Bass-Serre Theory: Groups acting on trees

Geometric Group Theory without Boundaries

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these notes are on my website! I'll put a link in the chat closer to

Lecture 1 7/20/20

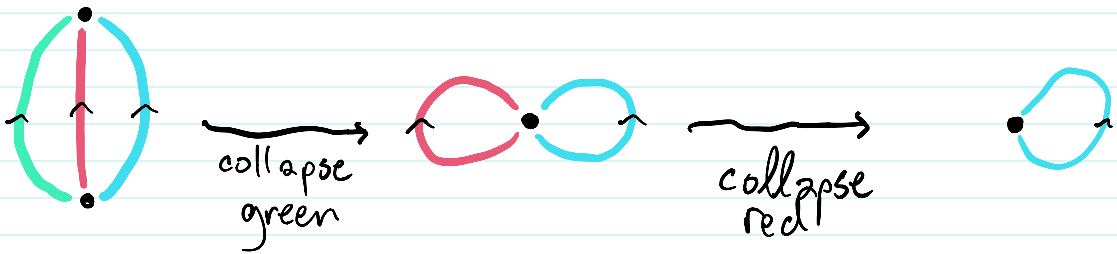
Today:
- "Why?"
- Examples (free groups, $SL_2(\mathbb{Z})$, surface groups)
- Definition of a graph of groups
- The "quotient" graph of groups
- The fundamental theorem of Bass-Serre theory

Examples: Free groups

Algebraic topology: free groups $\cong \pi_1(\text{graph})$

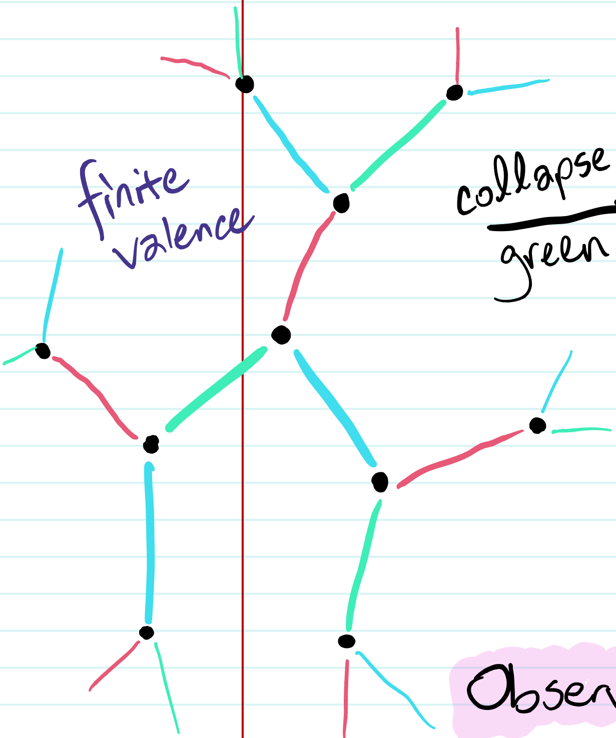
$\Rightarrow$ free groups $\curvearrowright$ trees $\cong$ universal cover

$F_2 \cong \pi_1$



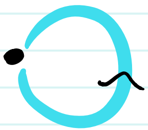
Examples: Free groups

finite valence

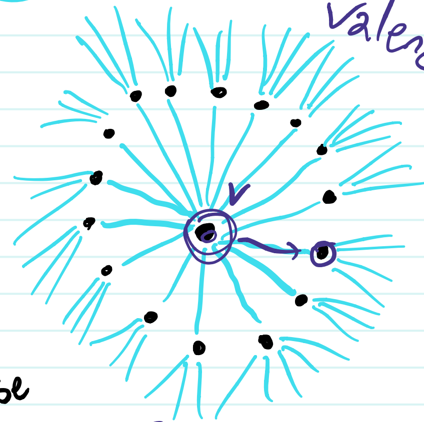


collapse green

$\mathrm{stab}(v) \cong \mathbb{Z}$



infinite valence



collapse red

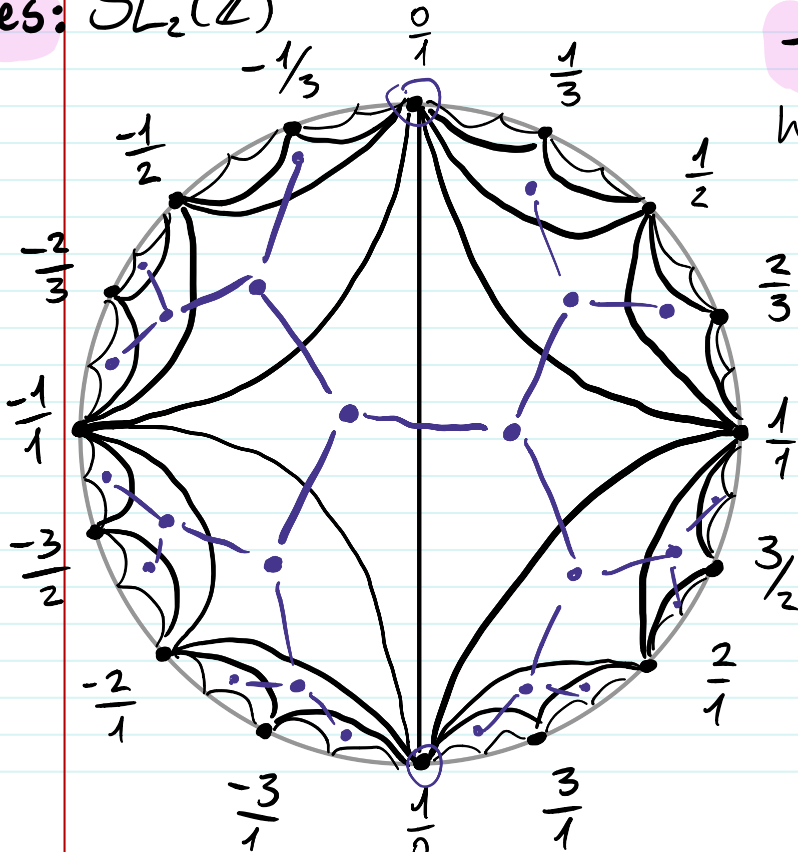
not free

Observations: the first two actions are free

trivial edge stabilizers

free splittings

Examples: $SL_2(\mathbb{Z})$



The Farey Diagram

has vertices fractions p/q ($p,q \in \mathbb{Z}$) in **lowest terms** (and $1/0$)

and an edge from p/q to m/n when

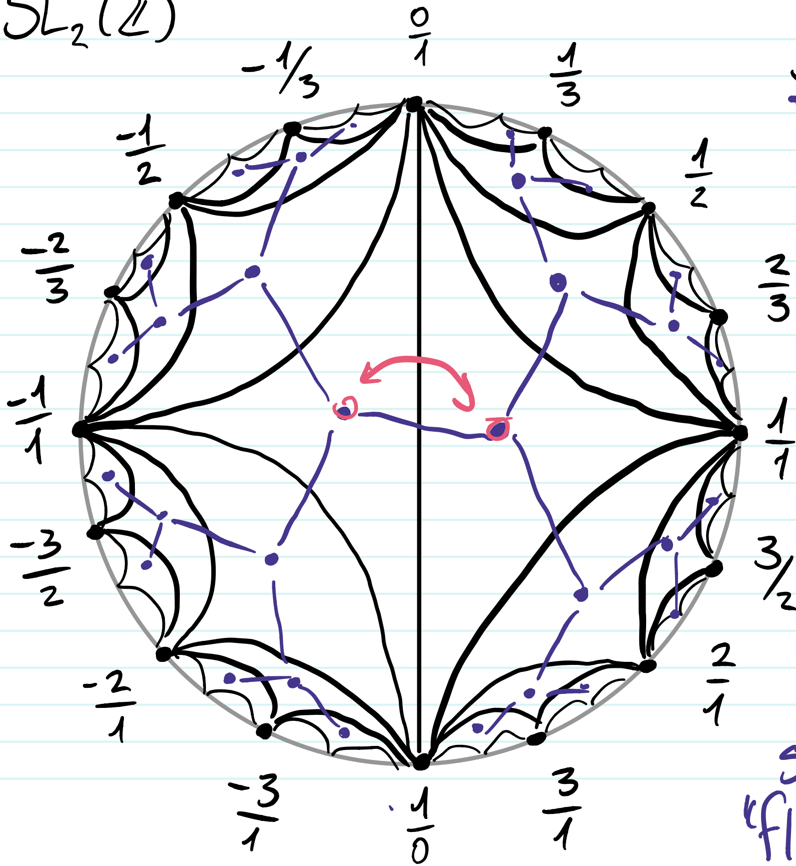
$$\det\begin{pmatrix} p & m \\ q & n \end{pmatrix} = \pm 1$$

Facts: every edge belongs to 2 triangles

edges do not cross

exercises?

Examples: $SL_2(\mathbb{Z})$



$SL_2(\mathbb{Z})$ acts on the Farey diagram via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{p}{q} = \left[\frac{ap+bq}{cp+dq}\right]$

(Easy) **exercise:** the action preserves adjacency.

Observations:

action on diagram $\rightsquigarrow$ action on tree

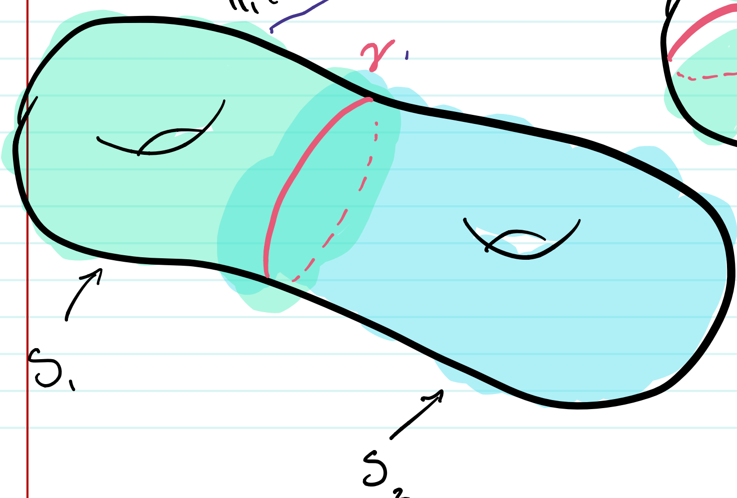
$\mathrm{stab}(v) \cong C_3$

"flips edges"

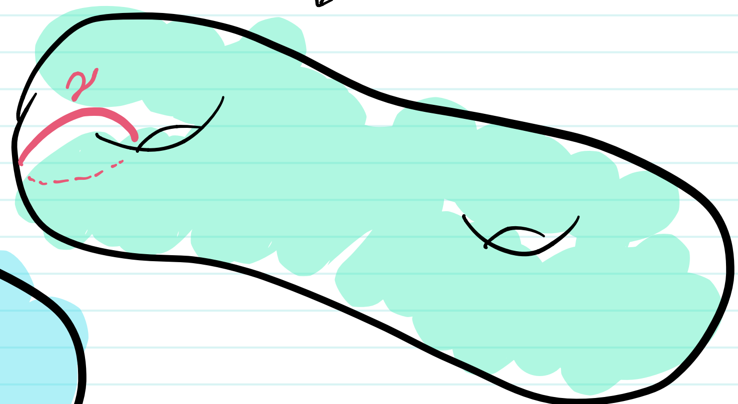
Examples: Surface groups

Let γ be an essential, simple closed curve on S, a surface of genus $g \geq 2$

Van Kampen: $\pi_1(S) \cong \pi_1(S_1) \ast_{\mathbb{Z} \cong \langle \gamma \rangle} \pi_1(S_2)$

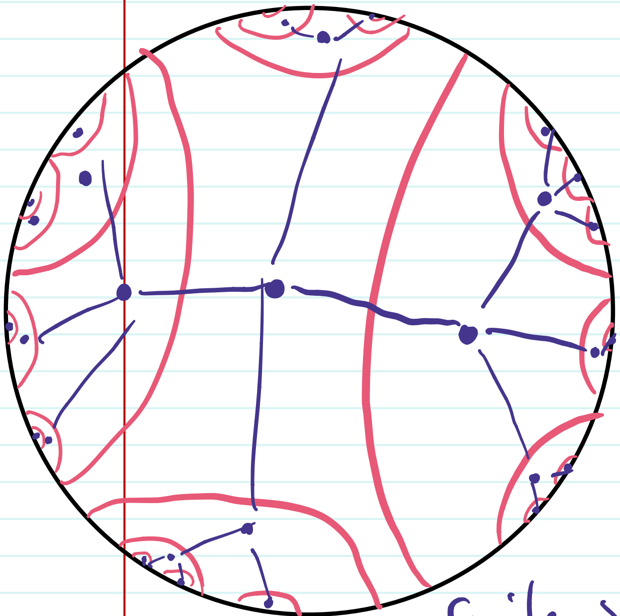


why can't we use Van Kampen?

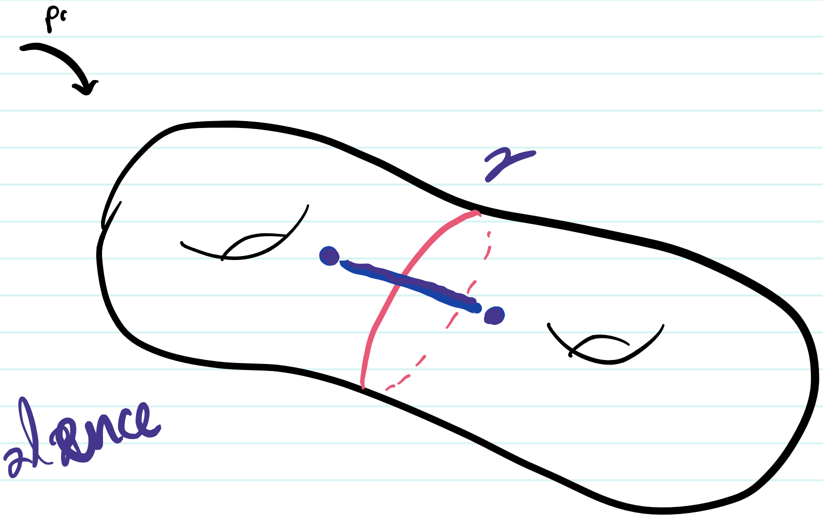


Examples: Surface groups

Identify the universal cover with $\mathbb{H}^2$

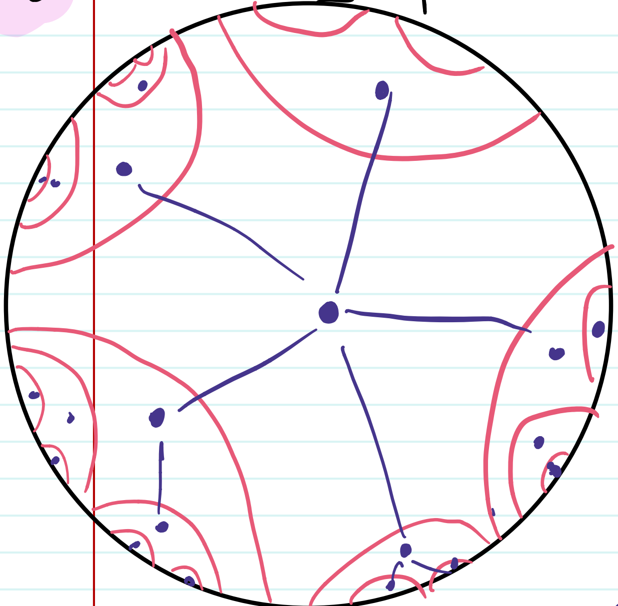


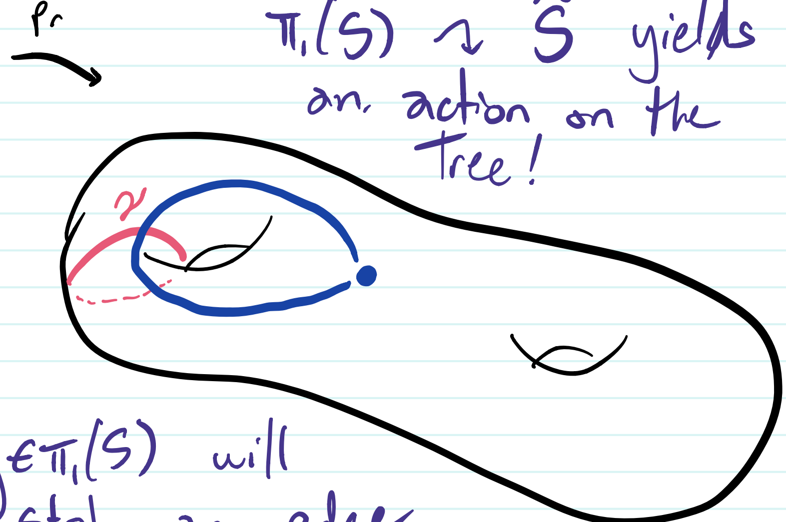
Observations:



infinite valence

Examples: Surface groups





Observations:

the action of $\pi_1(S) \curvearrowright \tilde{S}$ yields an action on the tree!

$\text{stab}(r) =$ conjugate of $\pi_1(S - \gamma)$

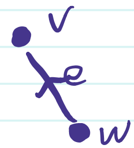
$g \in \pi_1(S)$ will stab. an edge

$\Longleftrightarrow$ preserves a lift of γ

$\Longleftrightarrow$ g is a power of a conjugate of γ

Three Ways to think about graphs

#1 **Def** A **graph** is a 1-dimensional CW-complex.



#2 **Def** (Gersten) a graph is a set Γ, an involution $\bar{\ }:\Gamma \longrightarrow \Gamma$, and a retraction $\tau:\Gamma \longrightarrow \Gamma$ onto the fixed-point set for $\bar{\ }$.

$\Gamma = \{v, w, e, \bar{e}\}$

$\tau(e) = v$

$\tau(\bar{e}) = w$

The fixed point set $V(\Gamma)$ is the set of **vertices**, its complement $E(\Gamma) = \Gamma - V(\Gamma)$ is the set of **oriented edges**.

$\bar{\ }:\Gamma \longrightarrow \Gamma$ reverses the orientation of edges, and τ sends an edge e to its **terminal vertex**.

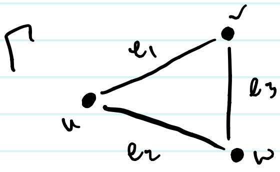
$$\begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, "\bar{\ }"'] & \Delta \arrow[d, "\bar{\ }"] \\ \Gamma \arrow[r, "f"] & \Delta \end{tikzcd} \qquad \begin{tikzcd} \Gamma \arrow[r, "f"] \arrow[d, "\tau"'] & \Delta \arrow[d, "\tau"] \\ \Gamma \arrow[r, "f"'] & \Delta \end{tikzcd}$$

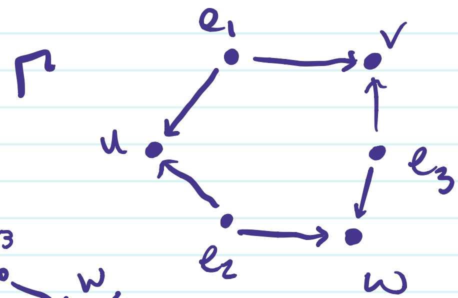
this allows us to formalize what a morphism is: $\Gamma \longrightarrow \Delta$

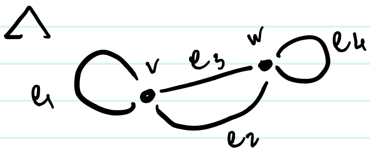
Three Ways to think about graphs

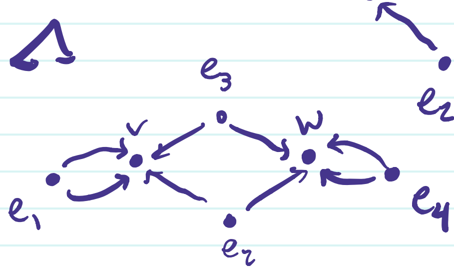
#3 A graph Γ determines a **small category** (without loops, if you'd like)

Ex









The rule for constructing the category is...

objects for vertices of the barycentric subdiv.

arrows from edges to incident vertices

Definition of a graph of groups

$\mathcal{G}_{\bar{e}} = \mathcal{G}_e$

$\mathcal{G}_v = \pi_1(S_1)$
$\mathcal{G}_w = \pi_1(S_2)$
$\mathcal{G}_e = \mathbb{Z}$
$\iota_e : \mathcal{G}_e \xrightarrow{\tau} \mathcal{G}_v$

Def A **graph of groups** is a pair $(\Gamma, \mathcal{G})$

- Γ is a connected graph
- $\mathcal{G}$ is a **functor** $\mathcal{G} : \Gamma \longrightarrow \text{Groups}^{\text{mono}}$

All this really means is an assignment:

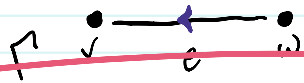
for every vertex v $\exists$ group $\mathcal{G}_v$
" " edge e $\exists$ group $\mathcal{G}_e$

if e is incident to v, $\tau(e) = v$, $\exists$ an injective homomorphism

$\iota_e : \mathcal{G}_e \hookrightarrow \mathcal{G}_v$

$\iota_{\bar{e}} : \mathcal{G}_e \longrightarrow \mathcal{G}_{\tau(\bar{e})}$

Ex.



$$\boxed{\pi_1(S_1) *_{\mathbb{Z}} \pi_1(S_2)}$$

The "quotient" graph of groups

Let G be a group, and suppose G acts on a tree T.
The action is **without inversions** [in edges]
if for every edge e of T with incident vertices v and w,
if $g.e = g$, then $g.v = v$ and $g.w = w$

Why is this useful? bc the quotient inherits a graph structure

this can be accomplished by barycentric subdivision

The "quotient" graph of groups

So: $G \curvearrowright T$ a tree w/o inversions. Let Γ be $\boxed{G \backslash T.}$

Choose a **spanning tree** $F \leq \Gamma$, and choose a lift $\tilde{F} \subset T$. If v is a vertex of Γ, write $\tilde{v}$ for the corresponding vertex in $\tilde{F}$.

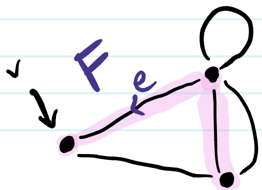
We will define $\mathcal{G}_v = \mathrm{stab}(\tilde{v})$

For e an edge in F, write $\tilde{e}$ for its lift in $\tilde{F}$.

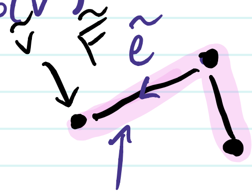
Define $\mathcal{G}_e = \mathrm{stab}(\tilde{e})$

$\iota_e : \mathrm{stab}(\tilde{e}) \subseteq \mathrm{stab}(\tilde{v})$

Ex. Γ



T



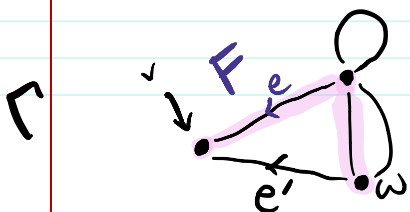
The "quotient" graph of groups

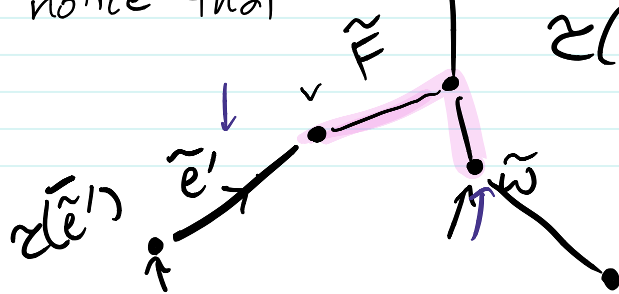
define $\iota_{\tilde{e}'} : G_{e'} \longrightarrow G_{w}$
$h \longmapsto ghg^{-1}$

Finally, suppose e' is an edge in $\Gamma \setminus F$.
Choose a lift $\tilde{e}'$ of e' so that $\tau(\tilde{e}') = \widetilde{\tau(e')}$

Define $G_{e'} = \mathrm{stab}(\tilde{e}')$ and $\iota_e : G_e \longrightarrow G_{\tau(e)}$
to be the inclusion as before.
By definition, there is some $g \in G$ such that
$g . \tau(\tilde{e}) \in \tilde{F}$.
If h stabilizes $\tilde{e}$, notice that

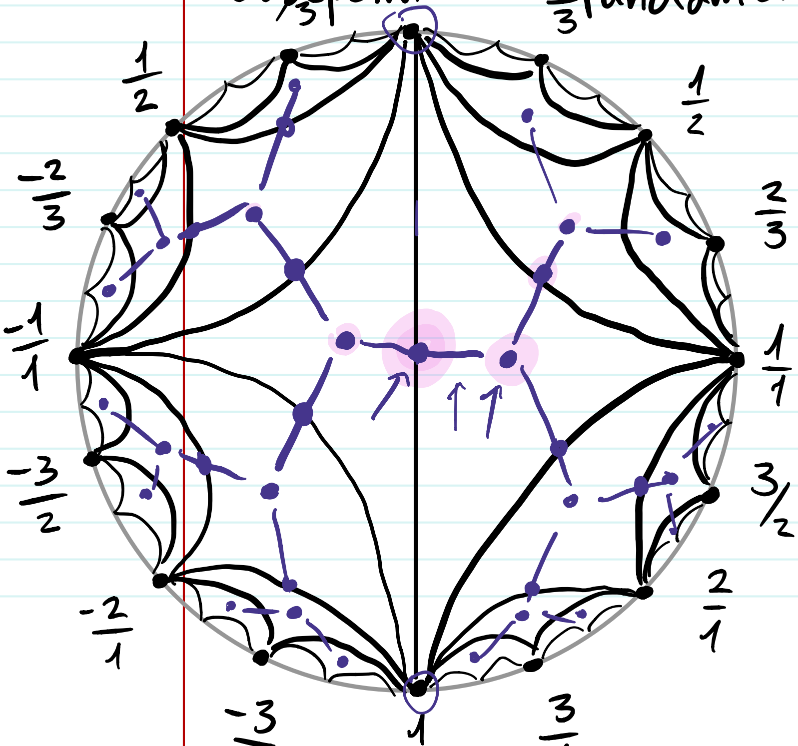
ghg^{-1} stabilizes $\widetilde{\tau(e')}$





Examples

Observation: the "quotient" graph of groups depends on choices — more than just a basepoint $\frac{1}{3}$ "$\frac{1}{3}$fundamental domain"



exercise: take any triangle to any other

same w/ edges

$$G\backslash T = \overset{b^2}{C_4} \hookleftarrow C_2 \hookrightarrow \overset{a^3}{C_6}$$

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$$

$$\begin{pmatrix}0&1\\-1&1\end{pmatrix}^3 = -Id$$

Fact: $\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ acts trivially

$1 \qquad \bar{0} \qquad 1$

Fundamental Theorem

Thm (Bass–Serre) If $(\Gamma, \mathcal{G})$ is a graph of groups and $p \in \Gamma$, there exists a group
$$G = \pi_1(\Gamma, \mathcal{G}, p)$$ the fundamental group of the graph of groups
and a tree T with an action of G without inversions such that the quotient graph of groups is $(\Gamma, \mathcal{G})$

every graph of groups arises as a quotient

the fundamental group has a combinatorially nice presentation, this turns this into an inductive tool