A family of CAT(0) outer automorphism groups of free products

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Abstract

Consider the free product of two nontrivial finite groups with an infinite cyclic group. We prove that the 2-dimensional spine of Outer Space for this free product supports an equivariant CAT(0) metric with infinitely many ends. The outer automorphism group of this free product is thus relatively hyperbolic. In the special case that both finite groups are cyclic of order two, we show that the outer automorphism is virtually a certain Coxeter group, and that the spine of Outer Space may be identified with its Davis–Moussong complex. These outer automorphism groups thus exhibit behavior extremely different from outer automorphism groups of free groups, and conjecturally, from other outer automorphism groups of free products of finite and cyclic groups.

Let A and B be finite groups and consider the free product $F = A * B * \mathbb{Z}$. The main result of this paper is the following theorem.

Theorem A. *The group* $\mathrm{Out}(F)$ *is CAT(0) and has infinitely many ends.*

The group $\mathrm{Out}(F)$ acts on a 2-dimensional simplicial complex called the *spine of reduced Outer Space* $L = L(F)$. We prove Theorem A by showing that L supports an $\mathrm{Out}(F)$-equivariant CAT(0) metric and exhibiting a compact subset of L that separates it into more than one component with noncompact closure.

We were surprised to find Theorem A while attempting to prove that for G a free product of finite and cyclic groups, the group $\mathrm{Out}(G)$ has one end whenever $\dim(L(G)) \geq 2$. Indeed, for the free group F_n of rank n, Vogtmann [Vog95] proved that whenever $\dim(L(F_n)) \geq 2$ (i.e. $n \geq 3$), the group $\mathrm{Out}(F_n)$ is one-ended. Das [Das18] in his thesis showed that for G the free product of n finite groups, the group $\mathrm{Out}(G)$ is *thick* in the sense of Behrstock–Druțu–Mosher [BDM09] and hence one-ended whenever $\dim(L(G)) \geq 2$ (i.e. $n \geq 4$). Since thick groups (which also include $\mathrm{Out}(F_n)$ for $n \geq 3$) are not nontrivially relatively hyperbolic, while groups with infinitely many ends are hyperbolic relative to the vertex groups of some nontrivial splitting over finite groups, Theorem A is again a surprise.

Theorem A may be compared to Bridson's [Bri91] and Cunningham's [Cun15] theses, where it is shown that the spine of Outer Space for F_n when $n \geq 3$ and McCullough–Miller Space for the free product of n copies of C_2 when $n \geq 4$ do not admit equivariant CAT(0) metrics, respectively. Indeed, we know that $\mathrm{Out}(F_n)$ is not a CAT(0) group when $n \geq 3$, and although the corresponding question for the free product of n copies of C_2 is still open, we suspect that the spine of Outer Space for the free product of four finite groups does not admit an equivariant CAT(0) metric, and that higher-complexity free products fail to be CAT(0) for similar reasons to $\mathrm{Out}(F_n)$.

The simplest example of Theorem A is when the groups A and B are cyclic of order two. In the CAT(0) metric produced in the proof of Theorem A, we observed that the complex $L(F)$ bears a striking resemblance to the Davis–Moussong complex of the Coxeter group

$$W = \langle x, y, z, w : x^2 = y^2 = z^2 = w^2 = (xy)^2 = (xz)^4 = (yz)^4 = (xw)^4 = (yw)^4 = 1 \rangle.$$

Letting $F = \langle a, b, t : a^2 = b^2 = 1\rangle$, the group $\mathrm{Out}(F)$ is generated by (the outer classes of) the following automorphisms defined by their action on the basis $\{a, b, t\}$

$$\sigma\begin{cases} a \mapsto b \\ b \mapsto a \\ t \mapsto t \end{cases} \quad \tau\begin{cases} a \mapsto a \\ b \mapsto b \\ t \mapsto t^{-1} \end{cases} \quad L_a\begin{cases} a \mapsto a \\ b \mapsto b \\ t \mapsto at \end{cases} \quad R_b\begin{cases} a \mapsto a \\ b \mapsto b \\ t \mapsto tb \end{cases} \quad \chi_t^b\begin{cases} a \mapsto a \\ b \mapsto t^{-1}bt \\ t \mapsto t. \end{cases}$$

(A reader familiar with, for example, Gilbert's [Gil87] finite presentation of $\mathrm{Aut}(F)$ might find it enjoyable to verify this claim.) It is an easy exercise to verify that the the map

$$\Phi\begin{cases} x \mapsto L_a \\ y \mapsto R_b \\ z \mapsto \tau \\ w \mapsto \tau(\chi_t^b)^2 \end{cases}$$

extends to a well-defined homomorphism $\Phi\colon W \to \mathrm{Out}(F)$. It is also not hard to show that the conjugation action of the generators of $\mathrm{Out}(F)$ on the image of the generators of $\mathrm{Im}(\Phi)$ yields elements of $\mathrm{Im}(\Phi)$, so $\mathrm{Im}(\Phi)$ is a normal subgroup of $\mathrm{Out}(F)$. Since τ, L_a and R_b are contained in $\mathrm{Im}(\Phi)$, the quotient $\mathrm{Out}(F)/\mathrm{Im}(\Phi)$ is generated by σ and χ_t^b, which have order at most 2 in the quotient. Since $\sigma\chi_t^b\sigma\chi_t^b$ is inner, the index of $\mathrm{Im}(W)$ in $\mathrm{Out}(F)$ is at most 4. In fact, using the geometry of L, we prove the following.

Theorem B. *With notation as above, the homomorphism $\Phi\colon W \to \mathrm{Out}(F)$ is injective and has image an index-4 subgroup of $\mathrm{Out}(F)$. Moreover, the Davis–Moussong complex for W may be identified with L.*

Since the Davis–Moussong complex for W is a 2-dimensional CAT(0) complex with infinitely many ends, Theorem B implies a special case of Theorem A. In fact, we find it instructive to consider the simpler case of Theorem B first in Section 2, turning to the proof of Theorem A in Section 3.

1 The Complex

We work in $L = L(F)$, the *spine of reduced Outer Space* for $F = A * B * \mathbb{Z}$. This is a simplicial complex in which a vertex corresponds to an action of F on a tree with finite stabilizers (and in fact trivial edge stabilizers). The purpose of this section is to introduce our perspective on this complex and build up tools for working with it. Although some of what we will say is tailored to the case of $F = A * B * \mathbb{Z}$, much of it applies with little or no change to F a free product or virtually free group.

Marked graphs of groups. In this paper, rather than working with tree actions, it will be more convenient to work in the quotient graph of groups $\mathcal{G} = F\backslash\backslash T$. We will assume a certain amount of familiarity with graphs of groups in this paper. The reader is referred to [Ser03], [Bas93], [SW79] and [Lym22b, Section 1] for more background on graphs of groups. We will follow the notation in [Lym22b]. In that paper, we defined a notion of a *map* of graphs of groups and of *homotopy* of maps. A map $f\colon \mathcal{G} \to \mathcal{G}'$ is a *homotopy equivalence* when there exists a *homotopy inverse;* a map $g\colon \mathcal{G}' \to \mathcal{G}$ such that fg and gf are homotopic to the respective identity maps.

A map $f\colon \mathcal{G} \to \mathcal{G}'$ induces, in particular, a continuous map of underlying graphs. We will say that the map f is a *collapse map* if it sends vertices to vertices and edges either to edges or collapses edges to vertices. A collapse map that does not collapse edges to vertices will be called a *morphism in the sense of Bass.* A morphism in the sense of Bass $f\colon \mathcal{G} \to \mathcal{G}'$ is thus determined by a collection of graph of groups edge paths $f(e) = g_0'e'g_1'$ in $\mathcal{G}'$ and homomorphisms $f_v\colon \mathcal{G}_v \to \mathcal{G}'_{f(v)}$ and $f_{e,e'}\colon \mathcal{G}_e \to \mathcal{G}'_{e'}$ of vertex and edge groups subject to certain compatibility conditions described in [Lym22b, p. 5]. If the map of underlying graphs is an isomorphism and each f_v and $f_{e,e'}$ are isomorphisms, then we say that $f\colon \mathcal{G} \to \mathcal{G}'$ is an *isomorphism.*

We will fix a finite graph of finite groups $\mathbb{G}$, a basepoint $\star \in \mathbb{G}$ and an identification $F \cong \pi_1(\mathbb{G}, \star)$. A *marked graph of groups* $\tau = (\mathcal{G}, \sigma)$ is a graph of groups $\mathcal{G}$ equipped with a homotopy equivalence $\sigma\colon \mathbb{G} \to \mathcal{G}$ called the *marking.* Two marked graphs of groups $(\mathcal{G}, \sigma)$ and $(\mathcal{G}', \sigma')$ are *equivalent* if there is an isomorphism $h\colon \mathcal{G} \to \mathcal{G}'$ such that the following diagram commutes up to homotopy

$$\begin{tikzcd} & \mathcal{G} \arrow[dd, "h"] \\ \mathbb{G} \arrow[ur, "\sigma"] \arrow[dr, "\sigma'"'] & \\ & \mathcal{G}'. \end{tikzcd}$$

If the map h is a collapse map but not an isomorphism, then we say that $(\mathcal{G}, \sigma)$ *collapses onto* $(\mathcal{G}', \sigma')$. The set of equivalence classes of marked graphs of groups is partially ordered, where $\tau' \prec \tau$ if the marked graph of groups τ collapses onto τ'. This partial order has minimal elements; these are *reduced* marked graphs of groups. An edge e of τ is *surviving* if there is a reduced marked graph of groups $\tau' \prec \tau$ in which the edge e is not collapsed.

The complex L. The complex $L = L(F)$ is the geometric realization of the subposet of the full poset of equivalence classes of marked graphs of groups that is spanned by those marked graphs of groups $(\mathcal{G}, \sigma)$ with the property that every edge of $\mathcal{G}$ is surviving. The group $\mathrm{Out}(F)$ acts on $L(F)$ on the right by precomposing the marking: the group of homotopy classes of homotopy equivalences $\mathbb{G} \to \mathbb{G}$ is isomorphic to $\mathrm{Out}(F)$, so the rule $(\mathcal{G}, \sigma).\varphi = (\mathcal{G}, \sigma\varphi)$ defines an action of $\mathrm{Out}(F)$ on $L(F)$.

A marked graph of groups $\tau = (\mathcal{G}, \sigma)$ has every edge surviving if and only if it satisfies the following conditions.

1. Edge groups of $\mathcal{G}$ are trivial.
2. Each valence-one and valence-two vertex of $\mathcal{G}$ has nontrivial vertex group.
3. If an (open) edge e separates $\mathcal{G}$, each component of the complement contains a vertex with nontrivial vertex group.

For the rest of the paper, we work exclusively with marked graphs of groups satisfying these conditions.

The group $F = A * B * \mathbb{Z}$ is a free product of finite and cyclic groups, and is in particular virtually free. In [Lym22a, Theorem 3.1] it is proved that L is isomorphic (as a poset, hence as geometric realizations) to the complex considered by Krstić–Vogtmann in [KV93]. Therefore by the main result of that paper, L is contractible. This also follows from the result of [GL07] that their Outer Space is contractible in the weak topology (in fact all three topologies coincide for F a free product of finite and cyclic groups) by considering L as (a deformation retraction of) the simplicial spine of their Outer Space. The dimension of $L(F)$ is 2; in general for G a free product of $n \geq 2$ finite groups with a free group of rank k, the dimension of $L(G)$ is $2k + n - 2$.

Collapsible and ideal edges. Let v be a vertex of a marked graph of groups $(\mathcal{G}, \sigma)$. We write $\mathrm{st}(v)$ for the set of oriented edges with initial vertex v. The set of *directions* at v is

$$D_v = \coprod_{e \in \mathrm{st}(v)} \mathcal{G}_v \times \{e\}.$$

(Recall that marked graphs of groups have trivial edge groups in this paper.) There is an obvious action of $\mathcal{G}_v$ on D_v; each orbit is an oriented edge $e \in \mathrm{st}(v)$. An *ideal edge* based at v is a subset α of D_v with the following properties.

1. The sets α and $D_v - \alpha$ have at least two elements.

2. The set α contains at most one element of each $\mathcal{G}_v$-orbit of directions.

3. There is a direction $(g, e) \in \alpha$ with the property that no direction with underlying oriented edge $\bar{e}$ belongs to α. (We may have that $\bar{e} \notin \mathrm{st}(v)$.)

The action of $\mathcal{G}_v$ on D_v descends to an action on the set of ideal edge based at v, and we say that α and α' are *equivalent* if they belong to the same $\mathcal{G}_v$-orbit. In the special case that $\mathcal{G}_v$ is trivial, we further require that $D_v - \alpha$ also satisfies the definition of an ideal edge; in this case we say that α is equivalent to $D_v - \alpha$. An ideal edge α is *contained in* an ideal edge β if both are based at a vertex v and α is equivalent to a subset of β. We say that α and β are *disjoint* if they are based at different vertices or if they are based at a vertex v and their $\mathcal{G}_v$-orbits are disjoint. We say that α and β are *compatible* if one is contained in the other or they are disjoint. Note that if $\mathcal{G}_v$ is trivial, then α and β are disjoint if and only if α is contained in $D_v - \beta$ and β is contained in $D_v - \alpha$; so the relation of compatibility makes sense. An *ideal forest* $\Phi = \{\alpha_1, \ldots, \alpha_I\}$ in a marked graph of groups τ is a set of pairwise compatible ideal edges containing at most one element of each equivalence class. The set of ideal forests in τ is preordered, where $\Phi \prec \Phi'$ if each $\alpha \in \Phi$ is equivalent to some $\alpha' \in \Phi'$, and $\Phi \sim \Phi'$ if, as usual, $\Phi \prec \Phi'$ and $\Phi' \prec \Phi$. We work with equivalence classes of ideal forests, but will typically confuse an equivalence class with a representative.

An edge e of $\mathcal{G}$ is *collapsible* if there is a collapse map colllapsing it. This happens if and only if the endpoints of e are distinct and at least one has trivial vertex group. A collection of edges $\Psi = \{e_1, \ldots, e_J\}$ in $\mathcal{G}$ is a *collapsible forest* if there is a collapse map that collapses all of them.

Given a marked graph of groups τ, write $\mathrm{Star}(\tau)$ for the combinatorial 1-neigbhorhood of τ in L; that is, a vertex τ' belongs to $\mathrm{Star}(\tau)$ when there is an edge in L connecting τ to τ' (or when $\tau' = \tau$), and a simplex belongs to $\mathrm{Star}(\tau)$ when each of its vertices does.

Proposition 1.1. *The complex* $\mathrm{Star}(\tau)$ *is isomorphic to the cone on the join of the poset of ideal forests in* τ *with the poset of collapsible edges in* τ*.*

Proof. It is clear that collapses of τ are in one-to-one correspondence with collapsible forests in τ. Because the collapse relation is transitive, any marked graph of groups that τ expands to will collapse onto any collapse of τ. In other words, $\mathrm{Star}(\tau)$ is the cone on the join of the poset of marked graphs of groups collapsing onto τ with the poset of collapsible forests in τ. Therefore the claim is that the former poset is isomorphic to the poset of ideal forests in τ. The argument for this latter claim is essentially [KV93, Proposition 5.9]. Since that paper is written in the language of graphs equipped with a finite group action, we will sketch a graph-of-groups proof.

Blowing up ideal edges. First we describe how to "blow up" an ideal edge α in τ producing a new marked graph of groups $\tau^\alpha = (\mathcal{G}^\alpha, \sigma^\alpha)$. For the graph of groups $\mathcal{G}^\alpha$, begin with $\mathcal{G}$ and add one new vertex v_α and one new edge α, which is oriented so that it has initial vertex v, the vertex α was based at, and terminal vertex v_α. An oriented edge e incident to v in $\mathcal{G}$ is now incident to v_α if there is a direction in the orbit e contained in α; all other oriented edges remain incident to the vertices they were incident to in $\mathcal{G}$. All edge groups remain trivial, all vertex groups remain what they were in $\mathcal{G}$, and the new vertex has trivial vertex group. There is a collapse map $\mathcal{G}^\alpha \to \mathcal{G}$ which collapses the edge α and sends an edge e incident to v_α to the edge path ge, where $(g, e) \in \alpha$. This collapse map is a homotopy equivalence; choose a homotopy inverse $f\colon \mathcal{G} \to \mathcal{G}^\alpha$ and define $\sigma^\alpha = f\sigma$.

The marked graph of groups τ^α represents a vertex of L: collapsing any edge e with the property that some direction in the orbit e is contained in α but no direction in the orbit $\bar{e}$ is contained in α along with the edges coming from some maximal collapsible forest in τ provides a reduced marked graph of groups τ^α_e in which the edge α is not collapsed.

If α and α' are equivalent, say at first $\alpha' = g\alpha$, then τ^α and $\tau^{\alpha'}$ are equivalent: one isomorphism $h\colon \mathcal{G}^\alpha \to \mathcal{G}^{\alpha'}$ is the "identity" on the common edges and vertices (and vertex groups) of $\mathcal{G}^\alpha$ and $\mathcal{G}^{\alpha'}$, sends v_α to $v_{\alpha'}$ and α to the edge path $g\alpha'$. If instead $\alpha' = D_v - \alpha$, one isomorphism is again the "identity" on the common edges and vertices other than v, but sends v to v'_α, v_α to v, and sends α to $\bar{\alpha}'$.

If α is contained in the ideal edge β, then α corresponds to an ideal edge of τ^β based at v_β. If α is based at v and disjoint from β, then α corresponds to an ideal edge α of τ^β based at v, and similarly β corresponds to an ideal edge of τ^α. In this latter situation we have $(\tau^\alpha)^\beta = (\tau^\beta)^\alpha$.

Given an ideal forest Φ, by repeatedly blowing up ideal edges in Φ that are maximal with respect to inclusion, we obtain a marked graph of groups τ^Φ. The edges α in Φ are a collapsible forest in τ^Φ and collapsing them recovers τ. Note that because blowing up ideal edges produces marked graphs of groups representing vertices of L with an increasing number of edges, $\tau^{\Phi'}$ is not equivalent to τ^Φ for any subset Φ' of Φ. Because L is finite-dimensional, the cardinality of any ideal forest must be finite.

Turns. A *nondegenerate turn* is the $\mathcal{G}_v$-orbit of a pair of distinct directions in D_v. If γ is a *circuit,* a graph-of-groups edge path of the form $e_1 g_1 \dots e_\ell g_\ell$ that cannot be shortened by a (free) homotopy, then we say that γ *takes* the (a fortiori nondegenerate) turns corresponding to the pairs $\{\bar{e}_i, g_i e_{i+1}\}$. Part of the condition of α being an ideal edge is that α contains a representative of some nondegenerate turn. If $\mathcal{G}_v$ is trivial, then $D_v - \alpha$ also contains a representative of a nondegenerate turn. If $\mathcal{G}_v$ is nontrivial, then α is determined up to equivalence by the set of nondegenerate turns it contains representatives of, while if $\mathcal{G}_v$ is trivial, then α is determined up to equivalence by the pair of sets of nondegenerate turns either contained in α or contained in $D_v - \alpha$.

Surjectivity. Suppose τ' is a marked graph of groups representing a vertex of L and collapsing onto τ via a map h, and write Ψ for the forest of edges which is collapsed to form τ. We will show that there exists an ideal forest Φ in τ such that τ^Φ is equivalent to τ'. For each edge e in Ψ, choose an orientation in the following way. Note that each component of Ψ is incident to at most one vertex with nontrivial vertex group. If there is no such vertex for some component of Ψ, choose a vertex of that component arbitrarily; call this vertex the source. Now orient each edge away from the specified source vertex. One can do this inductively if one likes. For each component C of Ψ and each edge e of C, removing the interior of e separates the vertices of C into a positive and a negative set e^+ and e^-; by construction the positive set contains only vertices with trivial vertex group. For each pair of distinct oriented edges d_1 and d_2 in $\tau' - \Psi$ with initial vertex in e^+, there is a circuit γ' which contains a subpath of the form $\bar{d}_1 c d_2$, where c is a possibly trivial path contained in C and not crossing e. (Note that there is at least one such pair.) The circuit γ' corresponds to a nontrivial conjugacy class in F by the marking, and the circuit in the free homotopy class of $h(\gamma')$ must cross the turn represented by $\{h(d_1), h(d_2)\}$ based at v_C, the vertex to which the component C collapses. Define an ideal edge $\alpha = \alpha(e)$ to be the ideal edge corresponding to the collection of all such turns. All the properties of an ideal edge are clear except the third: for this notice that if for every direction (g, d) contained in $\alpha(e)$, a direction of the form $(g', \bar{d})$ is contained in $\alpha(e)$, it follows that e separates τ' into two components, one of which contains no vertex with nontrivial vertex group. Since we assume that τ' represents a vertex of L this is impossible. Notice that if an edge e' is in the positive component of $C - e$, then $\alpha(e')$ is contained in $\alpha(e)$, and by construction if e' is in the negative component of $C - e$ and e is in the negative component of $C - e'$, then $\alpha(e)$ and $\alpha(e')$ are disjoint. Since all edges of C point away from a source vertex, this proves that $\Phi = \{\alpha(e) : e \in \Psi\}$ is an ideal forest. In the case we had a choice for the source vertex for a component C, we see that different choices correspond to replacing certain ideal edges α with the ideal edges $D_v - \alpha$.

We claim that τ^Φ is equivalent to τ'; indeed it is clear that there is an isomorphism H from $\mathcal{G}^\Phi$ to $\mathcal{G}'$ taking $\alpha(e)$ to e preserving chosen orientations and that is the "identity" on the common edges and vertices of $\mathcal{G}'$ and $\mathcal{G}^\Phi$. Since collapsing Ψ in τ' and Φ in τ^Φ recovers τ, this implies there exists a further automorphism H' of $\mathcal{G}'$ whose underlying graph map is the identity such that $H' H \sigma^\Phi = \sigma'$.

This proves that the correspondence $\Phi \mapsto \tau^\Phi$ is a surjection of posets.

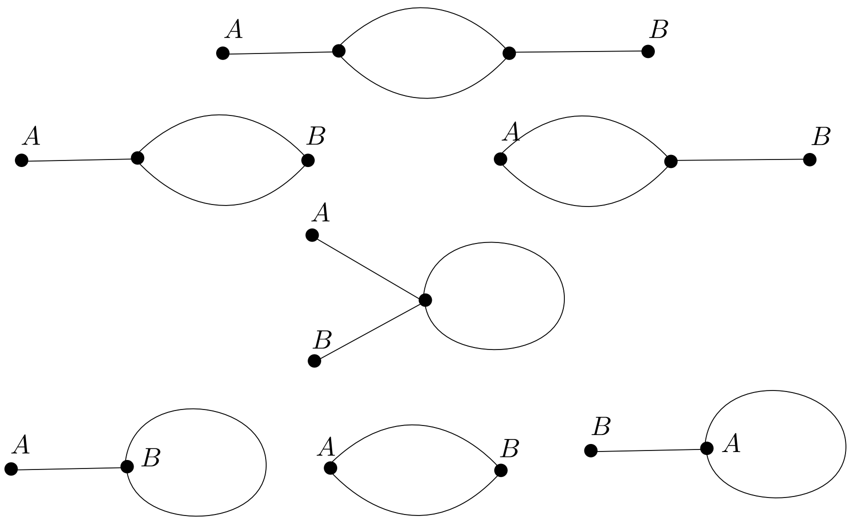

Figure 1: The combinatorial types of graphs of groups occuring in $L(F)$.

Injectivity. Now suppose τ^Φ is equivalent to τ^Ψ for ideal forests Φ and Ψ. Let h^Φ and h^Ψ be collapse maps to τ. There exists an isomorphism $f\colon \tau^\Phi \to \tau^\Psi$ such that the following diagram commutes

$$\begin{tikzcd} \tau^\Phi \arrow[r, "f"] \arrow[d, "h^\Phi"] & \tau^\Psi \arrow[d, "h^\Psi"] \\ \tau \arrow[r, equal] & \tau. \end{tikzcd}$$

In particular, f induces a bijection between the edges of Φ and those of Ψ. Choosing orientations as in the previous step, we see that in fact each $\alpha \in \Phi$ is equivalent to some $\alpha' \in \Psi$ if the orientations agree or to $D_v - \alpha'$ if the orientations disagree. In particular we have $\Phi \sim \Psi$, proving that the correspondence $\Phi \mapsto \tau^\Phi$ is an injection of posets. $\square$

The case $F = A * B * \mathbb{Z}$. In the case $F = A * B * \mathbb{Z}$, the complex L has dimension 2, and there are at most 7 combinatorial types of graphs of groups determining vertices of L (there are only 5 if $A \cong B$); one maximal with respect to expansion, and three each of intermediate and minimal. They are listed in Figure 1.

2 The case of $C_2 * C_2 * \mathbb{Z}$

The purpose of this section is to prove Theorem B; the whole section is given to its proof. In the particular case that $A = B = C_2$, we will show that the complex L has a very particular structure; a piece of it is drawn in Figure 3. This structure depends on two observations.

Intermediate graph of groups have two expansions. For the first, notice that each intermediate marked graph of groups has two expansions. Indeed, each intermediate marked graph of groups has a single vertex v that has either valence at least two (in fact equal) and $\mathcal{G}_v$ nontrivial or valence at least 4 (in fact equal) and $\mathcal{G}_v$ trivial. It's not hard to see from the definition of an ideal edge that every ideal edge of $\mathcal{G}$ is therefore based at v. In each case D_v has cardinality four and each ideal edge has exactly two elements. Up to the $\mathcal{G}_v = C_2$ action in the former case and complementation in the latter, we may assume that a fixed direction $d \in D_v$ belongs to every ideal edge in $\mathcal{G}$; there are three choices for the other direction but in each case one is forbidden.

Stars of minimal graphs of groups are polygons. For the second, we claim that for each minimal marked graph of groups τ, $\mathrm{Star}(\tau)$ in L is the first barycentric subdivision of a polygon. Indeed, this observation follows from the first: if τ^Φ is an expansion of τ which is maximal, then Φ has two ideal edges which can be collapsed independently to produce two intermediate collapses of τ^Φ which also collapse onto τ. Since intermediate marked graphs of groups have two expansions, the observation follows. We will think of the maximal marked graph of groups in $\mathrm{Star}(\tau)$ as the vertices of the polygon, the intermediate marked graphs of groups as barycenters of edges, and τ itself as the barycenter of the face.

Now consider the minimal marked graphs of groups again. If the graph of groups $\mathcal{G}$ has an edge e that forms a loop, then every ideal edge in $\mathcal{G}$ is based at the vertex v incident to that loop. By the third property in the definition of an ideal edge, if d is the other edge incident to v, then every ideal edge based at v contains, up to the action of $\mathcal{G}_v = C_2$, the direction $(1, d)$. There are four ideal edges of size two (If $\star$ denotes the nontrivial element of C_2, any choice of direction besides $(\star, d)$ is allowed) and a further four of size three (fixing a choice of $(1, e)$ or $(\star, e)$, we see that only the two directions with underlying oriented edge $\bar{e}$ may be chosen for the third).

If no edge forms a loop, then both vertices v and w of $\mathcal{G}$ support ideal edges. Since in this case v and w have valence two, there are as in the intermediate case only two choices of ideal edge based at v or w for a total of four. In other words, if τ is minimal, $\mathrm{Star}(\tau)$ is either a quadrilateral or an octagon. It is not hard to argue that L is tiled by these squares and octagons, which overlap in at most an edge.

L **is CAT(0).** Give L the piecewise-Euclidean metric in which each quadrilateral is a Euclidean unit square and each octagon is a regular Euclidean octagon with unit side-length. By [BH99, Theorem 5.4], since L is contractible, L equipped with this metric is CAT(0) if and only if it satisfies Gromov's *link condition* [BH99, Definition 5.1]. By construction, we need only check this at the vertices of each octagon and square. Since there is one combinatorial type of maximal marked graph of groups, there is one link to check.

Let τ be maximal. A vertex of $\mathrm{Link}(\tau)$ is an edge of a polygon with vertex τ; or in other words, a marked graph of groups obtained from τ by collapsing a single edge. τ has two separating edges and two nonseparating edges. The nonseparating edges cannot both be collapsed, so these give two vertices of $\mathrm{Link}(\tau)$ that are not connected by an edge. Every other pair of vertices of $\mathrm{Link}(\tau)$ can be collapsed simultaneously, and $\mathrm{Link}(\tau)$ contains an edge of length $\frac{\pi}{2}$ if the star of the collapsed graph of groups is a square and length $\frac{3\pi}{4}$ if it is an octagon. Collapsing both separating edges yields a square, while collapsing a separating edge and a nonseparating edge yields an octagon. The graph $\mathrm{Link}(\tau)$, pictured in Figure 2 thus has the property that every injective loop in the graph has length at least 2π, so by [BH99, Lemma 5.6], L satisfies the link condition and is thus CAT(0).

Labelled graphs of groups. In Figure 3, we have depicted the marked graphs of groups representing vertices and faces of polygons in L by *labelling* them. Now, F has the following presentation

$$F = \langle a, b, t \mid a^2 = b^2 = 1 \rangle.$$

Call a triple in the $\mathrm{Aut}(F)$-image of the set $\{a, b, t\}$ a *basis* for F. Under the marking, if T is a maximal tree in τ, the fundamental group $\pi_1(\tau, T)$ may be identified with F in the following way: the vertex groups of τ are each generated by an order-two element of a basis for F, and an edge outside of a maximal tree in τ may be identified with the infinite-order element of that basis. Recall that every marked graph of groups is obtained by collapsing edges of a maximal marked graph of groups

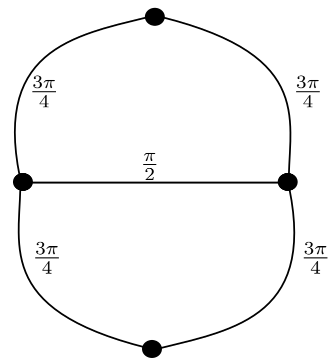

Figure 2: The link of a maximal marked graph of groups.

τ, which has four edges, which we will color red, orange, pink and blue in the following way: the pink and blue edges are nonseparating, the red edge connects to the vertex group generated by a conjugate of a and the orange edge to the vertex group generated by a conjugate of b.

Note that up to conjugating the basis as a whole, we may always assume that the conjugate of a chosen is actually a itself. We may furthermore assume that (up to the action of σ) the automorphism taking our original basis to the given one actually lies in the subgroup generated by the automorphisms τ, L_a, R_b and χ_t^b from the introduction since the outer classes of these automorphisms together with σ generate $\mathrm{Out}(F)$.

In fact, recalling the homomorphism $\Phi\colon W \to \mathrm{Out}(F)$ from the introduction, we claim that we may always choose the labelling automorphism from the normal subgroup $\mathrm{Im}(\Phi) < \mathrm{Out}(F)$. To see this, notice that the stabilizer of a maximal marked graph of groups in L under the $\mathrm{Out}(F)$-action is isomorphic to $C_2 \times C_2$ and is generated by an $\mathrm{Out}(F)$-conjugate of the outer class of the automorphisms $\tau\chi_t^b$ (this is the effect of swapping pink and blue edges) and $\tau\sigma$ (this is the effect of "reflecting" the graph of groups through the midpoints of the pink and blue edges). An arbitrary element of $\mathrm{Out}(F)$ may be written as

$$w, \qquad (\chi_t^b)^{-1}w, \qquad \sigma w \qquad \text{or} \qquad (\chi_t^b)^{-1}\sigma w$$

where $w \in \mathrm{Im}(\Phi)$. (We have not proved that $\mathrm{Out}(F)/\mathrm{Im}(\Phi)$ is all of $C_2 \times C_2$, so we do not claim uniqueness here) Since $\tau^2 = 1$, $(\chi_t^b)^{-1}\tau = \tau\chi_t^b$ and $\tau \in \mathrm{Im}(\Phi)$, we may introduce a pair of τ into the above equations to see that an arbitrary labelling automorphism may be represented by an element of the stabilizer of a fixed maximal marked graph of groups followed by an element of $\mathrm{Im}(\Phi)$.

The Davis–Moussong complex of W. The Davis–Moussong complex of W is a $\mathrm{CAT}(0)$ polygonal complex tiled by squares and octagons. It is built in the following way: begin with a variant of the (typically right, but for our purposes *left*) Cayley graph of W where rather than putting in a bigon between every pair of vertices that differ by a Coxeter generator, we put a single edge. Each Coxeter relation for W determines a family of loops in the Cayley graph of length four or eight; fill in these loops with 2-cells to form squares and octagons, and give the Davis–Moussong complex the piecewise-Euclidean metric that agrees with the metric we gave L on each cell.

The action isomorphism. We turn to proving that Φ is injective and that L is isomorphic to the Davis–Moussong complex of W. Observe that under Φ, each of the four Coxeter generators of W fixes the midpoint of a unique edge incident to the vertex represented by our fixed maximal graph of groups. This implies that there exists a Φ-equivariant cellular map from the Davis–Moussong complex of W to L that is an isometry on each 2-cell. Put another way, we may think of vertices of L as labelled

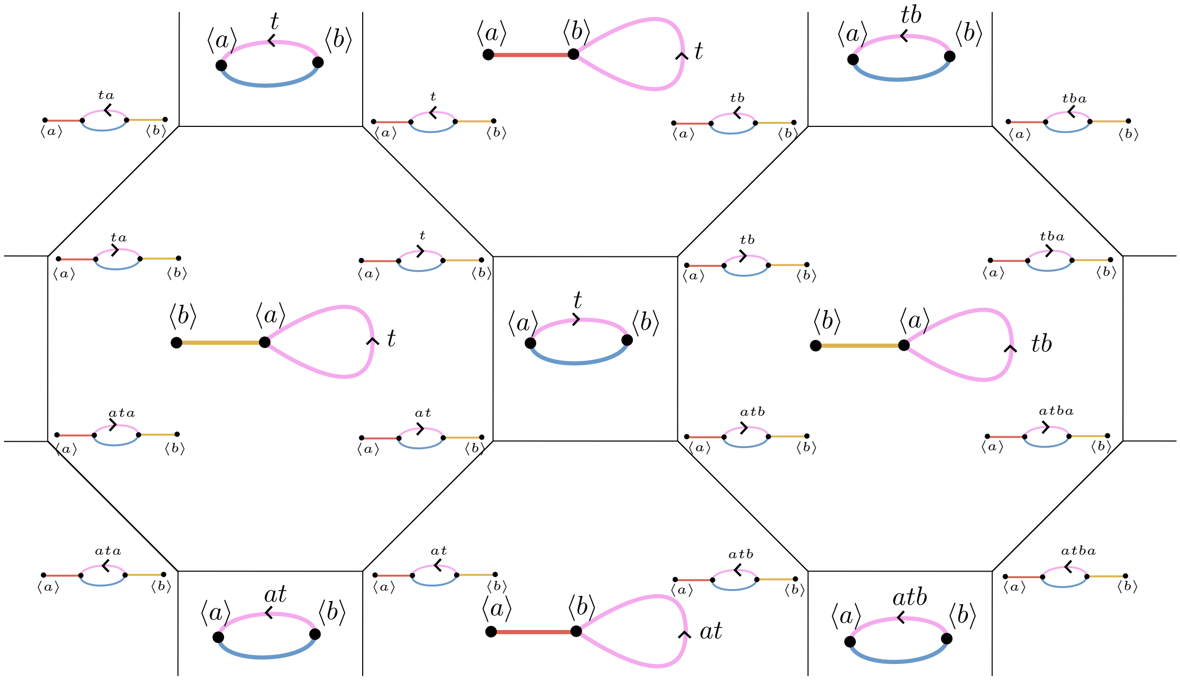

Figure 3: A portion of the complex L.

by elements of $\mathrm{Im}(\Phi)$, and a loop in the 1-skeleton of L corresponds to a word in the generators of $\mathrm{Im}(\Phi)$ (and thus W) that is trivial in $\mathrm{Out}(F)$. Since L is simply connected, there exists a van Kampen diagram for this word whose interior is tiled by squares and octagons. But each square and each octagon, which corresponds to a relator in $\mathrm{Im}(\Phi)$ is already a relator in W. This proves that Φ is injective and the Φ-equivariant cellular map is in fact an equivariant isometry. Since W therefore acts freely on the vertices of L via Φ, this also proves that $\mathrm{Im}(\Phi)$ has index four in $\mathrm{Out}(F)$. This completes the proof of Theorem B

3 The general case

The purpose of this section is to complete the proof of Theorem A; the entire section is devoted to its proof.

Recall from Figure 1 that there are seven combinatorial types of marked graphs of groups. In the general case it is not true that stars of minimal marked graphs of groups are polygons, but they may be, in a certain sense, thought of as being made by gluing a family of barycentrically subdivided squares or a family of barycentrically subdivided octagons together. We proceed through the combinatorial types.

Maximal links. If τ is maximal with respect to expansion, then any edge of τ may be collapsed, and a pair of edges may be collapsed if and only if they are not both nonseparating. Thus the combinatorial type of $\mathrm{Link}(\tau)$ is the same as in the special case $A = B = C_2$. The piecewise-Euclidean metric we give L will make $\mathrm{Link}(\tau)$ isometric to the special case, so these links will satisfy Gromov's link condition.

Intermediate links. There are three types of intermediate marked graphs of groups. The type which has one loop edge has the same link as in the special case, (a topological circle given the combinatorial structure of a bipartite graph on $2+2$ vertices) because every ideal edge is based at a vertex with trivial vertex group, just as in the special case. Also as in the special case, we will treat vertices corresponding to this type of marked graph of groups as being the barycenter of an edge, so each edge in $\mathrm{Link}(\tau)$ in this case will have length $\frac{\pi}{2}$; these links thus satisfy Gromov's link condition.

In the two other cases, every ideal edge is based at a vertex with valence two and vertex group A or B. A simple calculation shows that there are thus $|A|$ or $|B|$ ideal edges based at this vertex, respectively. Every edge of these graphs of groups may be collapsed (individually), so it follows from Proposition 1.1 that $\mathrm{Star}(\tau)$ is the cone on a bipartite graph with $3+|A|$ or $3+|B|$ vertices respectively. The piecewise-Euclidean metric we give L will make each edge in $\mathrm{Link}(\tau)$ have length $\frac{\pi}{2}$; these links thus satisfy Gromov's link condition.

Minimal links. There are three types of minimal marked graphs of groups. Consider the first the type with two nonseparating edges. Every ideal forest in such a marked graph of groups τ has one ideal edge based at a vertex with vertex group A and one at the other vertex; every such pair forms an ideal forest, so $\mathrm{Link}(\tau)$ is a bipartite graph with $|A|+|B|$ vertices. The piecewise-Euclidean metric we give L will make each edge in $\mathrm{Link}(\tau)$ have length $\frac{\pi}{2}$; these links thus satisfy Gromov's link condition.

In the final two cases, every ideal edge is based at a vertex with vertex group A or B. In each case, a similar argument as in the previous case shows that there are $2|A|$ or $2|B|$ ideal edges of size two and $|A|^2$ or $|B|^2$ ideal edges of size three respectively. A pair of ideal edges forms an ideal forest if and only if one is contained in the other, so each ideal edge of size two is compatible with $|A|$ or $|B|$ ideal edges of size three respectively, while each ideal edge of size three is compatible with two ideal edges of size two. Explicitly, label the oriented edges incident to the relevant vertex v as d, e and $\bar{e}$. Up to the $\mathcal{G}_v$-action we may assume that $(1,d)$ is contained in every ideal edge. The ideal edges of size two are thus in bijection with the set $\mathcal{G}_v \times \{e\} \sqcup \mathcal{G}_v \times \{\bar{e}\}$ and the ideal edges of size three are in bijection with $(\mathcal{G}_v \times \{e\}) \times (\mathcal{G}_v \times \{\bar{e}\})$. An injective loop in $\mathrm{Link}(\tau)$ of minimal length thus visits a sequence of size-three ideal edges of the form $((g,e),(g',\bar{e}))$, $((h,e),(g',\bar{e}))$, $((h,e),(h',\bar{e}))$, $((g,e),(h',\bar{e}))$ and thus has length sixteen (four size-three ideal edges, four size-two, and eight unions of two). The piecewise-Euclidean metric we give L will make each edge in $\mathrm{Link}(\tau)$ have length $\frac{\pi}{8}$; these links thus satisfy Gromov's link condition.

L **is CAT(0).** Consider the pieces in Figure 4. By the preceding discussion, we see that L is tiled by cells isomorphic to these pieces in the sense that every 2-simplex of L is contained in a unique such piece. The piece with five edges contains four 2-simplices of L; the piece with four contains two. Give L the piecewise-Euclidean metric in which the piece with five edges is a quarter of a regular octagon of side-length one and the piece with four edges is a quarter of a square of side-length one. By the preceding discussion we see that with this metric, L satisfies Gromov's link condition and is thus CAT(0).

L **has infinitely many ends.** Just as the Davis–Moussong complex of W had infinitely many ends, so too does L for general A and B. To see this, consider $\mathrm{Star}(\tau)$, where τ is a minimal marked graph of groups with two nonseparating edges. We claim that $\mathrm{Star}(\tau)$ separates L into at least two components with noncompact closure; since $\mathrm{Out}(F)$ is not virtually cyclic, this proves that L has infinitely many ends. To see that $\mathrm{Star}(\tau)$ separates, consider the three types of vertices on the boundary of $\mathrm{Star}(\tau)$. One has link given in Figure 2, in which the subset of that link contained in $\mathrm{Star}(\tau)$ is the closed edge of length $\frac{\pi}{2}$, and the other two types have links that are bipartite on $3+|A|$ or $3+|B|$ vertices, in which the subset of that link contained in $\mathrm{Star}(\tau)$ is the 1-neighborhood of one vertex from the group of 3 (thus it contains either $|A|+1$ or $|B|+1$ vertices). In either case, we see that the subset of the link contained in $\mathrm{Star}(\tau)$ separates the link of the corresponding vertex into two pieces: explicitly in the coloring scheme from Theorem B, marked graphs of groups in the link without a pink edge are separated from those without a blue edge. It follows that $\mathrm{Star}(\tau)$ separates a small tubular neighborhood of $\mathrm{Star}(\tau)$ into two pieces. Because L is CAT(0) and thus uniquely geodesic, it follows

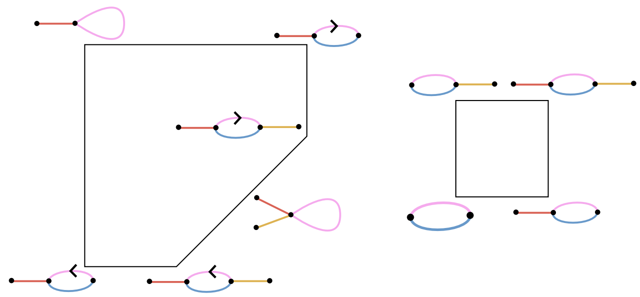

Figure 4: The pieces the complex L is tiled by.

that $\mathrm{Star}(\tau)$ separates L into two pieces. These pieces have noncompact closure: indeed, each contains a Euclidean plane tiled by squares and octagons minus one square.

Let us conclude by remarking that while the Davis–Moussong complex for W has isolated flats, the same does not appear to be true for L in general. Indeed, one can build Euclidean planes tiled by squares and octagons that have unbounded intersection.

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