Products of Scott topologies and spatiality

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A **DCPO** is a poset with directed joins. For a DCPO A, the **Scott topology** on A has closed sets which are closed downwards and under directed joins. It is well-known that for two DCPOs A, B, the Scott topology on $A \times B$ may be strictly finer than the product of Scott topologies; see [G$^+$, II-4.13]. In this note, we explain this phenomenon as an instance of non-spatiality of locales.

By convention, we distinguish between locales X and frames $\mathcal{O}(X)$. Recall that X is **spatial** if for every $U \not\subseteq V \in \mathcal{O}(X)$, there is a point in U but not in V.

For a DCPO A, we may interpret its Scott topology $\mathcal{O}(A)$ as the space of points of a locale with the same name, defined by declaring its frame of opens to be

$$\mathcal{O}(\mathcal{O}(A)) := \langle A \text{ qua } \mathsf{DCPO} \rangle_{\mathsf{Frm}} = \text{free frame over } A\text{, preserving directed joins.}$$

The injection of generators $[-] : A \to \mathcal{O}(\mathcal{O}(A))$ takes each $a \in A$ to "$[a] = \{U \in \mathcal{O}(A) \mid a \in U\}$". The space of points of $\mathcal{O}(A)$ is the space of frame homomorphisms $\mathcal{O}(\mathcal{O}(A)) \to 2$, or equivalently Scott-continuous $A \to 2$, i.e., Scott-open sets in A; and indeed a point U is in $[a]$ iff $a \in U$.

For DCPOs A, B, we define the product map $\times : \mathcal{O}(A) \times \mathcal{O}(B) \to \mathcal{O}(A \times B)$ (where $\mathcal{O}(A) \times \mathcal{O}(B)$ denotes the product locale, while $A \times B$ is the product poset) via its dual frame homomorphism

$$\begin{aligned} \times^* : \mathcal{O}(\mathcal{O}(A \times B)) &\longrightarrow \mathcal{O}(\mathcal{O}(A) \times \mathcal{O}(B)) = \mathcal{O}(\mathcal{O}(A)) \otimes \mathcal{O}(\mathcal{O}(B)) \\ [(a,b)] &\longmapsto [a] \times [b], \end{aligned}$$

i.e., "$U \times V \in [(a,b)] \iff (a,b) \in U \times V \iff (U,V) \in [a] \times [b]$". The Scott topology on $A \times B$ agrees with the product of Scott topologies iff for every $(a,b) \in W \in \mathcal{O}(A \times B)$, there is $(U,V) \in \mathcal{O}(A) \times \mathcal{O}(B)$ such that $(a,b) \in U \times V \subseteq W$; in other words, iff for every

$$[(a,b)] \not\subseteq \neg\overline{\{W\}} \in \mathcal{O}(\mathcal{O}(A \times B)), \tag{1}$$

there is a point witnessing the non-containment

$$\times^*([(a,b)]) \not\subseteq \times^*(\neg\overline{\{W\}}) \in \mathcal{O}(\mathcal{O}(A) \times \mathcal{O}(B)). \tag{2}$$

We would thus like to deduce this latter non-containment from the former.

A **preframe** is a DCPO which is also a (unital) meet-semilattice, with Scott-continuous $\wedge$; we denote the category of all such by PFrm. We also denote the category of **suplattices** (complete join-semilattices) by Sup.

Lemma 3. For DCPOs A, B and a preframe (resp., suplattice) C, Scott-continuous maps

$$A \times B \longrightarrow C$$

are in order-isomorphism with Scott-continuous maps

$$\langle A \rangle \times \langle B \rangle \longrightarrow C$$

preserving finite meets (resp., joins) separately in each variable, where $\langle - \rangle = \langle - \rangle_{\mathsf{PFrm}}$ (resp., $\langle - \rangle_{\mathsf{Sup}}$).

Proof. Maps of the latter form curry to morphisms $\langle A\rangle \to [\langle B\rangle, C]$, where $[-,-]$ denotes internal hom with pointwise partial order, in the respective category; these in turn correspond to Scott-continuous maps $A \to [\langle B\rangle, C] \cong \mathsf{DCPO}(B, C)$, or $A \times B \to C$. □

Corollary 4. For DCPOs A, B, the canonical map

$$A \times B \longrightarrow \langle A\rangle_{\mathsf{PFrm}} \times \langle B\rangle_{\mathsf{PFrm}}$$

given by the product of the injections of generators is an embedding for the Scott topologies.

Proof. Given a Scott-open $W \subseteq A \times B$, its characteristic function is a Scott-continuous $A \times B \to 2$, which extends by the previous lemma to a Scott-continuous (and separately meet-preserving) $\langle A\rangle \times \langle B\rangle \to 2$, which corresponds to a Scott-open $W' \subseteq \langle A\rangle \times \langle B\rangle$ restricting to W. □

Corollary 5. For DCPOs A, B, the canonical map

$$\langle A\rangle_{\mathsf{Sup}} \otimes \langle B\rangle_{\mathsf{Sup}} \longrightarrow \langle A\rangle_{\mathsf{Frm}} \otimes \langle B\rangle_{\mathsf{Frm}}$$

is an order-embedding.

Proof. Recall that for a DCPO C, $\langle C\rangle_{\mathsf{Sup}}$ can be constructed as the lattice of Scott-closed sets of C, and that the same construction also yields the free frame over a preframe; see [JV]. By Lemma 3 and the universal property of the suplattice tensor product,

$$\begin{aligned}
\langle A\rangle_{\mathsf{Sup}} \otimes \langle B\rangle_{\mathsf{Sup}} &\cong \langle A \times B\rangle_{\mathsf{Sup}}, \\
\langle A\rangle_{\mathsf{Frm}} \otimes \langle B\rangle_{\mathsf{Frm}} &\cong \langle \langle A\rangle_{\mathsf{PFrm}} \text{ qua } \mathsf{PFrm}\rangle_{\mathsf{Frm}} \otimes \langle \langle B\rangle_{\mathsf{PFrm}} \text{ qua } \mathsf{PFrm}\rangle_{\mathsf{Frm}} \\
&\cong \langle \langle A\rangle_{\mathsf{PFrm}} \text{ qua } \mathsf{DCPO}\rangle_{\mathsf{Sup}} \otimes \langle \langle B\rangle_{\mathsf{PFrm}} \text{ qua } \mathsf{DCPO}\rangle_{\mathsf{Sup}} \\
&\cong \langle \langle A\rangle_{\mathsf{PFrm}} \times \langle B\rangle_{\mathsf{PFrm}} \text{ qua } \mathsf{DCPO}\rangle_{\mathsf{Sup}}.
\end{aligned}$$

Thus the canonical map in question is isomorphic to image-closure under the canonical map $A \times B \to \langle A\rangle_{\mathsf{PFrm}} \times \langle B\rangle_{\mathsf{PFrm}}$, whose right adjoint, preimage, is surjective by the previous corollary. □

Corollary 6. For DCPOs A, B, whenever a non-containment as in (1) holds, then so does (2).

Proof. The composite

$$\langle A \times B\rangle_{\mathsf{Sup}} \hookrightarrow \langle A \times B\rangle_{\mathsf{Frm}} = \mathcal{O}(\mathcal{O}(A \times B)) \xrightarrow{\times^*} \mathcal{O}(\mathcal{O}(A) \times \mathcal{O}(B)) = \langle A\rangle_{\mathsf{Frm}} \otimes \langle B\rangle_{\mathsf{Frm}}$$

is the canonical map of the preceding corollary, hence an order-embedding. Thus it is enough to check that the open $\neg\overline{\{W\}}$ appearing in (1) belongs to $\langle A \times B\rangle_{\mathsf{Sup}} \subseteq \mathcal{O}(\mathcal{O}(A \times B))$, i.e., is a join of generators $[(a, b)]$, not just join of finite meets of such. Indeed, $\neg\overline{\{W\}} \in \mathcal{O}(\mathcal{O}(A \times B))$ is by definition the largest open not containing the point $W \in \mathcal{O}(A \times B)$; since $\mathcal{O}(\mathcal{O}(A \times B)) = \langle A \times B\rangle_{\mathsf{Frm}}$ consists of joins of finite meets of $[(a, b)]$, and such a finite meet does not contain W iff at least one of the $[(a, b)]$ does not, which means $(a, b) \notin W$, we have $\neg\overline{\{W\}} = \bigcup_{(a,b)\notin W} [(a, b)] \in \langle A \times B\rangle_{\mathsf{Sup}}$. □

Corollary 7. If $\mathcal{O}(A) \times \mathcal{O}(B)$ is spatial, then the Scott and product topologies on $A \times B$ agree. □

References

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