Practice 3: Random Variables' Simulation

1. Let U be the uniform distribution on $[0,1]$ and $p \in]0,1[$, set

$$X = \begin{cases} 1 & if\; U < p \\ 0 & if\; not \end{cases}$$

   i. Check that X is following a Bernoulli distribution with parameter p.

   ii. Use the latter to write an R-function that generates Bernoulli random numbers, call your function `Bernoulli(n,p)` where n and p denotes respectively the number of desired random numbers and the parameter of the Bernoulli distribution.

   iii. Write an R-function `SLLN(N,p)` that for a given $N \in \mathbb{N}$ and $p \in]0,1[$ generates N Bernoulli numbers $\{Bernoulli_i(p) : i = 1, \cdots, N\}$ and that compute for each $k = 1, \cdots, N$ the summation

$$\frac{1}{k}\sum_{i=1}^{k} Bernoulli_i.$$

   Plots the summations vector against `1:N`.
   Comment on.

2. Write your own generator of Binomial numbers, `Binomial(N,n,p)` where N is the desired number of Binomial numbers and (n,p) are the parameters of the Binomial distribution.

3. Let U be the uniform distribution on $[0,1]$ and $p_1, \cdots, p_n \in]0,1[$ for a given $n \in \mathbb{N}$, set

$$X = \begin{cases} 1 & if\;\; U < p_1 \\ 2 & if\;\; p_1 \le U < p_1 + p_2 \\ 3 & if\;\; p_1 + p_2 \le U < p_1 + p_2 + p_3 \\ \vdots & \vdots \\ n & if\;\; p_1 + \cdots + p_{n-1} \le U < 1 \end{cases}$$

   i. Check that X is distributed as

X	$\mathbb{P}_X$
1	p_1
$\vdots$	$\vdots$
n	p_n

   ii. Use the latter to write an R-function that generates random numbers following (3i), call your function `DiscreteGenerator(N,p)` where N is is the desired number of random numbers and $p = (p_1, \cdots, p_n)$.

4. Let U be the uniform distribution on $[0, 1]$ and let F be a given CDF function that is increasing. Set $X = F^{-1}(U)$.

   i. Check that $X \sim dF$.

   ii. Deduce an algorithm to generate random numbers following dF.

   iii. Write an R-function that generates Exponential numbers, call it `Exponential(N,λ)` where N is the desired number of random numbers and $\lambda > 0$ is the parameter of the Exponential distribution.

   iv. Check that the sum of n independent random variables following the same exponential distribution $\mathcal{E}(\lambda)$ is following a gamma distribution $\Gamma(n, \lambda)$.

   v. Deduce and write an R-function that generates Gamma numbers, name it `Gamma(N,n,λ)` where N is the desired number of random numbers and (n, λ) are parameters of the Gamma distribution.