Sample Problem 1 – 1D Collision

Question: An object of mass $m_1 = 2\,\mathrm{kg}$, moving with velocity $v_{i1} = 12\,\mathrm{m/s}$, collides head-on with a stationary object whose mass is $m_2 = 6\,\mathrm{kg}$. Given that the collision is elastic, what are the final velocities of the two objects. Neglect friction.
Answer: Momentum conservation yields

$$m_1\,v_{i1} = m_1\,v_{f1} + m_2\,v_{f2},$$

where v_{f1} and v_{f2} are the final velocities of the first and second objects, respectively. Since the collision is elastic, the total kinetic energy must be the same before and after the collision. Hence,

$$\frac{1}{2}\,m_1\,v_{i1}^2 = \frac{1}{2}\,m_1\,v_{f1}^2 + \frac{1}{2}\,m_2\,v_{f2}^2.$$

Let $x = v_{f1}/v_{i1}$ and $y = v_{f2}/v_{i1}$. Noting that $m_2/m_1 = 3$, the above two equations reduce to

$$1 = x + 3\,y,$$

and

$$1 = x^2 + 3\,y^2.$$

Eliminating x between the previous two expressions, we obtain

$$1 = (1 - 3\,y)^2 + 3\,y^2,$$

or

$$6\,y\,(2\,y - 1) = 0,$$

which has the non-trivial solution $y = 1/2$. The corresponding solution for x is $x = (1 - 3\,y) = -1/2$.
It follows that the final velocity of the first object is

$$v_{f1} = x\,v_{i1} = -0.5 \times 12 = -6\,\mathrm{m/s}.$$

The minus sign indicates that this object reverses direction as a result of the collision. Likewise, the final velocity of the second object is

$$v_{f2} = y\,v_{i1} = 0.5 \times 12 = 6\,\mathrm{m/s}.$$

Simulation Sample Output

Index	t (s)	sx1 (m)	sx2 (m)	vx1 (ms^{-1})	vx2 (ms^{-1})	Distance (m)
1	0	0.012	5	12	0	4.988
2	0.2686	3.2296	5.002	10.96	0.3467	1.7724
3	0.2786	3.3312	5.0081	9.2164	0.9279	1.6769
4	0.2886	3.4111	5.0214	6.652	1.7827	1.6103
5	0.2986	3.4624	5.0443	3.5586	2.8138	1.5819
6	0.3086	3.482	5.0778	0.3815	3.8728	1.5958
7	0.3186	3.4714	5.1214	-2.4094	4.8031	1.65
8	0.3286	3.4363	5.173	-4.4572	5.4857	1.7367
9	0.3386	3.3852	5.2301	-5.6156	5.8719	1.8449
10	0.3517	3.3081	5.3081	-5.9998	5.9999	2

Fig 1. Simulation 1 Sample Output. The table above summarizes ten lines of output by the simulation for the first sample problem.

Simulation x-t Graph

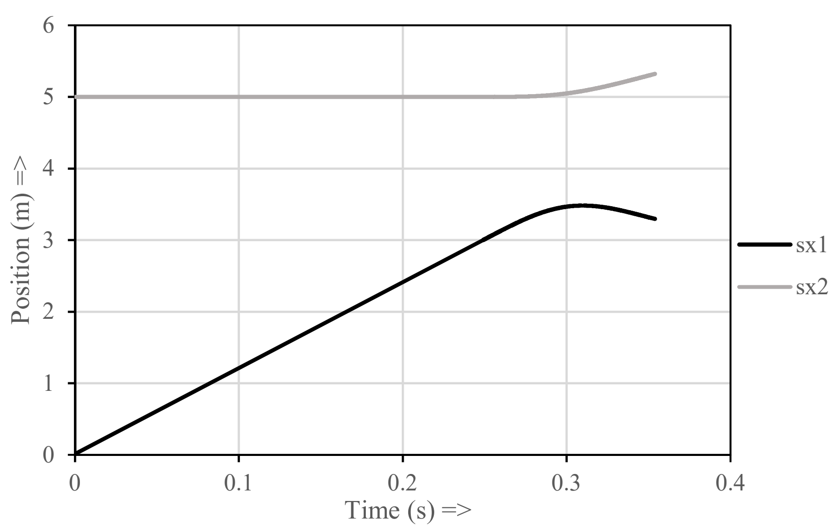

Fig 2. Position versus Time for 1D sample question. Positions are taken from the center of mass from each object, both of which had a radius of 1m. Graph depicts time from start of simulation until the collision has completely ended.

Comparison of Solution to Simulation:

	Worked Solution	Simulation	Residuals
Initial velocity of object 1 (ms^{-1})	12	12	0
Initial velocity of object 2 (ms^{-1})	0	0	0
Final velocity of object 1 (ms^{-1})	-6	-5.9998	0.0002
Final velocity of object 2 (ms^{-1})	6	5.9999	-0.0001

Fig 3. Velocity Comparison for Sample 1. A comparison of the initial and final velocities of each object in sample 1 between the worked solution and the simulation. Differences are shown in the right-most column.

Justification of Increment:

The time increment was dynamically set according to whether the objects had collided yet. If the objects were not in motion, then the increment would be calculated by dividing the distance between the two objects by the velocity of the system. This ratio would indicate approximately the amount of time required for the objects to make contact. The ratio was then divided by 50 to allow for a reasonable amount of points before and after collision, but a lower bound was set at 1E-3 seconds for when the two objects were near collision. When making contact, the increment was set to 1E-5 seconds in order to best mimic the continuous nature of the balls' motion while not returning an unnecessary number of data points. Given the constantly changing velocities and forces involved, a greater increment would incur greater error because its discontinuity. The data for the balls would then be printed for every 40 increments for brevity. Then, the stopping criteria for the simulation was a second of time passing following the end of the collision. This allowed for the simulated objects to display a reasonable displacement after the collision.

Sample Problem 2 – 2D Glancing Collision with Stationary Target

95 SSM In the arrangement of Fig. 9-21, billiard ball 1 moving at a speed of 2.2 m/s undergoes a glancing collision with identical billiard ball 2 that is at rest. After the collision, ball 2 moves at speed 1.1 m/s, at an angle of $\theta_2 = 60°$. What are (a) the magnitude and (b) the direction of the velocity of ball 1 after the collision? (c) Do the given data suggest the collision is elastic or inelastic?

95. The mass of each ball is m, and the initial speed of one of the balls is $v_{1i} = 2.2\,\text{m/s}$. We apply the conservation of linear momentum to the x and y axes, respectively:

$$mv_{1i} = mv_{1f}\cos\theta_1 + mv_{2f}\cos\theta_2$$
$$0 = mv_{1f}\sin\theta_1 - mv_{2f}\sin\theta_2.$$

The mass m cancels out of these equations, and we are left with two unknowns and two equations, which is sufficient to solve.

(a) Solving the simultaneous equations leads to

$$v_{1f} = \frac{\sin\theta_2}{\sin(\theta_1+\theta_2)}v_{1i}, \quad v_{2f} = \frac{\sin\theta_1}{\sin(\theta_1+\theta_2)}v_{1i}.$$

Since $v_{2f} = v_{1i}/2 = 1.1\,\text{m/s}$ and $\theta_2 = 60°$, we have

$$\frac{\sin\theta_1}{\sin(\theta_1+60°)} = \frac{1}{2} \quad \Rightarrow \quad \tan\theta_1 = \frac{1}{\sqrt{3}}$$

or $\theta_1 = 30°$. Thus, the speed of ball 1 after collision is

$$v_{1f} = \frac{\sin\theta_2}{\sin(\theta_1+\theta_2)}v_{1i} = \frac{\sin 60°}{\sin(30°+60°)}v_{1i} = \frac{\sqrt{3}}{2}v_{1i} = \frac{\sqrt{3}}{2}(2.2\,\text{m/s}) = 1.9\,\text{m/s}.$$

(b) From the above, we have $\theta_1 = 30°$, measured *clockwise* from the $+x$-axis, or equivalently, $-30°$, measured *counterclockwise* from the $+x$-axis.

(c) The kinetic energy before collision is $K_i = \frac{1}{2}mv_{1i}^2$. After the collision, we have

$$K_f = \frac{1}{2}m\left(v_{1f}^2 + v_{2f}^2\right).$$

Substituting the expressions for v_{1f} and v_{2f} found above gives

$$K_f = \frac{1}{2}m\left[\frac{\sin^2\theta_2}{\sin^2(\theta_1+\theta_2)} + \frac{\sin^2\theta_1}{\sin^2(\theta_1+\theta_2)}\right]v_{1i}^2.$$

Since $\theta_1 = 30°$ and $\theta_2 = 60°$, $\sin(\theta_1+\theta_2) = 1$ and $\sin^2\theta_1 + \sin^2\theta_2 = \sin^2\theta_1 + \cos^2\theta_1 = 1$, and indeed, we have $K_f = \frac{1}{2}mv_{1i}^2 = K_i$, which means that energy is conserved.

Simulation Sample Output

Index	t (s)	sx1 (m)	sy1 (m)	sx2 (m)	sy2 (m)	vx1 (ms^{-1})	vy1 (ms^{-1})	vx2 (ms^{-1})	vy2 (ms^{-1})
1	0	0.0022	0	3	0.0518	2.2	0	0	0
2	0.9094	2.8028	0	3	0.0518	2.1939	-0.0107	0.0061	0.0107
3	0.911	2.8063	-0.0001	3	0.0518	2.1487	-0.0892	0.0513	0.0892
4	0.9126	2.8097	-0.0003	3.0002	0.0519	2.0758	-0.2166	0.1242	0.2166
5	0.9142	2.8129	-0.0008	3.0005	0.052	1.9845	-0.3766	0.2155	0.3766
6	0.9158	2.816	-0.0015	3.0009	0.0521	1.8865	-0.5489	0.3135	0.5489
7	0.9174	2.819	-0.0025	3.0015	0.0522	1.7944	-0.7114	0.4056	0.7114
8	0.919	2.8218	-0.0038	3.0022	0.0535	1.7199	-0.8434	0.4801	0.8434
9	0.9206	2.8245	-0.0052	3.003	0.0601	1.6723	-0.928	0.5277	0.928
10	0.9341	2.8468	-0.0181	3.0103	1.031	1.6574	-0.9544	0.5425	0.9544

Fig 4. Simulation 2 Sample Output. The table above summarizes ten lines of output by the simulation for the first sample problem.

Simulation y-x Graph:

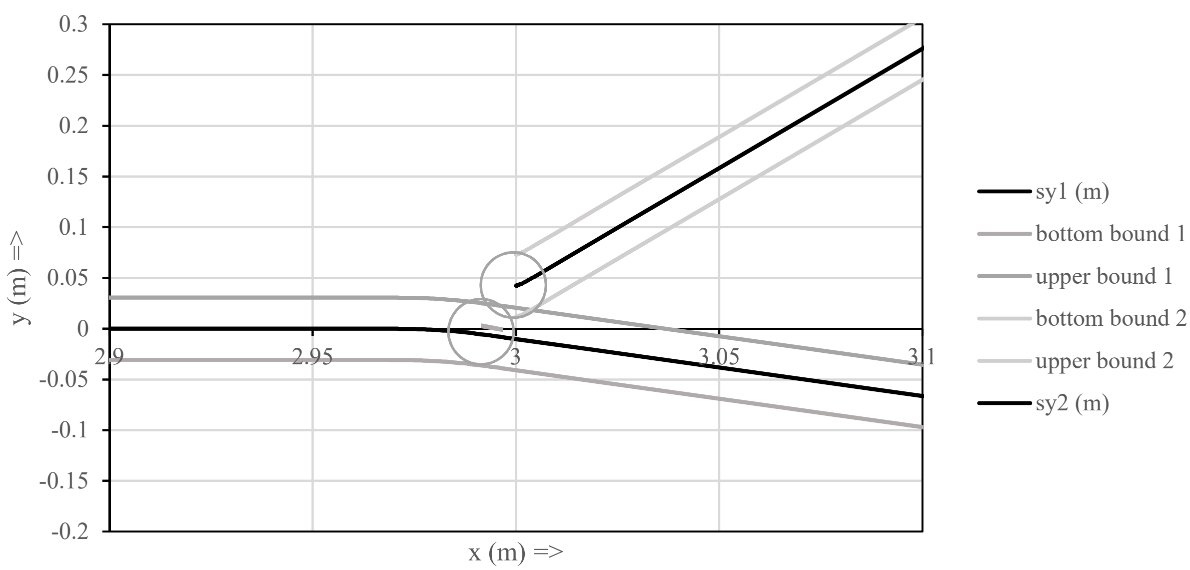

Fig 5. Position versus Time for 2D non-moving target. Positions are taken from the center of mass from each object, both of which had the experimentally measured radius and mass of the stress ball. Graph depicts positions starting from around 0.2m before the collision for clarity.

Comparison of Solution to Simulation:

	Worked Solution	Simulation	Residuals
V_{fx1} (ms^{-1})	1.645	1.657	0.012
V_{fx2} (ms^{-1})	0.55	0.543	-0.007
V_{fy1} (ms^{-1})	-0.953	-0.954	-0.001
V_{fy2} (ms^{-1})	0.953	0.954	0.001

Fig 6. Velocity Comparison for Sample 2. A comparison of the initial and final velocities of each object in sample 1 between the worked solution and the simulation. Differences are shown in the right-most column.

Justification of Increment:

The time increment was dynamically set according to whether the objects had collided yet. If the objects were not in motion, then the increment would be calculated by dividing the distance between the two objects by the velocity of the system. This ratio would indicate approximately the amount of time required for the objects to make contact. The ratio was then divided by 50 to allow for a reasonable amount of points before and after collision, but a lower bound was set at 1E-3 seconds for when the two objects were near collision. When making contact, the increment was set to 1E-5 seconds in order to best mimic the continuous nature of the balls' motion while not returning an unnecessary number of data points. Given the constantly changing velocities and forces involved, a greater increment would incur greater error because its discontinuity. The data for the balls would then be printed for every 40 increments for brevity. Then, the stopping criteria for the simulation was a second of time passing following the end of the collision. This allowed for the simulated objects to display a reasonable displacement after the collision.

Sample Problem 3 – 2D Glancing Collision with Moving Target

84 Figure 9-73 shows an overhead view of two particles sliding at constant velocity over a frictionless surface. The particles have the same mass and the same initial speed $v = 4.00$ m/s, and they collide where their paths intersect. An x axis is arranged to bisect the angle between their incoming paths, such that $\theta = 40.0°$. The region to the right of the collision is divided into four lettered sections by the x axis and four numbered dashed lines. In what region or along what line do the particles travel if the collision is (a) completely inelastic, (b) elastic, and (c) inelastic? What are their final speeds if the collision is (d) completely inelastic and (e) elastic?

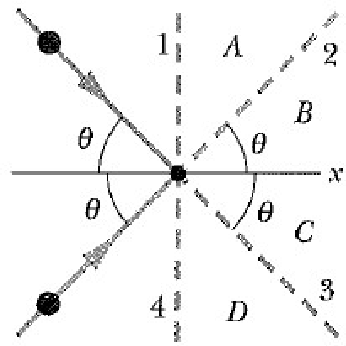

Fig. 9-73 Problem 84.

From the above photo, 84b) and e) were used. The solutions are shown below:

Part b:

For elastic collision, the component along the axis

$$m_1 v_1 \cos\theta + m_2 v_2 \cos\theta = m_1 v_{1f} \cos\theta_{1f} + m_2 v_{2f} \cos\theta_{2f}$$

Substitute the givens

$$2mv\cos\theta = mv_{1f}\cos\theta_{1f} + mv_{2f}\cos\theta_{2f}$$
$$2v\cos\theta = v_{1f}\cos\theta_{1f} + v_{2f}\cos\theta_{2f}$$

Noted that the in elastic collision there is no loss of energy and the speeds are the same, so that the angles will be the same as θ too.
The motion along lines 2 and 3.

Part e:

There is no speed loss so that

$$\boxed{v_f = 4 \text{ m/s}}$$

Simulation Sample Output

Index	t (s)	sx1 (m)	sy1 (m)	sx2 (m)	sy2 (m)	vx1 (ms^{-1})	vy1 (ms^{-1})	vx2 (ms^{-1})	vy2 (ms^{-1})	distance (m)
1	0	-3.8269	3.2114	-3.8269	-3.2114	3.064	-2.571	3.064	2.571	6.4229
2	0.6858	-1.7258	1.4484	-1.7258	-1.4484	3.064	-2.571	3.064	2.571	2.8967
3	0.9914	-0.7893	0.6625	-0.7893	-0.6625	3.064	-2.571	3.064	2.571	1.3251
4	1.1276	-0.3719	0.3123	-0.3719	-0.3123	3.064	-2.571	3.064	2.571	0.6246
5	1.1883	-0.1859	0.1562	-0.1859	-0.1562	3.064	-2.571	3.064	2.571	0.3124
6	1.2283	-0.0633	0.0534	-0.0633	-0.0534	3.064	-2.571	3.064	2.571	0.1067
7	1.2377	-0.0348	0.0296	-0.0348	-0.0296	3.064	-1.8407	3.064	1.8407	0.0592
8	1.2381	-0.0336	0.0296	-0.0336	-0.0296	3.064	0.9175	3.064	-0.9175	0.0592
9	1.2385	-0.0323	0.0304	-0.0323	-0.0304	3.064	2.5057	3.064	-2.5057	0.0608
10	2.0074	2.3239	2.1242	2.3239	-2.1242	3.064	2.5683	3.064	-2.5683	4.2484

Fig 7. Simulation 3 Sample Output. The table above summarizes ten lines of output by the simulation for the first sample problem.

Simulation y-x Graph

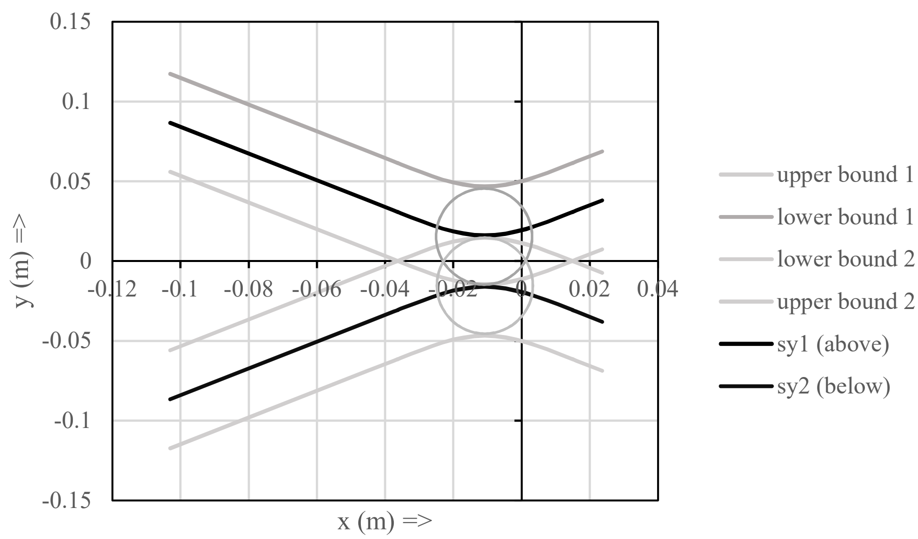

Fig 8. y versus x Positions for Sample 3. Positions are taken from the center of mass from each object, both of which have the measured radius and mass of the stress ball, which are (.0307±.0001)m and (.0198±.0001)kg, respectively.

Comparison of Solution to Simulation:

	Worked Solution	Simulation	Residuals
V_{fx1} (ms^{-1})	3.064	3.064	0
V_{fx2} (ms^{-1})	3.064	3.064	0
V_{fy1} (ms^{-1})	2.571	2.5683	-0.0027
V_{fy2} (ms^{-1})	-2.571	-2.5683	0.0027

Fig 9. Velocity Comparison for Sample 3. A comparison of the initial and final velocities of each object in sample 1 between the worked solution and the simulation. Differences are shown in the right-most column.

Justification of Increment:

The time increment was dynamically set according to whether the objects had collided yet. If the objects were not in motion, then the increment would be calculated by dividing the distance between the two objects by the velocity of the system. This ratio would indicate approximately the amount of time required for the objects to make contact. The ratio was then divided by 50 to allow for a reasonable amount of points before and after collision, but a lower bound was set at 1E-3 seconds for when the two objects were near collision. When making contact, the increment was set to 1E-5 seconds in order to best mimic the continuous nature of the balls' motion while not returning an unnecessary number of data points. Given the constantly changing velocities and forces involved, a greater increment would incur greater error because its discontinuity. The data for the balls would then be printed for every 40 increments for brevity. Then, the stopping criteria for the simulation was a second of time passing following the end of the collision. This allowed for the simulated objects to display a reasonable displacement after the collision.