A geometric proof that $S_0 \times S_0$ is not spatial

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Let $\widehat{S}_0^* := \mathbb{N}^{\le\mathbb{N}}$, ordered by the prefix relation $\preceq$, with subbasic closed sets $\uparrow s$ for $s \in \mathbb{N}^{<\mathbb{N}}$. Note that $\widehat{S}_0^*$ is quasi-Polish, copresented by the following relations between subbasic closed sets:

$$\begin{aligned} \uparrow s \subseteq \uparrow t &\quad \text{for } s \succeq t, \\ \uparrow s \cap \uparrow t = \varnothing &\quad \text{for } s, t \text{ incomparable.} \end{aligned}$$

Let $\widehat{S}_0 \subseteq \widehat{S}_0^*$ be the G_δ subspace on the complement of the countably many closed points given by the eventually all 0 infinite sequences. Let $S_0 \subseteq \widehat{S}_0$ be the subspace on $\mathbb{N}^{<\mathbb{N}}$. De Brecht proved [deB] that the product S_0^2 in the category of locales is non-spatial, via an analog of Johnstone's "C-ideals" proof [Joh, II 2.14] that $\mathbb{Q}^2$ is non-spatial. In this note, we give a different (though similar in spirit) proof modeled on Isbell's "geometric" proof [Joh, III 1.4] that $\mathbb{Q}^2$ is non-spatial.

Define the (discontinuous) bijection

$$\begin{aligned} f : \widehat{S}_0 &\longrightarrow [0,1) \\ s &\longmapsto 2^{-(1+s_0)} + 2^{-(2+s_0+s_1)} + 2^{-(3+s_0+s_1+s_2)} + \cdots \\ &= 0.\underbrace{0\cdots0}_{s_0}1\underbrace{0\cdots0}_{s_1}1\underbrace{0\cdots0}_{s_2}1\cdots \quad \text{in binary.} \end{aligned}$$

Thus $f(S_0) \subseteq [0,1)$ consists of the dyadic rationals; and for each $s \in \mathbb{N}^{<\mathbb{N}}$,

$$\begin{aligned} f(\uparrow s) &= [f(s), f(s) + \tfrac{1}{2^n}) \quad \text{where } 2^n = \text{denominator of } f(s) \\ &= \begin{cases} [f(s), f(s_0\hat{\ }s_1\hat{\ }\cdots\hat{\ }s_{k-1}\hat{\ }(s_k - 1))) & \text{if } s_k = \text{last nonzero term in } s, \\ [f(s), 1) & \text{if } s = \text{all 0's.} \end{cases} \end{aligned}$$

Let $E := \{(x,y) \in [0,1)^2 \mid x + y = \frac{1}{3}\}$.

Lemma 1. The closure of $f^{-1}(E) \subseteq \widehat{S}_0^2$ does not contain any point in S_0^2.

Proof. Let $(s,t) \in S_0^2$. For each $p \preceq s$, choose some $m_p \in \mathbb{N}$, and for each $q \preceq t$, choose some $n_q \in \mathbb{N}$, such that whenever $f(p) + f(q) < \frac{1}{3}$, then $f(p\hat{\ }m_p) + f(q\hat{\ }n_q) < \frac{1}{3}$. Then

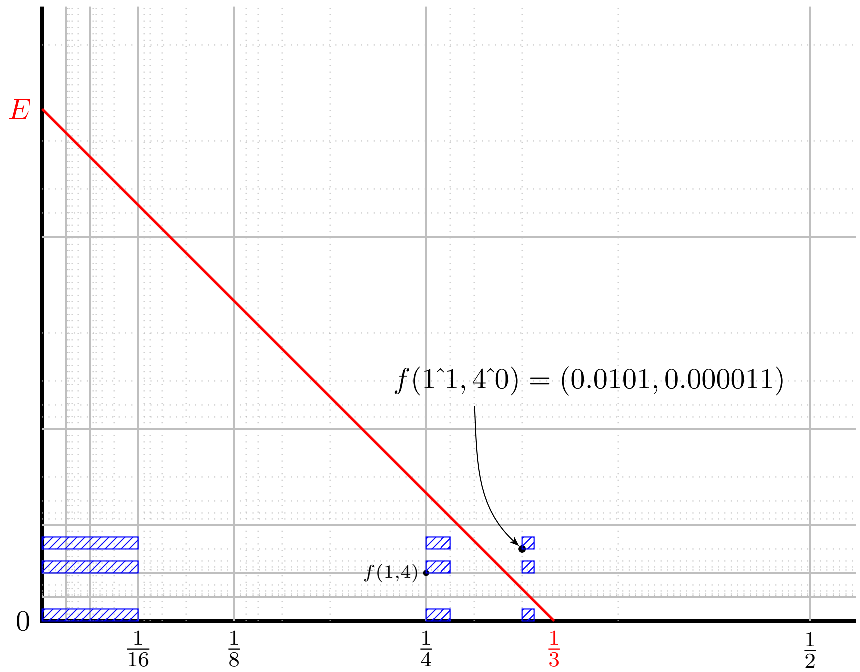
$$\begin{aligned} &\Big(\neg \textstyle\bigcup_{p\preceq s} \bigcup_{i \le m_p; p\hat{\ }i \not\preceq s} \uparrow(p\hat{\ }i)\Big) \times \Big(\neg \bigcup_{q\preceq t} \bigcup_{j \le n_q; q\hat{\ }j \not\preceq t} \uparrow(q\hat{\ }j)\Big) \\ &= \Big(\textstyle\bigcup_{p\preceq s}\big(\{p\} \cup \bigcup_{i > m_p} \uparrow(p\hat{\ }i)\big)\Big) \times \Big(\bigcup_{q\preceq t}\big(\{q\} \cup \bigcup_{j > n_q} \uparrow(q\hat{\ }j)\big)\Big) \subseteq \widehat{S}_0^2 \end{aligned}$$

is an open rectangle containing (s,t), which f maps to

$$\begin{aligned} &\Big(\textstyle\bigcup_{p\preceq s}\big(\{f(p)\} \cup \bigcup_{i>m_p}[f(p\hat{\ }i), f(p\hat{\ }(i-1)))\big)\Big) \times \Big(\bigcup_{q\preceq t}\big(\{f(q)\} \cup \bigcup_{j>n_q}[f(q\hat{\ }j), f(q\hat{\ }(j-1)))\big)\Big) \\ &= \Big(\textstyle\bigcup_{p\preceq s}[f(p), f(p\hat{\ }m_p))\Big) \times \Big(\bigcup_{q\preceq t}[f(q), f(q\hat{\ }n_q))\Big) \subseteq [0,1)^2 \end{aligned}$$

which is disjoint from E by our choice of m_p, n_q (and the fact that $f(p), f(q)$ are dyadic rationals). $\square$

The following picture shows what's going on in the above proof: in order to find a neighborhood of the indicated point disjoint from E, in the topology on $[0,1)^2$ induced by f, it is not enough to find a Euclidean rectangle around it; we must also find rectangles around its prefixes.



Lemma 2. $S_0 \times \widehat{S}_0$ is dense in $\overline{f^{-1}(E)} \subseteq \widehat{S}_0^2$.

Proof. For each subbasic open $\neg\uparrow s \subseteq \widehat{S}_0$, $f(\neg\uparrow s)$ is the union of two half-open intervals $[a,b)$ with dyadic rational endpoints a,b. Thus f takes each basic open rectangle in $\widehat{S}_0^2$ to a finite union of rectangles of the form $[a,b)\times[c,d)$ with dyadic rational a,b,c,d. If some such rectangle intersects E, then $a+c<\frac{1}{3}<b+d$; clearly then we may find $(x,y)\in[a,b)\times[c,d)$ with $x+y=\frac{1}{3}$ and x a dyadic rational, whence $(f^{-1}(x), f^{-1}(y)) \in S_0 \times \widehat{S}_0$. □

Corollary 3. The localic product $S_0 \times S_0$ is not spatial.

Proof. The sublocale $\overline{f^{-1}(E)} \cap (S_0 \times \widehat{S}_0) \subseteq \overline{f^{-1}(E)}$ is dense (as a point set, hence also as a sublocale), as is $\overline{f^{-1}(E)} \cap (\widehat{S}_0 \times S_0)$, hence so is their intersection, which is therefore a nonempty closed sublocale of $S_0 \times S_0$ without any points. □

References

[deB] M. de Brecht, *A note on the spatiality of localic products of countably based spaces*. Computability, Continuity, Constructivity – from Logic to Algorithms (CCC 2019), 2019.

[Joh] P. T. Johnstone, *Stone spaces*, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, Cambridge, England, 1982.