On Gaboriau's homological proof that treeings achieve cost

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This note is a self-contained exposition of Gaboriau's "ℓ^2 proof" that treeings achieve cost [Gab]. We are inspired by Bernshteyn's similar note on the same topic [Ber]; compared to it, the main feature of our treatment is the complete elimination of spectral theory (implicitly used in [Ber, 1.5]), replacing its role with an elementary argument based on orthogonal projections. We also place greater emphasis on a discrete linear-algebraic "dimension theorem" (Lemma 4) underlying the main result, and relegate the role of ℓ^2 to a calculation in the proof of that lemma.

Fix throughout a countable Borel equivalence relation E on a standard Borel space X.

Definition 1. A **standard Borel bundle of countable sets** over X/E is a standard Borel space Y equipped with a countable-to-1 Borel map $p : Y \to X/E$, in the sense that it has a Borel lift $\widetilde{p} : Y \to X$. For $C \in X/E$, we let $Y_C := p^{-1}(C)$ denote the fiber. We think of (Y, p) as a "Borel family of countable sets" $(Y_C)_{C \in X/E}$ indexed by X/E.

Given two such bundles $p : Y \to X/E$ and $q : Z \to X/E$, a **Borel map over** X/E between them is a Borel map $f : Y \to Z$ such that $q \circ f = p$, or equivalently, $f(Y_C) \subseteq Z_C$ for each $C \in X/E$.

For an E-invariant measure μ on X, the **fiberwise counting measure** μ_Y on Y is given by

$$\mu_Y(A) := \int_X |A \cap \widetilde{p}^{-1}(x)| \, d\mu(x)$$

for Borel $A \subseteq Y$ and any Borel lift $\widetilde{p} : Y \to X$ of p; invariance of μ ensures that this does not depend on the choice of $\widetilde{p}$. We omit the subscript Y when there is no risk of confusion; in particular,

$$\mu(Y) := \mu_Y(Y) = \int_X |\widetilde{p}^{-1}(x)| \, d\mu(x).$$

For two bundles $p : Y \to X/E$ and $q : Z \to X/E$, we write

$$Y \preceq_E Z$$

if there is a Borel injection $Y \hookrightarrow Z$ over X/E; this ensures $\mu(Y) \leq \mu(Z)$ for every E-invariant μ.

Remark 2. Any Borel $A \subseteq X$ gives rise to a standard Borel bundle over X/E, with projection $A \to X/E$ given by the quotient map $X \to X/E$ restricted to A. For $A, B \subseteq X$, a Borel map $f : A \to B$ between the two associated bundles over X/E is the same thing as one with graph contained in E. In particular, A, B are isomorphic over X/E iff they are E-equidecomposable.

If E is compressible, then every standard Borel bundle of countable sets $p : Y \to X/E$ is isomorphic over X/E to some Borel $A \subseteq X$. Thus the set of isomorphism types over X/E of such bundles (with the countable disjoint union operation) is isomorphic to the cardinal algebra $\mathcal{K}(E)$ of E-equidecomposability types of Borel sets. If E is not compressible, we may replace E with $E \times I_{\mathbb{N}}$. In particular, $Y \preceq_E Z$ iff $\mu(Y) \leq \mu(Z)$ for every E-invariant (σ-finite) μ, by Nadkarni's theorem; see [Ch] for details.

Definition 3. For a ring R and set Y, we let

$$R(Y) := R^{\bigoplus Y} \subseteq R^Y$$

denote the **free R-module** generated by Y. We usually identify $y \in Y$ with the corresponding basis vector in $R(Y)$. For a bundle $p : Y \to X/E$, we put

$$R_{X/E}(Y) := \bigsqcup_{C \in X/E} R(Y_C),$$

which is a bundle of R-modules, "Borel" if R, Y are. Rather than making precise what "Borel" means here, we only define, for a standard Borel ring R and Borel bundles of countable sets $p : Y \to X/E$ and $q : Z \to X/E$, a **Borel fiberwise R-linear map** $f : R_{X/E}(Y) \to R_{X/E}(Z)$ over X/E to mean one such that

$$\begin{aligned} Y \times_{X/E} Z &\longrightarrow R \\ (y, z) &\longmapsto f(y)(z) \end{aligned}$$

is Borel (where $f(y) \in R^{\bigoplus Z_{p(y)}} \subseteq R^{Z_{p(y)}}$ is a finitely supported function).

Entirely analogously, we have a "Borel bundle of Hilbert spaces" $\ell^2_{X/E}(Y)$; and we may speak of a **Borel fiberwise bounded linear map** $f : \ell^2_{X/E}(Y) \to \ell^2_{X/E}(Z)$ between two such bundles.

We now have the key result, which says that the "dimension" (i.e., isomorphism type of a basis) of a bundle of vector spaces is well-defined:

Lemma 4 (dimension theorem)**.** Let $p : Y \to X/E$ and $q : Z \to X/E$ be standard Borel bundles of countable sets over X/E, and let $g : \mathbb{C}_{X/E}(Z) \twoheadrightarrow \mathbb{C}_{X/E}(Y)$ be a Borel fiberwise linear surjection. Then $Y \preceq_E Z$, or equivalently, for any E-invariant σ-finite measure μ on X, $\mu(Y) \leq \mu(Z)$.

Proof. Let $f : \mathbb{C}_{X/E}(Y) \hookrightarrow \mathbb{C}_{X/E}(Z)$ be a Borel fiberwise linear section of g, with $g \circ f = 1$. (Lusin–Novikov suffices to find such f, since surjectivity of g implies that for $y \in Y$, there is $f(y) \in g^{-1}(y)$ with coordinates in the countable subfield of $\mathbb{C}$ generated by all coefficients of $g|\mathbb{C}(Z_{p(y)})$.)

We next reduce to the case where there is a single finite constant $N \in \mathbb{N}$ bounding all of:

(i) $\mu(Y)$ and $\mu(Z)$;

(ii) the absolute values of coordinates $|f(y)(z)|$ of $f(y) \in \mathbb{C}^{Z_{p(y)}}$ for each $y \in Y$ and $z \in Z$;

(iii) the cardinality of the support (i.e., number of nonzero coordinates) of each $f(y)$;

(iv) the chromatic number of the intersection graph on the supports of all $f(y)$;

and similarly for g. To achieve these conditions for f, let $c : Y \to \mathbb{N}$ be a Borel coloring of the intersection graph on the supports of all $f(y)$; then $Y = \Uparrow_n Y_n$ where

$$Y_n \subseteq \{ y \in Y \mid \|f(y)\|_\infty, |\mathrm{supp}(f(y))|, c(y) < n \}, \qquad \mu(Y_n) \leq n,$$

and each $f|\mathbb{C}_{X/E}(Y_n)$ satisfies the above bounds for $N := n$. Similarly, we may write $Z = \Uparrow_n Z_n$ such that each $g|\mathbb{C}_{X/E}(Z_n)$ satisfies the above bounds. Now each pair of maps

$$\mathbb{C}_{X/E}(Y_n \cap f^{-1}(\mathbb{C}_{X/E}(Z_n))) \underset{\mathrm{proj} \circ g}{\overset{f}{\rightleftarrows}} \mathbb{C}_{X/E}(Z_n)$$

forms a section-retraction pair, and both satisfy the above bounds. And it suffices to prove that each $Y_n \cap f^{-1}(\mathbb{C}_{X/E}(Z_n)) \preceq_E Z_n$, since the increasing unions of these sets are Y, Z respectively.

Now the above bounds ensure that f, g extend to bounded linear maps between the fiberwise ℓ^2-completions $\ell^2_{X/E}(Y), \ell^2_{X/E}(Z)$ of $\mathbb{C}_{X/E}(Y), \mathbb{C}_{X/E}(Z)$ respectively. Indeed, by finding a Borel N-coloring of the intersection graph as in (iv), we may write $f = f_0 + \cdots + f_{N-1}$ where each f_i has disjoint supports, hence maps the standard orthonormal basis of each fiber $\ell^2(Y_C)$ of $\ell^2_{X/E}(Y)$ to an orthogonal family of vectors in $\ell^2(Z_C)$, each of which has norm $\le N$ by (ii) and (iii); by orthogonality, it follows that $f_i : \mathbb{C}(Y_C) \to \mathbb{C}(Z_C)$ has ℓ^2-operator norm $\le N$, hence extends to a bounded linear map $\ell^2(Y_C) \to \ell^2(Z_C)$. Similarly, g extends to $\ell^2_{X/E}(Z) \to \ell^2_{X/E}(Y)$:

$$\ell^2_{X/E}(Y) \underset{g}{\overset{f}{\rightleftarrows}} \ell^2_{X/E}(Z)$$

We still have $g \circ f = 1$, since this holds on the dense subset $\mathbb{C}_{X/E}(Y)$.

By replacing f with $\mathrm{proj}_{\ker(g)^\perp} \circ f$, we may assume f lands in $\ker(g)^\perp$, so that for each $z \in Z$,

$$1 = \|z\|^2 = \|z - f(g(z))\|^2 + \langle z - f(g(z)), f(g(z)) \rangle + \langle f(g(z)), z \rangle \ge \langle f(g(z)), z \rangle = f(g(z))(z).$$

This gives by Fubini (using that these integrals are absolutely convergent by (i), (ii), and (iii))

$$\begin{aligned}
\mu(Y) &= \int_{y \in Y} 1 \, d\mu_Y = \int_{y \in Y} g(f(y))(y) \, d\mu_Y = \int_{y \in Y} \sum_{z \in Z_{p(y)}} f(y)(z) g(z)(y) \, d\mu_Y \\
&= \int_{(y,z) \in Y \times_{X/E} Z} f(y)(z) g(z)(y) \, d\mu_{Y \times_{X/E} Z} \\
&= \int_{z \in Z} \sum_{y \in Y_{q(z)}} f(y)(z) g(z)(y) \, d\mu_Z = \int_{z \in Z} f(g(z))(z) \, d\mu_Z \le \int_{z \in Z} 1 \, d\mu_Z = \mu(Z). \qquad \square
\end{aligned}$$

Remark 5. It is possible to carry out the above proof completely avoiding all mention of ℓ^2, by choosing each $f(y) \in \mathbb{C}(Z)$ to be suitably "ε-close" to the orthogonal complement of $\ker(g)$. This allows the graph coloring argument to achieve (iv) to be avoided, since there is then no need to extend f, g to bounded linear operators. However, we feel that the additional approximation arguments needed would have cost some clarity.

Remark 6. It is also possible to carry out the above proof entirely in the cardinal algebra $\mathcal{K}(E)$ from [Ch] (see Remark 2), or rather its completion under real multiples $\overline{\mathcal{K}}(E)$ (consisting of equidecomposability classes of $[0, \infty]$-valued functions, denoted $\mathcal{L}(E)$ in [Ch]), without appealing to Nadkarni's theorem. The basic idea is to replace all of the integrals above with the quotient map $h \mapsto [h]_{\sim_E}$ taking a $[0, \infty]$-valued function h (on $Y, Z, Y \times_{X/E} Z$ respectively, pushed forward to X) to its equidecomposability class in $\overline{\mathcal{K}}(E)$. Some care is needed to properly deal with cancellation between the positive and negative parts of the above functions; we omit the details.

Theorem 7 (Gaboriau, treeings achieve cost)**.** If $G \subseteq E$ is a (directed) graphing, and $T \subseteq E$ is a treeing, then $T \preceq_E G$, i.e., $\mathrm{cost}_\mu(T) := \mu(T) \le \mu(G) =: \mathrm{cost}_\mu(G)$ for every E-invariant σ-finite μ.

Proof. The map $\mathbb{C}_{X/E}(G) \to \mathbb{C}_{X/E}(T)$ taking a G-edge to the sum of the edges on the unique T-path between its endpoints is surjective, with a section given by taking a T-edge to any G-path between its endpoints. $\square$

As one more sample application of Lemma 4, we derive another basic fact about cost:

Theorem 8 (Levitt)**.** If E is aperiodic, then $\mathrm{cost}_\mu(E) \geq \mu(X)$ for every E-invariant σ-finite μ.

Proof. If $G \subseteq E$ is a graphing, then the map $\partial : \mathbb{C}_{X/E}(G) \to \mathbb{C}_{X/E}(X)$ taking an edge (x, y) to $y - x$ is "almost" surjective, with codimension 1 image consisting of all finite linear combinations of vertices with coefficients adding to 0. For any $\varepsilon > 0$, since E is aperiodic, we may find a complete section $Y \subseteq X$ with $\mu(Y) < \varepsilon$; then the map $\mathbb{C}_{X/E}(G) \oplus_{X/E} \mathbb{C}_{X/E}(Y) \to \mathbb{C}_{X/E}(X)$ which is ∂ on the first summand and inclusion on the second is surjective, whence $\mu(X) \leq \mu(G) + \mu(Y) < \mu(G) + \varepsilon$. $\square$

References

[Ber] A. Bernshteyn, *An ℓ_2-based proof of Gaboriau's theorem*, unpublished note, 2018.

[Ch] R. Chen, *Decompositions and measures on countable Borel equivalence relations*, Ergodic Theory Dynam. Systems **41** (2021), 3671–3703.

[Gab] D. Gaboriau, *Around the orbit equivalence theory of the free groups, cost and ℓ^2 Betti numbers*, unpublished note, 2016.