma_3$ required.

Suppose for a moment that $\sigma = E_i\gamma$. An immediate consequence of Step 1 above is that the last edge of σ is not eventually mapped to an almost periodic stratum. By replacing σ by some $f_\sharp^i(\sigma)$ if necessary, then we may assume that the last edge of $f_\sharp^k(\sigma)$ is contained in the same stratum for all $k \geq 0$. Since $f|_{G_{s-1}}$ is a CT, we have that one of the following conditions is satisfied.

(iv) The final edge of σ is a non-exponentially growing but not almost periodic edge E_j.

(v) The final edge of σ is contained in an exponentially growing stratum H_r.

If $\sigma = E_s\gamma\bar{E}_s$, then Item (iv) holds with $j = s$ without replacing σ. We suppose at first that Item (iv) holds.

Step 2: At least three blocks cancel. Write $\sigma = E_s\gamma'\bar{E}_j$, where $\gamma = \gamma'$ if $j = s$. Define the ray R_s to be the infinite path $u_s \cdot f_\sharp(u_s) \cdot f_\sharp^2(u_s) \cdots$ and let R_s^m be the initial segment $u_s \cdot f_\sharp(u_s) \cdots f_\sharp^{m-1}(u_s)$. We refer to the paths $f_\sharp^k(u_s)$ as *blocks* of R_s. By Lemma 6.6, we have may define the ray R_j and R_j^m analogously with u_j replacing u_s. We have that $f^m(\sigma) = gE_sR_s^m f^m(\gamma')\bar{R}_j^m\bar{E}_jh$, and we claim that if m is sufficiently large, then a subpath of R_i^m containing at least three blocks of R_i cancels with a subpath of $\bar{R}_j^m$ containing at least three blocks of $\bar{R}_j$ when $gE_iR_i^m f^m(\gamma')\bar{R}_j^m\bar{E}_jh$ is tightened to $f_\sharp^m(\sigma)$.

By Step 1, there exist a positive integer M and initial subpaths μ_M of R_s^M and $\bar{\nu}_M$ of $\bar{R}_j^M$ such that $f_\sharp^M(\sigma) = g'E_i\mu_M\bar{\nu}_M\bar{E}_jh'$. Since we have $g'E_s\mu_M = f_\sharp^i(E_sw_s)$ for some initial segment w_s of u_s, Item 5 above implies that $g'E_s\mu_M = g'E_s \cdot \mu_M$ and similarly $\bar{\nu}_M\bar{E}_jh' = \bar{\nu}_M \cdot \bar{E}_jh'$. Thus $f_\sharp^m(\sigma)$ is obtained from the concatenation of $g''E_iR_i^{m-M}f_\sharp^{m-M}(\mu_m)$ and $f_\sharp^{m-M}(\bar{\nu}_M)\bar{R}_j^{m-M}\bar{E}_jh''$ by cancelling at the juncture. Step 1 implies that for m sufficiently large, long cancellation must occur in both R_s^{m-M} and $\bar{R}_j^{m-M}$. The only way this can happen is if long segments of R_s^{m-M} and R_j^{m-M} cancel with each other, verifying the claim.

Step 3: When H_ℓ is non-exponentially growing. The third step is the following claim.

Claim 6.14 (cf. Lemma 5.5.1, Sublemma 1 of [BFH00])**.** *If E_s and E_j (with $s \geq j$) have distinct lifts $\tilde{E}_s$ and $\tilde{E}_j$ whose corresponding rays $\tilde{R}_s$ and $\tilde{R}_j$ have a subpath in common that contains at least three blocks in each ray and if H_ℓ is non-exponentially growing, then the path $\tilde{\delta}$ that connects the initial endpoint of $\tilde{E}_i$ to the terminal endpoint of $\tilde{E}_j$ projects to an exceptional path of height s.*

Proof. We follow the proof of [BFH00, Lemma 5.5.1, Sublemma 1]. The set-up is a variant of that in the paragraph "Restricting to G_{s-1} for non-exponentially growing strata." Let Γ_{s-1} be the component of the full preimage of G_{s-1} that contains $\tilde{u}_s$ and $\tilde{u}_j$, and write $h_s\colon \Gamma_{s-1} \to \Gamma_{s-1}$ and $h_j\colon \Gamma_{s-1} \to \Gamma_{s-1}$ for the restricted lifts of f fixing the initial endpoints of $\tilde{E}_s$ and $\tilde{E}_j$ respectively.

Suppose first that $h_s = h_j$. By the argument at the beginning of this lemma, we have that h_s has no fixed points, so in particular the initial endpoint of $\tilde{E}_j$ does not belong to Γ_{s-1} and we conclude $E_s = E_j$. The path $\tilde{\delta}$ projects to an almost Nielsen path for f. There is a loop $\gamma \in G_{s-1}$ lifting to the tight path in Γ_{s-1} from the terminal vertex of $\tilde{E}_s$ to the terminal vertex of $\tilde{E}_j$ such that the resulting automorphism $T_{[\gamma]}$ of the natural projection $\Gamma_{s-1} \to G_{s-1}$ takes $\tilde{u}_s$ to $\tilde{u}_j$. Indeed, if $\tilde{v}$ is the initial endpoint of $\tilde{u}_s$, we have $T_{[\gamma]}h_s(\tilde{v}) = h_sT_{[\gamma]}(\tilde{v})$ is the terminal endpoint of $\tilde{u}_j$. On the other hand, $h_sT_{[\gamma]} = T_{(h_s)_\sharp([\gamma])}h_s$, so we see that the element $[\gamma]^{-1}(h_s)_\sharp([\gamma]) = [\bar{\gamma}u_sf_\sharp(\gamma)\bar{u}_s]$ of $\pi_1(G_{i-1}, v)$ stabilizes $h(\tilde{v})$, and is thus homotopic to a loop of the form $u_sg\bar{u}_s$ for some $g \in \mathcal{G}_v$. Rewriting, we see that this implies that $f_\sharp(\gamma)$ is homotopic to $\bar{u}_s\gamma u_sg$. Because $E_s\gamma\bar{E}_s$ is an almost Nielsen path, we have $u_s\bar{u}_s\gamma u_sg\bar{u}_s$ is homotopic to γ, so we conclude that g is the identity, and thus that $T_{[\gamma]}$ and h commute. Note that if $T_{[\gamma]}$ were peripheral, its fixed point would be contained in Γ_{s-1} and preserved by h, so since h is fixed-point free, $T_{[\gamma]}$ is non-peripheral. By item 6, we have that u_s is a periodic Nielsen path. If the initial vertex of u_s is principal, then Lemma 5.18 implies that u_s has period one as an almost Nielsen path, so E_s is almost linear and the discussion after the definition of almost linear edges implies that E_s is linear, say with axis w_s. If the initial vertex of u_s is not principal, then u_s is contained in a dihedral pair and E_s is a linear or dihedral linear edge, say with axis w_s. In this case both $\tilde{R}_s$ and $\tilde{R}_j$ are contained in the axis of $T[\gamma]$ and γ itself takes the form w_s^p for some p. Thus the projection δ of $\tilde{\delta}$ is exceptional.

Thus we now assume that $h_s \neq h_j$. Let $\tilde{V}$ be the set of vertices in $\tilde{R}_s$ that are either the initial endpoint of a lift of E_ℓ or the terminal endpoint of a lift of $\bar{E}_\ell$. Order the elements of $\tilde{V}$ so that $\tilde{x}_p < \tilde{x}_{p+1}$ in the orientation on $\tilde{R}_s$. Lemma 5.15 implies that $h_s(\tilde{x}_p) = \tilde{x}_{p+n_0}$ for all p and fixed n_0. What's more, h_s takes the direction of the lift of E_ℓ to the direction of the lift of E_ℓ. Define $\tilde{W}$ and m_0 using $\tilde{R}_j$ and h_j instead of $\tilde{R}_s$ and h_s. The intersection of $\tilde{V}$ and $\tilde{W}$ contains by assumption at least $n_0 + m_0 + 1$ consecutive elements $\tilde{z}_0, \ldots, \tilde{z}_{n_0+m_0}$. We have $h_s h_j(\tilde{z}_0) = h_s(\tilde{z}_{m_0}) = \tilde{z}_{n_0+m_0} = h_j(\tilde{z}_{n_0}) = h_j h_s(\tilde{z}_0)$, and the two maps agree on the direction of E_ℓ at $\tilde{z}_0$, so we conclude that as lifts of f^2, we have $h_s h_j = h_j h_s$. There is a nontrivial element $c \in F$ such that T_c preserves Γ_{s-1} and $T_c h_s = h_j$, and there is an element $d \in F$ such that $T_d h_j h_s T_c$. We have $h_i h_j = h_i T_c h_s = T_d h_j h_i$, so we conclude that $d = 1$ and that h_s commutes with T_c. A symmetric argument shows that T_c commutes with h_j. The argument in the case $h_s = h_j$ shows that T_c must be non-peripheral. Item 6 above then implies that u_s and u_j are periodic Nielsen paths and that $\tilde{R}_s$ and $\tilde{R}_j$ are contained in the axis of T_c. If the initial vertex of u_s is principal, then so is the initial vertex of u_j, and we conclude that E_s and E_j are linear edges with the same axis w. If the initial vertex of u_s is not principal, then u_s and hence u_j are contained in a dihedral pair and E_s and E_j are both linear or both dihedral linear edges with the same axis w. The segment of the axis of T_c that separates the terminal endpoint of $\tilde{E}_s$ from the terminal endpoint of $\tilde{E}_j$ projects to w^p for some integer p. Supposing E_s and E_j are linear, since $\tilde{R}_s$ and $\tilde{R}_j$ have a common subpath containing blocks in both rays, we have $u_s = w^{d_s}$ and $u_j = w^{d_j}$, where d_s and d_j have the same sign, and we conclude that $\delta = E_s w^p \bar{E}_j$ is exceptional. If E_s and E_j are dihedral linear and $w = (\sigma\tau)$, again because $\tilde{R}_s$ and $\tilde{R}_j$ share blocks in both rays, we have that u_s is homotopic to $(\sigma\tau)^{d_s}\sigma$ and u_j is homotopic to $(\sigma\tau)^{d_j}\sigma$, where d_s and d_j have the same sign, and we again conclude that δ is exceptional. □

Step 4: When H_ℓ is exponentially growing. Suppose that m is chosen as in Step 2 and that H_ℓ is exponentially growing. We argue that $f^m_\sharp(\sigma)$ is an exceptional path of height s. Write $h_s \colon \Gamma_{s-1} \to \Gamma_{s-1}$ and $h_j \colon \Gamma_{s-1} \to \Gamma_{s-1}$ as in Step 3.

If the splitting of u_s given by Lemma 2.5 contains a path ρ_ℓ or $\bar{\rho}_\ell$ in P_ℓ, then the argument in Step 3 goes through with $\tilde{V}$ defined to be the set of lifts of ρ_ℓ or $\bar{\rho}_\ell$. These are "translated" by h_s and h_j and their initial and terminal directions are mapped to each other, so the argument in that step works without further change.

It remains to rule out the possibility that u_s is ℓ-legal, i.e. that there are no lifts of ρ_ℓ or $\bar{\rho}_\ell$. Since blocks of $\tilde{R}_s$ cancel with segments of $\tilde{R}_j$, we conclude that u_j is also ℓ-legal. By Step 2, we have that $f^m_\sharp(\sigma) = gE_s\mu_m\bar{\nu}_m\bar{E}_jh$, where μ_m and $\bar{\nu}_m$ are initial subpaths of R_s and $\bar{R}_j$ and so are ℓ-legal. Lemma 2.3 implies that μ_m and $\bar{\nu}_m$ take on only finitely many values as m varies, so some $f^m_\sharp(\sigma)$ is a periodic Nielsen path, say $f^{m+p}_\sharp(\sigma) = f^m_\sharp(\sigma)$. There is a lift $\tilde{f} \colon \Gamma \to \Gamma$ whose restriction to Γ_{s-1} equals h_s; we have that $\tilde{f}$ fixes the initial endpoint of $\tilde{E}_i$ and $\tilde{f}^p$ fixes the initial endpoint $\tilde{w}$ of $\tilde{E}_j$.

If $E_s \neq E_j$, then $\tilde{w} \in \Gamma_{s-1}$, and the fact that $\mathrm{Fix}(h_s) = \varnothing$ implies that $p > 1$. Write $\tilde{\gamma}$ for the path that connects $\tilde{w}$ to $h_s(\tilde{w})$. There is a projection γ of $\tilde{\gamma}$ that is a periodic Nielsen path with a principal endpoint and hence a Nielsen path. We have that $[\gamma^p] = [\gamma f(\gamma) \ldots f^{p-1}(\gamma)]$ lifts to the trivial path $[\tilde{\gamma} h_s(\tilde{\gamma}) \ldots h_s^{p-1}(\tilde{\gamma})]$, so the closed path γ represents a peripheral element of $\pi_1(G_{s-1}, w)$, so we may write $\gamma = \gamma' g \bar{\gamma}'$ for some vertex group element g and path γ' in G_{s-1}. Since $f_\sharp(\gamma) = f_\sharp(\gamma') f_\sharp(g) f_\sharp(\bar{\gamma}') = \gamma' g \bar{\gamma}$, we conclude that γ' is an almost Nielsen path. In the notation of Section 1, with respect to the basepoint w, the map h_s corresponds to the path γ, i.e. if $\tilde{x} \in \Gamma_{s-1}$ corresponds to the homotopy class of the path τ starting at w, the point $h_s(\tilde{x})$ corresponds to the homotopy class of the path $\gamma f(\tau)$. But if we let $\tilde{x}$ be the point corresponding to the path γ', we see that $\tilde{x}$ is fixed by h since γ' is an almost Nielsen path. This contradiction implies that $E_s = E_j$. The automorphism T of the natural projection that carries the direction of $\tilde{E}_s$ to the direction of $\tilde{E}_j$ commutes with $\tilde{f}^p$, so the restriction of T to Γ_{s-1} therefore commutes with h_s^p. But since $h_j^p = (T|_{\Gamma_{s-1}}) h_s^p (T|_{\Gamma_{s-1}})^{-1}$, we conclude $h_s^p = h_j^p$. Therefore the ray $\tilde{R}_s$ and $T(\tilde{R}_s) = \tilde{R}_j$ have an infinite end in common, and therefore T is non-peripheral and $\tilde{R}_s$ and the axis of T have an infinite end in common. We conclude by Item 6 that E_s is a linear edge, in contradiction to the assumption that u_s is ℓ-legal.

Step 5: Items (ii) and (iii) hold when (iv) does. Suppose Item (iv) holds, so that if we choose m as in Step 2, $f_\sharp^m(\sigma)$ begins with E_s and ends with some non-exponentially growing edge $\bar{E}_j$. Choose a lift of $f_\sharp^m(\sigma)$, and let $\tilde{R}_s$ and $\tilde{R}_j$ be the lifts of the rays R_s and R_j that begin at the terminal endpoints of $\tilde{E}_s$ and $\tilde{E}_j$ respectively. We have chosen m such that $\tilde{R}_i \cap \tilde{R}_j$ contains at least three blocks in each ray. By Steps 3 and 4, we have that $f_\sharp^m(\sigma)$ is an exceptional path of height s. If $s = j$, then $f_\sharp^m(\tilde{\sigma})$ is fixed by $f_\sharp$. Since σ and $f_\sharp^m(\sigma)$ have the same endpoints and the same image under $f_\sharp^m$, we conclude that $\sigma = f_\sharp^m(\sigma)$ is an exceptional path of height s.

Step 6: Item (v) does not occur. Suppose that the final edge of σ belongs to an exponentially growing stratum H_r. Step 1 implies that for all sufficiently large m, $f_\sharp^m(E_s\gamma) = gE_s\mu_m\nu_m$, where μ_m belongs to R_s and ν_m is r-legal. Furthermore, Step 1 implies that if $(M-m)$ is sufficiently large, then edges of H_r in $f_\sharp^{M-m}(\nu_m)$ cancel with edges of $f_\sharp^{M-m}(\mu_m)$, so $r \leq \ell$, and in fact a symmetric argument shows that $\ell = r$ and that μ_m is ℓ-legal. Lemma 2.3 implies that μ_m takes on only finitely many values, and the same is true for ν_m up to multiplication by vertex group elements at the end: the argument is essentially the same as that given in Lemma 2.4. Therefore we conclude that $f_\sharp^m(\sigma)$ is a periodic almost Nielsen path. But the argument in the last paragraph of Step 4 shows that this is impossible. Therefore we conclude that Item (v) does not occur, completing the proof. $\square$

We are ready to state and prove Theorem A.

Theorem 6.15. *Suppose $\varphi \in \mathrm{Out}(F, \mathcal{A})$ is rotationless and $\mathcal{C}$ is a nested sequence of φ-invariant free factor systems. Then φ is represented by a CT $f\colon \mathcal{G} \to \mathcal{G}$ and filtration $\varnothing = G_0 \subset G_1 \subset \cdots \subset G_m = G$ that realizes $\mathcal{C}$.*

Proof. We follow the outline of [FH11, Theorem 4.28]. We may assume that $\mathcal{C}$ is a maximal nested sequence with respect to $\sqsubset$, so that any filtration that realizes $\mathcal{C}$ is reduced. We claim that by Theorem 2.6 we may choose a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ with the following properties.

1. The topological representative f represents $\varphi \in \mathrm{Out}(F, \mathcal{A})$.

2. The filtration on $f\colon \mathcal{G} \to \mathcal{G}$ realizes $\mathcal{C}$.

3. Each contractible component of a filtration element is a union of zero strata.

4. The endpoints of all indivisible almost Nielsen paths of exponentially growing height are vertices.

Items 2 and 4 follow from Theorem 2.6. To see that item 3 does as well, suppose that C is a contractible component of some G_i. If C contains an edge in an irreducible stratum, observe that it is non-wandering and by property (NEG) the lowest stratum H_j that has an edge in C is either almost periodic or exponentially growing. It cannot be almost periodic by Lemma 2.11 and cannot be exponentially growing by Lemma 2.7. Therefore every edge in C is contained in a zero stratum. There is no loss in dividing up zero strata so that item 3 holds. (Notice as well that item 3 implies that contractible components of filtration elements are wandering, and thus contain no vertices with nontrivial vertex group.)

We will assume that all relative train track maps in this proof satisfy the above properties.

Step 1: (EG Almost Nielsen Paths). Let $N(f)$ be the number of indivisible almost Nielsen paths of exponentially growing height. We adapt Feighn–Handel's algorithm for proving (EG Almost Nielsen Paths): suppose some exponentially growing stratum does not satisfy (EG Almost Nielsen Paths) and let H_r be the highest such stratum. We will show in Lemma 6.16 that this implies that there is a sequence of proper folds at indivisible Nielsen paths of height r leading to a relative train track map with either a partial fold or an improper fold. If the fold is partial, then $N(f)$ may be decreased (Lemma 6.18) Since $N(f)$ is finite, eventually we may assume that all folds are full. We will show that improper folds allow us to decrease the number of edges in H_r (Lemma 6.20), while proper folds preserve the number of edges of H_r (Lemma 6.19). Both kinds of folds preserve the number of edges of each exponentially growing stratum H_s for $s > r$ (Lemma 6.20 and Lemma 6.19). Since this number is finite, eventually no improper folds occur, at which point H_r satisfies (EG Almost Nielsen Paths) by Lemma 6.16.

Lemma 6.16 (cf. Lemma 5.3.6 of [BFH00])**.** *Given a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$, an exponentially growing stratum H_r, and an indivisible almost Nielsen path ρ of height r, suppose that the fold at the illegal turn of each indivisible almost Nielsen path obtained by iteratively folding ρ is proper. Then H_r satisfies the conclusions of (EG Almost Nielsen Paths).*

In fact, by Lemma 6.8, (EG Almost Nielsen Paths) is equivalent to the statement that the fold at the illegal turn of each indivisible almost Nielsen path obtained by iteratively folding ρ is proper.

Proof. We follow the notation in the definition of folding the indivisible almost Nielsen path ρ. The proof is essentially identical to [BFH00, Lemma 5.3.6]. To ease notation, assume that $G = G_r$. Write ρ' for the indivisible almost Nielsen path of height r for $f'\colon \mathcal{G}' \to \mathcal{G}'$ determined by ρ, and let $\rho' = \alpha'\beta'$ be a decomposition into r-legal subpaths. Let $g'_1E'_1$ and $g'_2E'_2$ be the initial edges and vertex group elements of $\bar{\alpha}'$ and β' respectively and let $F'\colon \mathcal{G}' \to \mathcal{G}''$ be the extended fold determined by ρ'. We assume (up to reversing the orientation of ρ') that $f'(g'E'_1)$ is a proper subpath of $f'(g'_2E'_2)$, and we write $\alpha' = g'_1E'_1b'g'_3E'_3\ldots$ where b' is a maximal (possibly trivial) subpath in G'_{r-1} ending with trivial vertex group element and E'_3 is an edge of H'_r. Recall we have a map $g\colon \mathcal{G}' \to \mathcal{G}$ such that $f = gF$, where $F\colon \mathcal{G} \to \mathcal{G}'$ is the extended fold determined by ρ. Write E'_2 as a concatenation of subintervals $\mu'_1\mu'_2\mu'_3$ such that $f'(g'_2\mu'_1) = f'(g'_1E'_1)$ and $f'(\mu'_2) = f'_\sharp(b')$. (If b' is trivial, then μ'_2 is trivial.) We will show that if $g(E'_1)$ and $g(E'_2)$ have a nontrivial common initial segment, then $g(g'_2\mu'_1) = g(g'_1E'_1)$ and $g'(\mu'_2) = g'_\sharp(b')$. In other words, either the fold F' is a generalized fold factor of g, or g cannot be folded at the turn $\{(g'_1, E'_1), (g'_2, E'_2)\}$.

Suppose first that $g(g'_1E'_1)$ and $g(g'_2E'_2)$ have a common initial segment but that the fold for g is not full, i.e. that the maximal g-fold of $g'_1E'_1$ and $g'_2E'_2$ does not use all of μ'_1. Write $\hat{F}\colon \mathcal{G}' \to \hat{\mathcal{G}}$ for the maximal g-fold of $g'_1E'_1$ and $g'_2E'_2$, let $\hat{g}\colon \hat{\mathcal{G}} \to \mathcal{G}$ be the induced map satisfying $\hat{g}\hat{F} = g$ and let $\hat{G}_i = \hat{F}(G'_i)$ for $1 \le i \le r$. Since the fold is not full, $\hat{H}_r = \hat{F}(H'_r)$ has one more edge than H'_r and H_r.

Since the fold $\hat{F}$ is maximal, the map $\hat{g}$ cannot be folded at the newly created vertex. By Proposition 6.9, $\{(g'_1, E'_1), (g'_2, E'_2)\}$ is the only orbit of illegal turns for f' that involves an edge in H'_r. Note that every fold factor of g is a fold factor of $f' = Fg$, so we conclude that F' is the only fold for g that involves an edge of H'_r. Therefore all folds for $\hat{g}$ have both edges in $\hat{G}_{r-1}$. We claim that folding edges in $\hat{G}_{r-1}$ according to $\hat{g}|_{G_{r-1}}$ will not identify previously distinct vertices in $\hat{H}_r \cap \hat{G}_{r-1}$. To see this, observe that if $\hat{\gamma}$ is a nontrivial tight path in $\hat{G}_{r-1}$ with endpoints in $\hat{H}_r \cap \hat{G}_{r-1} = H'_r \cap G'_{r-1} = H_r \cap G_{r-1}$, then there exists a nontrivial tight path γ in G_r with the same endpoints satisfying $\hat{F}_\sharp F_\sharp(\gamma) = \hat{\gamma}$. It follows that $\hat{g}_\sharp(\hat{\gamma}) = f_\sharp(\gamma)$ is nontrivial. (Compare the argument in Lemma 1.5.) It follows that no new illegal turns involving edges of $\hat{H}_r$ are created by folding according to $\hat{g}$. Continue to fold edges in $\hat{G}_{r-1}$ according to $\hat{g}|_{\hat{G}_{r-1}}$ until no more folds are possible, and call the resulting composition of folds $f_{r-1}\colon \hat{\mathcal{G}} \to \mathcal{G}^*$. (By [Dun98], only finitely many folds are possible.) There is an induced immersion $\theta\colon \mathcal{G}^* \to \mathcal{G}$ satisfying $f = \theta f_{r-1}\hat{F}F$. Since $\mathcal{G}$ has no valence-one vertices with trivial vertex group, θ is an isomorphism of graphs of groups. Observe that $\theta(f_{r-1}(\hat{G}_{r-1})) = f(G_{r-1}) = G_{r-1}$, so we have $\theta(f_{r-1}(\hat{H}_r)) = H_r$. This is a contradiction, as $\theta f_{r-1}|_{\hat{H}_r}$ induces a bijection on the set of edges, but $\hat{H}_r$ has more edges than H_r. Therefore if g can be folded at the turn $\{(g'_1, E'_1), (g'_2, E'_2)\}$, then all of E'_1 can be folded with a proper initial segment of E'_2.

We turn now to completing the proof of the claim. Let E be the first edge of $f_\sharp(\beta)$ that is not part of the maximum common initial segment of $f_\sharp(\bar{\alpha})$ and $f_\sharp(\beta)$. Then the initial edge E' of $F(E) \subset \mathcal{G}'$ is the first edge of $g_\sharp(\beta')$ that is not part of the maximum common initial segment of $g_\sharp(\bar{\alpha}')$ and $g_\sharp(\beta')$. Since E is contained in H_r, the edges E' is contained in H'_r. In particular, E' is not contained in b', verifying the claim.

If g cannot be folded at the turn $\{(g'_1, E'_1), (g'_2, E'_2)\}$, then define $f_r = F$ and construct f_{r-1} and θ exactly as above. Otherwise let $F' \colon \mathcal{G}' \to \mathcal{G}''$ be the extended fold of ρ' with respect to $f' \colon \mathcal{G}' \to \mathcal{G}'$ and let $g' \colon \mathcal{G}'' \to \mathcal{G}$ be the induced map satisfying $g'F = g$. If g' cannot be folded at the illegal turn of ρ'', then define $f_r = F'F$ and construct f_{r-1} and θ as above. Otherwise, repeat the argument above to conclude that the extended fold of ρ'' is a fold factor of g'. We may continue this argument. Since there are finitely many folds in any factorization of $f \colon \mathcal{G} \to \mathcal{G}$, this process terminates, proving the lemma. $\square$

Lemma 6.17 (cf. Lemma 5.1.7 of [BFH00])**.** *Suppose that $f \colon \mathcal{G} \to \mathcal{G}$ is a relative train track map satisfying our standing assumptions. Suppose that H_r is an exponentially growing stratum and that ρ is an almost Nielsen path of height r that crosses some edge E of H_r exactly once and that the first and last edges of ρ are contained in H_r. Then the following hold.*

1. *The endpoints of ρ are distinct.*
2. *At most one endpoint of ρ has nontrivial vertex group.*
3. *If both endpoints are contained in G_{r-1}, then at least one endpoint is contained in a contractible component of G_{r-1}.*
4. *If one endpoint has nontrivial vertex group and the other is contained in G_{r-1}, it is contained in a contractible component of G_{r-1}.*

Proof. We follow the argument in [BFH00, Lemma 5.1.7]. Let $\mathcal{G}'$ be the graph of groups obtained from $\mathcal{G}$ by removing the edge E and adding a new edge E' with endpoints equal to the initial and terminal endpoints of ρ. Write $\rho = \alpha E \beta$. Define a homotopy equivalence $h \colon \mathcal{G} \to \mathcal{G}'$ that is the identity on all edges other than E and that satisfies $h(E) = \bar{\alpha} E' \bar{\beta}$.

Consider the free factor system $\mathcal{F}(G_{r-1} \cup E')$. It is φ-invariant. Clearly we have $\mathcal{F}(G_{r-1}) \sqsubset \mathcal{F}(G_{r-1} \cup E') \sqsubset \mathcal{F}(G_r)$. Suppose that the conclusions of the lemma fail. Then one of the following holds.

1. The endpoints of ρ are equal.
2. The endpoints of ρ are distinct and both have nontrivial vertex group.
3. The endpoints of ρ are distinct, exactly one has nontrivial vertex group and the other endpoint is contained in a noncontractible component of G_{r-1}.
4. Both endpoints are contained in noncontractible components of G_{r-1}.

In each case, we see that $\mathcal{F}(G_{r-1} \cup E')$ is strictly larger than $\mathcal{F}(G_{r-1})$. But since ρ is not r-legal, $\mathcal{F}(G_{r-1} \cup E')$ does not carry any line that is generic for the attracting lamination Λ^+ associated to H_r, (recall that f is eg-aperiodic by Lemma 4.7) so $\mathcal{F}(G_{r-1} \cup E')$ is strictly smaller than $\mathcal{F}(G_r)$. This contradicts the assumption that the filtration for $f \colon \mathcal{G} \to \mathcal{G}$ is reduced. $\square$

Lemma 6.18 (cf. Lemma 4.29 of [FH11])**.** *Suppose that H_r is an exponentially growing stratum of a relative train track map $f \colon \mathcal{G} \to \mathcal{G}$ and that ρ is an indivisible almost Nielsen path of height r. If the fold at the illegal turn of ρ is partial, then there is a relative train track map $f'' \colon \mathcal{G}'' \to \mathcal{G}''$ satisfying $N(f'') < N(f)$.*

Proof. We follow the arguments in [BFH00, Lemmas 5.2.3 and 5.2.4]. We may write $\rho = \alpha\beta$ as in Lemma 2.4. There exists a tight path τ in G_r such that $f_\sharp(\alpha) = g\alpha\tau$ and $f_\sharp(\bar\beta) = h\bar\beta\tau$ for vertex group elements g and h. Suppose that α is not a single edge of G. Then the final edge of α is entirely mapped into τ by f. The only way this edge is subdivided when folding ρ is if the first edge of β maps entirely into τ and in this case we have a full fold, in contradiction to our assumption.

Therefore suppose α and β are single edges. Notice that by the previous lemma, the endpoints of ρ are distinct and at most one endpoint of ρ has nontrivial vertex group. Up to reversing the orientation of ρ, suppose it is the initial endpoint, so we have $f(\alpha) = g\alpha\tau$ and $f(\bar\beta) = \bar\beta\tau$ for some vertex group element g. Change f via a homotopy so that $f(\alpha) = \alpha\tau$, then let $\mathcal{G}'$ be the graph of groups obtained from $\mathcal{G}$ by identifying α and $\bar\beta$ to a single edge E'. There is an induced map $f'\colon \mathcal{G}' \to \mathcal{G}'$; for each edge E of G we have $f'(q(E)) = q_\sharp(f(E))$. If there are any edges with trivial f'-image, they form a contractible forest; we collapse each component to a vertex. After tightening and collapsing finitely many times, we arrive at a topological representative still called $f'\colon \mathcal{G}' \to \mathcal{G}'$ and a quotient map still called $q\colon \mathcal{G} \to \mathcal{G}'$. We have that an edge of G is collapsed by q if and only if some iterate of $f_\sharp$ maps it to ρ or $\bar\rho$, so only edges in zero strata may be collapsed. Therefore $q\colon \mathcal{G} \to \mathcal{G}'$ is a homotopy equivalence.

The map $f'\colon \mathcal{G}' \to \mathcal{G}'$ is a topological representative preserving the filtration whose elements G'_i satisfy $G'_i = q(G_i)$, where if each component of a zero stratum H_j is collapsed to a point, then $q(G_j)$ is not added to the filtration. Notice that $\mathcal{F}(G_i) = \mathcal{F}(G'_i)$, so the filtration is reduced and still realizes $\mathcal{C}$.

If E is an edge of G_r, then $q(E)$ is an edge of G'. We have $f'(q(E)) = qf(E)$: this is clear if E in G_{r-1} and follows from the fact that an r-legal path in G_r cannot cross the illegal turn $(\bar\alpha, \beta)$ if $E \in H_r$. If E is an edge of $G \setminus G_r$ that is not collapsed by q, then the edge path $f'q(E)$ is obtained from $qf(E)$ by cancelling edges in H_r. Thus the stratum $H'_i = q(H_i)$ has the same type as H_i. In particular if H_i is exponentially growing, then so is H'_i and the Perron–Frobenius eigenvalues are equal.

Given a path σ in $\mathcal{G}$ and $k > 0$, we have $(f')^k_\sharp(q_\sharp(\sigma)) = q_\sharp f^k_\sharp(\sigma)$. So if σ is a periodic almost Nielsen path for f, then $\sigma' = q_\sharp(\sigma)$ is a periodic almost Nielsen path for f' and the period of σ' is at most the period of σ. If $\sigma \neq \rho$, then σ' is nontrivial.

Conversely suppose that $\sigma' \subset G'_i$ is a periodic Nielsen path for $f'\colon \mathcal{G}' \to \mathcal{G}'$, we will construct a path $\sigma \subset G_i$ satisfying $q_\sharp(\sigma) = \sigma'$. If the endpoints of σ' do not lie in $q(\alpha) = q(\bar\beta)$, then there is a unique path σ satisfying $q_\sharp(\sigma) = \sigma'$. If an endpoint of σ' lies in $q(\alpha) = q(\bar\beta)$ but is not the initial endpoint of $q(\alpha) = q(\bar\beta)$, then there is a unique path σ in $\mathcal{G}$ that has periodic endpoints such that $q_\sharp(\sigma) = \sigma'$. If σ' begins or ends at the initial endpoint of $q(\alpha) = q(\bar\beta)$, then there is a unique path σ that does not begin or end with ρ or $\bar\rho$ that satisfies $q_\sharp(\sigma) = \sigma'$. In all cases we see that σ is a periodic Nielsen path and that the period of σ' equals the period of σ. What's more, if σ' is indivisible, then σ is indivisible.

If $f'\colon \mathcal{G}' \to \mathcal{G}'$ is not a relative train track map, we may modify it to produce a relative train track map by using the moves "(invariant) core subdivision" and "collapsing inessential connecting paths" described in [Lym21, Lemmas 3.4 and 3.5] or [BH92, Lemmas 5.13 and 5.14] to restore properties (EG-i) and (EG-ii) for each exponentially growing stratum of $f'\colon \mathcal{G}' \to \mathcal{G}'$. Since $\mathrm{PF}(f') = \mathrm{PF}_{\min}$, Corollary 1.4 implies that (EG-iii) is satisfied.

We now have a relative train track map $f''\colon \mathcal{G}'' \to \mathcal{G}''$ and an identifying homotopy equivalence $q\colon \mathcal{G} \to \mathcal{G}''$. The graph of groups $\mathcal{G}''$ is obtained from $\mathcal{G}$ by subdivision, folding and collapsing pretrivial forests (which, again, are contained in zero strata). The resulting relative train track map still realizes $\mathcal{C}$ and thus its filtration is reduced. If C'' is a contractible component of some filtration element for $f''\colon \mathcal{G}'' \to \mathcal{G}''$, then $q^{-1}(C'')$ is a contractible component of some filtration element and is thus a union of zero strata. It follows that C'' is also a union of zero strata.

Collapsing inessential connecting paths, collapsing pretrivial forests and invariant core subdivision do not change the period of any indivisible periodic almost Nielsen path, nor do they alter the number of indivisible periodic almost Nielsen paths of exponentially growing height. It follows that $N(f'') = N(f') < N(f)$. $\square$

Lemma 6.19 (cf. Lemma 4.30 of [FH11]). *Suppose that H_r is an exponentially growing stratum of a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$, that ρ is an indivisible almost Nielsen path of height r and that the fold at the illegal turn of ρ is proper. Let $f'\colon \mathcal{G}' \to \mathcal{G}'$ be the relative train track map obtained from $f\colon \mathcal{G} \to \mathcal{G}$ by folding ρ. Then $N(f') = N(f)$ and there is a bijection $H_s \to H'_s$ between the exponentially growing strata of f and the exponentially growing strata of f' such that H'_s and H_s have the same number of edges for all s.*

Proof. The proof that f' is a relative train track map satisfying our standing assumptions is contained in Lemma 6.2. It is clear from the definition of $f'\colon \mathcal{G}' \to \mathcal{G}'$ and the proof of Lemma 6.2 that the bijection between exponentially growing strata exists and that H'_s and H_s have the same number of edges for all s. The fact that $N(f') = N(f)$ follows as in the previous lemma. $\square$

Lemma 6.20 (cf. Lemma 4.31 of [FH11]). *Suppose that H_r is an exponentially growing stratum of a relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ and that ρ is an indivisible almost Nielsen path of height r. If the fold at the illegal turn of ρ is improper, then there is a relative train track map $f''\colon \mathcal{G}'' \to \mathcal{G}''$ and a bijection $H_s \to H''_s$ between the exponentially growing strata of f and f'' such that the following hold.*

1. *$N(f'') = N(f)$.*
2. *H''_r has fewer edges than H_r.*
3. *If $s > r$, then H_s and H''_s have the same number of edges.*

Proof. Assume notation as in the definition of folding ρ; i.e. we have edges E_1 and E_2 and in H_r and vertex group elements g_1 and g_2 such that $f(g_1E_1) = f(g_2E_2)$. Let $F\colon \mathcal{G} \to \mathcal{G}''$ be the fold of E_1 with E_2. There is an induced map $g\colon \mathcal{G}'' \to \mathcal{G}$ such that $gF = f$. Define $f'\colon \mathcal{G}' \to \mathcal{G}'$ by tightening $Fg\colon \mathcal{G}' \to \mathcal{G}'$ and collapsing a maximal pretrivial forest. Observe that edges that are collapsed belong to zero strata. There is an identifying homotopy equivalence $q\colon \mathcal{G} \to \mathcal{G}'$. As in the proof of Lemma 6.18, the map $f'\colon \mathcal{G}' \to \mathcal{G}'$ is a topological representative preserving the filtration whose elements G'_i satisfy $G'_i = q(G_i)$, so the filtration is reduced and still realizes $\mathcal{C}$. If H_i is a stratum of $f\colon \mathcal{G} \to \mathcal{G}$ which is not entirely collapsed to a point, then H'_i is a stratum of the same type, and the Perron–Frobenius eigenvalues are equal if H_i was exponentially growing.

If $f'\colon \mathcal{G}' \to \mathcal{G}'$ is not a relative train track map, we may again perform invariant core subdivision and collapsing of inessential connecting paths to restore properties (EG-i) and (EG-ii). As in the proof of Lemma 6.18, the result is a relative train track map $f''\colon \mathcal{G}'' \to \mathcal{G}''$ satisfying our standing assumptions. When performing invariant core subdivision, the number of edges in the relevant exponentially growing stratum is unchanged. Thus the existence of the bijection $H_s \to H''_s$ is clear, as is the fact that H''_r has one fewer edge than H_r. If $s > r$, to show that H''_s has the same number of edges as H_s, we must show that no folding in H'_s occurs when collapsing inessential connecting paths. Since H'_s is not a zero stratum, the component of any filtration element it belongs to is noncontractible. (There is only one component by Lemma 4.7.) Observe that the map $q\colon \mathcal{G} \to \mathcal{G}'$ does not identify points that are not identified by some iterate of $f\colon \mathcal{G} \to \mathcal{G}$. Thus this component satisfies (EG-ii) by Lemma 1.5 for f' because f does, and we conclude that no folding in H'_s occurs. $\square$

Step 2: Theorem 2.6. The second step is to apply the construction in the proof of Theorem 2.6 to our relative train track map $f\colon \mathcal{G} \to \mathcal{G}$. As we remarked in the proof, the number of indivisible almost Nielsen paths of exponentially growing height is preserved and the number of edges of each exponentially growing stratum is unchanged. If the resulting relative train track map $f\colon \mathcal{G} \to \mathcal{G}$ does not satisfy (EG Almost Nielsen Paths), we can return to Step 1 and restore these properties. After starting over finitely many times we may assume that (EG Almost Nielsen Paths) is still satisfied.

Step 3: (Rotationless), (Filtration) and (Zero Strata). The properties (Rotationless) and (Filtration) follow from Proposition 5.19 and property (F) of Theorem 2.6. By property (Z) of Theorem 2.6, to arrange (Zero Strata) it suffices to show that each edge in a zero stratum H_i is r-taken. Each edge E in H_i is contained in an r-taken path $\sigma \subset H_i$, since $f|_{G_r}$ is a homotopy equivalence. If E itself is not r-taken, perform a tree replacement, removing E and insert a new edge E' that has the same endpoints as σ and such that the identifying homotopy equivalence $p' \colon \mathcal{G}' \to \mathcal{G}$ sends E' to σ. After finitely many tree replacements, (Zero Strata) is satisfied.

Step 4: (Almost Periodic Edges). Suppose at first that no component of $\mathrm{Per}(f)$ is either a topological circle nor the quotient of $\mathbb{R}$ by the standard action of $C_2 * C_2$ with exactly two periodic directions at each point. Then the endpoints of any almost periodic edge are principal, so by (Rotationless) each almost periodic edge is almost fixed and each almost periodic stratum H_r is a single edge E_r. The remainder of (Almost Periodic Edges) is the translation of (P) (or more accurately, Lemma 2.11) to this situation.

In general, we will suppose that $\mathrm{Per}(f)$ contains a component C of the form above and modify $f \colon \mathcal{G} \to \mathcal{G}$ to make C no longer problematic, either by adding periodic directions in the case of C a circle, or by arranging that C is a dihedral pair in the case that C is the quotient of $\mathbb{R}$ by the standard action of $C_2 * C_2$.

Item 1 of Proposition 5.20 implies that C is f-invariant, and that if C is a circle, then $g = f|_C$ is orientation preserving. By (Zero Strata) and the fact that there are no periodic directions based in C and pointing out of C, if E_j is an edge not in C that has an endpoint in C, then E_j is non-almost periodic, non-exponentially growing, and meets C only at its terminal vertex. Notice that since all non-periodic vertices are contained in exponentially growing strata (by (NEG) and (Zero Strata)) no vertex outside of C maps into C. By (NEG) again, C is a component of some filtration element.

Suppose now that C is a circle. First we arrange that C is fixed by f. Define a map $h \colon \mathcal{G} \to \mathcal{G}$ homotopic to the identity by extending the rotation $g^{-1} \colon C \to C$ such that h has support on a small neighborhood of C and such that $h(E_j) \subset E_j \cup C$ for each edge E_j with terminal endpoint in C. Redefine f on each edge E of G to be $h_\sharp f_\sharp(E)$ and observe that edges in C are now fixed. For each edge E_j with terminal vertex in C, the edge path $f(E_j) = E_j u_j$ differs from its original definition only by initial and terminal segments in C. The f-images of all other edges is unchanged; what's more, if σ is a path such that the endpoints of $f(\sigma)$ are not in the support of h, then $f_\sharp(\sigma)$ is unchanged. The map f remains a relative train track map, and all the properties we have previously established remain, with the possible exception of property (P), which will fail if edges (possibly in C or one of the E_j) is now a fixed edge that should be collapsed. If there is such an edge, collapse it; all previously established properties continue to hold. If after performing these collapses C now has outward-pointing periodic directions, then we are finished.

Suppose that C still does not have outward-pointing periodic directions. Then the first non-periodic edge E_m with terminal endpoint in C satisfies $f(E_m) = E_m C^d$ for some nonzero integer d. In this second step we modify f so that $f(E_m) = E_m$. Take a map $h' \colon \mathcal{G} \to \mathcal{G}$ which is the identity on C that satisfies $h'(E_j) = E_j C^{-d}$ for all edges E_j with terminal endpoint in C and that has support in a small neighborhood of C. Again, this map is homotopic to the identity. Redefine f on each edge by tightening $h'f$ and note that $C \cup E_m$ now belongs to $\mathrm{Fix}(f)$, so the component of $\mathrm{Per}(f)$ containing C is no longer a topological circle. If necessary, collapse fixed edges with an endpoint in C to restore property (P) and repeat this step.

So suppose instead that C is the quotient of $\mathbb{R}$ by the standard action of $C_2 * C_2$. By (P), C consists of at most two edges, and after subdividing at a fixed point in the center of

C we may assume there are exactly two. To arrange that C is a dihedral pair, slide each edge E_j with terminal endpoint in C so that its endpoint is now the center vertex of C.

Let us remark that we have shown that the endpoint of an almost periodic edge is either principal (and hence the edge is almost fixed) or part of a dihedral pair (and hence *not* principal).

Step 5: Induction, the NEG case. Let NI be the number of strata H_i such that each component of G_i is non-contractible. By properties (P), (Z) and (NEG), this is the count of the number of strata H_i that are irreducible, with the possible exception of those strata that are the bottom half of a dihedral pair. Given m satisfying $0 \leq m \leq NI$, let $G_{i(m)}$ be the mth filtration element satisfying the property that each component of G_i is non-contractible. If there are no dihedral pairs, this is the smallest filtration element containing the first m irreducible strata. We will prove by induction on m that one can modify f so that $f|_{G_{i(m)}}$, or more properly speaking the restriction of f to each component of $G_{i(m)}$, is a CT. The $m = 0$ case is vacuous, so we turn to the inductive step; assuming that $f|_{G_r}$ is a CT for $r = i(m)$, we will alter f so that $f|_{G_s}$ is a CT for $s = i(m+1)$. There are two cases, according to whether H_s is exponentially or non-exponentially growing. In this step we will assume that H_s is non-exponentially growing.

Suppose as a first case that H_s is almost periodic. Then either H_s consists of a single almost fixed edge with principal endpoints, or it is half or all of a dihedral pair. In the former case, (Completely Split) (Vertices), (Linear Edges) and (NEG Almost Nielsen Paths) follow from the inductive hypothesis. In the latter case, by (Almost Periodic Edges), the dihedral pair is a component of G_s, and the properties above again follow by the inductive hypothesis.

So suppose that H_s is not almost periodic, so it is a single edge E_s satisfying $f(E_s) = g_s E_s u_s$ for some vertex group element g_s and path u_s in G_{s-1}. By (Zero Strata), $r = s-1$. We will modify E_s and u_s by sliding along a path τ in G_{s-1} with initial endpoint equal to the terminal vertex of E_s. As we noted in Step 2, we may (after possibly starting over) assume that sliding preserves (EG Almost Nielsen Paths).

First suppose that after sliding, we have that E_s is almost fixed. By Lemma 2.10 this is equivalent to $[\bar{\tau} u_s f_\sharp(\tau)]$ being a trivial path and hence equivalent to $f_\sharp(E_s\tau) = g_s E_s [u_s f_\sharp(\tau)] = g_s E_s \tau g$ for some vertex group element g. In other words, this is equivalent to $E_s\tau$ being an almost Nielsen path.

Continuing to suppose that E_s is now almost fixed, observe that if both endpoints of E_s are contained in G_{s-1}, then (Almost Periodic Edges) is satisfied, as are all the conclusions of Theorem 2.6, with the possible exception of (P) which can fail if what was formerly a fixed dihedral pair is now no longer a dihedral pair (because one of the C_2 vertex groups now has valence two, say), in which case we may restore this property by collapsing an edge. All the properties from Step 3 are still satisfied, and the remaining properties of a CT follow from the inductive hypothesis.

If some endpoint of E_s is not contained in G_{s-1}, then collapse E_s to a point as in Step 4. None of the previously achieved properties are lost and the remainder of the properties of a CT follow by induction.

Assume now that there is no choice of τ such that $E_s\tau$ is an almost Nielsen path. By Proposition 6.13, after sliding, we have $f(E_s) = g_s E_s \cdot u_s$ where u_s is nontrivial, so $f|_{G_s}$ satisfies (Almost Periodic Edges) and all the properties from the first four steps of the proof. Items (Completely Split), (Vertices), (NEG Almost Nielsen Paths) and (Linear Edges) for $f|_{G_s}$ follow from those properties for $f|_{G_r}$ and Proposition 6.13. This completes the inductive step assuming that H_s is non-exponentially growing.

Step 6: Induction, the EG case. Suppose now that H_s is exponentially growing. Items (Vertices), (Linear Edges) and (NEG Almost Nielsen Paths) follow from (EG Almost Nielsen Paths) and the inductive hypothesis on $f|_{G_r}$. It remains to establish (Completely Split) for $f|_{G_s}$.

For each edge E in H_s, there is a decomposition $f(E) = \mu_1 \cdot \nu_1 \cdot \mu_2 \cdots \nu_{m-1}\mu_m$, where the μ_i belong to H_s^z and the ν_i are maximal subpaths in G_r. We let $\{\nu_\ell\}$ denote the collection of all such paths that occur as E varies over the edges of H_s. By (EG-ii), for all k and ℓ, the path $f_\sharp^k(\nu_\ell)$ is nontrivial. By Lemma 6.12, we may choose k so large that each $f_\sharp^k(\nu_\ell)$ is completely split. The endpoints of each ν_ℓ are periodic by Lemma 1.5 and thus principal by (Vertices) and hence fixed. There are finitely many connecting paths σ in the strata (if any) between G_r and H_s. Each $f_\sharp(\sigma)$ is either again one of these connecting paths or a nontrivial tight path in G_r whose endpoints are fixed by the observation above. Therefore we may (after increasing k) assume that each $f_\sharp^k(\sigma)$ is completely split for each such σ. After applying Lemma 6.11 k times with $j = r$, we have by item 7 of the proof of that lemma that $f|_{G_s}$ is completely split. This completes the inductive step and with it the proof of the theorem. □

Corollary 6.21. *If $f\colon \mathcal{G} \to \mathcal{G}$ is a CT, then f^k is a CT for all $k \geq 1$.*

Proof. Property (EG Almost Nielsen Paths) follows from Lemma 6.16 and Lemma 6.8. The rest of the properties are straightforward to check; we leave them to the reader. □

7 An index inequality

Gaboriau, Jaeger, Levitt and Lustig define in [GJLL98, Theorem 4] an *index* for (outer) automorphisms of free groups. Namely, suppose Φ is an automorphism of a free group F_n. Then, as in Section 3, Φ yields a homeomorphism $\hat\Phi$ of the Gromov boundary of F_n, which is equal to its Bowditch boundary. Let $a(\Phi)$ denote the number of $\mathrm{Fix}(\Phi)$-orbits of attracting fixed points for $\hat\Phi$ in $\partial_\infty(F_n)$. Gaboriau, Jaeger, Levitt and Lustig's *index* of the automorphism Φ is the quantity

$$i(\Phi) = \max\left\{0,\ \mathrm{rank}(\mathrm{Fix}(\Phi)) + \frac{1}{2}a(\Phi) - 1\right\}.$$

Observe that if $i(\Phi)$ is positive, then Φ is a principal automorphism, so there are only finitely many isogredience classes of automorphisms Φ such that $i(\Phi) > 0$. If $\varphi \in \mathrm{Out}(F_n)$ is the outer class of Φ, Gaboriau, Jaeger, Levitt and Lustig define the *index* of φ to be the sum

$$i(\varphi) = \sum i(\Phi),$$

where the sum may be equivalently defined over representatives of *all* isogredience classes of automorphisms Φ representing φ, or merely representatives of principal isogredience classes. Gaboriau, Jaeger, Levitt and Lustig prove that for all $\varphi \in \mathrm{Out}(F_n)$, the index $i(\varphi)$ satisfies $i(\varphi) \leq n-1$. The main result of Martino's paper [Mar99] is that if F_n is replaced by a free product $F = A_1 * \cdots * A_n * F_k$, then the index, defined exactly as above, of any $\varphi \in \mathrm{Out}(F, \mathcal{A})$ satisfies $i(\varphi) \leq n + k - 1$. Let us remark that Martino's result is stated for the Grushko decomposition of a finitely generated group, but that the proof does not use this assumption in any essential way.

The purpose of this section is to improve on Martino's result in two ways. The first is an innovation of Feighn–Handel: let $b(\Phi)$ denote the number of $\mathrm{Fix}(\Phi)$-orbits of attracting fixed points for $\hat\Phi$ in $\partial_\infty(F, \mathcal{A})$ associated to *eigenrays,* which are referred to as *NEG rays* in [FH18]; we follow the naming convention in [HM20]. We defer a precise definition for now, but the idea is that these attractors are associated in the sense of Lemma 5.14 to fixed directions coming from non-almost periodic non-exponentially growing strata of some relative train track map representing (a rotationless iterate of) $\hat\Phi$. The second is to use the data of the Bowditch boundary of $(F, \mathcal{A})$. Recall that the stabilizer of each point in $V_\infty(F, \mathcal{A})$ is an infinite group A conjugate into $\mathcal{A}$, and that a point in $V_\infty(F, \mathcal{A})$ is fixed by $\hat\Phi$ if and only if the associated group A is preserved by Φ. Let $c(\Phi)$ denote the number of $\mathrm{Fix}(\Phi)$-orbits of fixed points in $V_\infty(F,\mathcal{A})$ with the property that the intersection of the associated group A with $\mathrm{Fix}(\Phi)$ is trivial. Define the quantity $j(\Phi)$ as

$$j(\Phi) = \max\left\{0,\ \mathrm{rank}(\mathrm{Fix}(\Phi)) + \frac{1}{2}a(\Phi) + \frac{1}{2}b(\Phi) + c(\Phi) - 1\right\}.$$

Here and throughout the section, we use a slightly nonstandard definition of $\mathrm{rank}(H)$ for H a subgroup of F: If $(H, \mathcal{A}|_H)$ takes the form $C_2 * C_2$, define $\mathrm{rank}(H) = 1$; otherwise define it to be the usual Kurosh subgroup rank of H. This definition has the following advantage: if $j(\Phi)$ is positive, then Φ is a principal automorphism, so there are again only finitely many isogredience classes of automorphisms Φ for which $j(\Phi) > 0$, and we may define

$$j(\varphi) = \sum j(\Phi)$$

where again the sum may be equivalently defined over the isogredience classes of all automorphisms Φ representing φ, or merely the principal ones. The main result of this section is the following theorem.

Theorem 7.1. *Suppose $\varphi \in \mathrm{Out}(F,\mathcal{A})$ has a rotationless iterate, and that each $A \in \mathcal{A}$ is finitely generated. Then the index $j(\varphi)$ satisfies*

$$j(\varphi) \le n + k - 1.$$

Let us remark that although the conclusions of Theorem 7.1 are stronger than Martino's result, the assumption that φ has a rotationless iterate appears to be a nontrivial restriction. The strategy of the proof of Theorem 7.1 is to follow the strategy in [FH18, Section 15]. We firstly show that if $\psi = \varphi^K$ is a rotationless iterate of φ, then $j(\varphi) \le j(\psi)$. Then, using a CT $f\colon \mathcal{G} \to \mathcal{G}$ representing ψ, we construct a graph of groups $\mathcal{S}_N(f)$, invariants of which calculate $j(\psi)$, and argue by induction up through the filtration that $j(\psi) \le n + k - 1$.

Rays and attracting fixed points. Let $f\colon \mathcal{G} \to \mathcal{G}$ be a CT representing $\varphi \in \mathrm{Out}(F,\mathcal{A})$. Write $\mathcal{E}$ for the set of oriented non-almost periodic, non-linear edges E of G with the property that the initial vertex of E is principal and the property that the direction of E is almost fixed by Df. By Lemma 6.6 if E is non-exponentially growing and by Lemma 2.1 if E is exponentially growing, there is a path u such that $f_\sharp(E) = gE \cdot u$ is a splitting. If the length of $f_\sharp^k(u)$ does not go to infinity with k, then E is a non-linear almost linear edge. Define the ray R_E in $\mathcal{G}$ as

$$R_E = E \cdot u \cdot f_\sharp(u) \cdot f_\sharp^2(u) \cdots.$$

Each lift $\tilde{R}_E$ of R_E to Γ determines a point in $\partial_\infty(F,\mathcal{A})$, so R_E determines an F-orbit ∂R_E in $\partial(F,\mathcal{A})$.

Suppose $\Phi\colon (F,\mathcal{A}) \to (F,\mathcal{A})$ is an automorphism. Write $\mathrm{Fix}_+(\hat{\Phi})$ for the subset of $\mathrm{Fix}_N(\hat{\Phi})$ comprising the attracting fixed points for $\hat{\Phi}$ in $\partial_\infty(F,\mathcal{A})$. Recall from Section 5 that for $\varphi \in \mathrm{Out}(F,\mathcal{A})$, we write $P(\varphi)$ for the set of principal automorphisms representing φ.

Lemma 7.2 (cf. Lemma 3.10 of [FH18])**.** *Suppose that $f\colon \mathcal{G} \to \mathcal{G}$ is a CT and that E is an edge of $\mathcal{E}$. If $\tilde{E}$ is a lift of E and $\tilde{f}$ is the lift of f that fixes the initial direction of $\tilde{E}$, then the lift $\tilde{R}_{\tilde{E}}$ of R_E that begins with $\tilde{E}$ converges to an attracting fixed point in $\mathrm{Fix}_+(\hat{f})$. The map $E \mapsto \partial R_E$ defines a surjection $\mathcal{E} \to \left(\bigcup_{\Phi\in P(\varphi)} \mathrm{Fix}_+(\hat{\Phi})\right)/F$.*

Proof. Suppose that $\tilde{E}$ is a lift of $E \in \mathcal{E}$, and that $\tilde{f}\colon \Gamma \to \Gamma$ is a lift of f that fixes the direction determined by $\tilde{E}$. By Corollary 5.17, since $\mathrm{Fix}(\tilde{f})$ is principal and projects to a non-exceptional almost Nielsen class, we conclude that $\tilde{f}$ is principal. Item 1 of Lemma 6.5 implies that $\tilde{R}_{\tilde{E}}$ converges to a point $P \in \mathrm{Fix}_N(\hat{f})$. Either the length of $f_\sharp^k(u)$ goes to infinity with k, in which case the "superlinear attractors" piece of Proposition 3.4 implies that $P \in \mathrm{Fix}_+(\hat{f})$, or E is a non-linear almost linear edge. By Lemma 6.7, P is not a limit of points that are fixed by $\tilde{f}$. Since P is not the endpoint of an axis by the proof of Lemma 6.5, it follows that P is not in the boundary of the fixed subgroup, so Proposition 3.4 implies that $P \in \mathrm{Fix}_+(\hat{f})$. Item 2 of Lemma 6.5 implies that the map $E \mapsto \partial R_E$ is surjective. $\square$

Observe that because attracting fixed points of principal automorphisms are not the endpoints of axes by Proposition 3.4 and Lemma 3.2 and so are not fixed by T_c for any $c \in F$, it follows that the union $\bigcup_{\Phi \in P(\varphi)} \mathrm{Fix}_+(\hat{\Phi})$ is a disjoint union.

Lemma 7.3. *Suppose that $E \neq E'$ are distinct edges of $\mathcal{E}$ and that $\partial R_E = \partial R_{E'}$. Then E and E' belong to the same exponentially growing stratum H_r and are the initial and terminal edges of the unique equivalence class of indivisible almost Nielsen paths of height r.*

Proof. The argument is identical to [FH18, 12.1]. Suppose that E and E' are as in the statement. Choose a lift $\tilde{E}$ of E to Γ and let $\tilde{R}_{\tilde{E}}$ be the lift of R_E that begins with $\tilde{E}$. Let $P \in \partial(F,\mathcal{A})$ be the endpoint of $\tilde{R}_{\tilde{E}}$. By assumption there is a lift $\tilde{R}_{\tilde{E}'}$ of $R_{E'}$ that ends at P and begins at a lift $\tilde{E}'$ of E'. The lift $\tilde{f}$ that fixes the direction of $\tilde{E}$ fixes P. By the remark in the previous paragraph, $\tilde{f}$ is the only lift of f that fixes P, so it follows that $\tilde{f}$ fixes the direction of $\tilde{E}'$ as well, and thus the unique tight path beginning with $\tilde{E}$ and ending with $\tilde{E}'$ projects to an almost Nielsen path ρ in $\mathcal{G}$. Since there are no almost Nielsen paths of non-linear non-exponentially growing height, we conclude that E and E' are of exponentially growing height. In fact they each have exponentially growing height r, since R_E and $R_{E'}$ have height r and $\tilde{R}_{\tilde{E}}$ and $\tilde{R}_{\tilde{E}'}$ have a common terminal subray $\tilde{R}_{\tilde{E},\tilde{E}'}$. By construction, since R_E and $R_{E'}$ are r-legal, the path ρ has one illegal turn in H_r, so we conclude that ρ is indivisible, and is thus the unique (by Proposition 6.9) equivalence class of indivisible almost Nielsen paths of height r. If we write $\rho = \alpha\beta$ as in Lemma 2.4, then there is a terminal subray $R_{E,E'}$ of R_E and $R_{E'}$ such that we have $R_E = \alpha R_{E,E'}$ and $R_{E'} = \bar{\beta} R_{E,E'}$. $\square$

If P is an attracting fixed point for an automorphism Φ represented by the lifted ray $\tilde{R}_{\tilde{E}}$ as in Lemma 7.2, we say that P or $\tilde{R}_{\tilde{E}}$ is an *eigenray* for Φ if the stratum of $f\colon \mathcal{G} \to \mathcal{G}$ containing E is non-exponentially growing. The following lemma says that this definition is independent of the CT $f\colon \mathcal{G} \to \mathcal{G}$.

Lemma 7.4 (cf. Definitions 2.9 and 2.10 and Lemma 2.11 of Part II of [HM20])**.** *Suppose that $\varphi \in \mathrm{Out}(F,\mathcal{A})$ is rotationless, $\Phi \in P(\varphi)$ and that $P \in \mathrm{Fix}_+(\hat{\Phi})$. The following are equivalent.*

1. *For some CT $f\colon \mathcal{G} \to \mathcal{G}$ representing φ there is a non-linear non-exponentially growing edge E with a lift $\tilde{E}$ in Γ such that $\tilde{R}_{\tilde{E}}$ converges to P.*
2. *For* every *CT $f\colon \mathcal{G} \to \mathcal{G}$ representing φ there is a non-linear non-exponentially growing edge E with a lift $\tilde{E}$ in Γ such that $\tilde{R}_{\tilde{E}}$ converges to P.*

Proof. It is clear that item 2 implies item 1. Suppose item 1 holds. Let $\Lambda(P)$ be the limit set of P (see Section 4 before Lemma 4.10). The proof of Lemma 4.11 shows that $\Lambda(P)$ is carried by the conjugacy class of a free factor B of F of positive complexity. In fact, if E is the unique edge of H_r, then we have that $[[B]] \sqsubset \mathcal{F}(G_{r-1})$, so B is a proper free factor of F. Choose B within its conjugacy class so that $P \in \partial(B, \mathcal{A}|_B)$. By Lemma 6.7 and Lemma 5.13, the set $\mathrm{Fix}_N(\hat{\Phi}) \cap \partial(A, \mathcal{A}|_A)$ contains only P (uniqueness is established in Proposition 6.13).

Let $f'\colon \mathcal{G}' \to \mathcal{G}'$ be a CT. By Lemma 7.2, there is a non-almost fixed, non-linear edge $E' \in \mathcal{E}'$ such that $\tilde{R}_{\tilde{E}'}$ converges to P. Suppose that E' belonged to an exponentially growing stratum. In the proof of Lemma 4.10, we showed that there is a leaf of the attracting lamination $\Lambda^+ = \Lambda(P)$ with P as one of its endpoints. Both endpoints belong to $\mathrm{Fix}_N(\hat{\Phi})$, contradicting what we established in the previous paragraph. Therefore we conclude that item 2 holds. $\square$

If φ is not rotationless, and so not represented by a CT, then we say that P is an *eigenray* for Φ if it is an eigenray for ϕ^K, where φ^K is a rotationless iterate of φ. Notice that if φ is rotationless and $f\colon \mathcal{G} \to \mathcal{G}$ is a CT representing φ, then by Corollary 6.21, f^k is a CT representing φ^k for all $k \geq 1$, so it follows that P is an eigenray for Φ if and only if it is an eigenray for each Φ^k for $k \geq 1$. If φ is not rotationless but has rotationless iterates K and L, then P is an eigenray for Φ^K if and only if it is an eigenray for Φ^{KL} if and only if it is an eigenray for Φ^L, so this definition is independent of the rotationless iterate.

Let $\mathcal{R}(\varphi) = \bigcup_{\Phi\in P(\varphi)} \operatorname{Fix}_+(\Phi)$. Let $\mathcal{R}_{\mathrm{NEG}}(\Phi)$ denote the set of eigenrays for Φ, and let $\mathcal{R}_{\mathrm{NEG}}(\varphi) = \bigcup_{\Phi\in P(\varphi)} \mathcal{R}_{\mathrm{NEG}}(\Phi)$.

Lemma 7.5 (cf. Lemma 15.8 of [FH18])**.** *Suppose Φ and Ψ are principal automorphisms representing φ. The following hold.*

1. *If $\operatorname{Fix}_+(\Phi) \cap \operatorname{Fix}_+(\Psi) \neq \varnothing$, then $\Phi = \Psi$. We have*
$$\mathcal{R}(\varphi) = \coprod_{\Phi\in P(\varphi)} \operatorname{Fix}_+(\Phi)$$
and
$$\mathcal{R}_{\mathrm{NEG}}(\varphi) = \coprod_{\Phi\in P(\varphi)} \mathcal{R}_{\mathrm{NEG}}(\Phi).$$

2. *The stabilizers of $\operatorname{Fix}_+(\Phi)$ and $\mathcal{R}_{\mathrm{NEG}}(\Phi)$ respectively under the action of F on $\mathcal{R}(\varphi)$ and $\mathcal{R}_{\mathrm{NEG}}(\varphi)$ respectively are each $\operatorname{Fix}(\Phi)$.*

3. *If $\{\Phi_1, \ldots, \Phi_N\}$ is a set of representatives of isogredience classes in $P(\varphi)$, the natural maps*
$$\coprod_{i=1}^{N} \operatorname{Fix}_+(\Phi_i)/\operatorname{Fix}(\Phi_i) \to \mathcal{R}(\varphi)/F$$
and
$$\coprod_{i=1}^{N} \mathcal{R}_{\mathrm{NEG}}(\Phi_i)/\operatorname{Fix}(\Phi_i) \to \mathcal{R}_{\mathrm{NEG}}(\varphi)/F$$
are bijections.

Proof. The proof is identical to [FH18, Lemma 15.8]. For item 1, since Φ and Ψ both represent φ, we have that $\Phi\Psi^{-1} = i_c$ for some $c \in F$, where we remind the reader that i_c denotes the inner automorphism $x \mapsto cxc^{-1}$. If R belongs to $\operatorname{Fix}_+(\Phi) \cap \operatorname{Fix}_+(\Psi)$, then $\hat{T}_c(R) = R$. But since $R \in \partial_\infty(F, \mathcal{A})$ is not the endpoint of an axis, we conclude that $c = 1$.

For item 2, suppose first that $c \in \operatorname{Fix}(\Phi)$ and $R \in \mathcal{R}_{\mathrm{NEG}}(\Phi)$. Then
$$\hat{\Phi}\hat{T}_c(R) = T_{\Phi(c)}\hat{\Phi}(R) = \hat{T}_c(R),$$
so $\hat{T}_c(R) \in \mathcal{R}_{\mathrm{NEG}}(\Phi)$. Conversely, if we have $c \in F$ and $R \in \mathcal{R}_{\mathrm{NEG}}(\Phi)$ such that $\hat{\Phi}(\hat{T}_c(R)) = \hat{T}_c(R)$, then we have
$$\hat{T}_c(R) = \hat{\Phi}\hat{T}_c(R) = T_{\Phi(c)}\hat{\Phi}(R) = T_{\Phi(c)}(R).$$
As in the proof of item 1, we conclude that $\Phi(c) = c$. The same argument applies with $\mathcal{R}_{\mathrm{NEG}}(\Phi)$ replaced by $\operatorname{Fix}_+(\Phi)$.

Item 3 is a consequence of item 1 and item 2, combined with the observation that the action of F on $\mathcal{R}(\varphi)$ by permuting the terms of the disjoint union in item 1. The same holds for $\mathcal{R}_{\mathrm{NEG}}(\varphi)$. $\square$

Lemma 7.6 (cf. Lemma 15.9 of [FH18])**.** *For $k \geq 1$ and $\varphi \in \operatorname{Out}(F, \mathcal{A})$, we have $b(\varphi^k) \geq b(\varphi)$ and $a(\varphi^k) \geq a(\varphi)$.*

Proof. The proof is identical to [FH18, Lemma 15.9].

By the definition of $b(\varphi)$ and item 3 of the previous lemma, we have

$$b(\varphi) = \left| \coprod_{i=1}^{N} \mathcal{R}_{\mathrm{NEG}}(\Phi_i)/\operatorname{Fix}(\Phi_i) \right| = |\mathcal{R}_{\mathrm{NEG}}(\varphi)/F| \,.$$

By definition, if R is an eigenray for $\Psi \in P(\varphi)$, then R is also an eigenray for $\Psi^k \in P(\varphi^k)$ for $k \geq 1$, so we have $\mathcal{R}_{\mathrm{NEG}}(\varphi^k) \supset \mathcal{R}_{\mathrm{NEG}}(\varphi)$. Hence we conclude that

$$b(\varphi^k) = \left|\mathcal{R}_{\mathrm{NEG}}(\varphi^k)/F\right| \geq |\mathcal{R}_{\mathrm{NEG}}(\varphi)/F| = b(\varphi).$$

The above argument works for $a(\varphi)$ by replacing $\mathcal{R}_{\mathrm{NEG}}(\varphi)$ with $\mathcal{R}(\varphi)$ and $\mathcal{R}_{\mathrm{NEG}}(\Phi)$ with $\operatorname{Fix}_+(\Phi)$. ☐

Fixed subgroups. Given an automorphism $\Phi\colon (F,\mathcal{A}) \to (F,\mathcal{A})$, define

$$\hat{r}(\Phi) = \max\{0,\ \operatorname{rank}(\operatorname{Fix}(\Phi)) + c(\Phi) - 1\},$$

where we remind the reader that we use the convention that $\operatorname{rank}(C_2 * C_2) = 1$. Given $\varphi \in \operatorname{Out}(F,\mathcal{A})$, suppose $\{\Phi_1,\ldots,\Phi_N\}$ is a set of representatives of the isogredience classes of principal automorphisms representing φ. Define

$$\hat{r}(\varphi) = \sum_{i=1}^{N} \hat{r}(\Phi).$$

Our goal is the prove the following lemma.

Lemma 7.7 (cf. Lemma 15.11 of [FH18])**.** *Let $\varphi \in \operatorname{Out}(F,\mathcal{A})$ and suppose φ has a rotationless iterate $\psi = \varphi^k$ for some $k \geq 1$. If each $A \in \mathcal{A}$ is finitely generated, then $\hat{r}(\psi) \geq \hat{r}(\varphi)$.*

Before proceeding to the proof, we need a version of [Cul84, Theorem 3.1].

Proposition 7.8. *Suppose $\varphi \in \operatorname{Out}(F,\mathcal{A})$ has finite order, that each $A \in \mathcal{A}$ is finitely generated and that Φ represents φ. There exists a topological representative $f\colon \mathcal{G} \to \mathcal{G}$ representing φ which is an automorphism of graphs of groups, in the sense that the underlying graph map of f is an isomorphism. Either* $\operatorname{Fix}(\Phi)$ *is cyclic and non-peripheral, or* $\operatorname{Fix}(\Phi)$ *is conjugate to the fundamental group of a graph of groups that has an injective-on-edges immersion (in the sense of [Bas93]) into a component of* $\operatorname{Fix}(f)$*. Moreover $\sum \hat{r}(\Phi) \leq \operatorname{rank}(F) - 1$, where the sum is taken over isogredience classes of* all *automorphisms representing φ.*

Proof. Let us remark that the proposition is true if $F = C_2 * C_2$, so suppose that $F \neq C_2 * C_2$.

By [HK18, Theorem 4.1], the automorphism φ fixes a point in the *Outer Space* of $(F,\mathcal{A})$ as defined by Guirardel–Levitt [GL07]. This means there is a Grushko $(F,\mathcal{A})$-tree T, an automorphism $\Phi\colon (F,\mathcal{A}) \to (F,\mathcal{A})$ representing φ and a Φ-twisted equivariant homeomorphism $\tilde{f}\colon T \to T$. Let $\mathcal{G}$ be the marked graph of groups whose Bass–Serre tree is T. The resulting map of graphs of groups $f\colon \mathcal{G} \to \mathcal{G}$ (See [Lym21, Proposition 1.2] or [Bas93, 4.1–4.5]) is an automorphism of graphs of groups which represents φ.

Let us remark that [HK18, Theorem 4.1] is stated only for the Grushko decomposition of a finitely generated group, essentially because Guirardel–Levitt discuss only this case. The proof, like the definition of the Outer Space, does not use this assumption in any essential way.

Now fix Φ representing φ, and consider the corresponding lift $\tilde{f}\colon T \to T$. Suppose first that Φ is not principal, so that $\operatorname{Fix}_N(\hat{\Phi})$ contains either a single point or the endpoints of an axis A_c. Then $c(\Phi) \leq 1$ and $c(\Phi) = 1$ implies that $\operatorname{Fix}(\Phi)$ is trivial, for if it were not, then $\operatorname{Fix}_N(\hat{\Phi})$ would contain more than one point in $V_\infty(F,\mathcal{A})$. Similarly $\operatorname{rank}(\operatorname{Fix}(\Phi)) \leq 1$, (where again we have $\operatorname{rank}(C_2 * C_2) = 1$) and the previous argument shows that $\operatorname{rank}(\operatorname{Fix}(\Phi)) = 1$ implies that $c(\Phi) = 0$. To see the first assertion, note that if $\operatorname{rank}(\operatorname{Fix}(\Phi)) > 1$, then $\operatorname{Fix}_N(\hat{\Phi})$ must contain more than the endpoints of a single axis. Thus we see that $\hat{r}(\Phi) = 0$. If $\operatorname{Fix}(\Phi)$ is nontrivial and contained in a single $A \in \mathcal{A}$, then Φ-twisted equivariance implies that $\tilde{f}$ fixes the unique point in T fixed by A. This point projects to a fixed point for f in $\mathcal{G}$, so the assertion holds in this case.

Now suppose that Φ is principal. Then by Corollary 5.5, $\tilde{f}$ fixes a point $\tilde{v}$, and in fact $\operatorname{Fix}(\tilde{f})$ projects to a non-exceptional almost Nielsen class containing v. This also holds when Φ is not principal and $\operatorname{Fix}(\Phi) = C_2 * C_2$. Since $\tilde{f}$ is an isomorphism, we have $\tilde{f} = \tilde{f}_\sharp$. The fact that $\tilde{f}_\sharp$ preserves any ray $\tilde{R}$ beginning at $\tilde{v}$ and converging to some point $P \in \operatorname{Fix}_N(\hat{\Phi})$ therefore implies that $\tilde{R}$ is pointwise fixed. The same is true of the tight path between any pair of fixed points for $\tilde{f}$. This latter fact implies that $\operatorname{Fix}(\tilde{f})$ is connected and contains at least two points, while the former implies that it contains the $\operatorname{Fix}(\Phi)$-minimal subtree. Conversely, suppose T_c preserves $\operatorname{Fix}(\tilde{f})$. For $\tilde{x} \in \operatorname{Fix}(\tilde{f})$, we have $T_c\tilde{f}(\tilde{x}) = \tilde{f}T_c(\tilde{x})$, so since $\operatorname{Fix}(\tilde{f})$ contains at least two points, we conclude by Lemma 3.2 that $c \in \operatorname{Fix}(\Phi)$. It follows that the quotient of $\operatorname{Fix}(\tilde{f})$ by $\operatorname{Fix}(\Phi)$, call it $\mathcal{H}$, thought of as a graph of groups in its own right, is compact and immerses (in the sense of [Bas93]) into $\mathcal{G}$. This immersion is injective on edges, for if there were two edges of H that map to the same edge of $\mathcal{G}$, there would be a loop in $\mathcal{G}$ based at a point in the interior of an edge that lifts to a non-closed path in $\mathcal{H}$. This loop corresponds to an element of $\pi_1(\mathcal{G})$ that preserves $\operatorname{Fix}(\tilde{f})$ but does not belong to $\operatorname{Fix}(\Phi)$, a contradiction.

Let $e(\mathcal{H})$ be the number of edges of H and $v_0(\mathcal{H})$ the number of vertices of H with trivial vertex group and whose image in G has finite vertex group. An easy argument shows that $\hat{r}(\Phi)$ is bounded above by $e(\mathcal{H}) - v_0(\mathcal{H})$. Indeed, the only case where this computation overestimates $\hat{r}(\Phi)$ is the case where $\operatorname{Fix}(\Phi) = C_2 * C_2$.

Suppose that Φ and Φ' are not isogredient and correspond to the lifts $\tilde{f}$ and $\tilde{f}'$ such that $\operatorname{Fix}(\tilde{f})$ and $\operatorname{Fix}(\tilde{f}')$ contain at least two points. Then $\operatorname{Fix}(\tilde{f})$ and $\operatorname{Fix}(\tilde{f}')$ project to distinct non-exceptional almost Nielsen classes in $\operatorname{Fix}(f)$, and the images of $\mathcal{H}$ and $\mathcal{H}'$ in $\mathcal{G}$ share no edges. We have $\operatorname{rank}(F) - 1 = e(\mathcal{G}) - v_0(\mathcal{G})$. If a vertex v contributes to $v_0(\mathcal{G})$, then every vertex mapping to v in every $\mathcal{H}$ contributes to $v_0(\mathcal{H})$. The final assertion follows. $\square$

Proof of Lemma 7.7. We follow the argument in [FH18, Lemma 15.11]. Assume that $\hat{r}(\varphi) > 0$, let $\Phi_1, \ldots, \Phi_s, \ldots, \Phi_N$ be a set of representatives of isogredience classes of principal automorphisms in $P(\varphi)$, where $\hat{r}(\Phi_i) > 0$ if and only if $i \le s$. Let $\Psi_1, \ldots, \Psi_t, \ldots, \Psi_M$ be a set of representatives of isogredience classes of principal automorphisms in $P(\psi)$. By definition, up to reordering the representatives of $P(\psi)$ there is a function

$$p\colon \{1, \ldots, s\} \to \{1, \ldots, t\}$$

such that if $j = p(i)$, then Φ_i^k is isogredient to Ψ_j. By replacing Φ within its isogredience class, we may assume that $\Phi_i^k = \Psi_j$. It suffices to show that

$$\sum_{j=1}^{t} \hat{r}(\Psi_j) \ge \sum_{i=1}^{s} \hat{r}(\Phi_i),$$

for which it suffices to show that

$$\hat{r}(\Psi_j) \ge \sum_{i \in p^{-1}(j)} \hat{r}(\Phi_i)$$

for each j satisfying $1 \le j \le t$.

So fix j and write $\mathbb{F} = \operatorname{Fix}(\Psi_j)$ and C for the set of fixed points for Ψ_j in $V_\infty(F, \mathcal{A})$ whose $\mathbb{F}$-orbits contribute to $c(\Psi_j)$. For $i \in p^{-1}(j)$, we have $\Phi_i^k = \Psi_j$, so $\mathbb{F} = \operatorname{Fix}(\Phi_i^k)$. Thus Φ_i preserves $\mathbb{F}$, $\operatorname{Fix}(\Phi_i)$ is contained in $\mathbb{F}$, and the restriction of Φ_i to $\mathbb{F}$ is a finite order automorphism of $\mathbb{F}$. Similarly Φ_i permutes the elements of the set C. We claim that if i and $i' \in p^{-1}(j)$ are distinct (i.e. not isogredient) then $\operatorname{Fix}_N(\Phi_i) \cap C$ and $\operatorname{Fix}_N(\Phi_{i'}) \cap C$ project to distinct $\mathbb{F}$-orbits. Assuming the claim for now, let us use it to show that the displayed inequality holds. If $\mathrm{rank}(\mathbb{F}) = 1$ (where again $\mathrm{rank}(C_2 * C_2) = 1$, then every subgroup of $\mathbb{F}$ has rank at most 1, and by the claim, each $\mathbb{F}$-orbit in C is counted in at most one $c(\Phi_i)$, so the displayed inequality holds. If instead $\mathrm{rank}(\mathbb{F}) > 1$, then $\mathbb{F}$ is its own normalizer in F: to see this, take $\xi \in \partial_\infty(\mathbb{F}, \mathcal{A}|_{\mathbb{F}})$ that is not the endpoint of an axis. Then ξ is not fixed by any automorphism $\Psi' \neq \Psi_j$ representing ψ. Suppose that $c \in F$ normalizes $\mathbb{F}$. Then $T_c(\xi) \in \mathbb{F}$ and so ξ is fixed by $i_{c^{-1}}\Psi_j i_c = i_{c^{-1}\Psi-j(c)}\Psi_j$, so we conclude $c\Psi_j(c)$ and therefore $c \in \mathbb{F}$. Therefore the restriction of φ to $\mathbb{F}$ is well-defined and has finite order. Proposition 7.8 together with the claim proves the displayed inequality.

We now turn to the proof of the claim. Suppose that $\mathrm{Fix}_N(\Phi_i)$ and $\mathrm{Fix}(\Phi_{i'})$ contain ξ and ξ' in the same $\mathbb{F}$-orbit of C, say that there exists $c \in \mathbb{F}$ such that $\hat{T}_c(\xi) = \xi'$. We will show that $i_c\Phi_i i_c^{-1} = \Phi_{i'}$. Indeed, $i_c\Phi_i i_c^{-1}$ fixes ξ' and equals $i_g\Phi_{i'}$ for some g in F. By twisted equivariance, we have $g \in \mathrm{Stab}(\xi')$. Since $c \in \mathbb{F}$, we have

$$\Psi_j = i_c\Psi_j i_c^{-1} = i_c\Phi_i^k i_c^{-1} = (i_g\Phi_{i'})^k = i_{g\Phi_{i'}(g)\cdots\Phi_{i'}^{k-1}(g)}\Psi_j,$$

from which it follows that

$$\Psi_j(g) = \Phi_{i'}^k(g) = g\Phi_{i'}(g)\cdots\Phi_{i'}^k(g) = g.$$

But since $\mathrm{Stab}(\xi') \cap \mathrm{Fix}(\Psi_j)$ is trivial, we conclude that $g = 1$, proving the claim. ☐

The following lemma is an immediate consequence of Lemma 7.6 and Lemma 7.7.

Lemma 7.9. *Suppose that $\varphi \in \mathrm{Out}(F, \mathcal{A})$ has a rotationless iterate $\psi = \varphi^k$ and that each $A \in \mathcal{A}$ is finitely generated. Then $j(\varphi) \le j(\psi)$.* ☐

An Euler characteristic for graphs of groups. If $\mathcal{G}$ is a finite graph of groups with trivial edge groups, we define the *(negative) Euler characteristic* $\chi^-(\mathcal{G})$ of $\mathcal{G}$ to be the number of edges of G minus the number of vertices with trivial vertex group. Note that $\mathcal{G}$ defines a splitting of its fundamental group as a free product $B_1 * \cdots * B_m * F_\ell$. The ordinary negative Euler characteristic of G calculates the quantity $\ell - 1$. In our definition, the m vertices which have nontrivial vertex group are not counted, so we see that $\chi^-(\mathcal{G}) = m + \ell - 1$.

The core filtration. If $f\colon \mathcal{G} \to \mathcal{G}$ is a CT, property (Filtration) says that the core of a filtration element G_r is a filtration element unless H_r is the bottom half of a dihedral pair. The *core filtration*

$$\varnothing = G_0 = G_{\ell_0} \subset G_{\ell_1} \subset \ldots \subset G_{\ell_M} = G_m = G$$

is the coarsening of the filtration for $f\colon \mathcal{G} \to \mathcal{G}$ defined by only including those elements that are their own core. By (Almost Periodic Edges) and (Filtration), we have $\ell_1 = 1$ or 2. The ith stratum of the core filtration, $H^c_{\ell_i}$ is

$$H^c_{\ell_i} = \bigcup_{j=\ell_{i-1}+1}^{\ell_i} H_j.$$

The change in negative Euler characteristic is $\Delta_i\chi^- = \chi^-(G_{\ell_i}) - \chi^-(G_{\ell_{i-1}})$.

The index of a finite-type graph of groups. Let $f\colon \mathcal{G} \to \mathcal{G}$ be a CT, and suppose that $\mathcal{H}$ is a graph of groups equipped with an immersion $\mathcal{H} \to \mathcal{G}$ (so edge groups of $\mathcal{H}$ are trivial). We say that $\mathcal{H}$ is of *finite type* if it has finitely many connected components, each of which is a finite graph of groups with finitely many vertices marked and finitely many infinite rays attached; in particular there are only finitely many nontrivial vertex groups in each component. Let $\mathcal{H}_1, \ldots, \mathcal{H}_\ell$ be the components of $\mathcal{H}$. Let $\alpha(\mathcal{H}_i)$ denote the number of ends (infinite rays) of $\mathcal{H}_i$, and let $b(\mathcal{H}_i)$ denote the number of rays mapping to eigenrays in $\mathcal{G}$. Let $c(\mathcal{H}_i)$ be the number of marked vertices, which we assume to have trivial vertex group. Define the *index* of $\mathcal{H}_i$ to be the quantity

$$j(\mathcal{H}_i) = \operatorname{rank}(\pi_1(\mathcal{H}_i)) + \frac{1}{2}a(\mathcal{H}_i) + \frac{1}{2}b(\mathcal{H}_i) + c(\mathcal{H}_i) - 1.$$

Again, we use the convention that $\operatorname{rank}(C_2 * C_2) = 1$. Define $j(\mathcal{H}) = \sum_{i=1}^{\ell} j(\mathcal{H}_i)$.

Almost Nielsen paths in CTs. Let $f\colon \mathcal{G} \to \mathcal{G}$ be a CT representing $\psi \in \operatorname{Out}(F,\mathcal{A})$. Each almost Nielsen path with endpoints at vertices has a complete splitting into almost fixed edges and indivisible almost Nielsen paths. There are two kinds of indivisible almost Nielsen paths ρ. The first possibility, by (NEG Almost Nielsen Paths), is that E is a linear or dihedral linear edge with axis w_E, and $\rho = gEw_E^k\bar{E}h$ for some $k \neq 0$ and vertex group elements g and h. The other possibility is that ρ is an indivisible almost Nielsen path of exponentially growing height r. By Proposition 6.9, up to equivalence ρ and $\bar\rho$ are the only indivisible almost Nielsen paths of height r and the initial edges of ρ and $\bar\rho$ are distinct. Observe that by exhausting the possibilities, every almost Nielsen path has the property that its initial and terminal directions are almost fixed. It follows that a non-exceptional almost Nielsen class based at a vertex v with nontrivial vertex group corresponds to at least one almost fixed direction based at v.

The graph of groups $\mathcal{S}_N(f)$. Let $f\colon \mathcal{G} \to \mathcal{G}$ be a CT representing $\psi \in \operatorname{Out}(F,\mathcal{A})$. In the proof of Theorem 7.1, we build a finite-type graph of groups $\mathcal{S}_N(f)$, analogous to that considered by Feighn–Handel in [FH18, Section 12] equipped with an immersion $s\colon \mathcal{S}_N(f) \to \mathcal{G}$ such that $j(\mathcal{S}_N(f)) = j(\psi)$. Each nontrivial almost Nielsen path with principal endpoints is equivalent to an almost Nielsen path lifting to $\mathcal{S}_N(f)$, and each ray R_E for $E \in \mathcal{E}$ lifts to $\mathcal{S}_N(f)$. In the proof we build up $\mathcal{S}_N(f)$ in stages by induction up through the core filtration. Here we describe its construction without reference to the filtration.

Since each non-exceptional almost Nielsen class containing v corresponds to at least one oriented edge E beginning at v and determining an almost fixed direction at v, the set of such oriented edges—modulo the equivalence relation that says $E \sim E'$ if there exists g and $h \in \mathcal{G}_v$ such that $Df(1,E) = (g,E)$ and $g^{-1}Df(h,E') = (h,E')$—is in bijection with the set of non-exceptional almost Nielsen classes containing v. Let E_v be the set of equivalence classes of edges E as above and begin with $\mathcal{S}_N(f)$ equal to the following set of vertices

$$\{v_{[E]} : v \text{ is a principal vertex of } G \text{ and } [E] \in E_v\}.$$

Call these vertices the *principal vertices* of $\mathcal{S}_N(f)$ Since f is rotationless, if $\mathcal{G}_v$ is finite, then there is a single equivalence class. For each $[E] \in E_v$, choose a preferred representative edge E. Then $Df(1,E) = (g,E)$ for some $g \in \mathcal{G}_v$, and set the vertex group of $v_{[E]}$ equal to $\operatorname{Fix}(i_{g^{-1}}f_v)$. If $\mathcal{G}_v$ is infinite and this subgroup is trivial, then mark $v_{[E]}$. Define $Df_{[E]}$ to be the map $d \mapsto g^{-1}Df(d)$ for directions d based at v. By assumption, if $E' \sim E$, then there is a fixed direction for $Df_{[E]}$ in the $\mathcal{G}_v$-orbit E'. Observe as well that if the direction d is fixed for $Df_{[E]}$ and x belongs to $\operatorname{Fix}(i_{g^{-1}}f_v)$, then

$$Df_{[E]}(x.d) = g^{-1}f_v(x).Df(d) = x.g^{-1}Df(d) = x.d.$$

Conversely if $d = (x,E')$ and (h,E') are both fixed by $Df_{[E]}$ and $Df(1,E') = (g_{E'},E')$, then we have

$$g^{-1}f_v(xh^{-1})g = g^{-1}f_v(x)g_{E'}g_{E'}^{-1}f_v(h^{-1})g = xh^{-1}.$$

If E is an almost fixed edge with principal endpoints v and w, we attach an edge (abusing notation, call it E) with initial endpoint $v_{[E]}$ and terminal endpoint $w_{[\bar{E}]}$. As a map of graphs, the immersion $s\colon \mathcal{S}_N(f) \to \mathcal{G}$ sends this new edge to E; as a map of graphs of groups, define the map on the directions determined by E so that they are sent to fixed directions for the map $Df_{[E]}$.

Suppose now that E is a linear or dihedral linear edge with axis w and initial vertex v. To $v_{[E]}$ we attach a *lollipop:* a graph consisting of two edges sharing a vertex, exactly one of which forms a loop. The loop maps to w, and the other edge to E. Map the initial direction determined by the edge mapping to E to a fixed direction for the map $Df_{[E]}$.

Suppose next that ρ is an indivisible almost Nielsen path of exponentially growing height with initial edge E, initial vertex v, terminal edge $\bar{E}'$ and terminal vertex w. We attach an edge mapping to μ with initial vertex $v_{[E]}$ and terminal vertex $w_{[E']}$. Map the initial direction of μ to a fixed direction for the map $Df_{[E]}$ and the initial direction of $\bar{\mu}$ to a fixed direction for the map $Df_{[E']}$.

Finally, suppose that $E \in \mathcal{E}$ with initial vertex v. If E belongs to a non-exponentially growing stratum, or E belongs to an exponentially growing stratum and is not the initial edge of an indivisible almost Nielsen path, attach an infinite ray mapping to R_E to $v_{[E]}$ so that the initial direction of this ray is mapped to a fixed direction for the map $Df_{[E]}$. If E is the initial edge of an indivisible almost Nielsen path ρ and E' is the initial edge of $\bar{\rho}$, then by Lemma 7.3, R_E and $R_{E'}$ have a common terminal subray $R_{E,E'}$. If we write $\rho = \alpha\beta$ as in Lemma 2.4, we have $R_E = \alpha R_{E,E'}$ and $R_{E'} = \bar{\beta} R_{E,E'}$. In this case subdivide the edge mapping to ρ into two edges, one mapping to α and the other to β, and attach $R_{E,E'}$ at the newly added vertex. This completes the construction of $\mathcal{S}_N(f)$; one checks directly that the map $s\colon \mathcal{S}_N(f) \to \mathcal{G}$ is an immersion of graphs of groups, and that if a vertex of $\mathcal{S}_N(f)$ has valence one and trivial vertex group, then that vertex is marked. Marked vertices of $\mathcal{S}_N(f)$ map to vertices of $\mathcal{G}$ with infinite vertex group.

Lemma 7.10. *Every tight path in $\mathcal{S}_N(f)$ with endpoints at principal vertices of $\mathcal{S}_N(f)$ projects to an almost Nielsen path in $\mathcal{G}$. Conversely, if σ is an almost Nielsen path in $\mathcal{G}$ with endpoints at principal vertices, then a path equivalent to σ lifts to $\mathcal{S}_N(f)$. The lift $\tilde{\sigma}$ is closed if and only if σ is closed and a path σ' equivalent to σ satisfies that $\sigma'\sigma'$ is an almost Nielsen path.*

Proof. Let σ be a tight path in $\mathcal{S}_N(f)$ with endpoints at principal vertices and let $\sigma = \sigma_1 \dots \sigma_m$ be a subdivision of σ at the principal vertices of $\mathcal{S}_N(f)$. Then each σ_i projects to an almost fixed edge or an indivisible almost Nielsen path, so we need only check that the concatenation of two subpaths projects to an almost Nielsen path. By assumption, the initial directions of $\bar{\sigma}_i$ and σ_{i+1}, which are based at a principal vertex $v_{[E]}$ are mapped to fixed directions for the map $Df_{[E]} = g^{-1}Df$. It follows that $f_\sharp(\sigma_1\sigma_2) = h\sigma_1 g^{-1} g \sigma_2 h'$ for vertex group elements h and h', and we conclude that the concatenation is an almost Nielsen path.

Now suppose that σ is an almost Nielsen path in $\mathcal{G}$ with endpoints at principal vertices, and let $\sigma = \sigma_1 \dots \sigma_m$ be the complete splitting of σ into almost fixed edges and indivisible almost Nielsen paths. It is clear that if $m = 1$, then a path equivalent to $\sigma = \sigma_1$ lifts to $\mathcal{S}_N(f)$, and that this lift is unique up to equivalence. We only need check that if the concatenation $\sigma_i\sigma_{i+1}$ is an almost Nielsen path, and $\tilde{\sigma}_i$ and $\tilde{\sigma}_{i+1}$ are lifts of σ_i and σ_{i+1} with common vertex $v_{[E]}$ and associated map $Df_{[E]} = g^{-1}Df$, then there is some vertex group element $h \in \mathrm{Fix}(i_{g^{-1}} f_v)$ such that $\tilde{\sigma}_i h \tilde{\sigma}_{i+1}$ lifts $\sigma_i\sigma_{i+1}$. Indeed, if σ_i' and σ_{i+1}' are the projections of $\tilde{\sigma}_i$ and $\tilde{\sigma}_{i+1}$ respectively, then we have $\sigma_i\sigma_{i+1} = g'\sigma_i' g'' \sigma_{i+1}' g'''$, and

$$\begin{aligned} h'\sigma_i\sigma_{i+1}h'' &= f_\sharp(\sigma_i\sigma_{i+1}) = f_\sharp(g')f_\sharp(\sigma_i')f_v(g'')f_\sharp(\sigma_{i+1}')f_\sharp(g''') \\ &= f_\sharp(g')kf_\sharp(\sigma_i')g^{-1}f_v(g'')gf_\sharp(\sigma_{i+1}')k'f_\sharp(g''') = h'g'\sigma_i'g''\sigma_{i+1}'g'''h'' \end{aligned}$$

since the directions of $\bar{\sigma}_i'$ and σ_{i+1}' are fixed by $Df_{[E]}$. From this we conclude that $g'' \in \mathrm{Fix}(i_{g^{-1}} f_v)$, as required. It is clear that if σ is not closed, then its lift is not closed. Suppose that σ is closed and that $\sigma\sigma$ is an almost Nielsen path. The argument in the previous paragraph shows that a path equivalent to $\sigma\sigma$ has a lift to $\mathcal{S}_N(f)$, and the uniqueness of the lift of σ implies that the lift of σ must be closed. Conversely, if σ has a closed lift $\tilde{\sigma}$, then

$\tilde{\sigma}\tilde{\sigma}$ is a tight path with endpoints at principal vertices, so it projects to an almost Nielsen path of the form $\sigma'\sigma'$ for some path σ' equivalent to σ. □

Lemma 7.11 (cf. Lemma 12.4 of [FH18]). *Let $\tilde{f}\colon \Gamma \to \Gamma$ be a principal lift fixing a vertex $\tilde{v}$ and the direction of an oriented edge $\tilde{E}$ based at $\tilde{v}$. Let S be the component of $\mathcal{S}_N(f)$ containing $v_{[E]}$. There is a lift of the immersion $s|_S\colon S \to \mathcal{G}$ to an embedding of Bass–Serre trees $\tilde{s}\colon \tilde{S} \to \Gamma$ such that $\tilde{E}$ is in the image of $\tilde{s}$. The image of $\tilde{s}$ is the smallest subtree of Γ whose limit set contains $\mathrm{Fix}_N(\hat{f})$.*

Proof. Since $s|_S$ is an immersion, any lift $\tilde{s}$ is an embedding by [Bas93, Proposition 2.7]. By post-composing any lift with an element of F, we may arrange that $\tilde{E}$ is in the image of $\tilde{s}$. Suppose that the lift $\tilde{f}$ corresponds to the automorphism Φ. We show first that $\tilde{s}(\tilde{S})$ contains the $\mathrm{Fix}(\Phi)$-minimal subtree of Γ. Given $c \in \mathrm{Fix}(\Phi)$, we have that $\tilde{w} = T_c(\tilde{v})$ is fixed by $\tilde{f}$ by Lemma 3.2. The image of the tight path from $\tilde{v}$ to $\tilde{w}$ in $\mathcal{G}$ is an almost Nielsen path σ in the almost Nielsen class determined by $v_{[E]}$. By the previous lemma, we can lift all such almost Nielsen paths to S. If c is peripheral, it follows that the fixed point of T_c is contained in the image of $\tilde{s}$. If c is non-peripheral, it follows inductively that the axis of T_c is contained in the image of $\tilde{s}$. It follows that the $\mathrm{Fix}(\Phi)$-minimal subtree of Γ is contained in $\tilde{s}(\tilde{S})$.

If $P \in \mathrm{Fix}_N(\hat{f}) \cap V_\infty(F,\mathcal{A})$, then the tight path $\tilde{\sigma}$ from $\tilde{v}$ to P projects to an almost Nielsen path which we can lift to S and hence $\tilde{\sigma}$ is contained in the image of $\tilde{S}$. If $P \in \mathrm{Fix}_N(\hat{f}) \cap \partial_\infty(F,\mathcal{A})$ is not isolated, then it is in the boundary of the $\mathrm{Fix}(\Phi)$-minimal subtree and thus the ray from $\tilde{v}$ to P is contained in the image of $\tilde{S}$. Finally if $P \in \mathrm{Fix}_+(\hat{f})$, then by Lemma 6.5, there is an edge $\tilde{E}'$ projecting into $\mathcal{E}$ such that the ray from $\tilde{E}'$ to P is fixed-point free. The unique tight path from $\tilde{v}$ to the initial vertex of $\tilde{E}'$ projects to an almost Nielsen path which we can lift to S. Since the direction determined by $\tilde{E}'$ belongs to the almost Nielsen class determined by $v_{[E]}$, the ray $\tilde{R}_{\tilde{E}'}$ is in the image of $\tilde{s}$. This shows that the limit set of the image of $\tilde{s}$ contains $\mathrm{Fix}_N(\hat{f})$. Conversely, suppose $\tilde{R}$ is a ray in the image of $\tilde{s}$, and project it to a tight ray R in $\mathcal{S}_N(f)$. If the ray R does not have a common terminal subray with some R_E, then we may subdivide at principal vertices of $\mathcal{S}_N(f)$ and write R as an almost Nielsen concatenation of almost Nielsen paths, and we see that $\tilde{R}$ ends in the $\mathrm{Fix}(\Phi)$-minimal subtree, so its endpoint is in the limit set of $\mathrm{Fix}(\Phi)$. Otherwise we may write R as the concatenation of an almost Nielsen path with some R_E, and we see that $\tilde{R}$ ends in $\mathrm{Fix}_+(\hat{f})$. □

Corollary 7.12. *If $\psi \in \mathrm{Out}(F,\mathcal{A})$ is rotationless and $f\colon \mathcal{G} \to \mathcal{G}$ is a CT representing ψ, then $j(\mathcal{S}_N(f)) = j(\psi)$.* □

Example 7.13. Consider the following CT $f\colon \mathcal{G} \to \mathcal{G}$, where $\mathcal{G}$ is the graph of groups in Figure 4. The map on the C_2 vertex groups is the unique isomorphism, and we write for example $\hat{A}$ to represent the path $\bar{A}gA$, where g is the nontrivial element of C_2. Note that none of the C_2 vertices are principal.

$$f\begin{cases} A \mapsto E\hat{D}\hat{C}\hat{D}\hat{B}\hat{A} \\ B \mapsto A\hat{B} \\ C \mapsto B\hat{A}\hat{B} \\ D \mapsto C\hat{D} \\ E \mapsto D\hat{C}\hat{D}\hat{E}. \end{cases}$$

The turn $(\bar{B},\bar{C})$ is the unique illegal turn of $\mathcal{G}$ and there is an indivisible (almost) Nielsen path $\rho = \hat{A}\hat{B}\hat{C}\hat{D}\hat{E}$ and the vertex with trivial vertex group is principal. To this vertex we attach the rays $R_{\bar{B}} = \hat{B}\hat{A}\hat{B}\hat{A}\hat{B}\hat{D}\hat{C}\hat{D}\hat{E}\ldots$, $R_{\hat{D}} = \hat{D}\hat{C}\hat{D}\hat{B}\hat{A}\hat{B}\hat{A}\hat{B}\ldots$ and the indivisible almost Nielsen path ρ, which forms a loop. We write $\alpha = \hat{A}\hat{B}$ and $\beta = \hat{C}\hat{D}\hat{E}$, subdivide

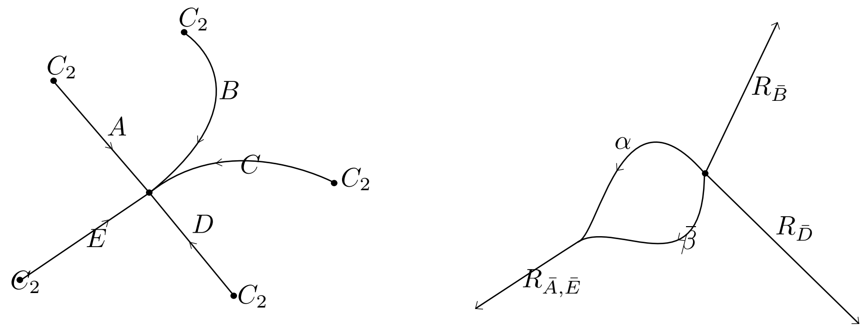

Figure 4: The CT $f\colon \mathcal{G} \to \mathcal{G}$ and the Stallings graph $\mathcal{S}_N(f)$.

ρ accordingly and we attach the ray $R_{\bar{A},\bar{E}} = \hat{D}\hat{C}\hat{D}\hat{E}\hat{D}\hat{C}\hat{D}\hat{B}\hat{A}\ldots$ to the added vertex. See Figure 4 We have the index formula $j(f) = \frac{3}{2} + 1 - 1 = \frac{3}{2}$.

Example 7.14. Consider the following CT $f\colon \mathcal{G} \to \mathcal{G}$, where $\mathcal{G}$ is the graph of groups in Figure 5. The map on the C_2 vertex groups is again the unique isomorphism, and we continue to write $\hat{A}$ to represent the path $\bar{A}gA$, where g is the nontrivial element of C_2. The C_2 vertex group incident to the edge C is principal, while the edges A and B form a dihedral pair.

$$f\begin{cases} A \mapsto B \\ B \mapsto A \\ C \mapsto C\hat{A}. \end{cases}$$

The edge C is a dihedral linear edge, so to its initial vertex (which has C_2 vertex group in

Figure 5: The CT $f\colon \mathcal{G} \to \mathcal{G}$ and the Stallings graph $\mathcal{S}_N(f)$.

$\mathcal{S}_N(f)$) we attach a lollipop. We have the index formula $j(f) = 2 - 1 = 1$.

Example 7.15. Consider the following CT $f\colon \mathcal{G} \to \mathcal{G}$ where $\mathcal{G}$ is the graph of groups in Figure 6. The groups A and A' are infinite (but somewhat arbitrary). Choose the automorphisms f_A and $f_{A'}$ to have no periodic points. The corresponding vertices form a single almost Nielsen class.

$$f\begin{cases} E \mapsto E \\ B \mapsto BEg\bar{E}g' \end{cases}$$

To them in $\mathcal{S}_N(f)$ we attach the fixed edge E. These vertices are marked. We also attach the eigenray $R_B = BEg\bar{E}g'Ef_A(g)\bar{E}f_{A'}(g')\ldots$. We have the index formula $j(f) = 2+1-1 = 2$, which is the maximum possible.

Figure 6: The CT $f\colon \mathcal{G} \to \mathcal{G}$ and the Stallings graph $\mathcal{S}_N(f)$.

Proof of Theorem 7.1. Let $f\colon \mathcal{G} \to \mathcal{G}$ be a CT representing $\psi \in \mathrm{Out}(F, \mathcal{A})$. We construct $\mathcal{S}_N(f)$ by inducting up through the core filtration of $\mathcal{G}$, following the outline of [FH18, Proposition 15.14]. For i satisfying $0 \le i \le M$, let $\mathcal{S}_N(i)$ denote the subgraph of groups of $\mathcal{S}_N(f)$ constructed by the ith term of the core filtration G_{ℓ_i}. We shall prove that

$$j(\mathcal{S}_N(i)) \le \chi^-(G_{\ell_i}),$$

from which the theorem follows. We prove this inequality by induction. The case $i = 0$ holds trivially, since $\mathcal{S}_N(0)$ and G_{ℓ_0} are empty. Let $\Delta_k j = j(\mathcal{S}_N(k)) - j(\mathcal{S}_N(k-1))$. Suppose that the inequality holds for $k-1$. We prove that $\Delta_k j \le \Delta_k \chi^-$, so that the inequality holds for k as well. Throughout we will in fact slightly overestimate $\Delta_k j$ by not considering the special case of $C_2 * C_2$. The proof has two cases.

Case 1. Suppose that $H^c_{\ell_k}$ contains no exponentially growing strata. We claim that one of the following occurs.

(a) We have $\ell_k = \ell_{k-1} + 1$ or $\ell_{k-1} + 2$. The stratum $H^c_{\ell_k}$ is disjoint from $G_{\ell_{k-1}}$ and forms a dihedral pair.

(b) We have $\ell_k = \ell_{k-1} + 1$. The unique edge in $H^c_{\ell_k}$ is disjoint from $G_{\ell_{k-1}}$ and is fixed or almost fixed with principal endpoints. Either it forms a loop, or both incident vertices have nontrivial vertex group.

(c) We have $\ell_k = \ell_{k-1} + 1$. The unique edge in $H^c_{\ell_k}$ either has both endpoints contained in $G_{\ell_{k-1}}$ or one endpoint contained in $G_{\ell_{k-1}}$ and the other has nontrivial vertex group.

(d) We have $\ell_k = \ell_{k-1} + 2$. The two edges in $H^c_{\ell_k}$ share an initial endpoint, are not almost periodic, and have terminal endpoints in $G_{\ell_{k-1}}$. (Note that the initial endpoint has trivial vertex group, otherwise we would be in case (c).)

The proof is similar to [FH09, Lemma 8.3]. First suppose that some edge in $H^c_{\ell_k}$ belongs to a dihedral pair. Then by (Filtration) if the dihedral pair is fixed, and by the definition of a dihedral pair otherwise, we see that we are in case (a). So suppose that no edge in $H^c_{\ell_k}$ belongs to a dihedral pair. Because $H^c_{\ell_k}$ contains no exponentially growing strata, it contains no zero strata by (Zero Strata). Since CTs satisfy the conclusions of Theorem 2.6, the initial endpoint of each non-almost periodic edge E_j in $H^c_{\ell_k}$ is principal, and thus E_j is the only edge in H_j for some j satisfying $\ell_{k-1} + 1 \le j \le \ell_k$. The same holds by (Almost Periodic Edges) for almost periodic (hence almost fixed) edges, so each H_j is a single edge E_j. If some E_j is almost fixed, then either (b) or (c) holds by (Almost Periodic Edges). If each E_j is not fixed or almost fixed and (c) does not hold, then E_j adds a valence-one vertex with trivial vertex group to $G_{\ell_{k-1}}$, and the terminal vertex of E_j belongs to $G_{\ell_{k-1}}$ by Lemma 6.6. Each new valence-one vertex must be an endpoint of E_{ℓ_k}, so we are in case (d). We analyze each subcase.

(a) Nothing is added to $\mathcal{S}_N(k)$ because no vertex of the dihedral pair is principal, so $\Delta_k j = 0$ while $\Delta_k \chi^- = 1$.

(b) Suppose that E_{ℓ_k} forms a loop with trivial incident vertex group. Then $\Delta_k j = \Delta_k \chi^- = 0$. In all other cases $\Delta_k \xi^- = 1$. The nontrivial vertex group(s) contribute new principal vertices to $\mathcal{S}_N(k)$ that either have nontrivial vertex group or are marked. The unique new edge in $\mathcal{S}_N(k)$ either forms a loop or does not. In all cases we see that $\Delta_k j = 1$.

(c) In this case we always have $\Delta_k \chi^- = 1$ and $\Delta_k j = 1$. There are several possibilities for the change to $\mathcal{S}_N(k)$, but they may be summarized as follows: at most two new vertices are added to $\mathcal{S}_N(k)$, each of which is either marked or has nontrivial vertex group. If E_{ℓ_k} is almost fixed, one new edge is added to $\mathcal{S}_N(k)$; if there are any new vertices, then the new edge is incident to them. If E_{ℓ_k} is linear, then we add a lollipop; the valence-one vertex of the lollipop is added to the new vertex, if there is one. If E_{ℓ_k} is nonlinear, then we add the eigenray E_{ℓ_k}, attached to the new vertex, if there is one. In all cases, we see that $\Delta_k j = 1$.

(d) In this case we have $\Delta_k \chi^- = 1$. The vertex in $\mathcal{G}_{\ell_k}$ not contained in $G_{\ell_{k-1}}$ is principal, so determines a new component of $\mathcal{S}_N(k)$. This vertex has trivial vertex group and is unmarked. To this new vertex we attach a lollipop for each linear edge, and an eigenray for each nonlinear edge, for a total of three possible combinations. In all cases $\Delta_k j = 1$.

Case 2. The argument again is similar to [FH18, Lemma 8.3]. Suppose that $H^c_{\ell_k}$ contains an exponentially growing stratum H_r. By the argument in Case 1, we see that $H^c_{\ell_k}$ contains no dihedral pairs. Therefore in this case the core of a filtration element is a filtration element, and because G_{r-1} does not carry the unique attracting lamination Λ^+ associated to H_r, we see that G_r is its own core. Thus H_{ℓ_k} is the unique exponentially growing stratum in $H^c_{\ell_k}$. We claim that there exists some integer u_k satisfying $\ell_{k-1} \leq u_k < \ell_k$ such that the following hold.

(a) For j satisfying $\ell_{k-1} < j \leq u_k$, the stratum H_j is a single edge which is not almost fixed whose terminal vertex is in $G_{\ell_{k-1}}$ and whose initial vertex has trivial vertex group and valence one in G_{u_k}.

(b) For j satisfying $u_k < j < \ell_k$, the stratum H_j is a zero stratum.

The existence u_k and that (b) holds follows from (Zero Strata). That (a) holds follows from (Almost Periodic Edges) and Lemma 6.6.

For j as in item (a) above, the contribution to $\mathcal{S}_N(k)$ is the addition of a new component, with new vertex unmarked and with trivial vertex group, to which we attach either a lollipop or an eigenray. Thus in each case $\Delta_k j = 0$, while $\Delta_k \chi^- = 0$. This completes the analysis up to G_{u_k}.

To calculate $\Delta\chi^-$, it will be useful to note that $\chi^-(\mathcal{G})$ is equal to the sum over the vertices v of G of the quantity $\frac{1}{2}\,\mathrm{valence}(v) - 1$ if v has trivial vertex group and $\frac{1}{2}\,\mathrm{valence}(v)$ otherwise. Note that each edge contributes valence to two vertices (which may be equal).

For each vertex $v \in H_{\ell_k}$, let $\Delta_k j(v)$ and $\Delta_k \chi^-(v)$ be the contributions to $\Delta_k j$ and $\Delta_k \chi$ from v that are not already considered. If v is principal, let $\kappa(v)$ denote the number of oriented edges of H_{ℓ_k} incident to v that do not determine almost fixed directions at v. If $v \in H_{\ell_k}$ is not principal, then it either has trivial vertex group or vertex group C_2. In the former case let $\kappa(v) = \mathrm{valence}(v) - 2$, and in the latter case let $\kappa(v) = \mathrm{valence}(v)$.

Suppose first that there no indivisible almost Nielsen paths of height ℓ_k. If the vertex v is not principal, then $\Delta_k j(v) = 0$. Such a vertex has link in G_{ℓ_k} entirely contained in $H^z_{\ell_k}$ and is thus new. We have $\Delta_k \chi^- = \frac{1}{2}\kappa(v)$ and we have $\Delta_k \chi^-(v) - \Delta_k j(v) = \frac{1}{2}\kappa(v) > 0$.

If instead the vertex v is principal, let $L(v)$ denote the number of oriented edges of H_{ℓ_k} based at v and contained in $\mathcal{E}$. We add $L(v)$ rays to $\mathcal{S}_N(k)$, possibly to new vertices of $\mathcal{S}_N(k)$. If v has nontrivial vertex group, then each $v_{[E]}$ in $\mathcal{S}_N(k)$ has nontrivial vertex group or is marked, and we see that $\Delta_k j(v) = \frac{1}{2}L(v)$ and $\Delta_k \chi^- = \frac{1}{2}(L(v) + \kappa(v))$. The same calculation holds if v has trivial vertex group and has already been added to $\mathcal{S}_N(k)$. Finally if v is new and has trivial vertex group, then $\Delta_k j(v) = \frac{1}{2}L(v) - 1$ and $\Delta_k \chi^-(v) = \frac{1}{2}(L(v) + \kappa(v)) - 1$. In all cases we see that $\Delta_k \chi^-(v) - \Delta_k j(v) = \frac{1}{2}\kappa(v) \geq 0$.

Suppose finally that there is an indivisible almost Nielsen path ρ of height ℓ_k. By Lemma 6.10, we have $\ell_k = u_k + 1$. Let w and w' be the endpoints of ρ. By Lemma 6.17, if $w \neq w'$, then at least one of w and w' is new and has trivial vertex group. In both cases, the initial edges of ρ and $\bar{\rho}$ are distinct, say e and e'.

Let $\mathcal{V}$ be the set of vertices of H_{ℓ_k} that are not endpoints of μ. Each vertex of $\mathcal{V}$ may be handled as in the case without indivisible almost Nielsen paths, so we conclude

$$\sum_{v \in \mathcal{V}} \Delta_k \chi^-(v) - \sum_{v \in \mathcal{V}} \Delta_k j(v) = \sum_{v \in \mathcal{V}} \frac{1}{2}\kappa(v) \geq 0.$$

Let $\Delta_k j(\rho)$ and $\Delta_k \chi^-(\rho)$ be the contributions to $\Delta_k j$ and $\Delta_k \chi^-$ coming from the endpoints of ρ and not already considered. There are two subcases to consider according to whether $w = w'$.

1. Suppose ρ is a closed path based at the vertex w. In $\mathcal{S}_N(k)$ we possibly add a new vertex, a loop mapping to ρ, a ray for each $E \in \mathcal{E}$ incident to w other than e and e', and then one ray corresponding to e and e'. The real picture is in fact slightly more complicated: the rays contributing to $L(v)$ may lie in distinct almost Nielsen classes, in which case multiple vertices mapping to w are added. In this case each such vertex either has nontrivial vertex group or is marked. Therefore the final calculation is that
$$\Delta_k j(\rho) = 1 + \frac{1}{2}(L(w) - 1) - 1 = \frac{1}{2}L(w) - \frac{1}{2}$$
if w is new and has trivial vertex group and $\Delta_k j(\mu) = \frac{1}{2}L(w) + \frac{1}{2}$ otherwise. Similarly we have
$$\Delta_k \chi^-(\mu) = \frac{1}{2}(L(w) + \kappa(w)) - 1$$
if w is new and has trivial vertex group and $\Delta_k \chi^-(\mu) = \frac{1}{2}(L(w) + \kappa(w))$ otherwise. Thus
$$\Delta_k \chi^-(\mu) - \Delta_k j(\mu) \geq \frac{1}{2}\kappa(w) - \frac{1}{2}.$$
Since there is always at least one illegal turn in H_{ℓ_k}—for instance, the one in ρ—and since illegal but nondegenerate turns are determined by distinct edges, there must be at least one vertex of H_{ℓ_k} with $\kappa(v) \neq 0$, so we conclude that $\Delta_k j \leq \Delta_k \chi^-$ as desired.

2. Suppose that w and w' are distinct. By Lemma 6.17, at least one vertex, say w, is new and has trivial vertex group. In $\mathcal{S}_N(k)$ we add the new vertex w, an edge connecting w to w', one ray for each $E \in \mathcal{E}$ based at w or w' other than e and e' and one ray corresponding to e and e'. We may add one or more vertices mapping to w'; if we add more than one, each said vertex has nontrivial vertex group or is marked. Thus we have
$$\Delta_k j(\rho) = \frac{1}{2}(L(w) + L(w')) - \frac{1}{2}$$
if w' is old or has nontrivial vertex group and $\Delta_k j(\rho) = \frac{1}{2}(L(w) + L(w')) - \frac{3}{2}$ if w' is new and has trivial vertex group. Similarly,
$$\Delta_k \chi^-(\rho) = \frac{1}{2}(L(w) + L(w') + \kappa(w) + \kappa(w')) - 1$$
if w' is old or has nontrivial vertex group and $\Delta_k \chi^-(\rho) = \frac{1}{2}(L(w) + L(w') + \kappa(w) + \kappa(w')) - 2$ if w' is new and has trivial vertex group. The argument concludes as in the previous subcase.

□

Corollary 7.16. *If in the definition of $j(\psi)$, we allowed* $\mathrm{rank}(C_2 * C_2) = 2$*, the conclusion*

$$j(\psi) \leq n + k - 1$$

holds, where now the sum defining $j(\psi)$ should be taken over the isogredience classes of all automorphisms Ψ representing ψ.

Proof. In the proof of Theorem 7.1, the only place where $C_2 * C_2$ is directly treated differently is the case of a dihedral pair, where we had $\Delta_k j = 0$ while $\Delta_k \chi^- = 1$. If we were to add the dihedral pair to $\mathcal{S}_N(k)$ in the case where the C_2 factors are fixed, we would get $\Delta_k j = 1$. The rest of the proof shows that this new $j(\psi)$ satisfies $j(\psi) \leq n+k-1$. If $\mathrm{Fix}(\Psi) = C_2 * C_2$ and Ψ is not principal, but corresponds to the lift $\tilde{f}$ of a CT $f\colon \mathcal{G} \to \mathcal{G}$ representing ψ, then $\mathrm{Fix}(\tilde{f})$ projects to the quotient of $\mathbb{R}$ by the standard action of $C_2 * C_2$; this quotient must be a dihedral pair, so we see that there are finitely many isogredience classes of automorphisms Ψ with this new $j(\Psi) > 0$, and that each one which is not principal corresponds to a dihedral pair. □

To avoid confusion, we will continue to use the index $j(\psi)$ from Theorem 7.1. The following corollary is immediate from the proof of Theorem 7.1.

Corollary 7.17. *Suppose $f\colon \mathcal{G} \to \mathcal{G}$ is a CT representing $\psi \in \mathrm{Out}(F, \mathcal{A})$. If f has a dihedral pair, then $j(\psi) \leq n + k - 2$. If f has no dihedral pairs and no exponentially growing strata, then $j(\psi) = n + k - 1$. If f has an exponentially growing stratum without an indivisible almost Nielsen path, then $j(\psi) < n + k - 1$.* □

Corollary 7.18 (cf. Corollaries 15.17 and 15.18 of [FH18])**.** *The following statements hold for a CT $f\colon \mathcal{G} \to \mathcal{G}$ representing $\psi \in \mathrm{Out}(F, \mathcal{A})$.*

1. *Each component of $\mathcal{S}_N(f)$ contributes at least $\frac{1}{2}$ to $j(\psi)$.*
2. *$\mathcal{S}_N(f)$ has at most $2j(\psi)$ components, and thus $P(\psi)$ has at most $2j(\psi)$ isogredience classes.*
3. $\left|\bigcup_{\Psi \in P(\psi)} \mathrm{Fix}_+(\hat{\Psi})/F\right| \leq 6n + 6k - 6.$

Proof. The proof is similar to [FH18, Corollaries 15.17 and 15.18]. Item 2 follows from item 1. The only possibility where a component of $\mathcal{S}_N(f)$ could contribute less than $\frac{1}{2}$ to $j(\psi)$ is the case where that component is topologically a line, no vertices of which have nontrivial vertex group and none of which are marked, and each end of the line is a ray in an exponentially growing stratum. It follows that if v is a principal vertex such that $v_{[E]}$ lies in our component, then v has trivial vertex group and is not the endpoint of an indivisible almost Nielsen path. Let w be the lowest principal vertex such that $w_{[E]}$ belongs to our putative component; we rule out each case of the proof of Theorem 7.1 to rule out the existence of $w_{[E]}$. Since w is principal we are not in Case (1a). Case (1b) is ruled out because w has trivial vertex group but is principal, so must have at least three fixed directions. Since w has trivial vertex group but is new in H_{ℓ_k}, we are not in case (1c). If we were in case (1d), then both edges incident to w must be non-linear, in contradiction to our assumption. In Case 2, if w belongs to H_{u_k}, then an edge incident to w is non-linear, in contradiction to our assumption. Therefore w is a new vertex in the exponentially growing stratum H_{ℓ_k}, and by Lemma 2.7, there are at least two fixed directions in H_{ℓ_k} based at w. Since w is principal but is not the endpoint of an indivisible almost Nielsen path, there must be at least *three* fixed directions in H_{ℓ_k} based at w; this contradiction completes the proof of item 1.

For item 3, we have

$$j(\psi) \geq \frac{|\text{ends of } \mathcal{S}_N(f)|}{2} - |\text{components of } \mathcal{S}_N(f)| \geq \frac{1}{2}\left|\bigcup_{\Psi \in P(\psi)} \mathrm{Fix}_+(\hat{\Psi})/F\right| - 2j(\psi).$$

So we see that $\left|\bigcup_{\Psi \in P(\psi)} \mathrm{Fix}_+(\hat{\Psi})/F\right| \leq 6j(\psi) \leq 6n + 6k - 6.$ □

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