INTRODUCTORY LECTURE

Mathematical logic is the mathematical study of mathematical *language* and its *meaning*.

It is useful to view mathematical logic almost as an "applied" area of math: there is a real-world phenomenon (namely, math itself); and we model and study this phenomenon using mathematical tools. Analogously, for instance:

Mathematical biology is the mathematical study of biological systems.

Mathematical physics is the mathematical study of physical systems, etc.

The following scenarios illustrate this analogy:

Person A. *drinks coffee, feels excited*

Biologist B. *creates a model of the human brain showing that caffeine induces excitement:*[1]

$$\frac{d(\text{Excitement})}{d(\text{Caffeine})} > 0$$

Theorem A. *coffee > tea.*

Theorem B (proved by a logician)**.** *There does not exist a proof of Theorem A. There also does not exist a proof that Theorem A is false.*

These examples serve to demonstrate the following major themes of logic, hence of this course:

- The distinction between the real-world phenomenon (a person; an informal theorem or proof), versus the mathematical model of it (Biologist B's differential equation; the formalized "Theorem A" and "a proof" of it that Logician B refers to). The tools of math can only be applied to the latter, not the former.
- The distinction between *language* (aka *syntax*) and its *meaning* (aka *semantics*). A formalized theorem or proof is basically a blob of meaningless symbols; the "real" theorem or proof that it represents is its meaning. Separating language from meaning means that we can change the meaning to something unintended:

  Proof of Theorem B. Let D be a proof of Theorem A. Swap all occurrences of the words "coffee" and "tea" in D to get a proof that Theorem A is false, a contradiction. $\square$
- The limited expressive power of language. In informal mathematical practice (the kind you do in every other math course besides logic), we are used to accepting any intuitively justified "reasoning principle" as sound, which leads to the idea that every mathematical statement is either true or false. Theorems like Theorem B, which are quite common in logic, show that once we carefully formalize our reasoning, some (in fact, "most") mathematical statements will have an indeterminate truth value. Similarly, "most" mathematical objects cannot be defined. (Philosophical exercise: if you can't define something, does it exist???)

Concretely, we will study these themes via two kinds of logic:

- **Propositional logic**: a "toy" logic that allows us to formalize "theorems" and "proofs", but is not nearly expressive enough to formalize any mathematically interesting statements.
- **First-order logic** or **predicate logic**: the "real" logic of mathematics, that allows us to make statements *about* mathematical entities (numbers, functions, . . .).

[1] I have no idea if a biologist would actually model this with a differential equation! Let's suppose they would for the sake of this example.