Bass-Serre Theory: Groups acting on trees

Geometric Group Theory without Boundaries

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Lecture 1 7/20/20

Today:

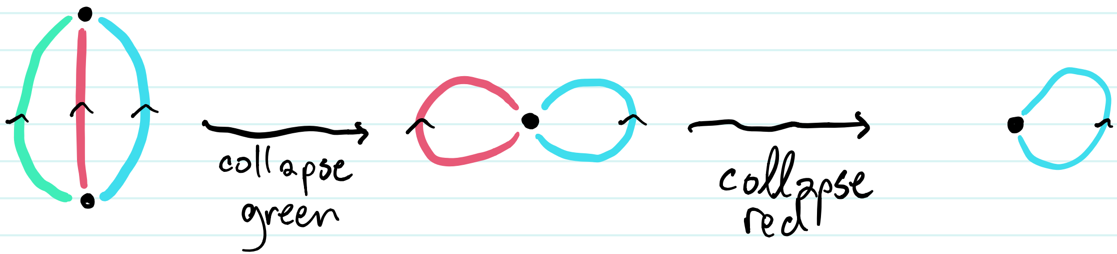
- Examples (free groups, $SL_2(\mathbb{Z})$, surface groups)
- Definition of a graph of groups
- The "quotient" graph of groups
- The fundamental theorem of Bass-Serre theory

Examples: Free groups

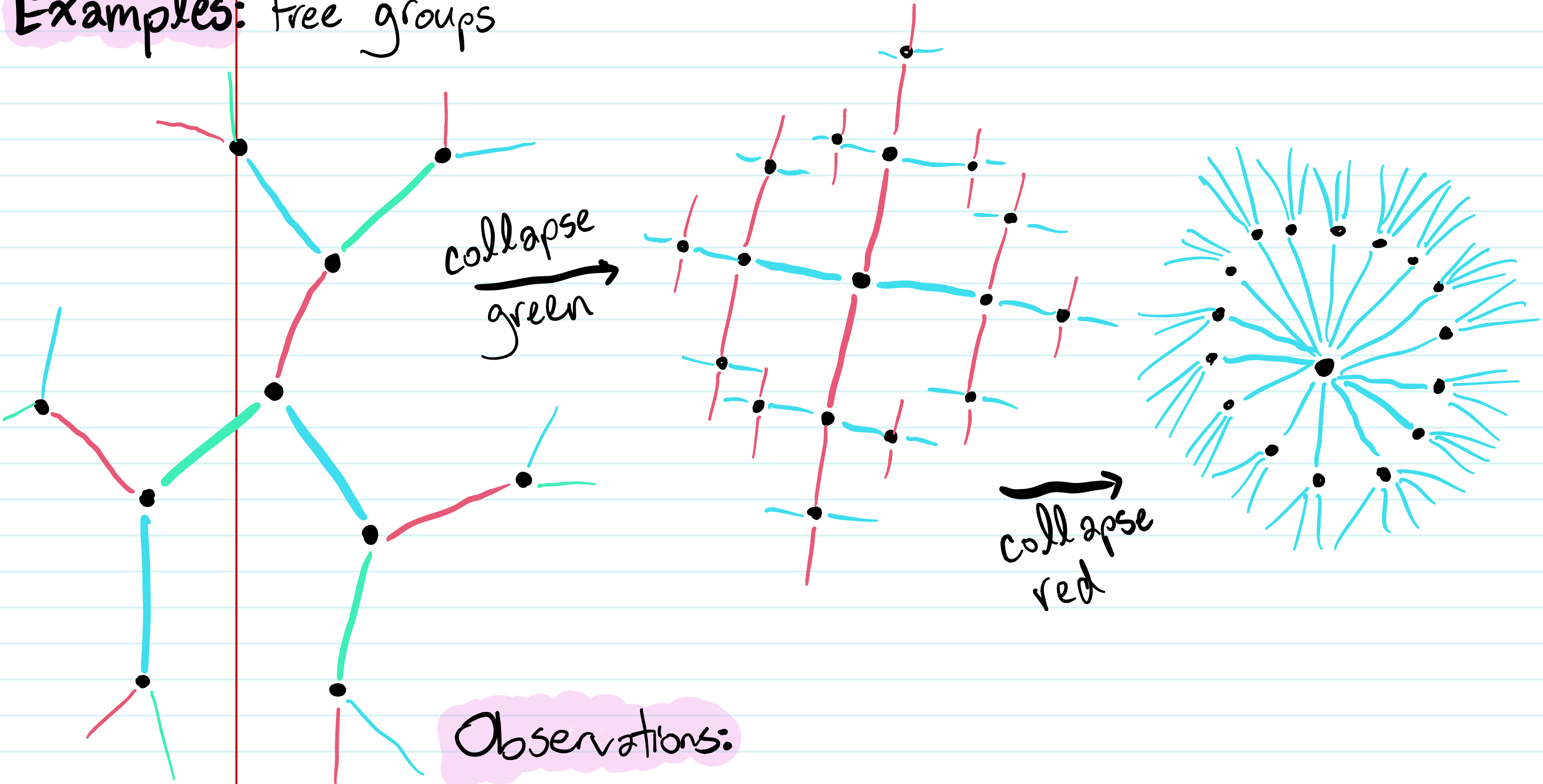
Algebraic topology: free groups $\cong \pi_1$(graph)

$\Rightarrow$ free groups $\curvearrowright$ trees $\cong$ universal cover

$F_2 \cong \pi_1$

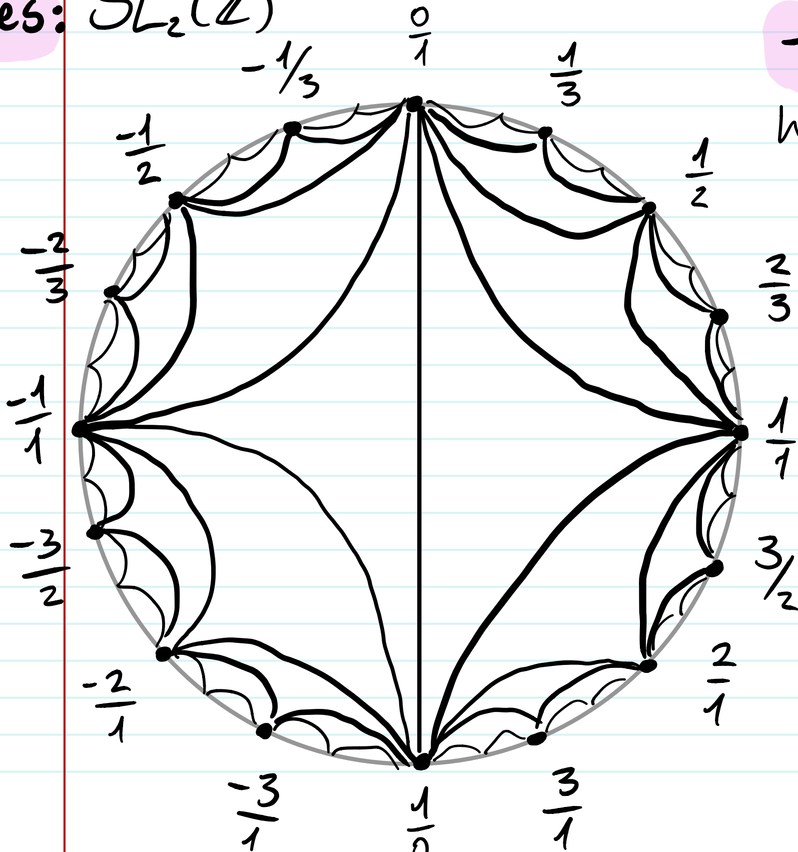


Examples: Free groups



Observations:

Examples: $SL_2(\mathbb{Z})$



The Farey Diagram

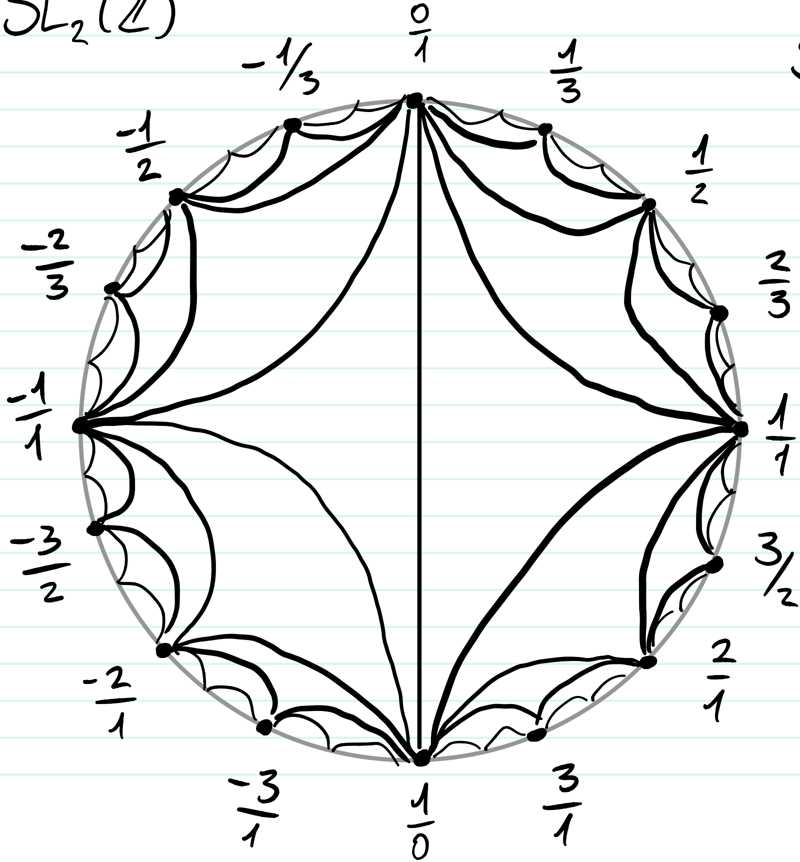
has vertices fractions p/q ($p,q \in \mathbb{Z}$) in **lowest terms** (and $1/0$) and an edge from p/q to m/n when

$$\det\begin{pmatrix} p & m \\ q & n \end{pmatrix} = \pm 1$$

Facts: every edge belongs to 2 triangles

edges do not cross

Examples: $SL_2(\mathbb{Z})$



$SL_2(\mathbb{Z})$ acts on the Farey diagram via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{p}{q} = \frac{ap+bq}{cp+dq}$$

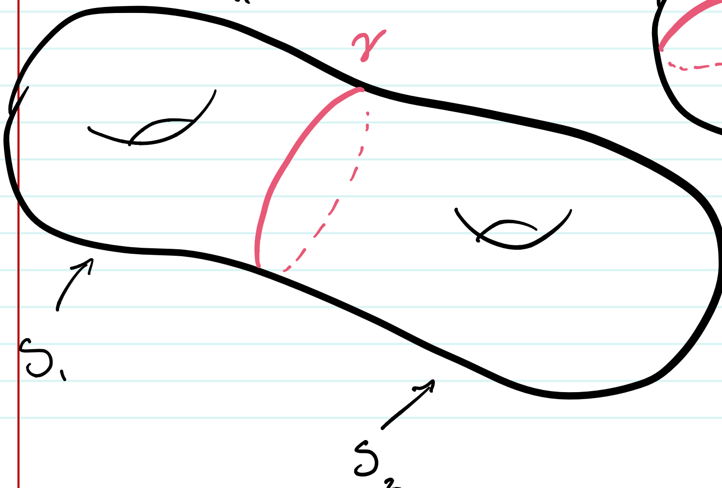
(Easy) **exercise**: the action preserves adjacency.

Observations:

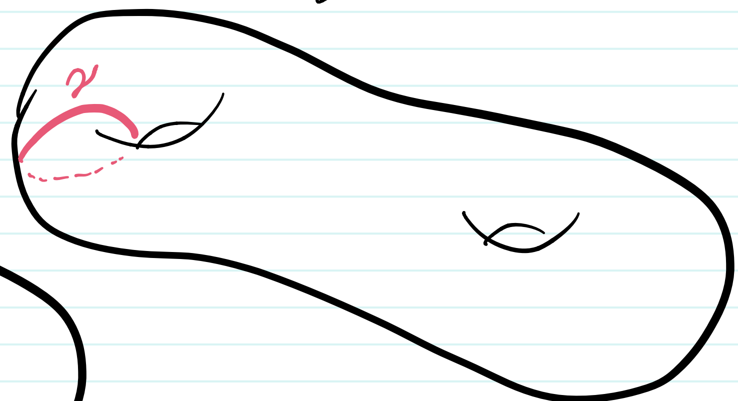
Examples: Surface groups

Let γ be an essential, simple closed curve on S, a surface of genus $g \geq 2$

Van Kampen: $\pi_1(S) \cong \pi_1(S_1) *_{\mathbb{Z}} \pi_1(S_2)$



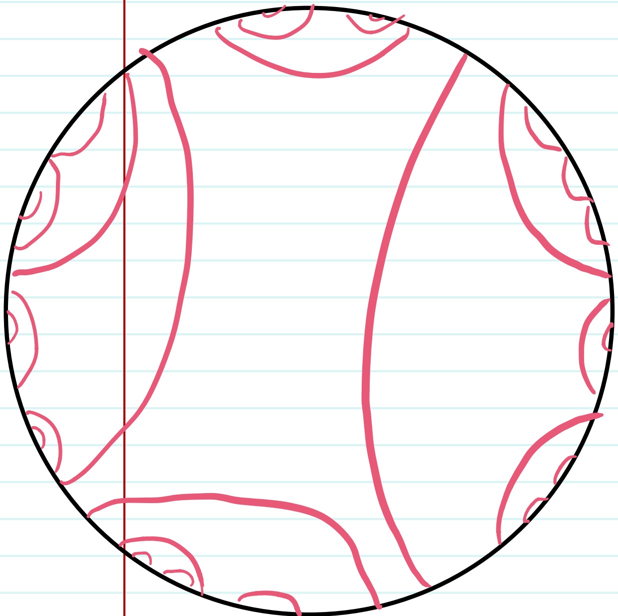
why can't we use Van Kampen?

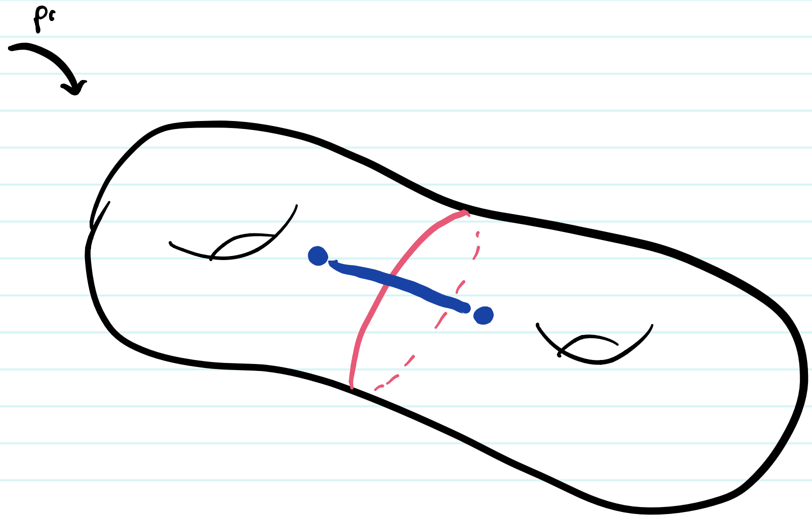


Examples: Surface groups

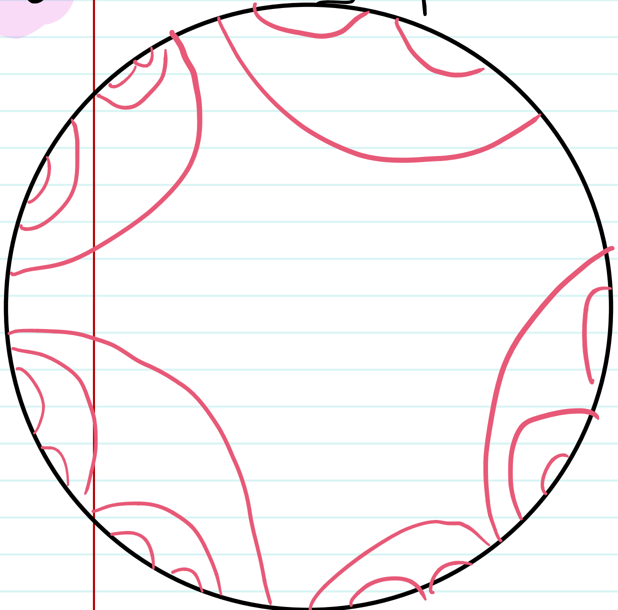
Identify the universal cover with $\mathbb{H}^2$

Observations:

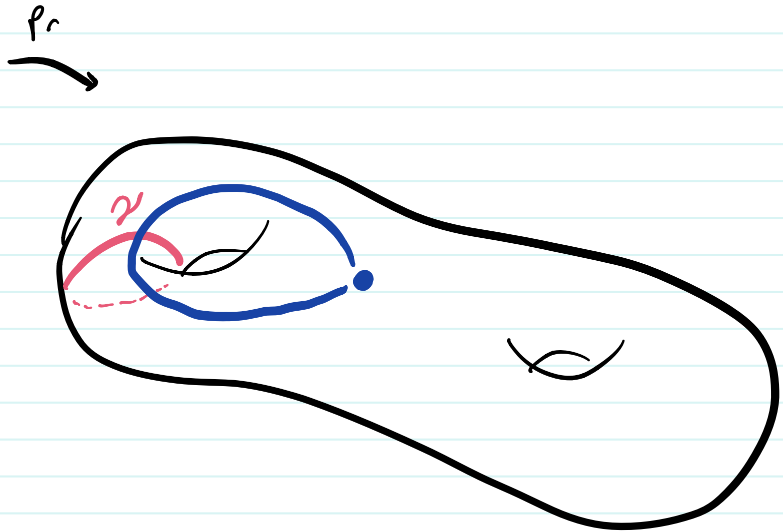




Examples : Surface groups



Observations:



Three ways to think about graphs

#1 **Def** A **graph** is a 1-dimensional CW-complex.

#2 **Def** (Gersten) a graph is a set Γ, an involution $\bar{\ }:\Gamma \longrightarrow \Gamma$, and a retraction $\tau:\Gamma \longrightarrow \Gamma$ onto the fixed-point set for $\bar{\ }$.

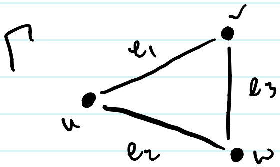
The fixed point set $V(\Gamma)$ is the set of **vertices**, its complement $E(\Gamma) = \Gamma - V(\Gamma)$ is the set of **oriented edges**.

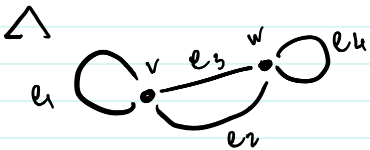
$\bar{\ }:\Gamma \longrightarrow \Gamma$ reverses the orientation of edges, and τ sends an edge e to its **terminal vertex**.

Three Ways to think about graphs

#3 A graph Γ (without loops, if you'd like) determines a **small category**

Ex





The rule for constructing the category is...

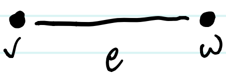
Definition of a graph of groups

Def A **graph of groups** is a pair $(\Gamma, \mathcal{G})$

- Γ is a connected graph
- $\mathcal{G}$ is a **functor** $\mathcal{G} : \Gamma \longrightarrow \text{Groups}^{\text{mono}}$

All this really means is an assignment:

Ex.

Γ 

The "quotient" graph of groups

Let G be a group, and suppose G acts on a tree T.
The action is **without inversions** [in edges]
if for every edge e of T with incident vertices v and w,
if $g.e = g$, then $g.v = v$ and $g.w = w$

Why is this useful?

The "quotient" graph of groups

So: $G \curvearrowright T$ a tree w/o inversions. Let Γ be $G \backslash T$.

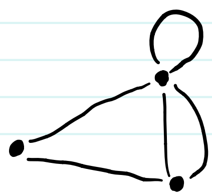
Choose a **spanning tree** $F \leq \Gamma$, and choose a lift $\hat{F} \subset T$. If v is a vertex of Γ, write $\tilde{v}$ for the corresponding vertex in $\tilde{F}$.

We will define $G_v =$

For e an edge in F, write $\tilde{e}$ for its lift in $\tilde{F}$.

Define $G_e =$

Ex. Γ



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The "quotient" graph of groups

Finally, suppose e is an edge in $\Gamma \backslash F$.
Choose a lift $\tilde{e}$ of e so that $\tau(\tilde{e}) = \widetilde{\tau(e)}$

Define $\mathcal{G}_e = \mathrm{stab}(\tilde{e})$ and $\iota_e : \mathcal{G}_e \longrightarrow \mathcal{G}_{\tau(e)}$
to be the inclusion as before.

By definition, there is some $g \in G$ such that
$g.\tau(\tilde{e}) \in \tilde{F}$.

If h stabilizes $\tilde{e}$, notice that

Examples

Fundamental Theorem

<u>Thm</u> (Bass–Serre) If $(\Gamma, \mathcal{G})$ is a graph of groups and $p \in \Gamma$, there exists a group

$$G = \pi_1(\Gamma, \mathcal{G}, p)$$

and a tree T with an action of G without inversions such that the quotient graph of groups is $(\Gamma, \mathcal{G})$