On sifted colimits in the presence of pullbacks

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Abstract

We show that in a category with pullbacks, arbitrary sifted colimits may be constructed as filtered colimits of reflexive coequalizers. This implies that "lex sifted colimits", in the sense of Garner–Lack, decompose as Barr-exactness plus filtered colimits commuting with finite limits. We also prove the refinements of these results for κ-small sifted and filtered colimits. Along the way, we prove a general technical result showing that the κ-small restriction of a saturated class of colimits is still "closed under iteration".

1 Introduction

A category is called **sifted** if the category of cocones over any finite discrete family of objects in it is connected. The significance of this notion is that sifted colimits are precisely those which commute with finite products in the category of sets. Thus, sifted colimits exist in any finitary universal-algebraic variety and are computed on the level of the underlying sets. For background on sifted colimits and their key role in categorical universal algebra, see [AR01], [ARV11].

The main examples of sifted colimits are filtered colimits and **reflexive coequalizers**, i.e., coequalizers of parallel pairs of morphisms $X \rightrightarrows Y$ with a common section $Y \to X$. It is well-known that these two types of colimits "almost" suffice to generate all sifted colimits. To state this precisely, recall that by general principles [Kel82, 5.35], every category C has a *free cocompletion* under any given class of colimits, which can be explicitly constructed as the full subcategory of the presheaf category $[\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ on the closure of the representables under said colimits. Let

$$\begin{aligned} \mathsf{Sind}(\mathsf{C}) &:= \text{free cocompletion of } \mathsf{C} \text{ under small sifted colimits,} \\ \mathsf{Ind}(\mathsf{C}) &:= \text{free cocompletion of } \mathsf{C} \text{ under small filtered colimits,} \\ \mathsf{Rec}(\mathsf{C}) &:= \text{free cocompletion of } \mathsf{C} \text{ under reflexive coequalizers.} \end{aligned}$$

Now Adámek–Rosický [AR01, 2.3(2)] (see also [ARV11, 7.3]) showed that for C with finite coproducts,

$$\mathsf{Sind}(\mathsf{C}) \simeq \mathsf{Ind}(\mathsf{Rec}(\mathsf{C})).$$

It follows that if C also has small sifted colimits, then those may be constructed as filtered colimits of reflexive coequalizers. This then implies that a functor $F : \mathsf{C} \to \mathsf{D}$ preserving these latter types of colimits also preserves all sifted ones, as shown by Joyal [Joy08, 33.24], Lack [LR11, 3.2], and Adámek–Rosický–Vitale [ARV10, 2.1]. However, some such assumption on C as existence of finite coproducts is needed in all of these results: Adámek–Rosický–Vitale [ARV10, §1] give counterexamples for a general C.

The main goal of this paper is to show that sifted colimits may be constructed as filtered colimits of reflexive coequalizers, in all the precise senses just described, assuming that C has pullbacks instead of finite coproducts. In fact, we prove a "relative" version of this, where all colimits are bounded in size by some regular cardinal $\kappa \leq \infty$; this can be useful in applications where size issues might arise. The precise statements are given by Theorem 5.1 and Corollary 5.2.

A consequence (Corollary 6.1) is that "lex sifted colimits" in the sense of Garner–Lack [GL12], meaning sifted colimits obeying all compatibility or "exactness" conditions with finite limits that hold in Set, are equivalent to the combination of Barr-exactness plus filtered colimits commuting with finite limits. Accessible such categories are related to Grothendieck toposes via a lax-idempotent 2-adjunction, which gives rise to Johnstone's upper bagdomain monad [Joh94]. This connection was our original motivation for the present work, and will be studied in detail in a future paper.

Sections 2 and 3 form the combinatorial core of the paper, in which we describe the elementary interactions between pullbacks and sifted colimits underlying our main result.

Section 4 is somewhat of a digression from the rest of the paper. In order to prove the κ-small version of our main result, we need the κ-small analog of the fact (due to [AR01]) that a sifted colimit of sifted colimits is a sifted colimit. It turns out that this can be deduced in an abstract manner from the known fact for $\kappa = \infty$, and for not just sifted colimits but an arbitrary "saturated class of colimits" Φ, in the sense of Albert–Kelly [AK88]. We prove this generalized result in Proposition 4.6 and Corollary 4.7, and give one other application: we rederive (in Corollary 4.13) Makkai–Paré's [MP89, 2.3.11] "retract-free" characterization of λ-presentable objects in κ-accessible categories.

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2 Reflexive coequalizers and pullbacks

Throughout this paper, "category" will mean locally small category, so that we have a Yoneda embedding, denoted $\mathsf{y} = \mathsf{y}_{\mathsf{C}} : \mathsf{C} \to [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$; we will sometimes treat y as an inclusion.

We begin by describing the free reflexive-coequalizer cocompletion $\mathsf{Rec}(\mathsf{C})$ of a category with pullbacks C. The construction is the same as that of Pitts (see [BC95, §2], [ARV11, 17.12]) for C with finite coproducts. Informally speaking, coproducts allow a coequalizer of coequalizers to be reduced to a single coequalizer, by taking the "union" of the edge-sets of the two graphs involved; when C instead has pullbacks, the "concatenation" graph may be used instead of the "union".

By a **graph** on an object X in a category C, we will mean an arbitrary parallel pair $p, q : G \rightrightarrows X$ with codomain X; the graph is **reflexive** if p, q have a common section $r : X \to G$ (i.e., $pr = qr = 1_X$). By abuse of terminology, we will often refer to the graph by G instead of p, q. For another graph $s, t : H \rightrightarrows X$, we say that G is **contained** in H if p, q jointly factor through s, t via some morphism $f : G \to H$, i.e., $sf = p$ and $tf = q$. If C has pullbacks, the **concatenation** of graphs $p, q : G \rightrightarrows X$ and $s, t : H \rightrightarrows X$ is the pullback

$$\begin{tikzcd}
X & & X & & X \\
& G \arrow[ul, "p"'] \arrow[ur, "q"] & & H \arrow[ul, "s"'] \arrow[ur, "t"] & \\
& & K = G \times_X H \arrow[ul, "v"] \arrow[ur, "w"'] & &
\end{tikzcd}$$

regarded as a graph via $pv, tw : K \rightrightarrows X$.

We record the following easy facts about graphs, which we will freely use:

Lemma 2.1.

(a) A graph $p, q : G \rightrightarrows X$ is reflexive iff it contains the identity graph $1_X, 1_X : X \rightrightarrows X$.

(b) If a graph G is contained in H, then a morphism coequalizing H also coequalizes G.

(c) For graphs G, H, K on X fitting into a diagram as above (without K necessarily being the pullback), any morphism coequalizing both G, H also coequalizes K.

(d) If G, H are graphs on X, and G is reflexive, then H is contained in $G \times_X H$ and $H \times_X G$.

(e) Thus, if G, H are both reflexive, then so is $G \times_X H$, and for any functor $F : \mathsf{C} \to \mathsf{D}$, a morphism coequalizes $F(G \times_X H) \rightrightarrows F(X)$ iff it coequalizes both $F(G), F(H)$.

Proof. (d) If p, q have common section $r : X \to G$, then s, t jointly factor through pv, tw via $(rs, 1_H) : H \to G \times_X H$; similarly for $H \times_X G$.

(e) The first claim follows from (a) and (d); the second follows from (b), (c), and (d). $\square$

For an arbitrary category C, as noted in the Introduction, the free reflexive-coequalizer cocompletion $\mathsf{Rec}(\mathsf{C})$ may be constructed as the full subcategory of $[\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ obtained by closing the representables under reflexive coequalizers, with the Yoneda embedding $\mathsf{y} : \mathsf{C} \to \mathsf{Rec}(\mathsf{C}) \subseteq [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ as unit. In particular, $\mathsf{Rec}(\mathsf{C})$ contains the coequalizers, in $[\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$, of all reflexive graphs in C.

Proposition 2.2. *For a category with pullbacks C, the full subcategory of $[\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ on the coequalizers of reflexive graphs in C is already closed under reflexive coequalizers, hence is $\mathsf{Rec}(\mathsf{C})$.*

Proof. Consider a reflexive parallel pair f, g, with common section h, between the coequalizers U, V in $[\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ of two reflexive graphs $p, q : G \rightrightarrows X$ and $s, t : H \rightrightarrows Y$ in C:

$$\begin{tikzcd}
\mathsf{y}G \arrow[d, shift right, "\mathsf{y}p"'] \arrow[d, shift left, "\mathsf{y}q"] & & \mathsf{y}H \arrow[d, shift right, "\mathsf{y}s"'] \arrow[d, shift left, "\mathsf{y}t"] \\
\mathsf{y}X \arrow[rr, shift left=2, "\mathsf{y}\widetilde{f}"] \arrow[rr, shift right=2, "\mathsf{y}\widetilde{g}"'] \arrow[d, two heads] & & \mathsf{y}Y \arrow[ll, dashed, "\mathsf{y}\widetilde{h}" description] \arrow[d, two heads] \\
U \arrow[rr, shift left=2, "f"] \arrow[rr, shift right=2, "g"'] & & V \arrow[ll, dashed, "h" description]
\end{tikzcd} \tag{2.3}$$

As in [BC95, §2], we may describe f, g, h explicitly as follows: f descends from a morphism $\mathsf{y}X \to V$ (coequalizing $\mathsf{y}p, \mathsf{y}q$), which corresponds by the Yoneda lemma to an element of $V(X)$, i.e., an equivalence class of morphisms $\widetilde{f} : X \to Y$ with respect to the equivalence relation generated by the graph $\mathsf{C}(X, s), \mathsf{C}(X, t) : \mathsf{C}(X, H) \rightrightarrows \mathsf{C}(X, Y)$. Similarly, g, h lift to some $\widetilde{g}, \widetilde{h}$ as shown. To say that $fh = 1_V$ means that $\widetilde{f}\widetilde{h} : Y \to Y$ is equivalent to 1_Y via the equivalence relation generated by $\mathsf{C}(Y, s), \mathsf{C}(Y, t) : \mathsf{C}(Y, H) \rightrightarrows \mathsf{C}(Y, Y)$, which means they are connected by a "homotopy in H":

$$\begin{tikzcd}
& Y & & Y & & Y & \cdots & Y \\
X \arrow[ur, "\widetilde{f}"] & & H \arrow[ul, "s"'] \arrow[ur, "t"] & & H \arrow[ul, "t"'] \arrow[ur, "s"] \\
Y \arrow[u, "\widetilde{h}"] \arrow[urr, dashed] \arrow[urrrr, dashed] \arrow[uurrrrrrr, bend right, "1_Y"']
\end{tikzcd}$$

Similarly, $gh = 1_V$ means that 1_Y is connected via a "homotopy" to $\widetilde{\widetilde{g}h}$. Pasting the latter "homotopy" to the left of the former one shows that the concatenation graph

$$K := \cdots \times_Y H \times_Y H^{\mathsf{op}} \times_Y X \times_Y H \times_Y H^{\mathsf{op}} \times_Y \cdots \rightrightarrows Y$$

(where H^{op} is H but with the roles of s, t swapped) is reflexive. Since H is reflexive, K contains the graph $\widetilde{f}, \widetilde{g} : X \rightrightarrows Y$. Now concatenating K once more with H yields a reflexive graph L on Y which contains both X and H and is also a concatenation of copies of X, H, H^{op}, hence has the same coequalizer in $[\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ as the joint coequalizer of $\widetilde{f}, \widetilde{g} : X \rightrightarrows Y$ and $s, t : H \rightrightarrows Y$, which is easily seen to be the same as the coequalizer of $f, g : U \rightrightarrows V$ (see diagram (2.3) above). $\square$

3 Sifted categories with pullbacks

Lemma 3.1. *In a sifted category with pullbacks C, every graph $p, q : G \rightrightarrows X$ is contained in a reflexive graph $s, t : H \rightrightarrows X$.*

Proof. Since C is sifted, there is a zigzag connecting the cospans $G \xrightarrow{p} X \xleftarrow{1} X$ and $G \xrightarrow{q} X \xleftarrow{1} X$:

$$\begin{tikzcd}
 & X_1 & & \cdots & & \\
X = X_0 \arrow[ur, dashed] & & X_2 \arrow[ul, dashed] \arrow[ur, dashed] & & X_{2n} = X & \\
 & G \arrow[ul, "p"] \arrow[uu, dashed] \arrow[ur, dashed] \arrow[urrr, "q"'] & & X \arrow[ulll, "1"] \arrow[uull, dashed] \arrow[ul, dashed] \arrow[ur, "1"'] & &
\end{tikzcd}$$

Repeatedly replace each "peak" $X_i \to X_{i+1} \leftarrow X_{i+2}$ by its pullback, to get a single "valley"

$$\begin{tikzcd}
X & & X \\
 & H \arrow[ul, dashed, "s"'] \arrow[ur, dashed, "t"] & \\
G \arrow[uu, "p"] \arrow[ur, dashed] \arrow[uurr, "q"'] & & X \arrow[uu, "1"'] \arrow[ul, dashed] \arrow[uull, "1"]
\end{tikzcd}$$

$\square$

The following forms the combinatorial core of our main result (Theorem 5.1):

Proposition 3.2. *For a sifted category with pullbacks C, $\mathsf{Rec}(\mathsf{C})$ is filtered.*

Proof. Clearly $\mathsf{Rec}(\mathsf{C})$ is nonempty because C is. Now let $U, V \in \mathsf{Rec}(\mathsf{C})$; by Proposition 2.2, they are the coequalizers of (reflexive) graphs $p, q : G \rightrightarrows X$ and $s, t : H \rightrightarrows Y$ in C. We may find a cospan over U, V by finding a cospan $X \to Z \leftarrow Y$ in C, finding reflexive graphs on Z containing the composite graphs $G \rightrightarrows X \to Z$ and $H \rightrightarrows Y \to Z$, concatenating them, and taking the reflexive coequalizer in $\mathsf{Rec}(\mathsf{C})$. Given a parallel pair $f, g : U \rightrightarrows V$, as in the proof of Proposition 2.2, we may lift them to $\widetilde{f}, \widetilde{g} : X \rightrightarrows Y$; we may find a morphism coequalizing f, g by finding a reflexive graph X' on Y containing $\widetilde{f}, \widetilde{g} : X \rightrightarrows Y$, concatenating it with H, and taking the reflexive coequalizer in $\mathsf{Rec}(\mathsf{C})$, yielding the joint coequalizer of X', H (see diagram (2.3)). $\square$

4 Saturated classes of κ-small colimits

As noted in the Introduction, this section is a digression from our main focus; the reader interested only in sifted colimits may take Corollary 4.9 as a black box, and skip to our main result in the next section. In order to use Proposition 3.2 above to prove our main result, we need an explicit description of the free sifted-cocompletion $\mathsf{Sind}(\mathsf{C})$; this is provided by [AR01, 2.6], which shows that $\mathsf{Sind}(\mathsf{C}) \subseteq [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ consists of the sifted colimits of representables. In other words, recalling that $\mathsf{Sind}(\mathsf{C})$ is by general principles [Kel82, 5.35] the *closure* of the representables under sifted colimits, [AR01, 2.6] shows that iteration is unnecessary: a sifted colimit of sifted colimits is again a sifted colimit. Since we wish to prove our main result also for κ-small sifted colimits, we need to know that these too are closed under iteration; this is the content of Corollary 4.9. It turns out that this can be deduced from the $\kappa = \infty$ case, and in an entirely abstract manner having nothing to do with siftedness *per se*: we will prove (see Proposition 4.6, Corollary 4.7 and following remarks) that for any "class of colimits Φ closed under iteration", the κ-small Φ-colimits are also "closed under iteration". This general result is itself the extension to $\kappa < \infty$ of a result of Albert–Kelly [AK88, 7.4] (which suffices for the $\kappa = \infty$ case of our main result in the next section).

We begin by recalling the relevant precise notion of a "class of colimits Φ". For details, see [AK88], [KS05]. However, our presentation differs slightly from these references, in that we do not initially restrict the weights in Φ to have small domain; we may therefore identify the saturation Φ^* with the free Φ-cocompletion monad. This is so that we may later discuss, in a uniform manner for all $\kappa \leq \infty$, the case where Φ is generated by κ-small weights, thereby making clear the connection between our results and [AK88]. (Details on our differing conventions are given in the footnotes.)

Recall [Kel82, §3.4] that given any category J and presheaf $\phi \in [\mathsf{J}^{\mathsf{op}}, \mathsf{Set}]$, we may take the **$\phi$-weighted colimit** $\phi \star F$ of a diagram $F : \mathsf{J} \to \mathsf{C}$, which is the same as the ordinary colimit of

$$\mathsf{y}_{\mathsf{J}}{\downarrow}\phi \to \mathsf{J} \xrightarrow{F} \mathsf{C},$$

i.e., F applied to the canonical diagram over the category of elements $\mathsf{y}_{\mathsf{J}}{\downarrow}\phi$ of ϕ. We will call ϕ a **small presheaf** if J is small; in that case, $\mathsf{y}_{\mathsf{J}}{\downarrow}\phi$ is small, so the weighted colimit $\phi \star F$ is a small colimit. More generally, we will call ϕ **small-presented** if it is a small colimit of representables, in which case we can always take ϕ to be the $\phi|\mathsf{K}^{\mathsf{op}}$-weighted colimit of the inclusion of a small full subcategory $\mathsf{K} \subseteq \mathsf{J}$; then a ϕ-weighted colimit $\phi \star F$ is the same as the small colimit $\phi|\mathsf{K}^{\mathsf{op}} \star F|\mathsf{K}$.[1]

For any category C, let

$$\mathsf{Psh}(\mathsf{C}) \subseteq [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$$

denote the full subcategory of small-presented presheaves, which is the free cocompletion of C under all small colimits by [Kel82, 5.35]. The universal property of Psh gives it the structure of a lax-idempotent 2-(pseudo)monad on the 2-category of all (locally small) categories, consisting of

(4.1) for each category C, the unit $\mathsf{y}_{\mathsf{C}} : \mathsf{C} \to \mathsf{Psh}(\mathsf{C})$;

(4.2) for each C, the multiplication $\mathrm{Lan}_{\mathsf{y}_{\mathsf{Psh}(\mathsf{C})}}(1_{\mathsf{Psh}(\mathsf{C})}) : \mathsf{Psh}(\mathsf{Psh}(\mathsf{C})) \to \mathsf{Psh}(\mathsf{C})$, taking $\phi \mapsto \phi \star 1_{\mathsf{Psh}(\mathsf{C})}$;

(4.3) for $F : \mathsf{C} \to \mathsf{D}$, the induced cocontinuous functor $\mathrm{Lan}_{F^{\mathsf{op}}} : \mathsf{Psh}(\mathsf{C}) \to \mathsf{Psh}(\mathsf{D})$, taking $\phi \mapsto \phi \star \mathsf{y}F$;

as usual for monads, given (4.1), we may combine (4.2) and (4.3) into

[1] Our "small-presented" is called "accessible" in [Kel82], [AK88], [KS05], and "small" in many other works. Our "small" is called such in [Kel82], but is called a "weight" (as opposed to general presheaf) in [KS05] and other works.

(4.4) for $F : \mathsf{C} \to \mathsf{Psh}(\mathsf{D})$, the Kleisli extension $\mathrm{Lan}_{\mathsf{y}_\mathsf{C}}(F) : \mathsf{Psh}(\mathsf{C}) \to \mathsf{Psh}(\mathsf{D})$, taking $\phi \mapsto \phi \star F$.

If y_C has a partial left adjoint defined at some $\phi \in \mathsf{Psh}(\mathsf{C})$, the value must be the colimit $\phi \star 1_\mathsf{C}$. Thus the algebras of the monad, i.e., those C for which y_C has a total left adjoint, are precisely the cocomplete categories. Similarly, the algebra homomorphisms are the cocontinuous functors.

Let Φ be a class of small-presented presheaves $\phi \in \mathsf{Psh}(\mathsf{C})$ on arbitrary categories C. We identify such a Φ with the map taking each C to the full subcategory[2]

$$\Phi(\mathsf{C}) := \mathsf{Psh}(\mathsf{C}) \cap \Phi \subseteq \mathsf{Psh}(\mathsf{C});$$

we therefore conversely use Psh to name the class of *all* small-presented presheaves. A **Φ-colimit** means a colimit weighted by some $\phi \in \Phi$; a category C is **Φ-cocomplete** if it has all Φ-colimits; and a functor is **Φ-cocontinuous** if it preserves all Φ-colimits. The **saturation** Φ^* of Φ is given by

$$\Phi^*(\mathsf{C}) := \text{closure of representables in } [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}] \text{ under } \Phi\text{-colimits.}$$

Since every ϕ is the ϕ-weighted colimit of representables $\phi \star \mathsf{y}$, we have $\Phi \subseteq \Phi^*$. By cocontinuity of $\star$ in the weight [Kel82, 3.23], the class of weights ψ for which a Φ-cocomplete category is ψ-cocomplete, respectively, for which a Φ-cocontinuous functor is ψ-cocontinuous, is closed under Φ-colimits; thus

$$\Phi\text{-cocomplete} \iff \Phi^*\text{-cocomplete}, \qquad \Phi\text{-cocontinuous} \iff \Phi^*\text{-cocontinuous}.$$

In particular, $\Phi^*(\mathsf{C})$, being by definition closed under Φ-colimits, is also closed under Φ^*-colimits; that is, $\Phi^{**} = \Phi^*$, so that $\Phi \mapsto \Phi^*$ is a closure operation on the lattice of subclasses of Psh. Note that an equivalent definition of this closure operation is

$$\Phi^* = \text{closure of } \Phi \text{ under the monad unit (4.1) and Kleisli extension (4.4) of } \mathsf{Psh}.$$

Thus the saturated classes Φ^* are precisely the **full submonads of Psh** (borrowing terminology from [GL12, §3]), i.e., each $\Phi^*(\mathsf{C}) \subseteq \mathsf{Psh}(\mathsf{C})$ is a full (replete) subcategory, and the monad operations of Psh restrict to Φ^*, making it into a lax-idempotent 2-monad in its own right. The Φ^*-algebras are those C for which $\mathsf{y} : \mathsf{C} \to \Phi(\mathsf{C})$ has a left adjoint, which means that $\phi \star 1_\mathsf{C}$ exists for each $\phi \in \Phi^*(\mathsf{C})$; since Φ^* is closed under (4.3), this implies that $\mathrm{Lan}_{F^{\mathsf{op}}}(\phi) \star 1_\mathsf{C} = \phi \star F$ exists for each $F : \mathsf{J} \to \mathsf{C}$ and $\phi \in \Phi^*(\mathsf{J})$, i.e., that C is Φ^*-cocomplete ($= \Phi$-cocomplete). Similarly, Φ-algebra homomorphisms are precisely the $\Phi^{(*)}$-cocontinuous functors.

Now let $\kappa \le \infty$ be an infinite regular cardinal (where ∞ is the bound on the fixed universe of small sets). By **κ-small**, we will generally mean of size $< \kappa$. A **κ-small presheaf** $\phi : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$ is one where (i) C is κ-small, and (ii) ϕ takes values in the full subcategory Set_κ of κ-small sets; note that these imply that $\mathsf{y}{\downarrow}\phi$ is κ-small, hence a ϕ-weighted colimit is a κ-small colimit.

Lemma 4.5. *If Φ is a class of κ-small presheaves, and C is κ-small, then $\Phi^*(\mathsf{C})$ consists of κ-small presheaves.*

Proof. $[\mathsf{C}^{\mathsf{op}}, \mathsf{Set}_\kappa]$ contains the representables and is closed under κ-small, hence Φ-, colimits. $\square$

[2] In [AK88], [KS05] and most other works on classes of colimits, our $\Phi(\mathsf{C})$ is denoted $\Phi[\mathsf{C}]$, while $\Phi(\mathsf{C})$ is used to denote our $\Phi^*(\mathsf{C})$, which for small C is also denoted $\Phi^*[\mathsf{C}]$ (and could also be denoted $\Phi^*(\mathsf{C})$) by the main result of [AK88]. Since we allow Φ to consist of small-presented, rather than just small, weights, there is no need for us to distinguish between the saturation Φ^* and free Φ-cocompletion submonad, so we have chosen this simpler notation.

We now have the main result of this section, which extends [AK88, 7.4] to the case $\kappa < \infty$. The main new difficulty compared to the case $\kappa = \infty$ is that C is only locally small (by our standing assumption), not locally κ-small; hence the need to consider non-full subcategories.

Proposition 4.6. *Let $\kappa \le \infty$ be uncountable regular, and let Φ be a class of κ-small presheaves. Then for any C and $\phi : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$, we have $\phi \in \Phi^*(\mathsf{C})$ iff it is the left Kan extension of some $\psi \in \Phi^*(\mathsf{D})$ (which is κ-small, by the preceding lemma) for some κ-small subcategory $\mathsf{D} \subseteq \mathsf{C}$.*

Proof. $\Longleftarrow$ is because Φ^* is closed under (4.3). For $\Longrightarrow$, since the conclusion is clearly satisfied by the representables, it suffices to check that if $\theta \in \Phi(\mathsf{J})$ and $F : \mathsf{J} \to [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ such that the conclusion holds for $F(J)$ for each $J \in \mathsf{J}$, then it also holds for $\theta \star F$. For each $J \in \mathsf{J}$, let $F(J)$ be the left Kan extension $\mathrm{Lan}_{\mathsf{D}_J^{\mathsf{op}} \hookrightarrow \mathsf{C}^{\mathsf{op}}}(\psi_J)$ of $\psi_J \in \Phi^*(\mathsf{D}_J)$ for some κ-small subcategory $\mathsf{D}_J \subseteq \mathsf{C}$. Since J is κ-small, the union of the D_J's generates a κ-small subcategory $\mathsf{D} \subseteq \mathsf{C}$; by replacing each ψ_J with its left Kan extension to D, we may assume all the D_J's are the same $\mathsf{D} \subseteq \mathsf{C}$ to begin with. For $J, K \in \mathsf{J}$, we have

$$\begin{aligned}
[\mathsf{C}^{\mathsf{op}}, \mathsf{Set}](F(J), F(K)) &\cong [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}](\mathrm{Lan}_{\mathsf{D}^{\mathsf{op}} \hookrightarrow \mathsf{C}^{\mathsf{op}}}(\psi_J), \mathrm{Lan}_{\mathsf{D}^{\mathsf{op}} \hookrightarrow \mathsf{C}^{\mathsf{op}}}(\psi_K)) \\
&\cong [\mathsf{D}^{\mathsf{op}}, \mathsf{Set}](\psi_J, \mathrm{Lan}_{\mathsf{D}^{\mathsf{op}} \hookrightarrow \mathsf{C}^{\mathsf{op}}}(\psi_K)|\mathsf{D}^{\mathsf{op}}) \\
&\cong [\mathsf{D}^{\mathsf{op}}, \mathsf{Set}](\psi_J, \varinjlim_{\kappa\text{-small } \mathsf{D} \subseteq \mathsf{E} \subseteq \mathsf{C}} \mathrm{Lan}_{\mathsf{D}^{\mathsf{op}} \hookrightarrow \mathsf{E}^{\mathsf{op}}}(\psi_K)|\mathsf{D}^{\mathsf{op}}) \\
(\text{since } \psi_J \in [\mathsf{D}^{\mathsf{op}}, \mathsf{Set}_\kappa]) \quad &\cong \varinjlim_{\kappa\text{-small } \mathsf{D} \subseteq \mathsf{E} \subseteq \mathsf{C}} [\mathsf{D}^{\mathsf{op}}, \mathsf{Set}](\psi_J, \mathrm{Lan}_{\mathsf{D}^{\mathsf{op}} \hookrightarrow \mathsf{E}^{\mathsf{op}}}(\psi_K)|\mathsf{D}^{\mathsf{op}}) \\
&\cong \varinjlim_{\kappa\text{-small } \mathsf{D} \subseteq \mathsf{E} \subseteq \mathsf{C}} [\mathsf{E}^{\mathsf{op}}, \mathsf{Set}](\mathrm{Lan}_{\mathsf{D}^{\mathsf{op}} \hookrightarrow \mathsf{E}^{\mathsf{op}}}(\psi_J), \mathrm{Lan}_{\mathsf{D}^{\mathsf{op}} \hookrightarrow \mathsf{E}^{\mathsf{op}}}(\psi_K));
\end{aligned}$$

and for $J, K, L \in \mathsf{J}$, a similar calculation shows that the composition map

$$\circ : [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}](F(K), F(L)) \times [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}](F(J), F(K)) \longrightarrow [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}](F(J), F(L))$$

is the κ-directed colimit, over all κ-small $\mathsf{D} \subseteq \mathsf{E} \subseteq \mathsf{C}$, of the composition maps

$$\circ : [\mathsf{E}^{\mathsf{op}}, \mathsf{Set}](\mathrm{Lan}(\psi_K), \mathrm{Lan}(\psi_L)) \times [\mathsf{E}^{\mathsf{op}}, \mathsf{Set}](\mathrm{Lan}(\psi_J), \mathrm{Lan}(\psi_K)) \longrightarrow [\mathsf{E}^{\mathsf{op}}, \mathsf{Set}](\mathrm{Lan}(\psi_J), \mathrm{Lan}(\psi_L)).$$

Thus the set of all functors $\mathsf{J} \to [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ agreeing with F on objects, which is some κ-small limit of the sets $[\mathsf{C}^{\mathsf{op}}, \mathsf{Set}](F(J), F(K))$ and these composition maps (as well as the "identity morphism" maps, expressing "a family of morphisms indexed by the morphisms of J, preserving $\circ$ and 1"), is the κ-directed colimit over all E of the set of functors $\mathsf{J} \to [\mathsf{E}^{\mathsf{op}}, \mathsf{Set}]$ taking the values $\mathrm{Lan}_{\mathsf{D}^{\mathsf{op}} \hookrightarrow \mathsf{E}^{\mathsf{op}}}(\psi_J)$ on objects $J \in \mathsf{J}$. In particular, there is some κ-small $\mathsf{D} \subseteq \mathsf{E} \subseteq \mathsf{C}$ and $G : \mathsf{J} \to [\mathsf{E}^{\mathsf{op}}, \mathsf{Set}]$ with object-values $\mathrm{Lan}_{\mathsf{D}^{\mathsf{op}} \hookrightarrow \mathsf{E}^{\mathsf{op}}}(\psi_J)$ (whence $G : \mathsf{J} \to \Phi^*(\mathsf{E})$) such that $F = \mathrm{Lan}_{\mathsf{E}^{\mathsf{op}} \hookrightarrow \mathsf{C}^{\mathsf{op}}} \circ G$. Then $\theta \star F = \theta \star \mathrm{Lan}_{\mathsf{E}^{\mathsf{op}} \hookrightarrow \mathsf{C}^{\mathsf{op}}}(G(-)) \cong \mathrm{Lan}_{\mathsf{E}^{\mathsf{op}} \hookrightarrow \mathsf{C}^{\mathsf{op}}}(\theta \star G)$ where $\theta \star G \in \Phi^*(\mathsf{E})$. ◻

The following equivalent form of Proposition 4.6 is more convenient for our purposes. For any class $\Phi \subseteq \mathsf{Psh}$, let $\Phi_\kappa \subseteq \Phi$ consist of the κ-small presheaves in Φ. We have an adjunction $(-)^* \dashv (-)_\kappa$ between classes of κ-small presheaves and saturated classes of small-presented presheaves; call the induced closure operation $((-)^*)_\kappa$ **κ-saturation**. Explicitly, for a class of κ-small presheaves Φ, by Lemma 4.5 we have

$$(\Phi^*)_\kappa(\mathsf{C}) = \begin{cases} \Phi^*(\mathsf{C}) & \text{if } \mathsf{C} \text{ is } \kappa\text{-small}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Thus Φ is κ-saturated iff for all κ-small C, $\Phi(\mathsf{C})$ contains the representables and is closed under Φ-colimits. (Taking $\kappa = \infty$ recovers the "small" notion of saturation used in [AK88].)

Corollary 4.7. *Let $\kappa \le \infty$ be uncountable regular, and let Φ be a κ-saturated class of κ-small presheaves. Then for any C and $\phi : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$, we have $\phi \in \Phi^*(\mathsf{C})$ iff it is the left Kan extension of some $\psi \in \Phi(\mathsf{D})$ for some κ-small subcategory $\mathsf{D} \subseteq \mathsf{C}$.*

Proof. This follows from Proposition 4.6, since $\Phi^*(\mathsf{D}) = (\Phi^*)_\kappa(\mathsf{D}) = \Phi(\mathsf{D})$ by κ-saturation. $\square$

We may interpret this result as follows. Note that the definition of "κ-saturated class of κ-small presheaves Φ" means equivalently that $\Phi = \Psi_\kappa$ for some saturated class of small-presented presheaves Ψ – or hence, also that $\Phi = \Psi_\kappa$ for some λ-saturated class of λ-small presheaves Ψ, for any $\lambda \ge \kappa$. Thus, Corollary 4.7 says that if Ψ-colimits are closed under iteration, then so are κ-small Ψ-colimits: that an iterated κ-small Ψ-colimit $\phi \star F$, where $\phi \in \Phi^* = (\Psi_\kappa)^*$ is an "iterated weight", is equivalent to the single κ-small Ψ-colimit $\psi \star F|\mathsf{D}$ where $\psi \in \Phi = \Psi_\kappa$.

Example 4.8. Consider the sifted colimits. For small C, it is known by [AR01, 2.6] that the free sifted-cocompletion $\mathsf{Sind}(\mathsf{C}) \subseteq [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ consists of exactly those presheaves ϕ whose category of elements $\mathsf{y}\downarrow\phi$ is sifted. Thus, in our terminology, letting Φ be the class of all such small presheaves, Φ is an ∞-saturated class of small presheaves, namely $\Phi = \Psi_\infty$ for $\Psi = \mathsf{Sind}$. It follows that Φ_κ is a κ-saturated class of κ-small presheaves. By definition, Φ_κ consists of exactly those κ-small presheaves whose category of elements is sifted, so that a Φ_κ-colimit is a κ-small sifted colimit. Thus

$$(\Phi_\kappa)^*(\mathsf{C}) = \mathsf{Sind}_\kappa(\mathsf{C}) := \text{free cocompletion of } \mathsf{C} \text{ under } \kappa\text{-small sifted colimits},$$

and so Corollary 4.7 yields

Corollary 4.9. *For uncountable regular $\kappa \le \infty$ and any category C, the following are equivalent for a presheaf $\phi : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$:*

(i) $\phi \in \mathsf{Sind}_\kappa(\mathsf{C})$, i.e., ϕ is an iterated κ-small sifted colimit of representables;

(ii) ϕ is the left Kan extension of some $\psi : \mathsf{D}^{\mathsf{op}} \to \mathsf{Set}_\kappa$ with sifted category of elements, for some κ-small $\mathsf{D} \subseteq \mathsf{C}$ (hence ϕ is a single κ-small sifted colimit of representables over $\mathsf{y}_{\mathsf{D}}\downarrow\psi$). $\square$

Example 4.10. We may of course similarly consider (κ-)filtered colimits. For any (infinite) regular $\kappa < \lambda \le \infty$ and category C, let

$$\kappa\mathsf{Ind}_\lambda(\mathsf{C}) := \text{free cocompletion of } \mathsf{C} \text{ under } \lambda\text{-small } \kappa\text{-filtered colimits}$$

(omitting λ when $\lambda = \infty$ and κ when $\kappa = \omega$). For small C, it is well-known (see e.g., [AR97, 2.24]) that $\kappa\mathsf{Ind}(\mathsf{C}) \subseteq [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$ consists of those ϕ with $\mathsf{y}\downarrow\phi$ κ-filtered. Arguing exactly as above, we get

Corollary 4.11. *For regular $\kappa < \lambda \le \infty$ and any C, the following are equivalent for $\phi : \mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$:*

(i) $\phi \in \kappa\mathsf{Ind}_\lambda(\mathsf{C})$, i.e., ϕ is an iterated λ-small κ-filtered colimit of representables;

(ii) ϕ is the left Kan extension of some $\psi : \mathsf{D}^{\mathsf{op}} \to \mathsf{Set}_\lambda$ with κ-filtered category of elements, for some λ-small $\mathsf{D} \subseteq \mathsf{C}$ (hence ϕ is a λ-small κ-filtered colimit of representables over $\mathsf{y}_{\mathsf{D}}\downarrow\psi$). $\square$

Remark 4.12. The preceding result can also be proved using accessible category theory, in the special case where the "sharply smaller" relation $\kappa \lhd \lambda$ holds (see [AR97, 2.12], [MP89, 2.3]). Indeed, every $\phi \in \kappa\mathsf{Ind}_\lambda(\mathsf{C}) \subseteq \kappa\mathsf{Ind}(\mathsf{C})$, being an iterated λ-small colimit of representables, is λ-presentable in $\kappa\mathsf{Ind}(\mathsf{C})$, hence by [MP89, 2.3.11] a λ-small κ-filtered colimit of κ-presentable objects in $\kappa\mathsf{Ind}(\mathsf{C})$, i.e., of representables (at least assuming C is Cauchy-complete).[3] Similar results on "relatively accessible" categories of the form $\kappa\mathsf{Ind}_\lambda(\mathsf{C})$ can be found in [Low16].

However, the argument given above works for all $\kappa < \lambda$, and avoids the combinatorial proof of [MP89, 2.3.11]. In fact, we can deduce [MP89, 2.3.11] from the above:

Corollary 4.13 (Makkai–Paré)**.** *For regular $\kappa \lhd \lambda < \infty$ and any C, every λ-presentable $\phi \in \kappa\mathsf{Ind}(\mathsf{C})$ is a λ-small κ-filtered colimit of representables.*

Proof. As in [MP89, proof of 2.3.10 and following remark], use $\kappa \lhd \lambda$ to write ϕ as a retract of a λ-small κ-filtered colimit of representables, which is in $\kappa\mathsf{Ind}_\lambda(\mathsf{C})$ since splitting an idempotent is a λ-small κ-filtered colimit, hence a λ-small κ-filtered colimit of representables by Corollary 4.11. □

5 The main result

Recall from [Kel82, 5.62] (see also [KS05, 4.2]) the following characterization of categories which are the free cocompletion of a subcategory under some given class of small colimits Φ. (Here Φ could be a class of weights as in the previous section; but we will only need one case, where the colimits are of diagrams of certain shapes.) For a category C cocomplete under Φ-colimits, an object $X \in \mathsf{C}$ is **Φ-atomic** if $\mathsf{C}(X, -) : \mathsf{C} \to \mathsf{Set}$ preserves Φ-colimits. Now such C is equivalent, via the restricted Yoneda embedding $\mathsf{C} \to [\mathsf{D}^{\mathsf{op}}, \mathsf{Set}]$, to the free Φ-cocompletion of a full subcategory $\mathsf{D} \subseteq \mathsf{C}$ iff

(i) every object in D is Φ-atomic in C, and

(ii) the (replete) closure of D under Φ-colimits is all of C.

This generalizes the standard characterization, for $\Phi =$ "filtered colimits", of finitely (class-)accessible categories $\mathsf{D} = \mathsf{Ind}(\mathsf{C})$ as those generated under filtered colimits by finitely presentable objects.

We now have the main result of the paper. As in the previous section, we write Sind_κ (resp., Ind_κ) to denote free cocompletion under κ-small sifted (resp., filtered) colimits.

Theorem 5.1. *For uncountable regular $\kappa \leq \infty$ and a category with pullbacks C, we have*

$$\mathsf{Sind}_\kappa(\mathsf{C}) \simeq \mathsf{Ind}_\kappa(\mathsf{Rec}(\mathsf{C}))$$

via the restricted Yoneda embedding $\mathsf{Sind}_\kappa(\mathsf{C}) \to [\mathsf{Rec}(\mathsf{C})^{\mathsf{op}}, \mathsf{Set}]$.

Proof. Since objects of $\mathsf{Rec}(\mathsf{C}) \subseteq [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$, being finite colimits of representables, are finitely presentable, hence also atomic with respect to κ-small filtered colimits (in $[\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$, hence also in the full subcategory $\mathsf{Sind}_\kappa(\mathsf{C})$ closed under κ-small filtered colimits), it suffices to show that every $\phi \in \mathsf{Sind}_\kappa(\mathsf{C})$ is a κ-small filtered colimit of objects in $\mathsf{Rec}(\mathsf{C})$. By Corollary 4.9, there is a κ-small subcategory $\mathsf{D} \subseteq \mathsf{C}$, which we may assume closed under pullbacks, and $\psi \in [\mathsf{D}^{\mathsf{op}}, \mathsf{Set}_\kappa]$ with sifted category of elements and left Kan extension ϕ, so that ϕ is the κ-small sifted colimit of

$$\mathsf{y}_\mathsf{D}{\downarrow}\psi \to \mathsf{D} \subseteq \mathsf{C} \xrightarrow{\mathsf{y}_\mathsf{C}} [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}].$$

[3]The proof of [MP89, 2.3.11] does not depend on the assumption that C is (essentially) small, i.e., that $\kappa\mathsf{Ind}(\mathsf{C})$ is κ-accessible instead of "class-κ-accessible" [CR12]. Cauchy-completeness of C is not needed either, since the proof of [MP89, 2.3.10] really shows that (in our notation) every λ-presentable $\phi \in \kappa\mathsf{Ind}(\mathsf{C})$ can be written as a retract of a λ-small κ-filtered colimit of (not just κ-presentables but) representables.

In other words, ϕ is the left Kan extension of this diagram along the unique functor $\mathsf{y_D}{\downarrow}\psi \to \mathbf{1}$, hence is also the colimit of its reflexive-coequalizer-preserving left Kan extension to a diagram

$$\mathsf{Rec}(\mathsf{y_D}{\downarrow}\psi) \to \mathsf{Rec}(\mathsf{C}) \subseteq [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}],$$

which is filtered by Proposition 3.2 since $\mathsf{y_D}{\downarrow}\psi$ inherits pullbacks from D, and is essentially κ-small since $\mathsf{y_D}{\downarrow}\psi$ is κ-small. □

Corollary 5.2. *For uncountable regular $\kappa \le \infty$, if a category with pullbacks C has reflexive coequalizers and κ-small filtered colimits, then it has κ-small sifted colimits, and these are preserved by any functor $F : \mathsf{C} \to \mathsf{D}$ preserving reflexive coequalizers and κ-small filtered colimits.*

Proof. Given a κ-small sifted diagram $G : \mathsf{J} \to \mathsf{C}$, a colimit of G is the same thing as a colimit of 1_{C} weighted by $\phi := \varinjlim(\mathsf{y_C}G) \in \mathsf{Sind}_\kappa(\mathsf{C})$; by writing ϕ as a κ-filtered colimit of reflexive coequalizers of representables (and using cocontinuity of weighted colimit $\star$ in the weight [Kel82, 3.23]), we get the colimit of G as the corresponding κ-filtered colimit of reflexive coequalizers in C, which is hence preserved by any $F : \mathsf{C} \to \mathsf{D}$ preserving these latter colimits. □

6 Lex sifted colimits

The statement of Corollary 5.2 is somewhat peculiar in that the pullbacks required to exist in C are not required to be compatible with the colimits in any way, nor are they required to be preserved by F. In the presence of such compatibility conditions, more can be said. Garner–Lack [GL12] have developed a general theory of what it means for a finitely complete category which is also cocomplete for a given class of colimits Φ to obey all "algebraic" compatibility conditions between finite limits and Φ-colimits as hold in Set (or more generally, a suitable enrichment base V), known as **Φ-exactness**. For small C and Φ, such as κ-small filtered or sifted colimits for $\kappa < \infty$, C is Φ-exact iff it admits a full embedding into a Grothendieck topos preserving finite limits and Φ-colimits [GL12, 4.1]; the case of an "unbounded" Φ, such as all small filtered or sifted colimits, can be understood as the conjunction of the κ-small cases for $\kappa < \infty$ [GL12, 4.4].

Corollary 6.1. *For uncountable regular $\kappa \le \infty$, a finitely complete category C is exact for the class of κ-small sifted colimits iff it is Barr-exact and admits κ-small filtered colimits commuting with finite limits; and a finitely continuous functor $F : \mathsf{C} \to \mathsf{D}$ between two such categories preserves κ-small sifted colimits iff it is regular and preserves κ-small filtered colimits.*

Proof. By the proof of [GL12, 5.10], C is exact for the class of κ-small filtered colimits iff it admits κ-small filtered colimits commuting with finite limits. By [GL12, 5.12], C is exact for reflexive coequalizers iff it is Barr-exact and admits pullback-stable colimits of certain countable sequences $R_0 \to R_1 \to \cdots$; this last condition is implied by commutativity of finite limits and countable (hence κ-small) filtered colimits, by considering $\varinjlim_i (R_i \times_{\varinjlim_j R_j} Y)$ for pullback-stability along $Y \to \varinjlim_j R_j$. Thus, C is Barr-exact and admits κ-small filtered colimits commuting with finite limits, iff it is exact for the union of the classes of κ-small filtered colimits and reflexive coequalizers. By Theorem 5.1, this union generates the class of κ-small sifted colimits (in the sense of the saturation $\Phi \mapsto \Phi^*$ from Section 4, hence also that of [GL12, §3] which further closes under finite limits), whence they determine the same exactness notion by [GL12, 3.4, 4.4].

Concretely, this argument reduces the computation of a κ-small sifted colimit in C to Barr-exactness and κ-small filtered colimits as follows: first use Theorem 5.1 to reduce to a κ-small filtered colimit of reflexive coequalizers; then (as in the proof of [GL12, 5.12]) reduce each such reflexive coequalizer of a graph $G \rightrightarrows X$ to the quotient of X by the equivalence relation generated by the image reflexive relation $R \subseteq X^2$ of G, which is the colimit of the countable sequence $R \subseteq R \circ R^{\mathsf{op}} \circ R \subseteq \cdots$ of composites of binary relations computed (as in any regular category) using pullback and image. Each step of this procedure is preserved by a regular $F : \mathsf{C} \to \mathsf{D}$ preserving κ-small filtered colimits, which proves the second claim. □

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