

A Gelfand duality for continuous lattices

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Abstract

We prove that the category of continuous lattices and meet- and directed join-preserving maps is dually equivalent, via the hom functor to $[0, 1]$, to the category of complete Archimedean meet-semilattices equipped with a finite meet-preserving action of the monoid of continuous monotone maps of $[0, 1]$ fixing 1. We also prove an analogous duality for completely distributive lattices. Moreover, we prove that these are essentially the only well-behaved “sound classes of joins Φ , dual to a class of meets” for which “ Φ -continuous lattice” and “ Φ -algebraic lattice” are different notions, thus for which a 2-valued duality does not suffice.

1 Introduction

The classical Gelfand duality asserts that a compact Hausdorff space X may be recovered from its ring of continuous functions $C(X)$, and moreover such rings are up to isomorphism precisely the commutative C^* -algebras. From a categorical perspective, $C(X)$ is best regarded as having “underlying set” given by its (positive) unit ball, i.e., consisting of continuous $\mathbb{I} := [0, 1]$ -valued functions, so that Gelfand duality falls under the umbrella of Stone-type dualities induced by two “commuting” structures on \mathbb{I} ; see [Joh82, VI §4]. Namely, \mathbb{I} is equipped with its usual compact Hausdorff topology, and also with all operations $\mathbb{I}^\kappa \rightarrow \mathbb{I}$ “commuting” with the topology, i.e., which are continuous. Thus, for another object in either category, the hom functor into \mathbb{I} yields a dual in the other category, and this gives a dual adjunction, which Gelfand duality asserts is an equivalence. An explicit axiomatization of the dual operations on the \mathbb{I} -valued $C(X)$ was recently given in [MR17]; see there for a detailed history of \mathbb{I} -valued Gelfand duality. In [HNN18], [Abb19], \mathbb{I} -valued Gelfand duality was further extended to compact partially ordered spaces (*a la* Nachbin).

In this note, we prove analogous Gelfand-type dualities for compact pospaces equipped with lattice operations. Recall that a **continuous lattice** is a compact topological meet-semilattice obeying a “local convexity under meets” condition, that each point has a neighborhood basis of subsemilattices. Equivalently, they can be defined purely order-theoretically as posets with arbitrary meets distributing over directed joins. An analog of Urysohn’s lemma, sometimes known as the Urysohn–Lawson lemma, states that every continuous lattice X admits enough morphisms to \mathbb{I} , i.e., the canonical evaluation map $X \rightarrow \mathbb{I}^{\text{Hom}(X, \mathbb{I})}$ is an embedding; see [G⁺03, IV-3.3], [Joh82, VII 3.2]. It is thus natural to ask whether, by equipping $\text{Hom}(X, \mathbb{I})$ with suitable structure commuting with the continuous lattice structure on \mathbb{I} , we may recover X as the double dual.

Let $\widehat{\mathbb{U}}$ denote the monoid of continuous monotone maps $\mathbb{I} \rightarrow \mathbb{I}$ fixing 1, i.e., all unary operations on \mathbb{I} commuting with the continuous lattice structure. Note that finite meets do as well. By a

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$\widehat{\mathbb{U}}$ -**module**, we mean a unital meet-semilattice equipped with an action of $\widehat{\mathbb{U}}$ preserving finite meets in both variables. In every $\widehat{\mathbb{U}}$ -module A , we have a canonical pseudoquasimetric

$$\rho(a, b) := \bigwedge \{r \in \mathbb{I} \mid a \leq b \dot{+} r\}.$$

We say A is **Archimedean** if $\rho(a, b) = 0 \implies a \leq b$, and **complete** if A is Archimedean and complete with respect to the induced metric $d(a, b) := \rho(a, b) \vee \rho(b, a)$. We prove

Theorem 1.1 (Corollary 5.9). *Hom into \mathbb{I} yields a dual equivalence of categories between continuous lattices and complete $\widehat{\mathbb{U}}$ -modules.*

There is a generalization of continuous lattice theory, with the role of directed joins replaced by an arbitrary “class of joins Φ ” obeying suitable axioms; see [WWT78], [BE83], [Xu95], as well as [AK88], [ABLR02], [KS05] for a further extension in enriched category theory. Other than $\Phi =$ “directed joins”, the most well-known case is $\Phi =$ “all joins”, for which Φ -continuous lattices are completely distributive lattices. As for continuous lattices, there is a Urysohn-type lemma, stating that all completely distributive lattices admit enough morphisms to \mathbb{I} ; see [G⁺03, IV-3.31–32], [Joh82, 1.10–14]. We likewise boost this to a Gelfand-type duality as follows.

Let $\mathbb{U} \subseteq \widehat{\mathbb{U}}$ denote the monoid of complete lattice morphisms, i.e., monotone surjections. A \mathbb{U} -**poset** is a poset with a monotone action of \mathbb{U} . There is a canonical way of defining a pseudoquasimetric on a \mathbb{U} -poset, agreeing with the above definition in $\widehat{\mathbb{U}}$ -modules; see Definition 4.2. A \mathbb{U} -poset A is **stackable** if, intuitively speaking, an element $a \in A$ may be specified via its “restrictions to sublevel and superlevel sets $a^{-1}([0, r]), a^{-1}([r, 1])$ ” for any $0 < r < 1$; see Definition 4.12.

Theorem 1.2 (Corollary 5.5). *Hom into \mathbb{I} yields a dual equivalence of categories between completely distributive lattices and complete stackable \mathbb{U} -posets.*

In fact, we prove a single result underlying Theorems 1.1 and 1.2, for a “class of joins Φ dual to a class of meets Ψ^{op} ”, more precisely for a *sound* class of joins in the sense of [ABLR02], [KS05]; see Section 3. This general result, Theorem 5.2, says that Φ -continuous lattices are dual to complete stackable \mathbb{U} - Ψ^{op} -inflattices, *provided that not all Φ -continuous lattices are Φ -algebraic*, i.e., already admit enough morphisms into 2 . This is a reasonable restriction, since for these other Φ , we instead have a simple 2-valued duality generalizing the classical Hofmann–Mislove–Stralka duality [HMS74] between algebraic lattices and meet-semilattices (see Corollary 3.7).

Part of the reason we work with general Φ is to hint at the possibility of generalizing to quantale-enriched posets, or even to enriched categories, which we plan to pursue in future work. However, in the original context of mere posets, it turns out that essentially the only Φ are the classical ones:

Theorem 1.3 (Theorem 3.9). *There are precisely 4 sound classes of joins Φ for which not every Φ -continuous lattice is Φ -algebraic: “directed joins”, “all joins”, and the minor variations including/excluding empty joins.*

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2 Φ -continuous lattices

We assume familiarity with basic category theory. For a category \mathcal{C} , $\mathcal{C}(X, Y)$ will denote the hom-set of morphisms from X to Y , while \mathcal{C}^{op} will denote the opposite category; this includes opposite

posets. We let \mathbf{Pos} denote the category of posets, \mathbf{Sup} denote the category of suplattices (i.e., complete lattices with join-preserving maps as morphisms), \mathbf{Inf} denote the category of inflattices, and $\mathbf{CLat} = \mathbf{Sup} \cap \mathbf{Inf}$ denote the category of complete lattices. These are all locally ordered categories: each hom-set is partially ordered pointwise, and composition is monotone on both sides. For $f : X \rightarrow Y \in \mathbf{Pos}$ left adjoint to $g : Y \rightarrow X$, we will write $f = g^+$ and $g = f^\times$. We will frequently use the “mate calculus”: for monotone h, k , we have $h \circ g \leq k \iff h \leq k \circ f$.

For a poset X , we let $\mathcal{L}(X)$ denote the poset of lower sets $\phi \subseteq X$, ordered via \subseteq . Then $\mathcal{L} : \mathbf{Pos} \rightarrow \mathbf{Pos}$ is the free suplattice monad, where the monad structure consists of;

- unit $\downarrow = \downarrow_X : X \rightarrow \mathcal{L}(X)$, where $\downarrow x = \{y \in X \mid y \leq x\}$ is the principal ideal below x ;
- multiplication $\bigcup : \mathcal{L}(\mathcal{L}(X)) \rightarrow \mathcal{L}(X)$;
- $f : X \rightarrow Y \in \mathbf{Pos}$ inducing $f_* = \mathcal{L}(f) : \mathcal{L}(X) \rightarrow \mathcal{L}(Y) \in \mathbf{Sup}$, where $f_*(\phi) = \bigcup_{x \in \phi} \downarrow f(x)$.

We now review the theory of “relative” suplattices for a “class of joins” Φ . This is a special case of the theory of “classes of colimits” in enriched category theory [AK88], [ABLR02], [KS05], and has also been well-studied in the order theory literature as “Z-completeness” [WWT78], [BE83]. We will use notation and terminology based on that from enriched categories.

Definition 2.1. A **join doctrine** is a class Φ of posets ϕ , thought of as indexing posets for certain joins $\bigvee_{x \in \phi} f(x)$ of monotone $f : \phi \rightarrow Y$. We require Φ to obey the following “saturation” conditions:

- The singleton poset $\mathbf{1}$ is in Φ .
- If ϕ is a poset which is a union $\bigcup \Psi$ of a set Ψ of subposets $\psi \subseteq \phi$, and $\Psi \in \Phi$, then $\phi \in \Phi$.
- If $f : \phi \rightarrow \psi$ is a monotone map with cofinal image, and $\phi \in \Phi$, then $\psi \in \Phi$.
- If $\phi \subseteq \psi$ is a cofinal subposet, and $\psi \in \Phi$, then $\phi \in \Phi$.

A **Φ -join** in a poset X is a join of a subset $\phi \subseteq X$ such that $\phi \in \Phi$. A **Φ -suplattice** is a poset with all Φ -joins; we denote the category of all such (and Φ -join-preserving maps) by $\Phi\mathbf{Sup}$. A **Φ -ideal** in a Φ -suplattice is a lower sub- Φ -suplattice. The **free Φ -suplattice** generated by a poset X is the subset $\Phi(X) \subseteq \mathcal{L}(X)$ of all lower sets in Φ ; we thereby identify Φ with the submonad $\Phi \subseteq \mathcal{L}$.

Remark 2.2. In [AK88] and [KS05] a more general notion of “class of colimits” is considered, consisting in the posets case of an arbitrary submonad $\Phi \subseteq \mathcal{L}$, i.e., an assignment to each poset X of a set of lower sets $\Phi(X) \subseteq \mathcal{L}(X)$ closed under the monad operations on \mathcal{L} . However, for the Φ suitable for duality that we will be interested in, this is no gain in generality; see Remark 3.5.

Definition 2.3. Let Φ be a join doctrine, X be a Φ -suplattice. We define, for $x, y \in X$,

$$\begin{aligned} \downarrow &= \downarrow_X^\Phi : X \longrightarrow \mathcal{L}(X) \\ x &\longmapsto \bigcap \{\phi \in \Phi(X) \mid x \leq \bigvee \phi\}, \\ x \ll y &:\iff x \ll^\Phi y :\iff x \in \downarrow y. \end{aligned}$$

We call $x \in X$ **Φ -compact** (Φ -atomic in [KS05]) if $x \ll^\Phi x$, i.e., whenever $\bigvee_i y_i$ is a Φ -join $\geq x$, then some $y_i \geq x$, i.e., the indicator function of $\uparrow x : X \rightarrow 2$ preserves Φ -joins. Denote these by

$$X_\Phi := \{x \in X \mid x \ll^\Phi x\}.$$

We call X **Φ -algebraic** if it is generated under Φ -joins by $X_\Phi \subseteq X$, in which case in fact each $X_\Phi \cap \downarrow x \in \Phi(X_\Phi)$ with join x , and this yields an order-isomorphism $X \cong \Phi(X_\Phi)$. Conversely, for any poset Y , we easily have that $\Phi(Y)$ is Φ -algebraic, with $\Phi(Y)_\Phi = \{\text{principal ideals}\} \cong Y$.

Proposition 2.4. *Let Φ be a join doctrine, X be a Φ -suplattice. The following are equivalent:*

- (i) *For each $x \in X$, there is a $\Phi(X) \ni \phi \subseteq \downarrow x$ such that $x \leq \bigvee \phi$, whence in fact $\phi = \downarrow x$.*
- (ii) *$\bigvee : \Phi(X) \rightarrow X$ has a left adjoint, namely \downarrow .*

If X is a complete lattice, these are further equivalent to:

- (iii) *$\bigvee : \Phi(X) \rightarrow X$ preserves meets.*
- (iv) *Arbitrary meets distribute over Φ -joins: if $\bigvee_{j \in J_i} x_{i,j}$ is a Φ -join for each $i \in I$, then*

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} x_{i,j} = \bigvee_{(j_i)_{i \in I} \in \prod_i J_i} \bigwedge_{i \in I} x_{i,j_i}.$$

All of these hold if X is algebraic, with $\downarrow = \downarrow_ : \Phi(X_\Phi) \rightarrow \Phi(\Phi(X_\Phi))$, i.e.,*

$$x \ll y \iff \exists z \in X_\Phi (x \leq z \leq y).$$

If (i), (ii) hold for a Φ -suplattice X , we call X **Φ -continuous**. If furthermore X is a complete lattice, we call X a **Φ -continuous lattice**, or a **Φ -algebraic lattice** if X is algebraic.

Proof. (i) \iff (ii) since it is easily seen that ϕ in (i) must be $\downarrow x$.

(ii) \iff (iii) by the adjoint functor theorem.

(iii) \iff (iv) because the latter says $\bigwedge_{i \in I} \bigvee_{j \in J_i} \downarrow x_{i,j} = \bigvee \bigcap_{i \in I} \bigcup_{j \in J_i} \downarrow x_{i,j}$. □

Proposition 2.5. *In every Φ -suplattice,*

- (a) *$\downarrow x \subseteq \downarrow x$, i.e., $y \ll x \implies y \leq x$.*
- (b) *$x' \leq x \ll y \leq y' \implies x' \ll y'$.*

In a Φ -continuous Φ -suplattice,

- (c) *(interpolation) $\downarrow = \bigcup \downarrow_* \downarrow$, i.e., $\downarrow x = \bigcup_{y \ll x} \downarrow y$, i.e.,*

$$z \ll x \iff \exists y (z \ll y \ll x).$$

Proof. The first two are obvious. For interpolation: since X is an algebra of the monad Φ , we have $\bigvee \bigcup = \bigvee \bigvee_* : \Phi(\Phi(X)) \rightarrow X$; taking left adjoints yields $\downarrow_* \downarrow = \downarrow_* \downarrow$; now take \bigcup . □

A **morphism of Φ -continuous lattices** is a meet-preserving, Φ -join-preserving map between Φ -continuous lattices. Let ΦCtsLat denote the category of Φ -continuous lattices and morphisms, and $\Phi\text{AlgLat} \subseteq \Phi\text{CtsLat}$ denote the full subcategory of Φ -algebraic lattices.

Proposition 2.6. *Let $f : X \rightarrow Y$ be a right adjoint between Φ -continuous Φ -suplattices, with left adjoint $f^+ : Y \rightarrow X$. Then f preserves Φ -joins iff f^+ preserves \ll . Thus*

$$\begin{aligned} \Phi\text{CtsLat}(X, Y)^{\text{op}} &\cong \ll^\Phi \text{Sup}(Y, X) := \{f^+ : Y \rightarrow X \mid f^+ \text{ preserves } \ll, \bigvee\} \\ f &\mapsto f^+. \end{aligned}$$

Proof. $f \bigvee = \bigvee f_* : \Phi(X) \rightarrow Y$ iff, taking left adjoints, $\downarrow f^+ = (f^+)_* \downarrow : Y \rightarrow \Phi(X)$. □

Proposition 2.7. *Let Φ be a join doctrine. The following are equivalent:*

- (i) *For every complete lattice X , $\Phi(X) \subseteq \mathcal{L}(X)$ is closed under meets.*
- (ii) *For every poset X , $\Phi(\mathcal{L}(X)) \subseteq \mathcal{L}(\mathcal{L}(X))$ is closed under meets.*
- (iii) *For every poset X , $\mathcal{L}(X)$ is Φ -continuous.*

If these conditions hold, we call Φ a **continuous** join doctrine.

Proof. (i) \implies (ii) is obvious.

(ii) \implies (iii) since $\bigcup : \Phi(\mathcal{L}(X)) \rightarrow \mathcal{L}(X)$ is the composite of the inclusion $\Phi(\mathcal{L}(X)) \hookrightarrow \mathcal{L}(\mathcal{L}(X))$ and $\bigcup : \mathcal{L}(\mathcal{L}(X)) \rightarrow \mathcal{L}(X)$, which both preserve meets, i.e., have left adjoints.

(iii) \implies (i) since the composite $\mathcal{L}(X) \xrightarrow{\downarrow_{\mathcal{L}(X)}} \Phi(\mathcal{L}(X)) \xrightarrow{\bigvee_*} \Phi(X)$ yields the $\Phi(X)$ -closure of each lower set ψ : we have $1_{\mathcal{L}(X)} \leq \bigvee_* \downarrow_{\mathcal{L}(X)}$ because $\bigcup \leq \bigvee_* : \Phi(\mathcal{L}(X)) \rightarrow \Phi(X) \subseteq \mathcal{L}(X)$, while $\bigvee_* \downarrow_{\mathcal{L}(X)}$ restricted to $\Phi(X) \subseteq \mathcal{L}(X)$ becomes $\bigvee_* \downarrow_* = 1_{\Phi(X)}$. \square

The following are the two main examples of continuous join doctrines:

Example 2.8. If Φ is the “class of directed joins”, i.e., the class of all directed posets, so that $\Phi(X) \subseteq X \in \mathbf{Pos}$ is the ideal completion of X , then \ll is the classical way-below relation, and Φ -continuity and Φ -algebraicity become classical continuity and algebraicity for DCPOs.

Similarly, for any infinite regular cardinal κ , one can consider κ -directed joins. But it turns out that for uncountable κ , continuity and algebraicity coincide; see Proposition 2.11.

Example 2.9. If Φ is the “class of all joins”, i.e., the class of all posets, so that $\Phi(X) = \mathcal{L}(X)$, then a Φ -continuous lattice is a completely distributive lattice, and \ll is the “way-way-below” relation sometimes denoted \lll ; see e.g., [G⁺03, IV-3.31].

Minor variations are to include/exclude empty joins, which only affects Φ -compactness of \perp .

Example 2.10 (the unit interval). For any join doctrine Φ , $\mathbb{I} := [0, 1]$ is a Φ -continuous lattice. Indeed, \ll contains $<$, since any $\phi \in \mathcal{L}(\mathbb{I})$ with $r \leq \bigvee \phi$ must clearly contain $[0, r)$; thus $r = \bigvee \downarrow r$.

We now completely characterize the \ll^Φ relation on \mathbb{I} , by determining which $r \in \mathbb{I}$ are Φ -compact.

Proposition 2.11. *Let Φ be a join doctrine.*

- (a) *For every Φ -suplattice X , $\perp \in X$ is Φ -compact iff $\emptyset \notin \Phi$. In particular, this holds for $0 \in \mathbb{I}$.*
- (b) *If $\omega \in \Phi$ (where ω has the usual linear order), then no $r > 0$ is Φ -compact in \mathbb{I} . Otherwise:*
 - (i) *For every $\phi \in \Phi$ and $x_0, x_1, \dots \in \phi$, there are $i_0 < i_1 < \dots$ such that x_{i_0}, x_{i_1}, \dots have an upper bound in ϕ . In particular, every $x_0 \leq x_1 \leq \dots \in \phi$ has an upper bound.*
 - (ii) *Every Φ -continuous Φ -suplattice X which also has countable increasing joins is Φ -algebraic, with the join of any $x_0 \ll x_1 \ll \dots \in X$ being Φ -compact. In particular, every $r > 0$ is Φ -compact in \mathbb{I} .*

Proof. (a) is clear from the definition of Φ -compact.

(b) If $\omega \in \Phi$, then no $r > 0$ is Φ -compact, since r is the join of a sequence in $[0, r)$. Now suppose $\omega \notin \Phi$. Then for $\phi \in \Phi$ and $x_0, x_1, \dots \in \phi$, if no infinite subfamily has an upper bound, then we have a monotone map $\phi \rightarrow \omega$ taking $\phi \setminus \bigcup_n \uparrow x_n$ to 0 and each $\uparrow x_n \setminus \bigcup_{m>n} \uparrow x_m$ to $n+1$; since

$\omega \notin \Phi$, this map must have finite image, whence there are $i_0 < i_1 < \dots$ with $x_{i_0} \geq x_{i_1} \geq \dots$, a contradiction, which proves (i). It follows that for a Φ -continuous Φ -suplattice X with countable increasing joins, every $\downarrow x \in \Phi(X)$ is closed under countable increasing joins. In particular, for $x_0 \ll x_1 \ll \dots \in X$, $x := \bigvee_n x_n$ has $x_n \ll x$ for each n , whence $x \ll x$. Now for any $y \in X$ and $x_0 \ll y$, by interpolation (Proposition 2.5(c)) we may find $x_0 \ll x_1 \ll \dots \ll y$, whence $x := \bigvee_n x_n$ is Φ -compact with $x_0 \leq x \leq y$; since $y = \bigvee \downarrow y$, it follows that X is Φ -algebraic, proving (ii). \square

3 Commuting meets and joins

We are interested in recovering Φ -continuous lattices from their dual algebras of morphisms (to 2 or \mathbb{I}). In order to do so, by general duality theory, the dual algebras must be equipped with all operations which commute with the Φ -continuous lattice operations of arbitrary meets and Φ -joins. Thus, we now review the theory of classes of commuting meets and joins, again due in the general enriched categories context to [KS05], although the posets case is much simpler.

It is convenient to treat a “class of meets” as simply the order-dual of a “class of joins”. Thus, given a join doctrine Φ , we will refer to $\Phi^{\text{op}} := \{\phi^{\text{op}} \mid \phi \in \Phi\}$ as a **meet doctrine**, and a meet indexed by $\phi^{\text{op}} \in \Phi^{\text{op}}$ as a Φ^{op} -**meet**. A poset with all Φ^{op} -meets is a Φ^{op} -**inflat**, with the category of all such denoted $\Phi^{\text{op}}\text{Inf}$. A Φ^{op} -**filter** is an upper sub- Φ^{op} -inflat. The free Φ^{op} -inflat generated by a poset X is $\Phi(X^{\text{op}})^{\text{op}}$.

Definition 3.1 (see [KS05]). For two join doctrines Φ, Ψ , where we regard Ψ^{op} as a meet doctrine, to say that Ψ^{op} -**meets commute with Φ -joins in 2** means that for any posets X, Y ,

$$\forall \phi \in \Phi(Y) \forall \psi \in \Psi(X) \forall F : X^{\text{op}} \times Y \rightarrow 2 \left(\bigwedge_{x \in \psi} \bigvee_{y \in \phi} F(x, y) = \bigvee_{y \in \phi} \bigwedge_{x \in \psi} F(x, y) \right)$$

(where F runs over monotone maps). By currying F , this is equivalent to

$$\begin{aligned} \forall \phi \in \Phi(Y) \forall \psi \in \Psi(X) \forall f : Y \rightarrow \mathcal{L}(X) \left(\psi \subseteq \bigcup_{y \in \phi} f(y) \iff \exists y \in \phi (\psi \subseteq f(y)) \right) \\ \iff \forall \psi \in \Psi(X) (\psi \in \mathcal{L}(X) \text{ is } \Phi\text{-compact}). \end{aligned}$$

We write $\Phi^*(X) := \mathcal{L}(X)_{\Phi}$ for the Φ -compact lower sets $\psi \subseteq X$, i.e., those indexing meets commuting with Φ -joins in 2 . Note that by order-duality, the roles of Φ, Ψ may be swapped. Thus

$$\Psi^{\text{op}}\text{-meets commute with } \Phi\text{-joins in } 2 \iff \Psi \subseteq \Phi^* \iff \Phi \subseteq \Psi^* \quad (\text{as submonads of } \mathcal{L}).$$

Remark 3.2. Φ -joins commute with Ψ^{op} -meets in 2 iff they do in the unit interval \mathbb{I} . This follows from the facts that 2 is a complete sublattice of the unit interval \mathbb{I} , while \mathbb{I} is a complete lattice quotient of a complete sublattice $\mathcal{L}(2_+^{\leq \omega}) \subseteq 2_+^{\leq \omega}$ of a power of 2 as in Example 2.10.

Remark 3.3. If $\phi \in \Psi^*(X)$ for a Ψ -suplattice X , then by considering the indicator function of $\leq \subseteq X^{\text{op}} \times X$, we get that ϕ must be a Ψ -ideal. (The converse is false in general: for $\Psi =$ directed posets, a Ψ -ideal is a Scott-closed subset, but only finite meets commute with directed joins.)

Proposition 3.4 ([KS05, 8.9, 8.11, 8.13]). *Let Φ, Ψ be two join doctrines such that Ψ^{op} -meets commute with Φ -joins in 2 . The following are equivalent:*

- (i) *For every poset X , $\mathcal{L}(X)$ is generated under Φ -joins by $\Psi(X) \subseteq \mathcal{L}(X)_{\Phi}$.*

(ii) For every Ψ -suplattice X , $\Phi(X)$ consists precisely of all Ψ -ideals in X .

(iii) For every poset X , there is a sub- Ψ -suplattice $\Psi'(X) \subseteq \mathcal{L}(X)$ containing all principal ideals $\downarrow x$ (e.g., $\Psi'(X) = \mathcal{L}(X)$ or $\Psi'(X) = \Psi(X)$) such that $\Phi(\Psi'(X))$ contains all Ψ -ideals in $\Psi'(X)$.

If these hold, then in fact $\Psi(X) = \mathcal{L}(X)_\Phi = \Phi^*(X)$, whence $\mathcal{L}(X) \cong \Phi(\Psi(X))$ is Φ -algebraic, whence Φ is a continuous join doctrine; and similarly $\Phi = \Psi^*$.

If these hold, we call Φ a **sound join doctrine, dual to the sound meet doctrine** Ψ^{op} . Thus, Φ is a sound join doctrine iff $\mathcal{L}(X) \cong \Phi(\Phi^*(X))$, iff $\Phi(X)$ contains every Φ^* -ideal in X .

Proof. (ii) \implies (iii) is obvious.

(iii) \implies (i): For any $\theta \in \mathcal{L}(X)$, clearly $\Psi'(X) \cap \downarrow \theta = \{\psi \in \Psi'(X) \mid \psi \subseteq \theta\}$ is a Ψ -ideal in $\Psi'(X)$, thus by (iii) is in $\Phi(\Psi'(X))$; and its union is θ , which is thus a Φ -join of elements of $\Psi(X)$.

(i) \implies (ii): For every $\theta \in \mathcal{L}(X)$, the Ψ -ideal $\langle \theta \rangle$ it generates is in $\Phi(X)$: this is true for $\theta \in \Psi(X)$ since $\langle \theta \rangle = \downarrow \bigvee \theta$, and is true for a Φ -join $\theta = \bigcup_i \theta_i$ if it is true for each θ_i since $\langle \theta \rangle = \bigcup_i \langle \theta_i \rangle$ (using that Ψ^{op} -meets commute with Φ -joins in 2), thus is true for all $\theta \in \mathcal{L}(X)$ by (i). Conversely, as noted above, every $\phi \in \Phi(X)$ is a Ψ -ideal.

The claims in the last sentence follow from (i) and (ii). \square

Remark 3.5. If Φ is a sound join doctrine, then for a poset X and $\psi \in \mathcal{L}(X)$, we have $\psi \in \Phi^*(X)$ iff $\psi \in \Phi^*(\psi)$, since a monotone $F : \psi^{\text{op}} \times Y \rightarrow 2$ may be extended to $F' : X^{\text{op}} \times Y \rightarrow 2$ (e.g., by 0 on $(X^{\text{op}} \setminus \psi^{\text{op}}) \times Y$) so that the ψ^{op} -meet and Φ -join of F commute iff those of F' do. This justifies our claim from Remark 2.2 that for our purposes, it suffices to take Φ to be a class of posets, rather than a class of presheaves (i.e., lower sets in posets) as in [KS05].

Remark 3.6. If Φ is a sound join doctrine, then $\omega \in \Phi$ iff $\omega \notin \Phi^*$. Indeed, ω cannot be in both Φ, Φ^* , since countable increasing joins do not commute with countable decreasing meets in 2; and if ω were in neither Φ nor Φ^* , then $\omega \in \mathcal{L}(\omega)$ would be a Φ^* -ideal not in Φ . (See also Theorem 3.9.)

Corollary 3.7 (generalized Hofmann–Mislove–Stralka duality). *Let Φ be a sound join doctrine, dual to the meet doctrine $\Psi^{\text{op}} = \Phi^{*\text{op}}$. We have a dual equivalence of categories*

$$\Phi\text{AlgLat}^{\text{op}} \xrightleftharpoons[\Psi^{\text{op}}\text{Inf}(-,2)]{\Phi\text{AlgLat}(-,2)} \Psi^{\text{op}}\text{Inf}.$$

We may replace ΦAlgLat with ΦCtsLat iff $\omega \notin \Phi$, i.e., $\omega \in \Psi$.

Proof. For a Φ -algebraic lattice X , a morphism $X \rightarrow 2$ is the indicator function of $\uparrow x$ for Φ -algebraic x . For a Ψ^{op} -inflattice A , a morphism $A \rightarrow 2$ is the indicator function of a Ψ^{op} -filter. So we have

$$\Phi\text{AlgLat}(X, 2) \cong X_\Phi^{\text{op}}, \quad \Psi^{\text{op}}\text{Inf}(A, 2) \cong \Phi(A^{\text{op}}).$$

Now the adjunction (co)unit on the left is given by, for $X \in \Phi\text{AlgLat}$, the evaluation map

$$\begin{aligned} X &\longrightarrow \Psi^{\text{op}}\text{Inf}(\Phi\text{AlgLat}(X, 2), 2) \\ x &\longmapsto (f \mapsto f(x)) \end{aligned}$$

via the above isomorphisms becomes the canonical isomorphism $X \cong \Phi(X_\Phi)$ characterizing algebraicity. Similarly, for $A \in \Psi^{\text{op}}\text{Inf}$, the unit $A \rightarrow \Phi\text{AlgLat}(\Psi^{\text{op}}\text{Inf}(A, 2), 2)$ is the canonical isomorphism $A^{\text{op}} \cong \Phi(A^{\text{op}})_\Phi$. By Proposition 2.11, $\Phi\text{AlgLat} = \Phi\text{CtsLat}$ iff \mathbb{I} is Φ -algebraic, iff $\omega \notin \Phi$. \square

Example 3.8. $\Phi = \text{directed posets}$ forms a sound join doctrine, dual to $\Psi^{\text{op}} = \text{“finite meets”}$, i.e., $\Psi = \text{the class of posets with finite cofinality}$. In this case, Corollary 3.7 becomes the classical Hofmann–Mislove–Stralka duality [HMS74] between (unital) meet-semilattices and algebraic lattices.

Similarly, the join doctrine Φ of κ -directed posets for an uncountable regular cardinal κ is sound, dual to κ -ary meets. But since $\omega \notin \Phi$ for uncountable κ , we get a duality between κ -meet-semilattices and κ -continuous lattices.

We now show that there are very few sound join doctrines $\Phi \ni \omega$, for which $\Phi\text{AlgLat} \neq \Phi\text{CtsLat}$: essentially, they are only the classical cases of continuous and completely distributive lattices (Examples 2.8 and 2.9), plus the minor variations including/excluding empty joins.

Theorem 3.9. *There are precisely 4 sound join doctrines $\Phi \ni \omega$, dual to Ψ^{op} :*

- (i) $\Phi = \text{directed posets}$, $\Psi = \text{posets with finite cofinality}$;
- (ii) $\Phi = \text{empty or directed posets}$, $\Psi = \text{nonempty posets with finite cofinality}$;
- (iii) $\Phi = \text{nonempty posets}$, $\Psi = \text{posets which are empty or have greatest element}$;
- (iv) $\Phi = \text{all posets}$, $\Psi = \text{posets with greatest element}$.

Proof. It is well-known and easily seen that each of these 4 cases is sound; we show the converse.

First, we show that Φ must contain every directed poset, i.e., every poset in Ψ must have finite cofinality. For every set X , Φ contains the finite powerset $\mathcal{P}_\omega(X)$, since this is a Ψ -ideal in the full powerset $\mathcal{P}(X)$, since by Proposition 2.11(i) (applied to $\Psi \not\ni \omega$), every $\psi \in \Psi(\mathcal{P}_\omega(X))$ can have neither a strictly increasing sequence nor infinitely many maximal elements, thus must be finite. Now for every join-semilattice X , we have a monotone surjection $\bigvee : \mathcal{P}_\omega(X) \twoheadrightarrow X$, whence $X \in \Phi$. Since every directed poset ϕ is cofinal in the free join-semilattice it generates, it follows that $\phi \in \Phi$.

So Ψ is determined by the finite antichains n in it. If some $n > 1$ is in Ψ , then by induction so is each $n^k \cong \bigsqcup_{i \in n} n^{k-1}$; now every $m \geq 1$ admits a surjection $n^k \twoheadrightarrow m$, whence $m \in \Psi$. \square

4 \mathbb{U} -posets

Henceforth, we assume $\Phi \ni \omega$ is a sound join doctrine, dual to Ψ^{op} , so one of the cases in Theorem 3.9. Then Hofmann–Mislove–Stralka duality does not apply to all Φ -continuous lattices, and so we would like to formulate a duality based on morphisms to \mathbb{I} instead of 2 .

By Remark 3.2, the dual algebra $\Phi\text{CtsLat}(X, \mathbb{I})$ will still be equipped with Ψ^{op} -meets. But these are not all the operations on \mathbb{I} commuting with the Φ -continuous lattice operations: clearly any complete lattice homomorphism $\mathbb{I} \rightarrow \mathbb{I}$ does as well. We thus introduce the following notions:

Definition 4.1. Let $\mathbb{U} := \text{CLat}(\mathbb{I}, \mathbb{I})$ denote the partially ordered monoid of all complete lattice homomorphisms $\mathbb{I} \rightarrow \mathbb{I}$, i.e., surjective monotone maps.

A **\mathbb{U} -poset** is a poset equipped with a monotone (in both variables) action of the monoid \mathbb{U} . Denote the category of these (and equivariant monotone maps) by \mathbb{UPos} .

A **\mathbb{U} - Ψ^{op} -inflattice** is a \mathbb{U} -poset which is also a Ψ^{op} -inflattice such that the action of each $u \in \mathbb{U}$ preserves Ψ^{op} -meets. Denote the category of these by $\mathbb{U}\Psi^{\text{op}}\text{Inf}$.

Definition 4.2. Let $\dot{+}, \dot{-}$ denote truncated $+, -$ on \mathbb{I} ; note that they obey the adjunction

$$(4.3) \quad r \dot{-} s \leq t \iff r \leq s \dot{+} t.$$

For a \mathbb{U} -poset A and $a, b \in A$, define

$$\begin{aligned} a \leq_r b &: \iff \forall u, v \in \mathbb{U} (u((-) \dot{+} r) \leq v \implies u(a) \leq v(b)), \\ \rho(a, b) &:= \bigwedge \{r \in \mathbb{I} \mid a \leq_r b\}, \\ d(a, b) &:= \rho(a, b) \vee \rho(b, a). \end{aligned}$$

Remark 4.4. In the definition of \leq_r , instead of testing $\forall u, v$, it is enough to test any particular $u \in \mathbb{U}$ which restricts to an order-isomorphism $u : [r, 1] \cong [0, 1]$ (e.g., the linear such isomorphism extended by 0 on $[0, r]$), so that $v := u((-) \dot{+} r) \in \mathbb{U}$. Indeed, for any other $u', v' \in \mathbb{U}$ with $u'((-) \dot{+} r) \leq v'$, there is $w \in \mathbb{U}$ with $u' = w \circ u$, whence $u'(a) = w(u(a)) \leq w(v(a)) \leq v'(a)$.

Remark 4.5. There is an evident order-duality for \mathbb{U} -posets A : let $u \in \mathbb{U}$ act on the order-dual A^{op} via $1 - u(1 - (-))$; this reverses each \leq_r , and turns ρ into $\rho^{\text{op}}(a, b) := \rho(b, a)$.

Intuitively, $a \leq_r b$ means “ $a \leq b \dot{+} r$ ”. The following properties justify this interpretation:

Proposition 4.6. In \mathbb{I} , we have $a \leq_r b \iff a \leq b \dot{+} r$, whence $\rho(a, b) = a \dot{-} b$ and $d(a, b) = |a - b|$.

Proof. If $a \leq b \dot{+} r$, then for every $u, v \in \mathbb{U}$ with $u((-) \dot{+} r) \leq v$, we have $u(a) \leq u(b \dot{+} r) \leq v(b)$.

For the converse, the case $r = 1$ is vacuous; thus we may assume $r < 1$. Note that $(-) \dot{+} r : \mathbb{I} \rightarrow \mathbb{I}$ can be written as $u^\times \circ v$ where $v := 1 \wedge (-)/(1 - r)$, $u := v((-) \dot{-} r)$, and u^\times is the right adjoint of u . Now from $a \leq_r b$ and $u((-) \dot{+} r) = v$, we get $u(a) \leq v(b)$, whence $a \leq u^\times(v(b)) = b \dot{+} r$. \square

Lemma 4.7. In every \mathbb{U} -poset A , we have the following, for $r, s, t \in \mathbb{I}$, $u, v \in \mathbb{U}$, $a, b, c \in A$:

- (a) $r \leq s \ \& \ a \leq_r b \implies a \leq_s b$.
- (b) \leq_0 is the same as \leq .
- (c) $a \leq_r b \leq_s c \implies a \leq_{r \dot{+} s} c$.
- (d) ρ is a pseudoquasimetric: $\rho(a, a) = 0$, and $\rho(a, b) + \rho(b, c) \geq \rho(a, c)$. Thus, d is a pseudometric.
- (e) $u((-) \dot{+} r) \leq v \dot{+} s \ \& \ a \leq_r b \implies u(a) \leq_s v(b)$. Thus, $\rho(u(a), v(b)) \leq \rho(u, v) := \bigvee (u \dot{-} v)$, i.e., the \mathbb{U} -action is 1-Lipschitz in the first variable with respect to the ℓ^∞ -quasimetric on \mathbb{U} . Moreover, if $u \in \mathbb{U}$ is uniformly continuous with modulus $\mu : \mathbb{I} \rightarrow \mathbb{I}$, i.e., $u(r) \dot{-} u(s) \leq \mu(r \dot{-} s)$, then the action of u is uniformly continuous with the same modulus: $\rho(u(a), u(b)) \leq \mu(\rho(a, b))$.
- (f) $u^\times((-) \dot{+} r) \leq v \dot{+} s \ \& \ u(a) \leq_r b \implies a \leq_s v(b)$ (where u^\times is the right adjoint of u).

In a \mathbb{U} - Ψ^{op} -inflatice, we moreover have, for $\psi, \psi' \in \Psi(A^{\text{op}})$:

- (g) $a \leq_r \bigwedge \psi \iff \forall b \in \psi (a \leq_r b)$. Thus, $\rho(\bigwedge \psi, \bigwedge \psi') \leq \bigwedge_{a \in \psi} \bigvee_{b \in \psi'} \rho(a, b)$.

Proof. (a) and (b) are straightforward, as is (d) given the previous parts.

(c) For $u, w \in \mathbb{U}$ with $u((-) \dot{+} (r \dot{+} s)) \leq w$, we have $v := u((-) \dot{+} r) \in \mathbb{U}$ with $u((-) \dot{+} r) \leq v$ and $v((-) \dot{+} s) \leq w$, whence $u(a) \leq v(b) \leq w(c)$.

(e) For $u', v' \in \mathbb{U}$ with $u'((-) \dot{+} s) \leq v'$, we have $u'(u((-) \dot{+} r)) \leq u'(v(-) \dot{+} s) \leq v' \circ v$, whence $u'(u(a)) \leq v'(v(b))$. For the last assertion: $u(r) \dot{-} u(s) \leq \mu(r \dot{-} s)$ means $u((-) \dot{+} r) \leq u(-) \dot{+} \mu(r)$.

(f) The assumption is equivalent to $(-) \dot{-} s \leq v(u(-) \dot{-} r)$; thus for $u', v' \in \mathbb{U}$ with $u'((-) \dot{+} s) \leq v'$, we have $u' \leq v'((-) \dot{-} s) \leq v'(v(u(-) \dot{-} r))$, whence $u'(a) \leq v'(v(u(a) \dot{-} r)) \leq v'(v(b))$.

(g) \implies and the last assertion follow from (c). For \Leftarrow : for $u, v \in \mathbb{U}$ with $u((-) \dot{+} r) \leq v$, we have $u(a) \leq \bigwedge_{b \in \psi} v(b) = v(\bigwedge \psi)$. \square

For general background on (pseudo)quasimetrics, see e.g., [Kün09]. A pseudoquasimetric ρ as above induces a topology, where a basic neighborhood of $a \in A$ is $\{b \in A \mid \rho(a, b) < r\}$ for some $r > 0$. Thus the closure of $B \subseteq A$ is the set of all $a \in A$ such that

$$\rho(a, B) = \bigwedge_{b \in B} \rho(a, b) = 0,$$

which is in particular a lower set. To avoid confusion, we will call a closed set in this topology a **ρ -closed lower set**, and denote the set of all such by $\overline{\mathcal{L}}(A) \subseteq \mathcal{L}(A)$. We will also say **ρ^{op} -closed upper set** $B \subseteq A$ for the order-dual notion, i.e., if $\rho(B, a) = 0$ then $a \in B$; the set of all such is thus $\overline{\mathcal{L}}(A^{\text{op}})$. For a \mathbb{U} - Ψ^{op} -inflatlattice A , recalling that $\Phi(A^{\text{op}})$ consists of Ψ^{op} -filters by soundness, let

$$\overline{\Phi}(A^{\text{op}}) := \Phi(A^{\text{op}}) \cap \overline{\mathcal{L}}(A^{\text{op}})$$

denote the **ρ^{op} -closed Ψ^{op} -filters** in A .

Lemma 4.8. *If $\phi \in \Phi(A^{\text{op}})$ is a Ψ^{op} -filter, then so is the ρ^{op} -closure $\overline{\phi}$.*

Proof. This follows from the facts that Ψ^{op} is a class of finite meets by Theorem 3.9, and that Ψ^{op} -meets are Lipschitz by Lemma 4.7(g). \square

As usual for actions, a subset $B \subseteq A$ of a \mathbb{U} -poset is **\mathbb{U} -invariant** if it is closed under the action. For a class of sets $\Gamma(A)$, we write $\Gamma^{\mathbb{U}}(A)$ for the \mathbb{U} -invariant members, e.g., $\mathcal{L}^{\mathbb{U}}(A), \overline{\Phi}^{\mathbb{U}}(A)$.

Lemma 4.9. *If $\phi \in \mathcal{P}^{\mathbb{U}}(A)$ is a \mathbb{U} -invariant set, then its ρ^{op} -closure $\overline{\phi}$ is a \mathbb{U} -invariant Ψ^{op} -filter, hence is the \mathbb{U} -invariant ρ -closed Ψ^{op} -filter generated by ϕ .*

Proof. By uniform continuity of the action of each u (Lemma 4.7(e)), $\overline{\phi}$ is \mathbb{U} -invariant. It is also upper, since every ρ^{op} -closed set is, thus it is also the ρ^{op} -closure of the upward closure of ϕ , which is a Ψ^{op} -filter by Theorem 3.9 since \mathbb{U}^{op} is directed, whence so is $\overline{\rho}$ by the preceding lemma. \square

Proposition 4.10. *For a \mathbb{U} - Ψ^{op} -inflatlattice A , we have an order-isomorphism*

$$\begin{aligned} \mathbb{U}\Psi^{\text{op}}\text{Inf}(A, \mathbb{I}) &\cong \overline{\Phi}^{\mathbb{U}}(A^{\text{op}}) = \{\mathbb{U}\text{-invariant } \rho^{\text{op}}\text{-closed } \Psi^{\text{op}}\text{-filters in } A\} \\ f &\mapsto f^{-1}(1) \\ 1 - \rho(\phi, -) &\leftarrow \phi. \end{aligned}$$

Proof. For ease of notation, we will prove the dual statement that for a \mathbb{U} - Ψ -suplattice A ,

$$\begin{aligned} \mathbb{U}\Psi\text{Sup}(A, \mathbb{I})^{\text{op}} &\cong \overline{\Phi}^{\mathbb{U}}(A) = \{\mathbb{U}\text{-invariant } \rho\text{-closed } \Psi\text{-ideals in } A\} \\ f &\mapsto f^{-1}(0) \\ \rho(-, \phi) &\leftarrow \phi. \end{aligned}$$

It is immediate from the definitions that for a \mathbb{U} -equivariant Ψ -join-preserving $f : A \rightarrow \mathbb{I}$, $f^{-1}(0) \subseteq A$ is \mathbb{U} -invariant ρ -closed lower, and also that a ρ -closed lower $\phi \subseteq A$ is equal to $\rho(-, \phi)^{-1}(0)$.

We now check that for a \mathbb{U} -invariant Ψ -ideal $\phi \subseteq A$, $\rho(-, \phi) : A \rightarrow \mathbb{I}$ is \mathbb{U} -equivariant Ψ -join-preserving (it is clearly monotone). For $\psi \in \Psi(A)$,

$$\begin{aligned} \rho(\bigvee \psi, \phi) &= \bigwedge_{b \in \phi} \bigvee_{a \in \psi} \rho(a, b) \quad \text{by the dual of Lemma 4.7(g)} \\ &= \bigvee_{a \in \psi} \bigwedge_{b \in \phi} \rho(a, b) \quad \text{because } \Phi \subseteq \Psi^* \text{ (Remark 3.2)} \\ &= \bigvee_{a \in \psi} \rho(a, \phi); \end{aligned}$$

thus $\rho(-, \phi)$ preserves Ψ -joins. To check \mathbb{U} -equivariance: let $u \in \mathbb{U}$ and $a \in A$. We have

$$\begin{aligned}\rho(u(a), \phi) &= \bigwedge_{b \in \phi} \rho(u(a), b) = \bigwedge \{r \in \mathbb{I} \mid u(a) \leq_r b \in \phi\}, \\ u(\rho(a, \phi)) &= u(\bigwedge_{b \in \phi} \rho(a, b)) = \bigwedge_{b \in \phi} u(\rho(a, b)) = \bigwedge \{u(r) \mid a \leq_r b \in \phi\}.\end{aligned}$$

For each $a \leq_r b \in \phi$, find

$$u((-) \dot{+} r) \dot{\div} u(r) \leq v \in \mathbb{U},$$

whence $u(a) \leq_{u(r)} v(b) \in \phi$ by Lemma 4.7(e); this proves $u(\rho(a, \phi)) \geq \rho(u(a), \phi)$. Conversely, for $u(a) \leq_r b \in \phi$ with $r < 1$, let u^\times be the right adjoint of u , and similarly to before, find

$$u^\times((-) \dot{+} r) \dot{\div} u^\times(r) \leq v \in \mathbb{U},$$

whence $a \leq_{u^\times(r)} v(b) \in \phi$ by Lemma 4.7(f), whence $u(\rho(a, \phi)) \leq r$; so $\rho(u(a), \phi) \geq u(\rho(a, \phi))$.

Finally, we check that for \mathbb{U} -equivariant monotone $f : A \rightarrow \mathbb{I}$, we have $f = \rho(-, f^{-1}(0))$. We have \leq since f is 1-Lipschitz. Conversely, for $a \in A$ with $f(a) < 1$, find $(-) \dot{\div} f(a) \leq u \in \mathbb{U}$ with $u(f(a)) = 0$; then $a \leq_{f(a)} u(a)$ by Lemma 4.7(e), so $\rho(a, f^{-1}(0)) \leq \rho(a, u(a)) \leq f(a)$. \square

The \mathbb{U} -poset \mathbb{I} obeys the following additional axioms, which must thus also hold in the dual of a Φ -continuous lattice:

Definition 4.11. We call a \mathbb{U} -poset A **Archimedean** if it obeys

$$\forall r > 0 (a \leq_r b) \implies a \leq b.$$

We call A **(Cauchy-)complete** if it is Archimedean and also complete in the metric d .

Definition 4.12. We call a \mathbb{U} -poset A **unstackable** if for any $0 < r < 1$ and $u, v \in \mathbb{I}$ restricting to order-isomorphisms $u : [0, r] \cong [0, 1]$ and $v : [r, 1] \cong [0, 1]$, we have

$$u(a) \leq u(b) \ \& \ v(a) \leq v(b) \implies a \leq b.$$

We call A **stackable** if it is unstackable and for r, u, v as above and $a, b \in A$ such that $v'(b) \leq u'(a)$ for all $u', v' \in \mathbb{U}$, there is a (unique, by unstackability) $c \in A$ with $u(c) = a$ and $v(c) = b$.

Intuitively, stackability means that, thinking of A as the dual of a Φ -continuous lattice X , we may specify $A \ni a : X \rightarrow \mathbb{I}$ via its restrictions to its sublevel and superlevel sets $a^{-1}([0, r]), a^{-1}([r, 1])$.

Remark 4.13. As in Remark 4.4, it is enough to take some particular u, v above. Also, it is enough to take some particular r (e.g., $1/2$), since we may move r around via an order-isomorphism $\mathbb{I} \cong \mathbb{I}$.

Lemma 4.14. *If A is (un)stackable, then more generally, for $0 = r_0 < r_1 < \dots < r_n = 1$ and $u_1, \dots, u_n \in \mathbb{U}$ restricting to $u_i : [r_{i-1}, r_i] \cong [0, 1]$, for $a_1, \dots, a_n \in A$ such that $v'(a_{i+1}) \leq u'(a_i)$ for all $u', v' \in \mathbb{U}$, there is (at most one, depending monotonically on (a_1, \dots, a_n)) $a \in A$ with $u_i(a) = a_i$.*

Proof. By a straightforward induction on n . \square

Lemma 4.15. *If A is unstackable, then more generally, for $0 \leq r = r_0 < r_1 < \dots < r_n = 1$ and $u_1, \dots, u_n \in \mathbb{U}$ with $u_i : [r_{i-1}, r_i] \cong [0, 1]$, so that $u_i((-) \dot{+} r) \in \mathbb{U}$, for any $a, b \in A$, we have*

$$u_1(a) \leq u_1(b \dot{+} r) \ \& \ \dots \ \& \ u_n(a) \leq u_n(b \dot{+} r) \implies a \leq_r b.$$

Proof. By Remark 4.4, it suffices to check that for $w \in \mathbb{U}$ with $w : [r, 1] \cong [0, 1]$, we have $w(a) \leq w(b \dot{+} r)$; this follows from applying the preceding lemma to $u_i \circ w^{-1} : [w(r_{i-1}), w(r_i)] \cong [0, 1]$. \square

5 The duality

Let $\mathbf{CSt}\mathbb{U}\Psi^{\text{op}}\mathbf{Inf} \subseteq \mathbb{U}\Psi^{\text{op}}\mathbf{Inf}$ denote the full subcategory of complete stackable \mathbb{U} - Ψ^{op} -inflattices. Since the Φ -continuous lattice and \mathbb{U} - Ψ^{op} -inflattice structures on \mathbb{I} commute, we have a dual adjunction

$$(5.1) \quad \Phi\mathbf{CtsLat}^{\text{op}} \xrightleftharpoons[\mathbb{U}\Psi^{\text{op}}\mathbf{Inf}(-, \mathbb{I})]{\Phi\mathbf{CtsLat}(-, \mathbb{I})} \mathbf{CSt}\mathbb{U}\Psi^{\text{op}}\mathbf{Inf} \subseteq \mathbb{U}\Psi^{\text{op}}\mathbf{Inf}.$$

Theorem 5.2. *For every Φ -continuous lattice X , the evaluation map*

$$\begin{aligned} \eta : X &\longrightarrow \mathbb{U}\Psi^{\text{op}}\mathbf{Inf}(\Phi\mathbf{CtsLat}(X, \mathbb{I}), \mathbb{I}) \\ x &\longmapsto (f \mapsto f(x)), \end{aligned}$$

which is the (co)unit on the left side of the above adjunction, is an order-isomorphism.

Proof. Via Propositions 2.6 and 4.10, η corresponds to the map

$$\begin{aligned} \tilde{\eta} : X &\longrightarrow \overline{\Phi}^{\mathbb{U}}(\ll^{\Phi}\mathbf{Sup}(\mathbb{I}, X)) \subseteq \mathcal{L}(\ll^{\Phi}\mathbf{Sup}(\mathbb{I}, X)) \\ x &\longmapsto \{f^+ \in \ll^{\Phi}\mathbf{Sup}(\mathbb{I}, X) \mid f^+(1) \leq x\} \end{aligned}$$

whose left adjoint is easily seen to be

$$\begin{aligned} \tilde{\eta}^+ : \mathcal{L}(\ll^{\Phi}\mathbf{Sup}(\mathbb{I}, X)) &\longrightarrow X \\ \phi &\longmapsto \bigvee_{f^+ \in \phi} f^+(1). \end{aligned}$$

That $x \leq \tilde{\eta}^+(\tilde{\eta}(x))$ is Urysohn's lemma for Φ -continuous lattices; see [G⁺03, IV-3.1, IV-3.32], [Joh82, VII 1.14, 3.2], [Xu95]. Since $x = \bigvee \downarrow x$, it suffices to show that for each $y \ll x$ there is $f^+ \in \ll^{\Phi}\mathbf{Sup}(\mathbb{I}, X)$ with $y \leq f^+(1) \leq x$. Let $\mathbb{I}_2 \subseteq \mathbb{I}$ be the dyadic rationals, define $g : \mathbb{I}_2 \rightarrow X$ by $g(0) := y$, $g(1) := x$, and inductively using interpolation (Proposition 2.5(c)) so that $r < s \implies g(r) \ll g(s)$; then $f^+(r) := \bigvee g(\mathbb{I}_2 \cap [0, r))$ works.

Now let $\phi \in \overline{\Phi}^{\mathbb{U}}(\ll^{\Phi}\mathbf{Sup}(\mathbb{I}, X))$; we must show $\tilde{\eta}(\tilde{\eta}^+(\phi)) \subseteq \phi$. Since $\tilde{\eta}$ preserves Φ -joins,

$$\tilde{\eta}(\tilde{\eta}^+(\phi)) = \bigvee_{f^+ \in \phi} \tilde{\eta}(f^+(1)).$$

For each $f^+ \in \phi$ and $g^+ \in \tilde{\eta}(f^+(1))$, i.e., $g^+(1) \leq f^+(1)$, we have $1 \leq g(f^+(1))$, thus there is $g \circ f^+ \geq u \in \mathbb{U}$, whence $g \geq u \circ f$, so $g^+ \leq (u \circ f)^+ \in \phi$ since ϕ is \mathbb{U} -invariant; thus $\tilde{\eta}(f^+(1)) \subseteq \phi$. \square

Theorem 5.3. *For every Archimedean unstackable \mathbb{U} - Ψ^{op} -inflattice A , the evaluation map*

$$\begin{aligned} \iota : A &\longrightarrow \Phi\mathbf{CtsLat}(\mathbb{U}\Psi^{\text{op}}\mathbf{Inf}(A, \mathbb{I}), \mathbb{I}) \\ a &\longmapsto (f \mapsto f(a)) \end{aligned}$$

is an embedding. If A is stackable, its image is dense; thus if A is also complete, it is an isomorphism.

Proof. Via Propositions 2.6 and 4.10, ι corresponds to the map

$$\begin{aligned} \tilde{\iota} : A &\longrightarrow \ll^{\Phi}\mathbf{Sup}(\mathbb{I}, \overline{\Phi}^{\mathbb{U}}(A^{\text{op}}))^{\text{op}} \\ a &\longmapsto (r \mapsto \min\{\phi \in \overline{\Phi}^{\mathbb{U}}(A^{\text{op}}) \mid r \leq 1 - \rho(\phi, a)\}). \end{aligned}$$

We claim that in fact, for $r > 0$, $\tilde{\iota}(a)(r)$ is the ρ^{op} -closure $\overline{U_r(a)}$ of

$$U_r(a) := \{u(a) \mid u \in \mathbb{U} \ \& \ u(r) = 1\}.$$

$\overline{U_r(a)}$ is a \mathbb{U} -invariant Ψ^{op} -filter by Lemma 4.9. Each $u(a) \in U_r(a)$ is in each $\phi \in \overline{\Phi}^{\mathbb{U}}(A^{\text{op}})$ with $r \leq 1 - \rho(\phi, a)$: if $u(s) = 1$ for some $s < r$, we may let $b \in \phi$ with $b \leq_{1-s} a$ to get $\phi \ni u(b \dot{-} (1-s)) \leq u(a)$, while if there is no such s , we may write u as a limit of u_0, u_1, \dots for which there are such s , then use that ϕ is closed. And $r \leq 1 - \rho(\overline{U_r(a)}, a)$: letting $(-) \dot{+} (1-r) \geq u \in \mathbb{U}$ with $u(r) = 1$, we have $U_r(a) \ni u(a) \leq_{1-r} a$ by Lemma 4.7(e). This proves the claim.

Now to show that $\tilde{\iota}$ is an order-embedding: let $\tilde{\iota}(a) \geq \tilde{\iota}(b) : \mathbb{I} \rightarrow \overline{\Phi}^{\mathbb{U}}(A^{\text{op}})$, i.e., $\overline{U_r(a)} \supseteq U_r(b)$ for every $r > 0$; since A is Archimedean, it suffices to show $a \leq_{2/n} b$ for all $n \geq 3$. For $i = 1, \dots, n$, let

$$(*) \quad v_i \in \mathbb{U}, \quad v_i : [(i-1)/n, i/n] \cong [0, 1].$$

Then $v_i(b) \in U_{i/n}(b)$, so there is $u_i \in \mathbb{U}$ with $u_i(i/n) = 1$ such that

$$u_i(a) \leq_{1/n} v_i(b).$$

Let $u', v' \in \mathbb{U}$ with $u'((-) \dot{+} 1/n) \leq v'$; then for $2 \leq i \leq n-1$, we have $v_{i+1}(a) \leq u'(u_i(a)) \leq v'(v_i(b)) \leq v_{i+1}(b \dot{+} 2/n)$ since $v_{i+1}(i/n) = 0$, $u'(u_i(i/n)) = 1$, $v'(v_i((i-1)/n)) = 0$, and $v_{i+1}((i-1)/n \dot{+} 2/n) = 1$. Thus since A is unstackable, by Lemma 4.15 we have $a \leq_{2/n} b$.

Finally, suppose A is stackable, and let $f^+ \in \ll^{\Phi} \text{Sup}(\mathbb{I}, \overline{\Phi}^{\mathbb{U}}(A^{\text{op}}))$, left adjoint to f ; we will find, for every $n \geq 2$, some $a \in A$ with $d(\iota(a), f) \leq 2/n$. For $i = 1, \dots, n$, we have $f^+((i-1)/n) \ll f^+(i/n) = \bigvee_{a \in f^+(i/n)} \overline{U_1(a)} = \bigvee_{a \in f^+(i/n)} \bigvee_{r < 1} \overline{U_r(a)}$ (again by Lemma 4.9), whence

$$f^+((i-1)/n) \subseteq \overline{U_{r_i}(a_i)}$$

for some $a_i \in f^+(i/n)$ and $r_i < 1$. Let $u_i \in \mathbb{U}$ with $u_i(r_i) = 0$, and let v_i as in (*). Then for $u' \in \mathbb{U}$,

$$f^+((i-1)/n) \subseteq \uparrow u'(u_i(a_i)) \subseteq \overline{U_1(u_i(a_i))},$$

since for $b \in f^+((i-1)/n) \subseteq \overline{U_{r_i}(a_i)}$, for every $s > 0$, there is $u'' \in \mathbb{U}$ with $u''(r_i) = 1$, whence $u' \circ u_i \leq u''$, such that $u'(u_i(a_i)) \leq u''(a_i) \leq_s b$, whence $u'(u_i(a_i)) \leq b$ since A is Archimedean. In particular, this holds for $b = v'(u_{i-1}(a_{i-1}))$ for every $v' \in \mathbb{U}$, so by Lemma 4.14, there is $a \in A$ with

$$v_i(a) = u_i(a_i)$$

for each i . Then

$$U_{i/n}(a) = U_1(v_i(a)) = U_1(u_i(a_i)),$$

since every $u \in \mathbb{U}$ with $u(i/n) = 1$ is $\geq u' \circ v_i$ for some $u' \in \mathbb{U}$. We now show that $d(f, \iota(a)) \leq 2/n$, in terms of the left adjoints $f^+, \tilde{\iota}(a)$: for each $t \in \mathbb{I}$, letting $1 \leq i \leq n$ with $t \leq i/n \leq t \dot{+} 1/n$,

$$\tilde{\iota}(a)(t) = \overline{U_t(a)} \subseteq \overline{U_{i/n}(a)} = \overline{U_1(u_i(a_i))} \subseteq f^+(i/n) \subseteq f^+(t \dot{+} 1/n),$$

$$f^+(t \dot{-} 1/n) \subseteq f^+((i-1)/n) \subseteq \overline{U_1(u_i(a_i))} = \overline{U_{i/n}(a)} \subseteq \overline{U_{t+1/n}(a)} = \tilde{\iota}(a)(t \dot{+} 1/n). \quad \square$$

Theorem 5.4. *The dual adjunction (5.1) is a dual equivalence of categories between Φ -continuous lattices and complete stackable \mathbb{U} - Ψ^{op} -inflattices.* \square

It is worth explicitly restating the duality for the two main examples of Φ :

Corollary 5.5. *Hom into \mathbb{I} yields a dual equivalence of categories between completely distributive lattices and complete stackable \mathbb{U} -posets.* \square

Let us say that a \mathbb{U} -meet-semilattice is a \mathbb{U} -poset with finite meets preserved by the \mathbb{U} -action.

Corollary 5.6. *Hom into \mathbb{I} yields a dual equivalence of categories between continuous lattices and complete stackable \mathbb{U} -meet-semilattices.* \square

We end by showing that in the presence of meets, stackability admits a simpler formulation:

Definition 5.7. Let $\widehat{\mathbb{U}} := \text{CtsLat}(\mathbb{I}, \mathbb{I}) \supseteq \mathbb{U}$ be the monoid of continuous lattice morphisms $\mathbb{I} \rightarrow \mathbb{I}$, i.e., continuous monotone maps preserving 1, but possibly not 0.

A $\widehat{\mathbb{U}}$ -module is a (unital) meet-semilattice with a $\widehat{\mathbb{U}}$ -action preserving finite meets on both sides.

Proposition 5.8. *The forgetful functor is an isomorphism of categories between complete $\widehat{\mathbb{U}}$ -modules and complete stackable \mathbb{U} -meet-semilattices. The \leq_r relations in a $\widehat{\mathbb{U}}$ -module are given by*

$$\rho(a, b) \leq r \iff a \leq_r b \iff a \leq b \dot{+} r.$$

Proof. The characterization of \leq_r is proved as in Proposition 4.6.

Next, an Archimedean $\widehat{\mathbb{U}}$ -module A is unstackable as a \mathbb{U} -poset: by Remark 4.13, it suffices to check that for $0 < r < 1$, $u := 1 \wedge (-)/r$, and $v := ((-) \div r)/(1 - r)$, if $u(a) \leq u(b)$ and $v(a) \leq v(b)$, then $a \leq b$. Let $s > 0$, and let $r(-) \leq w \in \mathbb{U}$ with equality on $[0, 1 - s]$. Then $1_{\mathbb{I}} \leq (w \circ u) \wedge (v^\times \circ v) \leq (-) \dot{+} rs$, whence from $u(a) \leq u(b)$ and $v(a) \leq v(b)$ we have $a \leq b \dot{+} rs$, i.e., $a \leq_{rs} b$ by the above. Since A is Archimedean, it follows that $a \leq b$.

If moreover A is a complete $\widehat{\mathbb{U}}$ -module, then it is stackable: for $a, b \in A$ such that $v'(b) \leq u'(a)$ for all $u', v' \in \mathbb{U}$, with the same s, u, v, w as above, letting $c_s := w(a) \wedge v^\times(b)$, we have $u(c_s) = u(w(a)) \wedge u(v^\times(b)) = u(w(a))$ which is within distance s of a since $1_{\mathbb{U}} \leq u \circ w \leq (-) \dot{+} s$, and $v(c_s) = v(w(a)) \wedge v(v^\times(b)) = v(v^\times(b)) = b$. In particular, by unstackability (using Lemma 4.15 and uniform continuity of u), the c_s form a Cauchy net as $s \searrow 0$, hence converge to some c such that $u(c) = a$ and $v(c) = b$. Thus the forgetful functor restricts to the claimed subcategories.

The forgetful functor is full on Archimedean $\widehat{\mathbb{U}}$ -modules: the action by $w \in \widehat{\mathbb{U}} \setminus \mathbb{U}$ can be recovered from the \mathbb{U} -action, since $w(a) = \top$ for $w(0) = 1$, while for $0 < w(0) < 1$, by unstackability, $w(a)$ is the unique element such that $u(w(a)) = \top$ and $v(w(a)) = (v \circ w)(a)$ where u, v are as above for $r := w(0)$. Thus \mathbb{U} -equivariance implies $\widehat{\mathbb{U}}$ -equivariance.

Conversely, in a complete stackable \mathbb{U} -meet-semilattice A , we may extend the \mathbb{U} -action to a $\widehat{\mathbb{U}}$ -action by defining $w(a)$ for $0 < w(0) < 1$ to be the unique element as above.

The \mathbb{U} -action on an Archimedean stackable \mathbb{U} -poset A preserves binary meets in \mathbb{U} : for piecewise linear $u, v \in \mathbb{U}$, we may show $(u \wedge v)(a) = u(a) \wedge v(a)$ by unstacking over a finite partition of $[0, 1]$ on each piece of which u, v are comparable; for arbitrary u, v , take piecewise linear approximations.

Finally, on a complete stackable \mathbb{U} -meet-semilattice, the extended $\widehat{\mathbb{U}}$ -action from above also preserves binary meets in $\widehat{\mathbb{U}}$, by a routine unstacking over $0 < w(0) < 1$. \square

Corollary 5.9 (of Corollary 5.6 and Proposition 5.8). *Hom into \mathbb{I} yields a dual equivalence of categories between continuous lattices and complete $\widehat{\mathbb{U}}$ -modules.* \square

Finally, we note that we currently do not know whether complete $\widehat{\mathbb{U}}$ -modules can be equationally axiomatized, perhaps along the lines of [Abb19], thereby showing that $\text{CtsLat}^{\text{op}}$ is a variety.

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