Train Tracks on Graphs of Groups & Outer Automorphisms of Hyperbolic Groups

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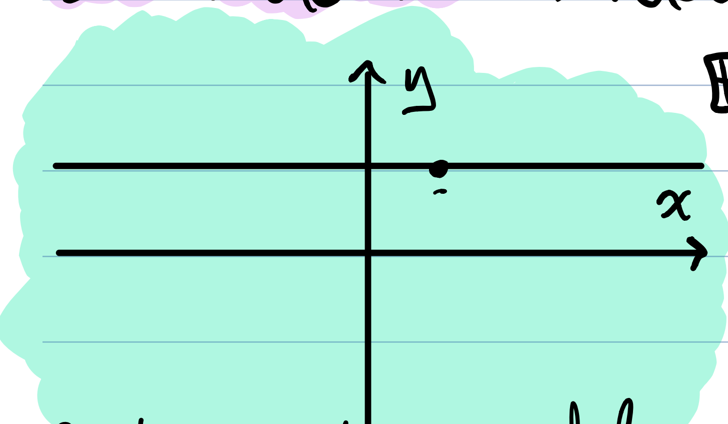
4/6/20

For non-experts: Three 2-dim'l geometries

Euclidean: middle-school, intuition



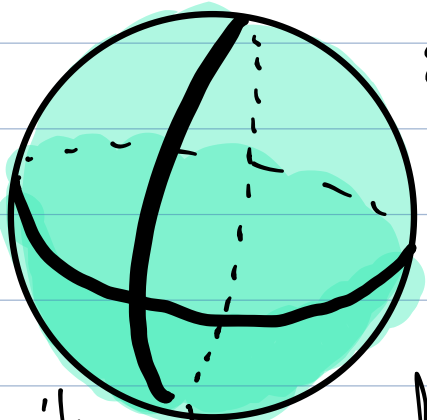
Cartesian plane model

$\mathbb{E}^2$

straight lines are straight lines

one parallel

Spherical: surface of earth at big scales


unit sphere model

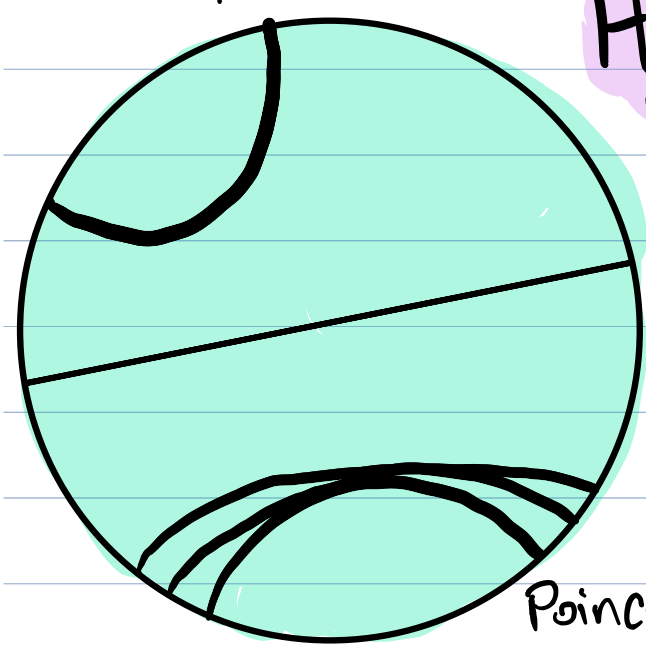
S^2

"straight lines" are great circles

0 parallels

Hyperbolic: kale leaves?

parameter spaces?

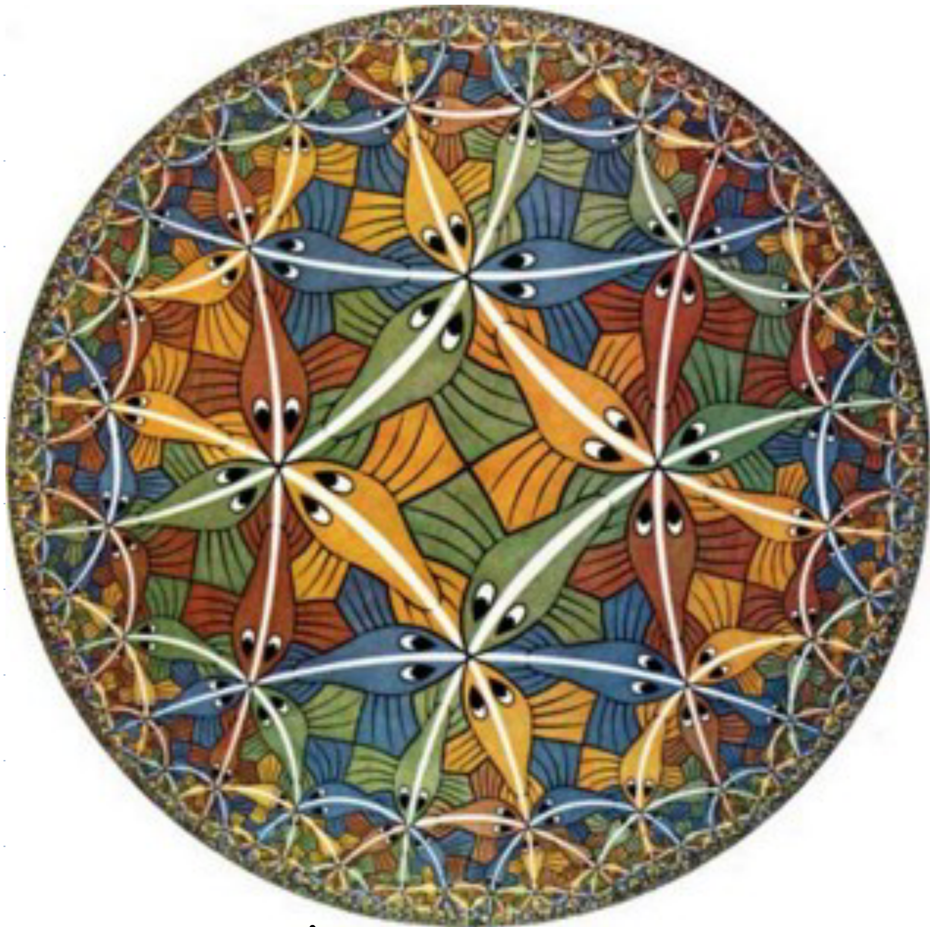

Poincaré unit disk model

$\mathbb{H}^2$

straight lines are arcs & lines $\perp$ to the ∂

∞-ly many parallels

What's up with the Poincaré disk?



M.C. Escher, Circle Limit III

The white lines are "straight" or geodesic — the fastest way to travel

All the fish are the same size — area is distorted in the model

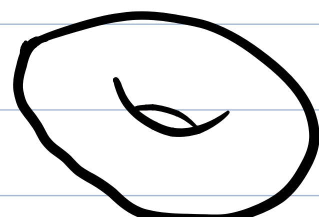
Thm (Uniformization, Poincaré, Koebe 1907)

A surface of genus g supports exactly one of the three geometries:

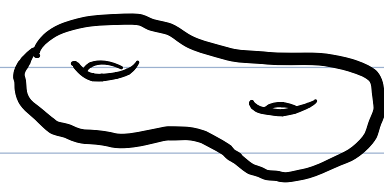
$g=0 \longleftrightarrow$ spherical

$g=1 \longleftrightarrow$ Euclidean

$g\geq 2 \longleftrightarrow$ hyperbolic



$g=1$



$g=2$

The **mapping class group** $\mathrm{Mod}(S_g)$ is the group of homotopy classes of orientation-preserving diffeomorphisms of S_g a surface of genus g.

$$\mathrm{Mod}(S_g) = \pi_0(\mathrm{Diff}^+(S_g))$$

Thm (Nielsen-Thurston classification) Thurston '76.

Every $\varphi \in \mathrm{Mod}(S_g)$ is either

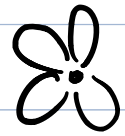
(1) **periodic**: $\exists k$, $\varphi^k = 1$

(2) **reducible**: $\exists$ finite set of isotopy classes of essential simple closed curves permuted by φ

(3) **pseudo-Anosov**: the contrary case "mixing"

The outer automorphism group of a free group $Out(F_n)$ is

$$Aut(F_n)/Inn(F_n).$$

$F_n = \pi_1($  $, \cdot)$; $Out(F_n) = \pi_0(HE_q($  $))$

n petals

A self-map $f : \Gamma \longrightarrow \Gamma$ of a graph Γ (sending vertices to vertices, edges to edge paths) is a **train track map** if $f^k|_e$ is locally injective $\forall k \geq 1$

Thm (Bestvina–Handel '92)

Every $\varphi \in Out(F_n)$ is represented by a relative train track map

$$f : \Gamma \longrightarrow \Gamma \quad \text{for some } \Gamma,\ \pi_1(\Gamma) \cong F_n$$

What is common to these?

Question: (Paulin '91) For $\Phi: G \to G$ an automphm of a hyperbolic group, does some $\Phi' \in [\Phi]$ satisfy one of the following cases?

(1) **periodic**: $[\Phi]$ has finite order in $\mathrm{Out}(G)$

(2) **reducible**: $\exists$ a G-tree T with "small" edge stabilizers and an automorphism $f: T \to T$ s.t. $\Phi'(g).x = f(g \cdot x)$

weaker condition

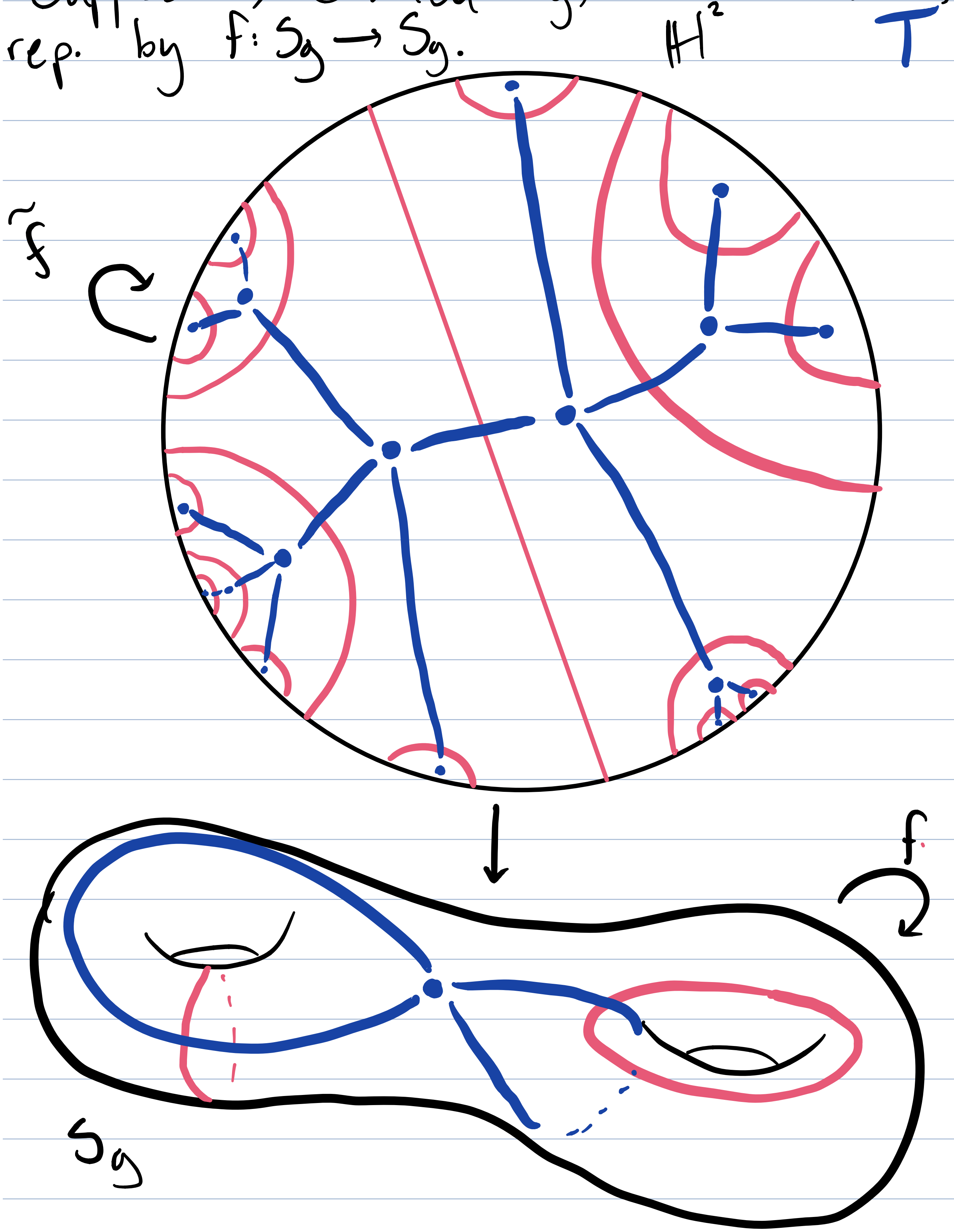
(3) **""pseudo-Anosov""** $\exists$ an $\mathbb{R}$-tree with small arc stabilizers T w/ isometric G-action, $H: T \to T$, $\lambda > 1$ an alg. integer and $x_0 \in T$ s.t.

$$d(H(x), H(y)) = \lambda d(x,y) \quad \forall x, y \in T$$

and $$\Phi'(g).x_0 = H(g.x_0).$$

Reducible is Reducible:

Suppose $\varphi \in \mathrm{Mod}(S_g)$ is reducible, rep. by $f: S_g \to S_g$.



Thm (L'20) If a gp G splits as $G = \pi_1(\Gamma, \mathcal{G})$ and $\varphi \in Out(G)$ is represented by

$$f : (\Gamma, \mathcal{G}) \to (\Gamma, \mathcal{G}),$$

then $\exists$ a train track map

$$f' : (\Gamma', \mathcal{G}') \to (\Gamma', \mathcal{G}')$$

representing φ, where $(\Gamma, \mathcal{G}) \geq (\Gamma', \mathcal{G}')$

might have collapsed stuff

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Cor Relative train track maps for free products, virtually free gps, etc.

Cor The answer to Paulin's question is yes.

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Tool in **Thm** (L'19) If A is a finite gp, and $\varphi\in Out(A*A*\cdots*A)$ is **polynomially-growing**

$$(A*A*\cdots*A)\rtimes_{\varphi^k}\mathbb{Z}\qquad k\geq 1$$

is a CAT(0) group

A **graph of groups** is a connected graph Γ, with a diagram $\mathcal{G}$ of groups and monomorphisms

$$\Gamma \quad \underset{v}{\bullet} \xleftarrow[e]{\quad} \underset{w}{\bullet} \qquad + \qquad \begin{tikzcd} v \arrow[d, mapsto] & e \arrow[l, "\iota_e"'] \arrow[r, "\iota_{\bar{e}}"] \arrow[d, mapsto, "\mathcal{G}"] & w \arrow[d, mapsto] \\ C_4 & C_2 \arrow[l, hook'] \arrow[r, hook] & C_6 \end{tikzcd}$$

$$= \quad (\Gamma, \mathcal{G}) \qquad \overset{C_4}{\bullet} \underset{C_2}{\text{———}} \overset{C_6}{\bullet} \qquad \pi_1(\Gamma, \mathcal{G}) \cong SL_2 \mathbb{Z}$$

Thm (Bass–Serre '70s)

Define $\pi_1(\Gamma, \mathcal{G}, *)$.

G-actions on trees w/o inversions

$\longleftrightarrow$ graphs of groups w/ $\pi_1(\Gamma, \mathcal{G}, *) \cong G$

Def (Bass '91) defines morphisms of graphs of groups

$f : (\Lambda, \mathcal{L}) \to (\Gamma, \mathcal{G})$ morphisms

$$\updownarrow$$

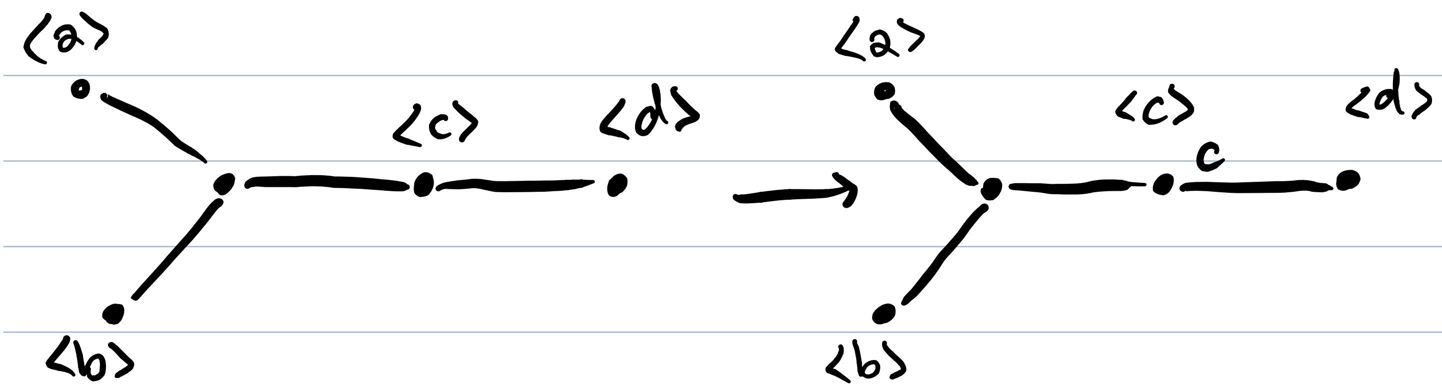
$f_{\#} : \pi_1(\Lambda, \mathcal{L}, p) \longrightarrow \pi_1(\Gamma, \mathcal{G}, f(p))$

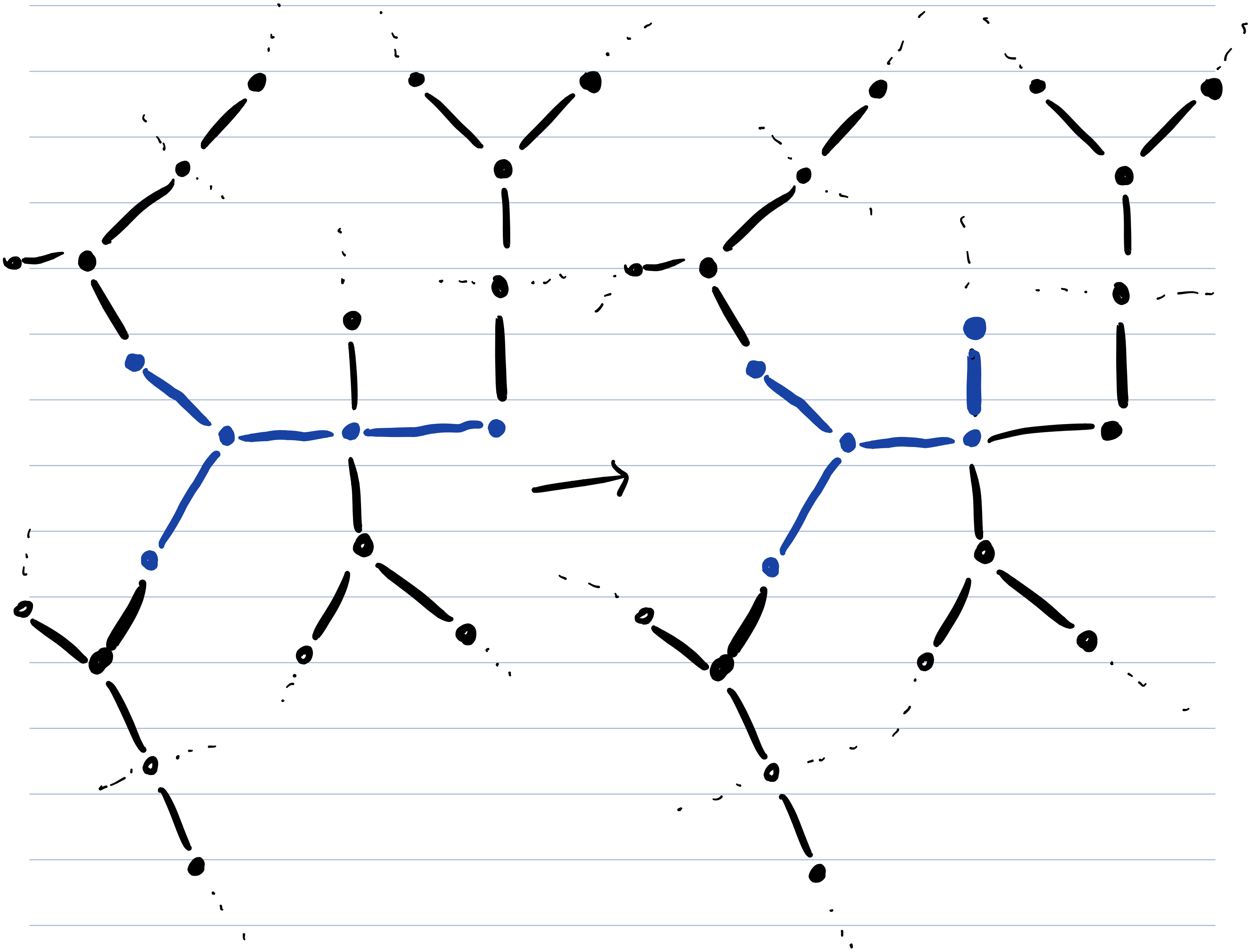
$\tilde{f} : (\tilde{\Lambda}, \tilde{p}) \longrightarrow (\tilde{\Gamma}, \widetilde{f(p)})$

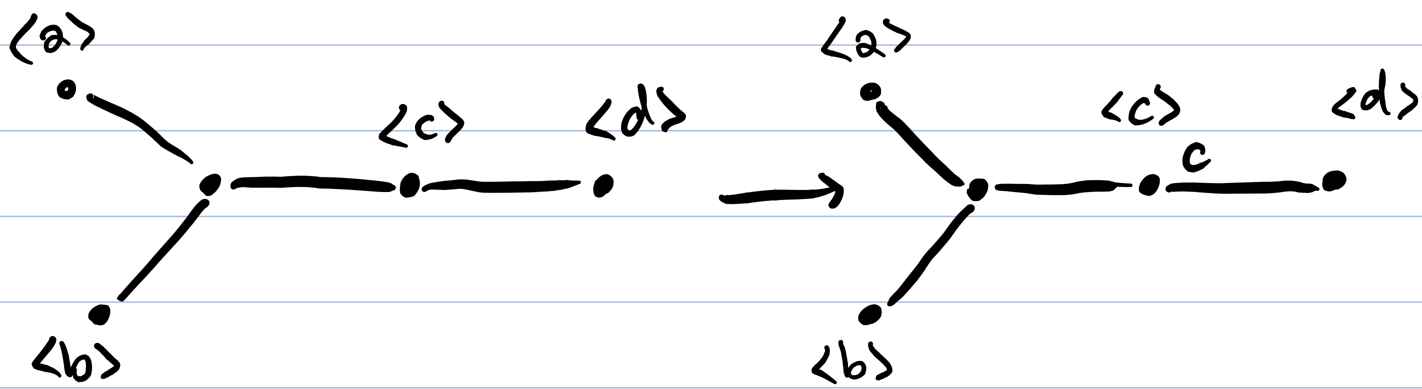
s.t. $\tilde{f}(g \cdot x) = f_{\#}(g) . \tilde{f}(x)$

Ex:

$W_4 = \langle a, b, c, d : a^2 = b^2 = c^2 = d^2 = 1 \rangle$






<a>
<b>
<c>
<d>
<a>
<c>
c
<d>
<b>

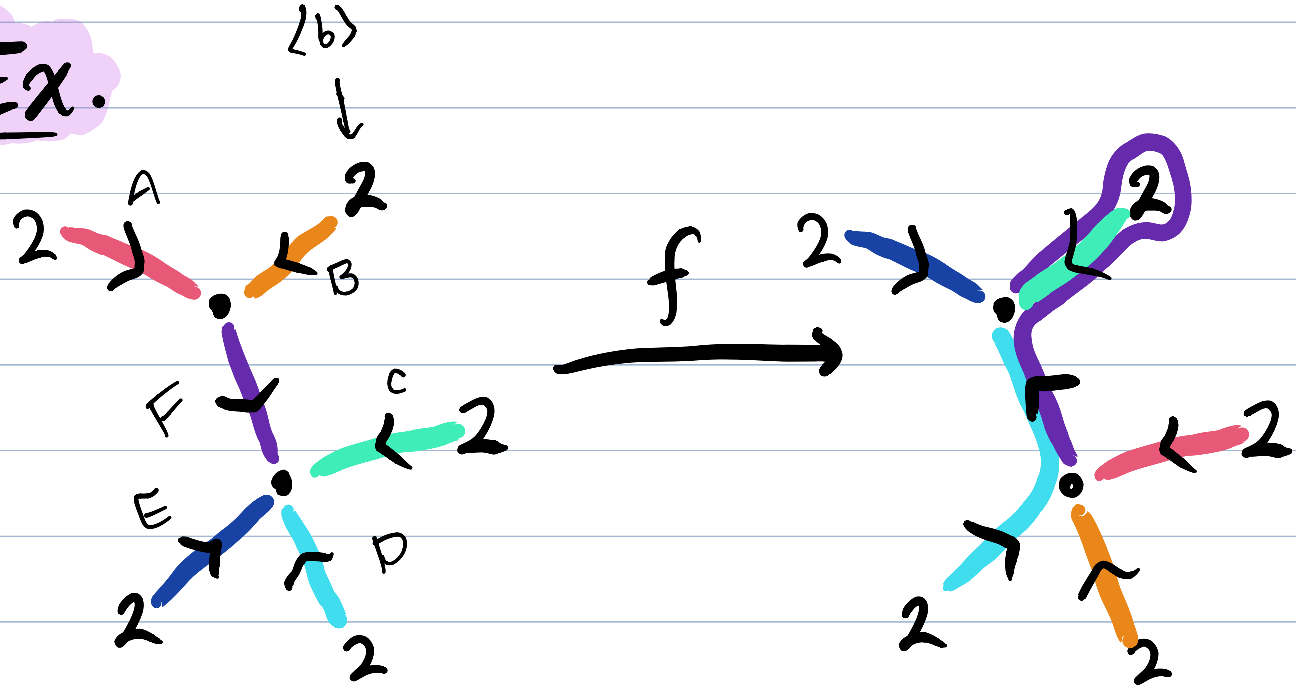
A map $f:(\Gamma, g) \longrightarrow (\Gamma, g)$

is a **train track map**

if it can be made a morphism after subdividing and

$f^k|_e$ is an immersion $\forall k \geq 1$

Ex.



$A \longmapsto C$

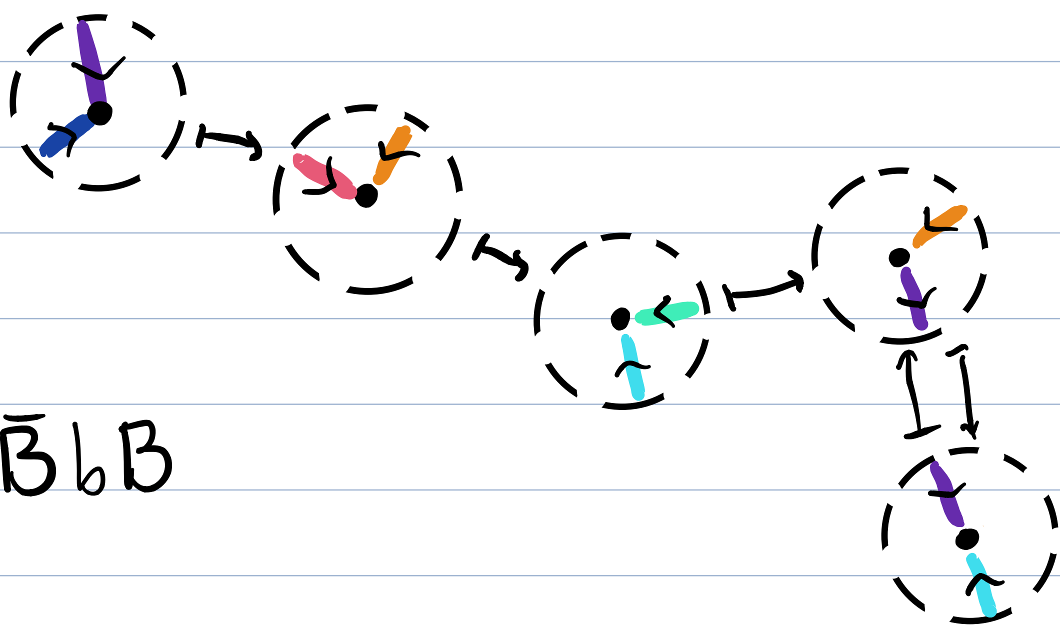
$B \longmapsto D$

$C \longmapsto B$

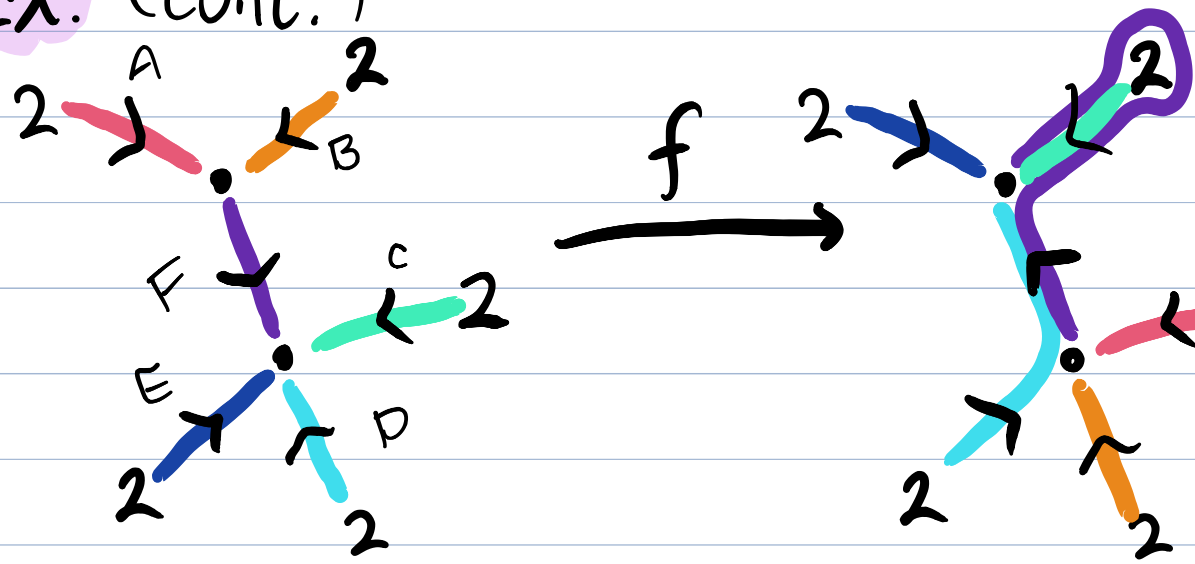
$D \longmapsto E\bar{F}$

$E \longmapsto A$

$F \longmapsto \bar{F}\bar{B}bB$



Ex. (cont.)



$$
\begin{array}{c|c}
 & \begin{matrix} A & B & C & D & E & F \end{matrix} \\
\begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}
\end{array}
$$

Perron-Frobenius

$\lambda > 1$

largest root of

$x^6 - x^5 - 2x^3 - x + 1$

≈ 1.722

λ has a positive eigenvector.

If we use it to assign lengths, f stretches paths by λ.

so get $\tilde{f} : \tilde{\Gamma} \longrightarrow \tilde{\Gamma}$ s.t.

$$d(\tilde{f}(x), \tilde{f}(y)) \leq \lambda\, d(x, y)$$

Recall:

(3) ""pseudo-Anosov"" $\exists$ an $\mathbb{R}$-tree T w/ isometric G-action, $H: T \to T$, $\lambda > 1$ an alg. integer and $x_0 \in T$ s.t.

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and $\Phi'(g) \cdot x_0 = H(g \cdot x_0)$.

So, how do we get a homothety from a train track map?

$\omega_5 \langle a, b, c, d, e \rangle$

$$\begin{aligned} a &\mapsto b \\ b &\mapsto c \\ c &\mapsto d \\ d &\mapsto e \\ e &\mapsto bab^{-1} \end{aligned}$$

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A: Pass to the limit! (GJLL + others)

$$d_\infty(x,y) = \lim_{k\to\infty} \frac{d(\tilde{f}^k(x), \tilde{f}^k(y))}{\lambda^k} \quad x, y \in \tilde{\Gamma}$$

(T, d) is $(\tilde{\Gamma}, d_\infty)/\sim$ $\quad$ $x \sim y$ if $d_\infty(x,y) = 0$

Answering Paulin's Question

Let G be a (f.g.) word hyperbolic gp.

Thm (Bestvina-Feighn '95, Paulin '91)

If $Out(G)$ is infinite, G acts on a tree with virtually cyclic edge stabilizers
(so all periodic if not)

Thm (Bowditch '98) If G is one-ended and not cocompact Fuchsian, $Aut(G)$ acts on the JSJ tree for $G \Rightarrow$ all reducible

Thm (Paulin '97) Nilpotent subgps of $Out(G)$ preserve a real tree

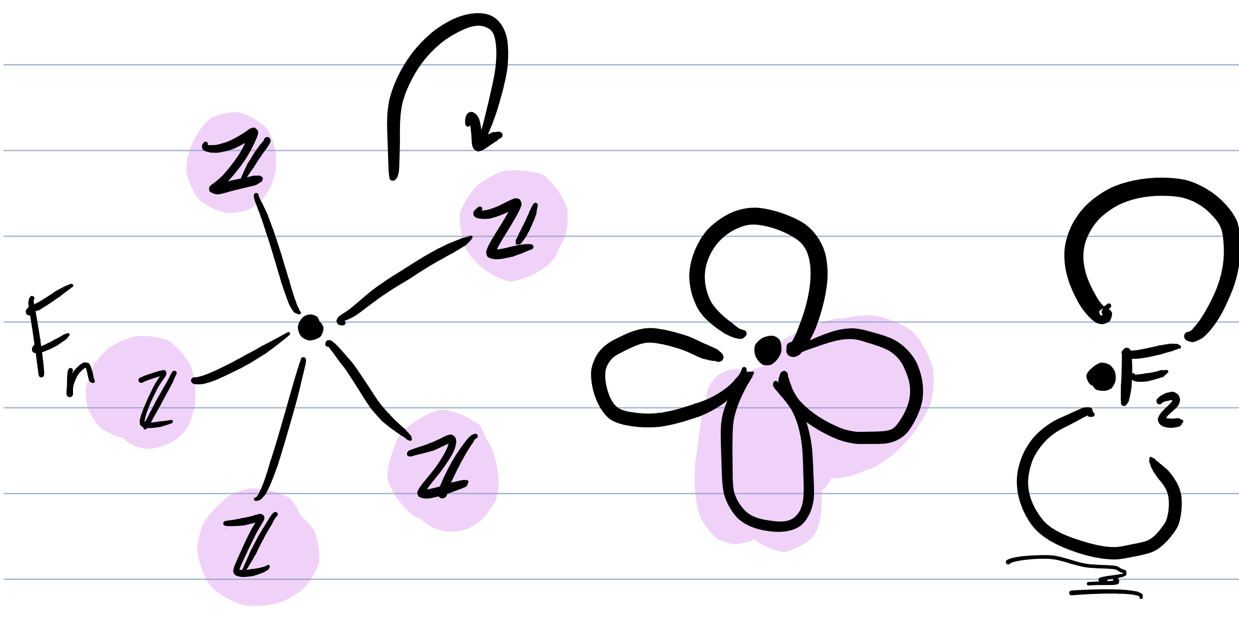
So the interesting case is ∞-ended.

Thm (Dunwoody '85) If G is finitely presented, $\exists (\Gamma, \mathcal{G})$, $\pi_1(\Gamma, \mathcal{G}) = G$ with finite edge groups and 1- or 2-ended vertex groups

The vertex groups are alg. distinguished*, so all $\varphi \in Out(G)$ can be realized. promote to a train track map.

- if $\lambda = 1$, periodic or reducible
- if $\lambda > 1$, get an $\mathbb{R}$-tree

$\square$



Ideas for next steps: Out(free product), Out(virtually free), Out(GBS), etc. etc. actual BS(2,4)

- Outer Space(s):
  - convexity of **KV**(virtually free) in Outer Space?
  - connectivity at infinity? duality gp?
  - connections to $\mathcal{M}_g^{trop}$?

- CTs Completely split relative train track maps

- (Non) linearity of Out(virtually free)?

- Dehn function of Out(free product) or Out(virtually free)?
  - Is hierarchical hyperbolicity ever possible?
  - Acylindrical hyperbolicity

- Coxeter groups?