Bass-Serre Theory: Groups acting on trees

Geometric Group Theory without Boundaries

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Lecture 3 7/22/20

Today: - A normal form for paths in $(\Gamma, \mathcal{G})$

- vocab (FA)

Applications of Bass Serre Theory

Normal Forms

if $g_0 = s_0 \iota_{\bar{e}_1}(h)$ in

$s_0 \in S_{\bar{e}_1}$

$g_0 e_1 g_1$

$s_0 e_1 \iota_{e_1}(h) g_1$

Let $(\Gamma, \mathcal{G})$ be a graph of groups.
For each **oriented** edge e, choose a set S_e of coset representatives for $\mathcal{G}_{\tau(e)} / \iota_e(\mathcal{G}_e)$ satisfying $1 \in S_e$.

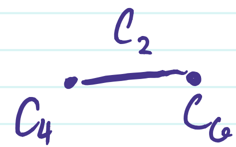
A path $g_0 e_1 g_1 \cdots e_n g_n$ is in **normal form** if each $g_i \in S_e$ for $0 \le i < n$.
Every path is homotopic rel endpoints to a path in normal form.

homotopically nontrivial loops have nontrivial normal form?

The Kernel

Prop. If G acts on T without inversion with quotient graph of groups $(\Gamma, \mathcal{G})$, the kernel of the action is the largest normal subgroup of G contained in

$$\bigcap_{e \in E(\Gamma)} \iota_e(\mathcal{G}_e)$$



Ex. C_2 is the kernel of $SL_2(\mathbb{Z}) \curvearrowright$ Farey tree

Vocab

Recall: G **splits** if $\exists\ (\Gamma, \mathcal{G})$ w/ $\pi_1(\Gamma, \mathcal{G}, p) \cong G$
s.t. for every vertex v, $\mathcal{G}_v \neq G$.
equivalently, G splits if it acts on a tree T
without global fixed point.

Def. A group G has **Serre's Property FA** (fixe arbre)
if it does not split.

Ex. $SL_n(\mathbb{Z})$ $n \geq 3$, most triangle groups, $\mathrm{Mod}(S_g)$ $g \geq 3$ (2?)

Property FA passes to quotients but not to subgroups.

Thm (Watani '81) If G has Kazhdan's Property (T),
then G and its finite-index subgroups have Property FA.

$\exists\ \pi_1(\Gamma, \mathcal{G}, p) \twoheadrightarrow \pi_1(\Gamma, p)$

Vocab

Thm (Serre) G having Property FA is equivalent to G satisfying the following three conditions

$(\Gamma, \mathcal{G})$ covers the case where Γ is a tree

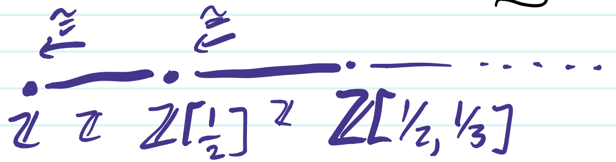
i) if $G \cong A *_C B$, then $G \cong A$ and $C = B$
or $G \cong B$ and $A = C$. not an amalgam

ii) $\mathbb{Z}$ is not a quotient of G. covers the case where Γ has a loop

iii) if $G = \bigcup_{i=1}^{\infty} G_i$ for $1 \leq G_1 \leq G_2 \leq \dots$,
then $G = G_n$ for some n.

(if G is countable, (iii) is equivalent to G being finitely generated.)

N.B. $\mathbb{Q}$ splits.



$PB_n \twoheadrightarrow PB_{n-1} \to PB_2 \cong \mathbb{Z}$

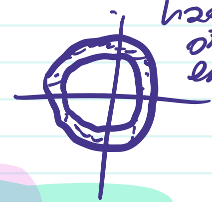
$$\begin{tikzcd} \mathbb{Z} \arrow[r] & \underset{\cong ?}{G} \arrow[r] & \text{Finite} \arrow[ll, bend left] \end{tikzcd} \qquad \operatorname{Out}(\mathbb{Z}) \cong C_2$$

Applications

Let X be a space. The **space of ends** of X is

$$\operatorname{Ends}(X) = \varprojlim \pi_0(X - K)$$

as K ranges over the compact subsets of X.



$\mathbb{R}^2$ has one end

If G is finitely generated (by S), set

$$\operatorname{Ends}(G) = \operatorname{Ends}(\Gamma(G,S)).$$

$D_\infty \times N$

$\geq_{\text{index } 2} \mathbb{Z}$

Thm (Freudenthal, Hopf '40s) If G is finitely generated,

$$\#(\operatorname{Ends}(G)) \in \{0, 1, 2, \infty\}.$$

$\#(\operatorname{Ends}(G)) = 0 \iff G$ is finite

Ex. $\mathbb{Z}^2$ has 1 end

$\#(\operatorname{Ends}(G)) = 2 \iff G$ is virtually $\mathbb{Z}$

Thm (Wall '67) G is virtually $\mathbb{Z} \iff \exists N$ finite $N \triangleleft G$, $G/N \cong \mathbb{Z}$ or $G/N \cong D_\infty$.

(generalizes to Euclidean groups)

Applications

Thm (Stallings '71) If G is finitely generated and $\mathrm{Ends}(G) = \infty$, then G **splits** as π_1 of a graph of groups with finite edge groups.

if G is torsion-free then G is a free product

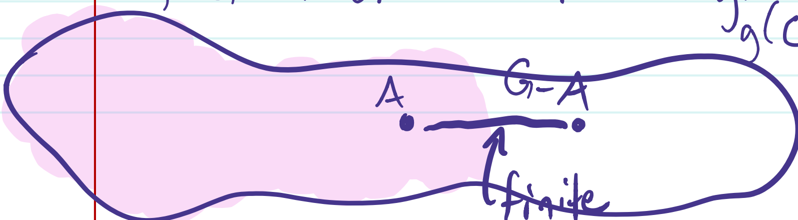
$x \Delta y := x \cup y - x \cap y$

"Idea" of proof:

"almost invariant subset $A \leq G$"

$gA \Delta A$ is finite

A, $G - A$ are infinite

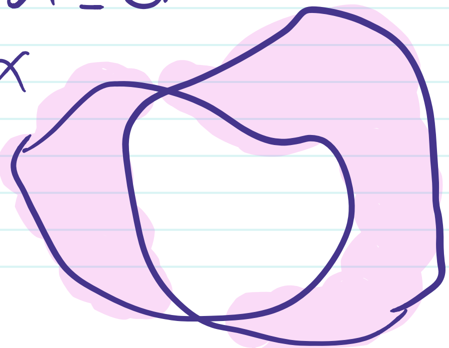


y

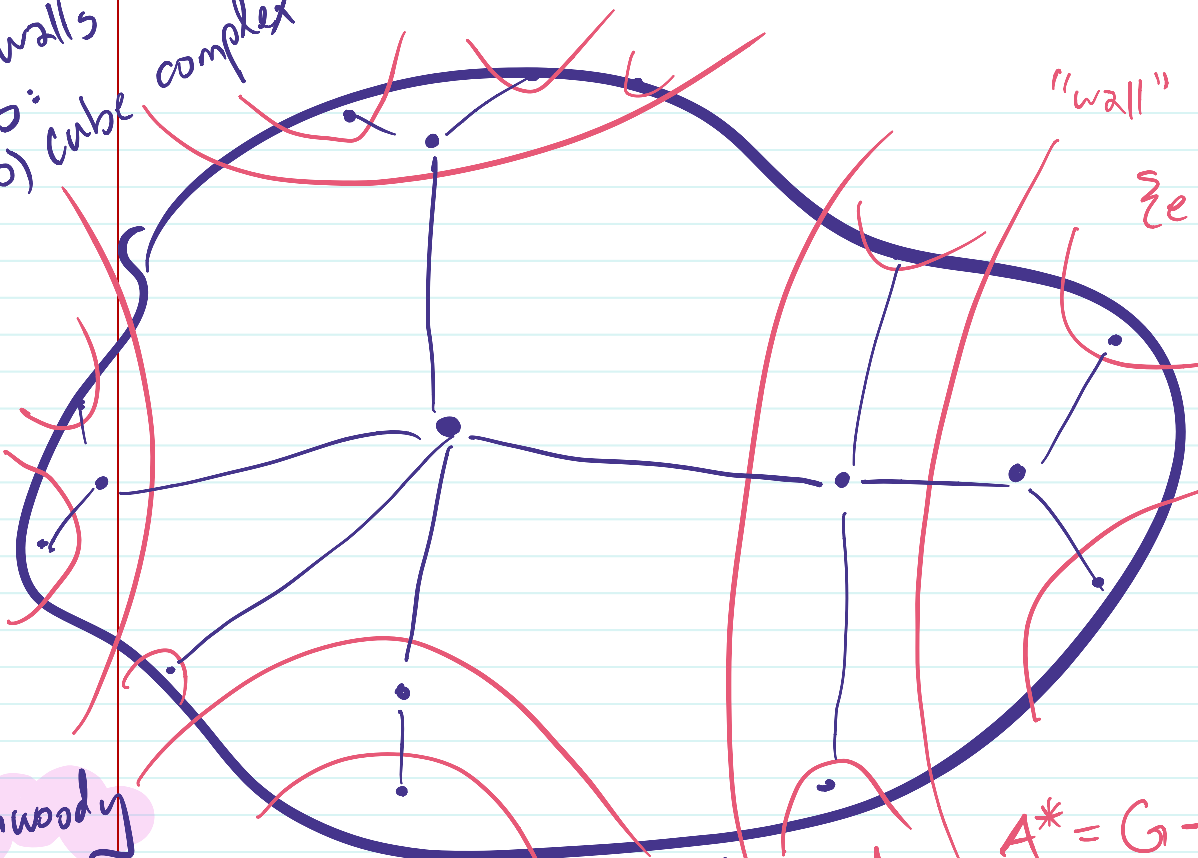
x

gA

$g(G-A)$



if walls cross: CAT(0) cube complex



$\Gamma(G,S)$

$\{e : \tau(e) \in A,\ \tau(\bar{e}) \in A^*\}$ is finite

$\Rightarrow$ stab "wall" is finite

$A^* = G - A$

Dunwoody

$\exists$ f.g. gps which can be split over finite groups forever

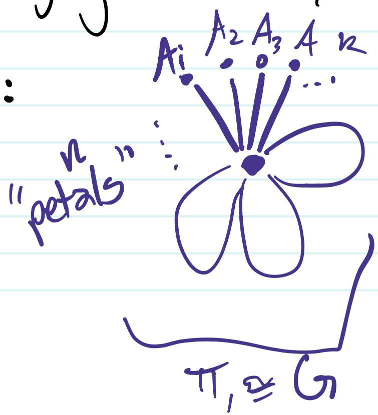
Applications

Thm (Grushko '40) Suppose $\overset{\text{f.g.}}{G} \cong \boxed{A_1 * \cdots * A_k * F_n}$, where F_n is free, and each A_i is **freely indecomposable** and not infinite cyclic.

does not act on a tree w/ trivial edge stabs.

If also $G \cong B_1 * \cdots * B_\ell * F_m$ (analogously), then $k=\ell$, $m=n$, and after reordering, B_i is conjugate to A_i.

"Idea":



$G \curvearrowright T$

$B_i \cong$ stab vertices

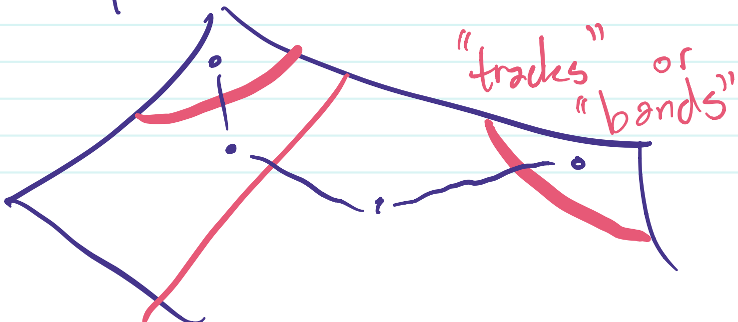
Applications

A group G is **accessible** if there is a splitting of G with finite edge groups and one-ended vertex groups.

f.g. of $G \Longrightarrow$ splitting $(\Gamma, \mathcal{G})$ is finite

Thm (Dunwoody '85)
Finitely presented groups are accessible.

"Idea": use a presentation complex for G
a compact 2-dim'l CW-cx with $\pi_1 \cong G$



every S^2 bounds a B^3

Thm (Jaco-Shalen, Johannson '79) JSJ decomposition

If M^3 is prime and orientable, there exists a finite collection of incompressible tori s.t. the complementary pieces are **atoroidal** or **seifert-fibered**

lots of T^2,

Every T^2 is ∂-parallel

In terms of trees:

$\exists$ a splitting of $\pi_1(M^3)$ where edge groups are $\mathbb{Z}^2$

"canonical"

JSJ Decompositions

Let $\mathcal{A}$ be a class of groups closed under taking subgroups.

Def. An $\mathcal{A}$-tree G is $(\Gamma, \mathcal{G})$ w/ $\pi_1 \cong G$ and edge groups in $\mathcal{A}$. (or the Bass-Serre tree T.)

Def. (Guirardel-Levitt) An $\mathcal{A}$-tree T is a **JSJ $\mathcal{A}$-tree** if

i) edge-stabilizers in T fix points in every $\mathcal{A}$-tree.

ii) if T' is an $\mathcal{A}$-tree, $\exists$ a G-equivariant map $T \longrightarrow T'$. T "refines" the splitting def. by T'

Thm. (Guirardel-Levitt '17) **JSJ $\mathcal{A}$-trees** exist for finitely-presented groups, but may not be canonical.

"deformation space"

← Haglund–Wise
Kahn–Markovič
Kleinian gps,

JSJ Decompositions

Thm (Bowditch '09) If $\mathcal{A}$ is the class of virtually cyclic groups and G is word-hyperbolic, there is a canonical JSJ $\mathcal{A}$-tree for G.

cube complexes

"Idea:" $\mathbb{Z} \hookrightarrow G$ are q.convex

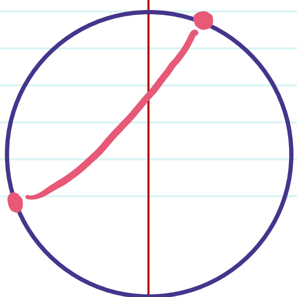
$\partial\mathbb{Z} \longrightarrow \partial G$

2 pts

if $G \cong \pi_1(M^3)$ hyp., its JSJ is a point

G splits over $\mathbb{Z}$

$\Longleftrightarrow$ $\partial\mathbb{Z}$ are a cutpair in ∂G



The circle has "too many" splitting

Applications

Thm (Karrass–Pietrowski–Solitar '72)
A finitely generated group G is virtually free $\iff$ G acts on a tree T with finite stabilizers and finite quotient.

"Idea":

Thm (Culler '84) Suppose $G < \mathrm{Out}(F_n)$ is finite.
Then G acts on a finite graph with $\pi_1 \cong F_n$.

Pf

$$\begin{tikzcd}
F_n \arrow[r] & \hat{G} \arrow[r] & G \\
\mathrm{inn}(F_n) \arrow[u] & \rho^{-1}(G) \in \mathrm{Aut}(F_n) \arrow[u] &
\end{tikzcd}$$

$\exists T$ for $\hat{G}$, and

$G \curvearrowright F_n \backslash T$

Applications Ch. 11

Thm (Bridson-Haefliger) Let $(\Gamma, \mathcal{G})$ be a graph of groups.
Suppose $X = (\Gamma, \mathcal{G}, X_v, X_e, f_e : X_e \longrightarrow X_{\tau(e)})$
is a graph of spaces such that

1) $\tilde{X}_e, \tilde{X}_v$ are complete, CAT(0) metric spaces w/ proper (and cocompact) isometric $\mathcal{G}_e, \mathcal{G}_v$ actions
2) each $\hat{f}_e : \hat{X}_e \longrightarrow \tilde{X}_v$ is an isometric embedding
3) The image of each $\tilde{f}_e$ is convex

Then $\tilde{X}$ is a complete CAT(0) metric space with a proper (and cocompact) isometric $\pi_1(\Gamma, \mathcal{G}, p)$ action.

Applications

I used the previous theorem to show

Thm (L '19) Let $\Phi : F_n \longrightarrow F_n$ be an automorphism.
If $\exists$ a free basis $x_1, \dots, x_n$ s.t.
Φ sends the x_i to **palindromes** in the x_i,
then $F_n \rtimes_\Phi \mathbb{Z}$ is virtually a CAT(0) group.

e.g.

$$\Phi \begin{cases} a \longmapsto a \\ b \longmapsto aba \\ c \longmapsto a^2 c a^2 \end{cases}$$

N.B. Gersten showed that $F_3 \rtimes_\Psi \mathbb{Z}$ is not a subgroup of a CAT(0) group, where

$$\Psi \begin{cases} a \longmapsto a \\ b \longmapsto ba \\ c \longmapsto ca^2 \end{cases}$$

Applications

Thm (Bestvina–Feighn '92) Let $(\Gamma, \mathcal{G})$ be a graph of hyperbolic groups and $X = (\Gamma, \mathcal{G}, X_v, X_e, f_e : X_e \longrightarrow X_{\tau(e)})$ a corresponding graph of spaces. Then if

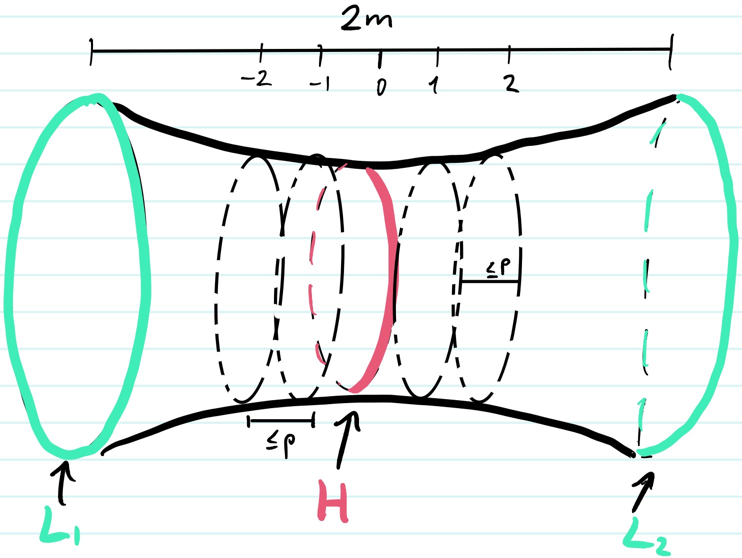
i) $\tilde{f}_e : \tilde{X}_e \longrightarrow \tilde{X}_v$ is a q.i. embedding

ii) X satisfies the **annuli flare** condition,

Then $\pi_1(\Gamma, \mathcal{G}, p)$ is word hyperbolic

$\Longleftrightarrow$ it contains no $BS(n,m)$ subgroup.

The **annuli flare** condition



$\exists\, \lambda > 1,\ m \geq 1$ s.t.

$\forall \rho\ \exists\, H = H(\rho)$ s.t.

if as above, then

$$\lambda H \leq \max\{L_1, L_2\}$$