Evaluating Fourier Transform of a Step Function

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We will assess a piecewise function that varies between 0 and 1, which repeats after a period of 2pi.

> $f := piecewise(0 < t < \mathrm{Pi}, 1, \mathrm{Pi} < t < 2\cdot\mathrm{Pi}, 0);$

$$f := \begin{cases} 1 & 0 < t \textbf{ and } t < \pi \\ 0 & \pi < t \textbf{ and } t < 2\pi \end{cases} \tag{1}$$

> $with(plots) : plot(f, t = 0..2\,\mathrm{Pi});$

In order to find the FFT of the function, we will first denote a[n] to be the coefficients of the nth cosine term, and b[n] to be the coefficients of the nth sine term. To find the appropriate a[n] and b[n] for f(t), we will use the formulas

$$\text{a[n]=}\frac{\int_0^{2\,\mathrm{Pi}} f(t)\cos(nt)\,dt}{\text{pi}}\ ;\ \text{b[n]=}\frac{\int_0^{2\,\mathrm{Pi}} f(t)\sin(nt)\,dt}{\text{pi}}$$

For the first term, which is the a[0], we get the average value of f(t), which is 1/2. In order to simplify integrals, we notice here that since f(t) = 0 from pi to 2pi, we can simply evaluate

$$\text{a[n]=}\frac{\int_0^{\text{pi}} \cos(nt)\,dt}{\text{pi}}\ ;\ \text{b[n]=}\frac{\int_0^{\text{pi}} \sin(nt)\,dt}{\text{pi}}\ ,$$

as f(t) is always 1 for t=0..pi. Solving for both a and b for n=1..10:

> $a := n \rightarrow \frac{int(\cos(n\cdot t), t = 0..\mathrm{Pi})}{\mathrm{Pi}};$

$$a := n \rightarrow \frac{\int_0^{\pi} \cos(n\,t)\,dt}{\pi} \tag{2}$$

> $seq(a[p] = a(p), p = 1..10);$

$$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0, a_7 = 0, a_8 = 0, a_9 = 0, a_{10} = 0 \tag{3}$$

> $b := \mathrm{n} \rightarrow \frac{int(\sin(n\cdot t), t = 0..\mathrm{Pi})}{\mathrm{Pi}};$

$$b := n \rightarrow \frac{\int_0^{\pi} \sin(n\,t)\,dt}{\pi} \tag{4}$$

> $seq(b[p] = b(p), p = 1..10);$

$$b_1 = \frac{2}{\pi}, b_2 = 0, b_3 = \frac{2}{3\,\pi}, b_4 = 0, b_5 = \frac{2}{5\,\pi}, b_6 = 0, b_7 = \frac{2}{7\,\pi}, b_8 = 0, b_9 = \frac{2}{9\,\pi}, b_{10} = 0 \tag{5}$$

From here, we notice that all coefficients are 0 except for odd intervals of b[n], where these coefficients are in the form 2/(n*pi). Thus, we can deduce that the FFT for the stepwise function takes the form

$$f = \frac{1}{2} + \sum_{n=1}^{N} \left(\frac{2}{(2n-1) \cdot \pi} \sin((2n-1) \cdot t) \right),$$

with N terms. This ensures that (2n-1) is always odd. Graphically, for N=5, 10, 20, 40, 80, 160, 320, 640, we get:

> $with(plots):$

> $Sn := \frac{1}{2} +'sum\left(\frac{2}{(2n-1) \cdot \pi} \sin((2n-1) \cdot t), n = 1..5 \cdot \left(2^{(N-1)}\right) \right)';$

$$Sn := \frac{1}{2} + \sum_{n=1}^{5\,2^{N-1}} \frac{2 \sin((2\,n-1)\,t)}{(2\,n-1)\,\pi} \qquad \textbf{(6)}$$

> $plots[display]([seq(animate(plot, [Sn, t = 0..x], x = 0..2\pi), N = 1..8)], insequence);$

> $plots[display]([seq(plot([Sn, \mathrm{f}], t = 0..2\,\mathrm{Pi}), N = 1..8)], insequence);$