One-endedness of outer automorphism groups of free products of finite and cyclic groups

Rylee Alanza Lyman

May 6, 2023

Abstract

In a previous paper, we showed that the group of outer automorphisms of the free product of two nontrivial finite groups with an infinite cyclic group has infinitely many ends, despite being of virtual cohomological dimension two. The main result of this paper is that aside from this exception, having virtual cohomological dimension at least two implies the outer automorphism group of a free product of finite and cyclic groups is one ended. As a corollary, the outer automorphism groups of the free product of four finite groups or the free product of a single finite group with a free group of rank two are virtual duality groups of dimension two, in contrast with the above example. We also prove that groups in this family are semistable at infinity (or at each end). Our proof is inspired by methods of Vogtmann, applied to a complex first studied in another guise by Krstić and Vogtmann.

Duality groups. A group G of type FP_∞ is a *duality group of dimension* n in the sense of Bieri–Eckmann [BE73] if there exists a G-module D such that for any G-module A and any integer k, we have an isomorphism

$$H^k(G; A) \cong H_{n-k}(G; D \otimes_{\mathbb{Z}} A),$$

where G acts on $D \otimes_{\mathbb{Z}} A$ diagonally. The dimension, n, turns out to be equal to the cohomological dimension of G, which is thus torsion free, and is therefore determined by G. Similarly, the dualizing module D turns out to be determined by G; it is isomorphic to $H^n(G; \mathbb{Z}G)$.

Many important examples of groups are duality groups, or more generally, *virtual duality groups,* having a duality group as a subgroup of finite index. (Virtual duality groups are thus allowed to have torsion, but they will be virtually torsion free.) Among these are, of course, groups having a closed manifold as a $K(\pi, 1)$, but also fundamental groups of knot complements, mapping class groups of closed surfaces, and, most relevant to the current paper, $\mathrm{Out}(F_n)$ [BF00].

This paper is motivated by the following question.

Question A. Let F be a free product of finite and cyclic groups. Is the outer automorphism group of F a virtual duality group?

The group $\mathrm{Out}(F)$ is of type FP_∞ and virtually torsion free, so the question makes sense. As we showed, but did not remark, in [Lym22a], the answer to this question is sometimes "no".

Our main result is the following.

Theorem B. *Let $F = A_1 * \cdots * A_n * F_k$ be a free product of the finite groups A_i with a free group of rank k. The invariants n and k determine the number of ends of $\mathrm{Out}(F)$ in the following way.*

$$\begin{cases} n \leq 1, \ k \leq 1 \ or \ (n,k) = (2,0) & \mathrm{Out}(F) \ is \ finite, \ i.e. \ has \ zero \ ends. \\ (n,k) = (3,0), \ (2,1), \ (0,2) & \mathrm{Out}(F) \ has \ infinitely \ many \ ends. \\ otherwise & \mathrm{Out}(F) \ has \ one \ end. \end{cases}$$

Ends of groups. Let us explain. Celebrated results of Freudenthal and Hopf say that a finitely generated group G has 0, 1, 2, or infinitely many *ends*. An *end* of a topological space X is a compatible choice, for every compact subset K of X, of a path component of the complement $X - K$. Put another way, if $K \subset K'$ are compact subsets of X, every path component of $X - K'$ determines a path component of $X - K'$, and in fact we have an inverse system of sets (or spaces, if we'd like to give them the discrete topology). The *space of ends* of X, is the inverse limit

$$\varprojlim_K \pi_0(X - K)$$

and the *number* of ends is its cardinality. What Freudenthal and Hopf prove is that the number of ends of a connected CW-complex, or if you'd like, a Cayley graph, that a finitely generated group G acts on with finite stabilizers and finite (i.e. compact) quotient is in fact an invariant of the group, and that if a group has at least three ends, it has infinitely many.

One way to think about one-endedness, which is in some sense generic, is to say that, supposing we have fixed an exhaustion of X by compact sets $K_0 \subset K_1 \subset \cdots$, for any two points "sufficiently far from K_n", say belonging to $X - K_N$ for N sufficiently larger than n, those two points can be connected by a path that avoids K_n. Thus for example the line is not one ended (in fact it has two ends), while the plane is.

End invariants of groups. This reframes one-endedness as 0-*connectivity at infinity,* and naturally leads one to generalize to questions about higher homotopy or homology groups "at infinity". The plane, then, would fail to be simply connected at infinity, since a loop encircling our compact set K_N is not nullhomotopic in $X - K_n$ as soon as K_n is nonempty, while $\mathbb{R}^n$ for larger n *would* be simply connected at infinity.

To ask these questions for groups, one needs to be slightly careful. After all, every finitely generated group has a Cayley graph, and graphs have no higher homotopy groups. In the case of the plane, what is interesting is that every loop *is* nullhomotopic in the plane as a whole, but may *not* remain so after removing a compact set. Thus we put restrictions on the spaces in question, asking them to be n-connected in the ordinary sense, before we ask whether they are n-connected at infinity. This has the effect for groups of restricting our attention to groups with certain *finiteness properties,* which generalize being finitely generated or finitely presented.

Semistability. Even with this restriction, there is a subtlety: the fundamental group—for instance—needs a basepoint, but as we work in the complements of a compact exhaustion no single basepoint will do. Making a choice of basepoint for $\pi_1(X - K_n, \star_n)$ for each n and a path from $\star_n$ to $\star_{n-1}$, we produce a *proper ray* in X. To show that resulting inverse limit

$$\varprojlim_n \pi_1(X - K_n, \star_n)$$

is independent of our choices, we would need to show that any two proper rays (ending in the same end if there is more than one) are properly homotopic. For locally finite complexes X,

so in particular for finitely presented groups, this is equivalent by [Geo08, Proposition 16.1.2] to the condition that this inverse system of groups is *semistable* or satisfies the *Mittag-Leffler condition:* for each n there exists N such that for $k \geq N$, the image of $\pi_1(X - K_N, \star_n)$ in $\pi_1(X - K_n, \star_n)$ is equal to the image of $\pi_1(X - K_k, \star_k)$ in $\pi_1(X - K_n, \star_n)$.

Unwinding this definition, we see that it is equivalent to the claim that every loop in $X - K_N$ based at $\star_N$ may be pushed off to $X - K_k$ (along our proper ray, if you'd like) by a homotopy that avoids K_n.

The relevance to group cohomology is that if G is *semistable at each end* in the above sense, then $H^2(G, \mathbb{Z}G)$ is a free abelian group [Geo08, Theorem 16.5.1]. It turns out that if G is $(n-2)$-connected at infinity, (as the plane and hence $\mathbb{Z}^2$ is 0-connected at infinity, for instance, although merely being $(n-2)$-acyclic will do) then $H^i(G, \mathbb{Z}G) = 0$ for $i < n$, and $H^n(G, \mathbb{Z}G)$ is torsion free [Geo08, BM01]. (Thus for a one-ended group semistability is somewhere between one-endedness and simple connectivity at infinity.) A group G of type FP_∞ is a duality group of dimension n if and only if $H^i(G, \mathbb{Z}G) = 0$ for $i \neq n$ and $H^n(G, \mathbb{Z}G)$ is nontrivial and torsion free. If additionally G has cohomological dimension n, then $H^i(G, \mathbb{Z}G) = 0$ for $i > n$ and $H^n(G, \mathbb{Z}G)$ is nontrivial.

Thus the combination of $(n - 2)$-connectivity at infinity with having virtual cohomological dimension n implies that a group G of type FP_∞ is a virtual duality group.

Additionally, let us remark that it is an open question whether every finitely presented group G satisfies that $H^2(G, \mathbb{Z}G)$ is free abelian, hence whether every finitely presented group is semistable at each end. We are able to answer this in the affirmative for $\mathrm{Out}(F)$ when F is a free product of finite and cyclic groups.

Theorem C. *Let $F = A_1 * \cdots * A_n * F_k$ be a free product of the finite groups A_i with a free group of rank k. The group $\mathrm{Out}(F)$ of outer automorphisms of F is semistable at each end.*

Corollary D. *For F as above, writing $G = \mathrm{Out}(F)$, we have that $H^2(G, \mathbb{Z}G)$ is free abelian.*

The case of $\mathrm{Out}(F)$**.** From here we discuss $\mathrm{Out}(F)$, where $F = A_1 * \cdots * A_n * F_k$ is the free product of the finite groups A_i with a free group of rank k, and $\mathrm{Out}(F)$ denotes its *outer automorphism group,* $\mathrm{Aut}(F)/\mathrm{Inn}(F)$. The virtual cohomological dimension of $\mathrm{Out}(F)$ was computed by Krstić and Vogtmann [KV93]; it is equal to the dimension of a certain contractible simplicial complex L that $\mathrm{Out}(F)$ acts on with finite stabilizers and finite quotient. We will call this complex the *spine of (reduced) Outer Space* for F. The groups $\mathrm{Out}(F)$, which are virtually torsion free, are thus virtually of type F, hence certainly of type FP_∞. In any case, the dimension of L can be computed from the invariants n and k by the following rule

$$\dim(L) = \begin{cases} 2k + n - 2 & n \geq 2, \\ \max\{2k + n - 3, 0\} & \text{otherwise.} \end{cases}$$

Thus, to prove that $\mathrm{Out}(F)$ is a virtual duality group, it suffices to show that it is $(2k+n-4)$-connected at infinity if $n \geq 2$ or $(2k-n-5)$-connected at infinity otherwise. In particular, when L has dimension 2, we need only show that $\mathrm{Out}(F)$ is one ended. There are three such cases: $F = A_1 * A_2 * A_3 * A_4$, $F = A_1 * A_2 * \mathbb{Z}$ and $F = A_1 * F_2$. In the first case, one-endedness follows from a result of Das [Das18], who shows that $\mathrm{Out}(F)$ is *thick* when $F = A_1 * \cdots * A_n$ and $n \geq 4$. The third case is covered by Theorem B, but in the second, Theorem B states that $\mathrm{Out}(F)$ has infinitely many ends: this is part of the main result of [Lym22a]. This is why the answer to Question A is "no" for some choices of F. We view Theorem B as partial progress towards a full answer to Question A.

3

In view of Theorem C and the fact that $\mathrm{Out}(A_1 * A_2 * A_3 * A_4)$ and $\mathrm{Out}(A_1 * F_2)$ are one-ended groups of virtual cohomological dimension two, these groups have a well-defined *fundamental group at infinity,* namely the inverse limit of the fundamental groups as discussed in the semistability paragraph above. Because $H^2(G, \mathbb{Z}G)$ is nontrivial, this fundamental group is nontrivial. What is it?

Question E. What is the fundamental group of $\mathrm{Out}(A_1 * A_2 * A_3 * A_4)$ or $\mathrm{Out}(A_1 * F_2)$ at infinity?

Our proof of Theorem B is inspired by Vogtmann's paper [Vog95]. There she analyzes the combinatorial structure of links of certain vertices of the Spine of Outer Space (for $F = F_k$) to push paths and homotopies off to infinity. The Spine of Outer Space for general F likewise has a rich combinatorial structure; the present paper is a first contribution towards an understanding of it in general.

Like the papers [CV86, KV93], we proceed by putting a kind of "height function" on certain vertices of the complex L of minimal complexity we term "briar patches". (A briar patch may have petals, like a rose, and thorns like a thistle, but may also sprawl somewhat.) Vertices of L correspond to marked graphs of groups (or if you'd rather, actions of F on trees), and the height function records the lengths of immersed loops associated to a predetermined finite set of conjugacy classes (or if you'd rather, the hyperbolic translation lengths of these conjugacy classes).

The proof of contractibility of L in [CV86, KV93] goes by showing, using "peak-reduction" techniques, that these height functions behave somewhat like non-singular Morse functions, allowing one to contract the whole complex onto the subcomplex where the function is minimized, and then arguing that by carefully choosing the finite set of conjugacy classes, this minimal subcomplex is manifestly contractible.

Like Vogtmann in [Vog95], we'd like to do the opposite: The height function provides an exhaustion of L; for well-chosen finite sets, this is a compact exhaustion, defining for us a sequence of compact subsets of L we call "balls". Analyzing the combinatorics of the spine allows us to understand how our height function changes as we move around the star of a single vertex of L.

Firstly, we restrict our attention to a family of "standard paths", which look like a sequence of one-edge expand–collapse moves (the term is "Whitehead move") between briar patches. In the case $F = F_k$, these moves between roses have the effect of applying Whitehead automorphisms to the fundamental group, hence the name. Supposing that τ_{2k} is a briar patch along such a path which is a kind of local minimum for the height function, the game is to find a new standard path between the adjacent briar patches $\tau_{2(k-1)}$ and $\tau_{2(k+1)}$ that passes through briar patches of height strictly greater than that of τ. By doing so while staying within the star of τ, we have a homotopy between the old path and the new one, and by iteratively applying this method to all local minima, we are able to push our path off to infinity. It turns out that when this strategy of proof works, it suffices to prove semistability as well; we turn to this in Section 3.

4

1   The Complex L

We work in the *spine of reduced Outer Space for F,* a simplicial complex which we denote as $L = L(F)$. A more detailed definition is given in [Lym22a]. A vertex of L corresponds to an action of F on a simplicial tree T with finite stabilizers (and in fact trivial edge stabilizers). In this paper it will be more convenient to work in the quotient graph of groups $\mathcal{G} = F\backslash\backslash T$. We assume a certain amount of familiarity with graphs of groups in this paper and refer the reader to [Ser03], [Bas93], [SW79] and [Lym22c, Section 1] for background on graphs of groups. We will follow the notation in [Lym22c], although almost everything we write is standard. The main innovation of [Lym22c] that we need is the notion of a *homotopy equivalence* between graphs of groups and homotopy of maps of graphs of groups.

Fix a finite graph of finite groups $\mathbb{G}$, a basepoint $\star \in \mathbb{G}$ and an identification $F \cong \pi_1(\mathbb{G}, \star)$. A *marked graph of groups* $\tau = (\mathcal{G}, \sigma)$ is a graph of finite groups $\mathcal{G}$ together with a homotopy equivalence $\sigma \colon \mathbb{G} \to \mathcal{G}$ called the *marking.* Two marked graphs of groups $(\mathcal{G}, \sigma)$ and $(\mathcal{G}', \sigma')$ are *equivalent* if there is an isomorphism $h \colon \mathcal{G} \to \mathcal{G}'$ such that the following diagram commutes up to homotopy

$$
\begin{array}{ccc}
 & & \mathcal{G} \\
 & \overset{\sigma}{\nearrow} & \downarrow h \\
\mathbb{G} & & \\
 & \underset{\sigma'}{\searrow} & \mathcal{G}'.
\end{array}
$$

The complex L.   A vertex of the complex L is determined by an equivalence class of marked graphs of groups $(\mathcal{G}, \sigma)$ satisfying the following conditions.

1. Edge groups of $\mathcal{G}$ are trivial.

2. Each valence-one and valence-two vertex of $\mathcal{G}$ has nontrivial vertex group.

3. If an (open) edge e separates $\mathcal{G}$, each component of the complement contains a vertex with nontrivial vertex group.

A marked graphs of groups $\mathcal{G}$ determining a vertex of L, is *reduced* if collapsing any edge of $\mathcal{G}$ yields a marked graph of groups not homotopy equivalent to $\mathcal{G}$ (because the collapse map cannot be "undone", for example because maybe the resulting graph of groups has an infinite vertex group). The third condition above is equivalent to asking that for each edge e of $\mathcal{G}$, there is a reduced marked graph of groups $\mathcal{G}'$ which may be obtained from $\mathcal{G}$ by collapsing edges in which the edge e is not collapsed. In the language of [Cla09, GL07], we say the edge e is *surviving.*

Two vertices $(\mathcal{G}, \sigma)$ and $(\mathcal{G}', \sigma')$ are connected by an oriented edge if a marked graph of groups equivalent to $(\mathcal{G}', \sigma')$ can be obtained from $(\mathcal{G}, \sigma)$ by collapsing certain edges in $\mathcal{G}$, and in general L possesses a simplex whenever its 1-skeleton is present. As observed in [Lym22b, Theorem 3.1], the main result of Krstić–Vogtmann [KV93] implies that L is contractible. The dimension of L is $2k + n - 2$ when $n \geq 2$ and $2k + n - 3$ when $n \leq 1$ and $k > 1$.

We assume, in view of [Vog95], throughout this paper that $n \geq 1$. The *edge number* of F is the quantity $\operatorname{en}(F) = 2k + n - 1$. If $\tau = (\mathcal{G}, \sigma)$ is a reduced marked graph of groups with exactly one vertex v of valence at least two, then the edge number of F is the valence of v.

The norm $\|\cdot\|$.   Given a finite set W of infinite order elements of F and a reduced marked graph of groups $\tau = (\mathcal{G}, \sigma)$, we define a norm on τ as

$$
\|\tau\| = \sum_{w \in W} \ell(w),
$$

where $\ell(w)$ is the hyperbolic translation length of w on the Bass–Serre tree T of $\mathcal{G}$. The *star* of a reduced marked graph of groups τ is the subcomplex of L spanned by those marked graphs of groups τ' which collapse onto τ. The *ball of radius r* is the union of the stars of reduced marked graphs of groups τ with norm at most r. It is contractible. Krstić–Vogtmann prove [KV93, Proof of Proposition 6.3] that for appropriate choice of W, balls of radius r are compact. We therefore choose W such that balls of radius r are compact. This gives us a compact exhaustion of L.

Directions and turns. Let v be a vertex of a marked graph of groups $(\mathcal{G}, \sigma)$. We write $\mathrm{st}(v)$ for the set of oriented edges with initial vertex v. The set of *directions* at v is

$$D_v = \coprod_{e \in \mathrm{st}(v)} \mathcal{G}_v \times \{e\}.$$

(Recall that marked graphs of groups have trivial edge groups in this paper.) There is an obvious action of $\mathcal{G}_v$ on D_v; each orbit is an oriented edge $e \in \mathrm{st}(v)$. A *turn* is the $\mathcal{G}_v$-orbit of a pair $\{(g_1, e_1), (g_2, e_2)\}$ of directions in D_v. A turn is *degenerate* if it is represented by a pair of identical directions and *nondegenerate* otherwise.

The star graph. Let $\tau = (\mathcal{G}, \sigma)$ be a marked graph of groups. For each $w \in W$ thought of as an element of the fundamental group $\pi_1(\mathbb{G}, \star)$, represent the conjugacy class of $\sigma_\sharp(w)$ by a graph-of-groups *edge path*

$$\gamma_w = g_0 e_1 g_1 \ldots e_k g_k$$

which is *cyclically reduced* in the sense that $e\bar{e}$ is not a subpath of any cyclic reordering of γ_w for any oriented edge e of $\mathcal{G}$. We say the path γ_w *takes* the turns $[\{(1, \bar{e}_i), (g_i, e_{i+1})\}]$ for $1 \le i \le k-1$ and $[\{(g_k^{-1}, \bar{e}_k), (g_0, e_1)\}]$. Since γ_w is cyclically reduced, these turns are nondegenerate. The *star graph of τ* with respect to W has as vertex set the disjoint union of D_v as v varies over the vertices of $\mathcal{G}$ and an edge connecting a pair of directions (necessarily based at the same vertex v) if the turn they determine is taken by some γ_w. Thus each turn of γ_w based at v corresponds to $|\mathcal{G}_v|$ edges of the star graph. Since turns taken by γ_w are nondegenerate, the star graph has no loop edges. If τ is reduced, then $\|\tau\|$ may be computed from the star graph as

$$\|\tau\| = \frac{1}{2} \sum_{v \in \mathcal{G}} \sum_{d \in D_v} \frac{\mathrm{valence}(d)}{|\mathcal{G}_v|}.$$

Ideal edges. An *ideal edge* based at v in $\mathcal{G}$ is a subset α of D_v with the following properties.

1. The sets α and $D_v - \alpha$ have at least two elements.

2. The set α contains at most one element of each $\mathcal{G}_v$-orbit of directions.

3. There is a direction $(g, e) \in \alpha$ with the property that no direction with underlying edge $\bar{e}$ belongs to α. (We may have that $\bar{e} \notin \mathrm{st}(v)$.)

Write $D(\alpha)$ for the set of directions satisfying item 3 of the above definition. We are mainly interested in the case where τ is reduced and $F = A_1 * \cdots * A_n * F_k$ where $n \ge 1$. In this situation, every vertex of $\mathcal{G}$ has nontrivial vertex group, so item 2 implies that $D_v - \alpha$ has at least two elements if α has at least two elements. There is an ideal edge based at v if and only if v has valence at least two, in which case we say that v is *active*. The action of $\mathcal{G}_v$ on D_v descends to an action on the set of ideal edges based at v, and we say α and α' are *equivalent*

if they belong to the same $\mathcal{G}_v$-orbit. An ideal edge α is *contained in* an ideal edge β if both are based at v and α is equivalent to a subset of β. We say α and β are *disjoint* if they are based at different vertices or if they are based at a vertex v and their $\mathcal{G}_v$-orbits are disjoint. We say α and β are *compatible* if one is contained in the other or they are disjoint.

In the case where $\mathcal{G}_v$ is trivial, the set $D_v - \alpha$ is also an ideal edge, and we say α and $D_v - \alpha$ are equivalent, but compatibility remains the same.

Blowing up ideal edges. Given an ideal edge α in a marked graph of groups τ, we describe a new marked graph of groups $\tau^\alpha = (\mathcal{G}^\alpha, \sigma^\alpha)$ obtained from $\tau = (\mathcal{G}, \sigma)$ by *blowing up* the ideal edge α. The graph $\mathcal{G}^\alpha$ is the same as the graph $\mathcal{G}$ with an additional vertex v_α and an additional edge α. The oriented edge α has initial vertex v and terminal vertex v_α. An oriented edge e incident to v in $\mathcal{G}$ is now incident to v_α if there is a direction in the orbit e in α; all other oriented edges remain incident to the vertices they were incident to in $\mathcal{G}$. All edge groups remain trivial, all vertex groups remain what they were in $\mathcal{G}$, and the new vertex has trivial vertex group. There is a collapse map $\mathcal{G}^\alpha \to \mathcal{G}$ which collapses the edge α and sends an edge e incident to v_α to the edge path ge, where $(g, e) \in \alpha$. This collapse map is a homotopy equivalence; choose a homotopy inverse $f \colon \mathcal{G} \to \mathcal{G}^\alpha$ and define $\sigma^\alpha = f\sigma$.

If τ is reduced, the marked graph of groups τ^α represents a vertex of L: collapsing any edge e in $D(\alpha)$ yields a reduced marked graph of groups τ_e^α in which the edge α is not collapsed. When τ is reduced, we say that τ_e^α is obtained from τ by the *elementary Whitehead move* (α, e). When it is convenient, we will sometimes write (α, d) where $d \in D(\alpha)$ is a direction, rather than an edge. The vertices τ and τ_e^α of L are connected by an edge path of length two in L called an *elementary Whitehead path.* Following [Vog95], we denote paths in L by listing their vertices; so the elementary Whitehead path from τ to τ_e^α is $(\tau, \tau^\alpha, \tau_e^\alpha)$.

If α and α' are equivalent, say $\alpha' = g\alpha$, then τ^α and $\tau^{\alpha'}$ are equivalent: one isomorphism $h \colon \mathcal{G}^\alpha \to \mathcal{G}^{\alpha'}$ is the identity on the common edges and vertices (and vertex groups) of $\mathcal{G}^\alpha$ and $\mathcal{G}^{\alpha'}$, sends v_α to v'_α and α to the edge path $g\alpha'$.

If α is contained in the ideal edge β, then α corresponds to an ideal edge α of τ^β based at v_β. If α is based at v and β is based at w and they are disjoint, then α corresponds to an ideal edge α of τ^β based at v and vice versa. In this latter situation we have $(\tau^\alpha)^\beta = (\tau^\beta)^\alpha$.

An *ideal forest* $\Phi = \{\alpha_1, \ldots, \alpha_I\}$ in a reduced marked graph of groups τ is a set of pairwise compatible ideal edges containing at most one element of each equivalence class. By repeatedly blowing up ideal edges in Φ that are maximal with respect to inclusion, we obtain a marked graph of groups $\tau^{\alpha_1, \ldots, \alpha_I}$. The edges $\{\alpha_1, \ldots, \alpha_I\}$ are a *forest* in the sense of [Lym22c], and collapsing them recovers τ.

The absolute value of an ideal edge. Following Culler and Vogtman [CV86], define the *dot product* of two subsets S and T of the set of vertices of a graph to be the number of (unoriented) edges with one vertex in S and the other in T. The *absolute value* of S is the dot product of S with its complement. An ideal edge corresponds to a subset of the set of vertices of the star graph, so we may compute $|\alpha|$. If $\alpha' = g.\alpha$, then we have $|\alpha| = |\alpha'|$. A direction d likewise has an absolute value $|d| = |\{d\}|$. If $d' = g.d$ for some $g \in \mathcal{G}_v$, then $|d'| = |d|$. Put another way, the absolute value of a direction is actually a property of the underlying oriented edge e, and we will sometimes write $|e|$ for $|d|$.

It follows from [KV93, Lemma 4.7 and Propositions 6.4, 6.5] that if $\Phi = \{\alpha_1, \ldots, \alpha_I\}$ is an ideal forest in a reduced marked graph of groups τ and $\{e_1, \ldots, e_I\}$ is a collection of edges in τ which form a collapsible forest in $\tau^{\alpha_1, \ldots, \alpha_I}$, after possibly reordering and reversing orientation of the e_i, we have directions $d_i \in D(\alpha_i)$ with underlying oriented edges e_i such that collapsing

the edges e_i yields a marked graph of groups $\tau^{\alpha_1,\ldots,\alpha_I}_{e_1,\ldots,e_I}$ such that

$$\|\tau^{\alpha_1,\ldots,\alpha_I}_{e_1,\ldots,e_I}\| = \|\tau\| + \sum_{i=1}^{I} |\alpha_i| - \sum_{i=1}^{I} |e_i|. \qquad (\star)$$

A Whitehead move (α, e) is *reductive* if $\|\tau^\alpha_e\| \leq \|\tau\|$ and *strictly reductive* if the inequality is strict. By Equation ($\star$), reductivity is equivalent to the condition that $|\alpha| \leq |e|$. An ideal edge α is reductive if every Whitehead move supported by that edge is reductive. The proof of contractibility in [CV86, KV93] goes by finding reductive ideal edges compatible with a given collection of ideal edges. In our case, we want to *avoid* reductive ideal edges if possible.

If we have $F = A_1 * \cdots * A_n * F_k$, let $s_1, \ldots, s_k$ be a free basis for F_k, and suppose W contains the elements

$$\{a_i a_j : a_i \in A_i - \{1\}, \ a_j \in A_j - \{1\}, \ i < j\}$$

assuming $n \geq 2$ along with the elements

$$\{s_i a_j, \ a_j s_i : a_j \in A_j\} \quad \text{and} \quad \{s_i s_j, \ s_i s_j^{-1} : i \neq j\}.$$

Note that W therefore contains the s_i. Throughout this paper, we assume that $\mathrm{Out}(F)$ is infinite, i.e. that $F \neq F_1$, A_1, $A_1 * F_1$ or $A_1 * A_2$.

Lemma 1.1 (cf. Lemma 2.2 of [Vog95]). *For W as above, the absolute value of each edge and each ideal edge in a reduced marked graph of groups τ is nonzero, and the star graph of τ with respect to W has exactly one connected component for each vertex of τ. Moreover, if $n \geq 2$, for each vertex v, there exists an oriented edge e with initial vertex v such that there exists an edge in the star graph between every pair of directions in D_v with the same underlying edge e.*

Proof. Let $\tau = (\mathcal{G}, \sigma)$ be our reduced marked graph of groups. The main idea of the proof is [Ser03, 6.5 Corollary 2], which states that if each $w \in W$ is elliptic in (the action of F on the Bass–Serre tree of) some graph of groups with fundamental group identified with F, then actually all of F is elliptic. We produce such splittings by possibly blowing up τ (although strictly speaking we do not always blow up *ideal edges*) and then collapsing all but one edge of the resulting graph of groups. As long as the blown-up graph of groups is minimal, we have a contradiction.

Let us apply this principle. If $|e| = 0$, then each $\sigma_\sharp(w)$ for $w \in W$ does not cross the edge e. Collapse the other edges of $\mathcal{G}$ to obtain a *free splitting* of F, that is, a graph of groups with trivial edge groups and fundamental group identified with F, in which F is elliptic. Since $\mathcal{G}$ was minimal, this is a contradiction, so we conclude that $|e| \neq 0$. If α is an ideal edge of τ with $|\alpha| = 0$, then thinking of α as an edge of τ^α, we can make the same argument to reach a contradiction. Therefore $|\alpha| \neq 0$.

Next we prove that the star graph of τ with respect to W has one component for each vertex v of τ. Write Γ_v for the induced subgraph of the star graph comprising all those edges between directions in D_v. If v has valence one in $\mathcal{G}$, then $n \geq 2$, and letting $\mathcal{G}_v \cong A_i$, the existence of the words $a_i a_j$ (or their inverses) in W implies Γ_v is connected.

So suppose v has valence at least two. If every component of Γ_v has one vertex, then in particular some edge of $\mathcal{G}$ satisfies $|e| = 0$, which we already saw was impossible. So we may suppose some component C of Γ_v has at least two vertices. Since $\mathrm{Out}(F)$ is infinite, assuming Γ_v is disconnected, $\Gamma_v \setminus C$ has at least two vertices. To see this, observe that if $\Gamma_v \setminus C$ is a single direction d, then $|d| = 0$, which implies that the $\mathcal{G}_v$-orbit of d cannot be contained in C, or, if $\mathcal{G}_v$ is trivial, that $|\bar{d}| = 0$, so $\bar{d}$ cannot be contained in C if $\mathcal{G}_v$ is trivial. In other words,

8

if $\Gamma_v \setminus C$ contains exactly one vertex, then it actually contains at least two. Write $\mathcal{G}_C$ for the stabilizer of C in the action of $\mathcal{G}_v$ on Γ_v. Although C does not form an ideal edge in general, we may still apply the procedure in the paragraph "Blowing up ideal edges" to produce a new graph of groups $\tau^C = (\mathcal{G}^C, \sigma^C)$ with one more edge, which we denote as C. The edge C has edge group $\mathcal{G}_C$. The only way $\mathcal{G}^C$ could fail to be minimal is if $\mathcal{G}_C = \mathcal{G}_v$ and either v or the new vertex v_C has valence one in $\mathcal{G}^C$. But together this would imply that either C is empty or all of Γ_v. Collapsing all the edges of $\mathcal{G}^C$ other than C produces a splitting (not necessarily free) of F in which each $w \in W$ is elliptic, a contradiction. Therefore we conclude that Γ_v is connected in this case.

For the final statement, observe that topologically the edge path representing $a_i a_j$ is like a rubber band held at two points: the map factors through a two-to-one branched cover of the circle over the interval. The branch points correspond to turns between a pair of directions with the same underlying edge. As a_i and a_j vary, every vertex of τ (which has nontrivial vertex group because τ is reduced) appears as the image of one of these branch points. In fact, fixing a_j but allowing a_i to vary in A_i, if the branch point corresponding to a_i yields a turn between a pair of directions with underlying oriented edge e, we see that *every* pair of directions with the same underlying oriented edge equal to e are connected by an edge in the star graph. $\square$

For the remainder of the paper, we fix W such that balls are compact and such that the hypotheses and conclusions of the previous lemma hold.

2 Connectivity at Infinity

We fix our free product $F = A_1 * \cdots * A_n * F_k$. Recall from the introduction our dimension formula

$$\dim(L(F)) = \begin{cases} 2k + n - 2 & n \geq 2 \\ \max\{2k + n - 3, 0\} & \text{otherwise.} \end{cases}$$

Thus it has dimension zero (i.e. is a single point) for $(n, k) = (0, 1), (1, 0), (1, 1)$, and $(2, 0)$. It follows that $\text{Out}(F)$ has zero ends (i.e. is finite) in these cases.

The complex L has dimension one, i.e. is a tree, for $(n, k) = (3, 0)$ and $(0, 2)$. Since it is easy to see that this tree is "bushy", i.e. is not quasi-isometric to the line, it follows that these groups, which have more than one end, actually have infinitely many. In [Lym22a], we showed that additionally the group $\text{Out}(A_1 * A_2 * \mathbb{Z})$ has infinitely many ends, even though for this choice of (n, k), the complex L has dimension two.

The goal of this section is to show that in all the remaining cases, the complex $L(F)$ is connected at infinity, hence the groups $\text{Out}(F)$ are one ended.

Let B_k denote the ball of radius k. Since we have chosen W such that balls of radius k are compact, it is the union of finitely many stars of reduced marked graphs of groups, where the *star* of a reduced marked graph of groups τ is the subcomplex spanned by the set of all τ' collapsing onto τ. Since L is locally finite and the star of τ is finite, there are only a finite number of reduced marked graphs of groups whose stars have nonempty intersection with B_k. Let C_k denote the union of all of these reduced marked graphs of groups, and let

$$N(k) = \max\{\|\tau\| : \tau \in C_k\}.$$

From here on out, we will say *briar patch* for *reduced marked graph of groups*. To show connectivity at infinity, we will show that any two briar patches with norm at least $N(k)$ can be connected by a path that lies outside of B_k.

To do this, we start with a path between our two briar patches of a given form, and then push it outside of B_k by a homotopy. We say that a path $P = (\gamma_0, \gamma_1, \ldots, \gamma_\ell)$ is *standard* if it has even length $\ell = 2m$, if γ_{2i} is a briar patch for $0 \leq i \leq m$, and if $(\gamma_{2i}, \gamma_{2i+1}, \gamma_{2i+2})$ is an elementary Whitehead path for each i satisfying $0 \leq i \leq m-1$. Since $\mathrm{Out}(F)$ is generated by Whitehead automorphisms in the sense of [CZ84] together with the finite group of *factor automorphisms* that fix a certain briar patch, given any two briar patches τ and τ', there is a standard path from τ to τ'.

We will push P toward infinity by eliminating local minima. Consider a briar patch $\tau = \gamma_{2i}$ of minimal norm along P, say $\|\tau\| = m$. The briar patches γ_{2i+2} and γ_{2i-2} are obtained from τ by elementary Whitehead moves (α, e) and (β, f) respectively. By choosing τ carefully, we may assume that $\|\tau_e^\alpha\| > m$, i.e. that $|\alpha| > |e|$ and that $\|\tau_f^\beta\| \geq m$, so that $|\beta| \geq |f|$. We replace the subpath $(\tau_f^\beta, \tau^\beta, \tau, \tau^\alpha, \tau_e^\alpha)$ of P by a new standard path from τ_f^β to τ_e^α that passes through only briar patches of norm strictly greater than m.

Lemma 2.1. *Suppose* $\dim(L(F)) \geq 2$ *and* F *is not of the form* $A_1 * A_2 * \mathbb{Z}$. *Suppose* τ *is a briar patch of norm* m *and* τ_e^α *and* τ_f^β *are briar patches obtained from* τ *by elementary Whitehead moves satisfying* $\|\tau_e^\alpha\| > \|\tau\|$ *and* $\|\tau_f^\beta\| \geq \|\tau\|$. *There is a standard path from* τ_e^α *to* τ_f^β *which passes through only briar patches of norm strictly greater than* m.

The bulk of this section is given to proving Lemma 2.1. For now we use it to prove the following piece of Theorem B.

Theorem 2.2. *If* $F = A_1 * \cdots * A_n * F_k$ *is such that* $\dim(L(F)) \geq 2$ *and not of the form* $A_1 * A_2 * \mathbb{Z}$, *then* $L(F)$ *is connected at infinity.*

Proof. In the situation of the statement, suppose ρ and ρ' are briar patches of norm at least $N(k)$, and let P be a standard path from ρ to ρ'. If P does not already lie outside B_k, then there is a briar patch τ of norm $m \leq k$ along P. By assumption on ρ and ρ', we may choose τ such that the adjacent briar patches along P, call them τ_e^α and τ_f^β, which we have already observed are obtained from τ by elementary Whitehead moves, satisfy the assumptions of Lemma 2.1. In fact, we may choose τ to have minimum norm along P and satisfying these assumptions. We may therefore replace the subpath $(\tau_f^\beta, \tau^b, \tau, \tau^\alpha, \tau_e^\alpha)$ by a new standard path from τ_f^β to τ_e^α with only briar patches of norm strictly greater than τ in its interior.

We may repeat this process. At each stage, our new standard path has fewer briar patches of minimum norm m, and since $N(k) > k$, we may even increase this minimum until it is bigger than k, at which point after finitely many repetitions of this process, we have a standard path from ρ to ρ' that avoids B_k, as desired. $\qquad\square$

The process in Lemma 2.1 is the opposite of Collins and Zieschang's "Peak Reduction". Like in [CZ84], the easy case is when the ideal edges α and β are compatible.

Lemma 2.3. *Suppose* α *and* β *are compatible ideal edges of the briar patch* τ, *and suppose* (α, e) *and* (β, f) *are Whitehead moves such that* (α, e) *strictly increases and* (β, f) *does not decrease norm. There is a standard path from* τ_e^α *to* τ_f^β *with at most one briar patch in its interior whose norm is strictly greater than the norm of* τ.

Proof. We follow [Vog95, Lemma 3.1]. Consider $(\mathcal{G}, \sigma) = \tau^{\alpha,\beta}$. We think of α and β as edges of $\mathcal{G}$. Our standard path lies in the link of $(\mathcal{G}, \sigma)$ in L and we describe it by listing the edges of $\mathcal{G}$ which must be collapsed. So, for example, we have $\tau_e^\alpha = \{\beta, e\}$.

If $\{e, f\}$ is a (collapsible) forest in $\mathcal{G}$, we may take our path to be

$$(\{\beta, e\}, \{e\}, \{e, f\}, \{f\}, \{\alpha, f\}).$$

10

If $f \neq e$, the middle vertex of this path is $\tau_{e,f}^{\alpha,\beta}$. Since $|\alpha| > |e|$ and $|\beta| \geq |f|$, Equation ($\star$) shows that this briar patch has norm greater than $\|\tau\|$. If instead $e = f$, this path degenerates to the elementary Whitehead path $(\tau_e^\alpha, \tau_e^{\alpha,\beta}, \tau_e^\beta)$.

Suppose now that $\{e, f\}$ is not a forest in $\mathcal{G}$. One case where this happens is when $\alpha = \beta$. In this case both e and f belong to $D(\alpha)$, so we may take the degenerate path $(\tau_e^\alpha, \tau^\alpha, \tau_f^\alpha)$.

Suppose instead that $\alpha \neq \beta$. Since e and f are individually collapsible in $\mathcal{G}$, they do not form loops and have at least one endpoint with trivial vertex group. If $\{e, f\}$ is not a forest, the subgraph of groups spanned by $\{e, f\}$ must be connected and have fundamental group of the form $A_1 * A_2$, $\mathbb{Z}$ or $A_1 * \mathbb{Z}$. In other words, the subgraph is either a loop on two edges or a path on two edges, and at least one vertex common to both edges has trivial vertex group. In fact, because α and β are distinct, the subgraph must contain both v_α and v_β, both of which have trivial vertex group. It follows that the subgraph is the loop on two edges with fundamental group $\mathbb{Z}$. We see that $\bar{e} \in D(\beta)$ and $\bar{f} \in D(\alpha)$. In other words, $\{\alpha, e\}$ and $\{\beta, f\}$ are forests. If $|e| \geq |f|$, then we may consider the path

$$(\{\beta, e\}\{\beta\}\{\beta, f\}, \{f\}, \{\alpha, f\}).$$

The briar patch τ_f^α corresponding to $\{\beta, f\}$ has norm

$$\|\tau_f^\alpha\| \geq \|\tau_e^\alpha\| > \|\tau\|.$$

if $|f| > |e|$, then we may consider instead the path $(\{\beta, e\}, \{e\}, \{\alpha, e\}, \{\alpha\}, \{\alpha, f\})$. The briar patch τ_e^β corresponding to $\{\alpha, e\}$ has norm

$$\|\tau_e^\beta\| > \|\tau_f^\beta\| \geq \|\tau\|. \qquad \square$$

As in [Vog95], the proof of Lemma 2.3 gives a simplicial map of a subdivided square into L which maps the center to $\tau^{\alpha,\beta}$ and the corners to τ, τ_e^α, τ_f^β and a briar patch $\tau_T^{\alpha,\beta}$ for some maximal forest T in $\tau^{\alpha,\beta}$. In the case where $\alpha \neq \beta$ and $e \neq f$, then $\tau_T^{\alpha,\beta}$ is a briar patch with norm strictly greate than τ. The square, see Figure 1, provides a homotopy rel endpoints between the old path and the new one.

Now, in general, we may not assume that α and β are compatible. The strategy for proving Lemma 2.1 is to find a sequence of ideal edges $\alpha = \gamma_0, \gamma_1, \ldots, \gamma_n = \beta$ such that each γ_i is compatible with its neighbors and supports a strictly increasing Whitehead move (γ_i, e_i). By Lemma 2.3, we obtain a standard path from $\tau_{e_i}^{\gamma_i}$ to $\tau_{e_{i+1}}^{\gamma_{i+1}}$, and the concatenation of all of these standard paths will be the standard path we seek. We will describe the outline of the argument in a moment.

Firstly, the *size* of an ideal edge is its cardinality, while the *size* of a Whitehead move is the size of its ideal edge. (Let us remark that in the case where $n = 0$ and an ideal edge α is equivalent to $D_v - \alpha$, we should take the size of α to mean the size of the smaller of the two.) Next, let the *edge number* of F, denoted $\text{en}(F)$, be the quantity $2k + n - 1$. Assuming that $n \geq 1$, if τ is a briar patch in $L(F)$ with one vertex $\star$ with valence greater than one, then the valence of $\star$ is $\text{en}(F)$. Finally, say that a vertex of τ is *active* if it has valence at least two. Active vertices support ideal edges.

In the course of our proof of Lemma 2.1, each ideal edge γ_i aside from α and β will have size two or size three. Our first step is to reduce to the case where τ has one active vertex.

Lemma 2.4. *Suppose τ has more than one active vertex and that the given ideal edges α and β are not already compatible. Then α and β are compatible with a size-two ideal edge γ (based at a different vertex) which strictly increases norm.*
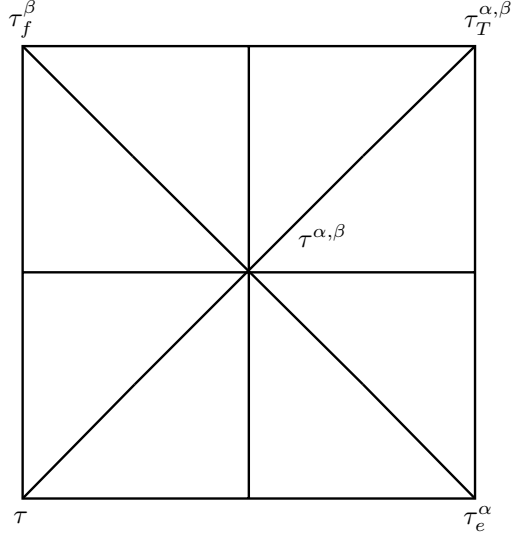
Figure 1: The subdivided square in the proof of Lemma 2.3.

Proof. This is a quick corollary of the fact (Lemma 2.8 below) that when $n \geq 2$, which is necessary for τ to have more than one active vertex, each active vertex supports a strictly increasing Whitehead move of size two, and that ideal edges based at different vertices are compatible. □

By Lemma 2.3, Lemma 2.1 follows in this case.

Supposing then instead that τ has only one active vertex, our first step is the following lemma.

Lemma 2.5. *If τ is a briar patch supporting the Whitehead move (α, e) such that $|\alpha| \geq |e|$, then either α has size two, or α is compatible with a size-two ideal edge which supports a strictly increasing Whitehead move.*

Proof of Lemma 2.5. Note that if an ideal edge α does not have size two, then it contains a *trio* (to be defined below, see the discussion before Lemma 2.9) unless its underlying edges are of the form $\{e, f, \bar{f}\}$. If it contains a trio, then α is compatible with a size-two ideal edge supporting a strictly increasing Whitehead move by Lemma 2.9, and if not the same holds by Lemma 2.10. □

Lemma 2.5 allows us to restrict our attention to the more well-behaved class of size-two Whitehead moves.

Here is our main step. Consider the simplicially subdivided polygons in Figure 2. Further subdivide each one by introducing a new vertex at the midpoint of each edge in the boundary of the polygon. Suppose (α, e) and (β, f) are Whitehead moves of size two based at τ such that $|\alpha| \geq |e|$ and $|\beta| > |f|$. A *good polygon from α to β* in the star of τ is a simplicial map of the further subdivided polygon into L with the following properties.

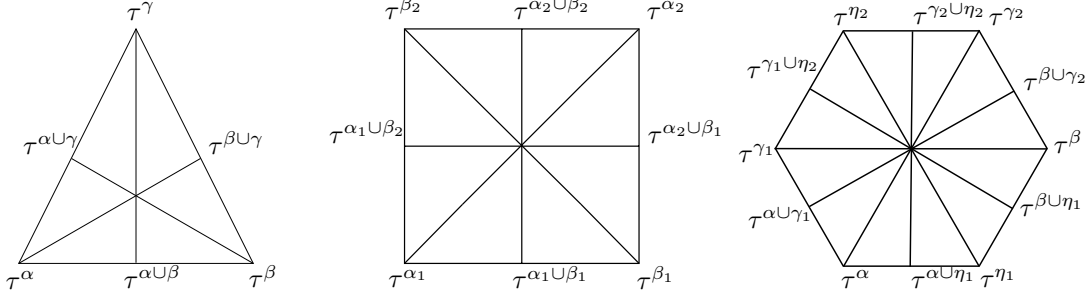1. The center of the polygon is mapped to τ.

Figure 2: The simplicially subdivided polygons in the proof of Theorem 2.2.

2. Each vertex of the polygon (three for the triangle, four for the rectangle, six for the hexagon) is mapped to a marked graph of groups τ^σ obtained from τ by blowing up an ideal edge σ of size two.

3. The marked graphs of groups τ^α and τ^β are mapped to vertices of the polygon.

4. Each midpoint of an edge (three for the triangle, four for the rectangle, six for the hexagon) is mapped to a marked graph of groups τ^σ obtained from τ by blowing up an ideal edge of size three.

5. Each size-three ideal edge σ is compatible with the size-two ideal edges corresponding to the endpoints of the edge which σ corresponds to the midpoint of.

6. At most one size-three ideal edge corresponding to a vertex of the polygon is reductive.

7. At most one size-two ideal edge (specifically α) corresponding to a vertex of the polygon is reductive.

Lemma 2.6. *Suppose τ has one active vertex supporting Whitehead moves (α, e) and (β, f) of size two such that $|\alpha| \geq |e|$ and $|\beta| > |f|$. Assume that $n \geq 1$, $\mathrm{en}(F) \geq 3$, and that $F \neq A_1 * A_2 * \mathbb{Z}$, and that α and β are not compatible. Either (after rechoosing α and β in their equivalence class) we have that $\alpha \cup \beta$ is an ideal edge which is not reductive, or there is a good polygon in the star of τ from α to β.*

With this lemma in hand, we complete the proof of Lemma 2.1.

Proof of Lemma 2.1. We begin with our given Whitehead moves (α, e) and (β, f) as in the statement. By Lemma 2.4 and Lemma 2.3, if τ has more than one active vertex, then there is a standard path from τ_e^α to τ_f^β with only briar patches of norm strictly greater than $\|\tau\|$ in its interior.

If instead τ has only one active vertex, then by Lemma 2.5, we may replace α and β by ideal edges of size two, call them α' and β' that support Whitehead moves (α', e') and (β', f') satisfying $|\alpha'| \geq |e'|$ and $|\beta'| > |f|$. Then by Lemma 2.6, there is a good polygon in the star of τ. Since there is at most one vertex of the polygon (aside from the vertex τ^α) corresponding to a reductive ideal edge σ, a path around the polygon from $\tau^{\alpha'}$ to $\tau^{\beta'}$ that avoids that vertex τ^σ gives a sequence of ideal edges $\alpha' = \gamma_1, \ldots, \gamma_m = \beta'$ such that each γ_i is compatible with its neighbors and supports a strictly increasing Whitehead move. By Lemma 2.3, we are done. $\square$

The remainder of this section is given to proving the statements left unproved above: Lemma 2.8, needed in the proof of Lemma 2.4, Lemma 2.5 and Lemma 2.6.

Reduction to one active vertex. We turn first to Lemma 2.8, for which our main tool is the following lemma, due to Vogtmann. We give a proof for completeness.

Lemma 2.7 (Lemma 3.2 of [Vog95])**.** *Let u, v and w be vertices of a simplicial graph (for example the star graph of τ) and S the set of other vertices. If $|\{u, v\}| \leq |v|$ and $|\{u, w\}| \leq |w|$, then $\{u\} \cdot \{v\} = \{u\} \cdot \{w\}$ and $\{u\} \cdot S = 0$.*

Proof. It is clear that if two sets S and T of vertices of a graph are disjoint, then for any set A of vertices, we have
$$A \cdot (S \cup T) = A \cdot S + A \cdot T.$$
Writing S for the set of other vertices as in the statement, we have
$$|\{u, v\}| = \{u\} \cdot S + \{u\} \cdot \{w\} + \{v\} \cdot \{w\} + \{v\} \cdot S$$
and
$$|v| = \{v\} \cdot S + \{v\} \cdot \{u\} + \{v\} \cdot \{w\}.$$
Our first inequality implies that
$$\{u\} \cdot S + \{u\} \cdot \{w\} \leq \{u\} \cdot \{v\}.$$
Calculating with w in place of v, we see that
$$\{u\} \cdot S + \{u\} \cdot \{v\} \leq \{u\} \cdot \{w\}.$$
Therefore we conclude that $\{u\} \cdot S = 0$ and $\{u\} \cdot \{v\} = \{u\} \cdot \{w\}$. $\qquad\square$

Using Lemma 2.7, we prove Lemma 2.8.

Lemma 2.8. *Let τ be a briar patch and suppose that $n \geq 2$. Let v be a vertex of τ of valence at least two. There is a strictly increasing Whitehead move of size two based at v.*

Proof. Since $n \geq 2$, by Lemma 1.1, the star graph of τ contains an edge betwen a pair of directions d and d' in D_v with the same underlying oriented edge e. Since v has valence at least two, there are a pair of ideal edges $\alpha = \{d, c\}$ and $\alpha' = \{d, c'\}$ where c and c' are distinct and have underlying oriented edges different from e and $\bar{e}$. If both of the Whitehead moves (α, c) and (α', c') are reductive, then by Lemma 2.7, we have $d \cdot S = 0$, where S denotes the set of remaining vertices in the star graph aside from d, c and c'. This contradicts the existence of the edge from d to d'. $\qquad\square$

Finding increasing Whitehead moves of size two. We will now turn to the proof of Lemma 2.5. Following Vogtmann, define a *trio* to be a set $\{e, f, g\}$ of three oriented edges with common initial vertex $\star$ of a briar patch τ which does not contain an edge twice, once in each orientation. A trio has the property that $(\{d, d'\}, d)$ is an elementary Whitehead move for any choice of directions d and d' representing edges of the trio. We say the trio *supports* these Whitehead moves.

Vogtmann proves the following lemma using Lemma 2.7.

Lemma 2.9 (Lemma 3.3 of [Vog95])**.** *For every choice of three directions $\{u, v, w\}$ representing the oriented edges of the trio $\{e, f, g\}$, at least one of the three size-two ideal edges $\{u, v\}$, $\{u, w\}$ and $\{v, w\}$ supports a strictly increasing Whitehead move.*

Proof. Suppose all Whitehead moves supported by the ideal edges in the statement are reductive. Repeatedly applying Lemma 2.7, we see that the directions u, v and w form an isolated component of the star graph. This contradictis Lemma 1.1. $\square$

Lemma 2.10. *Suppose (α, e) is a Whitehead move satisfying $|\alpha| \geq |e|$. If the underlying oriented edges of α are of the form $\{e, f, \bar{f}\}$, and we write $\alpha = \{u, v, \bar{v}\}$, then one of the size-two Whitehead moves $\{u, v\}$ or $\{u, \bar{v}\}$ supports a strictly increasing Whitehead move.*

Proof. Suppose towards a contradiction that the ideal edges $\{u, v\}$ and $\{u, \bar{v}\}$ are reductive. Applying Lemma 2.7, writing S for the set of directions not equal to u, v or $\bar{v}$, we see that $\{u\} \cdot S = 0$ and $\{u\} \cdot \{v\} = \{u\} \cdot \{\bar{v}\}$. Since

$$|v| = \{v\} \cdot \{u\} + \{v\} \cdot \{\bar{v}\} + \{v\} \cdot S = |\bar{v}|,$$

the latter equality implies that $\{v\} \cdot S = \{\bar{v}\} \cdot S$. We have by assumption $|\{u, v, \bar{v}\}| \geq |u|$, which implies that $\{v\} \cdot S \geq \{u\} \cdot \{v\}$. Now,

$$|\{u, v\}| = \{u\} \cdot \{\bar{v}\} + \{v\} \cdot S \quad \text{and} \quad |\{u\}| = \{u\} \cdot \{v\} + \{u\} \cdot \{\bar{v}\},$$

so since we assume $\{u, v\}$ is reductive, we conclude that $\{v\} \cdot \{\bar{v}\} = 0$ and that $\{v\} \cdot S = \{u\} \cdot \{v\}$. In particular $|v| = |u|$.

Now, either e, the underlying oriented edge of u, forms a loop, or it does not. If it does, because $\{u\} \cdot S = 0$, by $\mathcal{G}_\star$-equivariance of the star graph, every occurence of e (in either orientation) in each $w \in W$ is followed or preceded by an occurence of f, the underlying oriented edge of v. Since $|e| = |f|$, each word has an equal number of occurrences of e and f. Therefore, if we project to the subspace of $H_1(F, \mathbb{Z}/2\mathbb{Z})$ spanned by e and f, we see that each word $w \in W$ has image $(0,0)$ or $(1,1)$. Since these words do not form a basis for $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, this contradicts our assumption that the words $w \in W$ satisfy the hypotheses of Lemma 1.1.

Supposing instead that e does not form a loop, then every $w \in W$ contains an even number of occurrences of e and therefore an even number of occurences of f, which is again a contradiction mod 2. $\square$

We need the following elementary observation.

Lemma 2.11. *Suppose S is a nonempty subset of the vertices of a simplicial graph containing the vertex x. We have that $|S| \leq |x|$ if and only if we have that*

$$\{x\} \cdot T \geq \frac{1}{2}|T|,$$

where T is the set $S - \{x\}$. In particular equality holds in one equation if and only if it holds in the other.

Proof. Write T^c for the complement of T. On the one hand, we have $|x| = \{x\} \cdot T + \{x\} \cdot T^c$, and on the other, we have
$$|S| = \{x\} \cdot T^c + T \cdot (T^c - \{x\}).$$
A bit of algebra finishes the proof. $\square$

Corollary 2.12. *Suppose T is a nonempty subset of the vertices of a simplicial graph. There are at most two choices of a vertex $x \notin T$ such that $|T \cup \{x\}| \leq |x|$. If there are two choices, say x and x', then we have $|T \cup \{x\}| = |x|$ and $|T \cup \{x'\}| = |x'|$ and in fact elements of T are only connected to each other, x and x'.*

15

Proof. By Lemma 2.11, We have that $|T \cup \{x\}| \leq |x|$ if and only if $\{x\} \cdot T \geq \frac{1}{2}|T|$. If this holds for another $x' \notin T$, notice that we then have

$$|T| \leq \{x\} \cdot T + \{x'\} \cdot T \leq |T|.$$

Therefore equality holds throughout and the statement follows. □

For size-two ideal edges, we have the following strengthening.

Corollary 2.13. *Suppose u and v are directions based at a vertex $\star$ in a briar patch τ with valence at least three. Suppose that $\star$ has nontrivial vertex group $\mathcal{G}_\star$, and that $\{u, v\}$ is an ideal edge. There is at most one $h \in \mathcal{G}_\star$ such that $\{u, h.v\} \sim \{h^{-1}.u, v\}$ is reductive.*

Proof. By Corollary 2.12 applied to $T = \{u\}$, we see that there are at most two $h \in \mathcal{G}_\star$ such that $\{u, h.v\}$ is reductive. Supposing that there are two, we may without loss of generality suppose that one is v and the other is $h.v$. Corollary 2.12 allows us to conclude that u is connected only to v and $h.v$ in the star graph. But applying Corollary 2.12 to $T = \{v\}$, we see that v is only connected to u and $h^{-1}.u$. It follows that the $\mathcal{G}_\star$-orbits of u and v together form an isolated component of the star graph. This contradicts Lemma 1.1, since $\star$ has valence at least three. □

We also need the following elaboration of Corollary 2.12.

Lemma 2.14. *Suppose u, v, w, x, y are five distinct vertices of a simplicial graph. Suppose that*

$$|\{u, v, w\}| \leq \min\{|u|, |v|, |w|\}, \quad \text{that} \quad |\{u, v\}| \geq \min\{|u|, |v|\} \quad \text{and} \quad |\{u, w\}| \geq \min\{|u|, |w|\}|,$$

and that one of the above inequalities involving $\{u, v\}$ and $\{u, w\}$ is strict. If $|\{u, v, x\}| \leq |x|$, then $|\{u, w, y\}| > |y|$.

Proof. By two applications of Corollary 2.12, if there exists y such that $|\{u, w, y\}| \leq |y|$, we have $|w| = |\{u, v, w\}| = |v|$, and that u is connected only to v and w; that is, we have that $|u| = \{u\} \cdot \{v, w\}$. But notice that the first chain of equalities together with the inequalities in the statement implies that $|v| = |w| \leq |u|$. So the inequalities in the statement imply by Lemma 2.11 that $\{u\} \cdot \{v\} \leq \frac{1}{2}|u|$ and that $\{u\} \cdot \{w\} \leq \frac{1}{2}|u|$, and that one of the inequalities is strict. This is a contradiction. □

Finding good polygons: the proof of Lemma 2.6. For the remainder of this section, we turn to the proof of Lemma 2.6. Assuming (α, e) and (β, f) are size-two Whitehead moves satisfying $|\alpha| \geq |e|$ and $|\beta| > |f|$ which are not compatible, there are three cases based on the configurations of α and β. We briefly describe the cases.

1. **Case one: $\alpha = \{u, v\}$ and $\beta = \{u, w\}$.** In our first case, the directions $\{u, v, w\}$ represent a trio; that is, they are drawn from three distinct *unoriented* edges.

2. **Case two: $\alpha = \{u, v\}$ and $\beta = \{u, \bar{v}\}$.** In our second case, we have that $\{u, v, \bar{v}\}$ is an ideal edge, but the three directions u, v and $\bar{v}$ do not form a trio because v and $\bar{v}$ share an underlying *unoriented* edge.

3. **Case three: $\alpha = \{u, v\}$ and $\beta = \{u, v'\}$,** where $v' = g.v$ for some nontrivial $g \in \mathcal{G}_\star$. In our third case, the set of directions $\{u, v, v'\}$ is not an ideal edge.

Case one. In this case $\alpha \cup \beta = \{u, v, w\}$ is an ideal edge. If it is not reductive, we are done.

Supposing $\alpha \cup \beta$ is reductive, the good polygon we will build is the triangle in Figure 2 left. The ideal edge $\gamma = \gamma(g)$ will be of the form $\{v, g.w\} \sim \{g^{-1}.v, w\}$ for some nontrivial element $g \in \mathcal{G}_\star$. Recall that in view of Vogtmann [Vog95], we assume that $n \geq 1$. We have the ideal edges $\alpha \cup \gamma(g) = \{u, v, g.w\}$ and $\beta \cup \gamma(g) = \{u, g^{-1}.v, w\}$. By Lemma 2.14, at most one of the size-three ideal edges $\alpha \cup \gamma(g)$ and $\beta \cup \gamma(g)$ is reductive as g varies over the nontrivial elements of $\mathcal{G}_\star$.

First, observe that *no* ideal edge of the form $\gamma(g)$ is reductive when $g \in \mathcal{G}_\star$ is nontrivial. Indeed, if some $\gamma(g)$ were reductive, by Lemma 2.11, we have $\{v\} \cdot \{g.w\} \geq \frac{1}{2}|v|$. Notice that

$$|v| \leq \{v\} \cdot \{g.w\} + \{g^{-1}.v\} \cdot \{w\} \leq |\{u, v, w\}| \leq |v|,$$

so we conclude that equality holds throughout. In fact, by replacing v with w (and viewing $\gamma(g)$ as $\{g^{-1}.v, w\}$), we conclude that $|v| = |w| = |\{u, v, w\}| \leq |u|$, and in particular we have $|u| = \{u\} \cdot \{v, w\}$. As in the proof of Lemma 2.14, this is a contradiction: since at least one of α and β is not reductive, we have by Lemma 2.11 that

$$\{u\} \cdot \{v\} + \{u\} \cdot \{w\} < |u|.$$

Next we show that in fact no ideal edge of the form $\alpha \cup \gamma(g)$ or $\beta \cup \gamma(g)$ is reductive. Recall that by Lemma 2.14, since we assume that $\alpha \cup \beta$ is reductive, there is at most one such edge, suppose it is $\alpha \cup \gamma(g)$. The argument in the case it is of the form $\beta \cup \gamma(g)$ will be identical. By Lemma 2.11, we have that $\frac{1}{2}|\alpha| = \frac{1}{2}|\{u, v\}| = \{w\} \cdot \{u, v\} = \{g.w\} \cdot \{u, v\}$. We will show that $|\alpha \cup \gamma(g)| > |\alpha|$, contradicting our assumption that this ideal edge is reductive. Indeed, we claim that

$$|\{u, v, g.w\}| > 2\{u, v\} \cdot \{w\} = |\alpha|.$$

Indeed, observe that $\{g.w\} \cdot \{g.u, g.v\} = \{w\} \cdot \{u, v\}$ by $\mathcal{G}_\star$-equivariance of the star graph, so it is clear that we have inequality. That the inequality is *strict* follows in the case where $\mathrm{en}(F) > 3$ by Lemma 1.1, since the $\mathcal{G}_\star$-orbits of the directions u, v and w cannot form an isolated component of the star graph. If instead $\mathrm{en}(F) = 3$, so that $n \geq 2$, then again by Lemma 1.1, we must have $\{w\} \cdot \{g.w\} > 0$.

Thus *any* choice of $g \neq 1$ in $\mathcal{G}_\star$ yields a good polygon from α to β.

Case two. Recall that in this case we assume that $\alpha = \{u, v\}$ and $\beta = \{u, \bar{v}\}$, i.e. that $\alpha \cup \beta$ is an ideal edge, but v and $\bar{v}$ share an underlying *unoriented* edge. If $\alpha \cup \beta$ is not reductive, we are done. Supposing this ideal edge is reductive, the good polygon from α to β we will build will be the rectangle in Figure 2 center, where we set $\alpha_1 = \alpha$ and $\beta_1 = \beta$.

To construct this good polygon, we need size-two ideal edges α_2 and β_2 so that $\alpha_1 \cup \beta_2$, $\alpha_2 \cup \beta_1$ and $\alpha_2 \cup \beta_2$ are size-three ideal edges (where we may abuse notation slightly so that these unions may be taken) and so that none of these five ideal edges (including α_2 and β_2) are reductive.

Since we assume that $F \neq A_1 * A_2 * \mathbb{Z}$, there exists a direction w whose $\mathcal{G}_\star$-orbit is disjoint from $\{u, v, \bar{v}\}$; we choose one such. For $h \in \mathcal{G}_\star$, we focus on the ideal edges $\alpha(h) = \{u, h.v\}$, $\beta(h) = \{u, h.\bar{v}\}$, $\gamma(h) = \{\bar{v}, h.w\}$ and $\delta(h) = \{v, h.w\}$. We have $\alpha = \alpha(1)$ and $\beta = \beta(1)$ and will choose α_2 from $\{\alpha(g), \gamma(1), \gamma(g)\}$ and β_2 from $\{\beta(g), \delta(1), \delta(g)\}$ for $g \in \mathcal{G}_\star$ nontrivial. Observe that by abusing notation slightly, for any choice of α_2 and β_2, the size-three ideal edges $\alpha \cup \beta_2$, $\alpha_2 \cup \beta$ and $\alpha_2 \cup \beta_2$ make sense to define.

By Corollary 2.13, at most one of $\gamma(1)$ and $\gamma(g)$ is reductive, and similarly for $\delta(1)$ and $\delta(g)$. Since we assume that $\{u, v, \bar{v}\}$ is reductive, neither $\alpha(g)$ nor $\beta(g)$ can be reductive: if,

17

say, $|\alpha(g)| \leq |v|$, we have that $\{u\} \cdot \{g.v\} \geq \frac{1}{2}|u|$ by Lemma 2.11, but this would that

$$|u| \leq \{u\} \cdot \{g.v\} + \{v\} \cdot \{g^{-1}.u\} < |\{u, v, \bar{v}\}|,$$

contradicting our assumption that this ideal edge is reductive.

Notice that if $\gamma(1)$ is reductive, we have in particular that $\{\bar{v}\} \cdot \{w\} \geq \frac{1}{2}|w|$. Then all four of the ideal edges $\beta(1) \cup \gamma(g)$, $\beta(g) \cup \gamma(g)$, $\delta(1) \cup \gamma(g)$ and $|\delta(g) \cup \gamma(g)$ are greater in absolute value than $\{\bar{v}\} \cdot \{w\} + \{g.\bar{v}\} \cdot \{g.w\} \geq |w|$ and so are not reductive.

From this it follows that if there are two reductive size-two ideal edges under consideration, there is a good polygon from α to β. For instance, if $\gamma(1)$ and $\delta(g)$ are reductive, the relevant path around the good polygon reads

$$\alpha, \ \alpha \cup \delta(1), \ \delta(1), \ \gamma(g) \cup \delta(1), \ \gamma(g), \ \beta \cup \gamma(g), \ \beta.$$

We proceed continuing to assume that $\gamma(1)$ is reductive, but now assuming that neither $\delta(1)$ nor $\delta(g)$ is reductive. If one of $\alpha \cup \beta(g)$, $\alpha \cup \delta(1)$ or $\alpha \cup \delta(g)$ is not reductive, we may use that ideal edge together with ideas above to construct a good polygon from α to β. Explicitly, if $\alpha \cup \delta(1)$ is not reductive, the relevant path reads

$$\alpha, \ \alpha \cup \delta(1), \ \delta(1), \ \delta(1) \cup \gamma(g), \ \gamma(g), \ \beta \cup \gamma(g), \ \beta.$$

We will show that having all three ideal edges reductive leads to a contradiction. In fact, this does not use the assumption that $\gamma(1)$ is reductive. Indeed, by Corollary 2.12, we conclude assuming that $\alpha \cup \beta$ and $\alpha \cup \beta(g)$ are reductive that $|v| \leq |u|$. If we had equality, Corollary 2.12 would additionally imply that $\alpha \cup \delta(1)$ and $\alpha \cup \delta(g)$ are greater in norm than $|w|$, in contradiction to our assumption that these ideal edges are reductive. Again by Corollary 2.12, we conclude that $|w| \leq |v|$ as well. Therefore in particular by Lemma 2.11, we have $|\alpha| = \{w, g.w\} \cdot \{u, v\} \leq |w|$. Since therefore $|w| \geq |\alpha| \geq |v|$, we conclude that equality holds throughout. But it cannot, since this would imply that the $\mathcal{G}_\star$-orbits of the directions u, v and w form an isolated component of the star graph, in contradiction to Lemma 1.1.

Finally, assume that none of the ideal edges $\gamma(1)$, $\gamma(g)$, $\delta(1)$ or $\delta(g)$ are reductive. Now by Corollary 2.12, at most two of the four ideal edges $\alpha \cup \beta_2$, which take the form $\{u, v, x\}$, are less than $|x|$ in norm. If we have $|u| < |v|$, since $\alpha \cup \beta$ is reductive, this is the only ideal edge of that form which is reductive, and similarly for ideal edges of the form $\{u, \bar{v}, x\}$. It remains, then, just to show that one of the ideal edges of the form $\alpha_2 \cup \beta_2$ is not reductive. Suppose, for instance, then, that $\alpha(g) \cup \beta(g)$ is reductive. This, by Corollary 2.12, implies that $|\{v, \bar{v}\}| = \{u, g^{-1}.u\} \cdot \{v, \bar{v}\} \leq |u|$. Since $|u| < |v|$, this implies that $|\{v, \bar{v}\}| < |v|$, so that $\{v\} \cdot \{\bar{v}\} > \frac{1}{2}|v|$, and therefore $\gamma(1) \cup \delta(g)$ and $\gamma(g) \cup \delta(1)$ are both greater in norm than $|v|$, and thus are not reductive unless $|v| < |w|$. But notice that because $|u| < |v|$, reductivity of $\alpha(g) \cup \beta(g)$ implies that $\{u, g.v, g.\bar{v}\} < |g.\bar{v}| = |v|$; by Corollary 2.12 it is the only ideal edge of the form $\{u, g.v, x\}$ to be less than or equal to $|x|$ in norm. In particular $|\alpha(g) \cup \delta(1)| > |w|$ and so is not reductive. This provides a good polygon from α to β in this case.

We therefore assume that $|v| \leq |u|$. Now if $|w| < |v|$, for each of the ideal edges $\gamma(1)$, $\gamma(g)$, $\delta(1)$, and $\delta(g)$ at most of the relevant size-three ideal edges containing it can be reductive by Corollary 2.12. The only worry, then, is that both $\gamma(1) \cup \beta$ and $\gamma(g) \cup \beta$ are reductive or, symmetrically, both $\alpha \cup \delta(1)$ and $\alpha \cup \delta(g)$ are reductive. But in either case this implies that $|v| \leq |w|$: for instance in the latter case we have $|v| \leq |\{u, v\}| = \{w, g.w\} \cdot \{u, v\} \leq |w|$. This contradiction provides a good polygon from α to β in this case.

Continuing to assume that $|v| \leq |u|$, we now suppose that $|v| \leq |w|$. Since the argument in the previous case that not all three ideal edges of the form $\alpha \cup \beta_2$ are reductive did not use the assumption that $\gamma(1)$ was reductive, we conclude that at least one ideal edge of the form $\alpha \cup \beta_2$

is not reductive. Suppose at first that we may choose $\beta_2 = \delta(1)$. (By relabelling if necessary, this is equivalent to choosing $\delta(g)$.) Now by symmetry we know that at least one ideal edge of the form $\alpha_2 \cup \beta$ is not reductive, but we can show more: If both $\delta(1) \cup \beta$ and $\delta(g) \cup \beta$ are reductive, then $|v| = |w| = \{w, g.w\} \cdot \{u, \bar{v}\} = |\{u, \bar{v}\}|$, which is a contradiction to Lemma 1.1, (as well as the assumption that β is not reductive) since it implies that the $\mathcal{G}_\star$-orbits of u, $\bar{v}$ and w form an isolated component of the star graph.

Therefore either $\delta(1) \cup \beta$ or $\delta(g) \cup \beta$ is not reductive. Suppose at first the former is not reductive. If $\delta(1) \cup \gamma(1)$ is not reductive, we are done. Supposing it is, we see by Lemma 2.11 that $\{u, w\} \cdot \{v, \bar{v}\} = |\{v, \bar{v}\}|$. This implies that $|\{v, \bar{v}, g.w\}| = |w| + |\{v, \bar{v}\}|$, that $|\{g^{-1}.u, v, g.w\}| \geq |v| + |\{v, \bar{v}\}|$, that $|\{g^{-1}.u, \bar{v}, g.w\}| \geq |v| + |\{v, \bar{v}\}|$, and that $|\{g^{-1}.u, v, \bar{v}\}| = |u| + |\{v, \bar{v}\}|$ so these ideal edges are not reductive. If we can show that one of $\alpha \cup \beta(g)$ or $\alpha \cup \delta(g)$ is not reductive and that one of $\alpha(g) \cup \beta$ and $\gamma(g) \cup \beta$ is not reductive, we will have a good polygon from α to β.

Suppose, then, that $\alpha \cup \beta(g)$ and $\alpha \cup \delta(g)$ are reductive. From the former, we conclude that $\{u\} \cdot \{v\} = \{u\} \cdot \{v, g.\bar{v}\} \geq \frac{1}{2}|\{v, g.\bar{v}\}| = |v|$, so equality holds throughout. This implies that $|\bar{v}| = \{w\} \cdot \{\bar{v}\}$. From reductivity of the latter, we conclude using Lemma 2.11 that $|\{u, g.w\}| = \{u\} \cdot \{v\} + \{w\} \cdot \{\bar{v}\}$. In particular $|u| = \{u\} \cdot \{v, g.w\}$ and $|w| = \{w\} \cdot \{g^{-1}.u, \bar{v}\}$. This implies that $|\{u, g.v, \bar{v}\}| = |u| + 2|v|$ and that $|\{g^{-1}.u, v, w\}| = 3|v|$, so neither of these ideal edges are reductive, and we have a good polygon from α to β, the relevant path around which reads

$$\alpha, \ \alpha \cup \delta(1), \ \delta(1), \ \alpha(g) \cup \delta(1), \ \alpha(g) \cup \beta, \ \beta.$$

A symmetric argument dispenses with the case in which both $\alpha(g) \cup \beta$ and $\delta(g) \cup \beta$ are reductive. We have two nested assumptions that need to be dispensed with: the outer one, that $\alpha \cup \delta(1)$ is not reductive, and the inner one, that $\gamma(1) \cup \beta$ is not reductive.

So, still supposing that $\alpha \cup \delta(1)$ is not reductive, suppose that $\gamma(1) \cup \beta$ is reductive. As we noted above, this implies that $\gamma(g) \cup \beta$ is not reductive, and therefore if $\gamma(g) \cup \delta(1)$ is not reductive, we are done. We therefore suppose that it is reductive. We first show that $\gamma(g) \cup \delta(g)$ is not reductive. Indeed, if it were, we see by Corollary 2.12 that $\{u, g.w\} \cdot \{v, \bar{v}\} = |\{v, \bar{v}\}|$. Since $\gamma(g) \cup \delta(1)$ is reductive, we conclude that in fact $\{g.w\} \cdot \{\bar{v}\} = \{g.w\} \cdot \{g.v, \bar{v}\} \geq \frac{1}{2}|\{v, g^{-1}.v\}| = |\bar{v}|$, so equality holds throughout. But this contradicts our assumption that $\gamma(1) \cup \beta$ is reductive, since by Lemma 2.11 that implies that $\{\bar{v}\} \cdot \{u, w\} \geq \frac{1}{2}|\{u, w\}| > 0$.

Thus we are done if $\alpha \cup \delta(g)$ is not reductive. So suppose it is. Applying Lemma 2.11, we see that

$$|u| + |w| \leq \{u\} \cdot \{v, \bar{v}\} + \{w\} \cdot \{g^{-1}.v, \bar{v}\} + \{u\} \cdot \{w\} + \{u\} \cdot \{g.w\} \leq |u| + |v| - \{u\} \cdot \{w\} - \{g^{-1}.u\} \cdot \{w\},$$

so we have equality throughout; in particular $\{u\} \cdot \{w\} = \{g^{-1}.u\} \cdot \{w\} = 0$. But this contradicts reductivity of $\gamma(g) \cup \delta(1)$, since we must have $\{w\} \cdot \{v, g^{-1}.\bar{v}\} \geq \frac{1}{2}|\{v, g^{-1}.\bar{v}\}| > 0$. Therefore we again have a good polygon from α to β.

Finally, we supposed that we could choose $\alpha \cup \delta(1)$ to not be reductive. To negate this, in view of the possibility of relabelling, we may assume that $\alpha \cup \delta(1)$ and $\alpha \cup \delta(g)$ are both reductive. This implies that $|w| = |v|$, and thus that $\{w, g.w\} \cdot \{u, v\} = |\{u, v\}|$. As we saw, we must have that $\alpha \cup \beta(g)$ is not reductive. Since this is equal in norm to $|\{u, v\}| + |v|$, we have that these sum to more than $|u|$. By Lemma 2.11, this implies that $\{u\} \cdot \{v\} < |v|$. But reductivity of $\alpha \cup \beta$ actually implies that $\{u\} \cdot \{v\} = \{u\} \cdot \{v, \bar{v}\} \geq \frac{1}{2}|\{v, \bar{v}\}| = |v|$. This contradiction completes the proof in this case.

Case three. Recall that in this case we assume that $\alpha = \{u, v\}$ and $\beta = \{u, g.v\}$, i.e. α and β share both underlying oriented edges. In this case $\alpha \cup \beta$ fails to be an ideal edge. Since

en$(F) \geq 3$, there exists a direction w based at $\star$ whose $\mathcal{G}_\star$-orit is distinct from that of u and v; choose one.

The proof breaks into two cases: either we are able to choose w so that $\{u, v, w\}$ represents a trio, or we are not able to, in which case we may choose w to either be $\bar{u}$ or $\bar{v}$, since we assume that $F \neq A_1 * A_2 * \mathbb{Z}$.

The first subcase. Suppose at first that $\{u, v, w\}$ represents a trio. We consider the ideal edges $\gamma(1) = \{u, w\}$, $\gamma(g) = \{u, g.w\}$, $\delta(1) = \{v, w\}$ and $\delta(g) = \{v, g.w\}$. We will use these ideal edges to build a good polygon (the rectangle in Figure 2) from α to β.

Observe that we cannot have that $\alpha \cup \gamma(1)$ and $\alpha \cup \gamma(g)$ are both reductive. If we did, we conclude by Corollary 2.12 that $|\alpha| = \{w, g.w\} \cdot \{u, v\}$. An argument based on Lemma 1.1 shows that both $\alpha \cup \gamma(1)$ and $\alpha \cup \gamma(g)$ must be strictly greater in norm than this quantity, which in turn is not less than the minimum of $|u|$ and $|v|$, implying neither ideal edge is reductive after all. The same argument applies to all the pairs $\alpha \cup \delta(1)$ and $\alpha \cup \delta(g)$, $\beta \cup \gamma(1)$ and $\beta \cup \gamma(g)$ and $\beta \cup \delta(1)$ and $\beta \cup \delta(g)$.

In fact, provided that, say, $\gamma(1)$ is not reductive, the same argument above proves that $\alpha \cup \gamma(1)$ and $\beta \cup \gamma(1)$ cannot both be reductive. By Corollary 2.13, at most one of $\gamma(1)$ and $\gamma(g)$ is reductive, and similarly for $\delta(1)$ and $\delta(g)$.

So suppose for instance that $\gamma(1)$ and $\delta(g)$ are not reductive. At most one of $\alpha \cup \gamma(1)$ and $\alpha \cup \gamma(g)$ can be reductive; suppose the latter is not. If $\beta \cup \gamma(g) = \beta \cup \delta(1)$ is reductive, then we have that both $\alpha \cup \delta(g) = \alpha \cup \gamma(g)$ and $\beta \cup \delta(g)$ are not reductive. This actually suffices to prove the theorem, but we can work a little harder to complete this to a good polygon. If both $\gamma(g)$ and $\delta(1)$ are reductive, we have $\{w\} \cdot \{g^{-1}.u, v\} = |w|$. This implies that $|\beta \cup \gamma(1)| = |\{u, g.v, w\}| > 2|w|$, so is not reductive. (In fact, with a little more care, it can be shown that the ideal edge $\alpha \cup \gamma(1)$ is also not reductive, so the good polygon with corners α, $\gamma(1)$, β, and $\delta(g)$ has no reductive size-three ideal edges.) If, say, $\delta(1)$ is not reductive, continuing to assume that $\beta \cup \gamma(g) = \beta \cup \delta(1)$ is reductive, we conclude that $\alpha \cup \gamma(1) = \alpha \cup \delta(1)$ is not, and we have a good polygon with corners α, $\delta(1)$, β and $\delta(g)$. A variant of this argument works in all cases.

The second subcase. So suppose instead that $F = A_1 * F_2$: our only choices for w are $\bar{u}$ and $\bar{v}$. We will consider the ideal edges $\gamma(1) = \{u, \bar{v}\}$, $\gamma(g) = \{u, g.\bar{v}\}$, $\delta(1) = \{\bar{u}, v\}$, $\delta(g) = \{\bar{u}, g.v\}$, $\eta(1) = \{\bar{u}, \bar{v}\}$ and $\eta(g) = \{\bar{u}, g.\bar{v}\}$. The good polygon we build will be either the rectangle or the hexagon in Figure 2, right, with α and β seprarated by one vertex of the polygon.

Now, by Corollary 2.13, at most one γ ideal edge, one δ ideal edge and one η ideal edge can be reductive. Suppose that there are three such. For simplicity assume they are $\gamma(1)$, $\delta(1)$ and $\eta(1)$: the other cases are essentially identical. Then by Lemma 2.11, we have that $\{barv\} \cdot \{u, \bar{u}\} = |barv|$, while $\{\bar{u}\} \cdot \{v, \bar{v}\} = |\bar{u}|$. In particular $|u| = |v|$. Observe, then, that each of the ideal edges $\alpha \cup \gamma(g)$, $\beta \cup \gamma(g)$, $\alpha \cup \delta(g)$ and $\beta \cup \delta(g)$ are greater in norm than $\frac{3}{2}|v| = \frac{3}{2}|u|$, so none of these ideal edges are reductive and we have a good rectangle from α to β.

Suppose next that there are two reductive size-two ideal edges. Up to swapping u and v, we will consider the cases of $\gamma(1)$ and $\eta(1)$ as well as $\gamma(1)$ and $\delta(1)$; the others are essentially identical. If $\gamma(1)$ and $\eta(1)$ are reductive, then by Lemma 2.11, we have that $|\bar{v}| = \{\bar{v}\} \cdot \{u, \bar{u}\}$. We claim that we have a good rectangle with corners

$$\alpha, \ \gamma(g), \ \beta \text{ and } \delta(1).$$

To see this, suppose that one of the relevant size-three ideal edges is reductive. For example, if

20

$\alpha \cup \gamma(g)$ is reductive, we have

$$|v| \geq \{u\} \cdot \{v\} = \{u\} \cdot \{v, g.\bar{v}\} \geq \frac{1}{2}|\{v, g.\bar{v}\}| = |v|.$$

It follows that $|\beta \cup \gamma(g)| = |u| + |v| + |\bar{v}|$ and $|\beta \cup \delta(1)| = |v| + |\{g^{-1}.u, \bar{u}\}|$, so neither of these ideal edges is reductive. And additionally, by Lemma 2.11, if $\alpha \cup \delta(1)$ is reductive, then $|\bar{u}| = \{\bar{u}\} \cdot \{u, \bar{v}\} < \{u\} \cdot \{u, v, \bar{v}\} = |u|$, which is impossible. The argument if $\beta \cup \gamma(g)$ is reductive is analogous. On the other hand, if $\alpha \cup \delta(1)$ is reductive, it follows from Lemma 2.11 that $|v| = \{v\} \cdot \{u, \bar{u}\} = \frac{1}{2}|\{u, \bar{u}\}|$. Then $|\beta \cup \gamma(g)| = |u| + 2|v|$, so this ideal edge is not reductive. Reductivity of $\beta \cup \delta(1)$ or $\alpha \cup \gamma(g)$ would imply respectively that $\{v\} \cdot \{\bar{u}\} = |v| = |u|$ or $\{v\} \cdot \{u\} = |v|$. But again this would imply in either case that $|u|$ and $|\bar{u}|$ cannot be equal, but they must. The case that $\beta \cup \delta(1)$ is reductive is again analogous. The existence of a good rectangle follows.

If instead $\gamma(1)$ and $\delta(1)$ are reductive, we again have a good rectangle, this time with corners

$$\alpha, \ \gamma(g), \ \beta \ \text{ and } \delta(g).$$

Indeed, Lemma 2.11 implies that $\{\bar{v}\} \cdot \{u\} \geq \frac{1}{2}|u|$ and $\frac{1}{2}|\bar{v}|$, while $\{v\} \cdot \{\bar{u}\} \geq \frac{1}{2}|\bar{u}|$ and $\frac{1}{2}|v|$. We observe then, that $|\alpha \cup \gamma(g)| \geq \frac{3}{2}|u|$ because we have

$$\{u\} \cdot \{\bar{v}\} + \{v\} \cdot \{\bar{u}\} + \{g.\bar{v}\} \cdot \{g.u\} \leq |\{u, v, g.\bar{v}\}.$$

Similarly we see that $|\alpha \cup \delta(g)| \geq \frac{3}{2}|v|$ and $|\beta \cup \delta(g)| \geq \frac{3}{2}|v|$, so none of these ideal edges are reductive.

If there is only one reductive size-two ideal edge, up to swapping the roles of u and v it is either a γ or an η. Suppose that the reductive ideal edges is $\gamma(1)$, so that by Lemma 2.11, we have $\{u\} \cdot \{\bar{v}\} \geq \frac{1}{2}|v|$ and $\frac{1}{2}|u|$. Since the $\mathcal{G}_\star$-orbits of u, v and $\bar{v}$ cannot form an isolated component of the star graph, we conclude that $|\beta \cup \gamma(g)| > |u|$ and $|\alpha \cup \gamma(g)| > |u|$. The same argument applies to show that $|\alpha \cup \delta(g)| > |v|$ and $|\beta \cup \delta(g)| > |v|$, so we have a good rectangle from α to β. The argument supposing $\gamma(g)$ is reductive is identical.

If instead $\eta(1)$ is reductive, we see by Lemma 2.11 that $\{\bar{u}\} \cdot \{\bar{v}\} \geq \frac{1}{2}|v|$ and $\frac{1}{2}|u|$. This implies that $\alpha \cup \gamma(1)$ and $\beta \cup \gamma(g)$ cannot both be reductive, for if they were, we would have $|\{v, \bar{v}\}| = \{u, g^{-1}.u\} \cdot \{v, \bar{v}\}$ in contradiction to the fact that $\{\bar{v}\} \cdot \{\bar{u}\} > 0$. Similarly at most one of $\alpha \cup \gamma(g)$ and $\beta \cup \gamma(1)$, one of $\alpha \cup \delta(1)$ and $\beta \cup \delta(g)$ and one of $\alpha \cup \delta(g)$ and $\beta \cup \delta(1)$ are reductive.

Now by Corollary 2.12, at most two ideal edges of the form $\alpha \cup \{x\}$ satisfy $|\alpha \cup \{x\}| \leq |x|$, and if there are two, equality holds. If $|u| < |v|$, we see that $\alpha \cup \gamma(1)$ and $\alpha \cup \gamma(g)$ cannot both be reductive. If $|u| > |v|$, the same applies to $\alpha \cup \delta(1)$ and $\alpha \cup \delta(g)$, and if $|u| = |v|$, at most two of these four ideal edges may be reductive. The same holds for size-three ideal edges containing β.

Suppose $|u| < |v|$. If $\alpha \cup \gamma(1)$ and $\beta \cup \gamma(1)$ are reductive, then $\alpha \cup \gamma(g)$ and $\beta \cup \gamma(g)$ are not, and if $\alpha \cup \delta(1)$ is reductive, then $\beta \cup \delta(g)$ is not and regardless of whether $\alpha \cup \delta(g)$ is, we have a good rectangle from α to β. The same proof works if $\alpha \cup \gamma(g)$ and $\beta \cup \gamma(g)$ are reductive. If neither of these occurs, then at most one of the four size-three ideal edges is reductive and we again have a good rectangle.

If we suppose $|v| < |u|$, swapping the roles of u and v completes the proof as above. So suppose $|v| = |u|$. Now at most two of the four size-three ideal edges containing α are reductive, and similarly for β. If there are three or fewer, or more generally, if for some γ or some δ, the relevant size-three ideal edges are both not reductive, we have a good rectangle. If there are four, each involving a different γ or δ, we can build a hexagon. Indeed, because three of the

four directions cannot form an isolated component of the star graph, no size-three ideal edge containing $\eta(g)$ is reductive. For example if $\alpha \cup \gamma(1)$ and $\beta \cup \gamma(g)$ are not reductive, one good hexagon has corners

$$\alpha, \ \delta(1), \ \beta, \ \gamma(g), \ \eta(g) \text{ and } \gamma(1).$$

Now suppose no size-two ideal edge is reductive. Again we will distinguish between the case $|u| < |v|$ and the case of equality. In the former case, again we have that not both of $\alpha \cup \gamma(1)$ and $\alpha \cup \gamma(g)$ are reductive and similarly for $\beta \cup \gamma(1)$ and $\beta \cup \gamma(g)$. If only one of these four ideal edges is reductive, we have a good rectangle. If $\alpha \cup \gamma(1)$ and $\beta \cup \gamma(1)$ are both reductive, this would contradict Corollary 2.12 as well. Suppose, then, that $\alpha \cup \gamma(1)$ and $\beta \cup \gamma(g)$ are both reductive. Then $|v| > |u| \geq \{u, g^{-1}.u\} \cdot \{v, \bar{v}\} = |\{v, \bar{v}\}|$, so $\{v\} \cdot \{\bar{v}\} > \frac{1}{2}|v|$. From this it follows that $\delta(1) \cup \eta(1)$, $\delta(g) \cup \eta(1)$, $\delta(1) \cup \eta(g)$ and $\delta(g) \cup \eta(g)$ are all not reductive.

Observe that at most two of the size-three ideal edges $\alpha \cup \delta(1)$, $\beta \cup \delta(g)$ $\gamma(1)\eta(1)$ and $\gamma(g) \cup \eta(g)$ can be reductive by Lemma 2.11. The same holds true for $\alpha \cup \delta(g)$, $\beta \cup \delta(1)$, $\gamma(g) \cup \eta(1)$ and $\gamma(1) \cup \eta(g)$. If at most one of the four size-three ideal edges $\alpha \cup \delta(1)$, $\beta \cup \delta(1)$, $\alpha \cup \delta(g)$ and $\beta \cup \delta(g)$ is reductive, we have a good rectangle. If there are two, say $\alpha \cup \delta(1)$ and $\beta \cup \delta(1)$, we again have a good rectangle by including $\delta(g)$ and either γ ideal edge as corners. A similar statement holds if instead $\alpha \cup \delta(g)$ and $\beta \cup \delta(g)$ are reductive. If $\alpha \cup \delta(1)$ and $\beta \cup \delta(g)$ are reductive but $\alpha \cup \delta(g)$ and $\beta \cup \delta(1)$ are not, then $\gamma(g) \cup \eta(g)$ and $\gamma(1) \cup \eta(1)$ are not reductive and one good hexagon from α to β has corners

$$\alpha, \ \gamma(g), \ \beta, \ \gamma(1), \ \eta(1) \text{ and } \delta(g).$$

A similar statement holds if the two reductive ideal edges containing a δ ideal edge are $\alpha \cup \delta(g)$ and $\beta \cup \delta(1)$.

If three ideal edges containing a δ are reductive, say $\alpha \cup \delta(1)$, $\alpha \cup \delta(g)$ and $\beta \cup \delta(g)$, then $\{v, g.v\} \cdot \{u, \bar{u}\} = |\{u, \bar{u}\}|$, while $\{v\} \cdot \{g.\bar{u}, u\} = \{v\} \cdot \{u\} \geq \frac{1}{2}|\{g.\bar{u}, u\}| = |u|$. But this implies that $|\beta \cup \gamma(g)| = |u| + |\{g.v, g.\bar{v}\}|$, contradicting the assumption that this ideal edge is reductive.

Finally suppose that $|u| = |v|$. Then at most two ideal edges containing any given size-two ideal edge may be reductive by Corollary 2.12. Consider, for example, the γ and δ ideal edges. At most two of these together with α may be reductive and similarly for β. If for some γ or δ neither size-three ideal edge is reductive, we have a good rectangle from α to β, because for at least one of the remaining γ or δ ideal edges, at most one of the relevant size-three ideal edges is reductive.

So suppose that for all γ and δ, at least one of the size-three ideal edges containing it and either α or β is reductive. But then notice that at least one of the size-three ideal edges containing it and either $\eta(1)$ or $\eta(g)$ is not reductive. In fact, we have a good hexagon unless for each size-two ideal edge there are two others for which the corresponding size-three ideal edge is reductive and for, say, $\eta(1)$, either $\eta(1)$ and α have the same set of size-two ideal edges which yield (non)-reductive size-three ideal edges or $\eta(1)$ and β have this property (and therefore similarly for $\eta(g)$.)

Suppose for definiteness that $\alpha \cup \gamma(1)$ and $\alpha \cup \delta(g)$ are reductive, $\beta \cup \gamma(g)$ and $\beta \cup \delta(1)$ are reductive, $\eta(1) \cup \gamma(1)$ and $\eta(1) \cup \delta(g)$ are reductive, $\eta(g) \cup \gamma(g)$ and $\eta(g) \cup \delta(1)$ are reductive. (Notice that swapping for instance $\delta(1)$ and $\delta(g)$ above is not possible by Corollary 2.12.) We claim this is not possible. Indeed, by Lemma 2.11, we have that $\{u, g^{-1}.u\} \cdot \{v, \bar{v}\} = |\{v, \bar{v}\}|$, while $0 = \{\bar{u}\} \cdot \{\bar{v}, g.v\} \geq \frac{1}{2}|\{\bar{v}, g.v\}|$, which is impossible. A similar argument works in all cases; this completes the proof.

22

3 Semistability at infinity

The goal of this section is to prove Theorem C. Recall that we write B_k for the ball of radius k. We continue to let
$$N(k) = \max\{\|\tau\| : \tau \in C_k\},$$
where C_k denotes the set of reduced marked graphs of groups ("briar patches") whose stars have nonempty intersection with B_k. To prove semistability, we will show that when $L(F)$ is one ended, for a proper ray $\rho \colon \mathbb{R}_{\geq 0} \to L$, with $\rho(k)$ a briar patch such that the sequence $\|\rho(k)\| > N(k)$ is strictly increasing, the inverse sequence
$$\pi_1(L - B_k, \rho(k))$$
satisfies the Mittag-Leffler condition.

Theorem 3.1. *Suppose $L(F)$ is one ended. If $n \geq N(k)$, then every loop in $L - B_{N(k)}$ based at $\rho(N(k))$ is homotopic to a loop in $L - B_n$ based at $\rho(n)$ by a homotopy that avoids B_k.*

When $L(F)$ is not one ended, we will use our good algebraic understanding of $\mathrm{Out}(F)$ to show that each end of $L(F)$ is semistable.

To prove Theorem 3.1, we need the following proposition.

Proposition 3.2 (cf. Proposition 4.1 of [Vog95]). *Every path in $L - B_k$ is homotopic to a standard path in $L - B_k$ by a homotopy in $L - B_k$.*

To prove the proposition, we need a lemma.

Lemma 3.3. *Let $\mathcal{G}$ be a marked graph of groups representing a vertex of L with two maximal forests F and F'. For each edge $e' \in F' - F$, there exists an edge $e \in F - F'$ such that $F \cup \{e'\} - \{e\}$ is a maximal forest. Put another way, if $\mathcal{G}$ is not a briar patch, the collection of maximal forests in $\mathcal{G}$ forms a* matroid.

Recall that a *matroid* is simply a (nonempty) set, some of whose finite subsets are called *bases.* The bases of a matroid satisfy an *exchange condition* similar to the one in the statement of the lemma. Other examples include the set of spanning trees in a finite graph which is not a rose or the set of bases of a vector space of finite positive dimension.

Proof. Let F, F' and e' be as in the statement. Since F was maximal, $F \cup \{e'\}$ fails to be a maximal forest, either because as a graph it has rank one or because it connects a pair of vertices a and b with nontrivial vertex group. Let C be either the minimal subgraph supporting the (ordinary) fundamental group of $F \cup \{e'\}$ in the former case or the unique geodesic from a to b in the latter. Now because F' is a maximal forest, C cannot be completely contained in F', so any edge of $F - F'$ contained in C will do as e. $\qquad\square$

Proof of Proposition 3.2. If γ is a marked graph of groups which lies outside of B_k, then every marked graph of groups it collapses onto is also outside of B_k by definition. Thus if we have a path $P = (\gamma_0, \gamma_1, \ldots, \gamma_\ell)$, by collapsing forests in alternating marked graphs of groups, we may assume that each γ_k with k even is a briar patch. If ℓ is odd, we may extend P by some collapse of γ_ℓ to a path of even length. If γ_{k-1} and γ_{k+1} are briar patches obtained in this way, they are obtained from $\gamma_k = (\mathcal{G}, \sigma)$ by collapsing maximal forests F and F' in $\mathcal{G}$.

By repeatedly applying Lemma 3.3, we see that we may interpolate from F to F' through maximal forests by adding an edge of F' and removing one of F at each step. The path segment $(\gamma_{k-1}, \gamma, \gamma_{k+1})$ is homotopic to the standard path segment
$$(\gamma_{k-1} = \rho_0, \delta_1, \rho_1, \ldots, \delta_s, \rho_s = \gamma_{k+1})$$

where ρ_i is obtained from γ_k by collapsing the ith maximal forest F_i in the interpolation and δ_i is obtained by collapsing $F_{i-1} \cap F_i$. $\qquad\square$

With the proposition in hand, we are ready to prove the theorem.

Proof of Theorem 3.1. Let P be a loop in $L - B_{N(k)}$ based at $\rho(N(k))$ and let $n \geq N(k)$. By concatenating at both ends with an arbitrary path in $L - B_{N(k)}$ from $\rho(N(k))$ to $\rho(n)$, we obtain a loop P' based at $\rho(n)$. (Such a path exists by Theorem 2.2.) By Proposition 3.2, P' is homotopic relative to its basepoint to a standard path by a homotopy in $L - B_{N(k)}$. By the proof of Theorem 2.2, since $\|\rho(n)\| > N(n)$, we may push P' outside of B_n by a homotopy. This homotopy strictly increases the norm of briar patches along it. Since P' began in $L - B_{N(k)}$ and $N(k)$ is chosen so that the star of any briar patch in $L - B_{N(k)}$ has empty intersection with the ball B_k, this homotopy avoids B_k. $\qquad\square$

To complete the proof of Theorem C, we need only consider the cases where $\mathrm{Out}(F)$ has infinitely many ends. In the case where $\dim L = 1$, we see that $\mathrm{Out}(F)$ is virtually free. Since free groups are semistable at each end (consider the Mittag-Leffler condition in a tree) and by [Geo08, Proposition 16.5.3] semistability passes to and is inherited from finite-index subgroups, we see that $\mathrm{Out}(F)$ is semistable at each end in this case.

Thus it remains to consider the case of $F = A_1 * A_2 * \mathbb{Z}$. Observe that A_1 and A_2 are a complete set of representatives of the conjugacy classes of maximal finite subgroups of F, so these conjugacy classes are permuted by $\mathrm{Out}(F)$. There is a normal subgroup of index at most two in $\mathrm{Out}(F)$ comprising the pointwise stabilizer of these conjugacy classes. Call this subgroup G.

If A is a group, the group $\mathrm{Hol}(A) = A \rtimes \mathrm{Aut}(A)$ always admits an automorphism θ (which is an involution when it is nontrivial) sending an element (a, ψ), with $a \in A$ and $\psi \in \mathrm{Aut}(A)$, to the element $(a^{-1}, \iota_{a^{-1}}\psi)$, where $\iota_{a^{-1}}$ is the inner automorphism $x \mapsto axa^{-1}$.

Fix a generator t of a $\mathbb{Z}$ free factor. The group G contains the finite group $(\mathrm{Hol}(A_1) \times \mathrm{Hol}(A_2)) \rtimes C_2$, where the cyclic group of order two acts by θ on each direct factor. Explicitly, the subgroup A_1 acts by left multiplication on t, the subgroup A_2 acts by right multiplication on t, the subgroup $\mathrm{Aut}(A_1)$ acts on the A_1 free factor, $\mathrm{Aut}(A_2)$ on the A_2 free factor, and C_2 acts by the automorphism

$$\tilde{\theta} = \begin{cases} a_1 & \mapsto a_1 \\ a_2 & \mapsto ta_2t^{-1} \\ t & \mapsto t^{-1}. \end{cases}$$

One checks that the intersection of this group of *automorphisms* with the subgroup of inner automorphisms is trivial, so it injects into $\mathrm{Out}(F)$ and in fact into G.

We will also consider the infinite, one-ended subgroup of automorphisms generated by $\mathrm{Hol}(A_1) \times \mathrm{Hol}(A_2)$ and the automorphism τ which simply inverts t. This subgroup, which we will call H, also injects into G. To see that H is one-ended, observe that it has a map to the finite groups $\mathrm{Aut}(A_1 \times A_2)$ and C_2, defined by first using the universal property to induce an automorphism of $A_1 \times A_2 \times \mathbb{Z}$ and then looking alternately at the torsion subgroup or the action on $\mathbb{Z}$. The intersection H_0 of the kernels, which has finite index in H, comprises those automorphisms which (one checks) act trivially on the A_1 and A_2 free factors and sends t to an element gth, where g and h are elements of $A_1 * A_2$, so H_0 is isomorphic to $(A_1 * A_2) \times (A_1 * A_2)$, a one-ended group which is virtually the direct product of two free groups.

Proposition 3.4. *The group G is isomorphic to the amalgamated free product*

$$((\mathrm{Hol}(A_1) \times \mathrm{Hol}(A_2)) \rtimes C_2) *_{\mathrm{Hol}(A_1) \times \mathrm{Hol}(A_2)} H.$$

It follows from the proposition that G and hence $\mathrm{Out}(F)$ has a finite-index subgroup isomorphic to

$$(F_m \times F_m) * (F_m \times F_m),$$

where F_m is a finite-rank free group. This latter group is a right-angled Artin group, hence is semistable at each end by results of Mihalik [Mih96].

Proof. Firstly, we claim that the subgroup $\mathrm{Hol}(A_1) \times \mathrm{Hol}(A_2)$ together with τ and $\tilde{\theta}$ generate G. Indeed, it is not hard to see that $\mathrm{Aut}(F)$ is generated by these elements together with the family of automorphisms

$$\begin{cases} a_1 \in A_1 & \mapsto a_2 a_1 a_2^{-1} \\ a_2 \in A_2 & \mapsto a_2 \\ t & \mapsto t \end{cases} \quad \begin{cases} a_1 \in A_1 & \mapsto a_1 \\ a_2 \in A_2 & \mapsto a_1 a_2 a_1^{-1} \\ t & \mapsto t \end{cases} \quad \begin{cases} a_1 \in A_1 & \mapsto t a_1 t^{-1} \\ a_2 \in A_2 & \mapsto a_2 \\ t & \mapsto t, \end{cases}$$

as well as possibly some automorphism permuting A_1 and A_2 in the case they are isomorphic. (See for instance [Gil87] for a finite presentation of $\mathrm{Aut}(F)$.) Each of these first three differs from an automorphism in $\langle \mathrm{Hol}(A_1) \times \mathrm{Hol}(A_2), \tilde{\tau}, \tilde{\theta} \rangle$ by an inner automorphism, so the claim follows. To finish the proof, we will use a finite presentation for $\mathrm{Out}(F)$, essentially due to Fouxe-Rabinovitch [FR41], following notation of Marchand [Mar23].

Let us fix notation. For ϕ_i an element of $\mathrm{Aut}(A_i)$, we abuse notation and think of ϕ_i as an element of G via its action on the fixed A_i free factor of F. The subgroup of G generated by the ϕ_i as ϕ_i varies in $\mathrm{Aut}(A_i)$ and as i varies over $\{1, 2\}$ is isomorphic to $\mathrm{Aut}(A_1) \times \mathrm{Aut}(A_2)$. Given $i \neq j$ and $\gamma_i \in A_i$, the *partial conjugation* of A_j by γ_i, is the automorphism $\alpha_{ij}^{(\gamma_i)}$ defined as sending $\gamma_j \in A_j$ to $\gamma_i^{-1} \gamma_j \gamma_i$ and fixing A_i and t. We similarly define α_{ti} where we conjugate by t instead of γ_i, so $\gamma_i \in A_i \mapsto t^{-1} \gamma_i t$. Finally we define the *right* and *left transvections* $\rho_i^{(\gamma_i)}$ and $\lambda_i^{(\gamma_i)}$ as acting on t by right (respectively left) multiplication by $\gamma_i \in A_i$. The group generated by the $\lambda_1^{(\gamma_1)}$, the $\rho_2^{(\gamma_2)}$ and the Φ_i is the subgroup $\mathrm{Hol}(A_1) \times \mathrm{Hol}(A_2)$ described above.

As a special case of [Mar23, Theorem 5.2], we have that a defining system of relations for G with respect to the generating set

$$\{\mathrm{Aut}(A_1), \mathrm{Aut}(A_2), \alpha_{ij}^{(\gamma_i)}, \alpha_{ti}, \lambda_i^{(\gamma_i)}, \rho_i^{(\gamma_i)}, \tau\}$$

is given by the following. We follow the numbering in Marchand's paper, omitting relations that do not occur. Implicitly one includes relations specifying that, for instance, the subgroup generated by $\lambda_i^{(\gamma_i)}$ for fixed i but variable $\gamma_i \in A_i$ is isomorphic to A_i. (For the worried reader: think of automorphisms as acting on F on the left.)

1. $\varphi_1 \varphi_2 = \varphi_2 \varphi_1$, so that $\langle \mathrm{Aut}(A_1), \mathrm{Aut}(A_2) \rangle \cong \mathrm{Aut}(A_1) \times \mathrm{Aut}(A_2)$.

2. $\varphi_i \alpha_{tj} = \alpha_{tj} \varphi_i$ for $\varphi_i \in \mathrm{Aut}(A_i)$ and $i \neq j$.

3. $\varphi_i \alpha_{ij}^{(\gamma_i)} = \alpha_{ij}^{(\varphi_i \gamma_i)} \varphi_i$ for $\varphi_i \in \mathrm{Aut}(A_i)$ and $\gamma_i \in A_i$.

4. $\alpha_{t1} \alpha_{t2} = \alpha_{t2} \alpha_{t1}$.

5. $\alpha_{ti} \alpha_{ij}^{(\gamma_i)} \alpha_{ti}^{-1} = \alpha_{tj}^{-1} \alpha_{ij}^{(\gamma_i)} \alpha_{tj}$ for $\gamma_i \in A_i$.

12. $\lambda_i^{(\gamma_i)} \rho_j^{(\gamma_j)} = \rho_j^{(\gamma_j)} \lambda_i^{(\gamma_i)}$ for $\gamma_i \in A_i$ and $\gamma_j \in A_j$, with i and j not necessarily distinct. Also $\tau \lambda_i^{(\gamma_i)} = \rho_i^{(\gamma_i^{-1})} \tau$.

13. $\alpha_{ij}^{(\gamma_i)}\tau = \tau\alpha_{ij}^{(\gamma_i)}$.

14. $\alpha_{ti}\tau = \tau\alpha_{ti}^{-1}$.

15. $\rho_i^{(\gamma_i)}\varphi_j = \varphi_j\rho_i^{(\gamma_i)}$ for $\varphi_j \in \text{Aut}(A_j)$ and $i \neq j$.

16. $\rho_i^{(\varphi_i\gamma_i)}\varphi_i = \varphi_i\rho_i^{(\gamma_i)}$ for $\gamma_i \in A_i$, $\varphi_i \in \text{Aut}(A_i)$. This, together with some of the relations above implies that the subgroup generated by the $\lambda_1^{(\gamma_1)}$, $\rho_2^{(\gamma_2)}$ and the φ_i is isomorphic to $\text{Hol}(A_1) \times \text{Hol}(A_2)$.

19. $\rho_i^{(\gamma_i)}\alpha_{ij}^{(\gamma_i)} = \alpha_{ij}^{(\gamma_i)}\rho_i^{(\gamma_i)}$.

20. $\rho_i^{(\gamma_i)}\alpha_{tj} = \alpha_{tj}\rho_i^{(\gamma_i)}\alpha_{ij}^{(\gamma_i)}$.

21. $(\rho_i^{(\gamma_i)})^{-1}\rho_j^{(\gamma_j)}\rho_i^{(\gamma_i)}\alpha_{ij}^{(\gamma_i)} = \alpha_{ij}^{(\gamma_i)}\rho_j^{(\gamma_j)}$.

22. $\alpha_{ti}\rho_i^{(\gamma_i)} = \lambda_i^{(\gamma_i)}\alpha_{ti}\varphi_i$, where ϕ_i is the automorphism of A_i defined by $x \mapsto \gamma_i^{-1}x\gamma_i$.

23. $\tau\varphi_i = \varphi_i\tau$ for $\varphi_i \in \text{Aut}(A_i)$.

0. $\varphi_i\alpha_{ij}^{(\gamma_i)}\rho_i^{(\gamma_i)}(\lambda_i^{(\gamma_i)})^{-1} = 1$, where φ_i is the automorphism of A_i defined by $x \mapsto \gamma_i^{-1}x\gamma_i$. Also $\alpha_{t1}\alpha_{t2} = 1$.

We introduce a new generator $\tilde{\theta}$ and the relation $\tilde{\theta} = tau\alpha_{t2}$. As remarked already, the relations in items 0 and 12 allow us to discard generators down to the more minimal

$$\{\text{Hol}(A_1) \times \text{Hol}(A_2), \tau, \tilde{\theta}\}.$$

We have already seen that $\langle \text{Hol}(A_1) \times \text{Hol}(A_2), \tilde{\theta}\rangle$ is isomorphic to the semidirect product $(\text{Hol}(A_1) \times \text{Hol}(A_2)) \rtimes C_2$. Easy Tietze transformations yield a defining system of relations for G for the generating set

$$\{\text{Hol}(A_1) \times \text{Hol}(A_2), \tilde{\theta}, \tau\}.$$

Relevant to our purposes, we just need to check that every relation involving $\tilde{\theta}$ takes place in $(\text{Hol}(A_1) \times \text{Hol}(A_2)) \rtimes C_2$ and every relation involving τ takes place in H. Thus we need only examine relations 2, 5, 14, 20, and 22.

2. The relation $\varphi_1\alpha_{t2} = \alpha_{t2}\varphi_1$ becomes $\varphi_1\tau\tilde{\theta} = \tau\tilde{\theta}\varphi_1$. Because $\text{Hol}(A_1) \times \text{Hol}(A_2)$ is normal in $(\text{Hol}(A_1) \times \text{Hol}(A_2)) \rtimes C_2$, right-multiplying by $\tilde{\theta}^{-1}$ and then using a relation discussed before the proof shows that this relation involves only elements of H. A similar argument works for $\varphi_2\alpha_{t1} = \alpha_{t1}\varphi_2$, which becomes $\varphi_2\tilde{\theta}\tau = \tilde{\theta}\tau\varphi_2$.

5. This relation, observing that $\alpha_{t2} = \alpha_{t1}^{-1}$, is spurious.

14. This relation is also spurious: it says either that $\alpha_{t2}\tau\alpha_{t2} = \tilde{\theta}\tau\tau\tilde{\theta} = \tau$, which follows because τ and $\tilde{\theta}$ have order two, or that $\alpha_{t1}\tau\alpha_{t1} = \alpha_{t2}^{-1}\tau\alpha_{t2}^{-1} = \tau$, which follows from the previous relation.

20. Because $\text{Hol}(A_1) \times \text{Hol}(A_2)$ is normal in the semidirect product, the case of this relation of the form $\rho_2^{(\gamma_2)}\alpha_{t1} = \alpha_{t1}\rho_2^{(\gamma_2)}\alpha_{21}^{(\gamma_2)}$ may be written with only elements of H by writing $\alpha_{t1} = \tilde{\theta}\tau$ and left multiplying by $\tilde{\theta}^{-1}$. The same is true for the relation with the roles of 1 and 2 swapped, where instead we left multiply by $\tilde{\theta}^{-1}\tau^{-1}$ and observe that after conjugating $\rho_1^{(\gamma_1)}$ by τ we obtain an element of $\text{Hol}(A_1) \times \text{Hol}(A_2)$.

22. The same method of proof as in the previous case works: for example the relation $\alpha_{t1}\rho_1^{(\gamma_1)} = \lambda_1^{(\gamma_1)}\alpha_{t1}\varphi_1$ may be rewritten as $\tau\rho_1^{(\gamma_1)} = \tilde{\theta}^{-1}\lambda_1^{(\gamma_1)}\tilde{\theta}\tau\varphi_1$, which takes place in H.

Thus we have demonstrated that G splits as the amalgamated free product in the statement. $\square$

References

[Bas93]   Hyman Bass. Covering theory for graphs of groups. *J. Pure Appl. Algebra*, 89(1-2):3–47, 1993.

[BE73]   Robert Bieri and Beno Eckmann. Groups with homological duality generalizing Poincaré duality. *Invent. Math.*, 20:103–124, 1973.

[BF00]   Mladen Bestvina and Mark Feighn. The topology at infinity of $\text{Out}(F_n)$. *Invent. Math.*, 140(3):651–692, 2000.

[BM01]   Noel Brady and John Meier. Connectivity at infinity for right angled Artin groups. *Trans. Amer. Math. Soc.*, 353(1):117–132, 2001.

[Cla09]   Matt Clay. Deformation spaces of G-trees and automorphisms of Baumslag-Solitar groups. *Groups Geom. Dyn.*, 3(1):39–69, 2009.

[CV86]   Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.

[CZ84]   Donald J. Collins and Heiner Zieschang. Rescuing the Whitehead method for free products. I. Peak reduction. *Math. Z.*, 185(4):487–504, 1984.

[Das18]   Saikat Das. Thickness of $\text{Out}(A_1 * ... * A_n)$. Available at arXiv:1811.00435 [Math.GR], Nov 2018.

[FR41]   D. I. Fouxe-Rabinovitch. Über die Automorphismengruppen der freien Produkte. II. *Rec. Math. [Mat. Sbornik] N. S.*, 9 (51):183–220, 1941.

[Geo08]   Ross Geoghegan. *Topological methods in group theory*, volume 243 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[Gil87]   N. D. Gilbert. Presentations of the automorphism group of a free product. *Proc. London Math. Soc. (3)*, 54(1):115–140, 1987.

[GL07]   Vincent Guirardel and Gilbert Levitt. Deformation spaces of trees. *Groups Geom. Dyn.*, 1(2):135–181, 2007.

[KV93]   Sava Krstić and Karen Vogtmann. Equivariant outer space and automorphisms of free-by-finite groups. *Comment. Math. Helv.*, 68(2):216–262, 1993.

[Lym22a] Rylee Alanza Lyman. A family of CAT(0) outer automorphism groups of free products. Available at arXiv:2209.04711 [math.GR], September 2022.

[Lym22b] Rylee Alanza Lyman. Lipschitz metric isometries between Outer Spaces of virtually free groups. Available at arXiv:2203.09008 [math.GR], March 2022.

[Lym22c] Rylee Alanza Lyman. Train track maps on graphs of groups. Available at arXiv:2102.02848 [math.GR], March 2022.

[Mar23]   Alexis Marchand. Free representations of outer automorphism groups of free products via characteristic abelian coverings. *J. Group Theory*, 26(2):399–420, 2023.

[Mih96]   Michael L. Mihalik. Semistability of Artin and Coxeter groups. *J. Pure Appl. Algebra*, 111(1-3):205–211, 1996.

[Ser03]   Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[SW79]   Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge-New York, 1979.

[Vog95]   Karen Vogtmann. End invariants of the group of outer automorphisms of a free group. *Topology*, 34(3):533–545, 1995.