Evaluating Fourier Transform of a Step Function

Ryan Li

We will assess a piecewise function that varies between 0 and 1, which repeats after a period of 2pi.

> $f := piecewise(0 < t < \text{Pi}, 1, \text{Pi} < t < 2\cdot\text{Pi}, 0);$

> $with(plots) : plot(f, t = 0..2\,\text{Pi});$

In order to find the FFT of the function, we will first denote a[n] to be the coefficients of the nth cosine term, and b[n] to be the coefficients of the nth sine term. To find the appropriate a[n] and b[n] for f(t), we will use the formulas

$$\text{a[n]}=\frac{\int_0^{2\,\text{Pi}} f(t)\cos(nt)\,dt}{\text{pi}}\;;\;\text{b[n]}=\frac{\int_0^{2\,\text{Pi}} f(t)\sin(nt)\,dt}{\text{pi}}$$

For the first term a[0], we get 1/2. In order to simplify integrals, we notice here that since f(t) = 0 from pi to 2pi, we can simply evaluate the integrals $\int_0^{\text{pi}} \cos(nt)\,dt$ and $\int_0^{\text{pi}} \sin(nt)\,dt$ for a[n] and b[n], respectively, as f(t) is always 1 for t=0..pi. Solving for a and b for n=1..10:

> $a := n \rightarrow \frac{int(\cos(n\cdot t), t = 0..\text{Pi})}{\text{Pi}};$

> $seq(a[p] = a(p), p = 1..10);$

$$a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = 0,\ a_7 = 0,\ a_8 = 0,\ a_9 = 0,\ a_{10} = 0 \quad \textbf{(1)}$$

> $b := \text{n} \rightarrow \frac{int(\sin(n\cdot t), t = 0..\text{Pi})}{\text{Pi}};$

> $seq(b[p] = b(p), p = 1..10);$

$$b_1 = \frac{2}{\pi},\ b_2 = 0,\ b_3 = \frac{2}{3\,\pi},\ b_4 = 0,\ b_5 = \frac{2}{5\,\pi},\ b_6 = 0,\ b_7 = \frac{2}{7\,\pi},\ b_8 = 0,\ b_9 = \frac{2}{9\,\pi},\ b_{10} = 0 \quad \textbf{(2)}$$

From here, we notice that all coefficients are 0 except for odd intervals of b[n], where these coefficients are in the form 2/(n*pi). Thus, we can deduce that the FFT for the stepwise function takes the form

$$f = \frac{1}{2} + \sum_{n=1}^{N}\left(\frac{2}{(2n-1)\cdot\pi}\sin((2n-1)\cdot t)\right),$$

with N terms. This ensures that (2n-1) is always odd. Graphically, for N=5, 10, 20, 40, 80, 160, 320, 640, we get:

> $with(plots) :$

> $Sn := \frac{1}{2} + 'sum\left(\frac{2}{(2n-1)\cdot\pi}\sin((2n-1)\cdot t), n = 1..5\cdot\left(2^{(N-1)}\right)\right)';$

> $plots[display]([seq(animate(plot, [Sn, t = 0..x], x = 0..2\pi), N = 1..8)], insequence);$

> $plots[display]([seq(plot([Sn, \text{f}], t = 0..2\,\text{Pi}), N = 1..8)], insequence);$