EXPLORING THE CHARACTER OF SIMPLE PENDULUMS

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ABSTRACT

A simple pendulum was tested with three bob masses, three lengths, and seven release angles to determine the effects of each on period. The regression of the period against amplitude was found to be $T(L,\theta_0) = 2\sqrt{L}\left[1 + (0.072 \pm .008)\theta_0{}^2\right]$, and was shown to model the measured perturbation of large angles to an R^2 value of 0.97. This function suggests that a maximum amplitude of (21.3±.4)° for the small angle approximation to maintain a perturbation below 1% of the ideal period; the upper bound was rounded to 20° for this report. It was found that the damping coefficient of the pendulums was inversely proportional to the bob mass by a function of $k = \frac{(0.0030 \pm .0006)}{m} s^{-1}$.

I INTRODUCTION

Pendulum swings are an example of a regular, repeated, and measurable motion that can easily be modelled in ideal situations. However, when placed in an experimental setting, a pendulum's motion is perturbed by various factors. An analysis of these perturbations can depict the significance of small external forces that act upon practical structures, demonstrating the external considerations that must be accounted for when designing precise systems.

II THEORY

An ideal simple pendulum's period is:

$$T_0(L) = 2\pi\sqrt{\frac{L}{g}} \qquad [1]$$

(HRW, 2011)

Where:
- T_0 = Ideal period of the pendulum (s)
- L = Pendulum radius to centre of mass (m)
- g = Gravitational acceleration on Earth (ms^{-2})

The period of a practical pendulum can be approximated by:

$$T(L,\theta_0) = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{\theta_0{}^2}{16}\right) \qquad [2]$$

(Svirin, 2019)

Where:
- T = Period of pendulum (s)
- θ_0 = Amplitude (rad)

By rotational kinematics, the velocity of a mass on a pendulum is given by:

$$v = \omega L \qquad [3]$$

(HRW, 2011)

Where:
- v = Velocity of object (ms^{-1})
- ω = angular velocity (rad s^{-1})

For objects travelling at low speeds, the drag force can be approximated by:

$$F_D = -c_D v \qquad [4]$$

(PS, 2017)

Where:
- F_D = Drag force (N)
- c_D = Drag constant (kgs^{-1})

More generally, drag is given by:

$$F_D = -\frac{C\rho A v^2}{2} \quad [5]$$

(HRW, 2011)

Where:

C = Coefficient of drag
ρ = Density of the fluid (kgm^{-3})
A = Cross sectional area of the object (m^2)

In a damped pendulum, Svirin states that the total torque equals the sum of torques on the bob mass (2019). Thus:

$$-mgLsin\theta + LF_D = \tau \quad [6]$$

$$-mgLsin\theta + LF_D = mL^2\ddot{\theta} \quad [7]$$

Where:

m = Mass of pendulum (kg)
θ = Angular position of pendulum mass (rad)
τ = Net torque of the pendulum (Nm)

III Method

A test tube clamp was attached orthogonally to a retort stand. A fishing line was tied at two points on the clamp to allow pendulum bobs to swing on a plane as indicated by van Bemmel (2019). Three hook bob masses were obtained; their mass and dimensions were measured with a digital scale and Vernier caliper. The pendulum radius was measured from the top of the bob to the bottom of the test tube clamp, adding half the bob height to obtain the radius to its centre of mass. Measurements were effected 30 times in pursuit of the law of large numbers (Sternstein, 2017).

A metre stick was then placed such that it was coincident with the pendulum motion, starting directly under the bob mass rest position. Three arbitrary lengths were selected for the pendulum, and the release angles 5°, 10°, 15°, 20°, 30°, 45°, and 60° were tested. With trigonometric ratios, release distances that corresponded to the desired release angles were produced. The masses were held and released parallel to the string using tweezers as to impart minimal force on the bob on release. One Vernier Motion Detector 2 was placed in line with the pendulum to measure the displacement and amplitude of small angle releases. The period of the pendulum motion was measured with a Vernier Photogate secured under the bob mass resting position. Both sensors took 31 samples per second, the maximum sampling rate to avoid error as advised by the Logger Pro Program. The motion detector was tested prior to usage by calibrating its read distances against a metre stick. The Photogate was tested by checking if real time data matched up with the motion of the pendulum.

IV Data

The measurements of each bob mass and the three pendulum lengths are shown below. As it was noted that the heavier masses would slightly stretch the fishing line, adjustments were made to the line to maintain a consistent length for the masses.

	Light	Medium	Heavy
Height (cm)	5.145±.006	5.142±.008	5.143±.008
Diameter (cm)	2.435±.007	2.512±.008	2.521±.007
Mass (±.1 g)	17.0	70.4	207.6

Fig 1. Dimensions of the Pendulum Bobs. The chart above displays the averages of the measured heights, diameters, and masses of the 3 different pendulum bobs.

	Long	Medium	Short
Radius (cm)	46.7±.2	30.4±.1	14.4±.1

Fig 2. Pendulum Radius. The chart above displays the average measured radius of the pendulum to the bobs' centre of mass.

V Analysis

To derive the motion of a pendulum with a damping force, [7] was simplified and rearranged with [3], [4] and [5] to obtain:

$$\ddot{\theta} + k\dot{\theta} + \frac{g}{L}sin\theta = 0 \quad [8]$$

Where:

k = Coefficient of damping drag force (s^{-1})

The coefficient of the damping force in this case is a function of mass:

$$k = \frac{C\rho A}{m} \quad [9]$$

For smaller angles, the small angle approximation $\sin\theta = \theta$ is used so that [8] becomes a linear DE, which produces a general solution of the form:

$$\theta(t) = \theta_0 e^{\frac{-kt}{2}} \cos(\omega_0 t - \phi) \quad [10]$$

(HRW, 2011)

Where:

ω_0 = Angular frequency (rad s^{-1})
ϕ = Phase shift (rad)
t = Time (s)

The effect of large angles on the period was then measured. By subtracting the ideal period [1] from [2], it can be seen that the period correction factor is approximately a quadratic function of amplitude multiplied by ideal period. Therefore, the period data from the Photogate could be divided by the ideal period and the empirical period correction factor could then be regressed. Conversely, regressing against the bob mass produces coefficients within the same signal-noise range, so the period function is assumed to be independent of mass. The period function is shown below:

$$T(L, \theta_0) = 2\sqrt{L}\left[1 + (0.072 \pm .008){\theta_0}^2\right] \quad [11]$$

The form of the ideal period now assumes that $\pi \approx \sqrt{g}$ to simplify [1] and the regression. Comparing the measured period in terms of [1] yields:

$$T = T_0 + T_0(0.072 \pm .008){\theta_0}^2 \quad [12]$$

The second term of [12] describes the perturbation of the period with a coefficient that closely resembles the series in [2]. When this expected perturbation is plotted against the true perturbation for higher angles, Fig. 3 shows the accuracy of the function for modelling the empirical period.

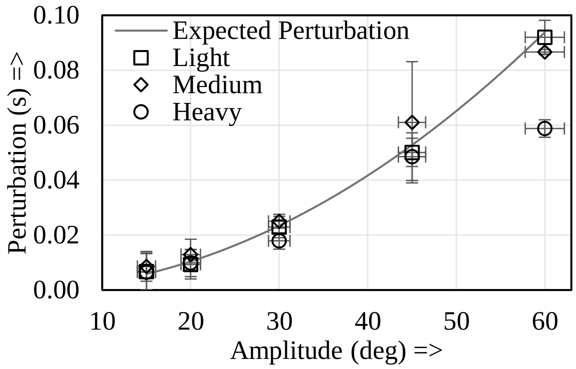

Fig 3. Measured and Expected Perturbation vs Amplitude. The average perturbations measured for higher angle releases, plotted with the expected perturbation from [10]. Points are shown for the light, medium, and heavy mass. The R^2 value was determined to be 0.97.

Based on the regressed period correction factor in [12], the threshold for the accepted small angle range can be selected. If the perturbation threshold is arbitrarily selected to be 1% of the ideal period, then the quadratic should equal, at most, 1.01. This result occurs at (21.3±.4)°. Thus, the small angle approximation is accepted up to 20° in this report, rounded down to decrease error while satisfying the selected threshold. Above this, the measured period is much greater than ideal for extended swing periods and the small angle approximation fails.

When the force of drag is considered to be the damping force on the pendulum, the damping coefficient can be approximated from [9]. From Fig 1, the average cross-sectional area of the masses was found to be (1.303±.006)E-3 m^2. Furthermore, the drag coefficient of a cylinder can be approximated as (1.00±.05) (Engineers Edge, 2017) and the air density is cited as (1.2041±.0005) kgm^{-3} (Helmenstine, 2019) in conditions similar to that of the lab room. Finally, the velocity of a pendulum is directly proportional to its radius. By multiplying these values in [9], the predicted coefficient is:

$$k_0 = \frac{(0.0016 \pm .0007)}{m} \quad [13]$$

Where:

k_0 = predicted value of damping coefficient (s^{-1})

To find the damping coefficients involved, each run with an amplitude within the small angle threshold of 20° was regressed using the Microsoft Excel Solver add-in fitting to [10]. To find the optimal fit, the sum of the squares of the residuals was minimized. The resulting R^2 values for the resulting fits were all seen to be over 0.95, establishing the validity of the regressions.

The constants found through the regression could then be analyzed. The amplitude values were used to find the true initial angle that the masses were released at. A linear regression was then found relating the pendulum mass to the damping coefficient in accordance with [13]. This resulted in the following equation, which is approximately double the predicted value.

$$k = \frac{(0.0030 \pm .0006)}{m} \quad [14]$$

This discrepancy is attributed to the force of friction between the string and pendulum pivot and the drag force on the fishing line, which were not taken into account in the prediction as they were believed to be negligible. Then, substituting [1] and [14] in [10], the small angle approximation solution of the pendulum motion with respect to time can be approximated by [15]. The ideal period [1] can be accepted to the upper bound of 20°.

$$\theta(t) = \theta_0 e^{\frac{(-0.0015 \pm .0003)t}{m}} \cos\left(\sqrt{\frac{g}{L}}\, t\right) \quad [15]$$

VI Sources of Error

The primary source of error came from the release of the mass. Since it was released manually, it may have been released at slightly different angles over the runs, and the hand may have imparted some degree of rotation or vibration onto the mass.

The bob mass was also observed at times to oppose the pendulum motion, either by wobbling, or rotating and winding the fishing line. This may have affected the motion sensor measurements, as the readings were assumed to be indicative of the bob's centre of mass location. The observed swaying may have caused measurements of the distance from the motion sensor to the pendulum mass to be lower than the actual value when the mass was on the closer side to the motion sensor. Likewise, the measured distance might have been slightly higher than the actual value when the mass is on the side of the pendulum further from the motion sensor.

VII Conclusion

The pendulum's period was found to be $T(L, \theta_0) = 2\sqrt{L}\left[1 + (0.072 \pm .008){\theta_0}^2\right]$. With this, a maximum amplitude of (21.3±.4)° was found to have a perturbation within 1% of the ideal period for small angle approximations. The damping coefficient was predicted to be $k_0 = \frac{(0.0016 \pm .0007)}{m}\, s^{-1}$ and was practically measured to be $k = \frac{(0.0030 \pm .0006)}{m} s^{-1}$ considering only drag. With these, the pendulum's motion at small angles was experimentally found to be $\theta(t) = \theta_0 e^{\frac{(-0.0015 \pm .0003)t}{m}} \cos\left(\sqrt{\frac{g}{L}}\, t\right)$.

VIII Sources

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