

Bass-Serre Theory: Groups acting on trees

Geometric Group Theory without Boundaries

Rylee Lyman

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Today:

- A normal form for paths in (Γ, g)
- vocab (FA)

Applications of Bass Serre Theory

Normal Forms

Let (Γ, \mathcal{G}) be a graph of groups.

For each oriented edge e , choose a set S_e of coset representatives for $\mathcal{G}_{r(e)} / c_e(\mathcal{G}_e)$ satisfying $1 \in S_e$.

A path $g_0 e_1 g_1 \dots e_n g_n$ is in **normal form** if each $g_i \in S_{e_i}$ for $0 \leq i < n$.

Every path is homotopic rel endpoints to a path in normal form.

homotopically nontrivial loops have nontrivial normal form.

The Kernel

Prop. If G acts on T with quotient graph of groups (Γ, G) ,

the kernel of the action is the largest normal subgroup of G contained in

$$\bigcap_{e \in E(\Gamma)} \text{ker}(g_e)$$

Ex. C_2 is the kernel of $SL_2(\mathbb{Z}) \curvearrowright$ Farey tree

Vocab

Recall: G splits if $\exists (\Gamma, g)$ w/ $\pi_1(\Gamma, g, p) \cong G$

s.t. for every vertex v , $G_v \neq G$.

equivalently, G splits if it acts on a tree T
without global fixed point.

Def.

A group G has **Serre's Property FA** (fixe arbre)
if it does not split.

Ex.

$SL_n(\mathbb{Z})$ $n \geq 3$, most triangle groups, $Mod(S_g)$ $g \geq 3$ (2?)

Property FA passes to quotients but not to subgroups.

Thm

(Watani '81) If G has Kazhdan's Property (T),
then G and its finite-index subgroups have Property FA.

Vocab

Thm (Serre) G having Property FA is equivalent to G satisfying the following three conditions

i) if $\begin{cases} G \cong A * C \\ \text{or } G \cong B \end{cases}$, then $G \cong A$ and $C = B$

ii) \mathbb{Z} is not a quotient of G .

iii) if $G = \bigcup_{i=1}^{\infty} G_i$ for $1 \leq G_1 \leq G_2 \leq \dots$,

then $G = G_n$ for some n .

(if G is countable, (iii) is equivalent to G being finitely generated.)

N.B. \mathbb{Q} splits.

Applications

Let X be a space. The **space of ends** of X is

$$\text{Ends}(X) = \varprojlim \pi_0(X - K)$$

as K ranges over the compact subsets of X .

If G is finitely generated (by S), set

$$\text{Ends}(G) = \text{Ends}(\Gamma(G, S)).$$

Thm (Freudenthal, Hopf '40s) If G is finitely generated,

$$\#\{\text{Ends}(G)\} \in \{0, 1, 2, \infty\}.$$

$$\#\{\text{Ends}(G)\} = 0 \iff G \text{ is finite}$$

$$\#\{\text{Ends}(G)\} = 2 \iff G \text{ is virtually } \mathbb{Z}$$

Thm (Wall '67) G is virtually $\mathbb{Z} \iff \exists N \text{ finite } N \triangleleft G, G/N \cong \mathbb{Z} \text{ or } G/N \cong D_\infty.$

Applications

Thm (Stallings '71) If G is finitely generated and $\text{Ends}(G) = \infty$, then G splits as π_1 of a graph of groups with finite edge groups.

"Idea" of proof:

Applications

Thm (Grushko '40) Suppose $G \cong A_1 * \dots * A_k * F_n$, where F_n is free, and each A_i is freely indecomposable and not infinite cyclic.

If also $G \cong B_1 * \dots * B_l * F_m$ (analogously), then $k=l$, $m=n$, and after reordering, B_j is conjugate to A_i .

“Idea”:

Applications

A group G is **accessible** if there is a splitting of G with finite edge groups and one-ended vertex groups.

Thm

(Dunwoody '85)

Finitely presented groups are accessible.

"Idea":

Thm

(Jaco-Shalen, Johannson '79)

If M^3 is prime and orientable,
there exists a finite collection of
incompressible tori s.t. the complementary
pieces are atoroidal or Seifert-fibered

In terms of trees:

JSJ Decompositions

Let \mathcal{A} be a class of groups closed under taking subgroups.

Def.

An \mathcal{A} -tree G is (Γ, G) w/ $\pi_1 \cong G$ and edge groups in \mathcal{A} . (or the Bass-Serre tree T .)

Def.

(Guirardel-Lovitt) An \mathcal{A} -tree T is a **JSJ \mathcal{A} -tree** if

- edge-stabilizers in T fix points in every \mathcal{A} -tree.
- if T' is an \mathcal{A} -tree, \exists a G -equivariant map

$$T \rightarrow T'$$

Thm.

(Guirardel-Lovitt '17) JSJ \mathcal{A} -trees exist for finitely-presented groups, but may not be canonical.

JSJ Decompositions

Thm (Bowditch '09) If \mathcal{A} is the class of virtually cyclic groups and G is word-hyperbolic, there is a canonical JSJ \mathcal{A} -tree for G .

"Idea:"

Applications

Thm (Karrass-Pietrowski-Solitar '72)

A finitely generated group G is virtually free

$\Leftrightarrow G$ acts on a tree T with finite stabilizers
and finite quotient.

"Idea":

Thm

(Culler '84) Suppose $G \subset \text{Out}(F_n)$ is finite.

Then G acts on a finite graph with $\pi_1 \cong F_n$.

Pf

Applications

Thm (Bridson-Haefliger) Let (Γ, \mathcal{G}) be a graph of groups.

Suppose $X = (\Gamma, \mathcal{G}, X_v, X_e, f_e : X_e \rightarrow X_{e(v)})$
is a graph of spaces such that

- 1) \tilde{X}_e, \tilde{X}_v are complete, $CAT(0)$ metric spaces
w/ proper (and cocompact) isometric g_e, g_v actions
- 2) each $\tilde{f}_e : \tilde{X}_e \rightarrow \tilde{X}_v$ is an isometric embedding
- 3) The image of each \tilde{f}_e is convex

Then \hat{X} is a complete $CAT(0)$ metric space
with a proper (and cocompact) isometric $\pi_1(\Gamma, \mathcal{G}, p)$
action.

Applications

I used the previous theorem to show

Thm (L'9) Let $\Phi: F_n \rightarrow F_n$ be an automorphism.

If \exists a free basis x_1, \dots, x_n s.t.

Φ sends the x_i to **palindromes** in the x_i ,

then $F_n \times_{\Phi} \mathbb{Z}$ is virtually a CAT(0) group.

e.g.

$$\Phi \begin{cases} a \mapsto a \\ b \mapsto ab^2a \\ c \mapsto a^2ca^2 \end{cases}$$

N.B. Gersten showed that $F_3 \times_{\Phi} \mathbb{Z}$ is not a subgroup of a CAT(0) group, where

$$\Phi \begin{cases} a \mapsto a \\ b \mapsto ba \\ c \mapsto ca^2 \end{cases}$$

Applications

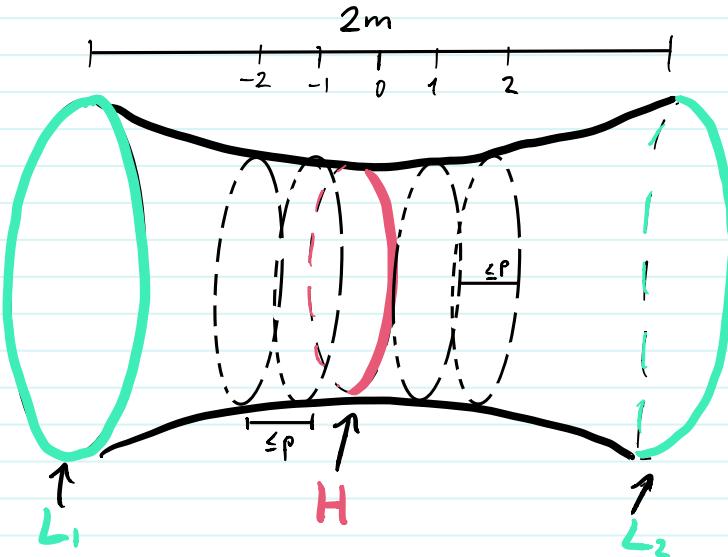
Thm (Bestvina-Feighn '92) Let (Γ, \mathcal{G}) be a graph of hyperbolic groups and $X = (\Gamma, \mathcal{G}, X_v, X_e, f_e: X_e \rightarrow X_{\text{rec}})$ a corresponding graph of spaces. Then if

- i) $f_e: \tilde{X}_e \longrightarrow \tilde{X}_v$ is a q.i. embedding
- ii) X satisfies the **annuli flare** condition,

Then $\pi_1(\Gamma, \mathcal{G}, \rho)$ is word hyperbolic

\iff it contains no $\text{BS}(n, m)$ subgroup.

The annuli flare condition



$\exists \lambda > 1, m \geq 1$ s.t.
 $\forall \rho \exists H = H(\rho)$ s.t.
if as above, then

$$\lambda H \leq \max \{L_1, L_2\}$$

