Bass-Serre Theory: Groups acting on trees

Geometric Group Theory without Boundaries

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Ref: my thesis, Train Track maps on Graphs of Groups and Outer Automorphisms of Hyperbolic Groups, Ch. 1

Lecture 2 7/21/20

Today: - A taste of constructing the tree
- A helpful aside: graphs of spaces
- The fundamental group of a graph of groups
  - w/ examples
- Covering space theory, morphisms (maybe)

The Bass–Serre Tree: general nonsense

Recall

Thm: Let $(\Gamma, \mathcal{G})$ be a graph of groups, and $p \in \Gamma$. Then there exists a group $G = \pi_1(\Gamma, \mathcal{G}, p)$, a tree T and an action of G on T s.t. the quotient graph of groups is $(\Gamma, \mathcal{G})$.

Bass-Serre tree

Note: If we assume G and T exist, then the orbit-stabilizer theorem says that vertices $\tilde{v}$ in T w/ $\pi(\tilde{v}) = v$ ...

is in G-equivariant bijection w/ $G/\mathcal{G}_v$

$$\begin{tikzcd} T \arrow[d, "\pi"] \\ G\backslash T = \Gamma \end{tikzcd}$$

Similarly edges $\tilde{e}$ w/ $\pi(\tilde{e}) = e$

" " " w/ $G/\iota_e(\mathcal{G}_e)$

The Bass-Serre Tree: first observations

Fix $(\Gamma, \mathcal{G})$. Let v be a vertex of Γ.

Def. Write

$$st(v) = \{\text{oriented edges } e : \tau(e) = v\}$$

Ex.

$$st(v) = \{e, \bar{e}\}$$

If $G \curvearrowright T$ exists and $\tilde{v} \in T$ is a vertex in $\pi^{-1}(v)$

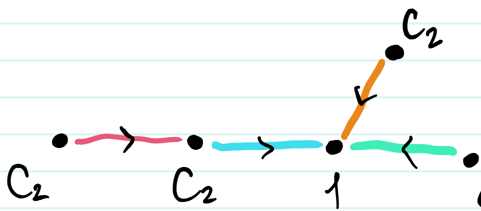
$$st(\tilde{v}) \cong \coprod_{e \in st(v)} \mathcal{G}_v / \iota_e(\mathcal{G}_e) \times \{e\}$$

$\mathcal{G}_v$-equivariant

The Bass-Serre Tree: construction using examples

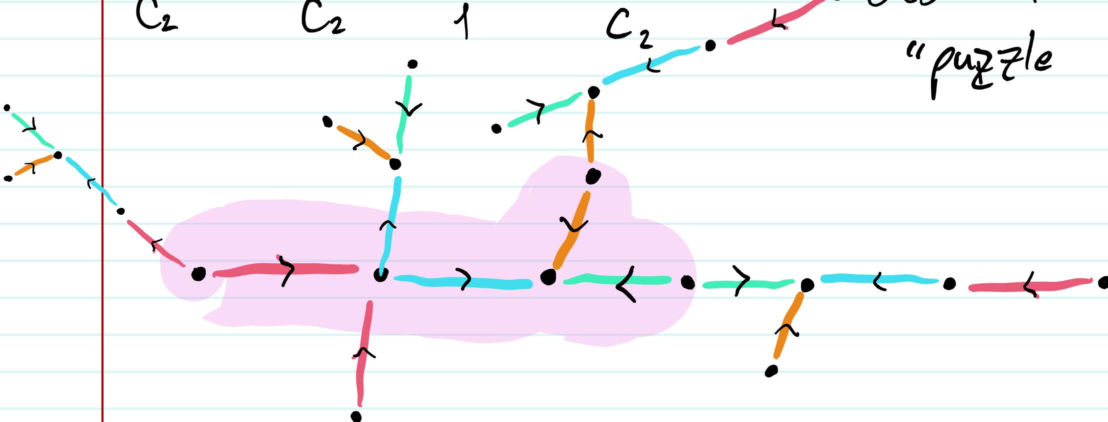
First case: Γ is a tree, $G_e = 1$ $\forall$ edges e.

Ex.



We will build T from copies of Γ so that the observation holds

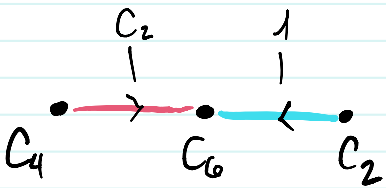
"puzzle pieces"

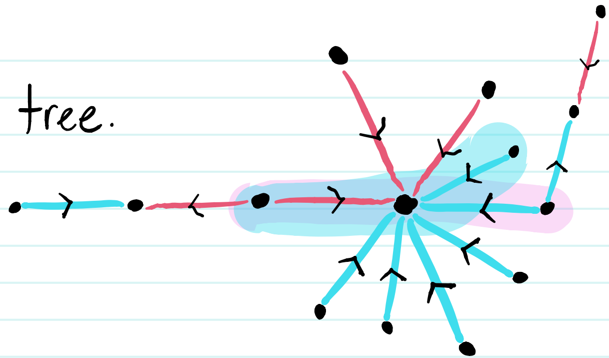


The Bass-Serre Tree

Second case: Γ is a tree.

Ex.



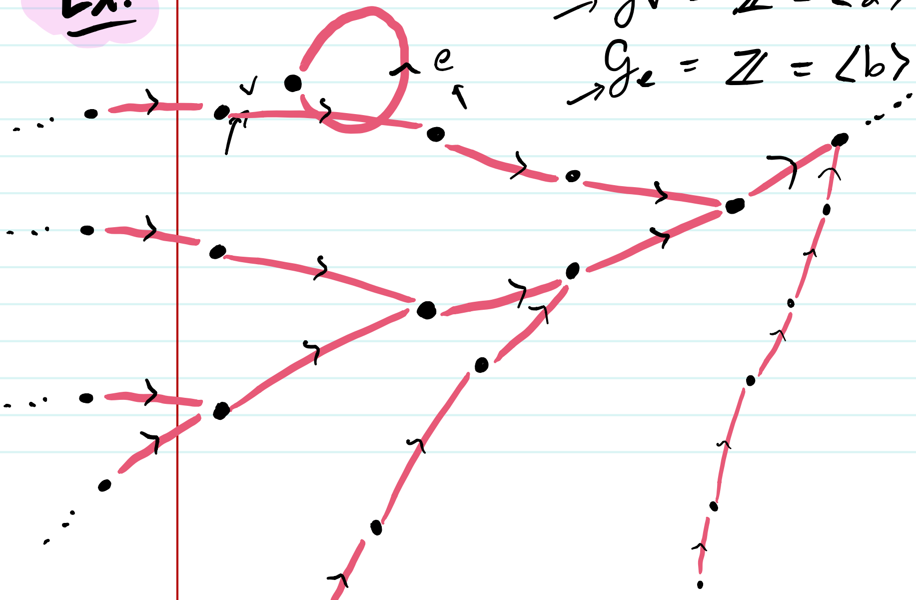


if you remove a vertex of a tree, you disconnect it.

The Bass-Serre Tree

General case: construct the (ordinary) universal cover $\tilde{\Gamma}$ of Γ, for $\tilde{v} \in \tilde{\Gamma}$, set $\tilde{\mathcal{G}}_{\tilde{v}} = \mathcal{G}_v$ etc. Now build the tree for $(\tilde{\Gamma}, \tilde{\mathcal{G}})$

Ex.



$\mathcal{G}_v = \mathbb{Z} = \langle a \rangle$

$\mathcal{G}_e = \mathbb{Z} = \langle b \rangle$

$\iota_e : b \longmapsto a^2$

$\iota_{\bar{e}} : b \longmapsto a$

$\pi_1(G, \mathcal{G}, v) \cong BS(1,2)$

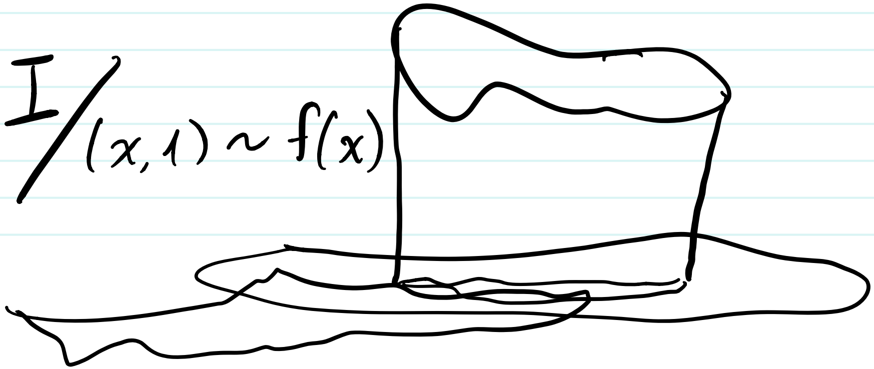
Scott-Wall Top. Methods in Gp Theory

A Helpful Aside: Graphs of spaces

Given a graph of groups $(\Gamma, \mathcal{G})$, choose, for each v, a space X_v w/ $\pi_1(X_v) \cong \mathcal{G}_v$, for each e, X_e w/ $\pi_1(X_e) \cong \mathcal{G}_e$ and maps $f_e, f_{\bar{e}} : X_e \longrightarrow X_{\tau(e)}, X_{\tau(\bar{e})}$ s.t. $f_{e*} : \mathcal{G}_e \longrightarrow \mathcal{G}_{\tau(e)}$ is ι_e.

Recall: Given $f : X \to Y$, the **mapping cylinder** of f is

$$Y \sqcup X \times I \Big/_{(x,1) \sim f(x)}$$



A Helpful Aside: Graphs of spaces

The **graph of spaces** $X = (\Gamma, \mathcal{G}, M_{f_e})$ is

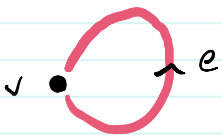
$$X = \coprod_{e \in E(\Gamma)} M_{f_e} \Big/ \sim$$

oriented $\rightarrow$ $e \in E(\Gamma)$

$\uparrow$ mapping cylinder of $f_e : X_e \to X_{\tau(e)}$

where each copy of X_v in M_{f_e} is identified together. Same with $X_e \times \{0\}$

Ex.

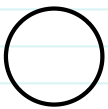


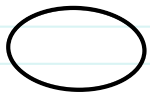
$\mathcal{G}_v = \mathbb{Z} = \langle a \rangle$ $\quad$ $\iota_e : b \longmapsto a^2$

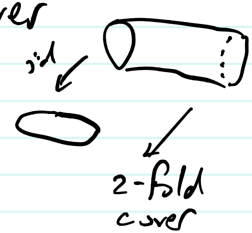
$\mathcal{G}_e = \mathbb{Z} = \langle b \rangle$ $\quad$ $\iota_{\bar{e}} : b \longmapsto a$

$X_v \simeq S^1$ $\quad$ $X_e \simeq S^1$ $\quad$ f_e : 2-fold cover

$f_{\bar{e}}$: id





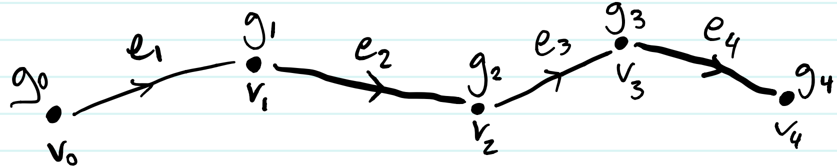


Useful: The universal cover $\tilde{X}$ maps to T. Graphs of aspherical spaces are aspherical.

$$g_1 \iota_{\bar{e}}(h) \xrightarrow{\;e\;} g_2 \;\cong\; g_1 \xrightarrow{\;e\;} \iota_e(h) g_2$$

The Fundamental Group

A **path** in $(\Gamma, \mathcal{G})$ is a sequence



$g_i \in \mathcal{G}_{v_i}$

$\tau(e_i) = \tau(\bar{e}_{i+1})$

If $\gamma = g_0 e_1 \cdots e_n g_n$, write $\bar{\gamma} = g_n^{-1} \bar{e}_n \cdots \bar{e}_1 g_0^{-1}$.

Def. Two paths γ and γ' are **homotopic** (rel endpoints) if γ can be transformed into γ' by a finite sequence of moves

1. add or subtract $\sigma\bar{\sigma}$ (σ a path)
2. if $h \in \mathcal{G}_e$, replace $e\,\iota_e(h)$ w/ $\iota_{\bar{e}}(h) e$

The Fundamental Group

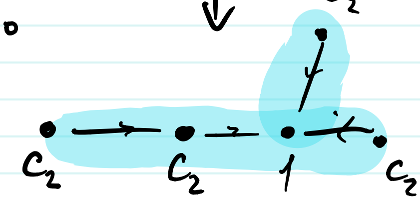
A path $\gamma = g_0 e_1 \cdots e_n g_n$ is a **loop** if $\tau(\bar{e}_1) = \tau(e_n)$.
it is **based** at $\tau(e_n)$.

Def. The **fundamental group** $\pi_1(\Gamma, \mathcal{G}, p)$ is the group of homotopy classes of loops based at p. The operation is concatenation.

$(\Gamma, \mathcal{G})$ is a **splitting** of $\pi_1(\Gamma, \mathcal{G}, p) = G$ if G is not equal to any $\mathcal{G}_v$.
G **splits** if it admits a splitting.

The Fundamental Group

$\pi_1 \cong C_2 * \cdots * C_2$
$\cong W_4$



Let $T_0 \subseteq \Gamma$ be a spanning tree $p \in T_0$

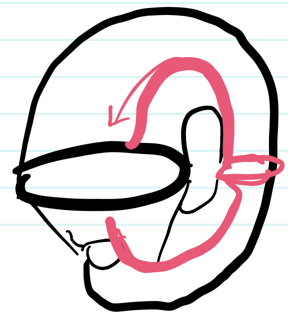
Thm $\pi_1(\Gamma, \mathcal{G}, p)$ is the quotient of

$$\mathop{\ast}_{v \in \Gamma} \mathcal{G}_v * F(E(\Gamma))$$

by the normal subgroup imposing the relations

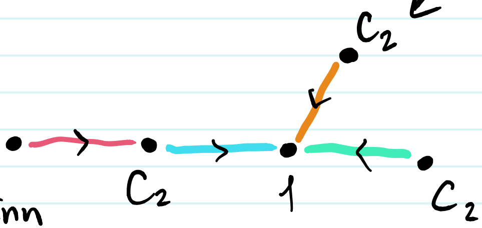
1. $\bar{e}^{-1} = \bar{e}$,
2. $e = 1$ if $e \in T_0$
3. $\iota_{\bar{e}}(h) e = e \iota_e(h)$
   $h \in \mathcal{G}_e$

Cor $\mathcal{G}_v \hookrightarrow \pi_1(\Gamma, \mathcal{G}, p)$



Examples:

π_1



$\overset{4}{\underset{i=1}{*}} C_2$

$\mathbb{Z} *_{\mathbb{Z}}$

$BS(m,n) = \langle a, e : e^{-1} a^m e = a^n \rangle$

$BS(1,1) \cong \mathbb{Z}^2$ $\quad BS(1,-1) =$ klein bottle

Higman Neumann Neumann

HNN extension

$G_v * G_e$

$v \bullet$ e

$G_v = \mathbb{Z} = \langle a \rangle$

$G_e = \mathbb{Z} = \langle b \rangle$

$\iota_e : b \longmapsto a^2$

$\iota_{\bar{e}} : b \longmapsto a$

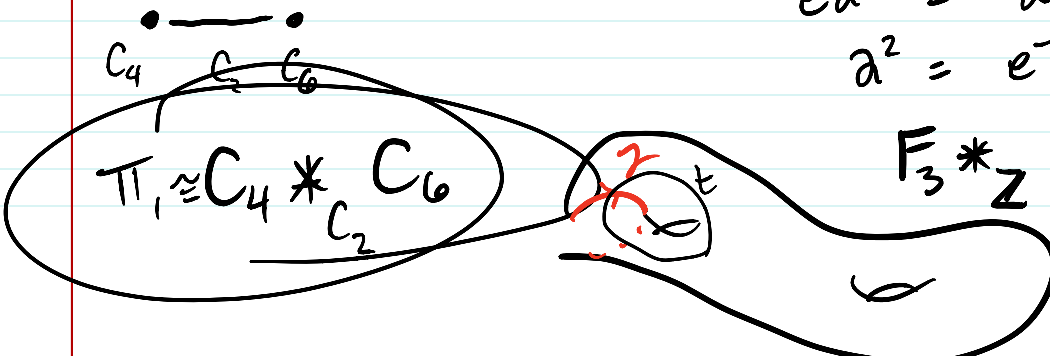
$\mathbb{Z} * F_2 / R \qquad e^{-1} = \bar{e}$

$e\,\iota_e(b) = \iota_{\bar{e}}(b)\,e$

$e a^2 = a e$

$a^2 = e^{-1} a e$

$SL_2(\mathbb{Z})$



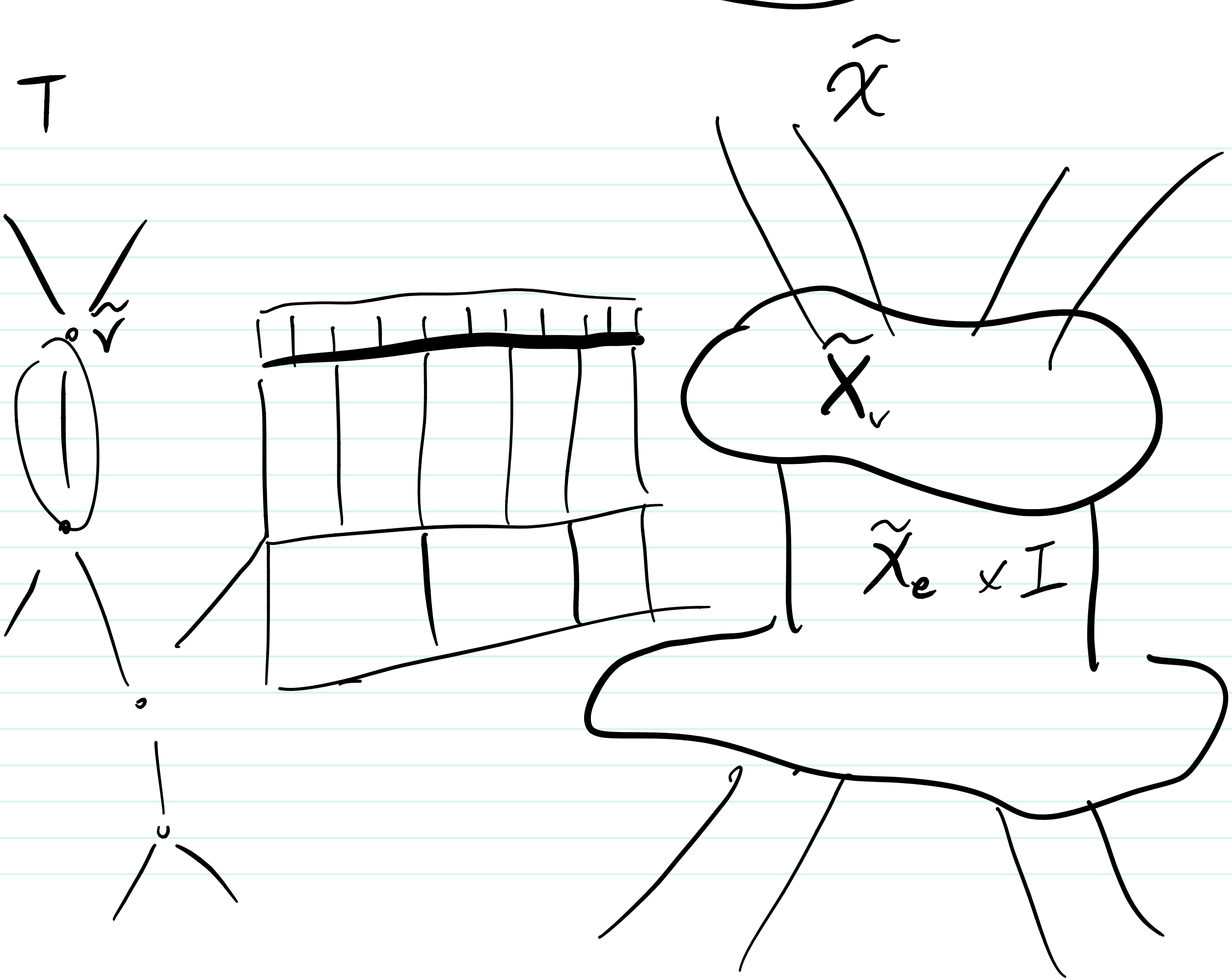
$\pi_1 \cong C_4 *_{C_2} C_6$

$F_3 * \mathbb{Z}$

$\mathbb{Z} \longmapsto \gamma$

$\mathbb{Z} \longmapsto \gamma^{-1}[\;,\;]$

$t \gamma t^{-1} = [\;,\;]^{-1}\gamma$


T
$\tilde{v}$
$\tilde{X}$
$\tilde{X}_v$
$\tilde{X}_e \times I$

A normal form

For each oriented edge e, choose a set S_e of representatives for $G_v / \iota_e(G_e)$ containing 1.

Each path γ is homotopic to a path of the form

$$g_0 e_1 \cdots e_n g_n$$

where $g_i \in S_{e_i}$ $1 \leq i \leq n$.

loops with nontrivial normal form are nontrivial in π_1

Morphisms *

Def. Since a graph of groups is a category Γ and a functor $\mathcal{G} : \Gamma \longrightarrow \text{Groups}^{\text{mono}}$, a **morphism** $f : (\Delta, \mathcal{L}) \longrightarrow (\Gamma, \mathcal{G})$ should be a **functor** $f : \Delta \longrightarrow \Gamma$ and a **morphism of functors** $f : \mathcal{L} \Longrightarrow \mathcal{G}f$.

It turns out the latter is a **pseudonatural transformation**

English: i.e. for each vertex, a homomorphism $f_v : \mathcal{L}_v \to \mathcal{G}_{f(v)}$ for each edge, a homomorphism $f_e : \mathcal{L}_e \longrightarrow \mathcal{G}_{f(e)}$ and an element $\delta_e \in \mathcal{G}_{\tau(f(e))}$ s.t. $f_v \iota_e(h) = \delta_e \iota_{f(e)} f_e(h) \delta_e^{-1}$

$$\begin{tikzcd}
\mathcal{L}_e \arrow[r, "f_e"] \arrow[d, "\iota_e"'] & \mathcal{G}_{f(e)} \arrow[d, "\mathrm{ad}(\delta_e) \cdot \iota_{f(e)}"] \\
\mathcal{L}_v \arrow[r, "f_v"'] & \mathcal{G}_{f(v)}
\end{tikzcd}$$

The Local Map

Given $f:(\Delta,\mathcal{L}) \longrightarrow (\Gamma,\mathcal{G})$, for $v\in\Gamma$, if f does not collapse edges, define

$$f_{st(v)}: \coprod_{e\in st(v)} \mathcal{L}_v / \iota_e(\mathcal{L}_e) \times \{e\} \longrightarrow \coprod_{e\in st(f(v))} \mathcal{G}_{f(v)} / \iota_e(\mathcal{G}_e) \times \{e\}$$

$$(g\,\iota_e(\mathcal{L}_e), e) \longmapsto (f_v(g)\,\delta_e\,\iota_{f(e)}(\mathcal{G}_{f(e)}), f(e))$$

Notice: $f_v(g\,\iota_e(h))\,\delta_e = f_v(g)\,f_v\,\iota_e(h)\,\delta_e$

$= f_v(g)\,\delta_e\,\iota_{f(e)}\,f_e(h)\,\delta_e^{-1}\,\delta_e$

$\in f_v(g)\,\delta_e\,\iota_{f(e)}(\mathcal{G}_{f(e)})$.

Coverings

A morphism $f:(\Delta, \mathcal{Z}) \longrightarrow (\Gamma, \mathcal{G})$ is an **immersion** if it does not collapse edges and each $f_{st(v)}$ is injective.
If $f:\Delta \longrightarrow \Gamma$ is surjective and each $f_{st(v)}$ is a bijection, then $f:(\Delta, \mathcal{Z}) \longrightarrow (\Gamma, \mathcal{G})$ is a **covering map**.

Ex.