Palindromic, Polynomially-Growing Free-by-Cyclic Groups are CAT(0)

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A Tale of Two Free-by-Cyclic Groups

Write $F_3\langle a, b, c\rangle$. Let φ and ψ in $\mathrm{Aut}(F_3)$ be the automorphisms

$$\varphi \begin{cases} a \mapsto a \\ b \mapsto ba \\ c \mapsto ca^2 \end{cases} \qquad \psi \begin{cases} a \mapsto a \\ b \mapsto aba \\ c \mapsto a^2ca^2 \end{cases}$$

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$$G_\varphi = F_3 \rtimes_\varphi \mathbb{Z} = \langle F_3, t \mid txt^{-1} = \varphi(x),\ x \in F_3\rangle$$
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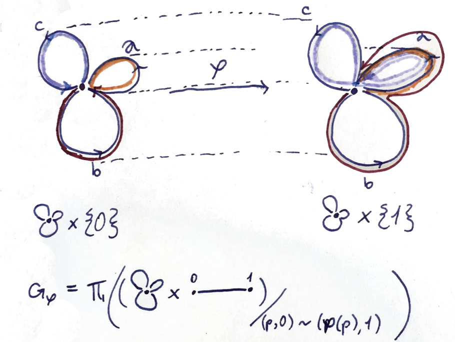
Free-by-Cyclic Groups as Mapping Tori



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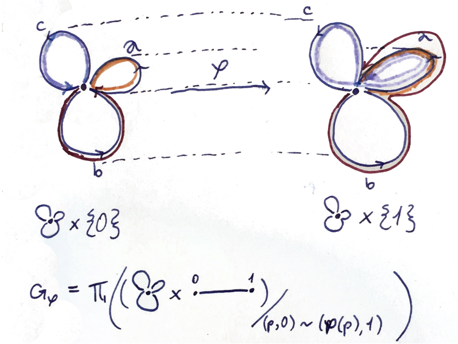

Although φ and ψ appear similar, G_φ and G_ψ turn out to have very different geometry!

Motivation

Theorem (Gersten, '94)

$\mathrm{Aut}(F_n)$, $\mathrm{Out}(F_{n+1})$ *are not* CAT(0) *groups for* $n \geq 3$

Key observation: G_φ cannot be a subgroup of a CAT(0) group, but $G_\varphi \leq \mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_{n+1})$ for $n \geq 3$.

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$\mathrm{Aut}(W_n)$, $\mathrm{Out}(W_{n+1})$ *are not* CAT(0) *groups for* $n \geq 4$.

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Actually, this is still wide open!

Results

Theorem (L, '19)

We fail! G_ψ *is a* CAT(0) *group. More generally, any polynomially-growing* $\theta \in \mathrm{Aut}(W_n)$ *has a power* $k < n!$ *such that* G_{θ^k} *is a* CAT(0) *group.*

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The k is needed to ensure that θ^k is in the kernel of the map $\mathrm{Aut}(W_n) \to S_n$. This restriction isn't good enough for Gersten's theorem.

Background

What is a W_n, anyway?

Let $W_n = \underbrace{\mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}}_{n \text{ times}} = \langle a_1, \cdots, a_n \mid a_i^2 \rangle$.

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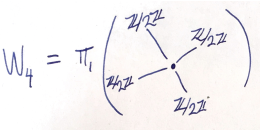
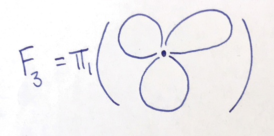
$W_n \cong F_{n-1} \rtimes \mathbb{Z}/2\mathbb{Z}$. We get $\mathrm{Aut}(W_n) \hookrightarrow \mathrm{Aut}(F_{n-1}$, and $\mathrm{Out}(W_n) \hookrightarrow \mathrm{Out}(F_{n-1})$.





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Facts about $\mathrm{Out}(F_n)$

Given $\Phi \in \mathrm{Out}(F_n)$ and $[g]$ a conjugacy class in F_n, the word length of the shortest representative of $\Phi^k([g])$ grows either polynomially or exponentially.

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Φ is *exponentially-growing* if at least one conjugacy class grows exponentially, and is *polynomially-growing* otherwise.

Examples: φ and ψ are polynomially (in fact linearly) growing.

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Pseudo-Anosov homeomorphisms of surfaces with nonempty boundary induce exponentially-growing $\Phi \in \mathrm{Out}(F_n)$.

(Up to finite index) elements in the image $\mathrm{Out}(W_n) \hookrightarrow \mathrm{Out}(F_{n-1})$ are *palindromic*—they send generators to palindromes.

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Our theorem is one of the first signs of a difference in structure between $\Phi \in \mathrm{Out}(W_n)$ and those in $\mathrm{Out}(F_n)$.

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Polynomially-growing $F_n \rtimes \mathbb{Z}$ admit a *hierarchy*, with all edge groups $\mathbb{Z}$ and vertex groups either $F_m \rtimes \mathbb{Z}$ with $m < n$, or (at the base) $\mathbb{Z}^2$.

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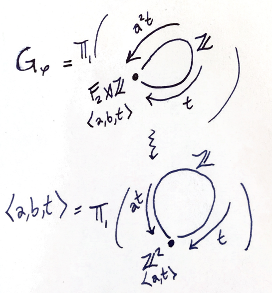
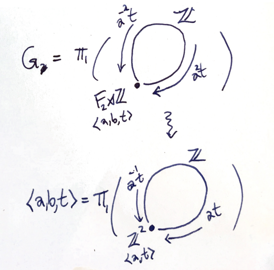
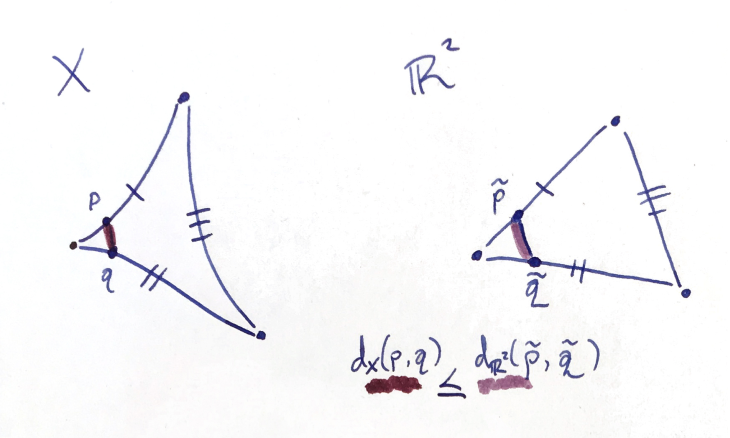




CAT(0) Geometry

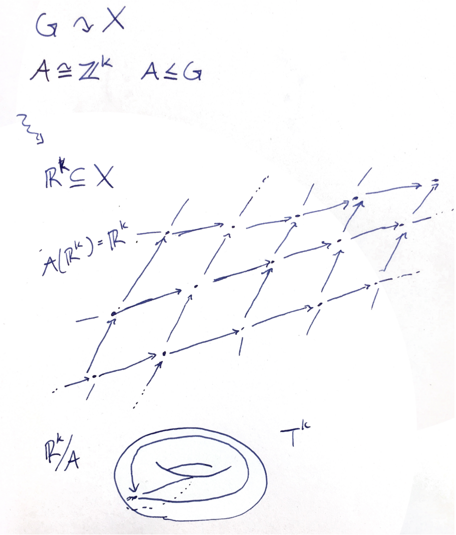


Theorem (Flat Torus Theorem; Bridson–Haefliger '99)

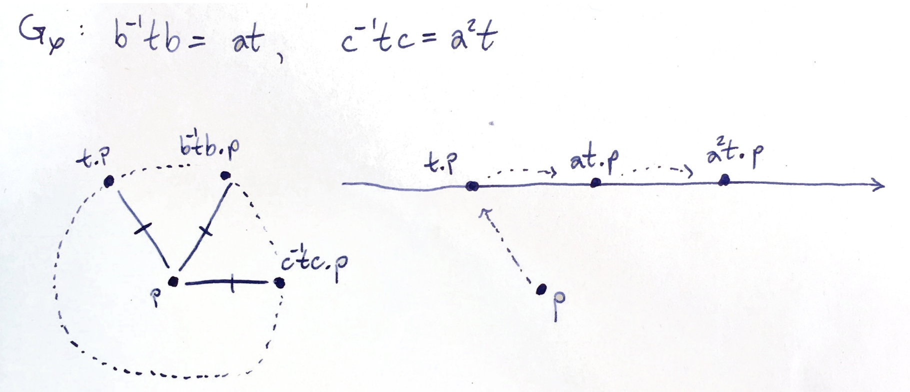
Given a "nice" action of G on a CAT(0) *space X and $\mathbb{Z}^k \cong A \leq G$, there is an isometric copy of $\mathbb{R}^k \subset X$ that A acts on by translation, and the quotient $\mathbb{R}^k/A$ is a k-torus.*

CAT(0) Geometry

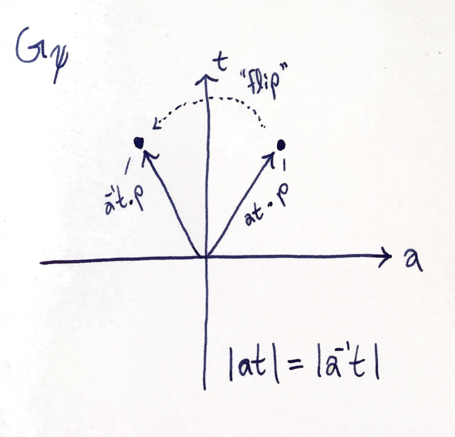
The Flat Torus Theorem



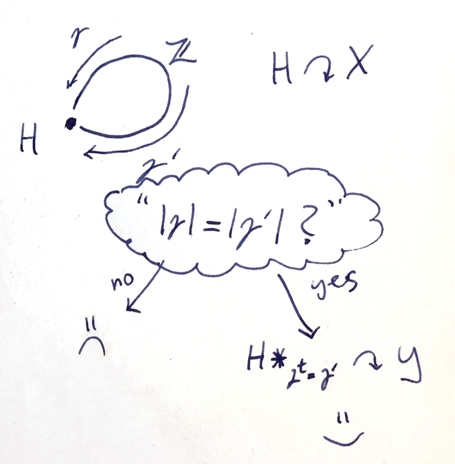
Gersten: G_φ is not CAT(0)



G_ψ fails to fail to be CAT(0)



Blowing up the Bass–Serre Tree



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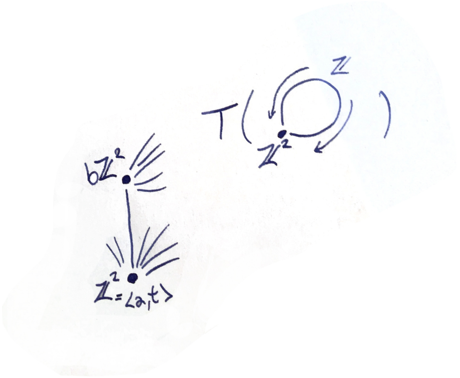

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