

Borel and analytic sets in locales

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Abstract

We systematically develop analogs of basic concepts from classical descriptive set theory in the context of pointless topology. Our starting point is to take the elements of the free complete Boolean algebra generated by the frame $\mathcal{O}(X)$ of opens to be the “ ∞ -Borel sets” in a locale X . We show that several known results in locale theory may be interpreted in this framework as direct analogs of classical descriptive set-theoretic facts, including e.g., the Lusin separation, Lusin–Suslin, and Baire category theorems for locales; we also prove several extensions of these results, such as an ordered Novikov separation theorem. We give a detailed analysis of various notions of image, and prove that a continuous map need not have an ∞ -Borel image. We introduce the category of “analytic ∞ -Borel locales” as the regular completion under images of the category of locales and ∞ -Borel maps (as a unary site), and prove analogs of several classical results about analytic sets, such as a boundedness theorem for well-founded analytic relations. We also consider the “positive ∞ -Borel sets” of a locale, formed from opens without using \neg .

In fact, we work throughout in the more refined context of κ -copresented κ -locales and κ -Borel sets for arbitrary regular $\omega_1 \leq \kappa \leq \infty$; taking $\kappa = \omega_1$ then recovers the classical context as a special case. The basis for the aforementioned localic results is a detailed study of various known and new methods for presenting κ -frames, κ -Boolean algebras, and related algebraic structures. In particular, we introduce a new type of “two-sided posite” for presenting (κ, κ) -frames A , meaning both A and A^{op} are κ -frames, and use this to prove a general κ -ary interpolation theorem for (κ, κ) -frames, which dualizes to the aforementioned separation theorems.

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1 Introduction

In this paper, we study the connection between *descriptive set theory* and *locale theory*, two areas which can each be thought of as providing a “non-pathological” version of point-set topology.

Descriptive set theory can be broadly described as the study of simply-definable sets and functions. The standard framework is to begin with a well-behaved topological space, namely a **Polish space**, i.e., a separable, completely metrizable space. One then declares open sets to be “simply-definable”, and builds up more complicated sets by closing under simple set-theoretic operations, e.g., countable Boolean operations (yielding the *Borel sets*), continuous images (yielding the *analytic sets*), etc. The definability restriction rules out many general point-set pathologies, yielding powerful structural and classification results which nonetheless apply to typical topological contexts encountered in mathematical practice. This has led in recent decades to an explosion of fruitful connections and applications to many diverse branches of mathematics, such as dynamical systems, operator algebras, combinatorics, and model theory. See [Kec95], [Kec99], [Mos09], [Gao09], [KM20] for general background on descriptive set theory and its applications.

Locale theory, also known as *pointless* or *point-free topology*, is the “dual” algebraic study of topological spaces and generalizations thereof via their lattices of open sets; see [Joh82], [PP12]. A **frame** is a complete lattice with finite meets distributing over arbitrary joins; the motivating example is the frame $\mathcal{O}(X)$ of open sets of a topological space X . In locale theory, one formally

regards an *arbitrary* frame $\mathcal{O}(X)$ as the “open sets” of a generalized “space” X , called a **locale**. One then proceeds to study algebraically-defined analogs of various topological notions. A key insight due to Isbell [Isb72] is that such analogies are not perfect, but that this is a feature, not a bug: where it differs from point-set topology, locale theory tends to be less pathological.

The starting point for this paper is an observation “explaining” this feature of locale theory: it seems to behave like a formal generalization of descriptive set theory, with countability restrictions removed. This observation is supported by many results scattered throughout the literature, e.g.:

- A fundamental construction of Isbell [Isb72] is to freely adjoin complements to an arbitrary frame $\mathcal{O}(X)$, yielding a bigger frame $\mathcal{N}(\mathcal{O}(X))$, whose elements are in canonical bijection with the sublocales of X . This is analogous to the Σ_2^0 (i.e., F_σ) sets in a Polish space X , the closure of the open and closed sets under finite intersections and countable unions; the complements of Σ_2^0 sets, the Π_2^0 (i.e., G_δ) sets, are precisely the Polish subspaces of X [Kec95, 3.11].
- Isbell [Isb72] also showed that dense sublocales of an arbitrary locale are closed under arbitrary intersections, just as dense Polish subspaces of Polish spaces are closed under countable intersections, by the Baire category theorem [Kec95, 8.4].
- Transfinitely iterating the $\mathcal{N}(-)$ construction yields a “Borel hierarchy” for an arbitrary locale X , the union of which is the free complete Boolean algebra generated by the frame $\mathcal{O}(X)$. By a classical result of Gaifman [Gai64] and Hales [Hal64], free complete Boolean algebras are generally proper classes, whence the “Borel hierarchy of locales is proper”.
- Ball [Bal18] developed a “Baire hierarchy of real-valued functions” on an arbitrary locale, via a transfinite construction closely related to the aforementioned \mathcal{N} functor.
- Isbell [Isb91] proved an analog of the classical Hausdorff–Kuratowski theorem [Kec95, 22.27]: any complemented sublocale (i.e., “ Δ_2^0 set”) can be written as $((F_0 \setminus F_1) \cup F_2) \setminus F_3 \cup \dots$ for some transfinite decreasing sequence $F_0 \supseteq F_1 \supseteq \dots$ of closed sublocales.
- The closed subgroup theorem of Isbell–Kříž–Pultr–Rosický [IKPR88] states that every closed localic subgroup of a localic group is closed, just as every Polish subgroup of a Polish group is closed [Gao09, 2.2.1]. Moreover, the proof of the closed subgroup theorem by Johnstone [Joh89] uses a localic version of Pettis’s theorem [Gao09, 2.3.2], itself proved via an analog of the classical proof based on (what we are calling) localic Baire category.
- The dual formulation of measure theory is well-known, via measure algebras. Explicit connections with general locales have been made by Simpson [Sim12] and Pavlov [Pav20].
- De Brecht [deB13] introduced **quasi-Polish spaces**, a well-behaved non-Hausdorff generalization of Polish spaces sharing many of the same descriptive set-theoretic properties. Heckmann [Hec15] proved that the category of these is equivalent to the category of locales X whose corresponding frame $\mathcal{O}(X)$ is countably presented. Thus,

“locale = (Polish space) – (countability restrictions) – (separation axioms)”.

- In [Ch19a], we explicitly used this correspondence to transfer a known result *from* locale theory (the Joyal–Tierney representation theorem [JT84] for Grothendieck toposes via localic groupoids) *to* the classical descriptive set-theoretic context (a Makkai-type strong conceptual completeness theorem for the infinitary logic $\mathcal{L}_{\omega_1\omega}$ in terms of Borel groupoids of models).

- In [Ch19b], we gave an intrinsic categorical characterization of the classical category **SBor** of **standard Borel spaces** (Polish spaces with their topologies forgotten, remembering only the Borel sets). This characterization was proved for “standard κ -Borel locales” for all regular cardinals $\kappa \geq \omega$, using algebraic techniques; taking $\kappa = \omega_1$ then yielded the classical result.

In spite of these and other results, the known analogy between locale theory and descriptive set theory has been on a largely *ad hoc* basis to date, without a complete or coherent locale-theoretic account of even basic descriptive set-theoretic notions such as Borel sets. The goal of this paper is to provide the beginnings of such an account.

1.1 Overview of localic results

Our main focus of study is on the two categories $\kappa\mathbf{Loc}$ of κ -**locales**, and $\kappa\mathbf{BorLoc}$ of κ -**Borel locales**, for each infinite regular cardinal κ (or $\kappa = \infty$). These are defined as the formal duals of the respective algebraic categories $\kappa\mathbf{Frm}$ of κ -**frames**, meaning frames but required only to have κ -ary joins, and $\kappa\mathbf{Bool}$ of κ -**(complete) Boolean algebras**. Thus, a κ -locale X is, formally, the same thing as a κ -frame $\mathcal{O}_\kappa(X)$, whose elements we call the κ -**open sets** in X ; while a κ -Borel locale X is formally a κ -Boolean algebra $\mathcal{B}_\kappa(X)$, whose elements we call the κ -**Borel sets** in X .¹ A morphism $f : X \rightarrow Y$ in any of these categories is a homomorphism f^* between the respective algebras, *in the opposite direction*, thought of as taking preimage of sets under f . When $\kappa = \omega_1$, we also write σ in place of κ . We define these categories in Section 3.1, and recall their connection with the classical categories of spaces in Section 3.2.

As κ varies, these categories are related by canonical forgetful functors

$$\begin{array}{ccccccc}
 \omega\mathbf{BorLoc} & \longrightarrow & \sigma\mathbf{BorLoc} & \longrightarrow & \cdots & \longrightarrow & \kappa\mathbf{BorLoc} & \longrightarrow & \cdots & \longrightarrow & \infty\mathbf{BorLoc} \\
 \uparrow & & \uparrow & & & & \uparrow & & & & \uparrow \\
 \omega\mathbf{Loc} & \longrightarrow & \sigma\mathbf{Loc} & \longrightarrow & \cdots & \longrightarrow & \kappa\mathbf{Loc} & \longrightarrow & \cdots & \longrightarrow & \mathbf{Loc}
 \end{array}$$

which are given by the *free* functors between the respective algebraic categories. Thus, for instance, given a σ -locale X , the underlying σ -Borel locale has σ -Borel sets $\mathcal{B}_\sigma(X)$ given by repeatedly adjoining complements to the σ -open sets $\mathcal{O}_\sigma(X)$ (see (1.1.2) below); while the ∞ -Borel sets $\mathcal{B}_\infty(X)$ are given by further freely completing under arbitrary “intersections” and “unions”.

We are particularly interested in the (full) subcategories of the above categories consisting of the κ -*copresented* objects X , meaning $\mathcal{O}_\kappa(X)$ (resp., $\mathcal{B}_\kappa(X)$) is κ -presented as an algebra of the respective type. We also call such X **standard**, and denote these subcategories by $\kappa\mathbf{Loc}_\kappa \subseteq \kappa\mathbf{Loc}$ and $\kappa\mathbf{BorLoc}_\kappa \subseteq \kappa\mathbf{BorLoc}$. When $\kappa = \omega_1$, by the aforementioned result of Heckmann [Hec15] as well as a classical result of Loomis–Sikorski (see [Sik69, 29.1], [Ch19b, 4.1]), we have equivalences with the classical descriptive set-theoretic categories:

$$\begin{aligned}
 (1.1.1) \quad & \text{standard } \sigma\text{-locales} = \sigma\mathbf{Loc}_\sigma \simeq \mathbf{QPol} := \text{quasi-Polish spaces,} \\
 & \text{standard } \sigma\text{-Borel locales} = \sigma\mathbf{BorLoc}_\sigma \simeq \mathbf{SBor} := \text{standard Borel spaces.}
 \end{aligned}$$

¹When $\kappa = \infty$, we allow $\mathcal{B}_\infty(X)$ to be a *large* (i.e., proper class) complete Boolean algebra, where “complete” means with respect to *small* (i.e., set-sized) joins; but we require $\mathcal{B}_\infty(X)$ to be *small-presented*, i.e., it must have a presentation (as a complete Boolean algebra) $\mathcal{B}_\infty(X) = \langle G \mid R \rangle$ with a *set* of generators G and of relations R , or equivalently, $\mathcal{B}_\infty(X)$ must be freely generated by a (small) κ -Boolean algebra for some $\kappa < \infty$.

This is described in Section 3.5 (see Theorem 3.5.8). Thus, as κ varies between ω_1 and ∞ , we interpolate between the contexts of classical descriptive set theory and classical locale theory.

As hinted above, for a κ -locale X , we may stratify the free κ -Boolean algebra $\mathcal{B}_\kappa(X)$ generated by the κ -frame $\mathcal{O}_\kappa(X)$ into the transfinite iterates $\mathcal{N}_\kappa^\alpha(\mathcal{O}_\kappa(X))$ of the functor $\mathcal{N}_\kappa : \kappa\mathbf{Frm} \rightarrow \kappa\mathbf{Frm}$ which freely adjoins a complement for each pre-existing element; we call this the **κ -Borel hierarchy of X** , described in Section 3.3. Each level $\mathcal{N}_\kappa^\alpha(\mathcal{O}_\kappa(X))$ consists of the $\kappa\Sigma_{1+\alpha}^0$ -sets of X , while their complements are the $\kappa\Pi_{1+\alpha}^0$ -sets:

$$(1.1.2) \quad \begin{aligned} \kappa\Sigma_{1+\alpha}^0(X) &:= \mathcal{N}_\kappa^\alpha(\mathcal{O}_\kappa(X)), \\ \kappa\Pi_{1+\alpha}^0(X) &:= \{\neg B \mid B \in \mathcal{N}_\kappa^\alpha(\mathcal{O}_\kappa(X))\}. \end{aligned}$$

(We should note that, unlike much of the locale theory literature (e.g., [Isb91]), our point of view is fundamentally Boolean: we always regard open sets, sublocales, etc., as embedded in the complete Boolean algebra $\mathcal{B}_\infty(X)$, with lattice operations taking place in there; see Convention 2.8.12. In fact, we will never refer to Heyting algebras or intuitionistic logic at all.)

κ -frames were introduced and studied by Madden [Mad91]; along with κ -Boolean algebras, they have since played a key role in much work on (what we are calling) the ∞ -Borel hierarchy of locales, by e.g., Madden–Molitor [MM91], Wilson [Wil94], Plewe [Pl00a]. Especially, they tend to be relevant whenever considering notions of “injectivity”, “surjectivity”, “image”, etc. It is well-known that such notions are a subtle area of locale theory. For instance, the usual notion of “image sublocale” of a continuous map $f : X \rightarrow Y \in \mathbf{Loc}$ is not well-behaved; see e.g., [Pl00b].

In Section 3.4, generalizing and extending these known results, we give a detailed analysis of “injectivity”, “surjectivity”, “image”, etc., in our main categories of interest $\kappa\mathbf{Loc}$, $\kappa\mathbf{BorLoc}$ as well as their subcategories of standard objects. This analysis shows that the subtlety in these notions is much clarified by the Boolean point of view, and ultimately arises from hidden interactions with topology when one insists on remaining in \mathbf{Loc} . The following is a summary of this analysis; see Corollary 3.4.5, Proposition 3.4.9, Proposition 3.4.14, Proposition 3.4.17, and Proposition 3.4.22.

Theorem 1.1.3 (structure of subobjects in $\kappa\mathbf{Loc}$, $\kappa\mathbf{BorLoc}$).

- (a) (LaGrange) *Monomorphisms (“injections”) in $\kappa\mathbf{BorLoc}$ are the same as regular monomorphisms (“embeddings”).*
- (b) (essentially Madden) *Regular subobjects of an object X (equivalence classes of “embeddings” into X) in $\kappa\mathbf{Loc}$ or $\kappa\mathbf{BorLoc}$ are canonically identified with certain ∞ -Borel sets $B \in \mathcal{B}_\infty(X)$: with arbitrary intersections of κ -Borel sets in the case of $\kappa\mathbf{BorLoc}$, and intersections of implications between κ -open sets in the case of $\kappa\mathbf{Loc}$.*
- (c) *In the subcategory $\kappa\mathbf{BorLoc}_\kappa \subseteq \kappa\mathbf{BorLoc}$ of standard κ -Borel locales, (regular) subobjects instead correspond to κ -Borel sets $B \in \mathcal{B}_\kappa(X) \subseteq \mathcal{B}_\infty(X)$. Similarly, in $\kappa\mathbf{Loc}_\kappa$, regular subobjects correspond to $\kappa\Pi_2^0$ sets.*
- (d) *For an arbitrary map $f : X \rightarrow Y$ in $\kappa\mathbf{Loc}$ or $\kappa\mathbf{BorLoc}$, its epi-regular mono (“image”) factorization $X \twoheadrightarrow Z \hookrightarrow Y$ in that category is given by the “ ∞ -Borel image”² $f(X) \in \mathcal{B}_\infty(Y)$ (the smallest ∞ -Borel set whose f -pullback is all of X), followed by closure into the respective class of sets as in (b).*

²assuming the ∞ -Borel image exists; otherwise, we need to take the ∞ -analytic image as described below

- (e) *Epimorphisms (“surjections”) in κBorLoc are well-behaved: they are regular epimorphisms (“quotient maps”), pullback-stable (so that κBorLoc is a regular category), and closed under products and codirected inverse limits of diagrams in which each morphism is an epimorphism.*

This result includes the localic analogs of several basic results in classical descriptive set theory. The combination of (a) and the first part of (c) yields a **Lusin–Suslin theorem** for standard κ -Borel locales (every κ -Borel injection has a κ -Borel image; see [Kec95, 15.1]). Likewise, as mentioned near the start of this Introduction, the second part of (c) generalizes the classical result that the Polish subspaces of a Polish space are precisely the Π_2^0 sets [Kec95, 3.11] as well as its quasi-Polish analog [deB13, Th. 23].

Classically, a key ingredient in the Lusin–Suslin theorem is the fundamental **Lusin separation theorem** [Kec95, 14.7]: if two Borel maps $f : Y \rightarrow X$ and $g : Z \rightarrow X$ between standard Borel spaces have disjoint images, then there is a Borel set $B \subseteq X$ separating those images. Note that this can be stated as follows, avoiding mention of the images: if the fiber product $Y \times_X Z$ is empty, then there is B such that $f^{-1}(B) = Y$ and $g^{-1}(B) = \emptyset$. The κ -Borel localic version of this was essentially proved, in a dual algebraic form, by LaGrange [LaG74] (and then used to prove Theorem 1.1.3(a); see [Ch19b, 3.2] for an explanation of this). The **Novikov separation theorem** [Kec95, 28.5] is the generalization of the Lusin separation theorem to countably many Borel maps $f_i : Y_i \rightarrow X$ with $\bigcap_i f(Y_i) = \emptyset$. We prove the κ -Borel localic generalization of this in Theorem 4.2.1:

Theorem 1.1.4 (Novikov separation theorem). *Let $f_i : Y_i \rightarrow X$ be $< \kappa$ -many κ -Borel maps between standard κ -Borel locales. If the fiber product $\prod_X (Y_i)_i$ of all of the Y_i over X is empty, then there are κ -Borel sets $B_i \in \mathcal{B}_\kappa(X)$ with $\bigcap_i B_i = \emptyset$ and $f_i^*(B_i) = Y_i$ for each i .*

In fact we also establish versions of most of the aforementioned results for a third category intermediate between κLoc and κBorLoc : the category $\kappa\text{Bor}^+\text{Loc}$ of **positive κ -Borel locales**, dual to the category κKFr of (κ, κ) -**frames**,³ meaning κ -frames whose order-duals are also κ -frames. The motivation for these is that given a κ -locale X , we may close the κ -open sets under *positive* κ -Boolean operations, i.e., not using \neg ; the resulting (κ, κ) -frame does not remember all of the topological structure on X , but still encodes the specialization order, with respect to which all of the positive κ -Borel sets must be upward-closed. In Section 3.7 (see Theorem 3.7.3 and Corollary 3.7.4), we make this precise by showing that

Theorem 1.1.5. *There is a canonical full and faithful “forgetful” functor $\kappa\text{Bor}^+\text{Loc} \rightarrow \kappa\text{BorPOLoc}$ into the category of (internally) partially ordered κ -Borel locales. In other words, positive κ -Borel locales may be faithfully identified with certain partially ordered κ -Borel locales.*

This result is in turn based on a positive κ -localic analog of a classical ordered version of the Lusin separation theorem [Kec95, 28.12]; we prove a simultaneous generalization of this and the Novikov separation theorem in Theorem 4.2.6.

The aforementioned results clearly hint at a theory of “ κ -analytic” or “ $\kappa\Sigma_1^1$ ” sets, meaning “images” of arbitrary κ -Borel maps $f : X \rightarrow Y \in \kappa\text{BorLoc}_\kappa$ between standard κ -Borel locales. In Section 4, we develop such a theory by “closing the category $\kappa\text{BorLoc}_\kappa$ under non-existent images”. This is done via two equivalent methods: externally, by closing under pre-existing images in the ambient supercategory κBorLoc of possibly nonstandard κ -Borel locales; or internally, by adjoining a “formal image” for each morphism $f : X \rightarrow Y \in \kappa\text{BorLoc}_\kappa$ (via an application of Shulman’s [Shu12]

³According to some literature, these would be called “ κ -biframes”; but that term also has a different meaning.

theory of *unary sites*). We call the result the category of **analytic κ -Borel locales**, denoted $\kappa\Sigma_1^1\text{BorLoc}$, with subobjects in it called **κ -analytic sets**. We prove in Theorem 4.1.1 that $\kappa\Sigma_1^1\text{BorLoc}$ shares many of the nice properties of κBorLoc and $\kappa\text{BorLoc}_\kappa$. These categories are related as follows:

$$\begin{array}{ccccccc}
\omega\Sigma_1^1\text{BorLoc} & \hookrightarrow & \sigma\Sigma_1^1\text{BorLoc} & \dashrightarrow^? & \cdots & \dashrightarrow^? & \kappa\Sigma_1^1\text{BorLoc} & \dashrightarrow^? & \cdots & \dashrightarrow^? & \infty\Sigma_1^1\text{BorLoc} \\
\parallel & & \uparrow & & & & \uparrow & & & & \uparrow \\
\omega\text{BorLoc}_\omega & \hookrightarrow & \sigma\text{BorLoc}_\sigma & \hookrightarrow & \cdots & \hookrightarrow & \kappa\text{BorLoc}_\kappa & \hookrightarrow & \cdots & \hookrightarrow & \infty\text{BorLoc}
\end{array}$$

Unlike in the standard case, we do not know if analytic κ -Borel locales are increasingly general as κ increases, i.e., if there are canonical forgetful functors $\kappa\Sigma_1^1\text{BorLoc} \rightarrow \lambda\Sigma_1^1\text{BorLoc}$ for $\kappa \leq \lambda$ (see Remark 4.1.11). In particular, we do not know if the resulting theory of “ ∞ -analytic sets” generalizes the classical ($\kappa = \omega_1$) one, or is merely an analog of it; hence the importance of our considering all $\kappa \leq \infty$ at once.

In Section 4.3, we generalize a fundamental aspect of classical analytic sets: their connection with well-foundedness. Classically, given a Borel map $f : X \rightarrow Y$ between standard Borel spaces, by representing each fiber $f^{-1}(y)$ for $y \in Y$ as the space of branches through a countably branching tree T_y in a uniform manner, we obtain a Borel family of trees $(T_y)_{y \in Y}$ such that the analytic set $\text{im}(f) \subseteq Y$ consists of precisely those y for which T_y has an infinite branch. Note that the space of infinite branches through a tree is the inverse limit of its (finite) levels. Thus, the following (Theorem 4.3.2) can be seen as a generalization of the classical result just mentioned:

Theorem 1.1.6 (“tree” representation of κ -analytic sets). *For any κ -Borel map $f : X \rightarrow Y$ between standard κ -Borel locales, there is a κ -ary codirected diagram $(X_i)_{i \in I}$ of standard κ -Borel locales $X_i \rightarrow Y$ over Y and κ -ary-to-one κ -Borel maps between them, such that $X = \varprojlim_i X_i$ (over Y).*

Thus, the κ -analytic set $\text{im}(f) \subseteq Y$ in $\kappa\Sigma_1^1\text{BorLoc}$ is given by “ $\{y \in Y \mid \varprojlim_i (X_i)_y \neq \emptyset\}$ ”.

As we recall in Section 3.6, a well-known technique from categorical logic endows every (sufficiently nice) category with an **internal logic** via which “pointwise” expressions like the above may be made precise. This yields a systematic way of translating many classical definitions into locale-theoretic analogs, including in the following.

Classically, given a Borel family of countably branching trees $(T_y)_{y \in Y}$ representing an analytic set $\text{im}(f) \subseteq Y$ as above, by repeatedly pruning the leaves of these trees, we obtain an ω_1 -length decreasing sequence of Borel sets $B_\alpha \subseteq Y$ whose intersection is $\text{im}(f)$; see [Kec95, 25.16]. Moreover, stabilization of this sequence below ω_1 is closely related to Borelness of the analytic set $\text{im}(f)$; see [Kec95, 35.D]. In Theorem 4.3.6, we prove the following localic generalization of these results:

Theorem 1.1.7 (pruning of “tree” representations). *Let $f : X \rightarrow Y \in \kappa\text{BorLoc}_\kappa$ be the inverse limit, over a standard κ -Borel locale Y , of a κ -ary codirected diagram $(X_i)_{i \in I}$ of standard κ -Borel locales over Y and κ -ary-to-one κ -Borel maps between them, as in Theorem 1.1.6.*

- (a) *The κ -analytic set $\text{im}(f) \subseteq Y$ is the κ -length decreasing meet (in the poset $\kappa\Sigma_1^1(Y)$) of the κ -Borel images of the α th prunings of the X_i .*
- (b) *If $X = \emptyset$ (i.e., $\text{im}(f) = \emptyset$), then the pruning stabilizes at the empty diagram by some $\alpha < \kappa$.*
- (c) *If $\text{im}(f) \subseteq Y$ is κ -Borel, then the sequence in (a) stabilizes (at $\text{im}(f)$) by some $\alpha < \kappa$.*

In fact, this result holds for *any* κ -ary codirected diagram over Y , not necessarily with κ -ary-to-one maps; the only difference is that the images in (a) need no longer be κ -Borel. Part (b) can then be seen as a generalization of the classical **boundedness theorem for Σ_1^1** ; see Remark 4.3.9. In Section 4.4 (see Theorem 4.4.3), using this boundedness theorem, we finally prove that

Theorem 1.1.8 (Borel \subsetneq analytic). *There is a continuous map $f : X \rightarrow Y$ between quasi-Polish spaces such that, regarded as a map in $\sigma\mathbf{Loc}_\sigma$ via (1.1.1), there is no smallest κ -Borel set $\text{im}(f) \in \mathcal{B}_\kappa(Y)$ whose f -pullback is all of X , for any $\omega_1 \leq \kappa \leq \infty$.*

In other words, $\text{im}(f) \subseteq Y$ is a “ σ -analytic, non- ∞ -Borel set”.

The map $f : X \rightarrow Y$ here is simply the usual projection to the space Y of binary relations on \mathbb{N} from the space X of binary relations equipped with a countable descending sequence, whose image would be the set of ill-founded relations. The key point is that, when we pass to the underlying ∞ -Borel locales, these same X, Y also have nonempty ∞ -Borel sets “of relations of arbitrary cardinality”. This is based on a well-known technique: a localic “cardinal collapse” (see [Joh02, C1.2.8–9], Theorem 3.8.8), proved using (what we are calling) the localic Baire category theorem of Isbell [Isb72] mentioned near the beginning of this Introduction. In Section 3.8, we give a self-contained exposition of localic Baire category from a descriptive point of view, including e.g., the **property of Baire** for ∞ -Borel sets (Proposition 3.8.6), as well as an inverse limit generalization of the Baire category theorem (Theorem 3.8.7) which is used in the proof of Theorem 1.1.7.

All of the aforementioned results reduce to their classical analogs when $\kappa = \omega_1$. For many of these results, this yields, modulo the equivalences of categories (1.1.1) (which are based on a simple Baire category argument), a purely lattice-theoretic proof of the classical result which is quite different from the classical proof (found in, e.g., [Kec95]).

1.2 Overview of algebraic results

While our main goal in this paper is descriptive set theory in locales, the technical heart of this paper lies largely on the dual algebraic side, in Section 2. Broadly speaking, we study there various methods for *presenting* the types of lattice-theoretic algebras A mentioned above (i.e., κ -frames, (λ, κ) -frames, and κ -Boolean algebras) via generators and relations:

$$A = \langle G \mid R \rangle.$$

We are particularly interested in presentation methods where G and/or R are more structured than mere sets; usually, they already incorporate part of the structure of the algebra A being presented. Such structured presentation methods can be used to prove many important facts about free functors between our various algebraic categories of interest (e.g., $\kappa\mathbf{Frm} \rightarrow \kappa\mathbf{Bool}$), hence about the *forgetful* functors between the corresponding categories of locales (e.g., $\kappa\mathbf{Loc} \rightarrow \kappa\mathbf{BorLoc}$).

For example, a standard method of presenting a frame A , due to Johnstone [Joh82, II 2.11] (although based on older ideas from sheaf theory), is via a **posite** (G, \triangleleft) , where G is a (finitary) meet-semilattice, while \triangleleft (called a **coverage** on G) is a set of relations of the form “ $a \leq \bigvee B$ ” where $B \subseteq G$. The presented frame $A = \langle G \mid \triangleleft \rangle$ is then freely generated by G as a meet-semilattice, subject to the relations specified by \triangleleft becoming true. The key point is that, since G already has finite meets, it is enough for \triangleleft to impose join relations. This can then be used to prove, for example, that a direct limit of frames is the direct limit of the underlying complete join-semilattices (see Corollary 2.7.2), from which one then deduces the results on inverse limits of locales mentioned

in the preceding subsection, such as Theorem 1.1.3(e) and the localic Baire category theorem (see Proposition 2.7.3, Corollary 2.8.7, Proposition 3.4.22, Theorem 3.8.7).

In the first parts of Section 2, after defining the basic algebraic categories in Section 2.1 and some general discussion of categorical properties of presentations in Section 2.2, we give a self-contained review of various standard constructions of free and presented frames, including posites in Section 2.6, as well as their generalizations to κ -frames. In many of these cases, the κ -ary generalization does not appear explicitly in the literature, but is largely routine. Our reason for the level of detail given is threefold: first, for the sake of completeness, especially since there are some occasional subtleties involved in the κ -ary generalizations. Second, we wish to make this paper accessible without assuming familiarity with standard locale theory. And third, this serves as motivation for some further generalizations of these constructions that we give later (such as (λ, κ) -distributive polyposets; see below).

We also take this opportunity to prove, via reasoning about presentations, some simple facts about κ -ary products in the categories $\kappa\mathbf{Frm}$, $\kappa\kappa\mathbf{Frm}$, $\kappa\mathbf{Bool}$ (see Section 2.5), which are quite possibly folklore, although we could not find them explicitly stated in this generality in the literature.

In Section 2.8, we consider the functor $\mathcal{N}_\kappa : \kappa\mathbf{Frm} \rightarrow \kappa\mathbf{Frm}$ that adjoins complements to a κ -frame, as well as its transfinite iterates $\mathcal{N}_\kappa^\alpha$. For $\kappa = \infty$, this is Isbell’s [Isb72] \mathcal{N} functor we mentioned near the start of this Introduction; for $\kappa < \infty$, this functor was studied by Madden [Mad91]. We recall several of his results, but using a new approach based on an explicit posite presentation for $\mathcal{N}_\kappa(A)$ (see Proposition 2.8.2). We also prove, in Propositions 2.8.13 and 2.8.15, the algebraic results underlying the generalizations (see Propositions 3.3.6 and 3.3.7) of the classical change of topology results for (quasi-)Polish spaces (see [Kec95, 13.1–5], [deB13, Th. 73]):

Theorem 1.2.1 (change of topology).

- (a) *Every κ -presented κ -Boolean algebra B is isomorphic to the free κ -Boolean algebra $\mathcal{N}_\kappa^\infty(A)$ generated by some κ -presented κ -frame A .*
- (b) *Given any such A , we may enlarge it to some $A \subseteq A' \subseteq \mathcal{N}_\kappa^\alpha(A)$ containing any $< \kappa$ -many prescribed elements of $\mathcal{N}_\kappa^\alpha(A)$, such that A' still freely generates B .*

Dually:

- (a) *Every standard κ -Borel locale X is the underlying κ -Borel locale of some standard κ -locale.*
- (b) *Given any standard κ -locale X and $< \kappa$ -many $\kappa\Sigma_\alpha^0$ -sets $C_i \in \kappa\Sigma_\alpha^0(X)$, we may find a standard κ -locale X' , such that $\mathcal{O}_\kappa(X) \cup \{C_i\}_i \subseteq \mathcal{O}_\kappa(X') \subseteq \kappa\Sigma_\alpha^0(X)$ and the canonical map $X' \rightarrow X$ is a κ -Borel isomorphism. In other words, X' is “ X with a finer topology making each C_i open”.*

While this result can easily be proved in the same way as the classical proof (in [Kec95, 13.1–5]), it is interesting to note that our proof (in Proposition 2.8.15) derives it from completely abstract universal-algebraic principles that have nothing to do with κ -frames or κ -Boolean algebras *per se*.

In Section 2.9, we consider κ -frames which are dual to **zero-dimensional** κ -locales, hence which are generated by complemented elements. Here, there is an extra subtlety which is invisible in the classical case $\kappa = \omega_1$: it is more natural to consider the **ultraparacompact** zero-dimensional κ -locales, meaning that there are “enough clopen partitions” (see Proposition 2.9.8). This is because ultraparacompactness, which is automatic for zero-dimensional Polish spaces, is precisely what allows every such Polish space to be represented as the space of infinite branches through a tree;

the $\kappa = \infty$ version of this fact was proved by Paseka [Pas92]. Classically, one may always change to a zero-dimensional Polish topology by making basic open sets clopen. When $\kappa > \omega_1$, that ultraparacompactness can be obtained via change of topology (in the sense of Theorem 1.2.1) is much less obvious, and was shown by Plewe [Pl02] when $\kappa = \infty$. We again adopt an approach based on presentations of ultraparacompact zero-dimensional κ -frames (via Boolean algebras and pairwise disjoint join relations), in order to give completely different proofs of the two results just cited for arbitrary κ (see Corollary 2.9.13, Theorem 2.9.19, Proposition 3.3.8, and Proposition 3.3.10):

Theorem 1.2.2 (ultraparacompactness and change of topology).

- (a) *Every ultraparacompact zero-dimensional (κ -presented) κ -frame is a (κ -ary) direct limit, in $\kappa\mathbf{Frm}$, of powersets of κ -ary sets.*
- (b) *Every κ -presented κ -frame A may be enlarged to an ultraparacompact zero-dimensional κ -presented κ -frame $A' \supseteq A$ freely generating the same κ -Boolean algebra $\mathcal{N}_\kappa^\infty(A') \cong \mathcal{N}_\kappa^\infty(A)$.*

Dually:

- (a) *Every ultraparacompact zero-dimensional (standard) κ -locale is a (κ -ary) inverse limit of discrete κ -ary sets.*
- (b) *Given any standard κ -locale X , we may find an ultraparacompact zero-dimensional standard κ -locale X' such that $\mathcal{O}_\kappa(X') \supseteq \mathcal{O}_\kappa(X)$ and the canonical map $X' \rightarrow X$ is a κ -Borel isomorphism.*

These results play a crucial role in the proof of the “tree” representation of κ -analytic sets (Theorem 1.1.6).

In Section 2.11, we introduce a new method for presenting (λ, κ) -**frames**, meaning κ -frames whose order-duals are λ -frames, for arbitrary λ, κ . A (λ, κ) -**distributive polyposet** (A, \triangleleft) is a “symmetric” generalization of a posite, consisting of an underlying set A equipped with a binary relation \triangleleft between λ -ary subsets and κ -ary subsets of A , thought of as a presentation of a (λ, κ) -frame $\langle A \mid \triangleleft \rangle$ with generating set A and relations “ $\bigwedge B \leq \bigvee C$ ” for $B \triangleleft C$. The \triangleleft relation is required to obey certain “saturation” conditions which capture implied relations in the presented algebra $\langle A \mid \triangleleft \rangle$; see Section 2.11. We prove in Theorem 2.11.4 that these “saturation” conditions indeed capture *all* such implied relations, and not just in the presented (λ, κ) -frame but in the presented complete Boolean algebra:

Theorem 1.2.3 (saturation of (λ, κ) -distributive polyposets). *Let (A, \triangleleft) be a (λ, κ) -distributive polyposet. For any λ -ary $B \subseteq A$ and κ -ary $C \subseteq A$, if $\bigwedge B \leq \bigvee C$ holds in the presented complete Boolean algebra $\langle A \mid \triangleleft \rangle$, then already $B \triangleleft C$.*

This easily implies, for instance, that every (λ, κ) -frame A embeds into the free $\max(\lambda, \kappa)$ -Boolean algebra it generates (by taking the canonical \triangleleft consisting of all relations that hold in A ; see Corollary 2.11.5), or dually, that the forgetful functor $\kappa\mathbf{Bor}^+\mathbf{Loc} \rightarrow \kappa\mathbf{BorLoc}$ from positive κ -Borel locales to κ -Borel locales is faithful (Proposition 3.1.10).

In Theorem 2.12.3, we use (κ, κ) -distributive polyposets to prove the most general algebraic form of the separation theorems described in the preceding subsection. This result can also be seen as an algebraic form of the Craig–Lyndon interpolation theorem for κ -ary infinitary propositional logic. We state here a somewhat simplified special case of the full result:

Theorem 1.2.4 (interpolation). *Let A, B_i, C_j be $< \kappa$ -many (κ, κ) -frames, with homomorphisms $f_i : A \rightarrow B_i$ and $g_j : A \rightarrow C_j$, and let D be their **bilax pushout**, i.e., D is presented by the disjoint union of all the elements of A, B_i, C_j , subject to the additional relations*

$$f_i(a) \leq a \leq g_j(a) \in D \quad \text{for } a \in A.$$

Thus D is the universal filler in the following diagram, equipped with homomorphisms from A, B_i, C_j satisfying the indicated order relations:

$$\begin{array}{ccc} B_i & \dashrightarrow & D \\ f_i \uparrow & \nearrow & \uparrow \\ A & \xrightarrow{g_j} & C_j \end{array}$$

If $b_i \in B_i$ and $c_j \in C_j$ satisfy

$$\bigwedge_i b_i \leq \bigvee_j c_j \in D,$$

then there are $a_i^L, a_j^R \in A$ such that

$$b_i \leq f_i(a_i^L), \quad \bigwedge_i a_i^L \leq \bigvee_j a_j^R, \quad g_j(a_j^R) \leq c_j.$$

The Novikov separation theorem (Theorem 1.1.4) follows by taking each of A, B_i, C_j to be κ -Boolean and then dualizing, while the more general positive κ -Borel version (Theorem 4.2.6) requires the full strength of Theorem 1.2.4. As we noted above Theorem 1.1.4, the Lusin separation theorem, which only requires Theorem 1.2.4 with three κ -Boolean algebras A, B, C , was essentially shown by LaGrange [LaG74]; this is already enough to imply many structural results about $\kappa\mathbf{Bool}$, e.g., that epimorphisms are regular and monomorphisms are regular and pushout-stable (see Corollaries 2.12.7, 2.12.9 and 2.12.10), which dualize to Theorem 1.1.3(a) and the first two parts of (e).

The proof of Theorem 1.2.3 is based on an explicit syntactic construction, in Section 2.10, of free complete Boolean algebras via the cut-free Gentzen sequent calculus for infinitary propositional logic (see Theorem 2.10.10). While this construction is new as far as we know, it has an obvious precedent in Whitman's construction of free lattices (see [Hal64, III]).

Finally, we note that from our proof that there exist σ -analytic, non- ∞ -Borel sets (Theorem 1.1.8), we have the following purely Boolean-algebraic consequence (see Corollary 4.4.6), which is somewhat surprising given the good behavior of monomorphisms in $\kappa\mathbf{Bool}$ in other respects (e.g., regularity and pushout-stability, as noted above):

Theorem 1.2.5. *There exist $\kappa < \lambda < \infty$, namely $\kappa = \omega_1$ and $\lambda = (2^{\aleph_0})^+$, such that the free functor $\kappa\mathbf{Bool} \rightarrow \lambda\mathbf{Bool}$ does not preserve injective homomorphisms.*

1.3 Future directions

In this paper, we have restricted attention to the “elementary” parts of classical descriptive set theory. We have not considered at all more advanced set-theoretic techniques such as determinacy, the projective hierarchy, or effective descriptive set theory (see [Kec95], [Mos09]), let alone large cardinals or forcing (although the “cardinal collapse” argument used in the proof of Theorem 1.1.8 could be understood as an instance of sheaf-theoretic forcing; see [MM94]). Nor have we given any

serious consideration to the modern area of *invariant descriptive set theory*, which studies definable (e.g., Borel) equivalence relations and group actions and their quotient spaces (see [Kec99], [Gao09]). All of these areas would be ripe for future investigation of potential connections with locale theory.

Even in “elementary” descriptive set theory, there remain many gaps in the localic analogy which we have not addressed. These include e.g., the perfect set theorem; universal and complete sets; a deeper study of Baire category and measure; and uniformization and selection theorems (see again [Kec95]). We plan to address several of these topics in a sequel paper.

Conversely, there are many parts of locale theory and adjacent areas which could probably be fruitfully placed into a descriptive set-theoretic framework. These include e.g., powerlocales (see [Sch93], [dBK19]), the theory of localic groups and groupoids, and connections with toposes and categorical logic (see [Joh02]). As we indicated near the start of this Introduction, some/all of these topics have already seen suggestive work that at least implicitly used descriptive set-theoretic ideas; what is missing is a more systematic account of the connection. For example, there is surely a link between the Scott analysis in invariant descriptive set theory and countable model theory (see [Gao09, Ch. 12]), the \mathcal{N} functor on frames, and its “first-order” topos-theoretic analog corresponding to Morleyization of $\mathcal{L}_{\infty\omega}$ -theories into geometric theories (see [Joh02, D1.5.13], [Ch19a, §4]).

Finally, as we noted in Section 1.1, in this paper we completely ignore a traditionally important aspect of locales: the use of intuitionistic logic, and the consequent possibility of internalization over an arbitrary base topos (see [Joh02, C1.6]). In fact, we make unrestricted use of the axiom of choice (e.g., in assuming that arbitrary subsets of frames may be well-ordered in Section 2.9 in order to disjointify elements). It would be interesting to develop a constructive version of the theory in this paper, perhaps as a bridge to effective descriptive set theory.

1.4 Structure of paper

This paper has three major sections.

Section 2 takes place entirely in the algebraic (i.e., lattice-theoretic) setting, and forms the foundations for the rest of the paper. Its contents are described in detail in Section 1.2. As indicated there, the broad theme of this section is to study various methods for presenting our main algebraic structures of interest (κ -frames, (λ, κ) -frames, κ -Boolean algebras), as well as other structural results proved using presentation-based techniques.

Section 3 introduces the dual localic viewpoint, and develops as much of the theory as is reasonable without needing to refer to images of arbitrary κ -Borel maps (including the analysis of existing κ -Borel images); Section 4 introduces the category $\kappa\Sigma_1^1\mathbf{BorLoc}$ of analytic κ -Borel locales as the “closure” of the standard κ -Borel locales $\kappa\mathbf{BorLoc}_\kappa$ under images. The contents of these two sections are described in detail in Section 1.1. Even though we adopt the localic viewpoint whenever possible, the algebraic preliminaries in Section 2 are of course used often in these later sections.

As we indicated above, we have tried to make this paper accessible without any prior locale theory background. For this reason, the first few subsections in Sections 2 and 3 give a self-contained (though terse) introduction to all of the basic notions needed. Likewise, we do not assume any prior descriptive set-theoretic background; all needed definitions are given in Sections 3 and 4.

However, we have found it necessary to make fairly liberal use of category theory throughout the paper, as an organizing framework for the various types of structures introduced. We therefore assume familiarity with basic notions of category theory, up to such notions as limits, colimits, and adjoint functors; see e.g., [Bor] for a general reference. In Appendix A, we give a “logician’s” introduction to the deeper theory of categories of well-behaved (infinitary) first-order structures.

Acknowledgments I would like to thank Matthew de Brecht for several stimulating discussions during the early stages of this work, as well as for providing some comments and corrections to a draft of this paper.

2 Frames and Boolean algebras

This section contains the order-theoretic foundations for the paper. In Section 2.1, we define the main categories of lattices and Boolean algebras of interest to us, discuss their relation to each other, and establish some notational conventions. In the rest of the subsections, we discuss various concrete methods (some well-known, some new) for constructing free structures, and more generally, presenting structures via generators and relations; these are then applied to study various other constructions, including certain types of limits and colimits.

Throughout this paper, κ, λ , and sometimes μ will denote infinite regular cardinals or the symbol ∞ (bigger than all cardinals). By κ -**ary**, we mean of size $< \kappa$. We will occasionally refer to structures which are class-sized, rather than set-sized; we call the former **large** and the latter **small** or ∞ -**ary**.⁴ All structures except categories are assumed to be small by default. In terminology in which “ κ ” appears, when $\kappa = \omega_1$, we write “ σ ” as a synonym, e.g., “ σ -frame” means “ ω_1 -frame”. It is occasionally helpful to regard $\kappa = 2, \{1\}$ as “degenerate regular cardinals”, where $\{1\}$ -ary means of size exactly 1.⁵

As indicated in the Introduction, we assume familiarity with basic notions of category theory, including limits, colimits, adjoint functors; a review of some more advanced notions is given in Appendix A. Categories will be denoted with symbols like \mathbf{C} . Hom-sets will be denoted $\mathbf{C}(A, B)$ for objects $A, B \in \mathbf{C}$; identity morphisms will be denoted $1_A \in \mathbf{C}(A, A)$. Products will be denoted $A \times B$ or $\prod_i A_i$; fiber products/pullbacks will be denoted $A \times_C B$; the equalizer of $f, g : A \rightrightarrows B$ will be denoted $\text{eq}(f, g)$, while their coequalizer will be denoted $\text{coeq}(f, g)$; and coproducts and pushouts will generally be denoted with the symbol \amalg , but also with other notation specific to certain categories (e.g., \otimes, \sqcup). The category of sets will be denoted \mathbf{Set} ; the category of posets and order-preserving maps will be denoted \mathbf{Pos} .

2.1 The main categories

A κ -**complete join-semilattice**, or κ - **\vee -lattice** for short, is a poset with κ -ary joins (i.e., suprema). An ∞ - \vee -lattice is also called a **complete join-semilattice** or a **\vee -lattice**. An ω - \vee -lattice is also called a **join-semilattice** or a **\vee -lattice**, with joins usually denoted $a_1 \vee \cdots \vee a_n$. Note that we require \vee -lattices to have a least element (nullary join), which we denote by \perp .

We regard κ - \vee -lattices A as infinitary algebraic structures with join operations $\bigvee : A^\lambda \rightarrow A$ for all $\lambda < \kappa$. Hence, terms like **homomorphism** and κ - **\vee -sublattice** have their usual meanings (map preserving κ -ary joins, and subset closed under κ -ary joins). The **category of κ - \vee -lattices** will be denoted $\kappa\mathbf{\vee Lat}$; when $\kappa = \omega, \omega_1, \infty$, we also use the names $\mathbf{\vee Lat}, \sigma\mathbf{\vee Lat}, \mathbf{\vee Lat}$.

The category $\kappa\mathbf{\wedge Lat}$ of κ -**complete meet-semilattices** is defined similarly. We adopt the obvious order-duals of the above conventions; the greatest element is denoted \top .

⁴To be more precise, we could work in NBG set theory instead of ZFC, or assume there to be an inaccessible cardinal called ∞ and take “set” to mean of size $< \infty$. None of our results will depend essentially on proper classes.

⁵These “possibly degenerate regular cardinals” are called “arity classes” in [Shu12].

A κ -**frame** [Mad91] is a poset which is a \wedge -lattice as well as a κ - \vee -lattice, and satisfies

$$a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$$

whenever the join is κ -ary. An ∞ -frame is also called a **frame**. An ω -frame is the same thing as a **distributive lattice**. The **category of κ -frames** is denoted κFrm (or $\text{DLat}, \sigma\text{Frm}, \text{Frm}$ when $\kappa = \omega, \omega_1, \infty$).

A κ -**complete Boolean algebra**, or just κ -**Boolean algebra**, is a κ -frame in which every element a has a (unique) complement $\neg a$, meaning $a \wedge \neg a = \perp$ and $a \vee \neg a = \top$. We will also use the implication operation $a \rightarrow b := \neg a \vee b$ and bi-implication $a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a)$. (We will never use \neg, \rightarrow to denote Heyting operations in a frame; see Convention 2.8.12.) An ∞ -Boolean algebra is a **complete Boolean algebra**, while an ω -Boolean algebra is a **Boolean algebra**. The **category of κ -Boolean algebras** is denoted κBool (or $\text{Bool}, \sigma\text{Bool}, \text{CBool}$ when $\kappa = \omega, \omega_1, \infty$).

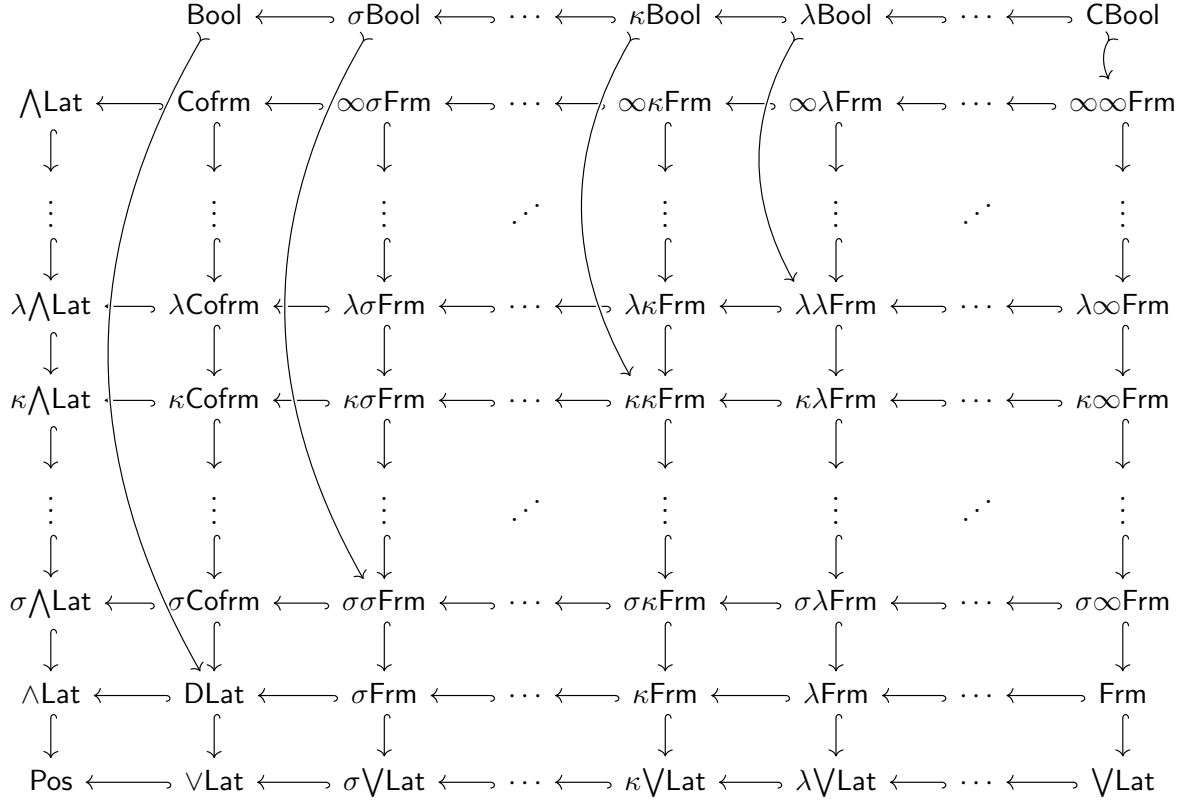
A κ -Boolean algebra also has κ -ary meets, over which finite joins distribute; it is sometimes useful to consider such meets in the absence of complements. A (λ, κ) -**frame**⁶ will mean a poset A with λ -ary meets and κ -ary joins, over which finite joins and meets distribute, respectively; in other words, A is a κ -frame, while the opposite poset A^{op} is a λ -frame. The category of these will be denoted $\lambda\kappa\text{Frm}$. Thus, $\kappa\text{Frm} = \omega\kappa\text{Frm}$, while $\kappa\text{Bool} \subseteq \kappa\kappa\text{Frm}$. A (κ, ω) -frame is also called a κ -**coframe**; the category of these is denoted $\kappa\text{Cofrm} = \kappa\omega\text{Frm}$. (One can also think of a κ - \vee -lattice as a “ $(\{1\}, \kappa)$ -frame”, of a \wedge -lattice as a “ $\{1\}$ -frame”, and of a poset as a “ $(\{1\}, \{1\})$ -frame”.)

The following commutative diagram of categories and forgetful functors depicts the relationships

⁶When $\kappa = \lambda = \infty$, these are sometimes called *biframes* (e.g., [Joh82, VII 1.15]), although that term now more commonly refers to a distinct notion ([PP12, XII 5.2]).

between these various categories, for $\kappa \leq \lambda$:

(2.1.1)



All of these forgetful functors are subcategory inclusions, in particular faithful. Moreover, the forgetful functors $\kappa\text{Bool} \rightarrow \kappa\kappa\text{Frm}$, as well as their composites with $\kappa\kappa\text{Frm} \rightarrow \kappa\lambda\text{Frm}$ or $\kappa\kappa\text{Frm} \rightarrow \lambda\kappa\text{Frm}$ for $\omega \leq \lambda \leq \kappa$, are also full; this is the meaning of the \rightarrow arrows.

The above categories with $\kappa, \lambda < \infty$, except for **Pos**, consist of algebraic structures, i.e., structures with (a small set of) infinitary operations obeying equational axioms. Such categories are extremely well-behaved: they are **monadic over Set**. The axioms of partial orders are also of a special type (namely, universal Horn), so that **Pos** belongs to the related class of **locally presentable** categories. See Appendix A for a review of the basic theory of such categories. In particular, they have all (small) limits and colimits, and all of the above forgetful functors between these categories have left adjoints, meaning that we have **free structures**: for example, any poset P generates a free distributive lattice

$$\langle P \rangle_{\text{DLat}} = \langle P \text{ qua Pos} \rangle_{\text{DLat}},$$

which is the universal distributive lattice obtained by adjoining finite meets and joins to P while respecting the distributive lattice axioms as well as pre-existing order relations in P . We adopt this “qua” notation from [JV91] in order to be specific, when not clear from context, what kind of pre-existing structure we want to preserve.

More generally, we may **present** structures via generators and relations: for example,

$$\langle P \text{ qua Pos} \mid a \wedge b \leq c \rangle_{\text{DLat}}$$

denotes the universal distributive lattice obtained from $\langle P \rangle_{\text{DLat}}$ (in this case, by taking a quotient) by imposing the further relation $a \wedge b \leq c$, where a, b, c are some elements of P . The general construction of presented structures is by a standard transfinite iteration; see Appendix A.1. For a category \mathbf{C} of structures as above, we let $\mathbf{C}_\kappa \subseteq \mathbf{C}$ denote the full **subcategory of κ -presented structures**, meaning those presented by κ -ary sets of generators and relations.

In the above categories of structures where κ or λ is ∞ , there is a proper class of meet and join operations, which means that the structure presented by a set of generators and relations may be a proper class. In the cases of \mathbf{Frm} , \mathbf{Cofrm} , $\mathbf{\vee Lat}$, $\mathbf{\wedge Lat}$, it is known via an explicit construction that small-generated free structures, and hence small-presented structures, are small; see Corollary 2.3.1 below. On the other hand, there is the following classical result of Gaifman [Gai64] and Hales [Hal64] (see [Joh82, I 4.10] and Corollary 4.4.4 below for different proofs):

Theorem 2.1.2 (Gaifman–Hales). *The free complete Boolean algebra on ω generators is a proper class. Thus, the forgetful functor $\mathbf{CBool} \rightarrow \mathbf{Set}$ does not have a left adjoint.*

Corollary 2.1.3. *None of the forgetful functors from \mathbf{CBool} or $\infty\infty\mathbf{Frm}$ to $\lambda\kappa\mathbf{Frm}$, $\kappa\mathbf{\vee Lat}$, $\kappa\mathbf{\wedge Lat}$ for $\kappa, \lambda < \infty$, or to \mathbf{Frm} , \mathbf{Cofrm} , $\mathbf{\vee Lat}$, $\mathbf{\wedge Lat}$, have left adjoints.*

Proof. For \mathbf{CBool} , this follows from Theorem 2.1.2 because these latter categories all have free functors from \mathbf{Set} (see Corollary 2.3.1 below for the last four). For $\infty\infty\mathbf{Frm}$, this likewise follows from the fact that the free algebra $\langle a_0, b_0, a_1, b_1, \dots \rangle_{\infty\infty\mathbf{Frm}}$ is a proper class; otherwise, its quotient by the relations $a_i \wedge b_i = \perp$, $a_i \vee b_i = \top$ would be the free complete Boolean algebra on a_0, a_1, \dots . \square

In this kind of situation, we denote the category of large structures in all-caps, and denote the full subcategory of small-presented large structures with a subscript ∞ . For example, \mathbf{CBOOL} is the category of large complete Boolean algebras, while \mathbf{CBOOL}_∞ is the full subcategory of small-presented algebras. Thus, the “left adjoints” to the forgetful functors in Theorem 2.1.2 and Corollary 2.1.3 land in \mathbf{CBOOL}_∞ (respectively, $\infty\infty\mathbf{FRM}_\infty$). One can treat categories like \mathbf{CBOOL} in the same way as $\kappa\mathbf{CBool}$, by taking κ to be inaccessible. However, we will only need to work with small-presented large structures, which are κ -presented for some $\kappa < \infty$; thus there are no serious set-theoretic issues involved. (We could have completely avoided large structures by only talking about κ -ary operations for $\kappa < \infty$, passing to a larger κ' and the free κ' -ary structure qua κ -ary structure whenever necessary; however, we feel that this would have cost some clarity.)

The above categories, other than the categories of Boolean algebras, consist of *ordered* algebraic structures, i.e., structures with an underlying poset and order-preserving operations. Such categories are **locally ordered** (i.e., \mathbf{Pos} -enriched), meaning each hom-set is equipped with a partial order (the pointwise order between homomorphisms), such that composition is order-preserving. Moreover, the various categorical constructions mentioned above, e.g., limits, colimits, free/forgetful adjoint functors, all have ordered analogs. See Appendix A.3 for details.

2.2 General remarks on presentations and adjunctions

In the following subsections, we will discuss various methods for constructing free and presented structures in the above categories, which are more structured, and therefore easier to work with, than the “naive” method via sets of generators and relations.

Broadly speaking, we can organize such methods as follows: given any forgetful functor $\mathbf{D} \rightarrow \mathbf{C}$ between two categories of structures, with a left adjoint free functor $\langle - \rangle_{\mathbf{D}} : \mathbf{C} \rightarrow \mathbf{D}$, we can think of any $A \in \mathbf{C}$ as a “presentation” for $\langle A \rangle_{\mathbf{D}} \in \mathbf{D}$. For example:

- A distributive lattice A presents a Boolean algebra $\langle A \rangle_{\mathbf{Bool}}$, where the generators are the elements of A and the relations are the pre-existing lattice relations in A .
- We can also understand “naive” presentations in this way: define the category $\mathbf{BoolPres}$ of Boolean algebra presentations, consisting of pairs (G, R) where G is any set and R is a set of equations between Boolean algebra terms; a morphism $(G, R) \rightarrow (H, S)$ is a map $G \rightarrow H$ which takes every relation in R to some relation in S . The forgetful functor $\mathbf{Bool} \rightarrow \mathbf{BoolPres}$ takes a Boolean algebra A to its underlying set together with all equations that hold in A .

Thus, broadly speaking, our goal is to study properties of free/forgetful adjunctions $\mathbf{C} \rightleftarrows \mathbf{D}$ between the categories \mathbf{D} in (2.1.1) and various other categories \mathbf{C} of structures (including other categories in (2.1.1)), which correspond to nice properties of the presentation method for \mathbf{D} given by \mathbf{C} . Here are some of the properties we will be interested in:

- If the counit $\langle A \rangle_{\mathbf{D}} \rightarrow A$ is an isomorphism for each $A \in \mathbf{D}$, then the presentation method is “complete” in that every $A \in \mathbf{D}$ has a canonical presentation, namely its own underlying object in \mathbf{C} . For example, this is the case for $\mathbf{Bool} \rightarrow \mathbf{DLat}$, but not $\mathbf{Bool} \rightarrow \mathbf{VLat}$.

By a standard fact about adjunctions (see e.g., [Bor, 3.4.1]), the counit is an isomorphism iff the right adjoint forgetful functor $\mathbf{D} \rightarrow \mathbf{C}$ is full and faithful; so we can regard \mathbf{D} as a reflective full subcategory of \mathbf{C} in this case.

- If the unit $A \rightarrow \langle A \rangle_{\mathbf{D}}$ is an embedding for each $A \in \mathbf{C}$, then presentations are “saturated” in that the relations are already closed under all implied relations in the presented structure. This is the case for all forgetful functors in (2.1.1), but not for $\mathbf{Bool} \rightarrow \mathbf{BoolPres}$, since e.g., $\langle a, b \mid a = \neg b \rangle$ does not already contain the implied relation $\neg a = b$.
- A general category of structures may have multiple natural notions of “embedding”. The weakest one is a monomorphism, which usually just means an injective homomorphism. In the case of the unit $A \rightarrow \langle A \rangle_{\mathbf{D}}$, this means that if a relation in the presentation forces two generators to become identified, then they are already required to be identified in the presenting structure itself. This is again false for $\mathbf{Bool} \rightarrow \mathbf{BoolPres}$, e.g., in $\langle a, b \mid a = b \rangle$.

Recall that an adjunction unit $A \rightarrow \langle A \rangle_{\mathbf{D}}$ is a monomorphism for each $A \in \mathbf{C}$ iff the left adjoint $\langle - \rangle_{\mathbf{D}} : \mathbf{C} \rightarrow \mathbf{D}$ is faithful. This means that every homomorphism $A \rightarrow B \in \mathbf{C}$ of presentations is determined by the induced homomorphism $\langle A \rangle_{\mathbf{D}} \rightarrow \langle B \rangle_{\mathbf{D}}$ between the presented structures.

- In good categories, the strongest reasonable notion of “embedding” is an **extremal monomorphism** (cf. Section 3.6), which usually means a homomorphism which is not only an embedding but whose image is also closed under all “positively definable elements”. For example, this is false for the unit of $\mathbf{DLat} \rightleftarrows \mathbf{Bool}$, since $A \in \mathbf{DLat}$ need not be closed under complements, which are defined by the equations $a \wedge \neg a = \perp$ and $a \vee \neg a = \top$.

Usually (see [Kel82, end of §3.4]), the unit $A \rightarrow \langle A \rangle_{\mathbf{D}}$ is an extremal monomorphism for each $A \in \mathbf{C}$ iff the left adjoint $\langle - \rangle_{\mathbf{D}} : \mathbf{C} \rightarrow \mathbf{D}$ is **conservative**, meaning that if a homomorphism of presentations $f : A \rightarrow B \in \mathbf{C}$ induces an isomorphism $\langle f \rangle_{\mathbf{D}} : \langle A \rangle_{\mathbf{D}} \cong \langle B \rangle_{\mathbf{D}}$, then f was already an isomorphism. Note that this is “orthogonal” to the first property listed above, of the counit being an isomorphism: if both hold, then the adjunction $\mathbf{C} \rightleftarrows \mathbf{D}$ is an equivalence.⁷

⁷There is a precise sense in which these two properties are “orthogonal complements”: any adjunction between sufficiently nice categories can be factored, essentially uniquely, into one whose left adjoint is conservative, followed by one whose right adjoint is full and faithful; see [Day74].

- If $\langle - \rangle_D : C \rightarrow D$ is faithful and **full on isomorphisms**, meaning each isomorphism $\langle A \rangle_D \cong \langle B \rangle_D$ is induced by an isomorphism $A \cong B$, this means that the presentation may be canonically recovered from the presented structure. This implies conservativity, but is much rarer. It tends to hold for order-theoretic “completions”, like $\mathbf{DLat} \rightleftarrows \mathbf{Frm}$ (see Corollary 2.3.4), where the generators become “atomic” or “compact” in the completed structure.⁸

2.3 Ideals

In this subsection, we recall the standard constructions of the free functors $\kappa\mathbf{VLat} \rightarrow \lambda\mathbf{VLat}$ and $\kappa\mathbf{Frm} \rightarrow \lambda\mathbf{Frm}$ for $\kappa \leq \lambda$.

A **lower set** in a poset A is a downward-closed subset $D \subseteq A$ (i.e., $a \leq b \in D \implies a \in D$). Let $\mathcal{L}(A)$ denote the poset of lower sets in A , ordered via \subseteq . Each lower set D can be regarded as a normal form for the \mathbf{V} -lattice term $\bigvee D$, so that $\mathcal{L}(A) \cong \langle A \text{ qua Pos} \rangle_{\mathbf{VLat}}$, i.e., $\mathcal{L} : \mathbf{Pos} \rightarrow \mathbf{VLat}$ is left adjoint to the forgetful functor. The adjunction unit is given by the principal ideal embedding

$$\begin{aligned} \downarrow : A &\longrightarrow \mathcal{L}(A) \\ a &\longmapsto \{b \in A \mid b \leq a\} \end{aligned}$$

for each poset A . More generally, let $\mathcal{L}_\kappa(A) \subseteq \mathcal{L}(A)$ denote the subset of κ -**generated lower sets** (i.e., those which are the downward-closure of a κ -ary subset); then $\mathcal{L}_\kappa(A) \cong \langle A \text{ qua Pos} \rangle_{\kappa\mathbf{VLat}}$, so that $\mathcal{L}_\kappa : \mathbf{Pos} \rightarrow \kappa\mathbf{VLat}$ is left adjoint to the forgetful functor. In particular, the free κ - \mathbf{V} -lattice generated by a set X is the κ -**ary powerset** $\mathcal{P}_\kappa(X) \subseteq \mathcal{P}(X)$ of κ -ary subsets of X .

A poset A is κ -**directed** if every κ -ary subset has an upper bound; when $\kappa = \omega$, we say **directed**, which equivalently means $A \neq \emptyset$ and every $a, b \in A$ have an upper bound. A κ -**ideal** in a poset A is a κ -directed lower set; an **ideal** is an ω -ideal. In a κ - \mathbf{V} -lattice, a κ -ideal is equivalently a lower set closed under κ -ary joins. Let $\kappa\mathcal{I}(A) \subseteq \mathcal{L}(A)$ denote the subset of κ -ideals. Then $\kappa\mathcal{I} : \kappa\mathbf{VLat} \rightarrow \mathbf{VLat}$ is left adjoint to the forgetful functor. More generally, for $\kappa \leq \lambda$, let $\kappa\mathcal{I}_\lambda(A) \subseteq \mathcal{L}(A)$ denote the λ -**generated κ -ideals**; then $\kappa\mathcal{I}_\lambda : \kappa\mathbf{VLat} \rightarrow \lambda\mathbf{VLat}$ is left adjoint to the forgetful functor.

When A is a \wedge -lattice, respectively a κ -frame, then $\mathcal{L}_\lambda(A)$, resp., $\kappa\mathcal{I}_\lambda(A)$, is a λ -frame, and is the free λ -frame generated by A . (See [Mad91, §2].) In particular, if A is the free \wedge -lattice generated by a (po)set, then $\mathcal{L}_\lambda(A)$ is the free λ -frame on that same (po)set. Taking $\lambda = \infty$ shows that despite there being a proper class of join operations,

Corollary 2.3.1 (Bénabou). *Small-generated free frames and free \mathbf{V} -lattices are small.*

Hence, so are arbitrary small-generated frames and \mathbf{V} -lattices, being quotients of free ones. \square

Summarizing, we have the following diagram of free/forgetful adjunctions, for $\kappa \leq \lambda$:

$$\begin{array}{ccccccc} \wedge\mathbf{Lat} & \xrightarrow{\mathcal{L}_\omega} & \mathbf{DLat} & \xrightarrow{\mathcal{I}_\sigma} & \sigma\mathbf{Frm} & \xrightarrow{\sigma\mathcal{I}_{\omega_2}} & \cdots & \xrightarrow{\perp} & \kappa\mathbf{Frm} & \xrightarrow{\kappa\mathcal{I}_\lambda} & \lambda\mathbf{Frm} & \xrightarrow{\lambda\mathcal{I}} & \mathbf{Frm} \\ \downarrow & \swarrow \perp & \downarrow & \swarrow \perp & \downarrow & \swarrow \perp & \cdots & \swarrow \perp & \downarrow & \swarrow \perp & \downarrow & \swarrow \perp & \downarrow \\ \mathbf{Pos} & \xrightarrow{\mathcal{L}_\omega} & \mathbf{VLat} & \xrightarrow{\mathcal{I}_\sigma} & \sigma\mathbf{VLat} & \xrightarrow{\sigma\mathcal{I}_{\omega_2}} & \cdots & \xrightarrow{\perp} & \kappa\mathbf{VLat} & \xrightarrow{\kappa\mathcal{I}_\lambda} & \lambda\mathbf{VLat} & \xrightarrow{\lambda\mathcal{I}} & \mathbf{VLat} \end{array}$$

Note that the squares with horizontal *left* adjoints (and vertical right adjoints) also commute; this fact does not extend beyond the bottom two rows of the diagram (2.1.1).

⁸A general theory of such “completions” is given by the theory of **KZ-monads**; see [Koc95], Remark 2.9.7.

The unit for all of these adjunctions is given by \downarrow , which is an (order-)embedding. By a standard fact about adjunctions (see e.g., [Kel82, end of §3.4]), this equivalently means that the left adjoints are (order-)faithful, i.e., restrict to (order-)embeddings on each hom-(po)set.

An unusual feature of these adjunctions, compared to arbitrary free/forgetful adjunctions between categories of algebras, is that the generators of free algebras are uniquely determined. For a λ - \vee -lattice B and $\kappa \leq \lambda$, an element $b \in B$ is called **κ -compact**⁹ if whenever $D \in \kappa\mathcal{I}_\lambda(B)$ with $b \leq \bigvee D$, then $b \in D$. This is easily seen to be equivalent to: every λ -ary cover of b has a κ -ary subcover, i.e., whenever $b \leq \bigvee C$ for a λ -ary $C \subseteq B$, then $b \leq \bigvee D$ for a κ -ary $D \subseteq C$. Let

$$B_\kappa := \{b \in B \mid b \text{ is } \kappa\text{-compact}\}.$$

We call B **κ -compactly based** if it is generated by B_κ under λ -ary joins. If B is a λ -frame which is κ -compactly based, and $B_\kappa \subseteq B$ is also closed under finite meets, then we call B **κ -coherent**. The following is routine; see [Mad91, 1.4].

Proposition 2.3.2.

(a) For $\kappa \leq \mu \leq \lambda$ and $A \in \kappa\mathcal{V}\text{Lat}$, we have $\kappa\mathcal{I}_\lambda(A)_\mu = \kappa\mathcal{I}_\mu(A)$, thus $\kappa\mathcal{I}_\lambda(A)$ is μ -compactly based (μ -coherent if $A \in \kappa\text{Frm}$).

In particular, $\kappa\mathcal{I}(A)_\lambda = \kappa\mathcal{I}_\lambda(A)$, and $\kappa\mathcal{I}_\lambda(A)_\kappa = \kappa\mathcal{I}_\kappa(A) = \downarrow(A) \cong A$ consists of the principal ideals, thus $\kappa\mathcal{I}_\lambda(A)$ is κ -compactly based (κ -coherent if $A \in \kappa\text{Frm}$).

(b) If $B \in \lambda\mathcal{V}\text{Lat}$ is κ -compactly based, then we have an isomorphism $\bigvee : \kappa\mathcal{I}_\lambda(B_\kappa) \cong B$.

Thus, the image of $\kappa\mathcal{I}_\lambda : \kappa\mathcal{V}\text{Lat} \rightarrow \lambda\mathcal{V}\text{Lat}$ (respectively, $\kappa\mathcal{I}_\lambda : \kappa\text{Frm} \rightarrow \lambda\text{Frm}$) consists, up to isomorphism, of precisely the κ -compactly based λ - \vee -lattices (resp., the κ -coherent λ -frames).

Remark 2.3.3. For $\kappa \leq \mu \leq \lambda$, whether a λ -frame B is κ -coherent can depend on whether B is regarded as a λ -frame or as a μ -frame. For example, if $\mu = \kappa$, clearly every κ -frame is κ -coherent, whereas not every λ -frame is κ -coherent.

What is true is that the notion of κ -coherent frame is preserved by the free functor $\mu\mathcal{I}_\lambda : \mu\text{Frm} \rightarrow \lambda\text{Frm}$ (since $\mu\mathcal{I}_\lambda \circ \kappa\mathcal{I}_\mu \cong \kappa\mathcal{I}_\lambda$), rather than the forgetful functor.

The categorical meaning of Proposition 2.3.2 is

Corollary 2.3.4. For $\kappa \leq \lambda$, the free functors $\kappa\mathcal{I}_\lambda : \kappa\mathcal{V}\text{Lat} \rightarrow \lambda\mathcal{V}\text{Lat}$ and $\kappa\mathcal{I}_\lambda : \kappa\text{Frm} \rightarrow \lambda\text{Frm}$ are full on isomorphisms.

Proof. Every isomorphism $\kappa\mathcal{I}_\lambda(A) \rightarrow \kappa\mathcal{I}_\lambda(B)$ is induced by its restriction to $A \cong \kappa\mathcal{I}_\lambda(A)_\kappa \rightarrow \kappa\mathcal{I}_\lambda(A)_\kappa \cong B$. \square

Recall from Section 2.1 that $\kappa\mathcal{V}\text{Lat}_\lambda \subseteq \kappa\mathcal{V}\text{Lat}$ and $\kappa\text{Frm}_\lambda \subseteq \kappa\text{Frm}$ denote the full subcategories of λ -presented structures. Clearly, free functors preserve λ -presentability. Furthermore, a κ -frame is λ -presented (resp., λ -generated) as a κ -frame iff it is so as a κ - \vee -lattice. For λ -generated, this is because if A is λ -generated as a κ -frame by $B \subseteq A$, then it is λ -generated as a κ - \vee -lattice by the closure of B under finite meets. For λ -presented, see Corollary 2.6.8 below.

⁹called a **κ -element** in [Mad91]

The categories $\kappa\mathbf{Frm}_\lambda$, as κ, λ vary, are related as in the following diagram in which $\kappa \leq \lambda$ (similarly for $\kappa\mathbf{V}\mathbf{Lat}_\lambda$):

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\perp} & \kappa\mathbf{Frm} & \xrightarrow[\perp]{\kappa\mathcal{I}_\lambda} & \lambda\mathbf{Frm} & \xleftarrow{\perp} & \cdots \xleftarrow{\perp} \mathbf{Frm} \\
& & \cup & & \cup & & \cup \\
& & \vdots & & \vdots & & \vdots \\
& & \cup & & \cup & & \cup \\
\cdots & \longrightarrow & \kappa\mathbf{Frm}_\lambda & \xrightarrow{\kappa\mathcal{I}_\lambda} & \lambda\mathbf{Frm}_\lambda & \xrightarrow{\simeq} & \cdots \xrightarrow{\simeq} \mathbf{Frm}_\lambda \\
& & \cup & & \cup & & \cup \\
\cdots & \longrightarrow & \kappa\mathbf{Frm}_\kappa & \xrightarrow[\simeq]{\kappa\mathcal{I}_\lambda} & \lambda\mathbf{Frm}_\kappa & \xrightarrow{\simeq} & \cdots \xrightarrow{\simeq} \mathbf{Frm}_\kappa \\
& & \cup & & \cup & & \cup \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

Note that the free functors in each row stabilize past the diagonal. This is due to the following simple fact, which says that in κ -generated κ - \mathbf{V} -lattices, κ -ary joins already determine arbitrary joins. Thus, “ κ -presented κ -frame” and “ κ -presented frame” mean exactly the same thing.

Proposition 2.3.5. *Let A be a κ -generated κ - \mathbf{V} -lattice. Then every κ -ideal in A is principal, i.e., $\downarrow : A \cong \kappa\mathcal{I}(A)$, whence A is a \mathbf{V} -lattice, and $A = A_\kappa$.*

Thus, for $\kappa \leq \lambda$, the free/forgetful adjunctions restrict to equivalences of categories

$$\kappa\mathbf{V}\mathbf{Lat}_\kappa \simeq \lambda\mathbf{V}\mathbf{Lat}_\kappa, \quad \kappa\mathbf{Frm}_\kappa \simeq \lambda\mathbf{Frm}_\kappa$$

(more generally, between the full subcategories of κ -generated objects).

Proof. For the first statement: letting $B \subseteq A$ be a κ -ary generating set, every κ -ideal $D \subseteq A$ is equal to the principal ideal $\downarrow \mathbf{V}(D \cap B)$. For the second statement: the first statement implies that the unit \downarrow of the adjunction is an isomorphism on κ -presented (resp., κ -generated) objects, thus the left adjoint $\kappa\mathcal{I}_\lambda$ is full and faithful on the full subcategory of such objects; it is also essentially surjective onto the full subcategory of κ -presented (resp., κ -generated) objects by the first statement. \square

We have the obvious order-duals of the above notions for the categories $\kappa\mathbf{\bigwedge}\mathbf{Lat}$ and $\kappa\mathbf{Cofrm}$. An **upper set** is an upward-closed set; a **κ -filter** is an upper set which is also **κ -codirected**, or equivalently in a κ - $\mathbf{\bigwedge}$ -lattice, closed under κ -ary meets. For $\kappa \leq \lambda$, the set of λ -generated κ -filters, ordered by *reverse* inclusion, yields the free functor $\kappa\mathbf{\bigwedge}\mathbf{Lat} \rightarrow \lambda\mathbf{\bigwedge}\mathbf{Lat}$, with unit given by the principal filter embedding $\uparrow a := \{b \mid b \geq a\}$.

2.4 Quotients

In this subsection, we recall some basic facts about quotients of frames, Boolean algebras, and related structures. This material is standard; in particular, see [Joh82, II §2] for the description of frame quotients via nuclei. However, our account takes a somewhat different viewpoint than what is usually found in the literature, focusing more on the ordered aspects of quotients, as well as on the general context of joins before also considering finite meets.

Recall that quotients of an algebraic structure A are determined by **congruences** \sim on A , i.e., equivalence relations which are also subalgebras of A^2 . Given a homomorphism $f : A \rightarrow B$, its **kernel** $\ker(f) := \{(a, a') \in A^2 \mid f(a) = f(a')\}$ is a congruence, whose quotient is isomorphic to the image of f . Conversely, any congruence \sim on A is the kernel of its quotient map $A \twoheadrightarrow A/\sim$. This gives a bijection between congruences on A and equivalence classes of surjective homomorphisms from A . See Appendix A.2 for details.

For ordered algebraic structures, in the sense of Appendix A.3, we have the following ordered analogs of these notions. The **order-kernel** of a homomorphism $f : A \rightarrow B$ is

$$\text{oker}(f) := \{(a, a') \in A^2 \mid f(a) \leq f(a')\},$$

and is an **order-congruence** on A , i.e., a preorder \lesssim which contains the partial order \leq_A on A and is also a subalgebra of A^2 . Given an order-congruence \lesssim , we always let $\sim := \lesssim \cap \lesssim^{-1}$ denote its symmetric part, which is a congruence such that \lesssim descends to a partial order on A/\sim making \lesssim the order-kernel of the quotient map $A \twoheadrightarrow A/\sim$. This again gives a bijection between order-congruences on A and equivalence classes of surjective homomorphisms from A . See Appendix A.3 for details.

For a κ - \vee -lattice A , it is easily seen that a binary relation $\lesssim \subseteq A^2$ is an order-congruence iff

- \lesssim is a preorder containing \leq_A ; and
- κ -ary joins $\bigvee_i a_i$ in A are also joins with respect to \lesssim , i.e., if $a_i \lesssim b$ for every i , then $\bigvee_i a_i \lesssim b$.

In terms of the second-coordinate fibers of $\lesssim \subseteq A^2$, i.e., the **principal \lesssim -ideals**

$$\Downarrow a := \{b \in A \mid b \lesssim a\},$$

the above conditions are equivalent to

- each $\Downarrow a$ is a κ -ideal, and $\Downarrow : A \rightarrow \kappa\mathcal{I}(A)$ is monotone;
- $\downarrow \leq \Downarrow$ (i.e., $\downarrow a \subseteq \Downarrow a$ for each a);
- $\Downarrow_* \circ \Downarrow \leq \Downarrow$ (i.e., $\bigcup_{b \in \Downarrow a} \Downarrow b \subseteq \Downarrow a$ for each a), where

$$\begin{aligned} \Downarrow_* : \kappa\mathcal{I}(A) &\longrightarrow \kappa\mathcal{I}(A) \\ D &\longmapsto \bigcup_{a \in D} \Downarrow a \end{aligned}$$

is the κ -directed-join-preserving extension of $\Downarrow : A \rightarrow \kappa\mathcal{I}(A)$ along \downarrow .

Call a map $\Downarrow : A \rightarrow \kappa\mathcal{I}(A)$ obeying these conditions a **κ -ideal closure operator** on A . These are in bijection with order-congruences \lesssim on A , with \lesssim recovered as $\text{oker}(\Downarrow)$.

When $\kappa = \infty$, for a \vee -lattice A , we have $\infty\mathcal{I}(A) \cong A$, so we may equivalently regard \Downarrow as a monotone map $A \rightarrow A$ which is a **closure operator**, i.e., satisfying

$$1_A \leq \Downarrow, \quad \Downarrow \circ \Downarrow \leq \Downarrow \quad (\text{thus } \Downarrow \circ \Downarrow = \Downarrow).$$

Each congruence class $[a]$ (for the order-congruence corresponding to \Downarrow) has a greatest element, namely $\Downarrow a$, so that the quotient \vee -lattice A/\sim may be isomorphically realized as the image $\Downarrow(A)$ of $\Downarrow : A \rightarrow A$, equivalently its set of fixed points, with join in $\Downarrow(A)$ given by join in A followed by applying \Downarrow . The quotient map $A \twoheadrightarrow A/\sim$ is then identified with $\Downarrow : A \twoheadrightarrow \Downarrow(A)$.

For $\kappa < \infty$, we may instead consider the κ -directed-join-preserving extension $\zeta_* : \kappa\mathcal{I}(A) \rightarrow \kappa\mathcal{I}(A)$ of ζ , which is a closure operator on $\kappa\mathcal{I}(A)$, whose fixed points are those κ -ideals $D \in \kappa\mathcal{I}(A)$ which are also \lesssim -downward-closed. The quotient A/\sim may be identified with the subset $\zeta(A) \subseteq \zeta_*(\kappa\mathcal{I}(A))$ consisting of the principal \lesssim -ideals, i.e., the sets ζa for $a \in A$; these are the κ -compact elements in $\zeta_*(\kappa\mathcal{I}(A))$. The quotient map $A \twoheadrightarrow A/\sim$ is again identified with ζ .

For a κ -frame A , a κ - \bigvee -lattice order-congruence $\lesssim \subseteq A^2$ is a κ -frame order-congruence iff \lesssim is closed under binary meets, which is easily seen to be equivalent to either of:

- binary meets in A are meets with respect to \lesssim ; or equivalently,
- $a \lesssim b \implies c \wedge a \lesssim c \wedge b$ (we say that \lesssim is \wedge -stable).

In terms of the corresponding κ -ideal closure operator $\zeta : A \rightarrow \kappa\mathcal{I}(A)$, this means

- $\zeta : A \rightarrow \kappa\mathcal{I}(A)$ preserves binary meets; or equivalently,
- $a \wedge \zeta b \subseteq \zeta(a \wedge b)$.

We call such ζ a κ -ideal nucleus. When $\kappa = \infty$, regarding $\zeta : A \rightarrow A$ as before, ζ is called a nucleus.

An arbitrary binary relation $\prec \subseteq A^2$ generates an order-congruence, which may be obtained by closing \prec under the above conditions via the usual transfinite recursion. The following results say that it suffices to close under a subset of the conditions:

Proposition 2.4.1. *Let A be a κ - \bigvee -lattice, \prec be a binary relation on A . Let $\lesssim \subseteq A^2$ be the smallest binary relation which is (i) reflexive, and satisfies (ii) for κ -many $a_i \lesssim b$, we have $\bigvee_i a_i \lesssim b$, (iii) $a \lesssim b \lesssim c \implies a \lesssim c$, and (iv) $a \prec b \lesssim c \implies a \lesssim c$. Then \lesssim is transitive, hence is the order-congruence generated by \prec .*

Proof. Given $a \lesssim b \lesssim c$, one may induct on the derivation of $a \lesssim b$ to show $a \lesssim c$.

Alternatively, let \lesssim' be the binary relation defined by $a \lesssim' b$ iff for every $c \in A$, if $b \lesssim c$, then $a \lesssim c$. Then one easily checks that \lesssim' obeys (i-iv), hence contains \lesssim . \square

The significance of conditions (i-iv) in this result is that they only involve \lesssim with a fixed right-hand side, hence translate to the following closure conditions on the fibers ζa :

Corollary 2.4.2. *Let A be a κ - \bigvee -lattice, \prec be a binary relation on A , \lesssim be the order-congruence generated by \prec , and $\zeta : A \rightarrow \kappa\mathcal{I}(A)$ be the corresponding κ -ideal closure operator with κ -directed-join-preserving extension $\zeta_* : \kappa\mathcal{I}(A) \rightarrow \kappa\mathcal{I}(A)$. Then for each $a \in A$, the principal \lesssim -ideal $\zeta a \subseteq A$ is the smallest κ -ideal containing a which is also \prec -downward-closed. Thus more generally, a κ -ideal $D \in \kappa\mathcal{I}(A)$ is a fixed point of ζ_* , i.e., \lesssim -downward-closed, iff it is \prec -downward-closed. \square*

Concerning κ -frame order-congruences, we have

Proposition 2.4.3. *Let A be a κ -frame, \prec be a \wedge -stable binary relation on A . Then the κ - \bigvee -lattice order-congruence \lesssim generated by \prec is still \wedge -stable, hence is also the κ -frame order-congruence generated by \prec .*

Proof. By transfinite induction, or by checking that $a \lesssim' b \iff \forall c \in A (c \wedge a \lesssim c \wedge b)$ is a κ - \bigvee -lattice order-congruence containing \prec . \square

For a (λ, κ) -frame A where $\lambda > \omega$, the first characterization of κ -frame order-congruences above generalizes straightforwardly, i.e., λ -ary meets in A should also be \lesssim -meets. Similarly, the corresponding κ -ideal closure operator $\downarrow : A \rightarrow \kappa\mathcal{I}(A)$ should preserve λ -ary meets. However, these notions are not as useful, due to the lack of an analog of the \wedge -stability characterization.

By an **upper (order-)congruence** on a (λ, κ) -frame A , we mean one generated by declaring $\top \sim a$ (equivalently, $\top \lesssim a$) for all a in some $U \subseteq A$; we will denote such a (order-)congruence by \sim^U (\lesssim^U). Clearly, $\sim^U = \sim^{U'}$ where U' is the λ -filter generated by U . If U is a λ -filter, it is *not* necessarily the case that there is no bigger $U' \supseteq U$ with $\sim^U = \sim^{U'}$ (see Remark 2.11.8 below), unless U is also a κ -filter (e.g., if $\lambda \geq \kappa$), in which case it is easily seen that

$$b \sim^U c \iff \exists a \in U (a \wedge b = a \wedge c), \quad b \lesssim^U c \iff \exists a \in U (a \wedge b \leq c).$$

It follows that U may be recovered from \sim^U as the congruence class $[\top]$. We call such \sim^U a **κ -filter congruence**, and call A/\sim^U a **κ -filterquotient** of A . In particular, for a principal filter $\uparrow a$, we have $b \sim^{\uparrow a} c \iff a \wedge b = a \wedge c$, hence $A/\sim^{\uparrow a}$ may be isomorphically realized as the principal ideal $\downarrow a \subseteq A$ with quotient map $a \wedge (-) : A \twoheadrightarrow \downarrow a$. We write $\sim^a := \sim^{\uparrow a}$, and call $\downarrow a$ the **principal filterquotient** at a when we are regarding it as the quotient by \sim^a .

For a κ -Boolean algebra A , owing to $a = b \iff a \leftrightarrow b = \top$, every congruence is a κ -filter congruence, corresponding to the κ -filter $[\top] \subseteq A$. Moreover, if \sim is a κ -generated congruence, then $[\top]$ is a κ -generated κ -filter, hence a principal filter, namely $\uparrow a$ for $a = \bigwedge_i (b_i \leftrightarrow c_i)$ for some κ -ary set of generators (b_i, c_i) for \sim .

2.5 Products and covers

In this subsection, we recall a standard characterization of products of frames and Boolean algebras, via pairwise disjoint covers. We then prove some preservation results for products as well as for general covers.

A **κ -ary partition of \top** in a κ -frame A is a κ -ary family of **pairwise disjoint** elements $a_i \in A$, i.e., $a_i \wedge a_j = \perp$ for $i \neq j$, which **cover** \top , i.e., $\bigvee_i a_i = \top$.

Given a κ -ary product $\prod_i A_i$ of κ -frames, the elements $\delta_i \in \prod_i A_i$ which are \top in the i th coordinate and \perp elsewhere form a κ -ary partition of \top . Moreover, each projection $\pi_i : \prod_j A_j \rightarrow A_i$ descends to an isomorphism between the principal filterquotient $\downarrow \delta_i$ and A_i . Conversely, given any κ -frame A with a κ -ary partition $(\delta_i)_i$ of \top , we have an isomorphism

$$\begin{aligned} A &\cong \prod_i \downarrow \delta_i \\ a &\mapsto (\delta_i \wedge a)_i \\ \bigvee_i a_i &\leftarrow (a_i)_i. \end{aligned}$$

This description of κ -ary products also applies to κ -Boolean algebras and (λ, κ) -frames (for $\lambda \geq \omega$). Using this, we have

Proposition 2.5.1. *κ -ary products in $\kappa\mathbf{Frm}, \lambda\kappa\mathbf{Frm}, \kappa\mathbf{Bool}$ are pushout-stable: given $< \kappa$ -many objects A_i in one of these categories, and a homomorphism $f : \prod_i A_i \rightarrow B$, letting*

$$\begin{array}{ccc} \prod_i A_i & \xrightarrow{\pi_i} & A_i \\ f \downarrow & & \downarrow f_i \\ B & \xrightarrow{g_i} & B_i \end{array}$$

be pushout squares, we have that the g_i exhibit B as an isomorphic copy of the product $\prod_i B_i$.

Proof. Let $\delta_i \in \prod_i A_i$ be the partition of \top defined above, so that each $A_i \cong \downarrow \delta_i$. Then the $f(\delta_i) \in B$ form a partition of \top , hence $B \cong \prod_i \downarrow f(\delta_i)$. Each $\downarrow f(\delta_i)$ is the quotient of B identifying $f(\delta_i)$ with $\top = f(\top)$, hence the quotient of B by the image under f of the congruence on $\prod_i A_i$ generated by $\delta_i \sim \top$ whose quotient is $A_i \cong \downarrow \delta_i$; this exactly describes the pushout of A_i across f . \square

We next derive a presentation of a product from presentations of the factors. More generally, given a κ -frame A with a κ -ary cover $(a_i)_i$ of \top , we derive a presentation of A from presentations of the principal filterquotients $\downarrow a_i$.

For this purpose, it is convenient to introduce the category $\lambda\kappa\text{LFrm}$ of **local** (λ, κ) -frames, meaning (λ, κ) -frames but possibly without a top element (thus, the operations are κ -ary joins and nonempty λ -ary meets). For example, every κ -ideal in a (λ, κ) -frame A is a local (λ, κ) -subframe; thus for a principal filterquotient $\downarrow a$, the inclusion $\downarrow a \hookrightarrow A$ is a local (λ, κ) -subframe homomorphism which is a section of the quotient map $a \wedge (-) : A \twoheadrightarrow \downarrow a$. Note that the facts about quotients from the preceding subsection all generalize easily from (λ, κ) -frames to local (λ, κ) -frames.

Proposition 2.5.2. *Let A be a local (λ, κ) -frame, $A_i \subseteq A$ be $< \kappa$ -many κ -ideals which together generate A under κ -ary joins. Then*

$$A = \langle A_i \text{ qua } \lambda\kappa\text{LFrm for each } i \mid (A_i \ni a_i) \wedge (a_j \in A_j) = a_i \wedge a_j \in A_i \rangle_{\lambda\kappa\text{Frm}}.$$

(More precisely, A is the quotient of the coproduct $\coprod_i A_i$ by the congruence \sim generated by

$$(*) \quad \iota_i(a_i) \wedge \iota_j(a_j) \sim \iota_i(a_i \wedge a_j) \quad \text{for } a_i \in A_i \text{ and } a_j \in A_j,$$

where the $\iota_i : A_i \rightarrow \coprod_j A_j$ are the cocone maps.)

Proof. For each $a_i \in A_i$, by considering the relations $(*)$ in the principal filterquotient $\downarrow \iota_i(a_i) \subseteq \coprod_i A_i$, we get that every generator $\iota_i(a_i) \wedge \iota_j(a_j)$, hence every element, of $\downarrow \iota_i(a_i)$ is $\sim \iota_i(b)$ for some $b \leq a_i$. Every $b \in \coprod_i A_i$ is $\leq \bigvee_i \iota_i(a_i)$ for some $a_i \in A_i$, since the set of such elements clearly contains the image of each ι_i and is a local (λ, κ) -subframe; thus $b = \bigvee_i (\iota_i(a_i) \wedge b) \sim \bigvee_i \iota_i(b_i)$ for some $b_i \leq a_i \in A_i$. For two elements of this latter form, $b = \bigvee_i \iota_i(b_i)$ and $c = \bigvee_i \iota_i(c_i)$, such that $\bigvee_i b_i = \bigvee_i c_i \in A$, we have $\iota_i(b_i) = \bigvee_j \iota_i(b_i \wedge c_j) \lesssim \bigvee_j \iota_j(c_j) = c$ by $(*)$ for each i , whence $b \lesssim c$; similarly $c \lesssim b$, whence $b \sim c$. This shows that \sim is the kernel of the canonical map $\coprod_i A_i \twoheadrightarrow A$. \square

Note that both sides of $(*)$ above are local (λ, κ) -frame homomorphisms as functions of either a_i or a_j . Thus, it is enough to consider $(*)$ where a_i, a_j are generators of A_i, A_j . This yields

Corollary 2.5.3. *In Proposition 2.5.2, if the A_i are μ -presented for some $\mu \geq \kappa$, then so is A . \square*

Remark 2.5.4. In Proposition 2.5.2, if the A_i are principal ideals $\downarrow a_i$ such that the a_i are pairwise disjoint, then \sim is generated simply by $\iota_i(a_i) \wedge \iota_j(a_j) \sim \perp$ for $i \neq j$.

We now connect local (λ, κ) -frame presentations with (λ, κ) -frame presentations. Let

$$A = \langle G \mid R \rangle_{\lambda\kappa\text{Frm}}$$

be a presented (λ, κ) -frame. We may turn this into a presentation of A as a local (λ, κ) -frame, as follows. Let $G' := G \sqcup \{\top_A\}$ where \top_A is a new symbol. For a (λ, κ) -frame term t over G , we

may turn it into a local (λ, κ) -frame term t' over G' by replacing every occurrence of the nullary meet operation \top in t with the new symbol \top_A . Now let R' be R with every term t in it replaced with t' just defined, together with the new relations $x \leq \top_A$ for every $x \in G$. These new relations ensure that \top_A is the top element of $\langle G' \mid R' \rangle_{\lambda\kappa\text{LFrm}}$, whence the latter is a (λ, κ) -frame; and the operation $t \mapsto t'$ on terms yields a (λ, κ) -frame homomorphism $f : A \rightarrow \langle G' \mid R' \rangle_{\lambda\kappa\text{LFrm}}$. Conversely, the relations in R' are clearly satisfied in A , whence we get a local (λ, κ) -frame homomorphism $g : \langle G' \mid R' \rangle_{\lambda\kappa\text{LFrm}} \rightarrow A$. It is easily seen that g is surjective and $f \circ g = 1$, whence $f = g^{-1}$, i.e.,

$$A = \langle G' \mid R' \rangle_{\lambda\kappa\text{LFrm}}.$$

Using this transformation $(G, R) \mapsto (G', R')$, we have

Proposition 2.5.5. *If a (λ, κ) -frame is μ -presented, then it is μ -presented as a local (λ, κ) -frame.* \square

Conversely, given a local (λ, κ) -frame A , the free (λ, κ) -frame generated by A is its **scone**

$$A_{\top} := A \sqcup \{\top\}$$

where $\top = \top_{A_{\top}}$ is a new top element, strictly greater than all elements of A . If A was already a (λ, κ) -frame, then $A \subseteq A_{\top}$ is the principal ideal $\downarrow \top_A$, i.e., the principal filterquotient by the congruence \sim^{\top_A} identifying \top_A with $\top_{A_{\top}}$. Along with Proposition 2.5.5, this yields

Corollary 2.5.6. *A (λ, κ) -frame is μ -presented iff it is μ -presented as a local (λ, κ) -frame.* \square

By Corollary 2.5.3, we get

Corollary 2.5.7. *If a (λ, κ) -frame A has a κ -ary cover $(a_i)_i$ of \top such that each $\downarrow a_i$ is μ -presented for some $\mu \geq \kappa$, then A is μ -presented.* \square

Corollary 2.5.8. *For $\mu \geq \kappa$, a κ -ary product of μ -presented (λ, κ) -frames is μ -presented.* \square

The above constructions of presentations also yield

Proposition 2.5.9. *For a (λ, κ) -frame A and $\lambda' \geq \lambda$, $\kappa' \geq \kappa$, we have $\langle A \text{ qua } \lambda\kappa\text{Frm} \rangle_{\lambda'\kappa'\text{Frm}} = \langle A \text{ qua } \lambda\kappa\text{LFrm} \rangle_{\lambda'\kappa'\text{LFrm}}$, i.e., the square*

$$\begin{array}{ccc} \lambda\kappa\text{Frm} & \xrightleftharpoons[\perp]{\langle - \rangle_{\lambda'\kappa'\text{Frm}}} & \lambda'\kappa'\text{Frm} \\ \uparrow \scriptstyle (-)_{\top} \downarrow \scriptstyle \dashv & & \uparrow \scriptstyle (-)_{\top} \downarrow \scriptstyle \dashv \\ \lambda\kappa\text{LFrm} & \xrightleftharpoons[\perp]{\langle - \rangle_{\lambda'\kappa'\text{LFrm}}} & \lambda'\kappa'\text{LFrm} \end{array}$$

consisting of the vertical right and horizontal left adjoints commutes (up to canonical isomorphism).

Proof. Let $A = \langle G \mid R \rangle_{\lambda\kappa\text{Frm}} = \langle G' \mid R' \rangle_{\lambda\kappa\text{LFrm}}$ where (G', R') is obtained from (G, R) via the procedure described above. Note that the above procedure works equally well for λ', κ' as for λ, κ . Thus $\langle A \text{ qua } \lambda\kappa\text{Frm} \rangle_{\lambda'\kappa'\text{Frm}} = \langle G \mid R \rangle_{\lambda'\kappa'\text{Frm}} = \langle G' \mid R' \rangle_{\lambda'\kappa'\text{LFrm}} = \langle A \text{ qua } \lambda\kappa\text{LFrm} \rangle_{\lambda'\kappa'\text{LFrm}}$. \square

Proposition 2.5.10. *For $\lambda' \geq \lambda$, $\kappa' \geq \kappa$, the free functor $\langle - \rangle_{\lambda'\kappa'\text{Frm}} : \lambda\kappa\text{Frm} \rightarrow \lambda'\kappa'\text{Frm}$ preserves κ -ary products.*

Proof. Let $A_i \in \lambda\kappa\text{Frm}$ be $< \kappa$ -many (λ, κ) -frames. Let $\prod_i A_i = \langle G \mid R \rangle_{\lambda\kappa\text{LFrm}}$ be the presentation given by Proposition 2.5.2, with R from Remark 2.5.4; it is clear that the same presentation yields $\langle G \mid R \rangle_{\lambda'\kappa'\text{LFrm}} = \prod_i \langle A_i \text{ qua } \lambda\kappa\text{LFrm} \rangle_{\lambda'\kappa'\text{LFrm}}$. Thus using Proposition 2.5.9 twice, we have

$$\begin{aligned} \langle \prod_i A_i \text{ qua } \lambda\kappa\text{Frm} \rangle_{\lambda'\kappa'\text{Frm}} &= \langle \prod_i A_i \text{ qua } \lambda\kappa\text{LFrm} \rangle_{\lambda'\kappa'\text{LFrm}} \\ &= \langle G \mid R \rangle_{\lambda'\kappa'\text{LFrm}} \\ &= \prod_i \langle A_i \text{ qua } \lambda\kappa\text{LFrm} \rangle_{\lambda'\kappa'\text{LFrm}} \\ &= \prod_i \langle A_i \text{ qua } \lambda\kappa\text{Frm} \rangle_{\lambda'\kappa'\text{Frm}}. \end{aligned} \quad \square$$

For the categories κBool , one may develop analogs of the above by replacing $\lambda\kappa\text{LFrm}$ with the category κLBool of **local κ -Boolean algebras**,¹⁰ meaning local (κ, κ) -frames such that each $\downarrow a$ is Boolean. However, we will be content with deducing the Boolean analogs of the main results above in an *ad hoc* manner. Note that a κ -Boolean algebra presentation $A = \langle G \mid R \rangle_{\kappa\text{Bool}}$ may be turned into a (κ, κ) -frame presentation by closing G under complements and adding relations

$$a \wedge \neg a = \perp, \quad a \vee \neg a = \top$$

for each $a \in G$. Thus Corollary 2.5.7 yields

Corollary 2.5.11. *If a κ -Boolean algebra A has a κ -ary cover $(a_i)_i$ of \top such that each $\downarrow a_i$ is μ -presented for some $\mu \geq \kappa$, then A is μ -presented.* \square

Corollary 2.5.12. *For $\mu \geq \kappa$, a κ -ary product of μ -presented κ -Boolean algebras is μ -presented.* \square

Proposition 2.5.13. *The free functor $\langle - \rangle_{\kappa\text{Bool}} : \kappa\kappa\text{Frm} \rightarrow \kappa\text{Bool}$ preserves κ -ary products.*

Proof. Let $A_i \in \lambda\kappa\text{Frm}$ be $< \kappa$ -many (κ, κ) -frames. For each i , let $G_i := A_i \sqcup \neg A_i$ where $\neg A_i$ consists of the symbols $\neg a$ for each $a \in A_i$, and let R_i consist of all relations between *local* (κ, κ) -frame terms over A_i which hold in A_i , together with the relations

$$(\dagger) \quad a \wedge \neg a = \perp, \quad a \vee \neg a = \top_{A_i}$$

for each $a \in A_i$; here \perp denotes the nullary join operation (as a local (κ, κ) -frame term), while \top_{A_i} denotes the top element of A_i , *not* the nullary meet operation. It is easily seen that $\langle G_i \mid R_i \rangle_{\kappa\kappa\text{LFrm}} = \langle A_i \rangle_{\kappa\text{Bool}}$. Now as in the proof of Proposition 2.5.10, let $\prod_i A_i = \langle G \mid R \rangle_{\lambda\kappa\text{LFrm}}$ be the presentation given by Proposition 2.5.2, with R from Remark 2.5.4; so $G = \bigsqcup_i A_i$, and R consists of all relations which hold in each A_i together with the relations

$$(\ddagger) \quad \top_{A_i} \wedge \top_{A_j} = \perp \quad \text{for } i \neq j.$$

Let $G' := \bigsqcup_i G_i$ and $R' := \bigsqcup_i R_i \cup (\ddagger)$. Then $\langle G' \mid R' \rangle_{\kappa\kappa\text{LFrm}} = \prod_i \langle A_i \rangle_{\kappa\text{Bool}}$, since (G', R') is obtained via Proposition 2.5.2 and Remark 2.5.4 from the presentations (G_i, R_i) for $\langle A_i \rangle_{\kappa\text{Bool}}$. On the other hand, by (\dagger) , $\langle G' \mid R' \rangle_{\kappa\kappa\text{LFrm}}$ is obtained from $\langle G \mid R \rangle_{\kappa\kappa\text{LFrm}} = \prod_i A_i$ by freely adjoining complements in each $\downarrow \top_{A_i} \subseteq \langle G \mid R \rangle_{\kappa\kappa\text{LFrm}}$, which is identified with $\downarrow \delta_i \subseteq \prod_i A_i$ (where δ_i are as defined at the beginning of this subsection); since the δ_i form a partition of \top , adjoining complements in each $\downarrow \top_{A_i}$ amounts to adjoining complements globally, so $\langle G' \mid R' \rangle_{\kappa\kappa\text{LFrm}} = \langle \prod_i A_i \rangle_{\kappa\text{Bool}}$. \square

Corollary 2.5.14. *For $\kappa \leq \kappa'$, the free functor $\kappa\text{Bool} \rightarrow \kappa'\text{Bool}$ preserves κ -ary products.*

Proof. The free functor is the composite $\kappa\text{Bool} \rightarrow \kappa\kappa\text{Frm} \rightarrow \kappa'\kappa'\text{Frm} \rightarrow \kappa'\text{Bool}$. \square

¹⁰also known as Boolean r(i)ngs, Boolean (pseudo)lattices

2.6 Posites

In this subsection, we discuss a method for presenting κ - \bigvee -lattices and κ -frames via generators and \bigvee -relations which are “saturated” (in the sense described in Section 2.2). This method is well-known in the case $\kappa = \infty$ (see [Joh82, II 2.11]), where it is derived from the more general theory of sites in sheaf theory (see [Joh02, C2.1]). The case $\kappa < \infty$ does not appear to have been spelled out before, although it is likewise a special case of the more general theory of κ -ary sites of Shulman [Shu12]. As in Section 2.4, we also make a point of working in the \bigvee -only context for as long as possible before introducing meets. Our presentation is partly chosen in order to highlight the analogy with some variants and generalizations of posites we will consider in Sections 2.9 and 2.11 below.

Let A be a poset, and consider an arbitrary κ - \bigvee -lattice presented by A (qua poset) together with additional relations imposed. In other words, we are considering an arbitrary quotient of the free κ - \bigvee -lattice $\mathcal{L}_\kappa(A)$ generated by A . As described in Section 2.4, such a quotient is determined by an order-congruence \lesssim on $\mathcal{L}_\kappa(A)$, which is completely determined by the relation

$$a \triangleleft C :\iff \downarrow a \lesssim C$$

between A and $\mathcal{L}_\kappa(A)$, read “ a is covered by C ”, via compatibility of \lesssim with κ -ary joins:

$$B \lesssim C \iff \forall a \in B (a \triangleleft C).$$

The other requirements on \lesssim translate to

- (reflexivity) $a \in C \implies a \triangleleft C$;
- (left-transitivity) $a \leq b \triangleleft C \implies a \triangleleft C$;
- (right-transitivity) if $a \triangleleft C$, and $c \triangleleft D$ for every $c \in C$, then $a \triangleleft D$.

Note that reflexivity and right-transitivity imply

- (monotonicity) $a \triangleleft C \subseteq D \implies a \triangleleft D$.

We call a relation $\triangleleft \subseteq A \times \mathcal{L}_\kappa(A)$ obeying these axioms a κ - \bigvee -**coverage** on A , or just a \bigvee -**coverage** when $\kappa = \infty$. We call the poset A equipped with a κ - \bigvee -coverage \triangleleft a κ - \bigvee -**posite**. We think of a κ - \bigvee -posite (A, \triangleleft) as a “saturated” presentation of a κ - \bigvee -lattice. Indeed, the order-congruence \lesssim corresponding to \triangleleft is clearly generated by declaring $\downarrow a \lesssim C = \bigvee_{c \in C} \downarrow c$ for $a \triangleleft C$, so that the quotient is the presented κ - \bigvee -lattice

$$\langle A \mid \triangleleft \rangle_{\kappa\bigvee\text{Lat}} := \mathcal{L}_\kappa(A) / \sim = \langle A \text{ qua Pos} \mid a \leq \bigvee C \text{ for } a \triangleleft C \rangle_{\kappa\bigvee\text{Lat}};$$

and \triangleleft is saturated in that it contains *all* relations of the form $a \leq \bigvee C$ that hold in $\langle A \mid \triangleleft \rangle_{\kappa\bigvee\text{Lat}}$.

Given a κ - \bigvee -coverage \triangleleft , we write $a \triangleleft C$ for an arbitrary subset $C \subseteq A$ to mean that there is a κ -ary $B \subseteq C$ which generates a lower set $\downarrow B \in \mathcal{L}_\kappa(A)$ such that $a \triangleleft \downarrow B$. It is easily seen that this extended \triangleleft relation continues to obey the three conditions above. In particular, taking $C \in \mathcal{L}_\lambda(A)$ for $\lambda \geq \kappa$ means that a κ - \bigvee -coverage may be regarded as a λ - \bigvee -coverage, hence as a \bigvee -coverage.

The κ -ideal closure operator $\downarrow : \mathcal{L}_\kappa(A) \rightarrow \kappa\mathcal{I}(\mathcal{L}_\kappa(A))$ corresponding to the order-congruence \lesssim corresponding to \triangleleft takes $C \in \mathcal{L}_\kappa(A)$ to the set of all $B \in \mathcal{L}_\kappa(A)$ such that $b \triangleleft C$ for all $b \in B$. Under the isomorphism $\kappa\mathcal{I}(\mathcal{L}_\kappa(A)) \cong \mathcal{L}(A)$, \downarrow becomes

$$\begin{aligned} \nabla : \mathcal{L}_\kappa(A) &\longrightarrow \mathcal{L}(A) \\ C &\longmapsto \{a \in A \mid a \triangleleft C\}. \end{aligned}$$

The κ -directed-join-preserving extension $\nabla_* : \mathcal{L}(A) \cong \kappa\mathcal{I}(\mathcal{L}_\kappa(A)) \rightarrow \mathcal{L}(A)$ is defined the same way as ∇ but regarding \triangleleft as a \vee -coverage as above. By Section 2.4, the presented \vee -lattice $\langle A \mid \triangleleft \rangle_{\vee\text{Lat}}$ may be realized as the set

$$\triangleleft\mathcal{I}(A) := \nabla_*(\mathcal{L}(A)) \subseteq \mathcal{L}(A)$$

of \triangleleft -ideals in A , i.e., lower sets $D \subseteq A$ which are closed under \triangleleft in that $a \triangleleft C \subseteq D \implies a \in D$, under the **principal \triangleleft -ideals** map

$$\begin{aligned} \nabla &:= \nabla \circ \downarrow : A \longrightarrow \triangleleft\mathcal{I}(A) \\ a &\longmapsto \nabla \downarrow a = \{b \in A \mid b \triangleleft \{a\}\}. \end{aligned}$$

The presented κ - \vee -lattice $\langle A \mid \triangleleft \rangle_{\kappa\vee\text{Lat}}$ may be realized as the image of $\nabla : \mathcal{L}_\kappa(A) \rightarrow \triangleleft\mathcal{I}(A)$, i.e., as the κ -generated, or equivalently κ -compact, \triangleleft -ideals $\triangleleft\mathcal{I}_\kappa(A) := \nabla(\mathcal{L}_\kappa(A)) = \triangleleft\mathcal{I}(A)_\kappa \subseteq \triangleleft\mathcal{I}(A)$.

The “injection of generators” $\nabla : A \rightarrow \triangleleft\mathcal{I}(A)$ is an order-embedding iff for all $a, b \in A$, whenever $\nabla a \subseteq \nabla b$, i.e., $a \triangleleft \{b\}$, then already $a \leq b$ in A ; in other words, this means $\nabla b = \downarrow b$, i.e., principal ideals are already \triangleleft -ideals. In this case, we call \triangleleft **separated** (or **subcanonical**). There is a largest separated κ - \vee -coverage, called the **canonical κ - \vee -coverage**, given by $a \triangleleft C$ (for $C \in \mathcal{L}_\kappa(A)$ or $C \in \mathcal{P}_\kappa(A)$) iff every upper bound of C is $\geq a$; if A is already a κ - \vee -lattice, this means

$$a \triangleleft C \iff a \leq \bigvee C,$$

i.e., \triangleleft corresponds to the canonical presentation $A \cong \langle A \mid \text{all relations which hold in } A \rangle_{\kappa\vee\text{Lat}}$. (For a general poset A , the canonical \vee -coverage presents the MacNeille completion of A .)

We regard κ - \vee -posites (A, \triangleleft) as infinitary first-order structures with a partial order together with $(1 + \lambda)$ -ary relations \triangleleft for all $\lambda < \kappa$. Thus, by a **homomorphism of κ - \vee -posites** $f : (A, \triangleleft_A) \rightarrow (B, \triangleleft_B)$, we mean a monotone map $f : A \rightarrow B$ which preserves covers, i.e., $a \triangleleft_A C \implies f(a) \triangleleft_B f(C)$. Let $\kappa\vee\text{Post}$ denote the **category of κ - \vee -posites**. Then $\kappa\vee\text{Post}$ is locally κ -presentable (see Appendix A.1). For $\kappa \leq \lambda$, we have a forgetful functor $\lambda\vee\text{Post} \rightarrow \kappa\vee\text{Post}$, whose left adjoint is given by regarding a κ - \vee -coverage as a λ - \vee -coverage as described above.

Regarding κ - \vee -lattices as κ - \vee -posites with the canonical κ - \vee -coverage, we have a full and faithful forgetful functor $\kappa\vee\text{Lat} \rightarrow \kappa\vee\text{Post}$, whose left adjoint is given by taking presented κ - \vee -lattices $(A, \triangleleft) \mapsto \langle A \mid \triangleleft \rangle_{\kappa\vee\text{Lat}} \cong \triangleleft\mathcal{I}_\kappa(A)$; thus $\kappa\vee\text{Lat}$ is a reflective subcategory of $\kappa\vee\text{Post}$. The adjunction unit $\nabla : (A, \triangleleft) \rightarrow \langle A \mid \triangleleft \rangle_{\kappa\vee\text{Lat}}$ “reflects covers”, i.e., $a \triangleleft C \iff \nabla a \triangleleft \nabla(C)$, which captures the “saturation” of \triangleleft as described above; \triangleleft is separated iff ∇ is also order-reflecting, hence an embedding of κ - \vee -posites.

Summarizing, we have the following commutative diagram of free/forgetful adjunctions:

$$(2.6.1) \quad \begin{array}{ccccccc} \text{Pos} & \xrightleftharpoons[\perp]{\mathcal{L}_\omega} & \vee\text{Lat} & \xrightleftharpoons[\perp]{\mathcal{I}_\sigma} & \sigma\vee\text{Lat} & \xrightleftharpoons[\perp]{\sigma\mathcal{I}_{\omega_2}} & \cdots & \xrightleftharpoons[\perp]{\perp} & \kappa\vee\text{Lat} & \xrightleftharpoons[\perp]{\sigma\mathcal{I}_\lambda} & \lambda\vee\text{Lat} & \xrightleftharpoons[\perp]{\lambda\mathcal{I}} & \vee\text{Lat} \\ \parallel & \begin{array}{c} \langle - \rangle_{\vee\text{Lat}} \uparrow \downarrow \\ \langle - \rangle_{\sigma\vee\text{Lat}} \uparrow \downarrow \end{array} & & \begin{array}{c} \langle - \rangle_{\sigma\vee\text{Lat}} \uparrow \downarrow \\ \langle - \rangle_{\kappa\vee\text{Lat}} \uparrow \downarrow \end{array} & & \begin{array}{c} \langle - \rangle_{\kappa\vee\text{Lat}} \uparrow \downarrow \\ \langle - \rangle_{\lambda\vee\text{Lat}} \uparrow \downarrow \end{array} & & \begin{array}{c} \langle - \rangle_{\lambda\vee\text{Lat}} \uparrow \downarrow \\ \langle - \rangle_{\vee\text{Lat}} \uparrow \downarrow \end{array} & & \begin{array}{c} \langle - \rangle_{\lambda\vee\text{Lat}} \uparrow \downarrow \\ \langle - \rangle_{\vee\text{Lat}} \uparrow \downarrow \end{array} & & \begin{array}{c} \langle - \rangle_{\vee\text{Lat}} \uparrow \downarrow \\ \langle - \rangle_{\vee\text{Lat}} \uparrow \downarrow \end{array} & \\ \text{Pos} & \xrightleftharpoons[\perp]{\perp} & \vee\text{Post} & \xrightleftharpoons[\perp]{\perp} & \sigma\vee\text{Post} & \xrightleftharpoons[\perp]{\perp} & \cdots & \xrightleftharpoons[\perp]{\perp} & \kappa\vee\text{Post} & \xrightleftharpoons[\perp]{\perp} & \lambda\vee\text{Post} & \xrightleftharpoons[\perp]{\perp} & \vee\text{Post} \end{array}$$

As before, the \rightarrow arrows denote full embeddings.

We now consider posites which present frames. Let \triangleleft be a κ - \vee -coverage on a \wedge -lattice¹¹ A , corresponding to a κ - \vee -lattice order-congruence \lesssim on $\mathcal{L}_\kappa(A)$ as described above. From Section 2.4, \lesssim is a κ -frame order-congruence iff it is \wedge -stable, which is easily seen to be equivalent to

¹¹For simplicity, we only consider posites based on a \wedge -lattice. There is a more general theory of “flat posites” based on arbitrary posets (but still presenting frames); see [Joh02, C1.1.16(e)], [Shu12].

- (\wedge -stability) $a \leq b \triangleleft C \implies a \triangleleft a \wedge C := \{a \wedge c \mid c \in C\}$.

Note that this (together with reflexivity and right-transitivity) implies left-transitivity. We call a \wedge -stable κ - \bigvee -coverage \triangleleft a **κ -coverage**, and call a κ - \bigvee -posite (A, \triangleleft) with $A \in \wedge\mathbf{Lat}$ and \wedge -stable \triangleleft a **κ -posite**. Thus a κ -posite (A, \triangleleft) can be seen as a κ -frame presentation which is also a κ - \bigvee -lattice presentation (a point of view due to Abramsky–Vickers [AV93]):

$$\begin{aligned} \langle A \mid \triangleleft \rangle_{\kappa\mathbf{Frm}} &:= \mathcal{L}_\kappa(A)/\sim = \langle A \text{ qua } \wedge\mathbf{Lat} \mid a \leq \bigvee C \text{ for } a \triangleleft C \rangle_{\kappa\mathbf{Frm}} \\ &= \langle A \text{ qua } \mathbf{Pos} \mid a \leq \bigvee C \text{ for } a \triangleleft C \rangle_{\kappa\bigvee\mathbf{Lat}} = \langle A \mid \triangleleft \rangle_{\kappa\bigvee\mathbf{Lat}}. \end{aligned}$$

By \wedge -stability and right-transitivity, a κ -coverage satisfies

$$a \triangleleft C \iff a \triangleleft a \wedge C,$$

hence is completely determined by its restriction to pairs (a, C) such that $C \subseteq \downarrow a$ (this is how coverages are usually presented in the literature). This restricted \triangleleft continues to satisfy reflexivity and \wedge -stability as stated (assuming all occurrences of \triangleleft obey the restriction), while right-transitivity is replaced with

- (right-transitivity') if $a \triangleleft C$, and $c \triangleleft c \wedge D$ for every $c \in C$, then $a \triangleleft D$.

As in the \bigvee -case, we also extend \triangleleft to (a, C) where $C \subseteq A$ is an arbitrary subset.

The largest separated κ -coverage on a \wedge -lattice A , called the **canonical κ -coverage**, is given by $a \triangleleft C$ (for κ -ary C) iff for every $b \leq a$, every upper bound of $b \wedge C$ is $\geq b$, i.e., $b = \bigvee(b \wedge C)$; if A is already a κ -frame, this just means $a \leq \bigvee C$ (by distributivity of $b \wedge \bigvee C$). This gives a full inclusion $\kappa\mathbf{Frm} \rightarrow \kappa\mathbf{Post}$ into the **category of κ -positives**, where by a homomorphism of κ -positives we mean a cover-preserving \wedge -lattice homomorphism. The left adjoint is given by taking presented κ -frames. We have the following commutative diagram of free/forgetful adjunctions, analogous to (2.6.1):

$$\begin{array}{ccccccccccc} \wedge\mathbf{Lat} & \xrightleftharpoons[\langle - \rangle_{\mathbf{DLat}}]{\mathcal{L}_\omega} & \mathbf{DLat} & \xrightleftharpoons[\langle - \rangle_{\sigma\mathbf{Frm}}]{\mathcal{I}_\sigma} & \sigma\mathbf{Frm} & \xrightleftharpoons[\langle - \rangle_{\kappa\mathbf{Frm}}]{\sigma\mathcal{I}_{\omega_2}} & \cdots & \xrightleftharpoons[\langle - \rangle_{\lambda\mathbf{Frm}}]{\kappa\mathcal{I}_\lambda} & \kappa\mathbf{Frm} & \xrightleftharpoons[\langle - \rangle_{\mathbf{Frm}}]{\lambda\mathcal{I}} & \lambda\mathbf{Frm} & \xrightleftharpoons[\langle - \rangle_{\mathbf{Post}}]{\mathbf{Frm}} & \mathbf{Frm} \\ \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ \wedge\mathbf{Lat} & \xrightleftharpoons[\omega\mathbf{Post}]{\omega} & \omega\mathbf{Post} & \xrightleftharpoons[\sigma\mathbf{Post}]{\sigma} & \sigma\mathbf{Post} & \xrightleftharpoons[\kappa\mathbf{Post}]{\kappa} & \cdots & \xrightleftharpoons[\lambda\mathbf{Post}]{\lambda} & \kappa\mathbf{Post} & \xrightleftharpoons[\mathbf{Post}]{\mathbf{Post}} & \lambda\mathbf{Post} & \xrightleftharpoons[\mathbf{Post}]{\mathbf{Post}} & \mathbf{Post} \end{array}$$

Moreover, this diagram “sits above” (2.6.1), in that for each κ , we have a square

$$(2.6.2) \quad \begin{array}{ccc} \kappa\bigvee\mathbf{Lat} & \xrightleftharpoons[\langle - \rangle_{\kappa\bigvee\mathbf{Lat}}]{\text{---}} & \kappa\mathbf{Frm} \\ \uparrow & & \uparrow \\ \kappa\bigvee\mathbf{Post} & \xrightleftharpoons[\langle - \rangle_{\kappa\bigvee\mathbf{Post}}]{\text{---}} & \kappa\mathbf{Post} \end{array}$$

in which not only do the forgetful functors commute, but also the two composites $\kappa\mathbf{Post} \rightarrow \kappa\bigvee\mathbf{Lat}$ agree.

We conclude this subsection with some simple consequences about presentations. First, translating Proposition 2.4.1, Corollary 2.4.2, and Proposition 2.4.3 about \lesssim to \triangleleft yields

Proposition 2.6.3. *Let A be a poset, \triangleleft_0 be a binary relation between A and $\mathcal{L}_\kappa(A)$ (or $\mathcal{P}_\kappa(A)$). Let \triangleleft be the smallest relation which satisfies reflexivity, left-transitivity, and “right-transitivity with respect to \triangleleft_0 ”, i.e., if $a \triangleleft_0 C$, and $c \triangleleft D$ for every $c \in C$, then $a \triangleleft D$. Then \triangleleft satisfies right-transitivity, hence is the κ - \bigvee -coverage generated by \triangleleft_0 . \square*

Corollary 2.6.4. *Let A be a poset, \triangleleft_0 be a binary relation between A and $\mathcal{L}_\kappa(A)$ (or $\mathcal{P}_\kappa(A)$), \triangleleft be the κ - \bigvee -coverage generated by \triangleleft_0 . Then a lower set $D \in \mathcal{L}(A)$ is a \triangleleft -ideal iff it is a \triangleleft_0 -ideal. \square*

Proposition 2.6.5. *Let A be a \wedge -lattice, \triangleleft_0 be a \wedge -stable binary relation between A and $\mathcal{L}_\kappa(A)$ (or $\mathcal{P}_\kappa(A)$). Then the κ - \bigvee -coverage \triangleleft generated by \triangleleft_0 is still \wedge -stable, hence is also the κ -coverage generated by \triangleleft_0 . \square*

Next, any presentation of a κ - \bigvee -lattice or κ -frame can be canonically turned into a κ -(\bigvee)-posite. For suppose $A = \langle G \mid R \rangle_{\kappa\mathbf{VLat}}$ is a κ - \bigvee -lattice presentation, where G is a set or more generally a poset, and R is a set of inequalities between κ - \bigvee -lattice terms. This means that we have a quotient map $\langle G \rangle = \mathcal{L}_\kappa(G) \twoheadrightarrow A$ whose order-kernel \lesssim is generated by the pairs $(B, C) \in \mathcal{L}_\kappa(G)^2$ which are the “normal forms” of the terms (s, t) in each inequality $s \leq t$ in R . We may replace $s \leq t$ with the inequalities $s_i \leq t$ whenever s is a κ -ary join term $s = \bigvee_i s_i$, until we are left with inequalities $b \leq t$ where $b \in G$; thus \lesssim is also generated by pairs $(\downarrow b, C)$. This means that the κ - \bigvee -coverage \triangleleft corresponding to \lesssim , so that $A \cong \langle G \mid \triangleleft \rangle_{\kappa\mathbf{VLat}}$, is generated by the pairs (b, C) .

If the original presentation (G, R) was λ -ary for some $\lambda \geq \kappa$, then so will be the new one. Moreover, it is a general algebraic fact (see Corollary A.2.2) that a λ -presented algebraic structure is λ -presented using any given λ -ary set of generators. We thus have

Proposition 2.6.6. *For $\lambda \geq \kappa$, a κ - \bigvee -lattice A is λ -presented iff it is presented by some λ -presented κ - \bigvee -posite, i.e., $A \cong \langle G \mid \triangleleft \rangle_{\kappa\mathbf{VLat}}$ for some λ -ary poset G and λ -generated κ - \bigvee -coverage \triangleleft .*

Moreover, in that case, we can find such \triangleleft for any λ -ary poset G and monotone $G \rightarrow A$ whose image generates A . \square

If $A = \langle G \mid R \rangle_{\kappa\mathbf{Frm}}$ is a κ -frame presentation, by replacing G with $\langle G \rangle_{\wedge\mathbf{Lat}}$, we can suppose $A = \langle G \text{ qua } \wedge\mathbf{Lat} \mid R \rangle_{\kappa\mathbf{Frm}}$, so that the quotient $\mathcal{L}_\kappa(G) \twoheadrightarrow A$ is a κ -frame quotient, whence its order-kernel \lesssim is the κ -frame order-congruence generated by R . By applying the distributive law, we may turn each inequality in R into one between joins of finite meets, and then evaluate the finite meets in G , so that we are left with κ - \bigvee -lattice inequalities $s \leq t$, corresponding to generators (B, C) for \lesssim . As above, we may reduce to inequalities $b \leq t$ with $b \in G$, corresponding to generators (b, C) for the κ -coverage \triangleleft . We may furthermore close R under \wedge -stability, so that the generators (b, C) for \triangleleft become \wedge -stable as well, and so by Proposition 2.6.5 they also generate \triangleleft as a κ - \bigvee -coverage.

Clearly these modifications also preserve λ -presentability for $\lambda \geq \kappa$, yielding

Proposition 2.6.7. *For $\lambda \geq \kappa$, a κ -frame A is λ -presented iff it is presented by some λ -presented κ -posite, i.e., $A \cong \langle G \mid \triangleleft \rangle_{\kappa\mathbf{Frm}}$ for some λ -ary \wedge -lattice G and λ -generated κ -coverage \triangleleft .*

Moreover, in that case, \triangleleft is λ -generated also as a κ - \bigvee -coverage, and G can be any λ -ary \wedge -lattice with a \wedge -homomorphism $G \rightarrow A$ whose image generates A . \square

Since $\langle G \mid \triangleleft \rangle_{\kappa\mathbf{Frm}} = \langle G \mid \triangleleft \rangle_{\kappa\mathbf{VLat}}$, this yields the aforementioned (in Section 2.3)

Corollary 2.6.8. *A κ -frame is λ -presented iff it is λ -presented as a κ - \bigvee -lattice. \square*

2.7 Colimits of frames

In this short subsection, we record some structural facts about colimits of κ -frames. These facts are well-known for $\kappa = \infty$ (see [JT84]); the same proofs can be easily adapted to work when $\kappa < \infty$, although we find it more natural to use a “presentational” approach based on posites.¹²

Consider the computation of the colimit $\varinjlim F = \varinjlim_{\kappa\mathbf{Post}} F$ of an arbitrary diagram $F : \mathbb{I} \rightarrow \kappa\mathbf{Post}$, where \mathbb{I} is a small indexing category. By the general abstract method for presenting colimits of structures axiomatized by universal Horn theories (see Appendix A.1), $\varinjlim F$ may be computed by taking the colimit of the underlying \wedge -lattices, then taking the image of the κ -coverage $\triangleleft_{F(I)}$ on each $F(I)$ under the cocone map $\iota_I : F(I) \rightarrow \varinjlim F$, and finally taking the κ -coverage $\triangleleft_{\varinjlim F}$ on $\varinjlim F$ generated by all of these images. An analogous description holds for the colimit $\varinjlim_{\kappa\mathbf{VPost}} F$ of F (composed with the forgetful functor) computed in $\kappa\mathbf{VPost}$.

Suppose the diagram F is such that the colimit of the underlying \wedge -lattices is preserved by the forgetful functor $\wedge\mathbf{Lat} \rightarrow \mathbf{Pos}$. For example, this holds if $\varinjlim F$ is a directed colimit (i.e., \mathbb{I} is a directed preorder). Then the first part of the computation of $\varinjlim_{\kappa\mathbf{Post}} F$ agrees with that of $\varinjlim_{\kappa\mathbf{VPost}} F$. Clearly, \wedge -stability of each $\triangleleft_{F(I)}$ implies \wedge -stability of the union of their images in $\varinjlim F$, which by Proposition 2.6.5 therefore also generates $\triangleleft_{\varinjlim F}$ as a $\kappa\mathbf{V}$ -coverage; thus the last part of the computation of $\varinjlim_{\kappa\mathbf{Post}} F$ is also preserved in $\kappa\mathbf{VPost}$. We thus have

Proposition 2.7.1. *The forgetful functor $\kappa\mathbf{Post} \rightarrow \kappa\mathbf{VPost}$ preserves directed colimits.* □

Now for a diagram $F : \mathbb{I} \rightarrow \kappa\mathbf{Frm}$, its colimit $\varinjlim_{\kappa\mathbf{Frm}} F$ is given by the reflection $\langle \varinjlim_{\kappa\mathbf{Post}} F \rangle_{\kappa\mathbf{Frm}}$ of the colimit in $\kappa\mathbf{Post}$, which is the same as $\langle \varinjlim_{\kappa\mathbf{Post}} F \rangle_{\kappa\mathbf{VLat}}$ (see the square (2.6.2)); while the colimit $\varinjlim_{\kappa\mathbf{VLat}} F$ in $\kappa\mathbf{VLat}$ is given by the reflection $\langle \varinjlim_{\kappa\mathbf{VPost}} F \rangle_{\kappa\mathbf{VLat}}$. Thus

Corollary 2.7.2. *The forgetful functor $\kappa\mathbf{Frm} \rightarrow \kappa\mathbf{VLat}$ preserves directed colimits.* □

For a directed diagram $F : \mathbb{I} \rightarrow \kappa\mathbf{VLat}$, the following gives us some control over the cocone maps $\iota_I : F(I) \rightarrow \varinjlim F$. For $I \leq J \in \mathbb{I}$, we write $F(I, J)$ for F applied to the unique $I \rightarrow J \in \mathbb{I}$.

Proposition 2.7.3. *Let $F : \mathbb{I} \rightarrow \kappa\mathbf{VLat}$ be a directed diagram, $I \in \mathbb{I}$, $a \in F(I)$. Suppose that for each $I \leq J \leq K \in \mathbb{I}$, $F(I, J)(a) \in F(J)$ is the greatest element of its $\text{oker}(F(J, K))$ -principal ideal (i.e., for all $b \in F(J)$, $b \leq F(I, J)(a) \iff F(J, K)(b) \leq F(I, K)(a)$). Then a is the greatest element of its $\text{oker}(\iota_I)$ -principal ideal.*

In particular, if each $F(I, J) : F(I) \rightarrow F(J)$ is injective, then so is each $\iota_I : F(I) \rightarrow \varinjlim F$.

Proof. By replacing \mathbb{I} with $\uparrow I$ (which is final in \mathbb{I}), we may assume I is the least element of \mathbb{I} . For each $J \in \mathbb{I}$, let $f_J : F(J) \rightarrow 2$ be the indicator function of $F(J) \setminus \downarrow F(I, J)(a)$. Then the assumptions ensure $(f_J)_J$ is a cocone over F , hence induce a homomorphism $f : \varinjlim F \rightarrow 2$ such that $f \circ \iota_I = f_I$. Then $\iota_I(b) \leq \iota_I(a) \implies f_I(b) \leq f_I(a) = 0 \implies b \in \downarrow a$. □

We also briefly recall the construction of coproducts of frames via posites. First, for \wedge -lattices A, B , the product $A \times B$ is a biproduct (“direct sum”) in $\wedge\mathbf{Lat}$, hence also the coproduct, with injections $\iota_1 : A \rightarrow A \times B$ given by $\iota_1(a) := (a, \top)$ and ι_2 given similarly.

¹²Such a “presentational” approach is usually used to prove the analogous results for Grothendieck toposes; see e.g., [Joh02, C2.5.14].

Now for κ -frames A, B , their \wedge -lattice coproduct $A \times B$ generates their κ -frame coproduct, whence the latter is presented by a κ -coverage \triangleleft on $A \times B$. Namely, \triangleleft imposes the relations that κ -ary joins in A must be preserved, as must be κ -ary joins in B , hence is generated by the pairs

$$(2.7.4) \quad \begin{aligned} (\bigvee C, b) &\triangleleft \{(c, b) \mid c \in C\} \quad \text{for } \kappa\text{-ary } C \subseteq A, \\ (a, \bigvee C) &\triangleleft \{(a, c) \mid c \in C\} \quad \text{for } \kappa\text{-ary } C \subseteq B, \end{aligned}$$

which are clearly \wedge -stable. Thus by Proposition 2.6.5 and Corollary 2.6.4, the κ -frame coproduct of A, B may be realized as the set of $D \in \mathcal{L}(A \times B)$ which are κ -generated ideals with respect to these generators for \triangleleft , i.e., D must be “closed under κ -ary joins separately in each coordinate”, and D must also be κ -generated as such.

As is standard, we denote the κ -frame coproduct of A, B by $A \otimes B$ (in addition to the category-agnostic notation $A \amalg B$). For $a \in A$ and $b \in B$, we write

$$a \times b := \iota_1(a) \wedge \iota_2(b) \in A \otimes B$$

(this is more commonly denoted $a \otimes b$); an arbitrary element of $A \otimes B$ is thus a κ -ary join of such “rectangles”. Under the above realization of $A \otimes B$ as all $D \subseteq \mathcal{L}(A \times B)$ closed under (2.7.4), it is easily seen that

$$a \times b = \downarrow(a, b) \cup (A \times \{\perp\}) \cup (\{\perp\} \cup B) \subseteq A \times B.$$

Corollary 2.7.5. *Let A, B be κ -frames, $a, a' \in A$, and $b, b' \in B$. If $a \times b \leq a' \times b' \in A \otimes B$, then either $a \leq a'$ and $b \leq b'$, or $a = \perp$, or $b = \perp$.*

In particular, if $a \times b = \perp$, then either $a = \perp$ or $b = \perp$.

Proof. If $a \times b \leq a' \times b'$, then $(a, b) \in \downarrow(a', b') \cup (A \times \{\perp\}) \cup (\{\perp\} \cup B) \subseteq A \times B$. □

See Corollary 2.12.2 below for a generalization of this to (λ, κ) -frames.

2.8 Adjoining complements

In this subsection, we discuss the process of completing a κ -frame to a κ -Boolean algebra by repeatedly adjoining complements. This is one of the central constructions of classical locale theory (known variously as the “frame of nuclei”, “dissolution locale”, or “assembly tower”; see e.g., [Joh82, II 2.5–10], [Wil94], [PI00a]), and corresponds to the Borel hierarchy in descriptive set theory (see Section 3.3 below). The case $\kappa < \infty$ was studied by Madden [Mad91]. One feature of our approach, which is new as far as we know, is an explicit posite presentation of the frame of nuclei (Proposition 2.8.2). At the end of the subsection, we also discuss the interaction between adjoining complements and κ -presentability.

For a κ -frame A , we let $\mathcal{N}_\kappa(A)$ denote the κ -frame resulting from freely adjoining a complement for each element of A :

$$\mathcal{N}_\kappa(A) := \langle A \text{ qua } \kappa\text{Frm}, \neg a \text{ for } a \in A \mid a \wedge \neg a = \perp, a \vee \neg a = \top \rangle_{\kappa\text{Frm}}.$$

Thus $\mathcal{N}_\kappa(A)$ is the universal κ -frame equipped with a homomorphism $\eta : A \rightarrow \mathcal{N}_\kappa(A)$ mapping each $a \in A$ to a complemented element. There is an obvious extension of \mathcal{N}_κ to a functor $\kappa\text{Frm} \rightarrow \kappa\text{Frm}$, such that η becomes a natural transformation $1_{\kappa\text{Frm}} \rightarrow \mathcal{N}_\kappa$. For a frame A , we write $\mathcal{N}(A) := \mathcal{N}_\infty(A)$.

By distributivity, every element of $\mathcal{N}_\kappa(A)$ can be written as a κ -ary join

$$(2.8.1) \quad \bigvee_i (a_i \wedge \neg b_i) \quad \text{for } a_i, b_i \in A$$

(where we are writing a_i instead of $\eta(a_i)$; see Convention 2.8.4 below). Thus $\mathcal{N}_\kappa(A)$ is generated as a κ - \bigvee -lattice by the image of the \wedge -lattice homomorphism

$$\begin{aligned} A \times \neg A &\longrightarrow \mathcal{N}_\kappa(A) \\ (a, \neg b) &\longmapsto a \wedge \neg b \end{aligned}$$

where $\neg A := A^{\text{op}}$ with elements written $\neg a$ for $a \in A$. The order-kernel of this map corresponds to a κ -coverage \triangleleft on $A \times \neg A$ such that $\mathcal{N}_\kappa(A) = \langle A \times \neg A \mid \triangleleft \rangle_{\kappa\text{Frm}}$.

Proposition 2.8.2. *The κ -coverage \triangleleft such that $\mathcal{N}_\kappa(A) = \langle A \times \neg A \mid \triangleleft \rangle_{\kappa\text{Frm}}$ is generated by the following pairs, which are \wedge -stable and hence also generate \triangleleft as a κ - \bigvee -coverage (by Proposition 2.6.5):*

- (i) $(\bigvee B, \neg c) \triangleleft \{(b, \neg c) \mid b \in B\}$ for κ -ary $B \subseteq A$ and $c \in A$,
- (ii) $(a, \neg b) \triangleleft \emptyset$ for $a \leq b \in A$,
- (iii) $(c, \neg b) \triangleleft \{(a, \neg b), (c, \neg d)\}$ for $a \vee b \geq c \wedge d \in A$.

Proof. First, we check that these pairs are indeed part of \triangleleft , i.e., that the corresponding relations hold in $\mathcal{N}_\kappa(A)$. Pair (i) says $(\bigvee B) \wedge \neg c \leq \bigvee_{b \in B} (b \wedge \neg c)$, while (ii) says $a \wedge \neg b \leq \perp$ if $a \leq b$; these are clearly true. Pair (iii) says $c \wedge \neg b \leq (a \wedge \neg b) \vee (c \wedge \neg d)$ if $a \vee b \geq c \wedge d$; indeed, we have

$$\begin{aligned} b \vee (a \wedge \neg b) \vee (c \wedge \neg d) &= b \vee a \vee (c \wedge \neg d) \\ &\geq (c \wedge d) \vee (c \wedge \neg d) = c, \end{aligned}$$

whence $(a \wedge \neg b) \vee (c \wedge \neg d) \geq c \wedge \neg b$.

From the presentation defining $\mathcal{N}_\kappa(A)$, it is clear that \triangleleft is generated by the pairs

- (i') $(\bigvee B, \neg \perp) \triangleleft \{(b, \neg \perp) \mid b \in B\},$
- (ii') $(a, \neg a) \triangleleft \emptyset,$
- (iii') $(\top, \neg \perp) \triangleleft \{(a, \neg \perp), (\top, \neg a)\},$

corresponding to the relations which say that (i') κ -ary joins in A are preserved, (ii') $a \wedge \neg a \leq \perp$, and (iii') $\top \leq a \vee \neg a$.

It is easily seen that the closure of (i') and (ii') under \wedge -stability yield (i) and (ii) respectively. For (iii'), taking the meet with an arbitrary $(c, \neg b) \in A \times \neg A$ yields $(c, \neg b) \triangleleft \{(a \wedge c, \neg b), (c, \neg(a \vee b))\}$, which is of the form (iii). Thus (i–iii) are included in \triangleleft and include the generators (i'–iii') of \triangleleft , and so generate \triangleleft . It remains to check that (iii) is \wedge -stable: taking meet with $(e, \neg f) \in A \times \neg A$ yields

$$(c \wedge e, \neg(b \vee f)) \triangleleft \{(a \wedge e, \neg(b \vee f)), (c \wedge e, \neg(d \vee f))\}$$

which is of the form (iii) since

$$\begin{aligned} (a \wedge e) \vee b \vee f &= ((a \vee b) \wedge (e \vee b)) \vee f \\ &\geq (c \wedge d \wedge (e \vee b)) \vee f \\ &\geq (c \wedge e \wedge d) \vee f \\ &\geq c \wedge e \wedge (d \vee f). \end{aligned}$$

□

It follows by Corollary 2.6.4 that $\mathcal{N}_\kappa(A)$ may be realized as the subset of $D \in \mathcal{L}(A \times \neg A)$ which are ideals with respect to the generators (i–iii) for \triangleleft from Proposition 2.8.2, and which are also κ -generated as such. In other words, (i–iii) describe precisely the saturation conditions characterizing uniqueness of the expressions (2.8.1): two such expressions denote the same element of $\mathcal{N}_\kappa(A)$ iff their sets of $(a_i, \neg b_i)$ agree when closed downward and under (i–iii). Using this, we have

Proposition 2.8.3. $\eta : A \rightarrow \mathcal{N}_\kappa(A)$ is injective.

Proof. $\eta(a) = a \wedge \neg \perp$, regarded as a \triangleleft -ideal in $A \times \neg A$ for \triangleleft given by Proposition 2.8.2, is the principal \triangleleft -ideal $\nabla(a, \neg \perp)$. We have $\nabla(a, \neg \perp) = \{(b, \neg c) \in A \times \neg A \mid b \leq a \vee c\}$, since this lower set contains $(a, \neg \perp)$ and is easily seen to be the smallest such closed under (i–iii) in Proposition 2.8.2. Thus $\eta(a) \leq \eta(b) \iff \nabla(a, \neg \perp) \subseteq \nabla(b, \neg \perp) \iff (a, \neg \perp) \in \nabla(b, \neg \perp) \iff a \leq b \vee \perp$. \square

Note that by definition of $\mathcal{N}_\kappa(A)$, η is surjective, hence an isomorphism, iff every $a \in A$ already has a complement, i.e., A is a κ -Boolean algebra. We henceforth adopt the following

Convention 2.8.4. For any κ -frame A , we regard $\eta : A \hookrightarrow \mathcal{N}_\kappa(A)$ as an inclusion.

Proposition 2.8.2 also shows that the same κ -coverage \triangleleft , regarded as an ∞ -coverage, presents the free frame generated by A qua κ -frame together with complements for elements of A , i.e., the frame $\kappa\mathcal{I}(\mathcal{N}_\kappa(A))$. This frame can thus also be realized as the set of all \triangleleft -ideals $D \subseteq A \times \neg A$. As an application, we give a simple direct proof of the following result of Madden [Mad91, 5.1] (due to Isbell [Isb72, 1.3] when $\kappa = \infty$), which was originally proved via a more abstract method (see also Remark 3.4.10):

Proposition 2.8.5 (Madden). *For any κ -frame A , we have order-isomorphisms*

$$\begin{aligned} \kappa\mathcal{I}(\mathcal{N}_\kappa(A)) &\cong \{\kappa\text{-frame order-congruences } \lesssim \subseteq A^2\}, \\ \mathcal{N}_\kappa(A) &\cong \{\kappa\text{-generated } \kappa\text{-frame order-congruences } \lesssim \subseteq A^2\} \\ c &\mapsto \{(a, b) \mid a \wedge \neg b \leq c\} \\ \bigvee_{a \lesssim b} (a \wedge \neg b) &\leftarrow \lesssim \end{aligned}$$

where the join on the last line can be taken over any generating set for \lesssim .

Proof. The first isomorphism is obtained by regarding a \triangleleft -ideal $D \subseteq A \times \neg A$ as a subset of $A \times A^{\text{op}} \cong A \times \neg A$, hence as a binary relation \lesssim on A . For D to be lower means that $a \leq b \lesssim c \leq d \implies a \leq d$. For D to be closed under (i–iii) in Proposition 2.8.2 mean that (i) joins in A are joins with respect to \lesssim , (ii) \lesssim contains \leq , and (iii) $a \lesssim b$, $c \lesssim d$, and $c \wedge d \leq a \vee b$ imply $c \lesssim b$; (iii) implies \lesssim is transitive by taking $a = d$ and \wedge -stable by taking $a = b = c' \wedge b'$, $c = c' \wedge a'$, and $d = b'$, while (iii) is also implied by these two conditions on \lesssim (along with the others), since from $c \lesssim d$, \wedge -stability, and (i), we have $c \lesssim c \wedge d \leq a \vee b \lesssim b$ whence $c \lesssim b$ by transitivity. Thus D is a \triangleleft -ideal iff \lesssim is an order-congruence.

It is clear from this correspondence that a join $\bigvee_i (a_i \wedge \neg b_i)$ in $\mathcal{N}_\kappa(A)$ or in $\kappa\mathcal{I}(\mathcal{N}_\kappa(A))$, with $a_i, b_i \in A$, corresponds to the \triangleleft -ideal generated by the pairs $(a_i, \neg b_i)$, which corresponds to the order-congruence \lesssim generated by $a_i \lesssim b_i$. In particular, a κ -ary such join corresponds to a κ -generated order-congruence, yielding the second isomorphism. \square

Now define inductively for each ordinal α

$$\begin{aligned}\mathcal{N}_\kappa^0(A) &:= A, \\ \mathcal{N}_\kappa^{\alpha+1}(A) &:= \mathcal{N}_\kappa(\mathcal{N}_\kappa^\alpha(A)), \\ \mathcal{N}_\kappa^\alpha(A) &:= \varinjlim_{\beta < \alpha} \mathcal{N}_\kappa^\beta(A) \quad \text{for } \alpha \text{ limit,}\end{aligned}$$

where the colimit is taken in $\kappa\mathbf{Frm}$ with respect to the inclusions $\mathcal{N}_\kappa^\beta(A) \hookrightarrow \mathcal{N}_\kappa^{\beta+1}(A) \hookrightarrow \dots$. By induction, using Proposition 2.8.3 for the successor stages and Proposition 2.7.3 for the limit stages, for $\alpha \leq \beta$, we have an injective homomorphism $\eta_{\alpha\beta} : \mathcal{N}_\kappa^\alpha(A) \rightarrow \mathcal{N}_\kappa^\beta(A)$. Since the operations of κ -frames are κ -ary, the forgetful functor $\kappa\mathbf{Frm} \rightarrow \mathbf{Set}$ preserves κ -directed colimits; thus every element of $\mathcal{N}_\kappa^\kappa(A)$ belongs to some earlier stage, and so already has a complement in $\mathcal{N}_\kappa^\kappa(A)$. Thus

$$\mathcal{N}_\kappa^\kappa(A) = \mathcal{N}_\kappa^\infty(A) = \langle A \text{ qua } \kappa\mathbf{Frm} \rangle_{\kappa\mathbf{Bool}}.$$

Corollary 2.8.6. *The unit $\eta = \eta_{0\kappa} = \eta_{0\infty} : A \rightarrow \mathcal{N}_\kappa^\infty(A) = \langle A \text{ qua } \kappa\mathbf{Frm} \rangle_{\kappa\mathbf{Bool}}$ of the free/forgetful adjunction $\kappa\mathbf{Frm} \rightleftarrows \kappa\mathbf{Bool}$ is injective. Equivalently, the left adjoint $\mathcal{N}_\kappa^\infty = \langle - \rangle_{\kappa\mathbf{Bool}}$ is faithful. \square*

This allows us to transfer results from Section 2.7 about colimits in $\kappa\mathbf{Frm}$ to $\kappa\mathbf{Bool}$:

Corollary 2.8.7. *Let $F : \mathbf{I} \rightarrow \kappa\mathbf{Bool}$ be a directed diagram, $I \in \mathbf{I}$, $a \in F(I)$. Suppose that for each $I \leq J \leq K \in \mathbf{I}$, $F(I, J)(a) \in F(J)$ is the greatest element of its $\text{oker}(F(J, K))$ -principal ideal. Then a is the greatest element of its $\text{oker}(\iota_I)$ -principal ideal.*

In particular, if each $F(I, J) : F(I) \rightarrow F(J)$ is injective, then so is each $\iota_I : F(I) \rightarrow \varinjlim F$.

Proof. The colimit $\varinjlim_{\kappa\mathbf{Bool}} F$ in $\kappa\mathbf{Bool}$ is the reflection $\mathcal{N}_\kappa^\infty(\varinjlim_{\kappa\mathbf{Frm}} F)$ of the colimit in $\kappa\mathbf{Frm}$; the result thus follows from Proposition 2.7.3, Corollary 2.7.2, and Corollary 2.8.6. \square

Corollary 2.8.8. *Let A, B be κ -Boolean algebras, $a, a' \in A$, and $b, b' \in B$. If $a \times b \leq a' \times b'$ in the coproduct $A \amalg B$ of A, B in $\kappa\mathbf{Bool}$, then either $a \leq a'$ and $b \leq b'$, or $a = \perp$, or $b = \perp$.*

In particular, if $a \times b = \perp$, then either $a = \perp$ or $b = \perp$.

Proof. This similarly follows from Corollary 2.7.5 and Corollary 2.8.6. \square

We can also use \mathcal{N}_κ to construct the free functor $\langle - \rangle_{\lambda\mathbf{Bool}} : \kappa\mathbf{Bool} \rightarrow \lambda\mathbf{Bool}$ for $\kappa \leq \lambda$. Indeed, since $\kappa\mathbf{Bool} \subseteq \kappa\mathbf{Frm}$ is a reflective subcategory,

$$\begin{aligned}\langle A \text{ qua } \kappa\mathbf{Bool} \rangle_{\lambda\mathbf{Bool}} &\cong \langle \langle A \text{ qua } \kappa\mathbf{Frm} \rangle_{\kappa\mathbf{Bool}} \rangle_{\lambda\mathbf{Bool}} \\ &\cong \langle \langle A \text{ qua } \kappa\mathbf{Frm} \rangle_{\lambda\mathbf{Frm}} \rangle_{\lambda\mathbf{Bool}} \\ &\cong \mathcal{N}_\lambda^\infty(\kappa\mathcal{I}_\lambda(A))\end{aligned}$$

(see the diagram (2.1.1)). Since the units for $\kappa\mathcal{I}_\lambda$ and $\mathcal{N}_\lambda^\infty$ are both injective,

Corollary 2.8.9. *For $\kappa \leq \lambda$, the unit of the free/forgetful adjunction $\kappa\mathbf{Bool} \rightleftarrows \lambda\mathbf{Bool}$ is injective. Equivalently, the left adjoint $\langle - \rangle_{\lambda\mathbf{Bool}}$ is faithful. \square*

Remark 2.8.10. The preceding result is also easily seen “semantically”, via MacNeille completion. On the other hand, Corollary 2.11.5 below yields yet another, purely “syntactic”, proof.

Remark 2.8.11. For $\kappa \leq \lambda$, unlike the free/forgetful adjunctions $\kappa\mathbf{Frm} \rightleftarrows \lambda\mathbf{Frm}$ and $\kappa\mathbf{VLat} \rightleftarrows \lambda\mathbf{VLat}$ (see Corollary 2.3.4), the free functor $\kappa\mathbf{Bool} \rightarrow \lambda\mathbf{Bool}$ is *not* full on isomorphisms. However, we will see below (Corollary 2.12.11) that it is at least conservative.

By Corollaries 2.8.6 and 2.8.9, every κ -frame or κ -Boolean algebra is embedded in the free (large) complete Boolean algebra it generates. We now extend Convention 2.8.4 to

Convention 2.8.12. For any κ -frame, κ -coframe, or κ -Boolean algebra A , we regard A as a subalgebra of the free complete Boolean algebra it generates. That is, we regard the unit $A \hookrightarrow \langle A \rangle_{\mathbf{CBool}}$ as an inclusion. Hence we also regard all of the intermediate units $A \hookrightarrow \langle A \rangle_{\lambda\mathbf{Frm}} = \kappa\mathcal{I}_\lambda(A)$ for $\lambda \geq \kappa$, $A \hookrightarrow \mathcal{N}_\kappa^\alpha(A)$, etc., as inclusions.

In particular, we adopt the (uncommon in locale theory) convention that **from now on, unless otherwise specified, all infinitary Boolean operations take place in $\langle A \rangle_{\mathbf{CBool}}$** . Thus for example, for elements a_i in a frame A , $\bigwedge_i a_i$ will by default no longer refer to their meet in the complete lattice A , but will instead refer to their meet in $\langle A \rangle_{\mathbf{CBool}}$. Similarly, \rightarrow, \neg will never be used to denote the Heyting implication or pseudocomplement in a frame A , but will instead denote Boolean implication or complement in $\langle A \rangle_{\mathbf{CBool}}$. Note that for $a, b, a_i \in A$, we have $a \rightarrow b, \neg a \in \mathcal{N}(A) \subseteq \mathcal{N}^\infty(A) = \langle A \rangle_{\mathbf{CBool}}$; thus $\bigwedge_i a_i = \neg \bigvee_i \neg a_i \in \mathcal{N}^2(A) \subseteq \mathcal{N}^\infty(A)$.

We also extend this convention to (λ, κ) -frames, once we know they also embed into their free complete Boolean algebras; see Corollary 2.11.5.

We conclude this subsection by showing that κ -Boolean algebras can be presented by “enough” κ -frames. First, from general categorical principles, we have

Proposition 2.8.13. *For every κ -presented κ -Boolean algebra B , there is a κ -presented κ -frame A such that $\mathcal{N}_\kappa^\infty(A) = \langle A \rangle_{\kappa\mathbf{Bool}} \cong B$.*

Proof. If $\kappa = \omega$, then B is finite, so we may take $A = B$. For uncountable κ , this follows from reflectivity and local κ -presentability of $\kappa\mathbf{Bool} \subseteq \kappa\mathbf{Frm}$; see Proposition A.1.6. \square

Remark 2.8.14. By Corollary 2.8.6, the conclusion above implies that (up to isomorphism) $A \subseteq B$ is a κ -subframe generating B as a κ -Boolean algebra. However, this condition is not sufficient for the conclusion to hold: for example, the topology $\mathcal{O}(2^\mathbb{N})$ of Cantor space $2^\mathbb{N}$ is a subframe of $\mathcal{P}(2^\mathbb{N})$ generating it as a complete Boolean algebra, but not freely (e.g., by the Gaifman–Hales Theorem 2.1.2, since $\mathcal{O}(2^\mathbb{N})$ is the ideal completion of the algebra of clopen sets in $2^\mathbb{N}$ which is the free Boolean algebra on \mathbb{N} , whence $\langle \mathcal{O}(2^\mathbb{N}) \text{ qua } \mathbf{Frm} \rangle_{\mathbf{CBool}} \cong \langle \mathbb{N} \text{ qua } \mathbf{Set} \rangle_{\mathbf{CBool}}$).

Unravelling the proof of Proposition 2.8.13 yields the following refined statement:

Proposition 2.8.15. *Let B be a κ -presented κ -Boolean algebra which is freely generated by a κ -subframe $A \subseteq B$. For any κ -ary subset $C \subseteq A$, there is a κ -presented κ -subframe $A' \subseteq A$ which contains C and still freely generates B .*

Proof. If $\kappa = \omega$, this is again trivial (take $A' := A$), so suppose κ is uncountable. As in the proof of Proposition A.1.6, write A as a κ -directed colimit $\varinjlim_{i \in I} A_i$ of κ -presented frames; we may assume that each $A_i \rightarrow A$ has image containing C (by taking the G_i in that proof to contain C). Then $\varinjlim_i \mathcal{N}_\kappa^\infty(A_i) \cong \mathcal{N}_\kappa^\infty(\varinjlim_i A_i) = B$, so as in that proof, we have $B \cong \varinjlim_j \mathcal{N}_\kappa^\infty(A_{i_j}) \cong \mathcal{N}_\kappa^\infty(A')$ where $A' := \varinjlim_j A_{i_j}$ for some $i_0 \leq i_1 \leq \dots$. From that proof, the unit $A' \rightarrow \mathcal{N}_\kappa^\infty(A') \cong B$ is the colimit comparison $A' \rightarrow A \subseteq B$, and is injective (Corollary 2.8.6), whence up to isomorphism, A' is a κ -presented κ -subframe of $A \subseteq B$, which contains C since the image of each $A_{i_j} \rightarrow A$ does. \square

Corollary 2.8.16. *Let A be a κ -presented κ -frame. For any α and κ -ary subset $C \subseteq \mathcal{N}_\kappa^\alpha(A)$, there is a κ -presented κ -subframe $A' \subseteq \mathcal{N}_\kappa^\alpha(A)$ such that $A \cup C \subseteq A'$ and $\mathcal{N}_\kappa^\infty(A) \cong \mathcal{N}_\kappa^\infty(A')$ (canonically).*

Proof. Take A in Proposition 2.8.15 to be $\mathcal{N}_\kappa^\alpha(A)$, and C to be the present C together with a κ -ary generating set for A . \square

Remark 2.8.17. If $C \subseteq \mathcal{N}_\kappa^\alpha(A)$ in Corollary 2.8.16 consists of complemented elements, then we may ensure they become complemented in A' , by adding the complements to C .

2.9 Zero-dimensionality and ultraparacompactness

In this subsection, which is somewhat technical and will not be used in an essential way until Section 4.3, we study κ -frames which have “enough” complemented elements, as well as a more subtle infinitary refinement of such a condition. The main results are towards the end of the subsection, concerning the connection of such frames with κ -Boolean algebras (Corollary 2.9.13 and Proposition 2.9.17) as well as with the \mathcal{N}_κ functor from the preceding subsection (Theorem 2.9.19). Many of these results are known for $\kappa = \infty$, due to work of Paseka [Pas92] and Plewe [Pl02]. However, in generalizing to $\kappa < \infty$, we once again follow a “presentational” approach (which was essentially introduced by van Name [vN13] when $\kappa = \infty$). Here, our approach yields proofs which are entirely different in spirit from the known ones for $\kappa = \infty$, and which (we believe) make the underlying algebraic ideas quite transparent.

For a κ -frame A , let $A_\perp \subseteq A$ denote the Boolean subalgebra of complemented elements. We call A **zero-dimensional** if it is generated by A_\perp , and let $\kappa\mathbf{ZFrm} \subseteq \kappa\mathbf{Frm}$ denote the full subcategory of zero-dimensional κ -frames.

It is easily seen that $\kappa\mathbf{ZFrm} \subseteq \kappa\mathbf{Frm}$ is a coreflective subcategory, with the coreflection of $A \in \kappa\mathbf{Frm}$ being the κ -subframe generated by A_\perp . We may view this coreflective adjunction $\kappa\mathbf{ZFrm} \rightleftarrows \kappa\mathbf{Frm}$ as a free/forgetful adjunction, provided we change the “underlying set” of a zero-dimensional κ -frame A to mean A_\perp , as we now explain (see Appendix A.1 for a general explanation of this method of regarding arbitrary adjunctions as free/forgetful). The following notion was essentially introduced by van Name [vN13] when $\kappa = \infty$, under different terminology.

Let A be a zero-dimensional κ -frame. Since $A_\perp \subseteq A$ is a \wedge -sublattice generating A , the canonical κ -coverage \triangleleft on A restricted to $B := A_\perp$ yields a κ -posite (B, \triangleleft) presenting A . Since B is Boolean, \triangleleft is completely determined by its restriction to left-hand side \top , via

$$(2.9.1) \quad b \triangleleft C \iff \top \triangleleft \{\neg b\} \cup C.$$

Since $B = A_\perp \subseteq A$ is closed under finite joins, we have

- (ω -supercanonicity) \triangleleft extends the canonical ω -coverage on B , i.e., for finite $C \subseteq B$, we have $\bigvee C = \top \implies \top \triangleleft C$.

The other axioms on a κ -coverage (see Section 2.6) imply

- (monotonicity) if $\top \triangleleft C \subseteq D$, then $\top \triangleleft D$;
- (transitivity) if $\top \triangleleft C \cup D$, and $\top \triangleleft \{\neg c\} \cup D$ for every $c \in C$, then $\top \triangleleft D$.

Lemma 2.9.2. *These three axioms on the unary predicate $\top \triangleleft$ on $\mathcal{P}_\kappa(B)$ imply that \triangleleft given by (2.9.1) is a κ -coverage on B .*

Proof. Reflexivity follows from ω -supercanonicity (whence $\top \triangleleft \{a, \neg a\}$) and monotonicity.

Right-transitivity of \triangleleft follows from transitivity of \triangleleft .

\wedge -stability: suppose $a \leq b \triangleleft C$, i.e., $\top \triangleleft \{\neg b\} \cup C$; we must show $a \triangleleft a \wedge C$, i.e., $\top \triangleleft \{\neg a\} \cup (a \wedge C)$. Since $a \leq b$, we have $\neg a \vee b = \top$, whence by ω -supercanonicity, $\top \triangleleft \{\neg a, b\}$. By monotonicity, we have $\top \triangleleft \{\neg a, b\} \cup C$ and $\top \triangleleft \{\neg a, \neg b\} \cup C$, whence by transitivity, $\top \triangleleft \{\neg a\} \cup C$, whence by monotonicity, $\top \triangleleft \{\neg a\} \cup C \cup (a \wedge C)$. For each $c \in C$, we have $\top \triangleleft \{\neg a, \neg c, a \wedge c\}$ by ω -supercanonicity whence $\top \triangleleft \{\neg a, \neg c\} \cup (a \wedge C)$ by monotonicity. Thus $\top \triangleleft \{\neg a\} \cup (a \wedge C)$ by transitivity. \square

We call a Boolean algebra B equipped with a unary predicate $\top \triangleleft$ on $\mathcal{P}_\kappa(B)$ obeying these three axioms a **Boolean κ -posite**, and denote the category of all such by $\kappa\mathbf{BPost}$. Via (2.9.1), this is the same thing as a κ -posite (B, \triangleleft) such that \triangleleft remembers finite joins in B , hence yields a κ -frame $A = \langle B \mid \triangleleft \rangle_{\kappa\mathbf{Frm}}$ together with a Boolean homomorphism $\eta : B \rightarrow A$ whose image generates A . Let $\kappa\mathbf{SBPost} \subseteq \kappa\mathbf{BPost}$ denote the full subcategory of separated Boolean κ -posites (B, \triangleleft) ; this means

$$\top \triangleleft C \implies \bigvee C = \top \in B.$$

We have a “forgetful” functor $\kappa\mathbf{Frm} \rightarrow \kappa\mathbf{SBPost} \subseteq \kappa\mathbf{BPost}$ taking A to A_\neg equipped with the restriction of the canonical \triangleleft on A , with left adjoint $(B, \triangleleft) \mapsto \langle B \mid \triangleleft \rangle_{\kappa\mathbf{Frm}}$, where the counit $\langle A_\neg \mid \triangleleft \rangle_{\kappa\mathbf{Frm}} \rightarrow A$ at $A \in \kappa\mathbf{Frm}$ is an isomorphism iff A was zero-dimensional. Thus, we have represented $\kappa\mathbf{ZFrm}$ as equivalent to a full reflective subcategory of $\kappa\mathbf{SBPost}$, consisting of all those separated Boolean κ -posites (B, \triangleleft) for which the unit $\eta : B \rightarrow (\langle B \mid \triangleleft \rangle_{\kappa\mathbf{Frm}})_\neg$ is an isomorphism. Since \triangleleft is assumed to be separated, η is always an embedding; thus η is an isomorphism iff it is surjective, which, by expressing a complemented element of $\langle B \mid \triangleleft \rangle_{\kappa\mathbf{Frm}}$ as a join of elements of the image of η , is easily seen to mean

- (complement-completeness) for $C, D \in \mathcal{P}_\kappa(B)$, if $\top \triangleleft C \cup D$, and $c \wedge d = \perp \in B$ for every $c \in C$ and $d \in D$, then there is $b \in B$ such that $\top \triangleleft \{\neg b\} \cup C$ and $\top \triangleleft \{b, \neg c\}$ for every $c \in C$ (i.e., $b \triangleleft C$ and $c \triangleleft \{b\}$ for every $c \in C$; by separatedness, this implies $b = \bigvee C$ in B).

We let $\kappa\mathbf{CCBPost} \subseteq \kappa\mathbf{SBPost}$ denote the full subcategory of complement-complete separated Boolean κ -posites. Summarizing, we have represented the coreflective adjunction $\kappa\mathbf{ZFrm} \rightleftarrows \kappa\mathbf{Frm}$ as equivalent to the free/forgetful adjunction $\kappa\mathbf{CCBPost} \rightleftarrows \kappa\mathbf{Frm}$:

$$(2.9.3) \quad \begin{array}{ccccc} & & \xrightarrow{\langle - \rangle_{\kappa\mathbf{Frm}}} & & \\ \kappa\mathbf{BPost} & \xrightleftharpoons{\perp} & \kappa\mathbf{CCBPost} \simeq \kappa\mathbf{ZFrm} & \xrightleftharpoons{\perp} & \kappa\mathbf{Frm} \\ & & \xleftarrow{(-)_\neg} & & \end{array}$$

Note that the above categories $\kappa\mathbf{BPost}, \kappa\mathbf{SBPost}, \kappa\mathbf{CCBPost}$ are all locally κ -presentable. For $\kappa\mathbf{CCBPost}$, this requires the use of unique existential quantifiers to axiomatize the existence of joins; see Example A.1.5.

We have the following weakenings of zero-dimensionality. A κ -frame A is **regular** [Mad91, §3] if every $a \in A$ is a κ -ary join of $b \in A$ for which there exist $c \in \neg A (= \{\neg d \mid d \in A\})$ with $b \leq c \leq a$ in $\langle A \rangle_{\mathbf{CBOOL}}$ (or in $\mathcal{N}_\kappa(A)$); **fit** if every $a \in A$ is a κ -ary join, in $\langle A \rangle_{\mathbf{CBOOL}}$ (or in $\mathcal{N}_\kappa(A)$), of $\neg A \ni b \leq a$; and **subfit** if every $a \in A$ is the least, in A , upper bound (in $\langle A \rangle_{\mathbf{CBOOL}}$) of a κ -ary set of $\neg A \ni b \leq a$. (The latter notions are due to Isbell [Isb72, §2] when $\kappa = \infty$; we have given an equivalent definition in terms of $\langle A \rangle_{\mathbf{CBOOL}}$.) Clearly,

$$\text{zero-dimensional} \implies \text{regular} \implies \text{fit} \implies \text{subfit}.$$

(Sometimes, zero-dimensional is called *ultraregular*; see e.g., [Ell67].) We also have the following strengthening. A κ -frame A is **ultranormal** if every disjoint $a, b \in \neg A$ are separated by some $c \in A_{\neg}$, i.e., $a \leq c \leq \neg b$; equivalently, every $\neg a, \neg b \in A$ which cover \top can be refined to a partition $\neg c, c$. If A is also subfit, then this implies that each $\neg b \in A$ is a κ -ary join of $c \in A_{\neg}$, i.e.,

$$\text{subfit} + \text{ultranormal} \implies \text{zero-dimensional}.$$

See [PP12, Ch. V] for an extended discussion of these and other localic separation axioms.

We next consider the κ -ary generalization of ultranormality. We say that A is **ultraparacompact** if every κ -ary cover $(b_i)_i$ of $\top \in A$ refines to a partition $(c_i)_i$, i.e., whenever $\top \leq \bigvee_i b_i$, then there are $c_i \leq b_i$ with $\top \leq \bigvee_i c_i$ and $c_i \wedge c_j = \perp$ for $i \neq j$ (whence in fact each $c_i \in A_{\neg}$). Clearly,

$$\text{ultraparacompact} \implies \text{ultranormal};$$

hence

$$\text{subfit} + \text{ultraparacompact} \implies \text{zero-dimensional}.$$

More generally, we say that $a \in A$ is **ultraparacompact** if the same condition defining ultraparacompactness of A holds replacing \top with a (without requiring $c_i \in A_{\neg}$), i.e., every κ -ary cover $(b_i)_i$ of a refines to a pairwise disjoint κ -ary cover $(c_i)_i$ (whence by replacing c_i with $a \wedge c_i$, we may ensure $a = \bigvee_i c_i$). We let $A_{\text{upk}} \subseteq A$ denote the ultraparacompact elements.

Lemma 2.9.4. *For any κ -frame A , $A_{\text{upk}} \subseteq A$ is closed under pairwise disjoint κ -ary joins.*

Proof. Let $a_i \in A_{\text{upk}}$ be $< \kappa$ -many pairwise disjoint ultraparacompact elements. If $\bigvee_i a_i \leq \bigvee_j b_j$ for $< \kappa$ -many b_j , then for each i , we have $a_i \leq \bigvee_j c_{ij}$ for some pairwise disjoint $c_{ij} \leq b_j$, whence $\bigvee_i a_i = \bigvee_{i,j} (a_i \wedge c_{ij})$ is a pairwise disjoint cover of $\bigvee_i a_i$. \square

As with zero-dimensionality, it is convenient to introduce an auxiliary category to facilitate the study of ultraparacompactness. By a **disjunctive κ -frame** (see [Ch19b]), we will mean a poset A which is a \wedge -lattice, has a least element \perp (whence we may speak of disjointness), and has pairwise disjoint κ -ary joins, which we also denote using \bigsqcup instead of \bigvee , over which finite meets distribute. Denote the category of all such by κDjFrm . Then κDjFrm is locally κ -presentable, where the partial join operations are again axiomatized using unique existentials (see Example A.3.2).

We have a forgetful functor $\kappa\text{Frm} \rightarrow \kappa\text{DjFrm}$, preserving the underlying \wedge -lattice but forgetting about non-pairwise-disjoint joins, whose left adjoint may be constructed as follows. Given a disjunctive κ -frame A , the **canonical disjunctive κ -coverage** on A is given by

$$\begin{aligned} a \triangleleft C &: \iff \exists \kappa\text{-ary pairwise disjoint } \{d_i\}_i \subseteq \downarrow C \text{ (} a \leq \bigsqcup_i d_i \text{) for } C \in \mathcal{P}_{\kappa}(A) \\ &\iff \exists \kappa\text{-ary pairwise disjoint } \{d_i\}_i \subseteq \downarrow C \text{ (} a = \bigsqcup_i d_i \text{).} \end{aligned}$$

It is readily seen that this defines a separated κ -coverage on A , such that $\triangleleft_{\kappa}(A) = \langle A \mid \triangleleft \rangle_{\kappa\text{Frm}} \cong \langle A \text{ qua } \kappa\text{DjFrm} \rangle_{\kappa\text{Frm}}$. Call the \triangleleft -ideals **disjunctive κ -ideals**, denoted $\text{dj}_{\kappa}\mathcal{I}(A) := \triangleleft\mathcal{I}(A)$, with κ -generated ideals $\text{dj}_{\kappa}\mathcal{I}_{\kappa}(A) \subseteq \text{dj}_{\kappa}\mathcal{I}(A)$. Note that the disjunctive κ -ideal generated by $C \subseteq A$ is

$$\nabla C = \{a \mid a \triangleleft C\} = \{a \mid \exists \kappa\text{-ary pairwise disjoint } \{d_i\}_i \subseteq \downarrow C \text{ (} a = \bigsqcup_i d_i \text{)}\}.$$

Thus, for $B \in \kappa\text{Frm}$, $b \in B$ is ultraparacompact iff whenever $C \in \mathcal{P}_{\kappa}(A)$ with $b \leq \bigvee C$, then $b \in \nabla C$. We now have the following, which is analogous to Proposition 2.3.2:

Proposition 2.9.5.

- (a) For $A \in \kappa\text{DjFrm}$, we have $\text{dj}\kappa\mathcal{I}_\kappa(A)_{\text{upk}} = \downarrow(A) \cong A$. In particular, $\text{dj}\kappa\mathcal{I}_\kappa(A)_{\text{upk}} \subseteq \text{dj}\kappa\mathcal{I}_\kappa(A)$ forms a generating disjunctive κ -subframe.
- (b) If $B \in \kappa\text{Frm}$ and $A \subseteq B_{\text{upk}}$ is a generating disjunctive κ -subframe of B , then we have an isomorphism $\bigvee : \text{dj}\kappa\mathcal{I}_\kappa(A) \cong B$, hence in fact $A = B_{\text{upk}}$.

Proof. We will need

Lemma 2.9.6. Let $B \in \kappa\text{Frm}$ and $A \subseteq B$ be closed under pairwise disjoint κ -ary joins and generate A under κ -ary joins. Let $c_i \in B_{\text{upk}}$ be $< \kappa$ -many ultraparacompact elements, and let $\bigvee_i c_i = \bigsqcup_i d_i$ with $c_i \geq d_i \in A$ be a pairwise disjoint refinement. Then each $d_i \in A$.

Proof. Since A generates B under κ -ary joins, each $d_i = \bigvee D_i$ for some $D_i \in \mathcal{P}_\kappa(A)$. Since each c_i is ultraparacompact and $c_i \leq \bigvee_i d_i = \bigvee \bigcup_i D_i$, we have $c_i = \bigsqcup_j d_{ij}$ for some d_{ij} each of which is a pairwise disjoint join of elements from D_j . For each $d \in D_i$, since $\bigvee D_i = d_i \leq c_i$, we have $d = d \wedge c_i = \bigsqcup_j (d \wedge d_{ij}) = d \wedge d_{ii} \leq d_{ii}$, since for $j \neq i$ we have $d \wedge d_{ij} \leq \bigvee D_i \wedge \bigvee D_j = d_i \wedge d_j = \perp$. Thus each $d_i = \bigvee D_i = d_{ii} \in A$. \square

(a) Clearly principal ideals are ultraparacompact in $\text{dj}\kappa\mathcal{I}_\kappa(A)$. Conversely, for $C \in \text{dj}\kappa\mathcal{I}_\kappa(A)_{\text{upk}}$ generated by κ -ary $\{c_i\}_i \subseteq C$, we have $C = \bigvee_i \downarrow c_i$ in $\text{dj}\kappa\mathcal{I}_\kappa(A)$, so using ultraparacompactness of C and Lemma 2.9.6 (with $A := \downarrow(A) \subseteq \text{dj}\kappa\mathcal{I}_\kappa(A) =: B$), we have pairwise disjoint $d_i \leq c_i$ with $C = \bigvee_i \downarrow d_i = \downarrow \bigsqcup_i d_i$.

(b) Since $A \subseteq B$ forms a generating \wedge -sublattice, we have $B \cong \langle A \mid \triangleleft \rangle_{\kappa\text{Frm}}$ for the restriction \triangleleft of the canonical κ -coverage on B ; thus it is enough to show that \triangleleft agrees with the canonical disjunctive κ -coverage \triangleleft' on A . Clearly the former contains the latter. Conversely, for $b \triangleleft \{c_i\}_i \in \mathcal{P}_\kappa(A)$, i.e., $b \leq \bigvee_i c_i$ in B , we have $b = \bigvee_i (b \wedge c_i)$ with $b \wedge c_i \in A \subseteq B_{\text{upk}}$ by assumption, so using ultraparacompactness of b and Lemma 2.9.6, we have pairwise disjoint $b \wedge c_i \geq d_i \in A$ with $b = \bigsqcup_i d_i$, which witnesses $b \triangleleft' \{c_i\}_i$. The last assertion follows from (a). \square

Remark 2.9.7. As with Corollary 2.3.4, it follows that the free functor $\text{dj}\kappa\mathcal{I}_\kappa : \kappa\text{DjFrm} \rightarrow \kappa\text{Frm}$ is full on isomorphisms. The κ -frames in its essential image, i.e., satisfying Proposition 2.9.5(b), might be called “ultraparacoherent”, by analogy with coherent frames.

The preceding discussion, along with Proposition 2.3.2, both fit into the categorical framework of KZ-monads; see [Joh02, B1.1.13–15], [Koc95], [Mes06]. Namely, we have a KZ-monad $\text{dj}\kappa\mathcal{I}_\kappa$ on κDjFrm , with category of algebras κFrm ; Proposition 2.9.5(a) shows that this monad has regular monic unit, whence κDjFrm embeds fully faithfully into the category of coalgebras for the induced comonad. These coalgebras are the retracts of “ultraparacoherent” frames, or equivalently frames with a \wedge -stable “relatively ultraparacompact” or “way-below” relation \ll , hence might be called “stably ultraparacompact” frames, in analogy with stably compact frames (see [Joh82, VII 4.6]), localic \vee -lattices (see [Sch93, §6.4]), etc. We do not study these in detail here, but make one preliminary observation: spaces of points of “stably ultraparacompact” frames have connected meets, since these commute with pairwise disjoint joins in $2 = \{0 < 1\}$; thus such frames can be thought of as dual to certain locales with connected meets.

Finally, we consider κ -frames which are both ultraparacompact and zero-dimensional (or equivalently subfit, as noted above). Let $\kappa\text{UPKZFrm} \subseteq \kappa\text{ZFrm}$ denote the full subcategory of these.

Proposition 2.9.8. *For a zero-dimensional κ -frame A , the following are equivalent:*

- (i) A is ultraparacompact;
- (ii) $A_{\text{upk}} \subseteq A$ forms a generating disjunctive κ -subframe (i.e., A is “ultraparacoherent”);
- (iii) every $a \in A_{\neg}$ is ultraparacompact;
- (iv) every κ -ary cover of $a \in A_{\neg}$ by complemented elements refines to a pairwise disjoint cover;
- (v) every κ -ary cover of $a \in A_{\neg}$ refines to a pairwise disjoint cover by complemented elements.

In this case, $A_{\text{upk}} \subseteq A$ is the closure of A_{\neg} under pairwise disjoint κ -ary joins.

Proof. Clearly (ii) \implies (i) and (v) \implies (iii) \implies (iv).

(iv) \implies (iii): Given a κ -ary cover $a \leq \bigvee_i c_i$ of $a \in A_{\neg}$ by $c_i \in A$, write each c_i as a κ -ary join of $c_{ij} \in A_{\neg}$, and refine these to $a = \bigsqcup_{i,j} d_{ij}$ with $d_{ij} \leq c_{ij}$; then $d_i := \bigsqcup_j d_{ij} \leq c_i$ with $a = \bigsqcup_i d_i$.

(iii) \implies (ii) follows from Proposition 2.9.5(b) (with $A :=$ closure of A_{\neg} under pairwise disjoint κ -ary joins).

(i) \implies (v): Given a κ -ary cover $a \leq \bigvee_i c_i$ of $a \in A_{\neg}$, letting $a = \bigsqcup_i d_i$ with $d_i \leq a \wedge c_i$, since a is complemented and the d_i are pairwise disjoint, each d_i has complement $\neg a \vee \bigvee_{j \neq i} d_j$.

The last statement follows from (iii) and Proposition 2.9.5(b) (with $A :=$ closure of A_{\neg} under pairwise disjoint κ -ary joins). \square

Given $A \in \kappa\text{UPKZFrm}$, as described above (2.9.3), we may equivalently represent A as the canonical Boolean κ -posite $(A_{\neg}, \triangleleft)$. For $< \kappa$ -many $a, c_i \in A_{\neg}$ with $a \leq \bigvee_i c_i$, we may find a pairwise disjoint refinement $a = \bigsqcup_i d_i$ with $d_i \in A_{\neg}$ (by Proposition 2.9.8(v)); in other words, \triangleleft is generated by its restriction to pairwise disjoint right-hand side. Call such a κ -posite **disjunctive**, and let $\kappa\text{DjBPost} \subseteq \kappa\text{BPost}$ be the full subcategory of disjunctive Boolean κ -posites, with the further full subcategories $\kappa\text{CCDjBPost} \subseteq \kappa\text{SDjBPost} \subseteq \kappa\text{DjBPost}$ of complement-complete and separated posites, respectively. Conversely, given any disjunctive Boolean κ -posite $(B, \triangleleft) \in \kappa\text{DjBPost}$, the presented zero-dimensional κ -frame $\langle B \mid \triangleleft \rangle_{\kappa\text{Frm}}$ is ultraparacompact: indeed, since \triangleleft is generated by its restriction to pairwise disjoint right-hand side, $\langle B \mid \triangleleft \rangle_{\kappa\text{Frm}}$ may be obtained by first taking the presented *disjunctive* κ -frame $\langle B \mid \triangleleft \rangle_{\kappa\text{DjFrm}}$, and then completing to a κ -frame as in Proposition 2.9.5. Thus, the equivalence of categories $\kappa\text{CCBPost} \simeq \kappa\text{ZFrm}$ from above (2.9.3) restricts to an equivalence $\kappa\text{CCDjBPost} \simeq \kappa\text{UPKZFrm}$ between complement-complete disjunctive Boolean κ -posites and ultraparacompact zero-dimensional κ -frames.

The following diagram, extending (2.9.3), depicts the relationship between all the categories we have considered in this subsection (as before, \rhd denotes full subcategory):

$$\begin{array}{ccccccc}
 \kappa\text{BPost} & \xleftrightarrow{\perp} & \kappa\text{CCBPost} & \xleftrightarrow[\langle - \rangle_{\kappa\text{Frm}}]{\simeq} & \kappa\text{ZFrm} & \xleftrightarrow{\perp} & \kappa\text{Frm} & \xleftrightarrow{\perp} & \kappa\text{Bool} \\
 \uparrow \scriptstyle \neg & & \uparrow \scriptstyle \neg & & \uparrow \scriptstyle \neg & & \uparrow \scriptstyle \neg & & \uparrow \scriptstyle \neg \\
 \kappa\text{DjBPost} & \xleftrightarrow{\perp} & \kappa\text{CCDjBPost} & \xleftrightarrow[\langle - \rangle_{\kappa\text{DjFrm}}]{\simeq} & \kappa\text{UPKZFrm} & \xleftrightarrow{\perp} & \kappa\text{DjFrm} & \xleftrightarrow{\perp} & \kappa\text{DjFrm}
 \end{array}$$

(2.9.9)

Most of these functors are the obvious ones, or else were described above. The right adjoint $\kappa\mathbf{BPost} \rightarrow \kappa\mathbf{DjBPost}$ takes a Boolean κ -posite (B, \triangleleft) and replaces \triangleleft with the κ -coverage generated by its restriction to pairwise disjoint right-hand side (this can be regarded as “forgetful”, if we view a disjunctive κ -coverage \triangleleft as already restricted to pairwise disjoint right-hand side). The right adjoint $\kappa\mathbf{DjFrm} \rightarrow \kappa\mathbf{CCDjBPost} \subseteq \kappa\mathbf{DjBPost}$ takes a disjunctive κ -frame A to $A_{\neg} \subseteq A$, same as $\kappa\mathbf{Frm} \rightarrow \kappa\mathbf{CCBPost} \subseteq \kappa\mathbf{BPost}$; the only thing to note is that pairwise disjoint joins are already enough to define complements and prove their basic properties (see e.g., [Ch19b]).

Remark 2.9.10. As κ varies, the above diagrams (2.9.9) fit together in the obvious manner. It follows that ultraparacompactness and zero-dimensionality are preserved by the *free* functors $\kappa\mathbf{Frm} \rightarrow \lambda\mathbf{Frm}$ for $\kappa \leq \lambda$ (but not necessarily by the forgetful functors, in analogy with Remark 2.3.3).

The above diagram (2.9.9) also indicates that the composite forgetful functor $\kappa\mathbf{Bool} \rightarrow \kappa\mathbf{UPKZFrm}$ (which takes a κ -Boolean algebra to itself) is full, which is just to say that $\kappa\mathbf{Bool} \subseteq \kappa\mathbf{UPKZFrm}$, i.e., every κ -Boolean algebra A is ultraparacompact and (clearly) zero-dimensional. Ultraparacompactness is because covers may be disjointified via the usual transfinite iteration: given $a = \bigvee_{i < \alpha} c_i$ for some ordinal $\alpha < \kappa$, we have $a = \bigsqcup_{i < \alpha} (c_i \wedge \neg \bigvee_{j < i} c_j)$.

Since $\kappa\mathbf{UPKZFrm} \subseteq \kappa\mathbf{Frm}$ is coreflective, it follows that colimits of κ -Boolean algebras in $\kappa\mathbf{Frm}$ remain ultraparacompact zero-dimensional. We now show that $\kappa\mathbf{UPKZFrm}$ is precisely the closure of $\kappa\mathbf{Bool} \subseteq \kappa\mathbf{Frm}$ under colimits. This result is essentially due to Paseka [Pas92] when $\kappa = \infty$.

Recall that a **complete atomic Boolean algebra** is an isomorphic copy of a full powerset $\mathcal{P}(X)$; let \mathbf{CABool} denote the category of these. By a **complete κ -atomic Boolean algebra**, we will mean a copy of $\mathcal{P}(X)$ for κ -ary X ; let $\kappa\mathbf{CABool}_{\kappa}$ denote the category of these. For κ -ary X , it is easily seen that $\mathcal{P}(X)$ is presented, as a κ -frame (and hence as a κ -Boolean algebra), by

$$(2.9.11) \quad \mathcal{P}(X) = \langle \{x\} \text{ for } x \in X \mid \{x\} \wedge \{y\} = \perp \text{ for } x \neq y, \top \leq \bigvee_{x \in X} \{x\} \rangle;$$

thus $\kappa\mathbf{CABool}_{\kappa} \subseteq \kappa\mathbf{Frm}_{\kappa}, \kappa\mathbf{Bool}_{\kappa}$. Moreover, the same presentation yields a κ -presented disjunctive Boolean κ -posite (A, \triangleleft) (namely $A :=$ all finite and cofinite subsets of X , and \triangleleft generated by $\top \triangleleft \{\{x\} \mid x \in X\}$) such that $\mathcal{P}(X) = \langle A \mid \triangleleft \rangle_{\kappa\mathbf{Frm}}$; thus $\kappa\mathbf{CABool}_{\kappa} \subseteq \kappa\mathbf{UPKZFrm}_{\kappa} \simeq \kappa\mathbf{CCDjBPost}_{\kappa}$.

Now given any ultraparacompact zero-dimensional κ -frame A , for κ -ary X , a homomorphism $f : \mathcal{P}(X) \rightarrow A$ is, by the above presentation for $\mathcal{P}(X)$, the same thing as a pairwise disjoint cover $\top \leq \bigsqcup_{x \in X} f(x)$ in A . Using this, we may convert a presentation of A into an expression of A as a colimit of complete κ -atomic Boolean algebras. Namely, suppose $A = \langle B \mid \triangleleft \rangle_{\kappa\mathbf{Frm}}$ for a disjunctive Boolean κ -posite (B, \triangleleft) (e.g., the canonical posite $(A_{\neg}, \triangleleft)$), and suppose that \triangleleft , or rather its one-sided restriction $\top \triangleleft$, is in turn generated by some unary relation $\top \triangleleft_0$ on pairwise disjoint κ -ary subsets of B . Then for each C such that $\top \triangleleft_0 C$, so that $\top = \bigsqcup C$ in A , we get a homomorphism $f_C : \mathcal{P}(C) \rightarrow A$ as above. Suppose, furthermore, that

- $\top \triangleleft_0 C \implies C \subseteq B \setminus \{\perp\}$;
- (ω -supercanonicity) for every finite pairwise disjoint $C \subseteq B \setminus \{\perp\}$, we have $\bigvee C = \top \implies \top \triangleleft_0 C$;
- (closure under common refinement) for every C, D such that $\top \triangleleft_0 C$ and $\top \triangleleft_0 D$, we have $\top \triangleleft_0 (C \wedge D) \setminus \{\perp\}$, where $C \wedge D := \{c \wedge d \mid c \in C \text{ \& } d \in D\}$.

Note that we can always modify \triangleleft_0 to satisfy these, by removing \perp from all C such that $\top \triangleleft_0 C$ and then closing under the last two conditions, which does not change the generated \triangleleft .

Theorem 2.9.12. *Under the above assumptions, the homomorphisms $f_C : \mathcal{P}(C) \rightarrow A$ for $\top \triangleleft_0 C$ form cocone maps exhibiting A as the directed colimit, in κFrm , of the $\mathcal{P}(C)$ together with the preimage maps $h_{CD}^{-1} : \mathcal{P}(C) \rightarrow \mathcal{P}(D)$ for $\top \triangleleft_0 C$ and $\top \triangleleft_0 D$ such that $D \subseteq \downarrow C$, where $h_{CD} : D \rightarrow C$ takes $d \in D$ to the unique element above it in C .*

Proof. By the general construction of colimits of presented structures (see Appendix A.1), the colimit $\varinjlim_{\top \triangleleft_0 C} \mathcal{P}(C)$ is presented by all the $\mathcal{P}(C)$ qua κ -frame, together with new relations which identify $c \in \mathcal{P}(C)$ with $h_{CD}^{-1}(c) = \downarrow c \cap D \in \mathcal{P}(D)$. Hence, incorporating the above presentations (2.9.11) of $\mathcal{P}(C)$, we get

$$\varinjlim_{\top \triangleleft_0 C} \mathcal{P}(C) = \left\langle f_C(c) \text{ for } \top \triangleleft_0 C \ni c \left| \begin{array}{ll} f_C(c) \wedge f_C(c') = \perp & \text{for } c \neq c' \in C \\ \bigvee_{c \in C} f_C(c) = \top & \text{for } \top \triangleleft_0 C \\ f_C(c) = \bigvee_{c \geq d \in D} f_D(d) & \text{for } D \subseteq \downarrow C \end{array} \right. \right\rangle_{\kappa\text{Frm}}.$$

On the other hand, since $A = \langle B \mid \triangleleft \rangle_{\kappa\text{Frm}}$ and \triangleleft_0 generates \triangleleft ,

$$A = \langle B \text{ qua Bool} \mid \top \leq \bigvee C \text{ for } \top \triangleleft_0 C \rangle_{\kappa\text{Frm}}.$$

Our goal is to show that these two presentations are equivalent.

First, note that every generator $b \in B$ in the second presentation, other than \perp, \top which are redundant since they are κ -frame operations, corresponds to some generator $f_C(b)$ in the first presentation, namely for $C := \{b, \neg b\}$ by ω -supercanonicity of \triangleleft_0 . Also, for each b , the $f_C(b)$ for different C are identified by the first presentation: indeed, for $\top \triangleleft_0 C, D \ni b$, we have $f_C(b) = \bigvee_{b \geq c \wedge d \in (C \wedge D) \setminus \{\perp\}} f_{(C \wedge D) \setminus \{\perp\}}(c \wedge d) = f_D(b)$ by the third relation. So the generators (other than \perp, \top) in the two presentations correspond to each other.

Since the relations in the first presentation clearly hold in A , it remains only to show that they imply the relations in the second presentation. For any $\top \triangleleft_0 C$, the first presentation says that C forms a partition of \top ; in particular, the relations $\top \leq \bigvee C$ for $\top \triangleleft_0 C$ are implied by the first presentation, as are finite partitions of \top in B by ω -supercanonicity of \triangleleft_0 . For each $b \in B$, by considering the partition $\top = b \sqcup \neg b$, we get that complements are preserved by the first presentation. For $b \leq b' \in B$, by considering the partition $\top = b \sqcup \neg b' \sqcup (b' \wedge \neg b)$, we get that the partial order on B is preserved. Finally, for a binary join $b \vee b' \in B$, by considering the partition $\top = b \sqcup (b' \wedge \neg b) \sqcup \neg(b \vee b')$ where $b' \wedge \neg b \leq b'$, we get that the join $b \vee b'$ is preserved. \square

Taking $B := A_{\neg}$ and \triangleleft_0 to be the canonical \triangleleft with all occurrences of \perp removed yields

Corollary 2.9.13. *Every ultraparacompact zero-dimensional κ -frame is a directed colimit, in κFrm , of complete κ -atomic Boolean algebras.* \square

Remark 2.9.14. For a zero-dimensional κ -frame A , of course the generating set $A_{\neg} \subseteq A$ is the directed union of all finite (hence complete atomic) Boolean subalgebras $B \subseteq A_{\neg}$. However, this does not generally yield an expression of A as the directed colimit $\varinjlim_{B \subseteq A_{\neg}} B$ in κFrm . Indeed, since each $B = \langle B \text{ qua Bool} \rangle_{\kappa\text{Frm}}$, this colimit $\varinjlim_{B \subseteq A_{\neg}} B$ in κFrm is the same as the free functor $\langle - \rangle_{\kappa\text{Frm}} : \text{Bool} \rightarrow \kappa\text{Frm}$ applied to the colimit in Bool , i.e., to A_{\neg} ; and of course A is usually a proper quotient of $\langle A_{\neg} \text{ qua Bool} \rangle_{\kappa\text{Frm}} = \mathcal{I}(A_{\neg})$. (The topological analog of $A \mapsto \mathcal{I}(A_{\neg})$ is the Banaschewski, i.e., universal zero-dimensional, compactification of a zero-dimensional space.)

Remark 2.9.15. It is well-known, and easily shown, that $\mathbf{CABool} \subseteq \mathbf{Frm}$ is closed under finite colimits (namely, $\mathcal{P}(X) \otimes \mathcal{P}(Y) \cong \mathcal{P}(X \times Y)$, and the coequalizer of $f^{-1}, g^{-1} : \mathcal{P}(X) \rightrightarrows \mathcal{P}(Y)$ for $f, g : Y \rightrightarrows X$ is $\mathcal{P}(\text{eq}(f, g))$). The same holds for $\kappa\mathbf{CABool}_\kappa \subseteq \kappa\mathbf{Frm}$.

It follows that for any $A \in \kappa\mathbf{Frm}$, the **canonical diagram** of all complete κ -atomic Boolean B equipped with a homomorphism $B \rightarrow A$ is filtered (see e.g., [AR97, 0.4, 1.4]). The proof of Theorem 2.9.12 easily adapts to show that for $A \in \kappa\mathbf{UPKZFrm}$, the colimit of all such B over the canonical diagram is A , i.e., $\kappa\mathbf{CABool}_\kappa \subseteq \kappa\mathbf{UPKZFrm}$ is a dense subcategory (see e.g., [AR97, 1.23]).

Now suppose A is a κ -presented ultraparacompact zero-dimensional κ -frame. Since $\kappa\mathbf{CCDjBPost} \simeq \kappa\mathbf{UPKZFrm} \subseteq \kappa\mathbf{Frm}$ is coreflective, this means that $A = \langle B \mid \triangleleft \rangle_{\kappa\mathbf{Frm}}$ for a κ -presented complement-complete disjunctive Boolean κ -posite (B, \triangleleft) (see Appendix A.1), which means the same without requiring complement-completeness, since $\kappa\mathbf{CCDjBPost} \subseteq \kappa\mathbf{DjBPost}$ is reflective (see Proposition A.1.6; for $\kappa = \omega$, use instead that finitely generated Boolean algebras are finite). So B is a κ -presented Boolean algebra, hence κ -ary; and \triangleleft is a κ -generated Boolean κ -coverage. Modifying the generators \triangleleft_0 to fulfill the assumptions of Theorem 2.9.12 clearly preserves the cardinality bound, yielding

Corollary 2.9.16. *Every κ -presented ultraparacompact zero-dimensional κ -frame is a κ -ary directed colimit, in $\kappa\mathbf{Frm}$, of complete κ -atomic Boolean algebras.* \square

We may combine this with Proposition 2.8.13 to get the following strengthening of the latter:

Proposition 2.9.17. *Every κ -presented κ -Boolean algebra is freely generated by a κ -presented ultraparacompact zero-dimensional κ -frame (hence is a κ -ary directed colimit, in $\kappa\mathbf{Bool}$, of complete κ -atomic Boolean algebras).*

Proof. As in the proof of Proposition 2.8.13, applied instead to the reflective inclusion of locally κ -presentable categories $\kappa\mathbf{Bool} \subseteq \kappa\mathbf{UPKZFrm} \simeq \kappa\mathbf{CCDjBPost}$. \square

As with Proposition 2.8.15, we have the following refinement:

Proposition 2.9.18. *Let B be a κ -presented κ -Boolean algebra which is freely generated by an ultraparacompact zero-dimensional κ -subframe $A \subseteq B$. For any κ -ary subset $C \subseteq A$, there is a κ -presented ultraparacompact zero-dimensional κ -subframe $A' \subseteq A$ which contains C and still freely generates B .*

Proof. As in the proof of Proposition 2.8.15, with the role of the reflective adjunction $\kappa\mathbf{Frm} \rightleftarrows \kappa\mathbf{Bool}$ in that proof replaced by $\kappa\mathbf{CCDjBPost} \rightleftarrows \kappa\mathbf{Bool}$, with $A \subseteq B$ replaced by the subposet $A_\perp \subseteq B$, and with $C \subseteq A$ replaced by some $C' \subseteq A_\perp$ which generates C under κ -ary joins. \square

The following result connects ultraparacompactness with the frames $\mathcal{N}_\kappa^\alpha(A)$ from the preceding subsection. The case $\kappa = \infty$ was proved by Plewe [P102, Th. 17], using very different methods.

Theorem 2.9.19. *For any κ -frame A , $\mathcal{N}_\kappa(A)$ is ultraparacompact (and zero-dimensional).*

Proof. Let $B \subseteq \mathcal{N}_\kappa(A)_\perp$ be the Boolean subalgebra generated by $A \subseteq \mathcal{N}_\kappa(A)_\perp$, and let \triangleleft be the restriction to B of the canonical κ -coverage on $\mathcal{N}_\kappa(A)$. Since B is a generating \wedge -sublattice of $\mathcal{N}_\kappa(A)$, we have $\mathcal{N}_\kappa(A) = \langle B \mid \triangleleft \rangle_{\kappa\mathbf{Frm}}$. Thus, it suffices to show that \triangleleft is disjunctive.

We claim that \triangleleft is generated by the canonical κ -coverage on A , together with the canonical ω -coverage on B . Let \triangleleft' be the κ -coverage on B generated by these. Clearly $\triangleleft' \subseteq \triangleleft$, whence we have a quotient map $f : \langle B \mid \triangleleft' \rangle_{\kappa\mathbf{Frm}} \twoheadrightarrow \langle B \mid \triangleleft \rangle_{\kappa\mathbf{Frm}} = \mathcal{N}_\kappa(A)$. From the definition of \triangleleft' , κ -ary joins

in A are preserved in $\langle B \mid \triangleleft' \rangle_{\kappa\text{Frm}}$, as are finite meets and joins in B whence so are complements of $a \in A$; thus the inclusion $A \hookrightarrow B \hookrightarrow \langle B \mid \triangleleft' \rangle_{\kappa\text{Frm}}$ extends to $g : \mathcal{N}(A) \rightarrow \langle B \mid \triangleleft' \rangle_{\kappa\text{Frm}}$. We have $g \circ f = 1$, since both restrict to the inclusion $A \hookrightarrow B \hookrightarrow \langle B \mid \triangleleft' \rangle_{\kappa\text{Frm}}$, which is a κ -frame epimorphism since $\langle B \mid \triangleleft' \rangle_{\kappa\text{Frm}}$ is generated by the sublattice B which is in turn generated by elements of A and their complements. It follows that f is an isomorphism, whence $\triangleleft' = \triangleleft$ (since both present the same κ -frame).

So it suffices to show that every κ -ary covering relation $a \triangleleft C$ which holds in A , as well as every finite covering relation which holds in B , is implied by the pairwise disjoint κ -ary covering relations which hold in B . Enumerate $C = \{c_i\}_{i < \alpha}$ for some ordinal $\alpha < \kappa$. In both cases, we have the pairwise disjoint cover $a \triangleleft \{c_i \wedge \neg \bigvee_{j < i} c_j\}_{i < \alpha}$ in B , which implies $a \triangleleft C$ by transitivity. \square

Corollary 2.9.20. *For any κ -frame A and $\alpha \geq 1$, $\mathcal{N}_\kappa^\alpha(A)$ is ultraparacompact zero-dimensional.*

Proof. Follows by induction from the preceding result and coreflectivity of $\kappa\text{UPKZFrm} \subseteq \kappa\text{Frm}$. \square

Corollary 2.9.21. *Let A be a κ -presented κ -frame. For any $\alpha \geq 1$ and κ -ary subset $C \subseteq \mathcal{N}_\kappa^\alpha(A)$, there is a κ -presented ultraparacompact zero-dimensional κ -subframe $A' \subseteq \mathcal{N}_\kappa^\alpha(A)$ such that $A \cup C \subseteq A'$ and $\mathcal{N}_\kappa^\infty(A) \cong \mathcal{N}_\kappa^\infty(A')$ (canonically).*

Proof. As in Corollary 2.8.16. \square

2.10 Free complete Boolean algebras

In this subsection, we give a “brute-force” syntactic construction of free complete Boolean algebras, using cut admissibility for a Gentzen sequent calculus for infinitary propositional logic. This is analogous to Whitman’s construction of free lattices (see [Hal64, III], [Joh82, I 4.6–7], [CS01]). We are not aware of a prior application of the technique to infinitary Boolean algebras, although it is possibly folklore. For background on sequent calculi in the finitary context, see [TS00].

Let X be a set (of generators), and let $\neg X$ be the set of symbols $\neg x$ for $x \in X$. Let $\text{Term}(X)$ be the class of **prenex complete Boolean terms** over X , constructed inductively as follows:

- $X \sqcup \neg X \subseteq \text{Term}(X)$;
- for any subset $A \subseteq \text{Term}(X)$, the symbols $\bigwedge A, \bigvee A$ are in $\text{Term}(X)$.

Define the involution $\neg : \text{Term}(X) \rightarrow \text{Term}(X)$ in the obvious way, interchanging $x, \neg x$ and \bigwedge, \bigvee .

Let \vdash be the unary predicate on *finite* subsets $A \subseteq \text{Term}(X)$ defined inductively as follows:

- (reflexivity/identity) if $x, \neg x \in A$ for some $x \in X$, then $\vdash A$;
- ($\bigwedge R$) if $\bigwedge B \in A$, and $\vdash A \cup \{b\}$ for every $b \in B$, then $\vdash A$;
- ($\bigvee R$) if $\bigvee B \in A$, and $\vdash A \cup \{b\}$ for some $b \in B$, then $\vdash A$.

For two finite sets $A, B \subseteq \text{Term}(X)$, we use the abbreviation¹³

$$A \vdash B :\iff \vdash \neg A \cup B$$

(where $\neg A := \{\neg a \mid a \in A\}$). Thus $\vdash A \iff \emptyset \vdash A$. The rules above imply their obvious generalizations when \vdash has nonempty left-hand side, and also imply

¹³For simplicity, we use a one-sided sequent calculus, with formulas in prenex form and \neg treated as an operation on terms. It is also possible to directly formulate a two-sided calculus, treating \neg as a primitive symbol, and prove all of the following results for two-sided sequents. See [TS00, Ch. 3].

- (reflexivity/identity) if $x \in A \cap B$ or $x, \neg x \in A$ for some $x \in X$, then $A \vdash B$;
- (\vee L) if $\vee B \in A$, and $A \cup \{b\} \vdash C$ for every $b \in B$, then $A \vdash C$;
- (\wedge L) if $\wedge B \in A$, and $A \cup \{b\} \vdash C$ for some $b \in B$, then $A \vdash C$.

From now on, we will freely switch between one-sided and two-sided \vdash without further comment; in particular, all of the following results stated for one-sided \vdash have two-sided generalizations.

Every term $a \in \text{Term}(X)$ can be represented as a well-founded tree, hence has an ordinal rank. Similarly, whenever $\vdash A$ holds, its derivation using the rules above has an ordinal rank.

Lemma 2.10.1 (generalized reflexivity/identity). *If $a, \neg a \in A$ for some $a \in \text{Term}(X)$, then $\vdash A$.*

Proof. By induction on a . The case $a \in X$ is immediate. Otherwise, by \neg -duality, it suffices to consider $a = \wedge B$. For every $b \in B$, we have $\vdash A \cup \{b, \neg b\}$ by the induction hypothesis, whence $\vdash A \cup \{b\}$ by (\vee R) since $\vee \neg B = \neg a \in A$. Thus $\vdash A$ by (\wedge R) since $\vee B = a \in A$. \square

Lemma 2.10.2 (monotonicity/weakening). *If $\vdash A \subseteq B$, then $\vdash B$. Moreover, the derivations have the same rank.*

Proof. By straightforward induction on the (rank of the) derivation of $\vdash A$. \square

Lemma 2.10.3 (transitivity/cut). *If $\vdash A \cup \{a\}$ and $\vdash A \cup \{\neg a\}$ for some $a \in \text{Term}(X)$, then $\vdash A$.*

Proof. By lexicographical induction on a followed by the ranks of the derivations of $\vdash A \cup \{a\}$ and $\vdash A \cup \{\neg a\}$. Up to \neg -duality, it suffices to consider the following cases:

- Suppose $\vdash A \cup \{a\}$ by reflexivity because $\neg a \in A$. Then $\vdash A = A \cup \{\neg a\}$ by hypothesis.
- Suppose $\vdash A \cup \{a\}$ by reflexivity because some $x, \neg x \in A$. Then $\vdash A$ for the same reason.
- Suppose $\vdash A \cup \{a\}$ by (\wedge R) because some $\wedge B \in A$ with $\vdash A \cup \{a, b\}$ for each $b \in B$. Then by weakening, $\vdash A \cup \{\neg a, b\}$ for each $b \in B$, derived with the same rank as $\vdash A \cup \{\neg a\}$. Thus by the induction hypothesis applied to $\vdash A \cup \{a, b\}$ and $\vdash A \cup \{\neg a, b\}$ where the rank of the former is less than that of $\vdash A \cup \{a\}$, we have $\vdash A \cup \{b\}$ for each $b \in B$. So by (\wedge R), $\vdash A$.
- Similarly if $\vdash A \cup \{a\}$ by (\vee R) because some $\vee B \in A$ with $\vdash A \cup \{a, b\}$ for some $b \in B$.
- Finally, suppose $a = \wedge B$,
 - (i) $\vdash A \cup \{a\}$ by (\wedge R) because
 - (ii) $\vdash A \cup \{a, b\}$ for each $b \in B$, and
 - (iii) $\vdash A \cup \{\neg a\}$ by (\vee R) because
 - (iv) $\vdash A \cup \{\neg a, \neg b\}$ for some $b \in B$.

By the induction hypothesis applied to (iv) and a weakening of (i) (where the former has lesser rank than $\vdash A \cup \{a, b\}$ and the latter has the same rank as (i)), we have $\vdash A \cup \{\neg b\}$ for some $b \in B$. Again by the induction hypothesis applied to (ii) and a weakening of (iii) (where again the former has lesser rank), we have $\vdash A \cup \{b\}$ for that b . Thus by the induction hypothesis (using that b is a subterm of a), $\vdash A$. \square

Note that by weakening, all of the rules above have equivalent (“multiplicative”) forms where the hypotheses are allowed to have different sets which are combined in the conclusion, e.g.,

- (\wedge R) if $\vdash A_b \cup \{b\}$ for every $b \in B$, and $\bigcup_b A_b$ is finite, then $\vdash \bigcup_{b \in B} A_b \cup \{\wedge B\}$;
- (cut) if $\vdash A \cup \{a\}$ and $\vdash B \cup \{\neg a\}$ for some $a \in \text{Term}(X)$, then $\vdash A \cup B$;

etc. We will freely use these forms from now on (without mentioning weakening).

Now let \leq be defined on $\text{Term}(X)$ by

$$a \leq b :\iff \{a\} \vdash \{b\}.$$

By (generalized) reflexivity and (multiplicative) cut, this is a preorder; let $\equiv := \leq \cap \geq$ be its symmetric part. By (\vee R), we have $\{a\} \vdash \{\vee A\}$, i.e., $a \leq \vee A$, for $a \in A$, while (\vee L) gives that $\vee A$ is the join of A with respect to \leq ; similarly, \wedge is meet. For every $a \in \text{Term}(X)$ and $B \subseteq \text{Term}(X)$, we have the derivation

$$\begin{array}{c} (\wedge R) \frac{\overline{\{a, b\} \vdash \{a\}} \quad \overline{\{a, b\} \vdash \{b\}}}{\overline{\{a, b\} \vdash \wedge \{a, b\}}} \\ (\vee R) \frac{\overline{\{a, b\} \vdash \wedge \{a, b\}}}{\overline{\{a, b\} \vdash \{\vee \{\wedge \{a, b\} \mid b \in B\}\}}} \quad \text{for each } b \in B \\ (\vee L) \frac{\overline{\{a, \vee B\} \vdash \{\vee \{\wedge \{a, b\} \mid b \in B\}\}}}{\overline{\{a, \vee B\} \vdash \{\vee \{\wedge \{a, b\} \mid b \in B\}\}}} \\ (\wedge L \times 2) \frac{\overline{\{a, \vee B\} \vdash \{\vee \{\wedge \{a, b\} \mid b \in B\}\}}}{\overline{\{\wedge \{a, \vee B\}\} \vdash \{\vee \{\wedge \{a, b\} \mid b \in B\}\}}} \end{array}$$

which shows $a \wedge \vee B = \vee_{b \in B} (a \wedge b)$ in $\text{Term}(X)/\equiv$, whence $\text{Term}(X)/\equiv$ is a (large) frame. And from $\vdash \{a, \neg a\}$, applying (\vee R) twice and (\wedge L) once yields $\{\wedge \emptyset\} \vdash \{\vee \{a, \neg a\}\}$, i.e., $\top \leq a \vee \neg a$; thus since \neg is an order-reversing involution, \neg descends to complement in $\text{Term}(X)/\equiv$. Thus

Lemma 2.10.4. $\text{Term}(X)/\equiv$ is a (large) complete Boolean algebra. □

Lemma 2.10.5 (soundness). *Whenever $\vdash A$, then $\vee A = \top$ holds in $\langle X \rangle_{\text{CBOOL}}$.*

Proof. By straightforward induction on the derivation of $\vdash A$. □

Theorem 2.10.6. $\langle X \rangle_{\text{CBOOL}} = \text{Term}(X)/\equiv$.

Proof. Clearly $\text{Term}(X)/\equiv$ is generated as a complete Boolean algebra by (the \equiv -classes of) $x \in X$, so it suffices to check that if two terms $a, b \in \text{Term}(X)$ obey $a \leq b$ in $\text{Term}(X)/\equiv$, i.e., $\{a\} \vdash \{b\}$, then they also obey $a \leq b$ in $\langle X \rangle_{\text{CBOOL}}$; this follows from the preceding lemma. □

The restriction of \vdash to *finite* sets of terms was needed for the inductive proof of Lemma 2.10.3 to go through. It is often more convenient to use an infinitary version of \vdash , which we now give.

We inductively define the relation \vdash on *arbitrary* subsets $A \subseteq \text{Term}(X)$ via the following rules:

- (reflexivity/identity) if $x, \neg x \in A$ for some $x \in X$, then $\vdash A$;
- (\wedge R) if $\wedge B \in A$, and $\vdash A \cup \{b\}$ for every $b \in B$, then $\vdash A$;
- (\vee R) if $\vee B \in A$, and $\vdash A \cup B$, then $\vdash A$.

As before, we also use the obvious two-sided generalization without further comment. It will turn out (by Theorem 2.10.10) that this relation restricted to finite sets is the same as defined before. Whenever we need to disambiguate, we will refer to these rules as the **infinitary sequent calculus** (and the previous ones as the **finitary calculus**; note that this does not refer to the arity of \wedge, \vee).

Lemma 2.10.7. *Weakening (Lemma 2.10.2) and soundness (Lemma 2.10.5) also hold for the infinitary sequent calculus.*

Proof. By induction on derivations in the infinitary calculus. \square

Lemma 2.10.8. *If $\vdash A$ in the finitary calculus, then $\vdash A$ in the infinitary calculus.*

Proof. By induction. In the only non-obvious case of $(\vee R)$, use weakening in the infinitary calculus to go from $\vdash A \cup \{b\}$ to $\vdash A \cup B$. \square

Lemma 2.10.9. *In the infinitary calculus, if $\vdash A \cup \{\vee B\}$, then $\vdash A \cup B$.*

Proof. By induction.

- If $\vdash A \cup \{\vee B\}$ by reflexivity, then there is some $x, \neg x \in A$, so $\vdash A \cup B$ by reflexivity.
- If $\vdash A \cup \{\vee B\}$ by $(\wedge R)$, then there is some $\wedge C \in A$ such that $\vdash A \cup \{\vee B\} \cup \{c\}$ for every $c \in C$; by the induction hypothesis, $\vdash A \cup B \cup \{c\}$ for every $c \in C$, whence by $(\wedge R)$, $\vdash A \cup B$.
- Suppose $\vdash A \cup \{\vee B\}$ by $(\vee R)$ because there is $\vee C \in A \cup \{\vee B\}$ such that $\vdash A \cup \{\vee B\} \cup C$. By the induction hypothesis, we have $\vdash A \cup B \cup C$. If $B = C$, this gives $\vdash A \cup B$; otherwise, we must have $\vee C \in A$, whence by $(\vee R)$, $\vdash A \cup B$. \square

Theorem 2.10.10. *For any subset $A \subseteq \text{Term}(X)$, the following are equivalent:*

- (i) $\vee A = \top$ in $\langle X \rangle_{\text{CBOOL}}$;
- (ii) $\vdash \{\vee A\}$ in the infinitary sequent calculus;
- (iii) $\vdash A$ in the infinitary sequent calculus;
- (iv) $\vdash \{\vee A\}$ in the finitary sequent calculus;
- (v) $\vdash A$ in the finitary sequent calculus, when A is finite.

Proof. (i) is implied by all the other conditions by soundness.

(i) $\iff \{\wedge \emptyset\} \vdash \{\vee A\}$ (in the finitary calculus) by Theorem 2.10.6; the latter implies (iv) by cut against $\vdash \{\wedge \emptyset\}$ which is immediate from $(\wedge R)$.

(iv) \implies (ii) by Lemma 2.10.8.

(ii) \iff (iii) by the preceding lemma and $(\vee R)$.

Similarly, when A is finite, (v) \implies (iv) by $(\vee R)$, and (iv) \implies (v) by cut against $\{\vee A\} \vdash A$ which follows from $(\vee L)$. \square

Corollary 2.10.11 (infinitary transitivity/cut). *In the infinitary calculus, if $\vdash A \cup B$, and $\vdash A \cup \{\neg b\}$ for every $b \in B$, then $\vdash A$.*

Proof. In $\langle X \rangle_{\text{CBOOL}}$, we have $\vee A \vee \vee B = \top$, and $\vee A \vee \neg b = \top$, i.e., $b \leq \vee A$, for every $b \in B$; this clearly implies $\vee A = \top$. \square

2.11 Distributive polyposets

In this subsection, we introduce a method for presenting (λ, κ) -frames for arbitrary λ, κ . This can be seen as an abstraction of the properties of the infinitary sequent calculus from the preceding subsection (see Example 2.11.1), or alternatively, as a “two-sided” generalization of posite presentations from Section 2.6 (see Example 2.11.6).

A (λ, κ) -**distributive prepolyposet**¹⁴ consists of an underlying set A equipped with a binary relation $\triangleleft \subseteq \mathcal{P}_\lambda(A) \times \mathcal{P}_\kappa(A)$ between λ -ary subsets and κ -ary subsets of A , obeying:

- (monotonicity) $B \supseteq C \triangleleft D \subseteq E \implies B \triangleleft E$;
- (reflexivity) $\{a\} \triangleleft \{a\}$ for all $a \in A$;
- (transitivity) if $B \cup C \triangleleft D \cup E$, $B \triangleleft \{c\} \cup E$ for every $c \in C$, and $B \cup \{d\} \triangleleft E$ for every $d \in D$, then $B \triangleleft E$.

Using monotonicity, transitivity is easily seen to be equivalent to the combination of its special cases with $D = \emptyset$ or $C = \emptyset$, which we call **left-transitivity** and **right-transitivity** respectively, or to the following “multiplicative” form; we will use all of these freely:

- if $B \cup C \triangleleft D \cup E$, $B_c \triangleleft \{c\} \cup E_c$ for every $c \in C$, and $B_d \cup \{d\} \triangleleft E_d$ for every $d \in D$, then $B \cup \bigcup_{d \in D} B_d \triangleleft \bigcup_{c \in C} E_c \cup E$, assuming $\bigcup_{d \in D} B_d$ is λ -ary and $\bigcup_{c \in C} E_c$ is κ -ary.

We refer to \triangleleft as a (λ, κ) -**distributive prepolyporder** on A . We define the preorder \leq on A by

$$a \leq b :\iff \{a\} \triangleleft \{b\},$$

and let $\equiv := \leq \cap \geq$. By transitivity, \triangleleft is invariant under \equiv on both sides. If \leq is a partial order, i.e., \equiv is equality, then we call \triangleleft **separated** or a (λ, κ) -**distributive polyorder**, and A a (λ, κ) -**distributive polyposet**. We denote the categories of (λ, κ) -distributive (pre)polyposets and \triangleleft -preserving maps by $\lambda\kappa\text{DP}(\text{r})\text{oly}$; for $\lambda, \kappa < \infty$, both are locally $\max(\lambda, \kappa)$ -presentable, and $\lambda\kappa\text{DPoly} \subseteq \lambda\kappa\text{DPoly}$ is reflective, with reflection given by quotienting by \equiv .

Example 2.11.1. Let X be any set, and let $A \subseteq \text{Term}(X)$ be a set of prenex complete Boolean terms over X , as in the preceding subsection. Declare $B \triangleleft C$ iff $B \vdash C$ in the infinitary sequent calculus from the preceding subsection. Then (A, \triangleleft) is a (λ, κ) -distributive prepolyposet, where the transitivity axiom is by (the two-sided version of) Corollary 2.10.11.

Example 2.11.2. Let A be a (λ, κ) -frame, and define the **canonical polyorder**

$$B \triangleleft C :\iff \bigwedge B \leq \bigvee C.$$

Then (A, \triangleleft) is a (λ, κ) -distributive polyposet. To check the left-transitivity axiom: if

- (i) $\bigwedge B \wedge \bigwedge C \leq \bigvee D$ and
- (ii) $\bigwedge B \leq c \vee \bigvee D$ for every $c \in C$, i.e., $\bigwedge B \leq \bigwedge C \vee \bigvee D$ (by distributivity),

¹⁴“Poly” derives from the theory of polycategories, and refers to the fact that both sides of \triangleleft may consist of multiple elements; see e.g., [Shu20]. “Distributive” refers to the fact that our axioms force existing finite meets/joins to distribute over existing κ -joins/ λ -meets; this follows from Proposition 2.11.3 and Theorem 2.11.4.

then $\bigwedge B = \bigwedge B \wedge (\bigwedge C \vee \bigvee D) \leq (\bigwedge B \wedge \bigwedge C) \vee \bigvee D = \bigwedge D$; right-transitivity is dual. This defines a forgetful functor $\lambda\kappa\mathbf{Frm} \rightarrow \lambda\kappa\mathbf{DPoly}$.

Proposition 2.11.3. *Let A be a (λ, κ) -distributive prepolyposet.*

- (a) *For $B \in \mathcal{P}_\kappa(A)$, a \leq -join $\bigvee B$, if it exists, is characterized by the relations $b \leq \bigvee B$ for each $b \in B$ together with $\{\bigvee B\} \triangleleft B$.*
- (b) *Dually, a meet of $B \in \mathcal{P}_\lambda(A)$ is characterized by $\bigwedge B \leq b$ for $b \in B$ together with $B \triangleleft \{\bigwedge B\}$.*
- (c) *Thus, any (λ, κ) -distributive prepolyposet homomorphism preserves existing λ -ary meets and κ -ary joins in the domain.*
- (d) *Thus, the forgetful functor $\lambda\kappa\mathbf{Frm} \rightarrow \lambda\kappa\mathbf{DPoly}$ is full (and faithful).*

Proof. (a) If $b \leq c$ for every $b \in B$, then by transitivity with $\{\bigvee B\} \triangleleft B$, $\bigvee B \leq c$.

The rest immediately follows. \square

Given a (λ, κ) -distributive prepolyposet (A, \triangleleft) , we let

$$\langle A \mid \triangleleft \rangle_{\lambda\kappa\mathbf{Frm}} := \langle (A, \triangleleft) \text{ qua } \lambda\kappa\mathbf{DProly} \rangle_{\lambda\kappa\mathbf{Frm}} = \langle A \text{ qua } \mathbf{Set} \mid \bigwedge B \leq \bigvee C \text{ for } B \triangleleft C \rangle_{\lambda\kappa\mathbf{Frm}}$$

be the **presented** (λ, κ) -**frame**. This is the left adjoint to the forgetful functor $\lambda\kappa\mathbf{Frm} \rightarrow \lambda\kappa\mathbf{DProly}$. We use similar notation for any of our main categories “above” $\lambda\kappa\mathbf{Frm}$ in the diagram (2.1.1): for $\lambda' \geq \lambda$, $\kappa' \geq \kappa$, and $\mu \geq \lambda, \kappa$, we let $\langle A \mid \triangleleft \rangle_{\lambda'\kappa'\mathbf{Frm}}$ and $\langle A \mid \triangleleft \rangle_{\mu\mathbf{Bool}}$ denote the corresponding presented structure in the respective category. We now have the main result of this subsection, which says that our three axioms on \triangleleft imply that it is already “saturated” under all relations that hold in the presented algebras, analogously to posites (Section 2.6):

Theorem 2.11.4. *For any (λ, κ) -distributive prepolyposet (A, \triangleleft) , the unit $\eta : A \rightarrow \langle A \mid \triangleleft \rangle_{\mathbf{CBool}}$ reflects \triangleleft , i.e., for any $B \in \mathcal{P}_\lambda(A)$ and $C \in \mathcal{P}_\kappa(A)$, we have $B \triangleleft C \iff \bigwedge \eta(B) \leq \bigvee \eta(C)$. Thus if \triangleleft is separated, then η is an embedding of (λ, κ) -distributive polyposets.*

Thus, the same holds for all of the intermediate units $A \rightarrow \langle A \mid \triangleleft \rangle_{\lambda'\kappa'\mathbf{Frm}}$ and $A \rightarrow \langle A \mid \triangleleft \rangle_{\mu\mathbf{Bool}}$.

Proof. By the general construction of presented algebras, $\langle A \mid \triangleleft \rangle_{\mathbf{CBool}}$ is the free complete Boolean algebra $\langle A \rangle_{\mathbf{CBool}} = \langle A \text{ qua } \mathbf{Set} \rangle_{\mathbf{CBool}}$ quotiented by the relations $\bigwedge B \leq \bigvee C$, i.e., $\bigwedge B \rightarrow \bigvee C = \top$, for $B \triangleleft C$. By the description of Boolean algebra quotients at the end of Section 2.4, an element $t \in \langle A \rangle_{\mathbf{CBool}}$ becomes \top in $\langle A \mid \triangleleft \rangle_{\mathbf{CBool}}$ iff it belongs to the principal ∞ -filter $\uparrow \bigwedge_{B \triangleleft C} (\bigwedge B \rightarrow \bigvee C)$. Thus for $D \in \mathcal{P}_\kappa(A)$ and $E \in \mathcal{P}_\lambda(A)$, we have

$$\bigwedge \eta(D) \leq \bigvee \eta(E) \iff \bigwedge_{B \triangleleft C} (\bigwedge B \rightarrow \bigvee C) \leq \bigwedge D \rightarrow \bigvee E \in \langle A \rangle_{\mathbf{CBool}},$$

which, using the infinitary sequent calculus from the preceding subsection, is by Theorem 2.10.10

$$\iff \{\bigvee(\neg B \cup C) \mid B \triangleleft C\} \vdash \neg D \cup E.$$

We induct on derivations to show that for every such D, E , we have $D \triangleleft E$.

- If the last rule used is reflexivity, then there must be some $a \in A$ with $a \in D \cap E$, whence $D \triangleleft E$ by reflexivity of \triangleleft .

- If the last rule used is $(\wedge R)$, i.e., $(\vee L)$, then there is some $B_0 \triangleleft C_0$ such that
 - $\{\vee(\neg B \cup C) \mid B \triangleleft C\} \vdash \{b\} \cup \neg(D) \cup E$ for every $b \in B_0$ and
 - $\{\vee(\neg B \cup C) \mid B \triangleleft C\} \vdash \{\neg c\} \cup \neg(D) \cup E$ for every $c \in C_0$.

By the induction hypothesis, this means $D \triangleleft \{b\} \cup E$ for every $b \in B_0$ and $D \cup \{c\} \triangleleft E$ for every $c \in C_0$. By transitivity with $B_0 \triangleleft C_0$, we get $D \triangleleft E$.

- The last rule cannot be $(\vee R)$, i.e., $(\wedge L)$, due to the form of the sequent. \square

Applying Theorem 2.11.4 to the canonical polyorder on a (λ, κ) -frame yields

Corollary 2.11.5. *For $\lambda' \geq \lambda$, $\kappa' \geq \kappa$, and $\mu \geq \lambda, \kappa$, the units of the free/forgetful adjunctions $\lambda\kappa\text{Frm} \rightleftarrows \lambda'\kappa'\text{Frm}$ and $\lambda\kappa\text{Frm} \rightleftarrows \mu\text{Bool}$ are injective. Equivalently, the left adjoints are faithful.* \square

Note that this gives alternative proofs of Corollaries 2.8.6 and 2.8.9, which are more “syntactic”, being ultimately based on the infinitary sequent calculus. A “semantic” proof of Corollary 2.11.5 in the case $\kappa = \lambda = \infty$ was essentially given by Picado [Pic94, 2.2].

We now know that all of our main categories defined in Section 2.1 admit faithful free functors to all of the ones “above” them, i.e., each of these types of structures embeds into the free complete Boolean algebra it generates. As promised in Convention 2.8.12, for a (λ, κ) -frame A , we henceforth also regard the unit $\eta : A \hookrightarrow \langle A \rangle_{\text{CBOOL}}$ (hence also all intermediate units) as an inclusion.

Next, we note that distributive polyposets are indeed a “two-sided” generalization of posites:

Example 2.11.6. Let (A, \triangleleft) be a $\kappa\text{-}\vee$ -posite (Section 2.6). We may extend \triangleleft to a (λ, κ) -distributive prepolymorder, for any λ , by defining $B \triangleleft C$ to mean that there is $b \in B$ with $b \triangleleft C$.

Proof of transitivity. Left-transitivity: from $B \cup C \triangleleft D$, we either have some $b \in B$ with $b \triangleleft D$ in which case $B \triangleleft D$, or some $c \in C$ with $c \triangleleft D$, in which case from $B \triangleleft \{c\} \cup D$, we have some $b \in B$ with $b \triangleleft \{c\} \cup D$, whence by right-transitivity of the original \triangleleft , we have $b \triangleleft D$ and so $B \triangleleft D$.

Right-transitivity: from $B \triangleleft C \cup D$, we have some $b \in B$ with $b \triangleleft C \cup D$, while for every $c \in C$, from $B \cup \{c\} \triangleleft D$, which means either $c \triangleleft D$ or there is $b_c \in B$ with $b_c \triangleleft D$; if the latter holds for some c then we immediately get $B \triangleleft D$, otherwise $B \triangleleft D$ by right-transitivity of the original \triangleleft . \square

The extended \triangleleft is clearly the (λ, κ) -distributive prepolymorder generated by declaring $\{a\} \triangleleft C$ whenever $a \triangleleft C$. Conversely, any (λ, κ) -distributive prepolymorder \triangleleft on A defines a preorder \leq on A , as well as a $\kappa\text{-}\vee$ -coverage by restricting the left-hand side of \triangleleft to be singletons, which is separated (as defined in Section 2.6, despite \leq being possibly not antisymmetric). Letting κVProst be the category of $\kappa\text{-}\vee$ -preposites, meaning $\kappa\text{-}\vee$ -posites where the underlying poset may instead be a preordered set, we thus have a free/forgetful adjunction $\kappa\text{VProst} \rightleftarrows \lambda\kappa\text{DPoly}$, such that the unit is \triangleleft -reflecting (but the preorder on the image is the restriction of \triangleleft to singletons). This restricts to a reflective adjunction $\kappa\text{VSPost} \rightleftarrows \lambda\kappa\text{DPoly}$ between the full subcategories of separated $\kappa\text{-}\vee$ -posites and (λ, κ) -distributive polyposets, respectively.

(It would perhaps have been more natural for us to only consider separated preposites and separated posites in the first place, since given a non-separated preposite (A, \triangleleft) , we may always replace the preorder \leq_A with \triangleleft on singletons, without affecting the presented $\kappa\text{-}\vee$ -lattice/ κ -frame. This would mirror our definition of prepolymorders via a single relation \triangleleft , from which \leq is then derived. However, in the literature, separation is usually regarded as an extra condition.)

In fact, we can view a $\kappa\text{-}\bigvee$ -(pre)poset as a “ $(\{1\}, \kappa)$ -distributive (pre)polyposet”, just as a $\kappa\text{-}\bigvee$ -lattice is a “ $(\{1\}, \kappa)$ -frame” (see Section 2.1). Regarding general (λ, κ) -distributive (pre)polyposets as presentations for (λ, κ) -frames then mirrors the role of (separated) $\kappa\text{-}\bigvee$ -(pre)posets as presentations for $\kappa\text{-}\bigvee$ -lattices, so that we have a commutative square of adjunctions, analogous to (2.6.1):

$$\begin{array}{ccc} \kappa\bigvee\text{Lat} & \xrightleftharpoons{\perp} & \lambda\kappa\text{Frm} \\ \langle - \rangle_{\kappa\bigvee\text{Lat}} \uparrow \downarrow \dashv & & \langle - \rangle_{\lambda\kappa\text{Frm}} \uparrow \downarrow \dashv \\ \kappa\bigvee(\text{S})\text{P}(\text{r})\text{ost} & \xrightleftharpoons{\perp} & \lambda\kappa\text{DP}(\text{r})\text{oly} \end{array}$$

The bottom left adjoint is full and faithful if we take $\kappa\bigvee\text{SP}(\text{r})\text{ost}$, and an isomorphism if $\lambda = \{1\}$.

Example 2.11.7. Now let (A, \triangleleft) be a κ -poset. If we regard (A, \triangleleft) as a (λ, κ) -distributive prepolyposet as in Example 2.11.6, we lose the information that finite meets in A are to be preserved. Instead, we regard (A, \triangleleft) as a (ω, κ) -distributive prepolyposet via

$$B \triangleleft C :\iff \bigwedge B \triangleleft C \quad \text{for } B \in \mathcal{P}_\omega(A).$$

Proof of transitivity. Left-transitivity: from $B \triangleleft \{c\} \cup D$ for every $c \in C$, we get $\bigwedge B \triangleleft \{c\} \cup D$ for every $c \in C$. Letting $c = \{c_1, \dots, c_n\}$, using \wedge -stability and right-transitivity, we have $\bigwedge B \triangleleft \{\bigwedge B \wedge c_1\} \cup D$, and $\bigwedge B \wedge c_1 \triangleleft \{\bigwedge B \wedge c_1 \wedge c_2\} \cup E$, whence $\bigwedge B \triangleleft \{\bigwedge B \wedge c_1 \wedge c_2\} \cup D$; continuing inductively, we get $\bigwedge B \triangleleft \{\bigwedge B \wedge c_1 \wedge \dots \wedge c_n\} \cup D$. Now from $B \cup C \triangleleft D$, we get $\bigwedge B \wedge \bigwedge C \triangleleft D$, whence by right-transitivity, $\bigwedge B \triangleleft D$. Right-transitivity is similar to Example 2.11.6. \square

This defines a functor $\kappa\text{Prost} \rightarrow \omega\kappa\text{DPoly}$, which, when restricted to κSPost , is a full and faithful *right* adjoint. Its left adjoint takes an (ω, κ) -distributive prepolyposet (A, \triangleleft) , takes the separated quotient, and then freely adjoins finite meets while preserving separatedness (hence does nothing for finite meets that already exist). Moreover, this adjunction is compatible (i.e., forms the obvious commutative triangle) with the previously defined forgetful functors $\kappa\text{Frm} \rightarrow \kappa\text{SPost}$ (from Section 2.6) and $\kappa\text{Frm} \rightarrow \omega\kappa\text{DPoly}$ (Example 2.11.2).

Remark 2.11.8. For $\lambda > \omega$, combining the preceding two examples, one might try to define the free (λ, κ) -distributive prepolyposet generated by a κ -poset (A, \triangleleft) by taking $B \triangleleft C$ iff there is a finite $B' \subseteq B$ with $\bigwedge B' \triangleleft C$. However, the resulting \triangleleft *need not* be transitive. When A is a κ -frame, this corresponds to the fact that given infinite $B, C \subseteq A$, $\bigwedge B \leq \bigvee C$ in $\langle A \rangle_{\text{CBOOL}}$ does not necessarily mean that there is a finite $B' \subseteq B$ with $\bigwedge B' \leq \bigvee C$. (For example, take $\kappa = \lambda = \omega_1$ and $A = \mathcal{O}(\mathbb{R})$; by Remark 3.5.9, countable meets are intersections, thus $\bigwedge_{n \in \mathbb{N}} (n, \infty) = \emptyset$.)

For similar reasons, for an arbitrary (λ, κ) -distributive polyposet (A, \triangleleft) , attempting to extend \triangleleft to arbitrary subsets by $B \triangleleft C$ iff there are λ -ary $B' \subseteq B$ and κ -ary $C' \subseteq C$ with $B' \triangleleft C'$ need not yield a transitive relation.

2.12 Interpolation

In this subsection, we apply distributive polyposets to study colimits in the categories $\kappa\kappa\text{Frm}$, κBool . Our main goal is to prove Theorem 2.12.3, a structural result about pushouts and their ordered analogs, which can be seen as an algebraic version of the Craig–Lyndon interpolation theorem for infinitary propositional logic.

First, we consider coproducts:

Proposition 2.12.1. *Let A_i for $i \in I$ be (λ, κ) -distributive prepolyposets. Then the coproduct of the A_i in $\lambda\kappa\text{DProly}$ is given by the disjoint union $\bigsqcup_i A_i$ with \triangleleft given by the closure under monotonicity of the union of the \triangleleft_{A_i} on each A_i , i.e.,*

$$B \triangleleft C :\iff \exists i (A_i \cap B \triangleleft_{A_i} A_i \cap C) \quad \text{for } B \in \mathcal{P}_\lambda(\bigsqcup_i A_i) \text{ and } C \in \mathcal{P}_\kappa(\bigsqcup_i A_i).$$

The coproduct in κBProly is given similarly, with \neg defined as in each A_i .

Proof. By the general construction of colimits (see Appendix A.1), \triangleleft is the closure of the union of the \triangleleft_{A_i} under monotonicity, reflexivity, and transitivity; since the union is clearly reflexive, it suffices to check that the closure under monotonicity is already transitive. So suppose

- (i) $B \cup C \triangleleft D$, i.e., there is some i such that $A_i \cap (B \cup C) \triangleleft_{A_i} A_i \cap D$;
- (ii) for each $c \in C$, $B \triangleleft \{c\} \cup D$, i.e., there is some i_c such that $A_{i_c} \cap B \triangleleft_{A_{i_c}} A_{i_c} \cap (\{c\} \cup D)$.

If there is some $c \in A_i \cap C$ such that $i_c \neq i$, then $c \notin A_{i_c}$, so (ii) gives $A_{i_c} \cap B \triangleleft_{A_{i_c}} A_{i_c} \cap D$, whence $B \triangleleft D$. Otherwise, left-transitivity of \triangleleft_{A_i} gives $A_i \cap B \triangleleft_{A_i} A_i \cap D$, whence $B \triangleleft D$. This proves left-transitivity; right-transitivity is dual. \square

We may apply this to deduce that only “trivial” relations hold in coproducts of (λ, κ) -frames:

Corollary 2.12.2. *Let A_i for $i \in I$ be (λ, κ) -frames, let $b_i \in A_i$ for each $i \in I$ with $b_i = \top$ for all but $< \lambda$ -many i , and let $c_i \in A_i$ for each $i \in I$ with $c_i = \perp$ for all but $< \kappa$ -many i . If $\bigwedge_i b_i \leq \bigvee_i c_i$ in the coproduct (λ, κ) -frame $\coprod_i A_i$, then some $b_i \leq c_i$.*

Proof. The coproduct $\coprod_i A_i$ may be computed as the reflection $\langle \bigsqcup_i A_i \mid \triangleleft \rangle_{\lambda\kappa\text{Frm}}$ into $\lambda\kappa\text{Frm}$ of the coproduct in $\lambda\kappa\text{DProly}$ given by Proposition 2.12.1. By Theorem 2.11.4, we have $\bigwedge_i b_i \leq \bigvee_i c_i$ in $\coprod_i A_i$ iff $B := \{b_i \mid b_i \neq \top\} \triangleleft \{c_i \mid c_i \neq \perp\} =: C$ in $\bigsqcup_i A_i$. By Proposition 2.12.1, the latter means there is some i such that $A_i \cap B \triangleleft_{A_i} A_i \cap C$ in A_i , which means $b_i \leq c_i$. \square

When $\kappa = \lambda$, we have the following generalization for coproducts with certain relations imposed, which says that the only relations which hold are those implied in one step by transitivity:

Theorem 2.12.3 (interpolation). *Let A, B_i, C_j be $< \kappa$ -many (κ, κ) -frames, with homomorphisms $f_i : A \rightarrow B_i$ and $g_j : A \rightarrow C_j$, and let D be their **bilax pushout**, i.e., the universal (κ, κ) -frame equipped with morphisms as in the diagram*

$$\begin{array}{ccc} B_i & \xrightarrow{\iota_i} & D \\ f_i \uparrow & \nearrow \iota & \uparrow \iota_j \\ A & \xrightarrow{g_j} & C_j \end{array}$$

obeying $\iota_i \circ f_i \leq \iota$ for each i and $\iota \leq \iota_j \circ g_j$ for each j . Let $a^L, a^R \in A$, $b_i^L, b_i^R \in B_i$ for each i , and $c_j^L, c_j^R \in C_j$ for each j . If

$$(*) \quad a^L \wedge \bigwedge_i b_i^L \wedge \bigwedge_j c_j^L \leq a^R \vee \bigvee_i b_i^R \vee \bigvee_j c_j^R \quad \in D$$

(where we omitted mentioning ι, ι_i, ι_j), then there are $a_i^L, a_j^R \in A$ such that

$$\begin{aligned} (\dagger) \quad & \bigwedge_i a_i^L \wedge a^L \leq a^R \vee \bigvee_j a_j^R \quad \in A, \\ & b_i^L \leq f_i(a_i^L) \vee b_i^R \quad \in B_i \quad \forall i, \\ & c_j^L \wedge g_j(a_j^R) \leq c_j^R \quad \in C_j \quad \forall j. \end{aligned}$$

Note that (\dagger) does indeed imply $(*)$ by transitivity; here our omission of the ι 's in $(*)$ somewhat obscures the significance of the directions of the \leq in the bilax pushout, which ensure that e.g., the a_i^L in the second line of (\dagger) is \leq the a_i^L in the first line. The bilax pushout is an instance of a **weighted colimit** in a locally ordered category; see Appendix A.3.

Proof. The bilax pushout D is presented by A, B_i, C_j qua $\kappa\kappa\mathbf{Frm}$ subject to the additional relations $f_i(a) \leq a$ and $a \leq g_j(a)$ for each $a \in A$. For $L, R \in \mathcal{P}_\kappa(A \sqcup \bigsqcup_i B_i \sqcup \bigsqcup_j C_j)$, define

$$L \triangleleft R :\iff \exists a_i^L, a_j^R \in A \text{ s.t. } \begin{cases} \bigwedge_i a_i^L \wedge \bigwedge (A \cap L) \leq \bigvee (A \cap R) \vee \bigvee_j a_j^R & \in A, \\ \bigwedge (B_i \cap L) \leq f_i(a_i^L) \vee \bigvee (B_i \cap R) & \in B_i \ \forall i, \\ \bigwedge (C_j \cap L) \wedge g_j(a_j^R) \leq \bigvee (C_j \cap R) & \in C_j \ \forall j. \end{cases}$$

We claim that \triangleleft is a (κ, κ) -distributive prepolorder on $A \sqcup \bigsqcup_i B_i \sqcup \bigsqcup_j C_j$ containing the canonical \triangleleft on A, B_i, C_j and also obeying $\{f_i(a)\} \triangleleft \{a\}$ and $\{a\} \triangleleft \{g_j(a)\}$ for each $a \in A$. This will imply that \triangleleft contains the relations needed to present D , thus whenever $(*)$ holds in D , then it also holds in $\langle A \sqcup \bigsqcup_i B_i \sqcup \bigsqcup_j C_j \mid \triangleleft \rangle_{\kappa\kappa\mathbf{Frm}}$, whence by Theorem 2.11.4, we have $\{a^L, b_i^L, c_j^L\}_{i,j} \triangleleft \{a^R, b_i^R, c_j^R\}_{i,j}$, which exactly gives (\dagger) .

To check that $L \triangleleft_A R \implies L \triangleleft R$: take $a_i^L := \top$ and $a_j^R := \perp$.

To check that $L \triangleleft_{B_i} R \implies L \triangleleft R$: take $a_i^L := \perp$, $a_{i'}^L := \top$ for all $i' \neq i$, and $a_j^R := \perp$.

Dually, $L \triangleleft_{C_j} R \implies L \triangleleft R$. Reflexivity follows, since $\triangleleft_A, \triangleleft_{B_i}, \triangleleft_{C_j}$ are reflexive. Monotonicity is obvious.

Finally, we check left-transitivity; right-transitivity is dual. Suppose $L \triangleleft R \cup S$, i.e., there are $\bar{a}_i^L, \bar{a}_j^R \in A$ such that

$$\begin{aligned} (a) \quad & \bigwedge_i \bar{a}_i^L \wedge \bigwedge (A \cap L) \leq \bigvee (A \cap (R \cup S)) \vee \bigvee_j \bar{a}_j^R, \\ (b) \quad & \bigwedge (B_i \cap L) \leq f_i(\bar{a}_i^L) \vee \bigvee (B_i \cap (R \cup S)), \\ (c) \quad & \bigwedge (C_j \cap L) \wedge g_j(\bar{a}_j^R) \leq \bigvee (C_j \cap (R \cup S)); \end{aligned}$$

and for each $s \in S$, we have $L \cup \{s\} \triangleleft R$, i.e., there are $\bar{a}_{s,i}^L, \bar{a}_{s,j}^R$ such that

$$\begin{aligned} (a_s) \quad & \bigwedge_i \bar{a}_{s,i}^L \wedge \bigwedge (A \cap (L \cup \{s\})) \leq \bigvee (A \cap R) \vee \bigvee_j \bar{a}_{s,j}^R, \\ (b_s) \quad & \bigwedge (B_i \cap (L \cup \{s\})) \leq f_i(\bar{a}_{s,i}^L) \vee \bigvee (B_i \cap R), \\ (c_s) \quad & \bigwedge (C_j \cap (L \cup \{s\})) \wedge g_j(\bar{a}_{s,j}^R) \leq \bigvee (C_j \cap R). \end{aligned}$$

We must show $L \triangleleft R$. Put

$$\begin{aligned} a_i^L &:= (\bar{a}_i^L \vee \bigvee_{s \in B_i \cap S} \bar{a}_{s,i}^L) \wedge \bigwedge_{s \in S \setminus B_i} \bar{a}_{s,i}^L, \\ a_j^R &:= (\bar{a}_j^R \wedge \bigwedge_{s \in C_j \cap S} \bar{a}_{s,j}^R) \vee \bigvee_{s \in S \setminus C_j} \bar{a}_{s,j}^R. \end{aligned}$$

Then

$$\begin{aligned}
f_i(a_i^L) \vee \bigvee (B_i \cap R) &= ((f_i(\bar{a}_i^L) \vee \bigvee_{s \in B_i \cap S} f_i(\bar{a}_{s,i}^L)) \wedge \bigwedge_{s \in S \setminus B_i} f_i(\bar{a}_{s,i}^L)) \vee \bigvee (B_i \cap R) \\
&\text{(by (b}_s\text{))} \quad \geq ((f_i(\bar{a}_i^L) \vee \bigvee_{s \in B_i \cap S} \bigwedge (B_i \cap (L \cup \{s\}))) \wedge \bigwedge (B_i \cap L)) \vee \bigvee (B_i \cap R) \\
&= ((f_i(\bar{a}_i^L) \vee (\bigwedge (B_i \cap L) \wedge \bigvee (B_i \cap S))) \wedge \bigwedge (B_i \cap L)) \vee \bigvee (B_i \cap R) \\
&\geq (f_i(\bar{a}_i^L) \vee \bigvee (B_i \cap R) \vee \bigvee (B_i \cap S)) \wedge \bigwedge (B_i \cap L) \\
&\text{(by (b))} \quad = \bigwedge (B_i \cap L), \\
\bigwedge (C_j \cap L) \wedge g_j(a_j^R) &= (\bigwedge (C_j \cap L) \wedge g_j(\bar{a}_j^R) \wedge \bigwedge_{s \in C_j \cap S} g_j(\bar{a}_{s,j}^R)) \vee \bigvee_{s \in S \setminus C_j} (\bigwedge (C_j \cap L) \wedge g_j(\bar{a}_{s,j}^R)) \\
&\text{(by (c), (c}_s\text{))} \quad \leq (\bigvee (C_j \cap (R \cup S)) \wedge \bigwedge (C_j \cap L) \wedge \bigwedge_{s \in C_j \cap S} g_j(\bar{a}_{s,j}^R)) \vee \bigvee (C_j \cap R) \\
&\leq \bigvee (C_j \cap R) \vee \bigvee_{s \in C_j \cap S} (s \wedge \bigwedge (C_j \cap L) \wedge g_j(\bar{a}_{s,j}^R)) \\
&\text{(by (c}_s\text{))} \quad = \bigvee (C_j \cap R),
\end{aligned}$$

and letting $t := \bigwedge_i \bigwedge_{s \in S \setminus B_i} \bar{a}_{s,i}^L \wedge \bigwedge (A \cap L)$,

$$\begin{aligned}
\bigwedge_i a_i^L \wedge \bigwedge (A \cap L) &= t \wedge \bigwedge_i (\bar{a}_i^L \vee \bigvee_{s \in B_i \cap S} \bar{a}_{s,i}^L) \\
&\text{(using } t\text{)} \quad = t \wedge \bigwedge_i (\bar{a}_i^L \vee \bigvee_{s \in B_i \cap S} (\bar{a}_{s,i}^L \wedge \bigwedge_{i' \neq i} \bar{a}_{s,i'}^L \wedge \bigwedge (A \cap L))) \\
&= t \wedge \bigwedge_i (\bar{a}_i^L \vee \bigvee_{s \in B_i \cap S} (\bigwedge_{i'} \bar{a}_{s,i'}^L \wedge \bigwedge (A \cap L))) \\
&\text{(by (a}_s\text{))} \quad \leq t \wedge \bigwedge_i (\bar{a}_i^L \vee \bigvee_{s \in B_i \cap S} (\bigvee (A \cap R) \vee \bigvee_j \bar{a}_{s,j}^R)) \\
&\text{(using } t\text{)} \quad \leq t \wedge ((\bigwedge_i \bar{a}_i^L \wedge \bigwedge (A \cap L)) \vee \bigvee (A \cap R) \vee \bigvee_i \bigvee_{s \in B_i \cap S} \bigvee_j \bar{a}_{s,j}^R) \\
&\text{(by (a))} \quad \leq t \wedge (\bigvee (A \cap R) \vee \bigvee (A \cap S) \vee \bigvee_j \bar{a}_j^R \vee \bigvee_i \bigvee_{s \in B_i \cap S} \bigvee_j \bar{a}_{s,j}^R) \\
&\text{(using } t\text{)} \quad = t \wedge (\bigvee (A \cap R) \vee \bigvee_{s \in A \cap S} (s \wedge \bar{a}_s^L \wedge \bigwedge (A \cap L)) \vee \bigvee_j \bar{a}_j^R \vee \bigvee_i \bigvee_{s \in B_i \cap S} \bigvee_j \bar{a}_{s,j}^R) \\
&\text{(by (a}_s\text{))} \quad = t \wedge (\bigvee (A \cap R) \vee \bigvee_{s \in A \cap S} \bigvee_j \bar{a}_{s,j}^R \vee \bigvee_j \bar{a}_j^R \vee \bigvee_i \bigvee_{s \in B_i \cap S} \bigvee_j \bar{a}_{s,j}^R) \\
&= t \wedge (\bigvee (A \cap R) \vee \bigvee_j \bar{a}_j^R \vee \bigvee_{s \in S \setminus \bigsqcup_j C_j} \bigvee_j \bar{a}_{s,j}^R) \\
&\text{(using } t\text{)} \quad \leq \bigvee (A \cap R) \vee \bigvee_j (\bar{a}_j^R \wedge \bigwedge_{s \in C_j \cap S} \bigwedge_i \bar{a}_{s,i}^L \wedge \bigwedge (A \cap L)) \vee \bigvee_{s \in S \setminus \bigsqcup_j C_j} \bigvee_j \bar{a}_{s,j}^R \\
&\text{(by (a}_s\text{))} \quad \leq \bigvee (A \cap R) \vee \bigvee_j (\bar{a}_j^R \wedge \bigwedge_{s \in C_j \cap S} (\bigvee (A \cap R) \vee \bigvee_{j' \neq j} \bar{a}_{s,j'}^R)) \vee \bigvee_{s \in S \setminus \bigsqcup_j C_j} \bigvee_j \bar{a}_{s,j}^R \\
&= \bigvee (A \cap R) \vee \bigvee_j (\bar{a}_j^R \wedge \bigwedge_{s \in C_j \cap S} (\bar{a}_{s,j}^R \vee \bigvee_{j' \neq j} \bar{a}_{s,j'}^R)) \vee \bigvee_{s \in S \setminus \bigsqcup_j C_j} \bigvee_j \bar{a}_{s,j}^R \\
&\leq \bigvee (A \cap R) \vee \bigvee_j (\bar{a}_j^R \wedge \bigwedge_{s \in C_j \cap S} \bar{a}_{s,j}^R) \vee \bigvee_j \bigvee_{s \in S \setminus C_j} \bar{a}_{s,j}^R \\
&= \bigvee (A \cap R) \vee \bigvee_j a_j^R.
\end{aligned}$$

Thus the a_i^L, a_j^R witness $L \triangleleft R$. □

We have the following special cases of Theorem 2.12.3:

Remark 2.12.4. If $A = 2 = \{\perp < \top\}$ is the initial (κ, κ) -frame, the relations $\iota_i \circ f_i \leq \iota$ and $\iota \leq \iota_j \circ g_j$ become trivial, so the bilax pushout in Theorem 2.12.3 is just the coproduct $\coprod_i B_i \amalg \coprod_j C_j$. If we take $a^L = \top$ and $a^R = \perp$, then (\dagger) says that either $a_i^L = \perp$ for some i in which case $b_i^L \leq b_i^R$, or $a_j^R = \top$ for some j in which case $c_j^L \leq c_j^R$. We thus recover Corollary 2.12.2 (for $\lambda = \kappa$).

Now consider Theorem 2.12.3 when A, B_i, C_j are κ -Boolean algebras. Then D , being generated by complemented elements, is also a κ -Boolean algebra. Moreover, the relation $f(a) \leq f(b) \iff$

$f(\neg a) \geq f(\neg b)$ means that imposing inequalities between morphisms is the same as imposing equalities, so that a bilax pushout is simply a pushout. Moving the inequalities in Theorem 2.12.3 to one side, we get the following simplified statement:

Corollary 2.12.5. *Let A, B_i be $< \kappa$ -many κ -Boolean algebras, with homomorphisms $f_i : A \rightarrow B_i$, and let D be their pushout. Let $a \in A$ and $b_i \in B_i$ for each i with $\bigwedge_i b_i \leq a \in D$. Then there are $a_i \in A$ such that $\bigwedge_i a_i \leq a \in A$ and $b_i \leq f_i(a_i) \in B_i$. \square*

The case of a single B_i and C_j in Theorem 2.12.3 is particularly useful:

Corollary 2.12.6. *Let*

$$\begin{array}{ccc} B & \xrightarrow{\iota_1} & D \\ f \uparrow & \searrow & \uparrow \iota_2 \\ A & \xrightarrow{g} & C \end{array}$$

*be a **cocomma** square in $\kappa\kappa\text{Frm}$, i.e., a universal square such that $\iota_1 \circ f \leq \iota_2 \circ g$. Let $b, b' \in B$ and $c, c' \in C$ with $b \wedge c \leq b' \vee c' \in D$. Then there is $a \in A$ with $b \leq f(a) \vee b'$ and $c \wedge g(a) \leq c'$.*

In particular, for $b \in B$ and $c \in C$ with $b \leq c \in D$, there is $a \in A$ with $b \leq f(a)$ and $g(a) \leq c$.

The same holds for pushout squares in κBool .

Proof. Since $\iota_1 \circ f \leq \iota_2 \circ g$ is implied by $\iota_1 \circ f \leq \iota \leq \iota_2 \circ g$ for $\iota : A \rightarrow D$, the cocomma D maps into the bilax pushout, thus $b \wedge c \leq b' \vee c' \in D$ implies that the same relation holds in the bilax pushout, whence Theorem 2.12.3 applies. As above, when $A, B, C \in \kappa\text{Bool}$, we get the Boolean version. \square

The κ -Boolean version of Corollary 2.12.6 was essentially proved by LaGrange [LaG74], via a different method, and in the form of the easily equivalent Corollary 2.12.8 (see [Ch19b, 3.2]).

Corollary 2.12.7 (LaGrange). *Pushout in κBool preserves monomorphisms, i.e., in a pushout*

$$\begin{array}{ccc} B & \xrightarrow{g'} & D \\ f \uparrow & & \uparrow f' \\ A & \xrightarrow{g} & C \end{array}$$

if f is injective, then so is f' . More generally, f' is injective as long as $\ker(f) \subseteq \ker(g)$.

Proof. Let $c \in C$ with $f'(c) = \perp$. Then $f'(c) \leq g'(\perp)$, so by Corollary 2.12.6, there is $a \in A$ with $c \leq g(a)$ and $f(a) \leq \perp$. Since $\ker(f) \subseteq \ker(g)$, we get $c \leq g(a) = \perp$. \square

A category has the **strong amalgamation property** if in the above pushout square, whenever f, g are both injective, then so are f', g' and the square is also a pullback.

Corollary 2.12.8 (LaGrange). *κBool has the strong amalgamation property.*

Proof. Let $(b, c) \in B \times_D C$, i.e., $g'(b) = f'(c)$. Then by Corollary 2.12.6, there are $a, a' \in A$ such that $f(a') \leq b \leq f(a)$ and $g(a) \leq c \leq g(a')$. Since f, g are injective, $a' \leq a$ and $a \leq a'$, whence $f(a) = b$ and $g(a) = c$. \square

A morphism is a **regular monomorphism** if it is the equalizer of its cokernel (i.e., pushout with itself); see Appendix A.2.

Corollary 2.12.9. *All monomorphisms in $\kappa\mathbf{Bool}$ are regular.*

Proof. If $f : A \rightarrow B$ is a monomorphism, then letting $\iota_1, \iota_2 : B \rightrightarrows C$ be its cokernel, for any $b \in \text{eq}(\iota_1, \iota_2)$, i.e., $\iota_1(b) = \iota_2(b)$, by the strong amalgamation property we have $b \in \text{im}(f)$. \square

Corollary 2.12.10 (LaGrange). *All epimorphisms in $\kappa\mathbf{Bool}$ are regular, i.e., surjective.*

Proof. $f : A \rightarrow B$ is an epimorphism iff its pushout with itself is the identity 1_B ; then for any $b \in B$, by Corollary 2.12.6, there is $a \in A$ with $b \leq f(a) \leq b$. \square

Corollary 2.12.11. *For $\kappa \leq \lambda$, the free functor $\langle - \rangle_{\lambda\mathbf{Bool}} : \kappa\mathbf{Bool} \rightarrow \lambda\mathbf{Bool}$ is conservative.*

Proof. Since $\langle - \rangle_{\lambda\mathbf{Bool}}$ is faithful (Corollary 2.8.9), it reflects monomorphisms and epimorphisms; since a morphism in $\kappa\mathbf{Bool}$ which is both a monomorphism and an epimorphism is a bijection by Corollary 2.12.10, hence an isomorphism, it follows that $\langle - \rangle_{\lambda\mathbf{Bool}}$ reflects isomorphisms. \square

We have analogous consequences of Corollary 2.12.6 for $\kappa\kappa\mathbf{Frm}$. Given a morphism $f : A \rightarrow B$ in a locally ordered category, consider its cocomma with itself:

$$\begin{array}{ccc} B & \xrightarrow{\iota_1} & C \\ f \uparrow & \searrow & \uparrow \iota_2 \\ A & \xrightarrow{f} & B \end{array}$$

We say that f is an **order-regular monomorphism** if f exhibits A as the **insertor** $\text{ins}(\iota_1, \iota_2)$, which is the universal subobject of B on which $\iota_1 \leq \iota_2$ holds; in categories of ordered algebraic structures, we have $\text{ins}(\iota_1, \iota_2) = \{b \in B \mid \iota_1(b) \leq \iota_2(b)\}$. We say that f is an **order-epimorphism** if $g \circ f \leq h \circ f \implies g \leq h$ for all $g, h : B \rightrightarrows D$, or equivalently $\iota_1 \leq \iota_2$, i.e., $\text{ins}(\iota_1, \iota_2) = B$. See Appendix A.3.

Corollary 2.12.12. *In any cocomma square in $\kappa\kappa\mathbf{Frm}$ as above, f surjects onto $\text{ins}(\iota_1, \iota_2)$.*

Proof. For any $b \in \text{ins}(\iota_1, \iota_2)$, by Corollary 2.12.6 there is $a \in A$ such that $b \leq f(a) \leq b$. \square

Corollary 2.12.13. *All monomorphisms in $\kappa\kappa\mathbf{Frm}$ are order-regular.* \square

Corollary 2.12.14. *All order-epimorphisms in $\kappa\kappa\mathbf{Frm}$ are surjective.* \square

Corollary 2.12.15. *For $\kappa \leq \lambda$, the free functor $\kappa\kappa\mathbf{Frm} \rightarrow \lambda\lambda\mathbf{Frm}$ is conservative.*

Proof. This follows similarly to Corollary 2.12.11 from Corollary 2.12.14 and the fact that the free functor is order-faithful, i.e., an order-embedding on each hom-poset (because the units are order-embeddings, by Corollary 2.11.5). \square

Remark 2.12.16. Epimorphisms in $\kappa\kappa\mathbf{Frm}$ are *not* necessarily surjective: the unit $A \rightarrow \langle A \rangle_{\kappa\mathbf{Bool}}$ is an epimorphism, essentially because \neg is uniquely defined in terms of \wedge, \vee . Indeed, the set of $b \in \langle A \rangle_{\kappa\mathbf{Bool}}$ at which the value of every $f : \langle A \rangle_{\kappa\mathbf{Bool}} \rightarrow C \in \kappa\mathbf{Bool}$ is determined by $f|_A$ is easily seen to be a κ -Boolean subalgebra containing A , hence is all of $\langle A \rangle_{\kappa\mathbf{Bool}}$.

It follows that monomorphisms in $\kappa\kappa\mathbf{Frm}$ are *not* necessarily regular: $A \rightarrow \langle A \rangle_{\kappa\mathbf{Bool}}$ is a monomorphism and an epimorphism; if it were a regular monomorphism, then (being an epimorphism) it would have to be an isomorphism.

Remark 2.12.17. Order-epimorphisms in $\lambda\kappa\mathbf{Frm}$ for $\lambda \neq \kappa$ are *not* necessarily surjective. Take $\lambda = \omega$ and $\kappa \geq \omega_1$, and consider the topology $\mathcal{O}(\overline{\mathbb{N}})$ of $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, the one-point compactification of the discrete space \mathbb{N} . The inclusion $\mathcal{O}(\overline{\mathbb{N}}) \hookrightarrow \mathcal{P}(\overline{\mathbb{N}})$ is a non-surjective κ -frame order-epimorphism. Indeed, every $a \in \mathcal{P}(\overline{\mathbb{N}}) \setminus \mathcal{O}(\overline{\mathbb{N}})$ is the join $a = (a \cap \mathbb{N}) \cup \{\infty\}$ of some $a \cap \mathbb{N} \in \mathcal{O}(\overline{\mathbb{N}})$ with an element which is a meet $\{\infty\} = \bigcap_{n \in \mathbb{N}} [n, \infty] = \neg \bigcup_n \neg[n, \infty]$ of complemented elements of $\mathcal{O}(\overline{\mathbb{N}})$. Thus for κ -frame homomorphisms $f, g : \mathcal{P}(\overline{\mathbb{N}}) \rightrightarrows B$, which automatically preserve complements, if $f \leq g$ on $\mathcal{O}(\overline{\mathbb{N}})$, then also $f(\{\infty\}) \leq g(\{\infty\})$, whence $f \leq g$ on all of $\mathcal{P}(\overline{\mathbb{N}})$.

It follows that Theorem 2.12.3 must also fail for $\lambda\kappa\mathbf{Frm}$ with $\lambda \neq \kappa$, or else the proof of Corollary 2.12.14 would go through.

Remark 2.12.18. The proof of Theorem 2.12.3 is inspired by the well-known technique of proving the Craig–Lyndon interpolation theorem by induction on derivations in the cut-free sequent calculus. See e.g., [TS00, §4.4]. However, we note that Theorem 2.12.3 does not immediately follow from applying this technique to the sequent calculus from Section 2.10: since we do not assume the algebras are free (or even κ -presented), we would need to include extra “non-logical axioms”, which the cut admissibility proof using *finitary* sequents from Lemma 2.10.3 does not easily handle.

3 Locales and Borel locales

In this section, we study the categories of “formal spaces” which are the duals of the algebraic categories $\kappa\mathbf{Frm}, \kappa\kappa\mathbf{Frm}, \kappa\mathbf{Bool}$ studied in the previous section. In Section 3.1, we define these categories and discuss their relations to each other as well as to the corresponding categories of spaces; in Section 3.2, we complete this discussion by defining the spatialization adjunctions between the corresponding categories of locales and spaces. In Section 3.3, we define the κ -Borel hierarchy on a κ -locale via the dual of the \mathcal{N}_κ functor from Section 2.8, and show how several algebraic results we proved in Sections 2.8 and 2.9 correspond to classical descriptive set-theoretic results on “change of topology”. In the crucial Section 3.4, we give a detailed analysis of notions of “sublocale”/“subobject”, “embedding”, “injection”, “surjection”, and “image” in our main categories.

The rest of the subsections cover somewhat more specialized topics. In Section 3.5, we review some known results which say that the spatialization adjunctions become equivalences when $\kappa = \omega_1$ and furthermore the algebras/spaces involved are restricted to being countably (co)presented; we then use these to give several important examples. In Section 3.6, we review some abstract categorical properties of our various categories of (Borel) locales, as well as the method from categorical logic of interpreting “pointwise” expressions in arbitrary categories obeying such properties. In Section 3.7, we show that our category of “positive κ -Borel locales” (dual to $\kappa\kappa\mathbf{Frm}$) can be meaningfully regarded as κ -Borel locales equipped with an intrinsic “specialization order”. Finally, in Section 3.8 we recall Isbell’s well-known localic Baire category theorem and discuss some of its consequences as well as a generalization.

3.1 The main categories

A topological space $X = (X, \mathcal{O}(X))$ can be defined as an underlying set X together with a subframe $\mathcal{O}(X) \subseteq \mathcal{P}(X)$; a continuous map $f : X \rightarrow Y$ between topological spaces is a map such that $f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ restricts to a frame homomorphism $\mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. A **locale** X is, formally, the same thing as an arbitrary frame $\mathcal{O}(X)$, but thought of as the frame of “open sets” in a generalized “space” without an underlying set. A **continuous map** $f : X \rightarrow Y$ between locales is

a frame homomorphism $f^* : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. The **category of locales** will be denoted \mathbf{Loc} ; thus by definition, $\mathbf{Loc} = \mathbf{Frm}^{\text{op}}$.

Convention 3.1.1. There are various commonly used conventions regarding the locale/frame distinction; see e.g., [PP12]. We follow Isbell [Isb72] in **strictly distinguishing between locales and frames**: the former are “geometric” objects, while the latter are their “algebraic” duals.

This convention means that we may unambiguously speak of notions such as **product locales** $X \times Y$ (coproduct frames $\mathcal{O}(X \times Y) := \mathcal{O}(X) \otimes \mathcal{O}(Y)$), which only implicitly refer to the direction of morphisms (the product projection $\pi_1 : X \times Y \rightarrow X$ corresponding to the coproduct injection $\iota_1 : \mathcal{O}(X) \rightarrow \mathcal{O}(X) \otimes \mathcal{O}(Y)$).

Convention 3.1.2. In the locale theory literature, it is common to refer to the elements of the frame $\mathcal{O}(X)$ corresponding to a locale X as *open parts*, *opens*, or some other name emphasizing that they need not be subsets in the usual sense; see e.g., [Isb91]. However, we will be dealing extensively with localic analogs of other types of sets, e.g., closed, Borel, and analytic sets, and we feel it would be rather awkward and confusing to similarly rename all of these. Therefore, we will freely refer to elements $U \in \mathcal{O}(X)$ as **open sets of X** . Note that there should be no ambiguity, since by Convention 3.1.1, there is no other notion of “subset” of X . When working in $\mathcal{O}(X)$, we will freely use set-theoretic notation interchangeably with order-theoretic ones, e.g.,

$$U \subseteq V :\iff U \leq V, \quad U \cap V := U \wedge V, \quad X := \top, \quad \emptyset := \perp.$$

For each of the other categories defined in Section 2.1, consisting of algebraic structures with a subset of the operations of a complete Boolean algebra, we can consider an analogous notion of “space” consisting of an underlying set X equipped with a subalgebra (of the specified type) of the powerset $\mathcal{P}(X)$. We can then define a pointless generalization of such “spaces” by dropping the underlying set and the requirement that the algebra be a subalgebra of a powerset.

Perhaps the most familiar example of such a “space”, other than topological, is a **Borel space** (or *measurable space*) $X = (X, \mathcal{B}(X))$, consisting of a set X together with a σ -algebra $\mathcal{B}(X)$, i.e., a σ -Boolean subalgebra $\mathcal{B}(X) \subseteq \mathcal{P}(X)$, whose elements are called **Borel sets** (or *measurable sets*). We analogously define a **(σ -)Borel locale** X to mean an arbitrary σ -Boolean algebra $\mathcal{B}(X) = \mathcal{B}_\sigma(X)$, whose elements are called **(σ -)Borel sets** of X . A **(σ -)Borel map** $f : X \rightarrow Y$ between σ -Borel locales is a σ -Boolean homomorphism $f^* : \mathcal{B}_\sigma(Y) \rightarrow \mathcal{B}_\sigma(X)$. The **category of (σ -)Borel locales** will be denoted $\sigma\mathbf{BorLoc} := \sigma\mathbf{Bool}^{\text{op}}$.

Analogously, by a **κ -Borel space** we mean a set X equipped with a κ -Boolean subalgebra $\mathcal{B}_\kappa(X) \subseteq \mathcal{P}(X)$. We then define a **κ -Borel locale** X to be an arbitrary κ -Boolean algebra $\mathcal{B}_\kappa(X)$, whose elements are called **κ -Borel sets** of X . A **κ -Borel map** $f : X \rightarrow Y$ is a κ -Boolean homomorphism $\mathcal{B}_\kappa(Y) \rightarrow \mathcal{B}_\kappa(X)$. The **category of κ -Borel locales** is $\kappa\mathbf{BorLoc} := \kappa\mathbf{Bool}^{\text{op}}$. We include the case $\kappa = \infty$ by taking an **∞ -Borel locale** X to be a *small-presented large* complete Boolean algebra $\mathcal{B}_\infty(X)$; thus $\infty\mathbf{BorLoc} := \mathbf{CBool}_\infty^{\text{op}}$.

Example 3.1.3. An ω -Borel space is a set X equipped with a Boolean algebra of subsets $\mathcal{B}_\omega(X)$. This can be seen as a description of a zero-dimensional compactification of X (namely the Stone space of $\mathcal{B}_\omega(X)$), hence might be called an “ultraproximity” on X . On the other hand, an ω -Borel locale is, equivalently by Stone duality, just a Stone space.

Example 3.1.4. An ∞ -Borel space is a set X equipped with the complete (atomic) Boolean algebra $\mathcal{B}_\infty(X)$ of invariant sets for some equivalence relation on X ; if $\mathcal{B}_\infty(X)$ separates points of X , then

it must be the full powerset $\mathcal{P}(X)$. On the other hand, an ∞ -Borel locale can, in particular, be given by an arbitrary small complete Boolean algebra, possibly atomless. For example, the regular open or Lebesgue measure algebra of $[0, 1]$ correspond to ∞ -Borel locales, which, intuitively, are the “intersections” of all dense open, respectively conull, sets in $[0, 1]$ (see Example 3.8.4).

Analogously in the topological setting, by a κ -**topological space** we will mean a set X equipped with a κ -subframe $\mathcal{O}_\kappa(X) \subseteq \mathcal{P}(X)$. We then define a κ -**locale** X to be an arbitrary κ -frame $\mathcal{O}_\kappa(X)$, whose elements are called κ -**open sets** of X ; a κ -frame homomorphism $\mathcal{O}_\kappa(Y) \rightarrow \mathcal{O}_\kappa(X)$ is a κ -**continuous map** $X \rightarrow Y$, and the **category of κ -locales** is denoted $\kappa\text{Loc} := \kappa\text{Frm}^{\text{op}}$. (Note that when we omit κ , we mean $\kappa = \omega_1$ in the Borel context but $\kappa = \infty$ in the topological context, in accord with standard terminology.)

The main example of a κ -topological space is a topological space which is κ -**based**, i.e., has a κ -ary (sub)basis; in such a space, arbitrary unions of open sets are determined by κ -ary ones, i.e., a κ -based topological space is the same thing as a κ -based κ -topological space (see Proposition 2.3.5). For example, a second-countable ($= \sigma$ -based) topological space can be faithfully regarded as a σ -topological space. The analogous class of κ -locales correspond to the κ -generated (κ -)frames.

A (second-countable σ -)topological space contains more structure than a Borel space, which may be forgotten by replacing the (σ -)topology with the Borel σ -algebra it generates. Motivated by this, we adopt the following

Convention 3.1.5. We regard all of the *free* functors from Section 2.1 as nameless *forgetful* functors between the dual categories of locales (and variations thereof). Thus, for example, a σ -locale X has an **underlying σ -Borel locale**, given by $\mathcal{B}_\sigma(X) := \langle \mathcal{O}_\sigma(X) \text{ qua } \sigma\text{Frm} \rangle_{\sigma\text{Bool}}$.

Note that this convention is consistent with our Convention 2.8.12, according to which $\mathcal{O}_\sigma(X) \subseteq \mathcal{B}_\sigma(X)$ is regarded as a σ -subframe, namely the σ -subframe of σ -open sets. We also have $\mathcal{O}_\sigma(X) \subseteq \mathcal{O}(X) := \langle \mathcal{O}_\sigma(X) \text{ qua } \sigma\text{Frm} \rangle_{\text{Frm}} \cong \sigma\mathcal{I}(\mathcal{O}_\sigma(X))$; note that $\mathcal{O}_\sigma(X) = \mathcal{O}(X)_\sigma$ (by Proposition 2.3.2), i.e., “ σ -open” = “ σ -compact open”. All of these lattices are contained in the (large) complete Boolean algebra $\mathcal{B}_\infty(X) := \langle \mathcal{O}_\sigma(X) \text{ qua } \sigma\text{Frm} \rangle_{\text{CBool}}$. It is often helpful to regard $\mathcal{B}_\infty(X)$ as playing role of the “powerset $\mathcal{P}(X)$ ” in the localic context; see Example 3.1.4. For example, a locale X consists of an underlying ∞ -Borel locale together with a “topology” $\mathcal{O}(X) \subseteq \mathcal{B}_\infty(X)$. (However, $\mathcal{O}(X)$ is not allowed to be an arbitrary subframe of $\mathcal{B}_\infty(X)$: it needs to be small and to freely generate $\mathcal{B}_\infty(X)$ qua frame. See Remark 2.8.14.)

Via this analogy, we can define the analogs of various common topological set notions in a locale X : a **closed set** will mean an element of $\neg\mathcal{O}(X) = \{\neg U \mid U \in \mathcal{O}(X)\} \subseteq \mathcal{B}_\infty(X)$; a **clopen set** is an element of $\mathcal{O}(X)_-$; the **interior** B° and **closure** \overline{B} of an arbitrary ∞ -Borel $B \in \mathcal{B}_\infty(X)$ are the greatest open set $\leq B$ and least closed set $\geq B$, respectively; B is **dense** if $\overline{B} = X (= \top \in \mathcal{B}_\infty(X))$. We also call X κ -**based** if $\mathcal{O}(X)$ is κ -generated (as a frame or \vee -lattice).

Remark 3.1.6. If X is a compact Hausdorff space, or more generally a regular Lindelöf space, then $\mathcal{O}(X)$ is a σ -coherent frame, whose σ -compact elements $\mathcal{O}_\sigma(X) \subseteq \mathcal{O}(X)$ are the cozero open sets in X . Thus, σ -locales can also be regarded as a generalization of regular Lindelöf spaces; see [MV86] for this point of view.

The **Baire σ -algebra** in a topological space is that generated by the (co)zero sets (see e.g., [Fre06, 4A3K]; this definition agrees with other commonly used definitions in compact Hausdorff spaces). Thus, it would perhaps have been more consistent for us to use “Baire set” instead of “Borel set” in our generalized terminology. We have chosen the latter, due to its far more ubiquitous use (as well as the various other unrelated uses of “Baire”) in classical descriptive set theory.

We also introduce analogous terminology corresponding to the categories $\kappa\kappa\mathbf{Frm}$. A **positive κ -Borel space** is a set X equipped with a (κ, κ) -subframe $\mathcal{B}_\kappa^+(X) \subseteq \mathcal{P}(X)$; a **positive κ -Borel locale** X is an arbitrary (κ, κ) -frame $\mathcal{B}_\kappa^+(X)$, whose elements are called **positive κ -Borel sets** of X ; and the category of these and **positive κ -Borel maps** (i.e., (κ, κ) -frame homomorphisms in the opposite direction) is denoted $\kappa\mathbf{Bor}^+\mathbf{Loc} := \kappa\kappa\mathbf{Frm}^{\text{op}}$. As with κ -Borel locales, we take a **positive ∞ -Borel locale** to mean a small-presented large (∞, ∞) -frame. These notions sit in between locales and Borel locales: every κ -locale X has an underlying positive κ -Borel locale (given by $\mathcal{B}_\kappa^+(X) := \langle \mathcal{O}_\kappa(X) \text{ qua } \kappa\mathbf{Frm} \rangle_{\kappa\kappa\mathbf{Frm}}$), while every positive κ -Borel locale X has an underlying κ -Borel locale (given by $\mathcal{B}_\kappa(X) := \langle \mathcal{B}_\kappa^+(X) \text{ qua } \kappa\kappa\mathbf{Frm} \rangle_{\kappa\mathbf{Bor}}$).

The main purpose of introducing positive Borel locales is to provide the localic analog of the specialization preorder on a topological space:

Example 3.1.7. A positive ∞ -Borel space is a set X equipped with a collection of subsets $\mathcal{B}_\infty^+(X)$ closed under arbitrary intersections and unions; such a collection is also known as an **Alexandrov topology** on X , and consists of all the upper sets with respect to some preorder \leq on X . The **specialization preorder** on an arbitrary topological space X is defined by

$$x \leq y :\iff x \in \overline{\{y\}} \iff \forall x \in U \in \mathcal{O}(X) (y \in U),$$

and corresponds to the Alexandrov topology generated by the topology on X . Thus, the specialization preorder can be regarded as the “underlying positive ∞ -Borel space” of a topological space, where positive ∞ -Borel sets are precisely the upper sets; the underlying positive ∞ -Borel locale of a locale is analogous.

Example 3.1.8. It is a classical theorem from descriptive set theory (see [Kec95, 28.11]) that a Borel subset of Cantor space $2^\mathbb{N}$ is upward-closed in the product of the usual ordering $2 = \{0 < 1\}$ iff it belongs to the closure under countable intersections and unions of the (subbasic) open sets for the Scott topology on $2^\mathbb{N}$, i.e., the product $\mathbb{S}^\mathbb{N}$ of the **Sierpinski space** $\mathbb{S} := 2$ with the topology where $\{1\}$ is open but not closed. In other words, the underlying positive (σ) -Borel space of $\mathbb{S}^\mathbb{N}$ remembers precisely the Borel σ -algebra together with the specialization preorder. We will prove the κ -localic analog of the theorem just cited in Theorem 3.7.3 below.

We locally order the categories $\kappa\mathbf{Bor}^+\mathbf{Loc}$, as well as $\kappa\mathbf{Loc}$, via the *same* pointwise partial order on the hom-sets of the dual categories $\kappa\kappa\mathbf{Frm}$ and $\kappa\mathbf{Frm}$. That is, for $f, g : X \rightarrow Y$, say in $\kappa\mathbf{Loc}$,

$$f \leq g :\iff \forall U \in \mathcal{O}_\kappa(Y) (f^*(U) \leq g^*(U)).$$

(Note that we do not flip the partial order on the hom-sets; in 2-categorical terminology, we flip the 1-cells but not the 2-cells, i.e., $\kappa\mathbf{Loc} = \kappa\mathbf{Frm}^{\text{op}} \neq \kappa\mathbf{Frm}^{\text{coop}}$.) To motivate this, note that for functions $f, g : X \rightarrow Y$ to a preorder Y , we have $f \leq g$ pointwise iff for every upper $U \subseteq Y$, we have $f^{-1}(U) \subseteq g^{-1}(U)$.

The following diagram, dual to part of the diagram (2.1.1), depicts the relationships between the categories introduced above, for $\kappa \leq \lambda$:

$$(3.1.9) \quad \begin{array}{ccccccc} \text{Stone} \simeq \omega\mathbf{BorLoc} & \xleftarrow{\tau} & \sigma\mathbf{BorLoc} & \xleftarrow{\tau} & \cdots & \xleftarrow{\tau} & \kappa\mathbf{BorLoc} & \xleftarrow{\tau} & \lambda\mathbf{BorLoc} & \longrightarrow & \infty\mathbf{BorLoc} \\ \uparrow \downarrow & & \uparrow \downarrow & & & & \uparrow \downarrow & & \uparrow \downarrow & & \uparrow \downarrow \\ \omega\mathbf{Bor}^+\mathbf{Loc} & \xleftarrow{\tau} & \sigma\mathbf{Bor}^+\mathbf{Loc} & \xleftarrow{\tau} & \cdots & \xleftarrow{\tau} & \kappa\mathbf{Bor}^+\mathbf{Loc} & \xleftarrow{\tau} & \lambda\mathbf{Bor}^+\mathbf{Loc} & \longrightarrow & \infty\mathbf{Bor}^+\mathbf{Loc} \\ \parallel & & \uparrow \downarrow & & & & \uparrow \downarrow & & \uparrow \downarrow & & \uparrow \\ \text{Spec} \simeq \omega\mathbf{Loc} & \xleftarrow{\tau} & \sigma\mathbf{Loc} & \xleftarrow{\tau} & \cdots & \xleftarrow{\tau} & \kappa\mathbf{Loc} & \xleftarrow{\tau} & \lambda\mathbf{Loc} & \longrightarrow & \mathbf{Loc} \end{array}$$

The functors pointing upward or to the right are the forgetful functors from Convention 3.1.5, corresponding to the *free* functors in (2.1.1); the left adjoints, when they exist, correspond to the *forgetful* functors in (2.1.1). (Most free functors from ∞BorLoc , $\infty\text{Bor}^+\text{Loc}$ do not exist, since they are dual to categories of small-presented large structures; but the free functor from the former category to the latter, i.e., the forgetful functor $\text{CBOOL}_\infty \rightarrow \infty\text{FRM}_\infty$, does exist, because a small-presented complete Boolean algebra is also small-presented as an (∞, ∞) -frame by adding complements of generators, as described before Corollary 2.5.11.) Here **Stone** denotes the category of Stone spaces, while **Spec** denotes the category of **spectral spaces** (or **coherent spaces**), which is dual to $\omega\text{Frm} = \text{DLat}$ by Stone duality for distributive lattices; see [Joh82, II 3.4].

By the various results in Section 2 on preservation properties of free functors, we have

Proposition 3.1.10.

- (a) *All of the forgetful functors in the above diagram (3.1.9) are faithful, as well as order-faithful for the functors with domain and codomain in the bottom two rows.*
- (b) *The horizontal forgetful functors are also conservative, as well as full-on-isomorphisms in the case of $\kappa\text{Loc} \rightarrow \lambda\text{Loc}$.*
- (c) *The forgetful functors whose domain is in column “ κ ” preserve κ -ary coproducts (as well as all small limits).*

Proof. By Corollaries 2.3.4, 2.8.6, 2.8.9, 2.11.5, 2.12.11 and 2.12.15, the results in Section 2.5, and the fact that free functors always preserve colimits. \square

Intuitively, the fact that the forgetful functors are faithful means that all of the above categories can be seen as consisting of ∞ -Borel locales equipped with additional structure, while the morphisms are ∞ -Borel maps which preserve that additional structure. Fullness of $\kappa\text{Loc} \rightarrow \lambda\text{Loc}$ on isomorphisms means that being a κ -locale can be regarded as a mere *property* of a locale, rather than additional structure; that is, we may treat κ -locales as particular kinds of locales.¹⁵ Conservativity of the other horizontal functors means that e.g., if a σ -Borel map $f : X \rightarrow Y$ between σ -Borel locales is an ∞ -Borel isomorphism, then its inverse (as an ∞ -Borel map) is automatically a σ -Borel map. This can be seen as an analog of the result from classical descriptive set theory that a Borel bijection between standard Borel spaces automatically has Borel inverse (see [Kec95, 14.12]).

We use the following terminology for (co)limits in the above categories. As already mentioned (Convention 3.1.1), **product locales** will mean categorical products, dual to coproducts in the dual algebraic categories. We will also refer to κ -ary coproducts in κLoc , κBorLoc , $\kappa\text{Bor}^+\text{Loc}$ as **disjoint unions**, denoted \bigsqcup , and corresponding to products of the dual algebras; the nullary coproduct is also called the **empty locale** \emptyset (with $\mathcal{O}(\emptyset) = \mathcal{B}_\infty(\emptyset) = 1$). In addition to Proposition 3.1.10(c), the dual of Proposition 2.5.1 tells us that such coproducts are well-behaved:

Proposition 3.1.11. *κ -ary disjoint unions in κLoc , κBorLoc , $\kappa\text{Bor}^+\text{Loc}$ are pullback-stable.* \square

A κ -**sublocale** of a κ -locale X will mean a *regular* subobject of $X \in \kappa\text{Loc}$, hence an equivalence class of κ -locales Y equipped with a κ -continuous map $f : Y \rightarrow X$ which is a regular monomorphism, i.e., $f^* : \mathcal{O}_\kappa(X) \rightarrow \mathcal{O}_\kappa(Y) \in \kappa\text{Frm}$ is a regular epimorphism, i.e., surjective (see Appendix A.2). In other words, every κ -open set in Y is the “restriction” $f^*(U)$ of a κ -open set U in X . We call such

¹⁵In the literature (e.g., [Joh82]), κ -locales are usually called κ -**coherent locales**.

an $f : Y \rightarrow X$ an **embedding of κ -locales**. By the usual abuse of notation for subobjects (again see Appendix A.2), we also refer to Y itself as a κ -sublocale of X , denoted $Y \subseteq X$, and refer to the embedding f as the “inclusion” $Y \hookrightarrow X$ (the notation $Y \subseteq X$ is consistent with our notation for κ -Borel sets $B \subseteq X$, by Convention 3.4.11 below). Let

$$\mathbf{RSub}_{\kappa\mathbf{Loc}}(X) := \{\text{regular subobjects of } X \in \kappa\mathbf{Loc}\} = \{\kappa\text{-sublocales of } X\}.$$

The terms **κ -Borel sublocale** and **positive κ -Borel sublocale** likewise mean regular subobjects in $\kappa\mathbf{BorLoc}$, $\kappa\mathbf{Bor}^+\mathbf{Loc}$ respectively, hence quotients in $\kappa\mathbf{Bool}$, $\kappa\mathbf{Frm}$ respectively. However, due to Corollaries 2.12.10 and 2.12.14, these are the same as ordinary subobjects in $\kappa\mathbf{BorLoc}$, respectively order-embedded subobjects in $\kappa\mathbf{Bor}^+\mathbf{Loc}$ (see Appendix A.3, and Proposition 3.4.25 below).

Finally, a (positive) κ -(Borel) locale will be called **λ -copresented** if its dual algebra is λ -presented, and the category of all these will be denoted $\kappa(\mathbf{Bor}^{(+)})\mathbf{Loc}_\lambda \subseteq \kappa(\mathbf{Bor}^{(+)})\mathbf{Loc}$. This implies that it is κ -based. By the dual of Proposition 2.3.5,

$$\mathbf{Loc}_\kappa = \kappa\mathbf{Loc}_\kappa = \lambda\mathbf{Loc}_\kappa \quad \text{for } \lambda \geq \kappa.$$

We also call a κ -copresented (positive) κ -(Borel) locale a **standard (positive) κ -(Borel) locale**. This is motivated by the equivalence between σ -copresented (σ -)Borel locales and **standard Borel spaces** (Borel subspaces of $2^{\mathbb{N}}$) from classical descriptive set theory (see Section 3.5 below). Note that by our definition, all (positive) ∞ -(Borel) locales are standard.

Proposition 3.1.12. *κ -ary limits and disjoint unions, as well as the forgetful functors in (3.1.9) from column “ κ ”, preserve κ -copresentability.*

Proof. κ -ary disjoint unions are by Corollaries 2.5.8 and 2.5.12; the rest are general facts about κ -presentable objects in the dual categories. \square

3.2 Spatialization

Given a topological space X , we may forget its underlying set and keep only the frame $\mathcal{O}(X)$ of open sets, yielding the **underlying locale** of X ; we treat this as a nameless forgetful functor

$$\mathbf{Top} \longrightarrow \mathbf{Loc},$$

which is formally the same as the functor $\mathcal{O} : \mathbf{Top} \rightarrow \mathbf{Frm}^{\text{op}}$.

Conversely, given a locale X , we may define a canonical topological space from it as follows. A **point** in X is a continuous map $x : 1 \rightarrow X$, or equivalently a frame homomorphism $x^* : \mathcal{O}(X) \rightarrow \mathcal{O}(1) = 2$. We may identify points $x : 1 \rightarrow X$ with the sets $(x^*)^{-1}(\top) \subseteq \mathcal{O}(X)$, which are **completely prime filters**, i.e., complements of principal ideals $(x^*)^{-1}(\perp)$, thought of as the neighborhood filter of x ; the complement (in $\mathcal{B}_\infty(X)$) of the greatest element of $(x^*)^{-1}(\perp)$ can be thought of as the closure $\overline{x} := \overline{\{x\}}$, and is an **irreducible** closed set, meaning $\uparrow \overline{x}$ is a prime filter in $\neg\mathcal{O}(X)$. The **spatialization** of X is the space of points

$$\mathbf{Sp}(X) := \mathbf{Loc}(1, X)$$

equipped with the topology consisting of the open sets

$$\mathbf{Sp}(U) := \{x \in \mathbf{Sp}(X) \mid x^*(U) = \top \in 2\}$$

for $U \in \mathcal{O}(X)$. In other words, $\text{Sp} : \mathcal{O}(X) \rightarrow \mathcal{P}(\text{Sp}(X))$ is easily seen to be a frame homomorphism; $\mathcal{O}(\text{Sp}(X))$ is by definition its image.

Given a topological space X and a locale Y , a continuous map $f : X \rightarrow Y$ induces a map $X \rightarrow \text{Sp}(Y)$ taking each point $x \in X$ to the composite map of locales $f(x) : 1 \xrightarrow{x} X \xrightarrow{f} Y$, given by $f(x)^*(V) = x^*(f^*(V)) = (\top \text{ iff } x \in V)$ for $V \in \mathcal{O}(Y)$. In other words, f is equivalently a frame homomorphism $f^* : \mathcal{O}(Y) \rightarrow \mathcal{O}(X) \subseteq \mathcal{P}(X) \cong 2^X$, hence transposes to a map $X \rightarrow 2^{\mathcal{O}(Y)}$ whose image lands in the set of frame homomorphisms, i.e., in $\text{Sp}(Y)$; this is the map induced by f . Conversely, a map $g : X \rightarrow \text{Sp}(Y)$ corresponds in this way to a frame homomorphism $(V \mapsto \{x \in X \mid g(x)^*(V) = \top\} = g^{-1}(\text{Sp}(V))) : \mathcal{O}(Y) \rightarrow \mathcal{P}(X)$, which lands in $\mathcal{O}(X)$ iff g is continuous. This yields a natural bijection

$$\text{Loc}(X, Y) \cong \text{Top}(X, \text{Sp}(Y)),$$

i.e., $\text{Sp} : \text{Loc} \rightarrow \text{Top}$ is right adjoint to the forgetful functor. (This is a Stone-type adjunction induced by the “commuting” structures of a topology and a frame on 2 ; see [Joh82, VI §4].)

The adjunction unit is given by, for each topological space X , the continuous map

$$\begin{aligned} \eta_X : X &\longrightarrow \text{Sp}(X) \cong \{\text{completely prime filters in } \mathcal{O}(X)\} \cong \{\text{irreducible closed sets}\} \subseteq \neg\mathcal{O}(X) \\ x &\longmapsto \{U \in \mathcal{O}(X) \mid x \in U\} && \mapsto \bar{x}, \end{aligned}$$

and is an isomorphism iff X is **sober**, meaning T_0 (whence η_X is injective) and every irreducible closed set has a dense point. Examples include any Hausdorff space (irreducible closed sets being singletons) as well as most naturally occurring non-Hausdorff spaces in “ordinary” mathematics, such as spectra of rings or continuous deposes (see [Joh82]). The adjunction counit is given by, for each locale Y , the map

$$\varepsilon_Y : \text{Sp}(Y) \longrightarrow Y$$

defined by

$$\varepsilon_Y^* := \text{Sp} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(\text{Sp}(Y)).$$

By definition, ε_Y^* is always surjective, i.e., ε_Y is a sublocale embedding; while ε_Y^* is injective iff Y is **spatial**, meaning it admits enough points to separate open sets. In particular, the underlying locale of a topological space X is always spatial; hence the adjunction $\text{Top} \rightleftarrows \text{Loc}$ is **idempotent** (see e.g., [Joh82, VI 4.5]), meaning it factors into the reflection into sober spaces followed by the coreflective inclusion of spatial locales:

$$\text{Top} \begin{array}{c} \xrightarrow{\text{Sp}} \\ \xleftarrow{\perp} \\ \xleftarrow{\text{Sp}} \end{array} \text{SobTop} \simeq \text{SpLoc} \begin{array}{c} \xrightarrow{\text{ }} \\ \xleftarrow{\perp} \\ \xleftarrow{\text{Sp}} \end{array} \text{Loc}$$

For each of the other categories of locale-like structures defined in the preceding subsection, dual to a category of lattice-theoretic algebras, we may set up an idempotent adjunction with the corresponding category of spaces equipped with a subalgebra of $\mathcal{P}(X)$ in exactly the same way; we refer to all of these right adjoints as **spatialization**. Thus we have idempotent adjunctions

$$\kappa\text{Top} \begin{array}{c} \xrightarrow{\text{ }} \\ \xleftarrow{\perp} \\ \xleftarrow{\text{Sp}} \end{array} \kappa\text{Loc}, \quad \kappa\text{Bor} \begin{array}{c} \xrightarrow{\text{ }} \\ \xleftarrow{\perp} \\ \xleftarrow{\text{Sp}} \end{array} \kappa\text{BorLoc}, \quad \kappa\text{Bor}^+ \begin{array}{c} \xrightarrow{\text{ }} \\ \xleftarrow{\perp} \\ \xleftarrow{\text{Sp}} \end{array} \kappa\text{Bor}^+\text{Loc}$$

between the respective categories of κ -topological spaces and κ -locales, of κ -Borel spaces and κ -Borel locales, and of positive κ -Borel spaces and positive κ -Borel locales. These adjunctions fit into the obvious three-dimensional commutative diagram together with (3.1.9), one “slice” of which is

$$(3.2.1) \quad \begin{array}{ccc} \kappa\mathbf{BorLoc} & \xrightleftharpoons[\tau]{\mathbf{Sp}} & \kappa\mathbf{Bor} \\ \uparrow \vdash \downarrow & & \uparrow \vdash \downarrow \\ \kappa\mathbf{Bor}^+\mathbf{Loc} & \xrightleftharpoons[\tau]{\mathbf{Sp}} & \kappa\mathbf{Bor}^+ \\ \uparrow \vdash \downarrow & & \uparrow \vdash \downarrow \\ \kappa\mathbf{Loc} & \xrightleftharpoons[\tau]{\mathbf{Sp}} & \kappa\mathbf{Top} \end{array}$$

Commutativity of the right adjoints means that e.g., for a κ -locale X , its κ -topological spatialization $\mathbf{Sp}(X)$, which is the set of points $\mathbf{Sp}(X) \cong \kappa\mathbf{Frm}(\mathcal{O}_\kappa(X), 2)$ equipped with the image $\mathcal{O}_\kappa(\mathbf{Sp}(X))$ of the κ -frame homomorphism $\mathbf{Sp} : \mathcal{O}_\kappa(X) \rightarrow \mathcal{P}(\mathbf{Sp}(X))$, has underlying κ -Borel space given by $\mathbf{Sp}(X)$ equipped with the κ -Boolean subalgebra of $\mathcal{P}(\mathbf{Sp}(X))$ generated by $\mathcal{O}_\kappa(\mathbf{Sp}(X))$, which is the same as $\mathbf{Sp}(X) \cong \kappa\mathbf{Frm}(\mathcal{O}_\kappa(X), 2) \cong \kappa\mathbf{Bool}(\langle \mathcal{O}_\kappa(X) \text{ qua } \kappa\mathbf{Frm} \rangle_{\kappa\mathbf{Bool}}, 2) = \kappa\mathbf{Bool}(\mathcal{B}_\kappa(X), 2)$ equipped with the image of $\mathbf{Sp} : \mathcal{B}_\kappa(X) \rightarrow \mathcal{P}(\mathbf{Sp}(X))$, which is the κ -Borel spatialization of X . Note that this justifies our using the same notation \mathbf{Sp} to denote all of these spatializations.

Remark 3.2.2. However, the horizontal *left* adjoints (the forgetful functors from spaces to locales) in the above diagram do not always commute with the vertical *right* adjoints (the forgetful functors from topological to positive Borel to Borel). Dually, this means that e.g., given a subframe $\mathcal{O}(X) \subseteq \mathcal{P}(X)$, the free complete Boolean algebra $\langle \mathcal{O}(X) \rangle_{\mathbf{CBOOL}}$ it generates need not be the complete Boolean subalgebra of $\mathcal{P}(X)$ generated by $\mathcal{O}(X)$. A counterexample is given by taking $\mathcal{O}(X) := \langle \mathbb{N} \rangle_{\mathbf{Frm}}$, which is a subframe $\mathcal{L}(\langle \mathbb{N} \rangle_{\wedge\mathbf{Lat}}) \subseteq \mathcal{P}(\langle \mathbb{N} \rangle_{\wedge\mathbf{Lat}})$ of a powerset (see Section 2.3); but $\langle \mathcal{O}(X) \rangle_{\mathbf{CBOOL}} = \langle \mathbb{N} \rangle_{\mathbf{CBOOL}}$ is not a complete Boolean subalgebra of a powerset.

A related issue is that many familiar (say, topological) properties and constructions need not be preserved by the horizontal forgetful functors above (even when starting with, say, sober spaces). For example, the localic product $\mathbb{Q} \times \mathbb{Q}$ is not spatial, hence is not the underlying locale of the topological product (see [Joh82, II 2.14]).

These issues mean that when we are regarding a space as a locale, we must be careful about *when* we apply the nameless horizontal forgetful functor above. In general, our convention will be to **pass to locales as soon as possible**. For example, “the ∞ -Borel locale \mathbb{R} ” will mean the underlying ∞ -Borel locale of the underlying locale, rather than the (discrete) underlying ∞ -Borel locale of the underlying ∞ -Borel space; while $\mathbb{Q} \times \mathbb{Q}$ will refer to the localic product.

We now list some well-behaved classes of topological spaces, for which many spatial notions agree with their localic analogs. Usually, we will only be regarding these kinds of spaces as locales, so that the above issues never arise.

- Compact Hausdorff spaces form a full subcategory of $\mathbf{SobTop} \simeq \mathbf{SpLoc} \subseteq \mathbf{Loc}$ which is closed under arbitrary limits in \mathbf{Loc} (see [Joh82, III 1.6–7]).
- Locally compact Hausdorff spaces are closed under finite limits in \mathbf{Loc} (see [Joh82, II 2.13]).
- In particular, identifying sets with discrete spaces, we may regard $\mathbf{Set} \subseteq \mathbf{SobTop}$ as a full subcategory of \mathbf{Loc} closed under finite limits (and arbitrary colimits), the **discrete locales**, dual to complete atomic Boolean algebras $\mathbf{CABool} \subseteq \mathbf{Frm}$ (see Section 2.9).

Similarly, we call κ -ary sets **κ -discrete locales**, regarded as a full subcategory $\mathbf{Set}_\kappa \subseteq \mathbf{Loc}_\kappa = \kappa\mathbf{Loc}_\kappa$ of standard κ -locales, dual to $\kappa\mathbf{CABool}_\kappa \subseteq \kappa\mathbf{Frm}_\kappa$ from Section 2.9. Note that \mathbf{Set}_κ is also a full subcategory of standard κ -Borel locales $\kappa\mathbf{BorLoc}_\kappa$.

- Completely metrizable spaces are closed under countable limits in \mathbf{Loc} (see [Isb75, 4.1]).
- The **Sierpinski space** $\mathbb{S} = \{0 < 1\}$, with $\{1\}$ open but not closed, is sober; and all topological and localic powers of it agree (with $\mathcal{O}(\mathbb{S}^X) = \langle X \rangle_{\mathbf{Frm}}$; see [Joh82, VII 4.9], Proposition 3.5.4).
- More generally, countably copresented subspaces of powers of \mathbb{S} are well-behaved, and also in the (positive) Borel contexts; see Section 3.5.

3.3 The Borel hierarchy

Classically, the Borel hierarchy in a metrizable topological space X is defined by letting the Σ_1^0 sets be the open sets, the Π_α^0 sets be the complements of Σ_α^0 sets for each ordinal α , the Δ_α^0 sets be those which are both Σ_α^0 and Π_α^0 , and the Σ_α^0 sets, for $\alpha \geq 2$, be countable unions of sets which are Π_β^0 for some $\beta < \alpha$; see [Kec95, 11.B]. Thus, for example, a Σ_2^0 set is an F_σ set (countable union of closed), while a Π_2^0 set is a G_δ set (countable intersection of open). In the non-metrizable setting, a better-behaved definition, due to Selivanov [Sel06], takes instead Σ_α^0 sets to be countable unions

$$\bigcup_{i \in \mathbb{N}} (A_i \setminus B_i) \quad \text{for } A_i, B_i \in \Sigma_{\beta_i}^0, \beta_i < \alpha.$$

It is easily seen by induction that this only makes a difference for Σ_2^0 , where the effect is to ensure that Σ_2^0 is the σ -subframe of $\mathcal{P}(X)$ generated by the open and closed sets.

Motivated by this definition and the formula (2.8.1) for elements of \mathcal{N}_κ , we define the **κ -Borel hierarchy** of a κ -locale X as follows:

$$\begin{aligned} \kappa\Sigma_{1+\alpha}^0(X) &:= \mathcal{N}_\kappa^\alpha(\mathcal{O}_\kappa(X)) \subseteq \mathcal{B}_\kappa(X), \\ \kappa\Pi_{1+\alpha}^0(X) &:= \neg \mathcal{N}_\kappa^\alpha(\mathcal{O}_\kappa(X)) \subseteq \mathcal{B}_\kappa(X), \\ \kappa\Delta_{1+\alpha}^0(X) &:= \kappa\Sigma_{1+\alpha}^0(X) \cap \kappa\Pi_{1+\alpha}^0(X) = \mathcal{N}_\kappa^\alpha(\mathcal{O}_\kappa(X))_\neg \subseteq \mathcal{B}_\kappa(X). \end{aligned}$$

We call the elements $B \in \kappa\Sigma_\alpha^0(X)$ the **$\kappa\Sigma_\alpha^0$ sets of X** ; similarly for the other classes.

Convention 3.3.1. Classically, boldface Σ, Π, Δ is used in order to distinguish from the *effective* Borel hierarchy, for which lightface is reserved (see [Mos09]). Since we will never consider effective notions in this paper, we have chosen to use lightface, in order to avoid cluttering the notation any more than it already is. To avoid any possible confusion, we will always write the prefix κ when denoting the localic Borel hierarchy (even when $\kappa = \omega_1$ or $\kappa = \infty$). However, we will continue to use boldface for the classical (σ -)Borel hierarchy in topological spaces.

We have the usual picture of the κ -Borel hierarchy (see [Kec95, 11.B]):

$$\begin{array}{ccccccc} \mathcal{O}_\kappa = \kappa\Sigma_1^0 & \subseteq & \kappa\Sigma_2^0 & \subseteq & \cdots & \subseteq & \kappa\Sigma_\alpha^0 \subseteq \cdots \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ (\mathcal{O}_\kappa)_\neg = \kappa\Delta_1^0 & \subseteq & \kappa\Delta_2^0 & \subseteq & \cdots & \subseteq & \kappa\Delta_\alpha^0 \subseteq \cdots \subseteq \mathcal{B}_\kappa(X) \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ \neg\mathcal{O}_\kappa = \kappa\Pi_1^0 & \subseteq & \kappa\Pi_2^0 & \subseteq & \cdots & \subseteq & \kappa\Pi_\alpha^0 \subseteq \cdots \end{array}$$

The σ -localic Borel hierarchy is connected to the classical one as follows: for a second-countable (σ -)topological space X , the Borel σ -algebra is the image in $\mathcal{P}(X)$ of the Borel σ -algebra of the underlying σ -locale of X (see Remark 3.2.2); this quotient map restricts to give

$$(3.3.2) \quad \Sigma_\alpha^0(X) = \text{im}(\sigma\Sigma_\alpha^0(X) \rightarrow \mathcal{P}(X)).$$

Similarly for Π_α^0 .

Remark 3.3.3. It is possible for the spatial $\Sigma_\alpha^0(X)$ to be a nontrivial quotient of the localic $\sigma\Sigma_\alpha^0(X)$ (even for a second-countable σ -topological X). Take $X = \mathbb{Q}$; by the localic Baire category theorem (see Theorem 3.8.2 below), the join of the closed singletons $\{x\}$ is $< \top$ in $\sigma\Sigma_2^0(X)$, but of course not in $\Sigma_2^0(X)$. In Section 3.5, we will see a large class of spaces for which $\Sigma_\alpha^0(X) \cong \sigma\Sigma_\alpha^0(X)$.

For a κ -locale X and $\lambda \geq \kappa$, we may regard X as a λ -locale, hence obtain a λ -Borel hierarchy of X . As λ varies, these are related as follows:

$$\begin{array}{ccccccccc} \infty\Sigma_1^0(X) & \subseteq & \infty\Sigma_2^0(X) & \subseteq & \cdots & \subseteq & \infty\Sigma_\alpha^0(X) & \subseteq & \cdots & \subseteq & \infty\Sigma_\kappa^0(X) & \subseteq & \cdots & \subseteq & \infty\Sigma_\lambda^0(X) & \subseteq & \cdots & \subseteq & \mathcal{B}_\infty(X) \\ \cup \downarrow & & \cup \downarrow & & & & \cup \downarrow & & & & \cup \downarrow & & & & \cup \downarrow & & & & \cup \downarrow \\ \vdots & & \vdots & & & & \vdots & & & & \vdots & & & & \vdots & & & & \vdots \\ \lambda\Sigma_1^0(X) & \subseteq & \lambda\Sigma_2^0(X) & \subseteq & \cdots & \subseteq & \lambda\Sigma_\alpha^0(X) & \subseteq & \cdots & \subseteq & \lambda\Sigma_\kappa^0(X) & \subseteq & \cdots & \subseteq & \lambda\Sigma_\lambda^0(X) & = & \mathcal{B}_\lambda(X) \\ \cup \downarrow & & \cup \downarrow & & & & \cup \downarrow & & & & \cup \downarrow & & & & \cup \downarrow & & & & \cup \downarrow \\ \kappa\Sigma_1^0(X) & \subseteq & \kappa\Sigma_2^0(X) & \subseteq & \cdots & \subseteq & \kappa\Sigma_\alpha^0(X) & \subseteq & \cdots & \subseteq & \kappa\Sigma_\kappa^0(X) & = & \mathcal{B}_\kappa(X) \end{array}$$

Note the differing lengths of the rows. Indeed, if X is the σ -locale with $\mathcal{O}(X) = \langle \mathbb{N} \rangle_{\text{Frm}}$ (namely $X = \mathbb{S}^{\mathbb{N}}$; see Proposition 3.5.4), then by the Gaifman–Hales Theorem 2.1.2, $\mathcal{B}_\infty(X) = \langle \mathbb{N} \rangle_{\text{CBOOL}}$ is a proper class, while each $\infty\Sigma_{1+\alpha}^0(X) = \mathcal{N}^\alpha(\langle \mathbb{N} \rangle_{\text{Frm}})$ is a set; thus

Corollary 3.3.4 (of Gaifman–Hales). *There is a σ -locale X whose ∞ -Borel hierarchy is strictly increasing.* \square

For a locale X , each $\infty\Sigma_\alpha^0(X) \subseteq \mathcal{B}_\infty(X)$ forms a small \vee -sublattice, while each $\infty\Pi_\alpha^0(X) \subseteq \mathcal{B}_\infty(X)$ forms a small \wedge -sublattice; thus these inclusions have right/left adjoints, i.e., for any ∞ -Borel set $B \in \mathcal{B}_\infty(X)$, we may define its $\infty\Sigma_\alpha^0$ -**interior** and $\infty\Pi_\alpha^0$ -**closure**

$$\begin{aligned} B^{\circ\alpha} &:= \max\{A \in \infty\Sigma_\alpha^0(X) \mid A \leq B\}, \\ \overline{B}^\alpha &:= \min\{C \in \infty\Pi_\alpha^0(X) \mid B \leq C\}. \end{aligned}$$

Thus $\overline{B}^1 = \overline{B}$, while $\overline{B}^\alpha = B \iff B \in \infty\Pi_\alpha^0(X)$. We say that B is $\infty\Pi_\alpha^0$ -**dense** if $\overline{B}^\alpha = X$. Note that even if X is a κ -locale and $B \in \mathcal{B}_\kappa(X)$, \overline{B}^α need not be in $\kappa\Pi_\alpha^0(X)$. Similarly, there is no useful spatial analog of these notions for $\alpha \geq 2$.

For a κ -Borel locale X , since each $\kappa\Sigma_{1+\alpha}^0(X)$ is a κ -frame, we may define a κ -locale by

$$\mathcal{O}_\kappa(\mathcal{D}_\kappa^\alpha(X)) := \kappa\Sigma_{1+\alpha}^0(X) = \mathcal{N}_\kappa^\alpha(\mathcal{O}_\kappa(X)).$$

This κ -locale $\mathcal{D}_\kappa^\alpha(X)$ is called the α **th dissolution** of X (a term due to Johnstone [Joh90]), and is equipped with a κ -continuous map

$$\varepsilon_\kappa : \mathcal{D}_\kappa^\alpha(X) \longrightarrow X,$$

corresponding to the unit $\varepsilon_\kappa^* : \mathcal{O}_\kappa(X) \hookrightarrow \mathcal{N}_\kappa^\alpha(\mathcal{O}_\kappa(X))$ (from Section 2.8), such that ε_κ is also a κ -Borel isomorphism and such that $\varepsilon_\kappa^*(\kappa\Sigma_{1+\alpha}^0(X)) = \mathcal{O}_\kappa(\mathcal{D}_\kappa^\alpha(X))$. In other words, we can think of $\mathcal{D}_\kappa^\alpha(X)$ as “ X with a finer κ -topology, inducing the same κ -Borel structure, in which $\kappa\Sigma_{1+\alpha}^0$ sets in X become open”.

Note that for $\kappa < \infty$, $\mathcal{D}_\kappa^\kappa(X) = \mathcal{D}_\kappa^\infty(X)$, which corresponds to $\mathcal{N}_\kappa^\infty(\mathcal{O}_\kappa(X)) = \langle \mathcal{O}_\kappa(X) \rangle_{\kappa\text{Bool}}$ but regarded as a κ -frame via the forgetful functor $\kappa\text{Bool} \rightarrow \kappa\text{Frm}$, is the “free κ -locale” on the underlying κ -Borel locale of X (see (3.1.9)). It can thus be thought of as a κ -localic analog of the discrete topology on a topological space. (However, the assumption $\kappa < \infty$ is needed, since the forgetful functor $\text{Loc} \rightarrow \infty\text{BorLoc}$ in (3.1.9) does not have a left adjoint.)

Remark 3.3.5. In [BWZ11], the authors construct a variant dissolution which only adds complements for cozero sets of completely regular locales; the Lindelöf case is an instance of the dissolution on σ -locales (see Remark 3.1.6).

A basic result in classical descriptive set theory says that Borel sets may be made (cl)open while preserving σ -copresentability of the topology (see [Kec95, 13.A] and Section 3.5 below). The localic analog of this is given by the duals of Proposition 2.8.13 and Corollary 2.8.16:

Proposition 3.3.6. *Every standard κ -Borel locale is the underlying κ -Borel locale of a standard κ -locale.* \square

Proposition 3.3.7. *Let X be a standard κ -locale. For any α and κ -ary $\mathcal{C} \subseteq \kappa\Sigma_\alpha^0(X)$, there is a standard κ -locale X' equipped with a κ -continuous κ -Borel isomorphism $f : X' \rightarrow X$ such that, identifying $\mathcal{B}_\kappa(X) = \mathcal{B}_\kappa(X')$ via f^* , we have $\mathcal{C} \subseteq \mathcal{O}_\kappa(X') \subseteq \kappa\Sigma_\alpha^0(X)$. (Thus we can also ensure $\mathcal{C} \cap \kappa\Delta_\alpha^0(A) \subseteq \mathcal{O}_\kappa(X')_{\neg\cdot}$.)* \square

We will henceforth refer to the conclusion of this result by saying that X' is a **partial dissolution** of X making each $C \in \mathcal{C}$ κ -open.

We call a κ -locale X **ultraparacompact**, respectively **zero-dimensional**, if the κ -frame $\mathcal{O}_\kappa(X)$ is, as defined in Section 2.9. Recall also κ -discrete locales $\text{Set}_\kappa \subseteq \text{Loc}_\kappa$ from Remark 3.2.2. By Corollaries 2.9.13 and 2.9.16,

Proposition 3.3.8. *A (standard) κ -locale is ultraparacompact zero-dimensional iff it is a (κ -ary) codirected limit, in κLoc , of κ -discrete locales.* \square

We now have the following strengthened “change of topology” results, dual to Proposition 2.9.17 and Corollary 2.9.21:

Proposition 3.3.9. *Every standard κ -Borel locale is the underlying κ -Borel locale of an ultraparacompact zero-dimensional standard κ -locale (hence a κ -ary codirected limit, in κBorLoc , of κ -ary sets).* \square

Proposition 3.3.10. *Let X be a standard κ -locale. For any $\alpha \geq 2$ and κ -ary $\mathcal{C} \subseteq \kappa\Sigma_\alpha^0(X)$, there is an ultraparacompact zero-dimensional partial dissolution $X' \rightarrow X$ such that $\mathcal{C} \subseteq \mathcal{O}_\kappa(X') \subseteq \kappa\Sigma_\alpha^0(X)$ (and $\mathcal{C} \cap \kappa\Delta_\alpha^0(A) \subseteq \mathcal{O}_\kappa(X')_{\neg\cdot}$).* \square

We also record here the dual of Corollary 2.9.20:

Proposition 3.3.11. *For any κ -locale X and $\alpha \geq 2$, if $C_i \in \kappa\Sigma_\alpha^0(X)$ are $< \kappa$ -many $\kappa\Sigma_\alpha^0$ sets which cover X , then there are pairwise disjoint $D_i \subseteq C_i$ (whence $D_i \in \kappa\Delta_\alpha^0(X)$) that still cover X .* \square

Remark 3.3.12. This is a weak form of the κ -ary analog of the **generalized reduction property** which holds classically for Σ_α^0 , $\alpha \geq 2$ in metrizable spaces (see [Kec95, 22.16]): any countable family of Σ_α^0 sets has a pairwise disjoint refinement with the same union. In other words, every Σ_α^0 set is ultraparacompact in the σ -frame Σ_α^0 . We do not know if analogously, every $\kappa\Sigma_\alpha^0$ set in every (standard) κ -locale is ultraparacompact.

For a κ -continuous map $f : X \rightarrow Y$ between κ -locales, the κ -Boolean homomorphism $f^* : \mathcal{B}_\kappa(Y) \rightarrow \mathcal{B}_\kappa(X)$ (extending the κ -frame homomorphism $f^* : \mathcal{O}_\kappa(Y) \rightarrow \mathcal{O}_\kappa(X)$) restricts to κ -frame homomorphisms $f^* : \kappa\Sigma_\alpha^0(Y) \rightarrow \kappa\Sigma_\alpha^0(X)$ for each α , i.e., “preimages of $\kappa\Sigma_\alpha^0$ sets are $\kappa\Sigma_\alpha^0$ ”.

If f is merely a κ -Borel map, then for each $V \in \mathcal{O}_\kappa(Y)$, we have $f^*(V) \in \mathcal{B}_\kappa(X)$, thus $f^*(V) \in \kappa\Sigma_\alpha^0(X)$ for some $\alpha < \kappa$. By taking a partial dissolution of X' making $f^*(V)$ κ -open for all (or only generating) $V \in \mathcal{O}_\kappa(Y)$ using Proposition 3.3.7 (or Proposition 3.3.10), we get the following counterpart “change of topology” results for maps:

Proposition 3.3.13. *Let $f : X \rightarrow Y$ be a κ -Borel map between κ -locales. Then there is a partial dissolution $X' \rightarrow X$ such that f lifts to a κ -continuous map $f' : X' \rightarrow Y$. Moreover:*

- (a) *if X, Y are standard, we can ensure X' is standard;*
- (b) *if $f^*(\mathcal{O}_\kappa(Y)) \subseteq \kappa\Sigma_\alpha^0(X)$, we can ensure $\mathcal{O}_\kappa(X') \subseteq \kappa\Sigma_\alpha^0(X)$;*
- (c) *for $\alpha \geq 2$, we can ensure X' is ultraparacompact zero-dimensional.* □

Remark 3.3.14. Classically, a map $f : X \rightarrow Y$ between topological spaces is **Baire class α** if it is in the α th iterated closure of the continuous maps under pointwise sequential limits, while a **Baire map** is a map which is Baire class α for some $\alpha < \omega_1$. For sufficiently nice spaces X, Y , f is Baire class α iff $f^*(\mathcal{O}(Y)) \subseteq \Sigma_{\alpha+1}^0(X)$, hence Baire iff it is Borel; see [Kec95, 24.3].

Our notion of κ -Borel map, as well as the above result, is thus related to Ball’s [Bal18] notion of Baire maps from a locale to \mathbb{R} , in the same way as in Remark 3.3.5: Ball’s notion in the regular Lindelöf case is an instance of our notion when $\kappa = \omega_1$.

Remark 3.3.15. We note the following abstract categorical consequence of the above (which is somewhat implicit in [MM91]): the forgetful functor $\kappa\text{Loc} \rightarrow \kappa\text{BorLoc}$ exhibits κBorLoc as a **localization** of κLoc , i.e., κBorLoc is obtained from κLoc by freely adjoining inverses for some morphisms. See e.g., [Bor, Ch. 5]. This follows (see e.g., [MP87, p. 486]) from the fact that $\kappa\text{Loc} \rightarrow \kappa\text{BorLoc}$ preserves finite limits, and that every κ -Borel map $f : X \rightarrow Y$ with $Y \in \kappa\text{Loc}$ factors as $X \cong X' \rightarrow Y$ for some κ -locale X' which is κ -Borel isomorphic to X and some κ -continuous map $X' \rightarrow Y$, by Propositions 3.3.6 and 3.3.13. Moreover, we can take X' standard if X, Y both are; thus $\kappa\text{BorLoc}_\kappa$ is also a localization of κLoc_κ .

Once we know we have a localization, we can of course describe $\kappa\text{BorLoc}_{(\kappa)}$ as the localization of $\kappa\text{Loc}_{(\kappa)}$ which adjoins an inverse for all those κ -continuous $f : X \rightarrow Y \in \kappa\text{Loc}_{(\kappa)}$ which do in fact become invertible in $\kappa\text{BorLoc}_{(\kappa)}$; these are exactly the partial dissolutions. In other words,

“(Standard) κ -Borel locales are what’s left of (standard) κ -locales after κ -continuous bijections are declared to be isomorphisms.”

(Note that “bijection” can be interpreted as either κ -Borel or ∞ -Borel isomorphism, by conservativity of $\kappa\text{BorLoc} \rightarrow \infty\text{BorLoc}$ (Proposition 3.1.10).)

3.4 Borel images and sublocales

We now give a detailed analysis of various notions of “image” for maps between locales.

For an ∞ -Borel map $f : X \rightarrow Y$ between ∞ -Borel locales, and any ∞ -Borel set $B \in \mathcal{B}_\infty(X)$, we define the ∞ -**Borel image** of B under f to be

$$f(B) := f^{\mathcal{B}_\infty}(B) := \min\{C \in \mathcal{B}_\infty(Y) \mid B \leq f^*(C)\}, \text{ if it exists.}$$

Thus, $f : \mathcal{B}_\infty(X) \rightarrow \mathcal{B}_\infty(Y)$ is the partial left adjoint to $f^* : \mathcal{B}_\infty(Y) \rightarrow \mathcal{B}_\infty(X)$. More generally, for any class of ∞ -Borel sets $\Gamma(Y) \subseteq \mathcal{B}_\infty(Y)$, we define the Γ -**image** of B under f to be

$$f^\Gamma(B) := \min\{C \in \Gamma(Y) \mid B \leq f^*(C)\}, \text{ if it exists.}$$

We also define the Γ -**image**, and in particular the ∞ -**Borel image**, of f to be

$$\begin{aligned} \text{im}^\Gamma(f) &:= f^\Gamma(X), \\ \text{im}(f) &:= f(X) = \text{im}^{\mathcal{B}_\infty}(f). \end{aligned}$$

(In Theorem 4.4.3 we will show that the ∞ -Borel image need not exist.)

If $\Gamma(Y) \subseteq \mathcal{B}_\infty(Y)$ is a small \wedge -sublattice, then the Γ -image always exists (by the adjoint functor theorem). For example, if Y is a locale, the $\infty\Pi_\alpha^0$ -**image** exists for every α ; we denote it by

$$\begin{aligned} \overline{f}^\alpha(B) &:= f^{\infty\Pi_\alpha^0}(B) = \bigwedge\{C \in \infty\Pi_\alpha^0(Y) \mid B \leq f^*(C)\}, \\ \overline{\text{im}}^\alpha(f) &:= \overline{f}^\alpha(X) = \bigwedge\{C \in \infty\Pi_\alpha^0(y) \mid f^*(C) = \top\}. \end{aligned}$$

There is a close relationship between $\infty\Pi_\alpha^0$ -image and $\infty\Pi_\alpha^0$ -closure (Section 3.3): the latter is the former under the identity map. Conversely, if the ∞ -Borel image exists, then its $\infty\Pi_\alpha^0$ -closure gives the $\infty\Pi_\alpha^0$ -image, hence our notation. (This will generalize to all $\infty\Pi_\alpha^0$ -images, once we define the notion of “ $\infty\Sigma_1^1$ -image” by adjoining nonexistent ∞ -Borel images in Section 4.1 below.)

Remark 3.4.1. In the locale theory literature (see e.g., [Joh02, C1.2]), the “image” of a continuous locale map usually refers to the image sublocale, i.e., what we are calling the “ $\infty\Pi_2^0$ -image” (see Convention 3.4.11 below). It is well-known that this notion of “image” behaves poorly in many ways, e.g., it is not pullback-stable (see [Joh02, C1.2.12] and Remark 3.4.18). This poor behavior is explained by the decomposition of “ $\infty\Pi_2^0$ -image” as “ $\infty\Sigma_1^1$ -image” (which is well-behaved; see Theorem 4.1.1) followed by $\infty\Pi_2^0$ -closure (which one does not expect to behave like an “image”).

To avoid confusion, we will henceforth avoid saying “image” without a prefix.

Clearly, if $\Gamma \subseteq \Gamma'$, and the Γ' -image exists and is in Γ , then it is also the Γ -image. Conversely, if Γ generates Γ' under arbitrary meets, then any Γ -image is also the Γ' -image; for example, if Y is a κ -locale, then a κ -closed image exists iff a closed image exists and is a κ -closed set. If the Γ -image and Γ' -image both exist, then the former is always bigger; e.g.,

$$\overline{f}(B) := \overline{f}^1(B) \geq \overline{f}^2(B) \geq \dots \geq \overline{f}^\alpha(B) \geq \dots \quad (\geq f(B), \text{ if it exists}).$$

Also, if Γ is a directed union of subclasses Γ_i , then the Γ -image is the eventual value of the Γ_i -images, if the latter eventually all exist and stabilize; e.g.,

$$f(B) = \min_{\alpha < \infty} \overline{f}^\alpha(B) = \text{eventual value of } \overline{f}^\alpha(B), \text{ if it exists.}$$

If $f : X \rightarrow Y$ with Y a κ -Borel locale, we denote the κ -**Borel image** by

$$f^\kappa(B) := f^{\mathcal{B}_\kappa}(B) = \min\{C \in \mathcal{B}_\kappa(Y) \mid B \leq f^*(C)\};$$

thus for $\kappa < \infty$,

$$f(B) = \text{eventual value of } f^\lambda(B) \text{ for } \kappa \leq \lambda < \infty, \text{ if it exists.}$$

(The above formulas are implicit in the work of Wilson [Wil94] and Madden–Molitor [MM91].)

The κ -Borel image has a simple dual algebraic meaning:

Proposition 3.4.2. *Let $f : X \rightarrow Y$ be an ∞ -Borel map from an ∞ -Borel locale to a κ -Borel locale, where $\kappa \leq \infty$. Then the κ -Borel image $f^\kappa(B)$ of some $B \in \mathcal{B}_\infty(X)$ exists iff the composite of $f^* : \mathcal{B}_\kappa(Y) \rightarrow \mathcal{B}_\infty(X)$ followed by the principal filterquotient $B \wedge (-) : \mathcal{B}_\infty(X) \twoheadrightarrow \downarrow B$ has kernel which is a κ -generated κ -Boolean congruence on $\mathcal{B}_\kappa(Y)$, hence a principal filter congruence \sim^C for some $C \in \mathcal{B}_\kappa(Y)$, in which case $f^\kappa(B) = C$.*

Thus, the κ -Borel image $\text{im}^\kappa(f) := f^\kappa(X)$ of f exists iff $f^ : \mathcal{B}_\kappa(Y) \rightarrow \mathcal{B}_\infty(X)$ has kernel which is a κ -generated congruence, hence a principal filter congruence $\sim^{\text{im}^\kappa(f)}$.*

Proof. This is a direct unravelling of the definitions: $f^\kappa(B) = C$ iff $C \in \mathcal{B}_\kappa(Y)$ is least such that $B \leq f^*(C)$, i.e., $B \wedge f^*(C) = B = \top_{\downarrow B}$, which means precisely that $\ker(B \wedge f^*(-) : \mathcal{B}_\kappa(Y) \rightarrow \downarrow B) = \sim^C$ by the general description of κ -Boolean algebra congruences (end of Section 2.4). \square

Recall from the end of Section 3.1 that by an **embedding** in one of our categories of locales, we mean a regular monomorphism, i.e., a map f such that f^* is a surjective homomorphism.

Corollary 3.4.3. *Let $f : X \rightarrow Y$ be an embedding in (i) κLoc , (ii) $\kappa\text{Bor}^+\text{Loc}$, or (iii) κBorLoc . Then the ∞ -Borel image $\text{im}(f)$ exists, and is given by the respective formulas*

- (i) $\text{im}(f) = \bigwedge\{B \rightarrow C \mid B, C \in \mathcal{O}_\kappa(Y) \text{ \& } f^*(B) \leq f^*(C)\},$
- (ii) $\text{im}(f) = \bigwedge\{B \rightarrow C \mid B, C \in \mathcal{B}_\kappa^+(Y) \text{ \& } f^*(B) \leq f^*(C)\},$
- (iii) $\text{im}(f) = \bigwedge\{B \in \mathcal{B}_\kappa(Y) \mid f^*(B) = \top\},$

where the meets may be restricted to any generating set for the (order-)kernel of f^* .

If X, Y are both standard, then the meets may be restricted to be κ -ary; thus $\text{im}(f) \in \mathcal{B}_\kappa(Y)$, and so $\text{im}(f)$ is also the κ -Borel image $\text{im}^\kappa(f)$.

(Note that when $\kappa = \infty$, these formulas still make sense, because we always assume (positive) ∞ -Borel locales are standard.)

Proof. Since f^* is surjective, so is its complete Boolean extension $\mathcal{B}_\infty(Y) \rightarrow \mathcal{B}_\infty(X)$, with kernel generated by any generators of the kernel of the original f^* . In case (i), for example, $f^* : \mathcal{O}_\kappa(Y) \twoheadrightarrow \mathcal{O}_\kappa(X)$ is the quotient of $\mathcal{O}_\kappa(Y)$ by the relations $B \leq C$ for all (generating) pairs (B, C) in the order-kernel of f^* , hence $\mathcal{B}_\infty(Y) \twoheadrightarrow \mathcal{B}_\infty(X)$ is the quotient by these same relations, which (again by the general description of Boolean congruences from Section 2.4) equivalently means $\mathcal{B}_\infty(X)$ is, up to isomorphism, the principal filterquotient $\mathcal{B}_\infty(Y) \twoheadrightarrow \downarrow \bigwedge_{f^*(B) \leq f^*(C)} (B \rightarrow C)$, which by Proposition 3.4.2 means $\text{im}(f) = \bigwedge_{f^*(B) \leq f^*(C)} (B \rightarrow C)$, which yields the formula (i). The cases (ii) and (iii) are similar.

If X, Y are both standard, then f^* is a quotient map between κ -presented algebras, hence has κ -generated kernel (see Lemma A.2.1). \square

Remark 3.4.4. The notions of monomorphism, regular monomorphism, and epimorphism in the categories of standard objects $\kappa(\mathbf{Bor}^{(+)})\mathbf{Loc}_\kappa$ are the restrictions of those in their respective parent categories $\kappa(\mathbf{Bor}^{(+)})\mathbf{Loc}$; this is a general algebraic fact about their dual categories (see Lemma A.2.3). Thus, there is no ambiguity when we say “standard κ -sublocale”, for example.

Let us define the following *ad hoc* classes of sets: for a (i) κ -locale, (ii) positive κ -Borel locale, or (iii) κ -Borel locale Y , respectively,

- (i) $\infty(\kappa\Pi_2^0)_\delta(Y) := \text{closure of } \kappa\Pi_2^0(Y) \subseteq \mathcal{B}_\infty(Y) \text{ under arbitrary } \bigwedge$
 $= \{\bigwedge_{i \in I} (B_i \rightarrow C_i) \mid I \in \mathbf{Set} \ \& \ B_i, C_i \in \mathcal{O}_\kappa(Y)\},$
- (ii) $\lambda I(\mathcal{B}_\kappa^+)_\delta(Y) := \{\bigwedge_{i \in I} (B_i \rightarrow C_i) \mid I \in \mathbf{Set}_\lambda \ \& \ B_i, C_i \in \mathcal{B}_\kappa^+(Y)\},$
- (iii) $\infty(\mathcal{B}_\kappa)_\delta(Y) := \{\bigwedge_{i \in I} B_i \mid I \in \mathbf{Set} \ \& \ B_i \in \mathcal{B}_\kappa(Y)\}.$

(The “ I ” in (ii) stands for “implication”.) The formulas in Corollary 3.4.3 clearly imply

Corollary 3.4.5. *In the situations of Corollary 3.4.3, $\text{im}(f)$ is the (i) $\infty(\kappa\Pi_2^0)_\delta$ -image (hence also $\infty\Pi_2^0$ -image), (ii) $\infty I(\mathcal{B}_\kappa^+)_\delta$ -image, or (iii) $\infty(\mathcal{B}_\kappa)_\delta$ -image of f , respectively.*

If X, Y are standard, then $\text{im}(f)$ is also the (i) $\kappa\Pi_2^0$ -image, (ii) $\kappa I(\mathcal{B}_\kappa^+)_\delta$ -image, or (iii) κ -Borel image of f , respectively. \square

The images in these different categories are related as follows. Since the forgetful functors $\kappa\mathbf{Loc} \rightarrow \kappa\mathbf{Bor}^+\mathbf{Loc} \rightarrow \kappa\mathbf{Bor}\mathbf{Loc}$ and $\kappa(\mathbf{Bor}^{(+)})\mathbf{Loc} \rightarrow \lambda(\mathbf{Bor}^{(+)})\mathbf{Loc}$ ($\kappa \leq \lambda$) from diagram (3.1.9) preserve limits, they preserve regular monomorphisms as well as pullbacks, and so restrict to \bigwedge -lattice homomorphisms on the regular subobject lattices of each object in the domain. Since the functors are also faithful (Proposition 3.1.10), these \bigwedge -lattice homomorphisms are injective (see Lemma A.2.4). Thus for $X \in \kappa\mathbf{Loc}$, say, we have embeddings

$$(3.4.6) \quad \begin{array}{ccccccc} \text{RSub}_{\kappa\mathbf{Bor}\mathbf{Loc}}(X) & \hookrightarrow & \text{RSub}_{\lambda\mathbf{Bor}\mathbf{Loc}}(X) & \hookrightarrow & \cdots & \hookrightarrow & \text{RSub}_{\infty\mathbf{Bor}\mathbf{Loc}}(X) \\ \uparrow & & \uparrow & & & & \uparrow \\ \text{RSub}_{\kappa\mathbf{Bor}^+\mathbf{Loc}}(X) & \hookrightarrow & \text{RSub}_{\lambda\mathbf{Bor}^+\mathbf{Loc}}(X) & \hookrightarrow & \cdots & \hookrightarrow & \text{RSub}_{\infty\mathbf{Bor}^+\mathbf{Loc}}(X) \\ \uparrow & & \uparrow & & & & \uparrow \\ \text{RSub}_{\kappa\mathbf{Loc}}(X) & \hookrightarrow & \text{RSub}_{\lambda\mathbf{Loc}}(X) & \hookrightarrow & \cdots & \hookrightarrow & \text{RSub}_{\mathbf{Loc}}(X) \end{array}$$

The dual of $\text{RSub}_{\kappa\mathbf{Loc}}(X) \hookrightarrow \text{RSub}_{\infty\mathbf{Bor}\mathbf{Loc}}(X)$, say, takes a κ -frame quotient $f^* : \mathcal{O}_\kappa(X) \twoheadrightarrow \mathcal{O}_\kappa(Y)$ to the induced complete Boolean quotient $f^* : \mathcal{B}_\infty(X) \twoheadrightarrow \mathcal{B}_\infty(Y)$. Thus, all of these embeddings commute with taking the ∞ -Borel image (by our definition of ∞ -Borel image).

The following is also immediate from Proposition 3.4.2:

Corollary 3.4.7. *Each $\text{im} : \text{RSub}_{\infty\mathbf{Bor}\mathbf{Loc}}(X) \rightarrow \mathcal{B}_\infty(X)$ is a \bigwedge -lattice embedding.*

Thus, each $\text{im} : \text{RSub}_{\kappa(\mathbf{Bor}^{(+)})\mathbf{Loc}}(X) \rightarrow \mathcal{B}_\infty(X)$ is a \bigwedge -lattice embedding.

Thus, each $\text{im} : \text{RSub}_{\kappa(\mathbf{Bor}^{(+)})\mathbf{Loc}_\kappa}(X) \rightarrow \mathcal{B}_\kappa(X)$ is a κ - \bigwedge -lattice embedding.

Proof. Clearly, a principal filterquotient of $\mathcal{B}_\infty(X)$ determines a unique element of $\mathcal{B}_\infty(X)$. Meets in $\text{RSub}_{\infty\mathbf{Bor}\mathbf{Loc}}(X)$ are given by pullback, i.e., pushout of the corresponding principal filterquotients of $\mathcal{B}_\infty(X)$, which corresponds to meet of the corresponding elements of $\mathcal{B}_\infty(X)$. \square

We now consider the inverses of im . Given, say, a κ -locale X , and an ∞ -Borel set $B \in \mathcal{B}_\infty(X)$, we define the **induced κ -sublocale** $Y \subseteq X$ on B by taking $\mathcal{O}_\kappa(Y)$ to be the image of the inclusion $\mathcal{O}_\kappa(X) \hookrightarrow \mathcal{B}_\infty(X)$ composed with the principal filterquotient $\mathcal{B}_\infty(X) \twoheadrightarrow \downarrow B$. In other words, $\mathcal{O}_\kappa(Y)$ is the quotient of $\mathcal{O}_\kappa(X)$ by the κ -frame congruence “identifying sets which are the same in B ”:

$$U \sim^B V :\iff U \cap B = V \cap B \in \mathcal{B}_\infty(X).$$

Similarly, an ∞ -Borel set in a (positive) κ -Borel locale induces a (positive) κ -Borel sublocale.

Remark 3.4.8. When $\kappa = \infty$, clearly small-presentability $\mathcal{B}_\infty(X)$ implies small-presentability of $\downarrow B$; thus the induced ∞ -Borel sublocale on B makes sense.

For the induced positive ∞ -Borel sublocale, we likewise need to require small-presentability of the image of the composite $\mathcal{B}_\infty^+(X) \hookrightarrow \mathcal{B}_\infty(X) \twoheadrightarrow \downarrow B$. We do not know if this is automatic.

The connection with ∞ -Borel images is given by

Proposition 3.4.9. *Let $X \in (i) \kappa\text{Loc}$, $(ii) \kappa\text{Bor}^+\text{Loc}$, or $(iii) \kappa\text{BorLoc}$.*

(a) *We have order-isomorphisms*

$$\begin{aligned} (i) \quad & \text{RSub}_{\kappa\text{Loc}}(X) \xrightleftharpoons[\text{induced } \kappa\text{-sublocale}]{\text{im}} \infty(\kappa\Pi_2^0)_\delta(X), \\ (ii) \quad & \text{RSub}_{\kappa\text{Bor}^+\text{Loc}}(X) \xrightleftharpoons[\text{induced positive } \kappa\text{-Borel sublocale}]{\text{im}} \infty I(\mathcal{B}_\kappa^+)_\delta(X), \\ (iii) \quad & \text{RSub}_{\kappa\text{BorLoc}}(X) \xrightleftharpoons[\text{induced } \kappa\text{-Borel sublocale}]{\text{im}} \infty(\mathcal{B}_\kappa)_\delta(X). \end{aligned}$$

(b) *For every other $B \in \mathcal{B}_\infty(X)$, the induced (positive) κ -(Borel) sublocale on B corresponds, via (a), to the closure of B in the respective class $\Gamma \subseteq \mathcal{B}_\infty$ on the right-hand side of (a).*

(c) *If X is standard, then the order-isomorphisms in (a) restrict to, respectively,*

$$\begin{aligned} (i) \quad & \text{RSub}_{\kappa\text{Loc}_\kappa}(X) \cong \kappa\Pi_2^0(X), \\ (ii) \quad & \text{RSub}_{\kappa\text{Bor}^+\text{Loc}_\kappa}(X) \cong \kappa I(\mathcal{B}_\kappa^+)_\delta(X), \\ (iii) \quad & \text{RSub}_{\kappa\text{BorLoc}_\kappa}(X) \cong \mathcal{B}_\kappa(X). \end{aligned}$$

Proof. We only do case (i); cases (ii) and (iii) are completely analogous.

(a) By Corollaries 3.4.5 and 3.4.7 it remains only to check that for every $B = \bigwedge_i (U_i \rightarrow V_i) \in \infty(\kappa\Pi_2^0)_\delta(X)$ where $U_i, V_i \in \mathcal{O}_\kappa(X)$, letting $f : Y \hookrightarrow X$ be the induced κ -sublocale on B , we have $B = \text{im}(f)$. By definition, $\mathcal{O}_\kappa(Y)$ is the quotient of $\mathcal{O}_\kappa(X)$ by the kernel of $\mathcal{O}_\kappa(X) \hookrightarrow \mathcal{B}_\infty(X) \twoheadrightarrow \downarrow B$. By the general description of Boolean congruences (Section 2.4), $\downarrow B$ is the quotient of $\mathcal{B}_\infty(X)$ by the relations $U_i \leq V_i$, hence is also the free complete Boolean algebra generated by the quotient Q of $\mathcal{O}_\kappa(X)$ by these same relations, i.e., $\mathcal{O}_\kappa(X) \hookrightarrow \mathcal{B}_\infty(X) \twoheadrightarrow \downarrow B$ also factors as $\mathcal{O}_\kappa(X) \twoheadrightarrow Q \hookrightarrow \langle Q \rangle_{\text{CBOOL}} \cong \downarrow B$, hence has kernel generated by the relations $U_i \leq V_i$. By Corollary 3.4.3, this means $\text{im}(f) = \bigwedge_i (U_i \rightarrow V_i) = B$.

(b) By definition of induced κ -sublocale and Corollary 3.4.3, the induced κ -sublocale $f : Y \hookrightarrow X$ on B has

$$\begin{aligned} \text{im}(f) &= \bigwedge \{U \rightarrow V \mid U, V \in \mathcal{O}_\kappa(X) \text{ \& } U \wedge B \leq V \leq B\} \\ &= \bigwedge \{U \rightarrow V \mid U, V \in \mathcal{O}_\kappa(X) \text{ \& } B \leq U \rightarrow V\}, \end{aligned}$$

which is clearly the $\infty(\kappa\Pi_2^0)_\delta$ -closure of B .

(c) is clear from Corollary 3.4.3 and the above proof of (a). \square

Remark 3.4.10. Proposition 3.4.9(a)(i) is a restatement of Proposition 2.8.5, in which the two maps are composed with the order-reversing bijections $\neg : \kappa\mathcal{I}(\mathcal{N}_\kappa(\mathcal{O}_\kappa(X))) = \langle \mathcal{N}_\kappa(\mathcal{O}_\kappa(X)) \rangle_{\text{Frm}} \cong \infty(\kappa\Pi_2^0)_\delta(X)$ and $\text{RSub}_{\kappa\text{Loc}}(X) \cong \{\text{order-congruences on } \mathcal{O}_\kappa(X)\}$. Thus, the above yields another proof of that result, which is somewhat closer in spirit to the original proof of Madden [Mad91, 5.1] rather than the concrete computation via posites in Proposition 2.8.5.

Convention 3.4.11. From now on, **we identify regular subobjects in κLoc , $\kappa\text{Bor}^+\text{Loc}$, κBorLoc with their ∞ -Borel images** via Proposition 3.4.9(a). Thus, we also regard the embeddings in (3.4.6) as inclusions. For a κ -locale X , say, by Proposition 3.4.9(b), the induced κ -sublocale on $B \in \mathcal{B}_\infty(X)$ becomes simply its $\infty(\kappa\Pi_2^0)_\delta$ -closure (hence its $\infty\Pi_2^0$ -closure if X is κ -based).

For an ∞ -Borel sublocale $Y \subseteq X$, i.e., $Y \in \mathcal{B}_\infty(X)$, **we also identify ∞ -Borel sets in Y with ∞ -Borel subsets of Y in X** , i.e., we treat $\mathcal{B}_\infty(Y) \cong \downarrow Y \subseteq \mathcal{B}_\infty(X)$ as an inclusion. Thus, the inclusion $f : Y \hookrightarrow X$ is given by $f^* = Y \wedge (-) : \mathcal{B}_\infty(X) \rightarrow \downarrow Y$. Note that if $Y \subseteq X$ is a κ -sublocale, say, then $\mathcal{O}_\kappa(Y)$ becomes identified with a subset of $\infty(\kappa\Pi_2^0)_\delta(X)$: namely, $V \in \mathcal{O}_\kappa(Y)$ is identified with $Y \wedge U$ for any $U \in \mathcal{O}_\kappa(X)$ such that $V = f^*(U)$.

(This is incompatible with another commonly used convention in locale theory, where for a sublocale $f : Y \hookrightarrow X$, one instead identifies $V \in \mathcal{O}(Y)$ with the *greatest* $U \in \mathcal{O}(X)$ such that $V = f^*(U)$; see e.g., [Joh82, II 2.3], [PP12, III 2.1].)

We next note that the above correspondence between regular subobjects and their image sets interacts well with other operations on either side. We have already noted in Corollary 3.4.7 that meet (pullback) of regular subobjects corresponds to meet of their images.

Proposition 3.4.12.

(a) *For*

- (i) $X \in \kappa\text{Loc}$ and $Y_i \in \mathcal{O}_\kappa(X)$,
- (ii) $X \in \kappa\text{Bor}^+\text{Loc}$ and $Y_i \in \mathcal{B}_\kappa^+(X)$, or
- (iii) $X \in \kappa\text{BorLoc}$ and $Y_i \in \mathcal{B}_\kappa(X)$

such that the Y_i are $<\kappa$ -many and pairwise disjoint, regarding them as regular subobjects $Y_i \subseteq X$ via Proposition 3.4.9(a), the disjoint union (i.e., coproduct) $\bigsqcup_i Y_i$ with the induced map $\bigsqcup_i Y_i \rightarrow X$ is a regular subobject, and corresponds to the join $\bigvee_i Y_i \in \mathcal{B}_\infty(X)$.

- (b) *For any $<\kappa$ -many objects Y_i in (i) κLoc , (ii) $\kappa\text{Bor}^+\text{Loc}$, or (iii) κBorLoc , their disjoint union $\bigsqcup_i Y_i$ is such that the cocone maps $\iota_i : Y_i \rightarrow \bigsqcup_j Y_j$ exhibit the $Y_i \subseteq \bigsqcup_j Y_j$ as pairwise disjoint regular subobjects, belonging to the respective class $\Gamma \subseteq \mathcal{B}_\infty$ in (a), with union $\bigsqcup_j Y_j$.*

Proof. This follows from the description of products in the dual algebraic categories from Section 2.5. For example, in case (i), the assumption in (a) says the $Y_i \in \mathcal{O}_\kappa(X)$ are pairwise disjoint, hence descend to a partition of \top in the principal filterquotient $\downarrow \bigvee_i Y_i$, which is thus isomorphic to $\prod_i \downarrow Y_i = \mathcal{O}_\kappa(\bigsqcup_i Y_i)$; the induced map $f : \bigsqcup_i Y_i \rightarrow X$ corresponds to the quotient map $f^* : \mathcal{O}_\kappa(X) \rightarrow \downarrow \bigvee_i Y_i$, hence f is a κ -locale embedding with ∞ -Borel image $\bigvee_i Y_i$ by Proposition 3.4.2. For (b), the $\iota_i : Y_i \rightarrow \bigsqcup_j Y_j$ are dual to the product projections $\iota_i^* : \mathcal{O}_\kappa(\bigsqcup_j Y_j) = \prod_j \mathcal{O}_\kappa(Y_j) \rightarrow \mathcal{O}_\kappa(Y_i)$, which are isomorphic to the principal filterquotients $\prod_j \mathcal{O}_\kappa(Y_j) \rightarrow \downarrow \delta_i$ where $\delta_i \in \prod_j \mathcal{O}_\kappa(Y_j)$ are as in Section 2.5; this easily yields the conclusions in (b). Cases (ii) and (iii) are similar. \square

Remark 3.4.13. Since $< \kappa$ -many κ -Borel sets may be disjointified, Proposition 3.4.12(a) allows the expression of arbitrary κ -ary joins in \mathcal{B}_κ in terms of coproducts of κ -Borel sublocales.

Proposition 3.4.14. *Let $f : X \rightarrow Y$ be a morphism in (i) κLoc , (ii) $\kappa\text{Bor}^+\text{Loc}$, or (iii) κBorLoc .*

- (a) *For a regular subobject $Z \subseteq Y$, identified with $Z \in \mathcal{B}_\infty(Y)$ via Proposition 3.4.9(a), $f^*(Z) \in \mathcal{B}_\infty(X)$ corresponds to the pullback of $Z \hookrightarrow Y$ along f .*
- (b) *Thus, $f^*(Z) = X$ iff f factors through $Z \hookrightarrow Y$.*
- (c) *The factorization $X \twoheadrightarrow Z \hookrightarrow Y$ of f into an epimorphism followed by a regular monomorphism is given by the (i) $\infty(\kappa\Pi_2^0)_\delta$ -image, (ii) $\infty I(\mathcal{B}_\kappa^+)_\delta$ -image, or (iii) $\infty(\mathcal{B}_\kappa)_\delta$ -image of f , with either existing if the other does (which is always if $\kappa < \infty$ or in case (i)).*

Proof. (a) If $Z = \bigwedge_i (U_i \rightarrow V_i)$, i.e., $\mathcal{O}_\kappa(Z)$ (say) is the quotient of $\mathcal{O}_\kappa(Y)$ by the relations $U_i \leq V_i$, then $f^*(Z) = \bigwedge_i (f^*(U_i) \rightarrow f^*(V_i))$, i.e., $\mathcal{O}_\kappa(f^*(Z))$ is the quotient of $\mathcal{O}_\kappa(X)$ by the relations $f^*(U_i) \leq f^*(V_i)$, which exactly describes the pushout $\mathcal{O}_\kappa(X) \otimes_{\mathcal{O}_\kappa(Y)} \mathcal{O}_\kappa(Z)$.

(b) follows since f factors through a subobject $Z \subseteq Y$ iff its pullback along f is all of X .

(c) follows since Z is by definition the least regular subobject through which f factors. \square

Remark 3.4.15. In Theorem 4.4.3 we will show that there is a single σ -Borel map $f : X \rightarrow Y$ between standard σ -Borel locales which does *not* have a κ -Borel image for any κ .

It follows from this that the $\infty(\mathcal{B}_\sigma)_\delta$ -image Z of f (which always exists) is *not* in $\mathcal{B}_\sigma(Y)$ (or else it would be the σ -Borel image). In other words, there is a morphism $f : X \rightarrow Y$ in the category σBor_σ of standard σ -Borel locales, whose epi-regular mono factorization $X \twoheadrightarrow Z \hookrightarrow Y$ is *not* standard.

Moreover, since $Z \subseteq Y$ is itself ∞ -Borel, hence λ -Borel for some $\omega_1 < \lambda < \infty$ (namely $\lambda = (2^{\aleph_0})^+$, since $|\mathcal{B}_\sigma(Y)| \leq 2^{\aleph_0}$ and so every meet of σ -Borel sets is $(2^{\aleph_0})^+$ -ary), Z cannot also be the $\infty(\mathcal{B}_\lambda)_\delta$ -image of f , or else it would be the λ -Borel image; thus the $\infty(\mathcal{B}_\lambda)_\delta$ -image must be strictly smaller. In other words, we have $\kappa = \omega_1 < \lambda < \infty$ such that the forgetful functor $\kappa\text{BorLoc} \rightarrow \lambda\text{BorLoc}$ does *not* preserve epi-regular mono factorizations, or equivalently does not preserve epimorphisms (since it does preserve regular monos). See Corollary 4.4.5.

For an ∞ -Borel map $f : X \rightarrow Y$ and a class of sets Γ , we call f Γ -**surjective** if $Y = \top \in \mathcal{B}_\infty(Y)$ is the Γ -image of f . (Traditionally, a “surjective” continuous map between locales usually means a “ $\infty\Pi_2^0$ -surjective” map in our terminology; see Remark 3.4.1.) By Proposition 3.4.14(c), and the fact that the middle classes below are the closures of the left classes under arbitrary meets,

$$\begin{aligned}
 (3.4.16) \quad & \kappa\Pi_2^0\text{-surjective} = \infty(\kappa\Pi_2^0)_\delta\text{-surjective} = \text{epimorphism in } \kappa\text{Loc}, \\
 & \kappa I(\mathcal{B}_\kappa^+)_\delta\text{-surjective} = \infty I(\mathcal{B}_\kappa^+)_\delta\text{-surjective} = \text{epimorphism in } \kappa\text{Bor}^+\text{Loc}, \\
 & \kappa\text{-Borel surjective} = \infty(\mathcal{B}_\kappa)_\delta\text{-surjective} = \text{epimorphism in } \kappa\text{BorLoc}.
 \end{aligned}$$

In this terminology, the epi-regular mono factorization in κLoc , say, consists of a $\kappa\Pi_2^0$ -surjection followed by a κ -locale embedding.

By Corollary 2.12.7, we have

Proposition 3.4.17. *Epimorphisms, hence epi-regular mono factorizations that exist, in κBorLoc are pullback-stable.* \square

Remark 3.4.18. The analogous statement in $\kappa\mathbf{Loc}$ is well-known to fail; likewise, the analogous statement in $\kappa\mathbf{Bor}^+\mathbf{Loc}$ fails. Let $X = [0, 1]$ with the Scott topology, where the open sets are $(r, 1]$ for each $r \in [0, 1]$; this is a quasi-Polish space (see Section 3.5). Let $Y = [0, 1]$ with the discrete topology (which is κ -based for $\kappa = (2^{\aleph_0})^+$). The continuous identity map $f : Y \rightarrow X$ is given by the inclusion $f^* : \mathcal{O}(X) = \mathcal{O}([0, 1]) \hookrightarrow \mathcal{P}([0, 1]) = \mathcal{O}(Y)$, whence f is an epimorphism of (κ) -locales. Moreover, by Example 3.5.14 below, $\mathcal{B}_\infty^+(X) = \mathcal{B}_\sigma^+(X)$ consists precisely of the open and closed intervals $(r, 1]$ and $[r, 1]$; thus $f^* : \mathcal{B}_\infty^+(X) \rightarrow \mathcal{P}([0, 1]) = \mathcal{B}_\infty^+(Y)$ is still injective, whence f is also an epimorphism of positive κ -Borel locales. Also, the image of f^* generates $\mathcal{P}([0, 1])$ as a complete Boolean algebra, i.e., $f^* : \mathcal{B}_\infty(X) \rightarrow \mathcal{P}([0, 1]) = \mathcal{B}_\infty(Y)$ is surjective, whence f is an ∞ -Borel locale embedding.

The ∞ -Borel image $f(Y) \subseteq X$ is the join of singletons $\bigvee_{x \in [0, 1]} \{x\} \in \kappa\Sigma_2^0(X)$, which is $< \top \in \mathcal{B}_\infty(X)$ by localic Baire category (Theorem 3.8.2, applied to $[0, 1]$ with the *Euclidean* topology which is σ -Borel isomorphic to X) or Gaifman–Hales (Theorem 2.1.2, which together with Remark 3.5.13 implies that $f^* : \mathcal{B}_\infty(X) \cong \langle \mathbb{N} \rangle_{\mathbf{CBOOL}} \twoheadrightarrow \mathcal{P}([0, 1]) = \mathcal{B}_\infty(Y)$ cannot be injective). Let $Z = \neg f(Y) \in \kappa\Pi_2^0(X)$; thus $Z \subseteq X$ is a κ -sublocale, hence also a positive κ -Borel sublocale, which is nonempty. But the pullback of f along $Z \hookrightarrow X$ is empty (since it corresponds to intersection of $f(Y)$ and Z by Corollary 3.4.7), hence not an epimorphism in either $\kappa\mathbf{Loc}$ or $\kappa\mathbf{Bor}^+\mathbf{Loc}$.

Indeed, we have the following characterization of pullback-stable epimorphisms in $\kappa\mathbf{Loc}$, $\kappa\mathbf{Bor}^+\mathbf{Loc}$, due to Wilson [Wil94, 28.5] in the case of \mathbf{Loc} :

Proposition 3.4.19. *A morphism $f : X \rightarrow Y$ in (i) $\kappa\mathbf{Loc}$ or (ii) $\kappa\mathbf{Bor}^+\mathbf{Loc}$, or their full subcategories of standard objects, is a pullback-stable epimorphism iff it becomes an epimorphism in $\kappa\mathbf{BorLoc}$.*

Proof. (\Leftarrow) If f becomes an epimorphism in $\kappa\mathbf{BorLoc}$, then since the forgetful functors $\kappa\mathbf{Loc} \rightarrow \kappa\mathbf{Bor}^+\mathbf{Loc} \rightarrow \kappa\mathbf{BorLoc}$ preserve pullbacks, every pullback of f becomes an epimorphism in $\kappa\mathbf{BorLoc}$, hence is already an epimorphism, since the forgetful functors are also faithful.

(\Rightarrow) (i) Suppose f is a pullback-stable epimorphism in $\kappa\mathbf{Loc}$ or $\kappa\mathbf{Loc}_\kappa$; we must show it is an epimorphism in $\kappa\mathbf{BorLoc}$, i.e., κ -Borel surjective. Let $B \in \mathcal{B}_\kappa(Y)$ such that $f^*(B) = \top$; we must show $B = \top$. By Proposition 3.3.7, let $Y' \rightarrow Y$ be a partial dissolution with $B \in \mathcal{O}_\kappa(Y')$, and with Y' standard if Y is. Let $f' : X' \rightarrow Y'$ be the pullback of f along $Y' \rightarrow Y$.

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

Then f' is an epimorphism in $\kappa\mathbf{Loc}_{(\kappa)}$, i.e., $\kappa\Pi_2^0$ -surjective. Since $Y' \rightarrow Y$ is a κ -Borel isomorphism and $\kappa\mathbf{Loc} \rightarrow \kappa\mathbf{BorLoc}$ preserves pullbacks, $X' \rightarrow X$ is also a κ -Borel isomorphism, i.e., we may identify $\mathcal{B}_\kappa(X') = \mathcal{B}_\kappa(X)$. So since $f'^*(B) = f^*(B) = \top \in \mathcal{B}_\kappa(X) = \mathcal{B}_\kappa(X')$ and $B \subseteq Y'$ is open, $B = \top \in \mathcal{B}_\kappa(Y') = \mathcal{B}_\kappa(Y)$, as desired.

The proof of (\Rightarrow) (ii) is analogous, using that we may “dissolve” a positive κ -Borel locale X to make any κ -Borel set $B \in \mathcal{B}_\kappa^+(X)$ positive, dually by taking $\langle \mathcal{B}_\kappa^+(X) \rangle_{\kappa\mathbf{Bbool}}$, and in the standard case, passing to a κ -presented subalgebra containing $\mathcal{B}_\kappa^+(X)$ and B using Proposition A.1.6 as in the proof of Corollary 2.8.16. \square

Remark 3.4.20. Thus far, we have mostly focused on images of maps $f : X \rightarrow Y$; but we can reduce images of arbitrary ∞ -Borel sets $B \subseteq X$ under f to images of maps. Indeed, B itself corresponds to an ∞ -Borel sublocale $g : B \hookrightarrow X$; hence the Γ -image of B under f , for any Γ , is the same as the Γ -image of $f \circ g$. Furthermore:

- (a) if X is a κ -Borel locale and $B \in \mathcal{B}_\kappa(X)$, then B is a κ -Borel sublocale of X , which is standard if X is (by Proposition 3.4.9);
- (b) if X is furthermore a κ -locale, then by Proposition 3.3.7 we can take a partial dissolution $X' \rightarrow X$, which is standard if X is, such that $B \in \mathcal{O}_\kappa(X')$, whence the composite $g : B \hookrightarrow X' \rightarrow X$ is a κ -continuous monomorphism with ∞ -Borel image B .

Applying this reduction to Proposition 3.4.17 yields the following “Beck–Chevalley equation”:

Corollary 3.4.21. *Let*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{f'} & Y \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

be a pullback square in $\kappa\mathbf{BorLoc}$. For any $\infty(\mathcal{B}_\kappa)_\delta$ -set $B \subseteq X$ such that $f^{\infty(\mathcal{B}_\kappa)_\delta}(B)$ exists, we have

$$f'^{\infty(\mathcal{B}_\kappa)_\delta}(g'^*(B)) = g^*(f^{\infty(\mathcal{B}_\kappa)_\delta}(B)).$$

Thus if $B \subseteq X$ is κ -Borel and the κ -Borel image $f^\kappa(B) \subseteq Z$ exists, then so does

$$f'^\kappa(g'^*(B)) = g^*(f^\kappa(B)).$$

Proof. The first equation follows from Proposition 3.4.17 applied to the epi-regular mono factorization of $B \hookrightarrow X \xrightarrow{f} Z$; the second follows because if $f^\kappa(B)$ exists, it must be $f^{\infty(\mathcal{B}_\kappa)_\delta}(B)$. \square

Another important aspect of epimorphisms in $\kappa\mathbf{BorLoc}$ is their nice behavior with respect to inverse limits:

Proposition 3.4.22. *Let $(X_i)_{i \in I}$ be a codirected diagram in $\kappa\mathbf{BorLoc}$, for a directed poset I , with morphisms $f_{ij} : X_i \rightarrow X_j$ for $i \geq j$. If each f_{ij} is an epimorphism, then so is each limit projection $\pi_i : \varprojlim_j X_j \rightarrow X_i$.*

Proof. By Corollary 2.8.7. \square

Corollary 3.4.23. *Epimorphisms in $\kappa\mathbf{BorLoc}$ are closed under arbitrary products.*

Proof. Closure under finite products follows from pullback-stability (Proposition 3.4.17). For an arbitrary family of epimorphisms $(f_i : X_i \twoheadrightarrow Y_i)_{i \in I}$, we can write $\prod_i X_i$ as the limit of all products of the form $\prod_{i \in F} X_i \times \prod_{i \notin F} Y_i$ for finite $F \subseteq I$, where for $F \subseteq G \subseteq I$ we have the map $\prod_{i \in G} X_i \times \prod_{i \notin G} Y_i \rightarrow \prod_{i \in F} X_i \times \prod_{i \notin F} Y_i$ which is the product of the f_i for $i \in G \setminus F$ and the identity on other coordinates; each such map is a finite product of epimorphisms, hence an epimorphism. Thus by Proposition 3.4.22, all of the limit projections $\prod_i X_i \rightarrow \prod_{i \in F} X_i \times \prod_{i \notin F} Y_i$ are epimorphisms; taking $F = \emptyset$ yields $\prod_i f_i : \prod_i X_i \rightarrow \prod_i Y_i$. \square

Remark 3.4.24. Note that even though Corollary 3.4.23 holds in **Set** by the axiom of choice, Proposition 3.4.22 only holds for *countable* inverse limits (consider the tree of bounded well-ordered sequences of rationals, whose empty set of branches is the inverse limit of its ω_1 levels). Thus, this is an example where locales are *better-behaved* than topological spaces. (The corresponding fact for epimorphisms in **Loc**, i.e., $\infty\Pi_2^0$ -surjective maps, is standard; see [JT84, IV 4.2].)

For the sake of completeness, we also record the following direct translations of results from Section 2.12. Recall that the **kernel** of a morphism $f : X \rightarrow Y$ in an arbitrary category is its pullback $\ker(f) := X \times_Y X$ with itself; f is a monomorphism iff $\ker(f) \subseteq X \times X$ is the diagonal $X \hookrightarrow X \times X$, and is a regular epimorphism iff it is the coequalizer of its kernel. Similarly, the **order-kernel** of f in a locally ordered category is the comma object $\text{oker}(f) := X \downarrow_Y X$ with itself; f is an **order-monomorphism** iff $\text{oker}(f) \subseteq X \times X$ is the **(internal) order \leq_X on X** , which is the universal object equipped with projections $\pi_1, \pi_2 : \leq_X \rightrightarrows X$ obeying $\pi_1 \leq \pi_2$; and f is an **order-regular epimorphism** iff it is the coinsserter of its order-kernel. An equivalence class of monomorphisms to X is a **subobject** of X , while an equivalence class of order-monomorphisms is a **order-embedded subobject**. See Appendices A.2 and A.3 for details.

Proposition 3.4.25. *In $\kappa\text{Bor}^{(+)}\text{Loc}$,*

- (a) *every (order-)monomorphism is regular, i.e., (order-embedded) subobject = regular subobject;*
- (b) *every epimorphism is (order-)regular.*

Proof. By Corollaries 2.12.9, 2.12.10, 2.12.13 and 2.12.14. □

Remark 3.4.26. Classically, the **Lusin–Suslin theorem** says that every injective Borel map $f : X \rightarrow Y$ between standard Borel spaces has Borel image; see [Kec95, 15.1]. Since injectivity is trivially equivalent to being a monomorphism, Proposition 3.4.25(a) in the case of κBorLoc , together with Corollary 3.4.3(iii), can be seen as a Lusin–Suslin theorem for κ -Borel locales: the former result says that every “injective” map is an embedding, while the latter says that every embedding has a κ -Borel image (which is also the ∞ -Borel image). Moreover, the proofs of Corollary 2.12.10 by LaGrange [LaG74] and Corollary 3.4.3(iii) can together be seen as an algebraic analog of a proof of the classical Lusin–Suslin theorem (see [Ch19b, 3.6] for an elaboration of this point).

Likewise, Proposition 3.4.25(a) for $\kappa\text{Bor}^+\text{Loc}$ says that every positive κ -Borel map $f : X \rightarrow Y$ which is “internally an order-embedding” is in fact an embedding of positive κ -Borel locales. Recalling from Example 3.1.7 that we regard positive κ -Borel locales as generalizations of the specialization preorder, this result can be seen as an analog of the following classical fact: given a Borel monotone map $f : X \rightarrow Y$ between quasi-Polish spaces, every open $U \subseteq X$ is the preimage of a positive Borel $B \subseteq Y$ (i.e., a set in the (σ, σ) -subframe generated by $\mathcal{O}(Y) \subseteq \mathcal{P}(Y)$).

(The analogous statement for κLoc is false by Remark 2.12.17: the identity $|\overline{\mathbb{N}}| \rightarrow \overline{\mathbb{N}}$ to the one-point compactification $\overline{\mathbb{N}}$ of \mathbb{N} from its discrete space is an order-embedding but not an embedding.)

Proposition 3.4.25(b) corresponds to the following classical fact and its ordered analog: given a Borel map $f : X \rightarrow Y$ between standard Borel spaces, if $f^{-1}(B) = \emptyset \implies B = \emptyset$ for all Borel $B \subseteq Y$ (which clearly just means f is surjective), then every Borel $A \subseteq X$ which is invariant under the equivalence relation $\ker(f) \subseteq X^2$ (i.e., the preimages of A under the two projections $\ker(f) \rightrightarrows A$ are the same) is $f^{-1}(B)$ for a unique Borel $B \subseteq Y$ (namely $B = f(A)$, which is Borel by the Lusin separation theorem [Kec95, 14.7] which corresponds to Corollary 2.12.6; see also Corollary 4.2.4).

Remark 3.4.27. Since the forgetful functors in (3.1.9) preserve limits and are faithful, they preserve and reflect (plain) monomorphisms. Thus, for example, a κ -continuous map $f : X \rightarrow Y \in \kappa\mathbf{Loc}$ is a monomorphism iff it is a κ -Borel monomorphism in $\kappa\mathbf{BorLoc}$, hence a κ -Borel embedding, in which case it has an $\infty(\kappa\Pi_2^0)_\delta$ -image $f(X) \subseteq Y$. In particular, any monomorphism of locales (hence κ -locales for sufficiently large κ) has an ∞ -Borel image. This fact was implicitly shown by Madden–Molitor [MM91], also ultimately using LaGrange’s interpolation theorem (Corollary 2.12.10).

Recall that a finite-limit-preserving functor is conservative iff it restricts to an injection on each subobject \wedge -lattice (see Lemma A.2.4). Since the horizontal forgetful functors in (3.1.9) are conservative (Proposition 3.1.10), they are thus injective on subobject \wedge -lattices. For $\kappa\mathbf{BorLoc} \rightarrow \lambda\mathbf{BorLoc}$, this is equivalent to injectivity on regular subobjects ((3.4.6), in turn equivalent to faithfulness) by Proposition 3.4.25(a); but for $\kappa\mathbf{Bor}^+\mathbf{Loc} \rightarrow \lambda\mathbf{Bor}^+\mathbf{Loc}$, the latter result only gives injectivity on order-embedded subobjects.

Remark 3.4.28. Finally, we record the interaction of sublocales with spatialization. The right adjoint functor $\mathbf{Sp} : \kappa\mathbf{Loc} \rightarrow \kappa\mathbf{Top}$ (say) preserves limits, hence preserves regular monomorphisms, which are (certain) κ -subspace embeddings in $\kappa\mathbf{Top}$; thus each κ -sublocale $Y \subseteq X$ yields a κ -subspace $\mathbf{Sp}(Y) \subseteq \mathbf{Sp}(X)$. Points $x \in \mathbf{Sp}(Y)$ are homomorphisms $x^* : \mathcal{O}_\kappa(Y) \rightarrow 2$, hence homomorphisms $\mathcal{O}_\kappa(X) \rightarrow 2$ which respect the kernel of $\mathcal{O}_\kappa(X) \twoheadrightarrow \mathcal{O}_\kappa(Y)$. If Y corresponds via Proposition 3.4.9 to an $\infty(\kappa\Pi_2^0)_\delta$ -set

$$Y = \bigwedge_i (U_i \rightarrow V_i) \subseteq X,$$

where $U_i, V_i \in \mathcal{O}_\kappa(X)$, then the (order-)kernel of $\mathcal{O}_\kappa(X) \twoheadrightarrow \mathcal{O}_\kappa(Y)$ is generated by the relations $U_i \leq V_i$; thus $x \in \mathbf{Sp}(X)$ is in $\mathbf{Sp}(Y)$ iff $x^*(U_i) \leq x^*(V_i)$, i.e., $x \in \mathbf{Sp}(U_i) \implies x \in \mathbf{Sp}(V_i)$ (recall Section 3.2) for each i , i.e., $x \in \bigcap_i (\mathbf{Sp}(U_i) \rightarrow \mathbf{Sp}(V_i))$; that is, we have

$$\mathbf{Sp}(Y) = \bigcap_i (\mathbf{Sp}(U_i) \rightarrow \mathbf{Sp}(V_i)) \subseteq \mathbf{Sp}(X).$$

In other words, letting $\infty(\kappa\Pi_2^0)_\delta(\mathbf{Sp}(X))$ consist of all sets in $\mathbf{Sp}(X)$ of the form $\bigcap_i (U_i \rightarrow V_i)$ for κ -open $U_i, V_i \subseteq \mathbf{Sp}(X)$, so that $\infty(\kappa\Pi_2^0)_\delta(\mathbf{Sp}(X))$ is the image of $\infty(\kappa\Pi_2^0)_\delta(X)$ under $\mathcal{B}_\infty(X) \twoheadrightarrow \mathcal{B}_\infty(\mathbf{Sp}(X)) = \mathcal{P}(\mathbf{Sp}(X))$ as in (3.3.2), we have that the following square commutes:

$$(3.4.29) \quad \begin{array}{ccc} \infty(\kappa\Pi_2^0)_\delta(X) & \cong & \mathbf{RSub}_\kappa(X) = \{\kappa\text{-sublocales of } X\} \\ \downarrow & & \downarrow \mathbf{Sp} \\ \infty(\kappa\Pi_2^0)_\delta(\mathbf{Sp}(X)) & \subseteq & \mathcal{P}(\mathbf{Sp}(X)) = \{\kappa\text{-subspaces of } X\} \end{array}$$

We clearly have analogs of this for $\kappa\mathbf{Bor}^{(+)}\mathbf{Loc}$, and for the full subcategories of standard objects.

3.5 Standard σ -locales

The standard objects in our main categories $\kappa(\mathbf{Bor}^{(+)})\mathbf{Loc}$, when $\kappa = \omega_1$, form subcategories which are particularly convenient when working with concrete examples: they are all spatial, corresponding to well-behaved classes of spaces; and moreover, other localic notions (such as products, sublocales, etc.), under suitable countability restrictions, also correspond to their spatial analogs. In this subsection, we review these correspondences, and use them to give several key examples.

Classical descriptive set theory takes place in the topological context of **Polish spaces**, i.e., second-countable completely metrizable spaces, and the Borel context of **standard Borel spaces**,

i.e., $(\sigma\text{-})$ Borel subspaces of Cantor space $2^{\mathbb{N}}$; see [Kec95]. Relatively recently, de Brecht [deB13] introduced **quasi-Polish spaces**, which are Π_2^0 (in the sense of Selivanov; see Section 3.3) subspaces of countable powers of Sierpinski space $\mathbb{S}^{\mathbb{N}}$, and showed that they are a robust generalization of Polish spaces to the non- T_1 context, with suitable analogs of most well-known descriptive set-theoretic properties of Polish spaces. Heckmann [Hec15] proved that the spatialization adjunction $\mathbf{Top} \rightleftarrows \mathbf{Loc}$ restricts to an equivalence between the full subcategories of quasi-Polish spaces and countably copresented locales (or equivalently in our terminology, standard σ -locales). As we noted in the Introduction, this result, and the consequent analogies it yields between descriptive set theory and locale theory, is one of the primary motivations behind the present paper.

As Heckmann [Hec15, §4.1] points out, the key ingredient in his result, namely that every countably copresented locale is spatial, has a much longer and more disguised history. It was shown by Fourman–Grayson [FG82, 3.12], in the related context of *formal topology*. If one takes for granted certain purely formal order-theoretic constructions, e.g., the injectivity of $A \rightarrow \langle A \rangle_{\mathbf{CBool}}$ for a frame A (shown already by Isbell [Isb72, 1.3, 1.5]), then the result can arguably be traced back even further; e.g., it follows from the Loomis–Sikorski representation theorem for σ -Boolean algebras, which is itself equivalent to the spatiality of standard σ -Borel locales (see [Ch19b, 4.1]). We may complete the triad by adding the spatiality of standard positive σ -Borel locales (see below). All of these results easily imply each other, and essentially boil down to a (classical) Baire category argument. In the following, we have arbitrarily chosen to take the Loomis–Sikorski theorem as fundamental, and to view the other results as consequences.

Theorem 3.5.1 (Loomis–Sikorski). *Every countably presented σ -Boolean algebra A admits enough homomorphisms to 2 to separate its elements.*

Proof. See [Sik69, 29.1] or [Ch19b, 4.1]. (Note that since every countably presented algebra is a principal filterquotient of a free algebra, it suffices to assume A is free.) \square

Corollary 3.5.2. *Every countably presented $(\sigma\text{-})$ frame or (σ, σ) -frame A admits enough homomorphisms to 2 to separate its elements.*

Proof. Follows from injectivity of the units $A \rightarrow \langle A \rangle_{\sigma\mathbf{Bool}}$ (Corollaries 2.8.6 and 2.11.5). \square

Corollary 3.5.3. *Every standard (positive) σ -(Borel) locale X is spatial.* \square

It follows that the spatialization adjunctions from Section 3.2, when $\kappa = \omega_1$, restrict to equivalences between $\sigma(\mathbf{Bor}^{(+)})\mathbf{Loc}_{\sigma}$ and full subcategories of the corresponding categories of spaces. In order to identify these full subcategories of spaces, we use the following well-known fact (see e.g., [Joh82, VII 4.9]):

Proposition 3.5.4 (folklore). *For any set X , the power \mathbb{S}^X of Sierpinski space is sober, and corresponds to the locale with $\mathcal{O}(\mathbb{S}^X) = \langle X \rangle_{\mathbf{Frm}}$ where the generators $x \in X \subseteq \langle X \rangle_{\mathbf{Frm}}$ are identified with the subbasic open sets in the product \mathbb{S}^X .*

Proof. The free frame on one generator is clearly $\langle 1 \rangle_{\mathbf{Frm}} \cong \mathcal{O}(\mathbb{S}) = \{\emptyset < \{1\} < \mathbb{S}\}$ where $\{1\}$ is the generator; and \mathbb{S} is clearly sober. It follows that the right adjoint spatialization functor $\mathbf{Sp} : \mathbf{Loc} \rightarrow \mathbf{Top}$ takes the localic power \mathbb{S}^X , corresponding to the frame coproduct $\mathcal{O}(\mathbb{S})^{\otimes X} \cong \langle X \rangle_{\mathbf{Frm}}$, to the spatial power \mathbb{S}^X , whence the latter is sober. But since $\langle X \rangle_{\mathbf{Frm}} = \mathcal{L}(\langle X \rangle_{\wedge\mathbf{Lat}}) \subseteq \mathcal{P}(\langle X \rangle_{\wedge\mathbf{Lat}}) \cong 2^{\langle X \rangle_{\wedge\mathbf{Lat}}}$ (see Section 2.3) is a subframe of a power of 2, it clearly admits enough homomorphisms to 2

(the projections), i.e., the localic power \mathbb{S}^X is spatial, hence agrees with the spatial power. Under this correspondence, each generator $x \in X$ corresponds to the coproduct injection $\iota_x : \mathcal{O}(\mathbb{S}) \rightarrow \mathcal{O}(\mathbb{S}) \otimes^X$, hence to the product projection $\pi_x : \mathbb{S}^X \rightarrow \mathbb{S}$, hence to the x th subbasic open set in \mathbb{S}^X . \square

Corollary 3.5.5. *For any countable set X , the countable power \mathbb{S}^X in $\sigma(\mathbf{Bor}^{(+)})\mathbf{Loc}$ is spatial, and corresponds to the spatial power of \mathbb{S} in $(\sigma)\mathbf{Top}$, $(\sigma)\mathbf{Bor}^+$, $(\sigma)\mathbf{Bor}$ respectively.*

Proof. Spatiality is by Corollary 3.5.3; the rest follows from chasing the product $\mathbb{S}^X \in \sigma\mathbf{Loc}$ through the various right adjoint forgetful and spatialization functors (3.2.1). \square

For a second-countable (σ) -topological space X to have a spatial underlying σ -Borel locale (of the underlying σ -locale) means that the localic σ -Borel algebra of X (which is *a priori* $\langle \mathcal{O}_\sigma(X) \rangle_{\sigma\mathbf{Bool}}$) agrees with the spatial Borel σ -algebra; this implies (see (3.3.2)) that we have levelwise agreement of the (σ) -Borel hierarchies, i.e.,

$$(3.5.6) \quad \sigma\Sigma_\alpha^0(X) \cong \Sigma_\alpha^0(X) \quad \forall \alpha < \omega_1.$$

Similarly, if a positive Borel space X has a spatial underlying σ -Borel locale, then countable Boolean combinations of positive Borel sets in X , as subsets, are in bijection with the respective combinations of positive Borel sets in the underlying σ -Borel locale; for example, if $\sigma I(\mathcal{B}_\sigma)_\delta$ is the class of countable meets of implications between positive σ -Borel sets defined above Corollary 3.4.5,

$$(3.5.7) \quad \sigma I(\mathcal{B}_\sigma)_\delta(X) \cong \{ \bigcap_{i \in \mathbb{N}} (B_i \rightarrow C_i) \mid B_i, C_i \in \mathcal{B}_\sigma^+(X) \} \subseteq \mathcal{P}(X).$$

Now for a standard σ -locale X , say, $\mathcal{O}(X)$ is a countably presented quotient of some countably generated free σ -frame $\mathcal{O}(\mathbb{S}^Y)$ with Y countable, i.e., $X \subseteq \mathbb{S}^Y$ is a standard σ -sublocale, hence a $\sigma\Pi_2^0$ set in \mathbb{S}^Y by Proposition 3.4.9(c), hence a Π_2^0 set in \mathbb{S}^Y by (3.5.6) and Remark 3.4.28, i.e., a quasi-Polish space. Similarly, a standard σ -Borel locale X corresponds to a Borel set in \mathbb{S}^Y (which is Borel isomorphic to 2^Y), i.e., a standard Borel space; while a standard positive σ -Borel locale X corresponds to a set of the form given by the right-hand side of (3.5.7) in some $X = \mathbb{S}^Y$. Call positive Borel spaces of the latter form **standard positive Borel spaces**. Let

$$\mathbf{QPol} \subseteq (\sigma)\mathbf{Top}, \quad \mathbf{SBor}^+ \subseteq \mathbf{Bor}^+, \quad \mathbf{SBor} \subseteq \mathbf{Bor}$$

denote the categories of quasi-Polish spaces, standard positive Borel spaces, and standard Borel spaces, respectively. We have thus recovered the equivalences of categories mentioned above:

Theorem 3.5.8. *The spatialization adjunctions from Section 3.2 restrict to equivalences*

$$\begin{array}{ll} \text{(Heckmann)} & \mathbf{QPol} \simeq (\sigma)\mathbf{Loc}_\sigma, \\ & \mathbf{SBor}^+ \simeq \sigma\mathbf{Bor}^+\mathbf{Loc}_\sigma, \\ \text{(Loomis–Sikorski)} & \mathbf{SBor} \simeq \sigma\mathbf{Bor}\mathbf{Loc}_\sigma. \end{array} \quad \square$$

In the rest of this subsection, we record some other key notions whose spatial and localic versions agree via the above equivalences, as well as some important examples:

Remark 3.5.9. As noted above (3.5.6), for a quasi-Polish space X , the localic and spatial σ -Borel hierarchy agree levelwise.

For example, this tells us that in \mathbb{R} , a countable intersection of open sets $\bigcap_{n \in \mathbb{N}} U_n$ is also a countable meet in $\mathcal{B}_\infty(\mathbb{R})$ (which is not true for uncountable intersections; see Example 3.8.4).

Also, standard σ -sublocales of X correspond to localic $\sigma\Pi_2^0$ sets (Proposition 3.4.9), hence to spatial Π_2^0 sets, hence to quasi-Polish subspaces (see [deB13, Th. 23], or [Ch18b, 4.2] for a proof via a spatial analog [Ch18b, 4.1] of Lemma A.2.1 used in the proof of Corollary 3.4.3).

Remark 3.5.10. Of course, all existing categorical limits and colimits transfer across the equivalences in Theorem 3.5.8. In particular, this applies to κ -ary limits and κ -ary coproducts, which on the left-hand side are the usual limit and disjoint union of spaces, while on the right-hand side are the localic notions in $\sigma(\text{Bor}^{(+)})\text{Loc}$ (since $\sigma(\text{Bor}^{(+)})\text{Loc}_\sigma \subseteq \sigma(\text{Bor}^{(+)})\text{Loc}$ is closed under these operations).

Remark 3.5.11. Also, (regular) monomorphisms and epimorphisms transfer across Theorem 3.5.8.

- Monomorphisms are clearly injective maps in all three spatial categories, which hence correspond to monomorphisms in the localic categories.
- Regular monomorphisms $f : X \rightarrow Y$ are embeddings in all three spatial categories; this can be seen *via* Theorem 3.5.8, since f^* being surjective says precisely that each open/(positive) Borel set in X is the restriction of such a set in Y .
- Epimorphisms in SBor are easily seen to be surjective Borel maps (given a non-surjective $f : X \rightarrow Y$, duplicate some $y \in Y \setminus f(X)$ to get two distinct maps from Y equalized by f); these thus correspond by (3.4.16) to σ -Borel surjective maps in $\sigma\text{BorLoc}_\sigma$.

(We do *not* know if all such maps are ∞ -Borel surjective; see Remark 4.1.11.)

- Epimorphisms in QPol need not be surjective: consider $\overline{\mathbb{N}} = \mathbb{N} \sqcup \{\infty\}$ with the Scott topology, with open sets $\emptyset, [n, \infty]$ for all $n \in \mathbb{N}$ (which is indeed quasi-Polish; see [deB13, §8]), and the inclusion $\mathbb{N} \hookrightarrow \overline{\mathbb{N}}$ where \mathbb{N} has the discrete topology. Every continuous map $\overline{\mathbb{N}} \rightarrow X$ to a sober space X is determined by its restriction to \mathbb{N} , since directed joins are topologically definable in sober spaces (see [Joh82, II 1.9]).

Remark 3.5.12. Polish spaces are precisely the regular (i.e., T_3) quasi-Polish spaces (see [Ch18b, 5.3]). Also, a space is clearly regular iff its underlying locale is, as defined in Section 2.9. Thus, we have an equivalence of categories between Polish spaces and regular standard σ -locales.

Remark 3.5.13. It is a classical result that the only standard Borel spaces, up to Borel isomorphism, are $0, 1, 2, \dots, \mathbb{N}$ with the discrete Borel structure, and \mathbb{R} with the usual Borel structure (see [Kec95, 15.6]). Thus, any two uncountable standard Borel spaces are Borel isomorphic.

It follows that for any uncountable standard Borel space X , we have

$$\mathcal{B}_\infty(X) \cong \langle \mathbb{N} \rangle_{\text{CBOOL}}$$

(which is a proper class, by the Gaifman–Hales Theorem 2.1.2). Indeed, $\mathcal{B}_\infty(X) = \langle \mathcal{B}_\sigma(X) \rangle_{\text{CBOOL}} \cong \langle \mathcal{B}_\sigma(\mathbb{S}^\mathbb{N}) \rangle_{\text{CBOOL}} = \langle \langle \mathbb{N} \rangle_{\sigma\text{BorLoc}_\sigma} \rangle_{\text{CBOOL}} = \langle \mathbb{N} \rangle_{\text{CBOOL}}$, since $X \cong \mathbb{S}^\mathbb{N} \in \sigma\text{BorLoc}_\sigma$ and by Corollary 3.5.5. (Several instances of this were already pointed out by Isbell [Isb72], including $X = \mathbb{S}^\mathbb{N}, 2^\mathbb{N}, \mathbb{R}$.)

Example 3.5.14. Consider the interval $X = [0, 1]$ with the Scott topology, with open sets $(r, 1]$; this is a quasi-Polish space (see [deB13, §8]), whose specialization preorder is the usual linear order on $[0, 1]$. Since it is uncountable, we thus have $\mathcal{B}_\infty(X) \cong \langle \mathbb{N} \rangle_{\text{CBOOL}}$.

However, $\mathcal{B}_\infty^+(X)$ is much simpler: it is the same as $\mathcal{B}_\sigma^+(X)$, the closure of the open sets $\mathcal{O}(X)$ under countable unions and intersections, i.e., consisting of all open and closed intervals $(r, 1]$ and $[r, 1]$. Indeed, note that every lower set $D \subseteq \mathcal{B}_\sigma^+(X)$ is countably generated: it is either a principal ideal, or else contains a strictly increasing cofinal sequence $[r_0, 1] \subsetneq [r_1, 1] \subsetneq \dots$ for some $r_0 > r_1 > \dots$. Thus, every (σ, σ) -frame homomorphism $\mathcal{B}_\sigma^+(X) \rightarrow A$ already preserves arbitrary joins, and dually, already preserves arbitrary meets, which implies $\mathcal{B}_\infty^+(X) = \langle \mathcal{B}_\sigma^+(X) \rangle_{\infty\infty\text{FRM}} = \mathcal{B}_\sigma^+(X)$.

Example 3.5.15. On the other hand, if X is Polish (or just fit quasi-Polish; see Section 2.9), then $\mathcal{B}_\sigma^+(X) = \mathcal{B}_\sigma(X)$, since every closed set is a countable intersection of opens. Hence in this case, $\mathcal{B}_\infty^+(X) = \langle \mathcal{B}_\sigma^+(X) \rangle_{\infty\infty\text{FRM}} = \langle \mathcal{B}_\sigma(X) \rangle_{\infty\infty\text{FRM}} = \langle \mathcal{B}_\sigma(X) \rangle_{\text{CBOOL}} = \mathcal{B}_\infty(X) (\cong \langle \mathbb{N} \rangle_{\text{CBOOL}}, \text{ if } X \text{ is uncountable}).$

3.6 The internal logic

In categorical logic, there is a method for formally interpreting “pointwise” expressions, such as

$$(3.6.1) \quad \ker(f) := \{(x, y) \in X^2 \mid f(x) = f(y)\},$$

in arbitrary categories. This can be used to give intuitive-looking definitions of the localic analogs of various spatial notions such as \ker . In this subsection, we briefly review this method, concentrating on the categories κBorLoc ; see [Joh02, D1.2] for a more comprehensive reference.

Let \mathbf{C} be a category with κ -ary limits; we have in mind $\mathbf{C} = \kappa\text{BorLoc}_{(\kappa)}$. The **internal language** of \mathbf{C} is the κ -ary multi-sorted infinitary first-order language $\mathcal{L}(\mathbf{C})$ with

- a sort for each object $X \in \mathbf{C}$;
- a κ -ary function symbol for each morphism $f : \prod_i X_i \rightarrow Y \in \mathbf{C}$, where $\vec{X} = (X_i)_i$ is a κ -ary family of objects in \mathbf{C} ;
- a κ -ary relation symbol for each subobject $R \subseteq \prod_i X_i$ in \mathbf{C} .

Now let $\phi(\vec{x})$ be a formula in the κ -**ary infinitary first-order logic** $\mathcal{L}_{\kappa\kappa}(\mathbf{C})$ over $\mathcal{L}(\mathbf{C})$, with $< \kappa$ -many free variables \vec{x} with sorts \vec{X} (for a κ -ary family of objects \vec{X} in \mathbf{C}), using κ -ary conjunctions \bigwedge and disjunctions \bigvee and \exists over κ -ary families of variables; we treat \forall as an abbreviation for $\neg\exists\neg$. For certain such formulas ϕ , we define its **interpretation in \mathbf{C}** , which is a certain subobject

$$\{\vec{x} \in \prod \vec{X} \mid \phi(\vec{x})\}_{\mathbf{C}} \subseteq \prod \vec{X} \in \mathbf{C};$$

the formulas which have an interpretation are called **interpretable**.

- First, each term $t(\vec{x})$ of sort Y with variables \vec{x} of sort \vec{X} has an interpretation as a morphism $t_{\mathbf{C}} : \prod \vec{X} \rightarrow Y$, defined by induction on t in the obvious way.
- Atomic formulas are interpretable: for variables $\vec{x} = (x_i)_{i \in I}$ (where I is a κ -ary set) of sorts $\vec{X} = (X_i)_{i \in I}$, for terms $\vec{t}(\vec{x}) = (t_j(\vec{x}))_{j \in J}$ (where J is κ -ary) of sorts $(Y_j)_{j \in J}$, and for a relation $R \subseteq \prod_{j \in J} Y_j$, the atomic formula $R(\vec{t})$ is interpreted as the pullback

$$\begin{array}{ccc} \{\vec{x} \in \prod_i X_i \mid R(\vec{t})\} & \longrightarrow & R \\ \downarrow & & \downarrow \\ \prod_i X_i & \xrightarrow{((t_j)_{\mathbf{C}})_j} & \prod_j Y_j \end{array}$$

(the binary equality relation $=$ is treated as the diagonal $X \hookrightarrow X^2$ for each $X \in \mathbf{C}$).

- κ -ary conjunction $\bigwedge_i \phi_i$ is interpreted as the meet (i.e., wide pullback) of the interpretations of the ϕ_i .
- κ -ary disjunction $\bigvee_i \phi_i$ is interpreted as the join, in the subobject lattice, of the interpretations of the ϕ_i , *provided the join exists and is pullback-stable*.

In particular, the nullary disjunction \perp is interpreted as a pullback-stable least subobject, which is the same thing (see [Joh02, A1.4.1]) as a **strict initial object** $\emptyset \in \mathbf{C}$, i.e., an initial object such that every morphism to it is an isomorphism.

- If $\phi(\vec{x})$ has an interpretation $\{\vec{x} \mid \phi(\vec{x})\} \subseteq \prod \vec{X}$, and this interpretation has a *pullback-stable complement*, meaning another subobject $\neg\{\vec{x} \mid \phi(\vec{x})\} \subseteq \prod \vec{X}$ whose meet with $\{\vec{x} \mid \phi(\vec{x})\}$ is the pullback-stable least subobject \emptyset and whose join with $\{\vec{x} \mid \phi(\vec{x})\}$ is all of $\prod \vec{X}$ and such that this join is pullback-stable, then this complement $\neg\{\vec{x} \mid \phi(\vec{x})\}$ is defined to be $\{\vec{x} \mid \neg\phi(\vec{x})\}$.

(Note that pullback-stability of a join implies that meets distribute over it; thus pullback-stable complements that exist are unique.)

- Finally, for \exists : if $\phi(\vec{x}, \vec{y})$ has an interpretation $\{(\vec{x}, \vec{y}) \mid \phi(\vec{x}, \vec{y})\} \subseteq \prod \vec{X} \times \prod \vec{Y}$, and the projection $\pi_{\vec{X}} : \prod \vec{X} \times \prod \vec{Y} \rightarrow \prod \vec{X}$ is such that there is a least subobject $\pi_{\vec{X}}(\{(\vec{x}, \vec{y}) \mid \phi(\vec{x}, \vec{y})\}) \subseteq \prod \vec{X}$ whose pullback is $\geq \{(\vec{x}, \vec{y}) \mid \phi(\vec{x}, \vec{y})\}$, and furthermore it remains least when pulled back along any $Z \rightarrow \prod \vec{X}$, then we call $\pi_{\vec{X}}(\{(\vec{x}, \vec{y}) \mid \phi(\vec{x}, \vec{y})\})$ the **pullback-stable image** of $\{(\vec{x}, \vec{y}) \mid \phi(\vec{x}, \vec{y})\}$ under $\pi_{\vec{x}}$, and define it to be $\{\vec{x} \mid \exists \vec{y} \phi(\vec{x}, \vec{y})\}$.

Alternatively, say that an arbitrary morphism $f : A \rightarrow X$ has **pullback-stable image** $\text{im}(f) \subseteq X$ when $\text{im}(f)$ is least such that $f^*(\text{im}(f)) = A$, and remains so when pulled back along any $Z \rightarrow X$, i.e., $A \twoheadrightarrow \text{im}(f) \hookrightarrow X$ is a pullback-stable factorization of f into an **extremal epimorphism** (a morphism not factoring through any proper subobject of its codomain) followed by a monomorphism. Then the image of a subobject, as defined above, is the same thing as the image of the composite

$$(3.6.2) \quad \{(\vec{x}, \vec{y}) \mid \phi(\vec{x}, \vec{y})\} \hookrightarrow \prod \vec{X} \times \prod \vec{Y} \xrightarrow{\pi_{\vec{X}}} \prod \vec{X}.$$

If ϕ is an interpretable formula with no free variables, then its interpretation $\{() \mid \phi\}$ is a subobject of the terminal object $1 \in \mathbf{C}$; if it is all of 1, then we say that \mathbf{C} **satisfies** ϕ , denoted

$$\mathbf{C} \models \phi :\iff \{() \mid \phi\} = 1.$$

For interpretable $\phi(\vec{x})$ with free variables of sorts \vec{X} , if $\forall \vec{x} \phi(\vec{x})$ is also interpretable, then it is easily seen that $\{\vec{x} \mid \phi(\vec{x})\} \subseteq \prod \vec{X}$ is all of $\prod \vec{X}$ iff $\mathbf{C} \models \forall \vec{x} \phi(\vec{x})$; thus more generally, regardless of whether $\forall \vec{x} \phi(\vec{x})$ is interpretable, we put

$$\mathbf{C} \models \forall \vec{x} \phi(\vec{x}) :\iff \{\vec{x} \mid \phi(\vec{x})\} = \prod \vec{X}.$$

More generally still, for interpretable $\phi(\vec{x}), \psi(\vec{x})$ both with free variables of sorts \vec{X} , we put

$$\mathbf{C} \models \forall \vec{x} (\phi(\vec{x}) \rightarrow \psi(\vec{x})) :\iff \{\vec{x} \mid \phi(\vec{x})\} \subseteq \{\vec{x} \mid \psi(\vec{x})\} \subseteq \prod \vec{X};$$

again if $\phi(\vec{x}) \rightarrow \psi(\vec{x})$ is in fact interpretable, this is consistent with the above.

Example 3.6.3. It is straightforward to see that for any morphism $f : X \rightarrow Y \in \mathbf{C}$, (3.6.1) above does define the same subobject of X^2 as the usual kernel, i.e., the pullback $X \times_Y X$.

Formulas such as (3.6.1), whose interpretation only uses $(\kappa\text{-ary})$ limits in \mathbf{C} , are called **$(\kappa\text{-})$ limit formulas** (cf. Appendix A.1); they are always interpretable (provided \mathbf{C} has $\kappa\text{-ary}$ limits), and moreover one may reason about them exactly as in \mathbf{Set} (since the Yoneda embedding $\mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ preserves limits). Atomic formulas are clearly limit formulas, as are conjunctions thereof. Certain existentials are also limit formulas: namely, if the composite (3.6.2) above is already a monomorphism (which is expressible via pullbacks), then it is its own pullback-stable image, hence is the interpretation of $\exists \vec{y} \phi(\vec{x}, \vec{y})$ using only limits (and composition). Such existentials are called **provably unique**.

In order for *all* $\mathcal{L}_{\kappa\kappa}(\mathbf{C})$ -formulas to be interpretable, \mathbf{C} needs to have (i) all pullback-stable images, or equivalently pullback-stable extremal epi-mono factorizations; and (ii) all pullback-stable joins and complements of subobjects. Categories with (i) are called **regular** (cf. Appendix A.2); in such categories, all extremal epimorphisms are in fact regular epimorphisms (see [Joh02, A1.3.4]), hence one may equivalently speak of regular epi-mono factorizations. By Propositions 3.4.17 and 3.4.25(b),

Proposition 3.6.4. *For $\kappa < \infty$, κBorLoc is a regular category.* □

Remark 3.6.5. $\kappa\text{BorLoc}_\kappa$ is *not* regular, not being closed under images in κBorLoc ; see Theorem 4.4.3 below. In particular, ∞BorLoc ($= \infty\text{BorLoc}_\infty$) is *not* regular. Furthermore, for $\kappa \leq \lambda$, images in κBorLoc need not be preserved in λBorLoc , i.e., the forgetful functor from the former category to the latter is not regular (see Remark 3.4.15, Corollary 4.4.5).

A category with (finite limits and) all pullback-stable $\kappa\text{-ary}$ joins of *pairwise disjoint* subobjects is called **$\kappa\text{-subextensive}$** in [Ch19b]. A **Boolean $\kappa\text{-subextensive}$** category is one which furthermore has complements (which can be defined using meets and pairwise disjoint joins; cf. Section 2.9); this implies arbitrary $\kappa\text{-ary}$ joins, by disjointifying. There does not seem to be a standard name for a category with pullback-stable $\kappa\text{-ary}$ joins in general; however, if such a category is also regular, it is called **$\kappa\text{-coherent}$** (see [Joh02, A1.4]). In a $\kappa\text{-subextensive}$ category \mathbf{C} , $\kappa\text{-ary}$ pairwise disjoint joins of subobjects are also coproducts; thus if every $\kappa\text{-ary}$ family of objects X_i embeds as pairwise disjoint subobjects of a common object Y , then \mathbf{C} has $\kappa\text{-ary}$ coproducts, which can be defined purely in terms of (meets and) $\kappa\text{-ary}$ pairwise disjoint joins of subobjects, hence are pullback-stable. Such a category \mathbf{C} is also called **$\kappa\text{-extensive}$** , and can be equivalently defined as a category with (finite limits and) pullback-stable $\kappa\text{-ary}$ coproducts in which coproduct injections are pairwise disjoint and the initial object is strict. (For these last facts, see [CLW93], [Joh02, A1.4.3], [Ch19b, §2].) A $\kappa\text{-extensive}$ regular category is $\kappa\text{-coherent}$, i.e., again has arbitrary $\kappa\text{-ary}$ joins, by taking an image of a coproduct.

Proposition 3.6.6 (see [Ch19b, 3.8–9]).

- (a) κBorLoc and $\kappa\text{BorLoc}_\kappa$ are $\kappa\text{-extensive}$.
- (b) For $\kappa < \infty$, κBorLoc is $\kappa\text{-coherent}$.
- (c) $\kappa\text{BorLoc}_\kappa$ is *Boolean*.

Proof. (a) and (c) are by Proposition 3.1.11 and the fact that subobjects in these categories are the same as $\infty(\mathcal{B}_\kappa)_\delta$, respectively $\kappa\text{-Borel}$, sets (by Propositions 3.4.9 and 3.4.25), with pullback along $f : X \rightarrow Y$ corresponding to $f^* : \mathcal{B}_\infty(Y) \rightarrow \mathcal{B}_\infty(X)$ (Proposition 3.4.14) and $\kappa\text{-ary}$ coproduct corresponding to pairwise disjoint union of subobjects (Proposition 3.4.12).

(b) follows from Proposition 3.6.4 and $\kappa\text{-extensivity}$. □

It follows that large fragments of $\mathcal{L}_{\kappa\kappa}$ may be interpreted in the categories $\kappa\mathbf{BorLoc}_{(\kappa)}$. Namely, starting in $\kappa\mathbf{BorLoc}_{\kappa}$, we may interpret any formula, as long as we ensure that whenever an \exists appears, its interpretation still lands in $\kappa\mathbf{BorLoc}_{\kappa}$; while arbitrary \exists may be interpreted by passing to $\kappa\mathbf{BorLoc}$, as long as we no longer take complements outside of such arbitrary \exists . We will continue and extend this discussion of interpreting \exists when we introduce $\kappa\Sigma_1^1$ sets in Section 4 below.

Remark 3.6.7. In [Ch19b] we showed that $\kappa\mathbf{BorLoc}_{\kappa}$ is the *initial* Boolean κ -extensive category with κ -ary limits (in the 2-category of all such categories).

Remark 3.6.8. Proposition 3.4.22 also gives a compatibility condition between *arbitrary* limits and images in $\kappa\mathbf{BorLoc}$. In particular, it implies Makkai’s [Mak90, 2.2] *principle of dependent choices* (for all lengths λ), whence by the main result of that paper, we may also interpret ∞ -regular formulas (built using arbitrary \bigwedge and \exists) in $\kappa\mathbf{BorLoc}$, with the same implications between such formulas holding as in **Set**. (This is stronger than Carboni–Vitale’s [CV98, §4.5] notion of **completely regular** category, i.e., a regular category in which regular epimorphisms are closed under arbitrary products, which $\kappa\mathbf{BorLoc}$ is, by Corollary 3.4.23; see also Appendix A.2.)

Remark 3.6.9. For a κ -subextensive category, κ -extensivity imposes the existence of additional structure (disjoint coproducts) which is uniquely defined in terms of the κ -subextensive structure, if it exists. There is a similar counterpart to regularity, called **(Barr-)exactness** (cf. Appendix A.2): every equivalence relation $\sim \subseteq X^2$ (i.e., $\mathbf{C} \models “\sim \text{ is an equivalence relation on } X”$) has a coequalizer whose kernel is \sim . In contrast to Proposition 3.6.4, the category $\kappa\mathbf{BorLoc}$ is *not* exact. Its “exact completion” is related to the theory of *Borel equivalence relations* in descriptive set theory; see e.g., [Gao09]. We leave a detailed study of the exact completion for future work.

3.7 Positive Borel locales and partial orders

Recall (see Example 3.1.7) that our main motivation for positive κ -Borel locales is to provide a pointless approach to the specialization order on a locale. We now make this precise.

By an **(internal) partial order** on an object X in a category \mathbf{C} with finite limits, we mean a subobject $\leq \subseteq X^2$ which internally (see Section 3.6) obeys the axioms of reflexivity, transitivity, and antisymmetry (which are implications between finite limit formulas, thus interpretable in \mathbf{C}). Thus, an internal partial order on a κ -Borel locale $X \in \kappa\mathbf{BorLoc}$ is an $\infty(\mathcal{B}_{\kappa})_{\delta}$ binary relation $\leq \subseteq X^2$ obeying these axioms, which explicitly means

$$\begin{aligned} (X \xrightarrow{\delta} X^2)^*(\leq) &= X, \\ (X^3 \xrightarrow{\pi_{12}} X^2)^*(\leq) \cap (X^3 \xrightarrow{\pi_{23}} X^2)^*(\leq) &\subseteq (X^3 \xrightarrow{\pi_{13}} X^2)^*(\leq) \subseteq X^3, \\ (\leq) \cap (X^2 \xrightarrow{\sigma} X^2)^*(\leq) &= (=_X) \subseteq X^2 \end{aligned}$$

where δ is the diagonal, $\pi_{ij} : X^3 \rightarrow X^2$ is the projection onto the i th and j th coordinates, σ is the “swap” map, and $=_X$ is the image of δ ; while an internal partial order on a standard κ -Borel locale $X \in \kappa\mathbf{BorLoc}_{\kappa}$ is a κ -Borel such \leq . By a **κ -Borel polocale**, we mean a κ -Borel locale X equipped with a $\infty(\mathcal{B}_{\kappa})_{\delta}$ partial order $\leq = \leq_X$, while by a **standard κ -Borel polocale**, we mean a standard κ -Borel locale equipped with a κ -Borel partial order. Let $\kappa\mathbf{BorPOLoc}$, $\kappa\mathbf{BorPOLoc}_{\kappa}$ be

the categories of these and κ -Borel order-preserving maps. We have forgetful functors

$$(3.7.1) \quad \begin{array}{ccc} \kappa\text{BorPOLoc}_\kappa & \xrightarrow{\quad} & \kappa\text{BorPOLoc} \\ \downarrow & & \downarrow \\ \kappa\text{BorLoc}_\kappa & \xrightarrow{\quad} & \kappa\text{BorLoc} \end{array}$$

Moreover, for $\kappa \leq \lambda$, since the forgetful functor $\kappa\text{BorLoc} \rightarrow \lambda\text{BorLoc}$ preserves finite limits, it preserves internal partial orders; thus we have forgetful functors $\kappa\text{BorLoc}_{(\kappa)} \rightarrow \lambda\text{BorLoc}_{(\lambda)}$, compatible with (3.7.1) in the obvious sense.

Recall that for an object X in a locally ordered category, such as $\kappa\text{Bor}^+\text{Loc}$, the **order** $\leq_X \subseteq X^2$ **on** X is the weighted limit defined as the universal object equipped with projections $\pi_1, \pi_2 : \leq_X \rightrightarrows X$ such that $\pi_1 \leq \pi_2$, which is to say the order-kernel $\leq_X = \text{oker}(1_X)$ of the identity, or equivalently the power $\leq_X = X^{\mathbb{S}}$ of X by the poset $\mathbb{S} = \{0 < 1\}$ (see paragraph before Proposition 3.4.25, Appendix A.3). In $\kappa\text{Bor}^+\text{Loc}$, we also call \leq_X the **specialization order** on X . By a straightforward calculation (or by the enriched Yoneda lemma [Kel82]), the order on X is indeed an internal partial order on X in the above sense. Moreover, every morphism $f : X \rightarrow Y$ is order-preserving. Thus, by assigning to every (standard) positive κ -Borel locale X its specialization order \leq_X in $\kappa\text{Bor}^+\text{Loc}_{(\kappa)}$, and then applying the forgetful functor to $\kappa\text{BorLoc}_{(\kappa)}$, we get a (faithful) forgetful functor

$$\kappa\text{Bor}^+\text{Loc}_{(\kappa)} \longrightarrow \kappa\text{BorPOLoc}_{(\kappa)}$$

which factors the forgetful functor $\kappa\text{Bor}^+\text{Loc}_{(\kappa)} \rightarrow \kappa\text{BorLoc}_{(\kappa)}$ through (3.7.1).

Recall from (3.1.9) that the forgetful functor $\kappa\text{Bor}^+\text{Loc} \rightarrow \kappa\text{BorLoc}$ has a left adjoint “free functor” $D : \kappa\text{BorLoc} \rightarrow \kappa\text{Bor}^+\text{Loc}$, dual to the forgetful functor $\kappa\text{Bool} \rightarrow \kappa\kappa\text{Frm}$ (or $\text{CBOOL}_\infty \rightarrow \infty\infty\text{FRM}_\infty$ when $\kappa = \infty$), which exhibits κBorLoc as equivalent to a coreflective subcategory of $\kappa\text{Bor}^+\text{Loc}$. For $X \in \kappa\text{BorLoc}$, we can think of $D(X)$ as the “discrete” positive κ -Borel locale on X . For $X \in \kappa\text{Bor}^+\text{Loc}$, the counit $\varepsilon_X : D(X) \rightarrow X$ is dual to the inclusion $\varepsilon_X^* : \mathcal{B}_\kappa^+(X) \hookrightarrow \mathcal{B}_\kappa^+(D(X)) = \mathcal{B}_\kappa(X)$; hence ε_X is an epimorphism.

Lemma 3.7.2. *For $X \in \kappa\text{Bor}^+\text{Loc}$, the specialization order $\leq_X \subseteq X^2$, regarded as a subobject in κBorLoc , is also the order-kernel in $\kappa\text{Bor}^+\text{Loc}$ of the counit $\varepsilon_X : D(X) \rightarrow X$ regarded as in κBorLoc .*

Proof. For any $Y \in \kappa\text{BorLoc}$, a κ -Borel map $Y \rightarrow \leq_X \in \kappa\text{BorLoc}$ is equivalently a positive κ -Borel map $D(Y) \rightarrow \leq_X \in \kappa\text{Bor}^+\text{Loc}$, which by definition of \leq_X is equivalently a pair $f, g : D(Y) \rightrightarrows X \in \kappa\text{Bor}^+\text{Loc}$ such that $f \leq g$, which (by coreflectivity) is equivalently a pair $f', g' : D(Y) \rightrightarrows D(X) \in \kappa\text{Bor}^+\text{Loc}$ such that $f = \varepsilon_X \circ f' \leq \varepsilon_X \circ g' = g$, which is equivalently a map $D(Y) \rightarrow \text{oker}(\varepsilon_X) \in \kappa\text{Bor}^+\text{Loc}$, which is equivalently a map $Y \rightarrow \text{oker}(\varepsilon_X) \in \kappa\text{BorLoc}$. \square

Given any κ -Borel polocale X , we say that a κ -Borel set $B \subseteq X$ is **upper** if it is internally so (with respect to the order \leq_X) in κBorLoc , i.e.,

$$(\leq_X \xrightarrow{\pi_1} X)^*(B) \subseteq (\leq_X \xrightarrow{\pi_2} X)^*(B) \subseteq \leq_X.$$

(Note that since $\kappa\text{BorLoc} \rightarrow \infty\text{BorLoc}$ is conservative (Proposition 3.1.10), i.e., injective on subobjects (Lemma A.2.4), a κ -Borel set B is upper in κBorLoc iff it is so as an ∞ -Borel set in ∞BorLoc .)

We now have the main result connecting positive κ -Borel locales with κ -Borel polocales:

Theorem 3.7.3. *For a positive κ -Borel locale X , a κ -Borel set $B \in \mathcal{B}_\kappa(X)$ is upper (with respect to the specialization order) iff it is a positive κ -Borel set.*

Proof. Letting $D(X)$ as above be the “discrete” positive κ -Borel locale on the underlying κ -Borel locale of X , the counit $\varepsilon_X : D(X) \rightarrow X$ is an epimorphism, hence an order-regular epimorphism by Proposition 3.4.25. This means ε_X is the coinserter of its order-kernel, which by Lemma 3.7.2 is the underlying κ -Borel locale of the specialization order \leq_X in $\kappa\text{Bor}^+\text{Loc}$. Dually, this means

$$\begin{aligned}\mathcal{B}_\kappa^+(X) &= \text{ins}(\pi_1^*, \pi_2^* : \mathcal{B}_\kappa^+(D(X)) \rightrightarrows \mathcal{B}_\kappa(\leq_X)) \\ &= \text{ins}(\pi_1^*, \pi_2^* : \mathcal{B}_\kappa(X) \rightrightarrows \mathcal{B}_\kappa(\leq_X)) \\ &= \{B \in \mathcal{B}_\kappa(X) \mid \pi_1^*(B) \leq \pi_2^*(B)\},\end{aligned}$$

which consists precisely of the upper sets by definition. \square

Since the preimage of an upper set under an order-preserving map is clearly upper,

Corollary 3.7.4. *The forgetful functor $\kappa\text{Bor}^+\text{Loc} \rightarrow \kappa\text{BorPOLoc}$ is full.* \square

Remark 3.7.5. Given any κ -Borel polocale X , the upper κ -Borel sets form a (κ, κ) -subframe of $\mathcal{B}_\kappa(X)$, hence when $\kappa < \infty$ define a positive κ -Borel locale X' equipped with a positive κ -Borel epimorphism $\eta : X \rightarrow X'$ (with η^* given by the inclusion). This X' is the reflection of X into $\kappa\text{Bor}^+\text{Loc}$: indeed, for any order-preserving κ -Borel map $f : X \rightarrow Y$ into a positive κ -Borel locale Y , $f^* : \mathcal{B}_\kappa(Y) \rightarrow \mathcal{B}_\kappa(X)$ must preserve upper sets, i.e., take positive κ -Borel maps in Y to positive κ -Borel maps in X' , which yields a factorization of f through X' . So $\kappa\text{Bor}^+\text{Loc} \subseteq \kappa\text{BorPOLoc}$ is an epireflective subcategory, for $\kappa < \infty$.

Example 3.7.6. Consider the interval $[0, 1]$ with the Scott topology (see Example 3.5.14), a quasi-Polish space, i.e., standard σ -locale. Let $X = [0, 1] \times \mathbb{S}$, regarded as a standard Borel space (standard σ -Borel locale) equipped with the *lexicographical* order \leq (where $\mathbb{S} = \{0 < 1\}$). This is a linear order, i.e., it satisfies the axiom $\forall x, y ((x \leq y) \vee (y \leq x))$ internally in \mathbf{SBor} , hence also in ∞BorLoc since the forgetful functor preserves unions of subobjects (being given by join in $\mathcal{B}_\sigma \subseteq \mathcal{B}_\infty$).

We claim that $A := [0, 1] \times \{0\} \subseteq X$ is not in the complete Boolean subalgebra of $\mathcal{B}_\infty(X)$ generated by the upper ∞ -Borel sets. To see this, first recall from Example 3.5.14 that $\mathcal{B}_\infty^+([0, 1])$ consists only of the intervals $(r, 1]$ and $[r, 1]$. Next, any upper ∞ -Borel $B \subseteq X$ is of the form

$$(*) \quad B = (C \times \{0\}) \cup (D \times \{1\}),$$

where $C, D \in \mathcal{B}_\infty([0, 1])$ are the preimages of B under the two “cross-section” inclusions $[0, 1] \hookrightarrow X$ which are order-preserving, whence C, D are upper, hence intervals of the aforementioned form; by definition of the lexicographical order, it is easily seen that the only possibilities are (i) $C = D$ or (ii) $C = (r, 1]$ and $D = [r, 1]$ for some r . Thus, the complete Boolean subalgebra of $\mathcal{B}_\infty(X)$ generated by the upper sets is contained in the complete Boolean subalgebra consisting of all B as in $(*)$ such that $C \triangle D \in \mathcal{B}_\infty([0, 1])$ is a union of points (and conversely, it is easy to see that every such B can be generated from $\mathcal{B}_\infty^+(X)$). Since $[0, 1] \in \mathcal{B}_\infty([0, 1])$ is not a union of points (by localic Baire category or Gaifman–Hales; cf. Remark 3.4.18), it follows that A is not in this complete Boolean subalgebra.

Thus, X is a standard σ -Borel polocale which is not a positive κ -Borel locale, for any $\kappa \leq \infty$; that is, the full inclusion $\kappa\text{Bor}^+\text{Loc} \rightarrow \kappa\text{BorPOLoc}$ is not essentially surjective, for any $\kappa \geq \omega_1$.

Indeed, for $\kappa < \infty$, the positive κ -Borel locale reflection $\eta : X \rightarrow X'$ of X , as described in the preceding remark, is such that $\eta^* : \mathcal{B}_\kappa(X') \rightarrow \mathcal{B}_\kappa(X)$ is not surjective (since its image does not contain A), i.e., η is not a monomorphism.

Moreover, when $\kappa = \omega_1$, the same argument as above shows that the σ -Boolean subalgebra of $\mathcal{B}_\sigma(X)$ generated by the upper sets consists of all B as in (*) such that $C \triangle D$ is a *countable* union of points. This algebra is not countably generated, since countably many such B can only generate other such B for which $C \triangle D$ is contained in a fixed countable set (the union of $C \triangle D$ for the generating B). Thus, *there is a standard σ -Borel polocale whose positive σ -Borel locale reflection is not standard*; that is, the free functor $\sigma\text{BorPOLoc} \rightarrow \sigma\text{Bor}^+\text{Loc}_\sigma$ does not preserve standard objects. It follows easily from this that *the forgetful functor $\sigma\text{Bor}^+\text{Loc}_\sigma \rightarrow \sigma\text{BorPOLoc}_\sigma$ does not have a left adjoint F* : if it did, letting $X \rightarrow X'$ be the reflection of X in $\sigma\text{Bor}^+\text{Loc}$ (so that $\mathcal{B}_\sigma^+(X') \subseteq \mathcal{B}_\sigma(X)$ is the non-countably generated σ -Boolean subalgebra generated by the upper sets as above), X' would be a countably cofiltered limit of standard positive σ -Borel locales X'_i , and each $X \rightarrow X' \rightarrow X'_i$ would have to factor uniquely through $X \rightarrow F(X)$; these factorizations would yield a retraction $F(X) \rightarrow X'$ of the comparison map $X' \rightarrow F(X)$, whence X' must already be standard.

A similar argument shows that the forgetful functor $\kappa\text{Bor}^+\text{Loc}_\kappa \rightarrow \kappa\text{BorPOLoc}_\kappa$ does not have a left adjoint, for any κ which is *not inaccessible*: replace $[0, 1]$ above with a κ -standard linear order with $\geq \kappa$ -many points, e.g., \mathbb{S}^λ lexicographically ordered for some $\lambda < \kappa$ with $2^\lambda \geq \kappa$. However, we do not know what happens for κ inaccessible; in particular, we do not know if the forgetful functor $\infty\text{Bor}^+\text{Loc} \rightarrow \infty\text{BorPOLoc}$ has a left adjoint.

For more information on standard Borel posets in the classical context, especially the special role played by lexicographical orders such as X above, see [HMS88], [Kan98].

3.8 Baire category

The classical **Baire category theorem** states that in sufficiently nice topological spaces X , such as locally compact sober spaces (see [G⁺03, I-3.40.9]), completely metrizable spaces (see [Kec95, 8.4]), or quasi-Polish spaces (see [deB13, Cor. 52], [Ch18b, 7.3]), the intersection of countably many dense open sets is dense. It follows that the dense open sets generate a σ -filter $\text{COMGR}(X)$, consisting of the **comeager** sets, all of which are dense, hence nonempty whenever the space is. This σ -filter is equivalently generated by the dense Π_2^0 -sets (in the sense of Selivanov; see Section 3.3), due to

Lemma 3.8.1. *In a topological space or locale X , if a union $U \cup F$ of an open set U and closed set F is dense, then so is the open subset $U \cup F^\circ$.*

Proof. We have $X = \overline{U \cup F} = \overline{U} \cup \overline{F} = \overline{U} \cup F$, i.e., $\neg \overline{U} \subseteq F$, whence $\neg \overline{U} \subseteq F^\circ$ since $\neg \overline{U}$ is open, whence $X = \overline{U} \cup F^\circ \subseteq \overline{U} \cup \overline{F^\circ}$. \square

A major result of locale theory, due to Isbell [Isb72, 1.5], yields the localic analog of the Baire category theorem:

Theorem 3.8.2 (Isbell). *In every locale X , the intersection of all dense $\infty\Pi_2^0$ sets (i.e., sublocales, by Proposition 3.4.9) is dense, i.e., there is a smallest dense $\infty\Pi_2^0$ set.*

Remark 3.8.3. Unlike most localic analogs of classical results that we consider, the proof of the above localic Baire category theorem has nothing to do with the classical proof. This is because the classical proof instead underlies the entire correspondence between locales and spaces in the countably presented setting (see Section 3.5).

For a locale X , we denote the smallest dense $\infty\Pi_2^0$ set by $\text{Comgr}(X) \subseteq X$, and call any other set $B \subseteq X$ **comeager** if $\text{Comgr}(X) \subseteq B$, and **meager** if $\text{Comgr}(X) \cap B = \emptyset$; thus $\text{Comgr}(X)$ is the smallest comeager set, an improvement over the σ -filter $\text{COMGR}(X)$ in the spatial context. Explicitly, $\text{Comgr}(X)$ is given by the equivalent formulas

$$\begin{aligned} \text{Comgr}(X) &= \neg \bigcup_{U \in \mathcal{O}(X)} \partial U \quad \text{where } \partial U := \overline{U} \setminus U \\ &= \bigcap_{U \in \mathcal{O}(X)} (\overline{U} \rightarrow U) \\ &= \bigcap_{U \in \mathcal{O}(X)} (\overline{U}^\circ \rightarrow U). \end{aligned}$$

The last formula means that $\mathcal{O}(X) \twoheadrightarrow \mathcal{O}(\text{Comgr}(X))$ is the quotient of $\mathcal{O}(X)$ identifying each open set U with its **regularization** \overline{U}° . Hence, $\mathcal{O}(\text{Comgr}(X))$ is isomorphic to the complete Boolean algebra of regular open sets $\{U \in \mathcal{O}(X) \mid U = \overline{U}^\circ\}$, sometimes called the **(Baire) category algebra** of X (see e.g., [Kec95, 8.32]).

Example 3.8.4. For $X = \mathbb{R}, 2^{\mathbb{N}}$, or more generally any nonempty perfect (i.e., without isolated points) Polish space, $\mathcal{O}(\text{Comgr}(X))$ is the usual Cohen algebra (the unique nontrivial atomless complete Boolean algebra countably generated under joins; see [Kec95, 8.32]). $\text{Comgr}(X) \subseteq X$ is a dense, pointless sublocale; but its complement contains much more than just points, e.g., $\neg \text{Comgr}(\mathbb{R})$ contains the usual Cantor set in \mathbb{R} (since it is nowhere dense), which is a perfect closed set, hence contains its own dense, pointless smallest comeager set which is disjoint from $\text{Comgr}(\mathbb{R})$.

Example 3.8.5. For $X = [0, 1]$ with the *Scott* topology (see Example 3.5.14), every nonempty open set is dense; thus $\text{Comgr}(X) = \{1\}$.

Since $\mathcal{O}(\text{Comgr}(X))$ is Boolean, $\mathcal{B}_\infty(\text{Comgr}(X)) = \mathcal{O}(\text{Comgr}(X))$, i.e., every ∞ -Borel set $B \subseteq \text{Comgr}(X)$ is open, i.e., of the form $U \cap \text{Comgr}(X)$ for some open $U \subseteq X$. Thus, we have the following analog of the **property of Baire** (see [Kec95, 8.F]) for ∞ -Borel sets:

Proposition 3.8.6. *Every ∞ -Borel set in a locale differs from an open set by a meager set.* \square

We also have the following strengthening of the localic Baire category Theorem 3.8.2, which follows from it by taking the X_i to consist of all dense open sets in a fixed locale X . For the classical analog (as well as a different “geometric” proof using Theorem 3.8.2), see [Ch18a, A.1].

Theorem 3.8.7. *Let $(X_i)_{i \in I}$ be a codirected diagram of locales, for a directed poset I , with morphisms $f_{ij} : X_i \rightarrow X_j$ for $i \geq j$. If each f_{ij} has dense image (i.e., its closed image $\overline{\text{im}}(f_{ij}) \in \infty\Pi_1^0(X_j)$ is all of X_j), then so does each limit projection $\pi_i : \varprojlim_j X_j \rightarrow X_i$.*

Proof. Each f_{ij} having dense image means $\perp \in \mathcal{O}(X_j)$ is the only element of its $\ker(f_{ij}^*)$ -congruence class; by Proposition 2.7.3, the same holds for each π_i . \square

We conclude by giving, for the sake of completeness, a self-contained proof of a well-known application of localic Baire category: a localic “cardinal collapse”, which says that any two infinite sets have a nonempty locale “parametrizing bijections between them”. See e.g., [Joh02, C1.2.8–9].

Theorem 3.8.8 (folklore). *Let X, Y be κ -ary sets, with Y infinite, regarded as κ -discrete locales.*

- (a) *There is a nonempty standard κ -locale Z together with a κ -locale embedding $f : Z \times X \hookrightarrow Z \times Y$ over Z (i.e., commuting with the projections to Z).*

(b) If $X \neq \emptyset$, then f in (a) has a κ -Borel retraction $g : Z \times Y \rightarrow Z \times X$ over Z with $g \circ f = 1_{Z \times X}$; in particular, there is an ∞ -Borel surjective κ -Borel map $g : Z \times Y \rightarrow Z \times X$ over Z .

(c) If X is infinite, then f in (a) can be taken to be a κ -locale isomorphism.

Proof. Consider the standard κ -locale $\mathbb{S}^{X \times Y}$. For $x \in X$ and $y \in Y$, let $[x \mapsto y] \in \mathcal{O}(\mathbb{S}^{X \times Y})$ denote the subbasic open set at the (x, y) th coordinate. Let

$$P := \bigwedge_{x \in X; y \neq y' \in Y} \neg([x \mapsto y] \wedge [x \mapsto y']) \wedge \bigwedge_{x \neq x' \in X; y \in Y} \neg([x \mapsto y] \wedge [x' \mapsto y]) \in \kappa\Pi_1^0(\mathbb{S}^{X \times Y})$$

be the closed set “of graphs of partial injections $X \hookrightarrow Y$ ”. Clearly P has a point, namely the empty function; hence $P \neq \emptyset$. Let

$$D_x := \bigvee_{y \in Y} [x \mapsto y] \in \mathcal{O}(\mathbb{S}^{X \times Y}), \quad E_y := \bigvee_{x \in X} [x \mapsto y] \in \mathcal{O}(\mathbb{S}^{X \times Y}).$$

Then each $P \wedge D_x \subseteq P$ is dense, since for a basic open set $U = [x_1 \mapsto y_1] \wedge \cdots \wedge [x_n \mapsto y_n] \subseteq \mathbb{S}^{X \times Y}$ which intersects P , we must have $x_i = x_j \iff y_i = y_j$ for all $1 \leq i, j \leq n$ by definition of P ; taking $y = y_i$ if $x = x_i$ for some i and $y \in Y \setminus \{y_1, \dots, y_n\}$ otherwise, the partial injection mapping $x_i \mapsto y_i$ and $x \mapsto y$ is a point of $P \wedge D_x \wedge U$. Similarly, if X is infinite, then each $P \wedge E_y \subseteq P$ is dense.

To prove (a), take $Z := P \wedge \bigwedge_{x \in X} D_x \subseteq P$, a dense and hence nonempty standard κ -sublocale. Note that for each $x \in X$, the join defining D_x , when intersected with P , is pairwise disjoint, i.e., Z is the disjoint union of open sublocales

$$Z = \bigsqcup_{y \in Y} (Z \wedge [x \mapsto y]).$$

Thus $Z \times X$ is the disjoint union of open sublocales

$$Z \times X = \bigsqcup_{x \in X} (Z \times \{x\}) = \bigsqcup_{x \in X} \bigsqcup_{y \in Y} ((Z \wedge [x \mapsto y]) \times \{x\}).$$

Define $f : Z \times X \rightarrow Z \times Y$ by taking its restriction to each $U_{x,y} := (Z \wedge [x \mapsto y]) \times \{x\}$ to be the inclusion $U_{x,y} \cong V_{x,y} := (Z \cap [x \mapsto y]) \times \{y\} \subseteq Z \times Y$. Then f restricted to each $U_{x,y}$ is an embedding; while each $U_{x,y}$ is itself equal to $f^*(V_{x,y})$: \subseteq is by definition of f , while for $(x', y') \neq (x, y)$, $f(U_{x',y'}) = V_{x',y'}$ is disjoint from $V_{x,y}$, since either $y \neq y'$ in which case $\{y\} \cap \{y'\} = \emptyset$, or else $y = y'$ and $x \neq x'$ in which case $Z \cap [x \mapsto y]$, $Z \cap [x' \mapsto y]$ are disjoint by definition of P ; thus $\neg U_{x,y} = \bigsqcup_{(x',y') \neq (x,y)} U_{x',y'} \subseteq f^*(\neg V_{x,y})$. Thus f is an embedding, and clearly commutes with the projections to Z .

To prove (c), take $Z := P \wedge \bigwedge_{x \in X} D_x \wedge \bigwedge_{y \in Y} E_y$, define $f : Z \times X \rightarrow Z \times Y$ as before, and define $g : Z \times Y \rightarrow Z \times X$ in a symmetric manner, so that $f = g^{-1}$.

To prove (b), fix some $x \in X$, define g on the ∞ -Borel image $f(Z \times X) \subseteq Z \times Y$ (which is κ -Borel, since $Z \times X$ is standard κ -Borel) to be the inverse of the κ -Borel isomorphism $f : Z \times X \cong f(Z \times X)$, and define g on $\neg f(Z \times X)$ to be the projection $\neg f(Z \times X) \subseteq Z \times Y \rightarrow Z$ followed by the inclusion $Z \cong Z \times \{x\} \subseteq Z \times X$. \square

4 Analytic sets and locales

In this section, we study κ -analytic (i.e., $\kappa\Sigma_1^1$) sets in standard κ -Borel locales. These are defined by “completing” the category $\kappa\mathbf{BorLoc}_\kappa$ of standard κ -Borel locales under non-existing κ -Borel images. There are two equivalent ways of performing this “completion”: externally, by closing under images

in the regular supercategory $\kappa\mathbf{BorLoc}$ of possibly nonstandard κ -Borel locales, or internally, by freely “adjoining” an image for each morphism in $\kappa\mathbf{BorLoc}_\kappa$. In Section 4.1, we give both definitions and derive some basic properties of the resulting category $\kappa\Sigma_1^1\mathbf{BorLoc}$ of “analytic κ -Borel locales”.

In the remaining subsections, we prove the analogs of some key classical results about analytic sets in standard Borel spaces. In Section 4.2, we state the Lusin–Novikov separation theorem for κ -analytic sets, as well as an ordered analog; these are simply translations of the algebraic interpolation theorems from Section 2.12. In Section 4.3, we discuss the connection between κ -analytic sets and well-foundedness; in particular, we prove that “every κ -analytic set can be reduced to ill-foundedness of (generalized) trees” (Theorem 4.3.2), that “every κ -analytic set is a κ -length decreasing intersection of κ -Borel sets”, and that “every κ -analytic well-founded (generalized) forest has bounded rank $< \kappa$ ” (Theorem 4.3.6). Finally, in Section 4.4 we apply this boundedness theorem to prove that indeed, there exist κ -analytic, non- κ -Borel sets, thus justifying the careful treatment of κ -Borel images we have given up to this point.

4.1 The main categories

Classically, an **analytic set** or Σ_1^1 **set** in a standard Borel space X is a set which is the image of a Borel map $f : Y \rightarrow X$ from some standard Borel space Y . An **analytic Borel space** is an analytic set regarded as an abstract Borel space, equipped with the subspace Borel structure; these form a well-behaved category $\Sigma_1^1\mathbf{Bor}$, containing the category \mathbf{SBor} of standard Borel spaces as a full subcategory. See [Kec95, III].

In the localic setting, for $\kappa < \infty$, recall from Proposition 3.6.4 that the category $\kappa\mathbf{BorLoc}$ of all κ -Borel locales is regular, hence has a well-behaved notion of image (i.e., $\infty(\mathcal{B}_\kappa)_\delta$ -image). We define an **analytic κ -Borel locale** to mean a κ -Borel locale X which is the image of a κ -Borel map $p : \tilde{X} \rightarrow X' \in \kappa\mathbf{BorLoc}_\kappa$ between standard κ -Borel locales. Equivalently, X admits both a (regular) epimorphism (κ -Borel surjection) $p : \tilde{X} \twoheadrightarrow X$ from a standard κ -Borel locale, as well as a monomorphism (κ -Borel embedding) $X \hookrightarrow X'$ into a standard κ -Borel locale. Dually, this means $\mathcal{B}_\kappa(X)$ is a κ -generated κ -Boolean algebra which embeds into a κ -presented algebra. We let $\kappa\Sigma_1^1\mathbf{BorLoc} \subseteq \kappa\mathbf{BorLoc}$ denote the full subcategory of analytic κ -Borel locales.

The following result summarizes the key properties of the category $\kappa\Sigma_1^1\mathbf{BorLoc}$:

Theorem 4.1.1.

- (a) $\kappa\Sigma_1^1\mathbf{BorLoc} \subseteq \kappa\mathbf{BorLoc}$ is closed under κ -ary limits, κ -ary coproducts, and images of morphisms.
- (b) $\kappa\Sigma_1^1\mathbf{BorLoc}$ is a κ -extensive, κ -coherent category.
- (c) Let $(X_i)_{i \in I}$ be a κ -ary codirected diagram in $\kappa\Sigma_1^1\mathbf{BorLoc}$, for a directed poset I , with morphisms $f_{ij} : X_i \rightarrow X_j$ for $i \geq j$. If each f_{ij} is a (regular) epimorphism, then so is each limit projection $\pi_i : \varprojlim_j X_j \rightarrow X_i$.
In particular, κ -ary products of epimorphisms in $\kappa\Sigma_1^1\mathbf{BorLoc}$ are epimorphisms, i.e., $\kappa\Sigma_1^1\mathbf{BorLoc}$ is a κ -**completely regular** category.
- (d) The subobject lattices in $\kappa\Sigma_1^1\mathbf{BorLoc}$ are (κ, κ) -frames.
- (e) All epimorphisms in $\kappa\Sigma_1^1\mathbf{BorLoc}$ are regular.

Proof. (a) Closure under κ -ary products follows from the fact that product in $\kappa\mathbf{BorLoc}$ preserves regular epimorphisms (i.e., complete regularity; see Remark 3.6.8), whence the κ -ary product of $X_i = \text{im}(p_i : \tilde{X}_i \rightarrow X'_i) \in \kappa\Sigma_1^1\mathbf{BorLoc}$, where $p_i \in \kappa\mathbf{BorLoc}_\kappa$, is the image of $\prod_i p_i : \prod_i \tilde{X}_i \rightarrow \prod_i X'_i$. Likewise, closure under κ -ary coproducts follows from the fact that κ -ary coproduct in $\kappa\mathbf{BorLoc}$ preserves monomorphisms (which follows from κ -extensivity, or more concretely from the dual fact that product in $\kappa\mathbf{Bool}$ preserves surjective homomorphisms). Closure under images is obvious. Closure under pullback also follows easily from regularity of $\kappa\mathbf{BorLoc}$: the pullback $X \times_Z Y$ of $f : X \rightarrow Z \in \kappa\Sigma_1^1\mathbf{BorLoc}$ and $g : Y \rightarrow Z \in \kappa\Sigma_1^1\mathbf{BorLoc}$ admits a regular epimorphism from a standard κ -Borel locale, namely $\tilde{X} \times_{Z'} \tilde{Y}$ for any $\tilde{X} \twoheadrightarrow X$, $\tilde{Y} \twoheadrightarrow Y$, and $Z' \supseteq Z$ with $\tilde{X}, \tilde{Y}, Z' \in \kappa\mathbf{BorLoc}_\kappa$, and also embeds into $X \times Y$ which embeds into a standard κ -Borel locale since X, Y do. This is enough, since κ -products and (binary) pullbacks generate κ -ary limits (see e.g., [Bor, 2.8.1–2]).

(b) and (c) now follow from the corresponding properties of $\kappa\mathbf{BorLoc}$ (Propositions 3.4.22 and 3.6.6). We record the following consequence of (c):

Lemma 4.1.2. *For $< \kappa$ -many morphisms $f_i : X_i \rightarrow Y \in \kappa\Sigma_1^1\mathbf{BorLoc}$, the meet of their images is given by the image of their wide pullback*

$$\bigwedge_i \text{im}(f_i) = \text{im}(\prod_Y (f_i)_i : \prod_Y (X_i)_i \rightarrow Y).$$

Proof. For two morphisms, hence finitely many, this follows from pullback-stability of image (cf. Corollary 3.4.21). In general, the \geq inequality is trivial; so it suffices to prove \leq . By considering the pullback of all the f_i to $\bigwedge_i \text{im}(f_i) \subseteq Y$, it is enough to assume that each f_i is an epimorphism, and prove that $\prod_Y (X_i)_i \rightarrow Y$ is still an epimorphism. For finite sets of indices $K \subseteq K'$, the projection $\prod_Y (X_i)_{i \in K'} \rightarrow \prod_Y (X_i)_{i \in K}$ is an epi by pullback-stability. Thus the claim follows from (c). \square

(d) Let $X \in \kappa\Sigma_1^1\mathbf{BorLoc}$. That finite meets in $\text{Sub}(X)$ distribute over κ -ary joins follows from pullback-stability of κ -ary joins by (b). Let $Y, Z_i \subseteq X$ be $< \kappa$ -many subobjects; we must show $\bigwedge_i (Y \vee Z_i) \leq Y \vee \bigwedge_i Z_i$. Again by replacing X with $\bigwedge_i (Y \vee Z_i)$, we may assume each $Y \vee Z_i = X$, and prove $Y \vee \bigwedge_i Z_i = X$. Now X is an image of some $\tilde{X} \in \kappa\mathbf{BorLoc}_\kappa$; by pulling everything back to \tilde{X} , we may assume X was already standard κ -Borel. And $Y, Z_i \subseteq X$ are the images of some $f : \tilde{Y} \rightarrow X$ and $g_i : \tilde{Z}_i \rightarrow X$ with \tilde{Y}, \tilde{Z}_i standard κ -Borel. By Lemma 4.1.2, $\bigwedge_i Z_i = \text{im}(\prod_X (g_i)_i : \prod_X (\tilde{Z}_i)_i \rightarrow X)$; while by the general construction of joins of subobjects in extensive categories (see before Proposition 3.6.6), each $Y \vee Z_i \subseteq X$ is the image of the map $[f, g_i] : \tilde{Y} \sqcup \tilde{Z}_i \rightarrow X$ induced by f, g_i , with a similar description of $Y \vee \bigwedge_i Z_i \subseteq X$. All told, we are given κ -Borel maps $f : \tilde{Y} \rightarrow X$ and $g_i : \tilde{Z}_i \rightarrow X$ between standard κ -Borel locales, such that each $[f, g_i] : \tilde{Y} \sqcup \tilde{Z}_i \rightarrow X$ is κ -Borel surjective; and we must show that $[f, \prod_X (g_i)_i] : \tilde{Y} \sqcup \prod_X (\tilde{Z}_i)_i \rightarrow X$ is κ -Borel surjective.

Let $B \in \mathcal{B}_\kappa(X)$ such that $[f, \prod_X (g_i)_i]^*(B) = \top$, i.e., $f^*(B) = \top$ and also $(\prod_X (g_i)_i)^*(B) = \top$. The latter preimage map $(\prod_X (g_i)_i)^* : \mathcal{B}_\kappa(X) \rightarrow \mathcal{B}_\kappa(\prod_X (\tilde{Z}_i)_i)$ is the wide pushout of the $g_i^* : \mathcal{B}_\kappa(X) \rightarrow \mathcal{B}_\kappa(\tilde{Z}_i) \in \kappa\mathbf{Bool}$, so by Corollary 2.12.5 (applied to $b_i := \top$ and $a := \neg(\prod_X (g_i)_i)^*(B)$), there are $C_i \in \mathcal{B}_\kappa(X)$ such that $B \wedge \bigwedge_i C_i = \emptyset$ and $g_i^*(C_i) = \tilde{Z}_i$ for each i ; in particular, $g_i^*(B) \leq g_i^*(C_i)$ for each i . But since each $[f, g_i]$ is κ -Borel surjective, and we also have $f^*(B) = \emptyset \leq f^*(C_i)$ for each i , we get $B \leq C_i$ for each i , whence $B = B \wedge \bigwedge_i C_i = \emptyset$. Thus $[f, \prod_X (g_i)_i]$ is κ -Borel surjective, as desired.

(e) Given an epimorphism $f : X \rightarrow Y \in \kappa\Sigma_1^1\mathbf{BorLoc}$, f must also be an epimorphism in $\kappa\mathbf{BorLoc}$, since the discrete κ -Borel locale 2 is in $\kappa\Sigma_1^1\mathbf{BorLoc}$, whence by considering the composite of f with

maps $Y \rightarrow 2$ we get that $f^* : \mathcal{B}_\kappa(Y) \rightarrow \mathcal{B}_\kappa(X)$ is injective. Thus f must be a regular epimorphism in κBorLoc (by Proposition 3.4.25), i.e., the coequalizer of its kernel which is in $\kappa\Sigma_1^1\text{BorLoc}$ by (a). \square

Remark 4.1.3. We record the following useful fact about limits in $\kappa\Sigma_1^1\text{BorLoc}$, which was implicitly used in the proof of Theorem 4.1.1(a) for pullbacks. Given a κ -ary diagram $F : \mathbb{I} \rightarrow \kappa\Sigma_1^1\text{BorLoc}$, for some κ -ary category \mathbb{I} , if every $I \in \mathbb{I}$ admits a morphism from some $\tilde{I} \in \mathbb{I}$ such that $F(\tilde{I})$ is a standard κ -Borel locale (i.e., the set of such \tilde{I} is cofinal in \mathbb{I}^{op} in the weaker order-theoretic sense, not necessarily the categorical sense of a final functor), then $\varprojlim F$ is also standard κ -Borel.

To see this, first note that F has the same limit as its restriction F' to the subcategory $\mathbb{I}' \subseteq \mathbb{I}$ consisting of all nonidentity morphisms $f : I \rightarrow J$ such that $F(I)$ is standard κ -Borel (this subcategory is initial in the categorical sense). Now F' is a diagram consisting only of (nonidentity) morphisms whose domain is standard κ -Borel; each such morphism can be composed with an embedding from its codomain to a standard κ -Borel locale, again without changing the limit.

We call a subobject A of an analytic κ -Borel locale $X \in \kappa\Sigma_1^1\text{BorLoc}$ a κ -**analytic set** or $\kappa\Sigma_1^1$ **set** of X , and denote the (κ, κ) -frame of all such by

$$\kappa\Sigma_1^1(X) := \text{Sub}_{\kappa\Sigma_1^1\text{BorLoc}}(X).$$

When X is a standard κ -Borel locale, $\mathcal{B}_\kappa(X) \subseteq \kappa\Sigma_1^1(X)$ is a κ -Boolean subalgebra. We call images in $\kappa\Sigma_1^1\text{BorLoc}$ $\kappa\Sigma_1^1$ -**images**, denoted $\text{im}^\kappa(f)$ for the image of a κ -Borel map f or $f^\kappa(A)$ for the image of a κ -analytic set A under a map f . Note that this is the same as the $\infty(\mathcal{B}_\kappa)_\delta$ -image from Section 3.4, for a κ -Borel map that happens to be between analytic κ -Borel locales. In particular, it agrees with existing κ -Borel images.

Remark 4.1.4. When $\kappa = \omega_1$, the spatialization functor $\text{Sp} : \sigma\text{BorLoc} \rightarrow \sigma\text{Bor}$ restricts to an equivalence of categories

$$\sigma\Sigma_1^1\text{BorLoc} \simeq \Sigma_1^1\text{Bor}$$

between analytic σ -Borel locales and analytic Borel spaces. Upon considering subobjects, it follows that the localic notion of σ -analytic set corresponds to the classical notion.

To see this, recall (Theorem 3.5.8) that Sp restricts to an equivalence $\sigma\text{BorLoc}_\sigma \simeq \text{SBor}$ between standard σ -Borel locales and standard Borel spaces. It follows that analytic σ -Borel locales are spatial, since if $\tilde{X} \twoheadrightarrow X$ is an epimorphism from a standard σ -Borel locale, then $\mathcal{B}_\sigma(X) \subseteq \mathcal{B}_\sigma(\tilde{X})$ is a σ -Boolean subalgebra, hence admits enough homomorphisms to 2 since $\mathcal{B}_\sigma(\tilde{X})$ does; thus Sp restricts to a full and faithful functor on $\sigma\Sigma_1^1\text{BorLoc}$. Now given a Borel map $f : X \rightarrow Y$ between standard σ -Borel locales, we just need to check that the regular epi-mono factorization $X \twoheadrightarrow \text{im}(f) \hookrightarrow Y$ of f in σBorLoc is preserved by Sp . Since Sp is right adjoint, it preserves the mono. To check that it preserves the epi $X \twoheadrightarrow \text{im}(f)$, we need to check that the induced Borel map $\text{Sp}(f) : X \rightarrow \text{Sp}(\text{im}(f))$ is surjective. Indeed, for a point y in the standard Borel space Y , if $y \notin \text{im}(\text{Sp}(f))$, then the singleton $\{y\} \in \mathcal{B}_\sigma(Y)$ witnesses that the σ -Boolean homomorphism $y^* : \mathcal{B}_\sigma(Y) \rightarrow 2$ does not descend to the image $\mathcal{B}_\sigma(\text{im}(f))$ of $f^* : \mathcal{B}_\sigma(Y) \rightarrow \mathcal{B}_\sigma(X)$, i.e., that $y \notin \text{Sp}(\text{im}(f))$, as desired.

We will show below (Theorem 4.4.3) that for every κ , there are κ -analytic sets in standard κ -Borel locales which are not κ -Borel. In particular, there are analytic κ -Borel locales which are not standard κ -Borel, i.e., the inclusion $\kappa\text{BorLoc}_\kappa \subseteq \kappa\Sigma_1^1\text{BorLoc}$ is proper. This generalizes the classical fact when $\kappa = \omega_1$ (see [Kec95, 14.2]).

Remark 4.1.5. In contrast to Proposition 3.4.25(b), *not every monomorphism in $\kappa\Sigma_1^1\mathbf{BorLoc}$ is regular*. This follows from the existence of κ -analytic, non- κ -Borel sets in standard κ -Borel locales X , since every equalizer of some $X \rightrightarrows Y \in \kappa\Sigma_1^1\mathbf{BorLoc}$ must be a κ -Borel set (being the preimage of the diagonal in some standard κ -Borel Y' containing Y).

Remark 4.1.6. The only reason why the above definition of $\kappa\Sigma_1^1\mathbf{BorLoc}$ requires $\kappa < \infty$ is that we defined (in Section 3.1) ∞ -Borel locales X to always be standard, i.e., $\mathcal{B}_\infty(X)$ must be a *small-presented* large complete Boolean algebra. As a result, the category $\infty\mathbf{BorLoc}$ is not regular, so we could not take $\infty\Sigma_1^1\mathbf{BorLoc}$ to be the closure under existing images. By relaxing this restriction, we could define an **analytic ∞ -Borel locale** X to mean the dual of a *small-generated* large complete Boolean algebra $\mathcal{B}_\infty(X)$ which embeds into a small-presented algebra $\mathcal{B}_\infty(\tilde{X})$ (note that small-generatability is still enough to ensure only a small set of ∞ -Borel maps $Y \rightarrow X$ for any Y). However, some foundational care would be needed to work with $\infty\Sigma_1^1\mathbf{BorLoc}$ in this way.

Using generalities about regular categories, we now give an equivalent “synthetic” description of $\kappa\Sigma_1^1\mathbf{BorLoc}$, as a “completion” of the subcategory $\kappa\mathbf{BorLoc}_\kappa$, which does not require the ambient category $\kappa\mathbf{BorLoc}$ in which to take images. In particular, this description works equally well when $\kappa = \infty$, thereby circumventing some of the aforementioned foundational issues.

- An object $X \in \kappa\Sigma_1^1\mathbf{BorLoc}$ is the $\kappa\Sigma_1^1$ -image of some $p : \tilde{X} \rightarrow X' \in \kappa\mathbf{BorLoc}_\kappa$, thus may be represented by such p .

In other words, we can *define* an **analytic κ -Borel locale** to mean a κ -Borel map $p : \tilde{X} \rightarrow X'$ between standard κ -Borel locales, thought of formally as representing its “ $\kappa\Sigma_1^1$ -image”.

- Given analytic κ -Borel locales $X, Y \in \kappa\Sigma_1^1\mathbf{BorLoc}$, represented as the “formal $\kappa\Sigma_1^1$ -images” of $p : \tilde{X} \rightarrow X' \in \kappa\mathbf{BorLoc}_\kappa$ and $q : \tilde{Y} \rightarrow Y' \in \kappa\mathbf{BorLoc}_\kappa$ respectively, a morphism $f : X \rightarrow Y$ can be represented as the composite $f' : \tilde{X} \xrightarrow{p} X' \xrightarrow{f} Y \subseteq Y'$, which uniquely determines f since p is an epimorphism and $Y \hookrightarrow Y'$ is a monomorphism.

$$(4.1.7) \quad \begin{array}{ccc} \tilde{X} \times_{Y'} \tilde{Y} & & \\ q' \downarrow & \searrow f'' & \\ \tilde{X} & & \tilde{Y} \\ p \downarrow & \searrow f' & \downarrow q \\ X' \supseteq X & \xrightarrow{f} & Y \subseteq Y' \end{array}$$

Moreover, any $f' : \tilde{X} \rightarrow Y'$ descends to such $f : X \rightarrow Y$ iff

- $\ker(p) \subseteq \ker(f')$ (which ensures f' descends along p to $X \rightarrow Y'$, by the universal property of the coequalizer $\ker(p) \rightrightarrows \tilde{X} \twoheadrightarrow X$), and
- the pullback of q along f' is a regular epi, i.e., κ -Borel surjection, $q' : \tilde{X} \times_{Y'} \tilde{Y} \twoheadrightarrow \tilde{X}$ (which ensures that f' lands in $Y \subseteq Y'$, since q' lands in the pullback $f'^*(Y) \subseteq \tilde{X}$ of $Y \subseteq Y'$, which must hence be all of \tilde{X} since q' is an extremal epi).

Thus, we can *define* a κ -Borel map $f : X \rightarrow Y$ to mean an $f' : \tilde{X} \rightarrow Y'$ obeying (i) and (ii).

- The identity at $X = \mathrm{im}^\kappa(p : \tilde{X} \rightarrow X')$ is represented by p .

- To compose $f : X \rightarrow Y$ represented by $f' : \tilde{X} \rightarrow Y'$ as above with $g : Y \rightarrow Z$ represented by $g' : \tilde{Y} \rightarrow Z'$, form the pullback $f'' : \tilde{X} \times_{Y'} \tilde{Y} \rightarrow \tilde{Y}$ of f' as in (4.1.7), then compose it with g' ; since $\ker(q') \subseteq \ker(f' \circ q') = \ker(q \circ f'') = f''^*(\ker(q)) \subseteq f''^*(\ker(g')) = \ker(g' \circ f'')$, the composite $g' \circ f'' : \tilde{X} \times_{Y'} \tilde{Y} \rightarrow Z'$ descends to $\tilde{X} \rightarrow Z'$ which represents $g \circ f$.

We may also describe subobjects in $\kappa\Sigma_1^1\text{BorLoc}$, i.e., κ -analytic sets, purely in terms of $\kappa\text{BorLoc}_\kappa$:

- κ -analytic sets $A \subseteq X$ in a standard κ -Borel locale $X \in \kappa\text{BorLoc}_\kappa$ are represented by κ -Borel maps $f : \tilde{A} \rightarrow X$ from some other $\tilde{A} \in \kappa\text{BorLoc}_\kappa$, thought of as their “formal $\kappa\Sigma_1^1$ -images”, preordered by $A \subseteq B = \text{im}^\kappa(g : \tilde{B} \rightarrow X)$ iff the pullback $\tilde{A} \times_X \tilde{B} \rightarrow \tilde{A}$ of g along f is κ -Borel surjective (by pullback-stability of regular epis).

κ -analytic sets in an analytic κ -Borel locale X are the κ -analytic sets in a standard κ -Borel $X' \supseteq X$ which are contained in X .

- κ -ary meets are given by $\bigwedge_i \text{im}^\kappa(f_i) = \text{im}^\kappa(\prod_X (f_i)_i)$ (by Lemma 4.1.2).
- κ -ary joins are given by image of disjoint union (by the general construction of joins of subobjects in extensive regular categories; see before Proposition 3.6.6).
- Preimage is given by pullback (by pullback-stability of image).
- κ -Borel sets in an analytic κ -Borel locale $X = \text{im}^\kappa(p : \tilde{X} \rightarrow X')$ are the restrictions of κ -Borel sets $B' \subseteq X'$, with two such sets having the same restriction iff they have the same p -preimage.

Alternatively, a κ -Borel set in X may be uniquely represented by its p -preimage, which is a κ -Borel set $\tilde{B} \subseteq \tilde{X}$ which is invariant with respect to the internal equivalence relation $\ker(p) \subseteq \tilde{B}^2$ (see Corollary 4.2.4 below).

Remark 4.1.8. The above construction of $\kappa\Sigma_1^1\text{BorLoc}$ from $\kappa\text{BorLoc}_\kappa$ is an instance of the general technique of *regular completion of unary sites*; see [Shu12], [CV98]. A **unary site** is a category \mathbf{C} with finite limits¹⁶ equipped with a subcategory $\mathbf{E} \subseteq \mathbf{C}$ containing all objects in \mathbf{C} and whose morphisms are closed under pullback and right-cancellation with arbitrary morphisms in \mathbf{C} ; one thinks of \mathbf{E} as morphisms which are “declared to be regular epimorphisms”. The **regular completion** of a unary site (\mathbf{C}, \mathbf{E}) is the free regular category \mathbf{C}' generated by \mathbf{C} qua category with finite limits, subject to every morphism in \mathbf{E} becoming a regular epi in \mathbf{C}' . If every morphism in \mathbf{E} is already a regular epi in \mathbf{C} , then the canonical unit functor $\eta : \mathbf{C} \rightarrow \mathbf{C}'$ is full and faithful, and \mathbf{C}' can be constructed explicitly as the category of “formal images” of morphisms in \mathbf{C} , exactly as above (see [Shu12, 8.22, 10.19]). Moreover, \mathbf{C}' inherits various other existing limit and colimit properties from \mathbf{C} , such as κ -ary limits and κ -extensivity (see [Shu12, 11.13], [CV98, §4.5]).

By applying this to $\mathbf{C} = \kappa\text{BorLoc}_\kappa$ and $\mathbf{E} = \kappa$ -Borel surjections (i.e., all regular epimorphisms in \mathbf{C}), we obtain $\kappa\Sigma_1^1\text{BorLoc}$ as the free regular category obeying Theorem 4.1.1(b–e) generated by $\kappa\text{BorLoc}_\kappa$ while remembering κ -ary limits, κ -ary coproducts, and regular epimorphisms. In particular, Theorem 4.1.1(b–e) can all be proved without referring to κBorLoc , hence remaining in the small-presented context when $\kappa = \infty$.

Remark 4.1.9. We can also describe $\kappa\Sigma_1^1\text{BorLoc}$ as the free regular category obeying Theorem 4.1.1(b–e) generated by κLoc_κ while remembering κ -ary limits, κ -ary coproducts, and that

¹⁶In [Shu12], a more general theory is developed where \mathbf{C} needs only have “weak finite limits with respect to \mathbf{E} ”.

κ -Borel surjections in $\kappa\mathbf{Loc}_\kappa$ (equivalently by Proposition 3.4.19, pullback-stable epimorphisms) become regular epimorphisms in $\kappa\Sigma_1^1\mathbf{BorLoc}$. This follows from combining the universal property of the regular completion as in Remark 4.1.8 with the description of $\kappa\mathbf{BorLoc}_\kappa$ as the localization of $\kappa\mathbf{Loc}$ inverting the κ -continuous bijections (Remark 3.3.15).

Remark 4.1.10. For $\kappa \leq \lambda < \infty$, the forgetful functor $\kappa\mathbf{BorLoc} \rightarrow \lambda\mathbf{BorLoc}$ need not restrict to $\kappa\Sigma_1^1\mathbf{BorLoc} \rightarrow \lambda\Sigma_1^1\mathbf{BorLoc}$. This follows from the existence of σ -Borel maps with no ∞ -Borel image (see Corollary 4.4.5 below, and also Remark 3.4.15).

We can interpret this as saying that the embedding $\kappa\Sigma_1^1\mathbf{BorLoc} \subseteq \kappa\mathbf{BorLoc}$, despite offering a convenient first definition of $\kappa\Sigma_1^1\mathbf{BorLoc}$, is in fact “unnatural”. In other words, a κ -analytic set $A \subseteq X$ should not really be identified with an $\infty(\mathcal{B}_\kappa)_\delta$ -set, i.e., subobject in $\kappa\mathbf{BorLoc}$. Indeed, Theorem 4.4.3 below says that there is a “ σ -analytic, non- ∞ -Borel” set; this only makes sense if we do not regard $\sigma\Sigma_1^1$ as a subset of $\infty(\mathcal{B}_\sigma)_\delta$ (hence of \mathcal{B}_∞). We will therefore mostly avoid using the embedding $\kappa\Sigma_1^1\mathbf{BorLoc} \subseteq \kappa\mathbf{BorLoc}$ henceforth.

Remark 4.1.11. Despite the preceding remark, we do not know if for $\kappa \leq \lambda$, the forgetful functor $\kappa\mathbf{BorLoc}_\kappa \rightarrow \lambda\mathbf{BorLoc}_\lambda$ between the categories of standard Borel locales preserves regular epimorphisms. If this were so, we would nonetheless have a natural forgetful functor

$$\kappa\Sigma_1^1\mathbf{BorLoc} \longrightarrow \lambda\Sigma_1^1\mathbf{BorLoc},$$

namely the extension of the forgetful functor $\kappa\mathbf{BorLoc}_\kappa \rightarrow \lambda\mathbf{BorLoc}_\lambda$ to the regular completions of the respective unary sites as in Remark 4.1.8.

In particular, for a standard Borel space X , we do not know if every classical analytic set $A \subseteq X$ yields an ∞ -analytic set. We can take the ∞ -analytic image of any Borel $f : \tilde{A} \rightarrow X$ with classical image A ; but we do not know if the result depends on f (even if $A = X$).

4.2 Separation theorems

The classical **Novikov separation theorem** [Kec95, 28.5] states that countably many analytic sets $A_i \subseteq X$ in a standard Borel space (or analytic Borel space [Kec95, 28.6]) with empty intersection $\emptyset = \bigcap_i A_i$ must be contained in Borel sets $A_i \subseteq B_i \subseteq X$ with empty intersection $\emptyset = \bigcap_i B_i$ (the **Lusin separation theorem** [Kec95, 14.7] is the special case of two analytic sets). The localic analog is given by the interpolation theorems from Section 2.12:

Theorem 4.2.1. Let X be an analytic κ -Borel locale, $A_i \subseteq X$ be $< \kappa$ -many κ -analytic sets. If $\bigcap_i A_i = \emptyset$, then there are κ -Borel $A_i \subseteq B_i \subseteq X$ with $\bigcap_i B_i = \emptyset$.

In particular, if $A_1, A_2 \subseteq X$ are disjoint κ -analytic sets, then there is a κ -Borel set $B \subseteq X$ with $A_1 \subseteq B$ and $A_2 \cap B = \emptyset$.

Proof. Let $X \subseteq X'$ with X' standard κ -Borel, and let $f_i : \tilde{A}_i \rightarrow A_i$ be κ -Borel surjections from standard κ -Borel locales \tilde{A}_i . Then (by Lemma 4.1.2) $\bigcap_i \tilde{A}_i$ is the κ -analytic image of the wide pullback $\prod_{X'} (f_i)_i$, which must hence be empty. By Corollary 2.12.5 applied to $a := \perp \in \mathcal{B}_\kappa(X')$ and $b_i := \top \in \mathcal{B}_\kappa(\tilde{A}_i)$, we get that there are $B_i \in \mathcal{B}_\kappa(X')$ with $\bigcap_i B_i \subseteq \emptyset$ and $\top \leq f_i^*(B_i)$, i.e., $A_i = f_i^\kappa(\tilde{A}_i) \subseteq B_i$. \square

Remark 4.2.2. In particular, this gives, modulo the Baire category-based spatiality results from Section 3.5, a completely lattice-theoretic proof of the classical Lusin–Novikov separation theorems.

(However, the classical proof from [Kec95, 28.5] and especially [Kec95, 28.1, proof II] is somewhat analogous to the proof of Theorem 2.12.3, with the tree $\mathbb{N}^{<\mathbb{N}}$, which forms a posite presenting $\mathcal{O}(\mathbb{N}^{\mathbb{N}})$, playing a role analogous to that of the distributive polyposets in Theorem 2.12.3.)

Corollary 4.2.3 (Suslin’s theorem for locales). *Let X be an analytic κ -Borel locale. Then $\mathcal{B}_\kappa(X) = \kappa\Sigma_1^1(X)_\neg$, i.e., the κ -Borel sets in X are precisely the complemented κ -analytic sets.* \square

Corollary 4.2.4. *Let $f : X \rightarrow Y \in \kappa\Sigma_1^1\mathbf{BorLoc}$ be a κ -Borel map between analytic κ -Borel locales. Then a κ -Borel set $B \subseteq X$ is $f^*(C)$ for some κ -Borel $C \subseteq Y$ iff B is invariant with respect to the internal equivalence relation $\ker(f) \subseteq X^2$.*

The classical analog of this is a theorem of Blackwell (see [Kec95, 14.16]).

Proof. This follows from regarding $\kappa\Sigma_1^1\mathbf{BorLoc} \subseteq \kappa\mathbf{BorLoc}$, then replacing Y with $\mathrm{im}^\kappa(f)$, so that $f : X \twoheadrightarrow Y$ is a regular epi, i.e., $\mathcal{B}_\kappa(Y)$ is the equalizer of $\mathcal{B}_\kappa(X) \rightrightarrows \mathcal{B}_\kappa(\ker(f))$, which is easily seen to consist of precisely the $\ker(f)$ -invariant sets $B \in \mathcal{B}_\kappa(X)$.

We also give a different proof, avoiding mention of $\kappa\mathbf{BorLoc}$. (\implies) is a general fact about categories with finite limits (e.g., reason as in **Set** using the Yoneda lemma). For (\impliedby): in any regular category, $\ker(f)$ -invariance of B implies $B = f^*(f^\kappa(B))$; this easily yields that $f^\kappa(B) \cap f^\kappa(\neg B) = \emptyset$ (by Frobenius reciprocity), whence there is a κ -Borel $C \subseteq Y$ separating them, whence $B = f^*(C)$. \square

Remark 4.2.5. For many other classical consequences of the Lusin separation theorem, we have already seen their localic analog, which “secretly” used the localic Lusin separation theorem in the form of the various consequences of the interpolation theorem from Section 2.12. This includes, e.g., the Lusin–Suslin theorem that injective Borel maps have Borel image (see Remark 3.4.26).

We also have the following dual version of the general interpolation Theorem 2.12.3, which generalizes Theorem 4.2.1 to the positive setting. Recall from Section 3.7 that a standard positive κ -Borel locale X has a specialization order $\leq_X \subseteq X^2$, which is a κ -Borel partial order on X .

Theorem 4.2.6. *Let X be a standard positive κ -Borel locale, and let $A_i, B_j \subseteq X$ be $< \kappa$ -many κ -analytic sets such that “ $\bigcap_i \uparrow A_i \cap \bigcap_j \downarrow B_j = \emptyset$ ”, i.e.,*

$$\kappa\Sigma_1^1\mathbf{BorLoc} \models \nexists x \in X, a_i \in A_i, b_j \in B_j (\bigwedge_i (a_i \leq x) \wedge \bigwedge_j (x \leq b_j)).$$

Then there are positive κ -Borel $A_i \subseteq C_i \subseteq X$ and negative (= complement of positive) κ -Borel $B_j \subseteq D_j \subseteq X$ such that $\bigcap_i C_i \cap \bigcap_j D_j = \emptyset$.

Proof. Note first that the above formula in the internal logic should be interpreted as “ $\exists \dots \implies \perp$ ” (which makes sense in the coherent category $\kappa\Sigma_1^1\mathbf{BorLoc}$), or equivalently

$$\{(x, a_i, b_j)_{i,j} \in X \times \prod_i A_i \times \prod_j B_j \mid \bigwedge_i (a_i \leq x) \wedge \bigwedge_j (x \leq b_j)\}_{\kappa\Sigma_1^1\mathbf{BorLoc}} = \emptyset.$$

Let $f_i : \tilde{A}_i \twoheadrightarrow A_i, g_j : \tilde{B}_j \twoheadrightarrow B_j$ be κ -Borel surjections from standard κ -Borel \tilde{A}_i, \tilde{B}_j . Then using that $1_X \times \prod_i f_i \times \prod_j g_j$ is κ -Borel surjective (by κ -complete regularity, Theorem 4.1.1(d)), the above is equivalent to

$$(*) \quad \{(x, \tilde{a}_i, \tilde{b}_j)_{i,j} \in X \times \prod_i \tilde{A}_i \times \prod_j \tilde{B}_j \mid \bigwedge_i (f_i(\tilde{a}_i) \leq x) \wedge \bigwedge_j (x \leq g_j(\tilde{b}_j))\}_{\kappa\mathbf{BorLoc}_\kappa} = \emptyset.$$

Regard \tilde{A}_i, \tilde{B}_j as “discrete” positive κ -Borel locales via the free functor $\kappa\mathbf{BorLoc} \rightarrow \kappa\mathbf{Bor}^+\mathbf{Loc}$, as in Section 3.7 (there denoted D). Then the left-hand side of $(*)$, but interpreted instead in $\kappa\mathbf{Bor}^+\mathbf{Loc}$, yields an order-embedded subobject of $X \times \prod_i \tilde{A}_i \times \prod_j \tilde{B}_j \in \kappa\mathbf{Bor}^+\mathbf{Loc}$ which the forgetful functor takes to the left-hand side of $(*)$, since the specialization order \leq_X is by definition the forgetful functor applied to the internal order on $X \in \kappa\mathbf{Bor}^+\mathbf{Loc}$. But this subobject interpreted in $\kappa\mathbf{Bor}^+\mathbf{Loc}$ is easily seen (e.g., as in \mathbf{Pos} , using the enriched Yoneda lemma) to be the **bilax pullback**

$$\begin{array}{ccc} \tilde{A}_i & \xleftarrow{\pi_i} & \{(x, \tilde{a}_i, \tilde{b}_j)_{i,j} \mid \bigwedge_i (f_i(a_i) \leq x) \wedge \bigwedge_j (x \leq g_j(b_j))\}_{\kappa\mathbf{Bor}^+\mathbf{Loc}} \\ \downarrow f_i & \swarrow \pi & \downarrow \pi_j \\ X & \xleftarrow{g_j} & \tilde{B}_j \end{array}$$

which is dual to the bilax pushout from Theorem 2.12.3. Thus, $(*)$ implies that this bilax pullback is empty, i.e., $\bigwedge_i \pi_i^*(\top) \leq \bigvee_j \pi_j^*(\perp)$. By Theorem 2.12.3, there are $C_i, \neg D_j \in \mathcal{B}_\kappa^+(X)$ such that $\bigwedge_i C_i \leq \bigvee_j \neg D_j$, $\top \leq f_i^*(C_i)$, and $g_j^*(\neg D_j) \leq \perp$, which exactly yields the conclusion. \square

The classical analog is a simultaneous generalization of the Novikov separation theorem and [Kec95, 28.12] (see Example 3.1.8), the latter of whose localic analog (the binary case of Theorem 4.2.6, or dually Corollary 2.12.6) we already used (via Proposition 3.4.25) to prove Theorem 3.7.3.

4.3 Inverse limit representations

Classically, a basic representation for a standard Borel space is as the space of infinite branches through a countably branching tree of height ω ; see [Kec95, 7.8–9, 13.9]. A suitable “fiberwise” version of this yields a representation of any analytic set $A \subseteq X$ via a Borel family of trees $(T_x)_{x \in X}$ such that $x \in A$ iff T_x has an infinite branch; see [Kec95, 25.2–13]. In this subsection, we consider localic generalizations of such tree representations.

First, note that given a tree T , if we let T_n be the nodes on level n , equipped with the predecessor maps $T_{n+1} \rightarrow T_n$ for each n , then the inverse limit $\varprojlim_n T_n$ is precisely the space of infinite branches. Thus an arbitrary codirected diagram of sets $(T_i)_{i \in I}$, for a directed poset I and “predecessor” functions $p_{ij} : T_i \rightarrow T_j$ for each $i \geq j \in I$, might be viewed as a generalization of a tree, with $\varprojlim_i T_i$ as the space of “branches”. Note that if I is countable, then it has a cofinal monotone sequence $(i_n)_{n \in \mathbb{N}}$, whence the subdiagram $(T_{i_n})_{n \in \mathbb{N}}$ is a genuine tree with the same inverse limit as $(T_i)_{i \in I}$. For $\kappa > \omega_1$, general codirected diagrams are more convenient than trees in the localic setting.

The generalization of the “tree representation” for a single standard κ -Borel locale is given by Proposition 3.3.9, which we recall here:

Proposition 4.3.1. *Every standard κ -Borel locale X is a limit, in $\kappa\mathbf{BorLoc}_\kappa$, of a κ -ary codirected diagram of κ -ary sets.* \square

Now given a family of trees $(T_x)_{x \in X}$, we can let each T_n be the set of pairs (x, t) where $x \in X$ and t is a node at level n in T_x . Then each projection $p_n : T_n \rightarrow X$ is a countable-to-one function whose fiber over each $x \in X$ is the n th level of T_x , and the system of all such T_n, p_n , with the projections $p_{(n+1)n} : T_{n+1} \rightarrow T_n$, forms a codirected **diagram over X** (i.e., $p_n \circ p_{(n+1)n} = p_{n+1}$), such that $\varprojlim_n T_n$ has fibers over each $x \in X$ consisting of the branches through T_x . Thus, we can

regard the system of T_n, p_n as a diagrammatic representation of the family of trees $(T_x)_{x \in X}$.

$$\begin{array}{ccccccc}
 \varprojlim_n T_n & \xrightarrow{\pi_n} & \cdots & \xrightarrow{p_{32}} & T_2 & \xrightarrow{p_{21}} & T_1 & \xrightarrow{p_{10}} & T_0 \\
 & & & & \searrow p_2 & & \searrow p_1 & & \downarrow p_0 \\
 & & & & & & & & X \\
 & & \searrow \pi & & & & & &
 \end{array}$$

For the localic generalization, we say that a κ -Borel map $f : X \rightarrow Y$ between κ -Borel locales is **κ -ary-to-one**, or that X is **fiberwise κ -ary over Y** , if X has a κ -ary cover by κ -Borel sets to which the restriction of f is a monomorphism (this condition in the classical setting follows from the Lusin–Novikov uniformization theorem; see [Kec95, 18.10]). We now have

Theorem 4.3.2. *Let $f : X \rightarrow Y$ be a κ -Borel map between standard κ -Borel locales. Then there is a κ -ary codirected diagram $(f_i : X_i \rightarrow Y)_{i \in I}$ over Y , for a κ -directed poset I and κ -Borel maps $f_{ij} : X_i \rightarrow X_j$ for each $i \geq j \in I$, such that each f_i is κ -ary-to-one and $X \cong \varprojlim_i X_i$ over Y .*

In particular, every κ -analytic set $A \subseteq Y$ is the κ -analytic image of a κ -ary codirected limit of fiberwise κ -ary standard κ -Borel locales over Y .

Proof. By Propositions 3.3.9 and 3.3.13, we may assume X, Y are ultraparacompact zero-dimensional standard κ -locales and f is (κ) -continuous. Write $X = \varprojlim_i A_i$ and $Y = \varprojlim_j B_j$ as κ -ary codirected limits of κ -ary sets, with projections $p_i : X \rightarrow A_i$ and $q_j : Y \rightarrow B_j$.

For each j , we claim that the maps $(p_i, q_j \circ f) : X \rightarrow A_i \times B_j$ exhibit X as the limit over i , in κLoc , of the images $\text{im}(p_i, q_j \circ f) \subseteq A_i \times B_j$ (where the image can be either closed or ∞ -Borel, since $A_i \times B_j$ is a κ -ary set). Indeed, the composite $(p_i, q_j \circ f)_i : X \rightarrow \varprojlim_i \text{im}(p_i, q_j \circ f) \subseteq \varprojlim_i (A_i \times B_j) \cong X \times B_j$ is the usual embedding $(1_X, q_j \circ f) : X \hookrightarrow X \times B_j$ of X as the **graph** of $q_j \circ f$. The graph is a closed set in $X \times B_j$, since it is the continuous preimage, under $(q_j \circ f) \times 1_{B_j}$, of the diagonal in B_j which is closed (since B_j is a set). Hence, the graph is an intersection of complements of basic clopen rectangles $\neg(p_i^*(\{a\}) \times \{b\}) = (p_i \times 1_{B_j})^*(\neg\{(a, b)\})$ for $a \in A_i$ and $b \in B_j$. Each such basic closed set, in order to contain the graph of $q_j \circ f$, must be such that $(a, b) \notin \text{im}(p_i, q_j \circ f)$, and so must contain $(p_i \times 1_{B_j})^*(\text{im}(p_i, q_j \circ f)) \supseteq \varprojlim_i \text{im}(p_i, q_j \circ f)$. Thus the graph of $q_j \circ f$ must be equal to $\varprojlim_i \text{im}(p_i, q_j \circ f) \subseteq Y \times A_i$, which is hence isomorphic to X .

We thus have

$$X \cong \varprojlim_i \varprojlim_j \text{im}(p_i, q_j \circ f) \subseteq \varprojlim_i \varprojlim_j (A_i \times B_j) = \varprojlim_i (A_i \times Y) = X \times Y.$$

Let $X_i := \varprojlim_j \text{im}(p_i, q_j \circ f) \subseteq \varprojlim_j (A_i \times B_j) = A_i \times Y$, and let $f_i : X_i \subseteq A_i \times Y \rightarrow Y$ be the projection. Since A_i is a κ -ary set, clearly each f_i is κ -ary-to-one. And clearly, the limit projections $X = \varprojlim_{i'} X_{i'} \rightarrow X_i$, as well as the diagram maps $X_i \rightarrow X_{i'}$ for $i \geq i'$, commute with the projections to Y . \square

Remark 4.3.3. Note that κ -ary-to-one maps in $\kappa\text{BorLoc}_\kappa$ have the nice property that (unlike for general κ -Borel maps) their κ -Borel images always exist and are **absolute**, in the sense that they are preserved by any functor $\kappa\text{BorLoc}_\kappa \rightarrow \mathbf{C}$ that preserves monomorphisms, in particular by the forgetful functors to $\lambda\text{BorLoc}_\lambda$ for $\lambda \geq \kappa$. Equivalently, κ -ary-to-one epimorphisms $f : X \twoheadrightarrow Y$ in $\kappa\text{BorLoc}_\kappa$ are preserved by any functor whatsoever. This is because, given a κ -ary cover $X = \bigcup_i X_i$ such that each $f|_{X_i}$ is a monomorphism, disjointifying the $f(X_i) \subseteq Y$ and then pulling back along the $f|_{X_i}$ yields a κ -Borel $X' \subseteq X$ such that $f|_{X'} : X' \rightarrow Y$ is a κ -Borel isomorphism, whose inverse is a κ -Borel section $Y \hookrightarrow X$ of f .

Let $(X_i)_{i \in I}$, with κ -Borel maps $f_{ij} : X_i \rightarrow X_j$ for $i \geq j$, be an arbitrary κ -ary codirected diagram of analytic κ -Borel locales. We say that the diagram is **pruned** if each f_{ij} is κ -Borel surjective. By Theorem 4.1.1(c), this implies that each projection $\pi_i : \varprojlim_j X_j \rightarrow X_i$ is κ -Borel surjective. Thus if the diagram is over $Y \in \kappa\Sigma_1^1\text{BorLoc}$, then each $X_i \rightarrow Y$ already has the same $\kappa\Sigma_1^1$ -image as $\varprojlim_i X_i$. There is a unique pruned levelwise subdiagram $(X'_i)_i$ of $(X_i)_i$, i.e., each $X'_i \subseteq X_i$, having the same inverse limit, namely $(\text{im}^\kappa(\pi_i))_i$.

Let us also assume \prec is a “successor” relation on I which generates \leq as a preorder and obeys

$$(4.3.4) \quad j \geq i \prec k \implies \exists l (j \prec l \geq k).$$

Motivating examples are the successor relation on \mathbb{N} , or the relation of “having exactly one more element” on a finite powerset $\mathcal{P}_\omega(A)$. Note that we can always take $\prec := \leq$. We define the **canonical pruning** (with respect to \prec) of the codirected diagram $(X_i)_i$ to be the levelwise subdiagram $(X'_i)_i$ given by

$$X'_i := \bigwedge_{j \succ i} f_{ji}^\kappa(X_j).$$

Intuitively, we remove all “leaves” in the original diagram $(X_i)_i$.

Lemma 4.3.5. *Each $f_{ij} : X_i \rightarrow X_j$ for $i \geq j$ restricts to $X'_i \rightarrow X'_j$.*

Proof. By definition of X'_j , this means that for each $k \succ j$, we must show $f_{ij}^\kappa(X'_i) \subseteq f_{kj}^\kappa(X'_k)$. By (4.3.4), there is some $i \prec l \geq k$, whence $f_{ij}^\kappa(X_i^{\alpha+1}) \subseteq f_{ij}^\kappa(f_{li}^\kappa(X_l^\alpha)) = f_{kj}^\kappa(f_{lk}^\kappa(X_l^\alpha)) \subseteq f_{kj}^\kappa(X_k^\alpha)$. \square

Clearly, $(X'_i)_i = (X_i)_i$ iff f_{ji} is κ -Borel surjective for each $j \succ i$; but since \prec generates \leq , this means the same for all $j \geq i$, i.e., the original diagram $(X_i)_i$ is pruned. Also, the canonical pruning $(X'_i)_i$ has the same limit as the original diagram $(X_i)_i$, since each projection $\pi_i : \varprojlim_j X_j \rightarrow X_i$ lands in $X'_i \subseteq X_i$, i.e., in $f_{ji}^\kappa(X_j)$ for each $j \succ i$, since π_i factors through π_j .

We define the **canonical pruning sequence** of $(X_i)_i$ to be the decreasing κ -length sequence of diagrams given by transfinitely iterating the canonical pruning:

$$\begin{aligned} X_i^{(0)} &:= X_i, \\ X_i^{(\alpha+1)} &:= X_i^{(\alpha)'} , \\ X_i^{(\alpha)} &:= \bigwedge_{\beta < \alpha} X_i^{(\beta)} \quad \text{for } \alpha \text{ limit.} \end{aligned}$$

By induction, each diagram $(X_i^{(\alpha)})_i$ in the canonical pruning sequence has the same limit as the original diagram $(X_i)_i$. If the sequence stabilizes (at all i) past some $\alpha < \kappa$, then the resulting diagram $(X_i^{(\alpha)})_i$ is a pruned diagram having the same limit as $(X_i)_i$, hence must be $(\text{im}^\kappa(\pi_i : \varprojlim_j X_j \rightarrow X_i))_i$.

We also say that a **pruning** of $(X_i)_i$ is any levelwise subdiagram containing the canonical pruning, and that a **pruning sequence** is any decreasing κ -length sequence of subdiagrams starting with $(X_i)_i$ and whose limit stages are given by meet, while a successor stage may be any pruning of the previous stage. Intuitively, this means we remove *some* “leaves” at each stage. Every diagram in a pruning sequence has the same limit as the original diagram (since it is sandwiched between the limits of the original diagram and the canonical pruning sequence). A non-canonical pruning sequence may stabilize at a non-pruned diagram; but if it stabilizes at a pruned diagram, then so must the canonical sequence (by the same or an earlier stage).

We now have the main result connecting pruning with κ -analytic sets, which intuitively says that the failure of pruning to stabilize before κ is due to “essential κ -analyticity”, i.e., non- κ -Borelness.

$$\begin{array}{ccccccc}
& & & & \vdots & & \vdots \\
& & & & \sqcap & & \sqcap \\
& & & \cdots \longrightarrow & X'_i & \longrightarrow & X'_j \\
& & & & \sqcap & & \sqcap \\
X = \varprojlim_i X_i & \xrightarrow{\pi_i} & \cdots \longrightarrow & X_i & \xrightarrow{f_{ij}} & X_j \\
& & \searrow p & \searrow p_i & \downarrow p_j & \\
& & & & Y &
\end{array}$$

- Proof.* (a) is most easily seen via the embedding $\kappa\Sigma_1^1\mathbf{BorLoc} \subseteq \kappa\mathbf{BorLoc}$ into the category of all κ -Borel locales from Section 4.1. Since the dual algebraic category $\kappa\mathbf{Bool}$ is locally κ -presentable, all of the κ -ary limit and colimit operations in $\kappa\mathbf{BorLoc}$ used to compute the canonical pruning $(X'_i)_i$ commute with κ -codirected limits (see [AR97, 1.59]). Thus, if we continue the canonical pruning sequence (in $\kappa\mathbf{BorLoc}$) past κ by defining $X_i^{(\kappa)} := \bigwedge_{\alpha < \kappa} X_i^{(\alpha)}$, then $X_i^{(\kappa)'} = \bigwedge_{\alpha < \kappa} X_i^{(\alpha)'} = X_i^{(\kappa)}$, i.e., the sequence necessarily stabilizes past κ . Now similarly to above, Proposition 3.4.22 implies that the stabilized value $X_i^{(\kappa)}$ must be $\mathrm{im}^\kappa(\pi_i)$.

(For $\kappa = \infty$, one can either work with large ∞ -Boolean algebras in a bigger universe, as in Remark 4.1.6, or unravel the above argument to remove the need for κBorLoc .)

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sequence, except replacing κ -analytic images with closed images:

$$\begin{aligned}\overline{X}_i^{(0)} &:= X_i, \\ \overline{X}_i^{(\alpha+1)} &:= \bigwedge_{j \succ i} \overline{f}_{ji}(\overline{X}_j^{(\alpha)}), \\ \overline{X}_i^{(\alpha)} &:= \bigwedge_{\beta < \alpha} \overline{X}_i^{(\beta)} \quad \text{for } \alpha \text{ limit.}\end{aligned}$$

Lemma 4.3.7. *Let $(a_\alpha)_{\alpha < \gamma}$ be a strictly increasing or decreasing transfinite sequence in a κ -generated \bigvee -lattice or \bigwedge -lattice A . Then $\gamma < \kappa$.*

Proof. Let $B \subseteq A$ be a κ -ary generating set. If A is a \bigvee -lattice, then $(\{b \in B \mid b \leq a_\alpha\})_\alpha$ is a strictly increasing or decreasing sequence in $\mathcal{P}(B)$, hence has length $\gamma < \kappa$. Dually if A is a \bigwedge -lattice. \square

Since each X_i is a standard κ -locale, and each $\overline{X}_i^{(\alpha)}$ is closed, it follows from this lemma that the pruning sequence $(\overline{X}_i^{(\alpha)})_{i,\alpha}$ stabilizes past some $\alpha < \kappa$. By definition of $\overline{X}_i^{(\alpha+1)}$, this means each $f_{ij} : \overline{X}_i^{(\alpha)} \rightarrow \overline{X}_j^{(\alpha)}$ has dense image. By Theorem 3.8.7, so does each projection $\pi_i : X = \varprojlim_j \overline{X}_j^{(\alpha)} \rightarrow \overline{X}_i^{(\alpha)}$. Since $X = \emptyset$, this means each $\overline{X}_i^{(\alpha)} = \emptyset$. Hence the canonical pruning sequence also stabilizes by or before α at \emptyset .

To prove (b) in the general case where each X_i is an analytic κ -Borel locale: for each i , let $q_i : \tilde{X}_i \twoheadrightarrow X_i$ be a κ -Borel surjection from a standard κ -Borel \tilde{X}_i . Let \hat{X}_i be the limit of the diagram consisting of the full subdiagram of $(X_j)_j$ consisting of X_j for all $j \leq i$, as well as the \tilde{X}_j and q_j for all $j \geq i$, i.e.,

$$\hat{X}_i := \left\{ (\tilde{x}_j)_j \in \prod_{j \leq i} \tilde{X}_i \mid \bigwedge_{j \leq i} (q_j(\tilde{x}_j) = f_{ij}(q_i(\tilde{x}_i))) \right\}_{\kappa \Sigma_1^1 \text{BorLoc}}.$$

Since each \tilde{X}_i is standard κ -Borel, so is \hat{X}_i (see Remark 4.1.3). We have obvious projections $\hat{f}_{ij} : \hat{X}_i \rightarrow \hat{X}_j$ for $i \geq j$, as well as $r_i : \hat{X}_i \rightarrow \tilde{X}_i$. Let $\hat{q}_i := q_i \circ r_i : \hat{X}_i \rightarrow X_i$. Then the \hat{X}_i together with the maps \hat{f}_{ij} form a codirected diagram of standard κ -Borel locales; and the \hat{q}_i form a natural transformation $(\hat{X}_i)_i \rightarrow (X_i)_i$. Let $\hat{X} := \varprojlim_i \hat{X}_i$.

$$(*) \quad \begin{array}{ccccccc} \hat{X} & \xrightarrow{\hat{\pi}_i} & \cdots & \longrightarrow & \hat{X}_i & \xrightarrow{\hat{f}_{ij}} & \hat{X}_j \\ \downarrow & & & & \hat{q}_i \downarrow & & \downarrow \hat{q}_j \\ X & \xrightarrow{\pi_i} & \cdots & \longrightarrow & X_i & \xrightarrow{f_{ij}} & X_j \end{array}$$

We claim that each \hat{q}_i is κ -Borel surjective, and moreover, for each $i \geq j$,

$$(\dagger) \quad (\hat{q}_i, \hat{f}_{ij}) : \hat{X}_i \twoheadrightarrow X_i \times_{X_j} \hat{X}_j \text{ is } \kappa\text{-Borel surjective.}$$

To see this, for each $J \subseteq I$ with a greatest element $i \in J$, define \hat{X}_J in a similar manner to \hat{X}_i above, except restricting to $j \in J$. Thus $\hat{X}_i = \hat{X}_{\downarrow i}$, $\tilde{X}_i = \hat{X}_{\{i\}}$, and r_i is the projection $\hat{X}_{\downarrow i} \rightarrow \hat{X}_{\{i\}}$. We now claim that for $K \subseteq J \subseteq I$ with the same greatest element i , the projection $\hat{X}_J \rightarrow \hat{X}_K$ is κ -Borel surjective; this is because, by Proposition 3.4.22, it suffices to consider the case where $J = K \cup \{j\}$ for some $j \leq i$, in which case the projection is the pullback of $q_i : \tilde{X}_j \twoheadrightarrow X_j$ along $\tilde{X}_J \rightarrow X_i \xrightarrow{f_{ij}} X_j$. It follows that r_i is κ -Borel surjective, whence so is $\hat{q}_i = q_i \circ r_i$. It also follows

that for $j < i$, the projection $\widehat{X}_i = \widehat{X}_{\downarrow i} \rightarrow \widehat{X}_{\{i\} \cup \downarrow j} = \widetilde{X}_i \times_{X_j} \widehat{X}_{\downarrow j} = \widetilde{X}_i \times_{X_j} \widehat{X}_j$ is κ -Borel surjective; composing with the pullback of $q_i : \widetilde{X}_i \twoheadrightarrow X_i$ yields (\dagger) .

It now follows that each of the squares in $(*)$ for $i \geq j$ obeys the “Beck–Chevalley equation”

$$\widehat{f}_{ij}^\kappa(\widehat{q}_i^*(A)) = \widehat{q}_j^*(f_{ij}^\kappa(A)) \quad \forall A \in \kappa\Sigma_1^1(X_i).$$

Indeed, the pullback square for f_{ij}, \widehat{q}_j obeys the Beck–Chevalley equation by regularity of $\kappa\Sigma_1^1\mathbf{BorLoc}$ (see Corollary 3.4.21); and by (\dagger) , we may insert the preimage from the pullback to \widehat{X}_i followed by the $\kappa\Sigma_1^1$ -image into the left-hand side with no effect. This implies that the computation of the canonical pruning of $(X_i)_i$, which only involves $\kappa\Sigma_1^1$ -image and \bigwedge , may be pulled back to $(\widehat{X}_i)_i$, i.e.,

$$\widehat{X}_i^{(\alpha)} = \widehat{q}_i^*(X_i^{(\alpha)})$$

for all i, α . Since \widehat{q}_i is κ -Borel surjective, this implies

$$\widehat{q}_i^\kappa(\widehat{X}_i^{(\alpha)}) = X_i^{(\alpha)}.$$

Now since $X = \emptyset$ and there is a map $\widehat{X} \rightarrow X$, also $\widehat{X} = \emptyset$, whence by the proof of (b) for the standard κ -Borel case, there is some $\alpha < \kappa$ by which all the $\widehat{X}_i^{(\alpha)}$ become empty, whence so do all the $X_i^{(\alpha)}$, completing the proof of (b).

(c) follows by pulling everything back along $\neg \text{im}^\kappa(p) \subseteq Y$, noting that the definition of the canonical pruning is pullback-stable (by regularity of $\kappa\Sigma_1^1\mathbf{BorLoc}$), then applying (b). \square

Corollary 4.3.8. *Every κ -analytic set in a standard κ -Borel locale Y is a meet, in $\kappa\Sigma_1^1(Y)$, of a κ -length decreasing sequence of κ -Borel sets.*

Proof. Apply Theorem 4.3.6(a) to a diagram obtained from Theorem 4.3.2, where each of the X_i is a fiberwise κ -ary standard κ -Borel locale over Y , whence all of the stages of the canonical pruning sequence are κ -Borel, as are their images in Y (see Remark 4.3.3). \square

The above corollary can be seen as a generalization of the classical **Lusin–Sierpinski theorem** that every analytic set in a standard Borel space is an ω_1 -length decreasing intersection of Borel sets; see [Kec95, 25.16].

Remark 4.3.9. Theorem 4.3.6(b) can be seen as a generalization of the classical **boundedness theorems for Σ_1^1** ; see [Kec95, 31.A]. For example, one version says that any analytic well-founded binary relation has rank $< \omega_1$. For $\kappa \geq \omega_1$, we have the following generalization: take a κ -analytic binary relation $R \subseteq Y^2$ in an analytic κ -Borel locale Y , which is internally well-founded in that

$$\kappa\Sigma_1^1\mathbf{BorLoc} \models \forall (y_0, y_1, \dots) \in Y^\mathbb{N} (R(y_1, y_0) \wedge R(y_2, y_1) \wedge \dots \implies \perp).$$

The set $\{(x_0, x_1, \dots) \mid R(y_1, y_0) \wedge R(y_2, y_1) \wedge \dots\}_{\kappa\Sigma_1^1\mathbf{BorLoc}} \subseteq Y^\mathbb{N}$ is the limit of the inverse sequence of sets of finite sequences $X_n := \{(x_0, x_1, \dots, x_n) \mid R(y_1, y_0) \wedge \dots \wedge R(y_n, y_{n-1})\}_{\kappa\Sigma_1^1\mathbf{BorLoc}} \subseteq Y^{n+1}$ under the projection maps $X_{n+1} \rightarrow X_n$, whence by Theorem 4.3.6(b), we get that the pruning of this inverse sequence must stop before κ , which can be thought of as an internal way of saying that “ R has rank $< \kappa$ ”.

4.4 Ill-founded relations

We now prove that there exist κ -analytic, non- ∞ -Borel sets. Classically, the existence of analytic, non-Borel sets may be shown using (one version of) the boundedness theorem for Σ_1^1 : if the set of ill-founded binary relations $R \subseteq \mathbb{N}^2$ were Borel, then the pruning of minimal elements for all $R \subseteq \mathbb{N}^2$ must stop by some fixed stage $\alpha < \omega_1$, contradicting that there are well-founded R with arbitrarily high rank $< \omega_1$. Our goal is to generalize a version of this proof.

Let X be an infinite set. For each $n \in \mathbb{N}$, let

$$\text{DESC}(X, n) := \{(R, x_0, \dots, x_{n-1}) \in \mathbb{S}^{X^2} \times X^n \mid x_{n-1} R \cdots R x_0\}$$

be the space of binary relations on X together with a descending sequence of length n , an open subspace of $\mathbb{S}^{X^2} \times X^n$ (where X is discrete). For $n \geq 1$, let $p_n : \text{DESC}(X, n) \rightarrow \text{DESC}(X, n-1)$ be the projection omitting the last coordinate. If X is κ -ary, then clearly (the underlying locale of) each $\text{DESC}(X, n)$ is a standard κ -locale, and p_n is κ -ary-to-one as witnessed by the clopen partition

$$\text{DESC}(X, n) = \bigsqcup_{x \in X} \{(R, x_0, \dots, x_{n-1}) \in \text{DESC}(X, n) \mid x_{n-1} = x\}$$

each piece to which the restriction of p_n is an open embedding. In particular, every κ -Borel $B \subseteq \text{DESC}(X, n)$ has a κ -Borel image under p_n which is also the ∞ -Borel image (see Remark 4.3.3). It follows that the κ -length canonical pruning sequence, as defined in the preceding subsection, of $(\text{DESC}(X, n))_n$, regarded as a diagram in $\kappa\Sigma_1^1\text{BorLoc}$ with the maps p_n , consists of standard κ -Borel locales at each stage $< \kappa$, and forms an initial segment of the ∞ -length canonical pruning sequence when $(\text{DESC}(X, n))_n$ is instead regarded as a diagram in $\infty\Sigma_1^1\text{BorLoc}$.

Lemma 4.4.1. *For any two infinite κ -ary sets X, Y , there is a nonempty standard κ -locale Z together with a natural isomorphism in κLoc of diagrams $(f_n : Z \times \text{DESC}(X, n) \cong Z \times \text{DESC}(Y, n))_{n \in \mathbb{N}}$ over Z .*

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & Z \times \text{DESC}(X, 2) & \xrightarrow{p_2} & Z \times \text{DESC}(X, 1) & \xrightarrow{p_1} & Z \times \text{DESC}(X, 0) \\
 & & \cong \downarrow f_2 & & \cong \downarrow f_1 & & \cong \downarrow f_0 \\
 (*) & \cdots & \longrightarrow & Z \times \text{DESC}(Y, 2) & \xrightarrow{p_2} & Z \times \text{DESC}(Y, 1) & \xrightarrow{p_1} & Z \times \text{DESC}(Y, 0) \\
 & & & & \searrow & \swarrow & \swarrow & \nwarrow \\
 & & & & & & Z &
 \end{array}$$

Proof. By Theorem 3.8.8, there is a nonempty standard κ -locale Z “collapsing $|X|, |Y|$ ”, so that there is an isomorphism $g : Z \times X \cong Z \times Y \in \text{Loc}$ over Z . The result now follows by internalizing the definitions of $\text{DESC}(X, n), p_n$ over Z . Namely, let $g_Z^n : Z \times X^n \cong Z \times Y^n$ be the n -fold fiber product of g over Z , and let

$$\mathbb{S}_Z^{g_Z^n} : Z \times \mathbb{S}^{X^n} \longrightarrow Z \times \mathbb{S}^{Y^n} \in \text{Loc}$$

be the “fiberwise exponential of g_Z^n over Z ”, i.e., $\mathbb{S}_Z^{g_Z^n}$ commutes with the projections to Z , while for the \vec{y} th subbasic open set in \mathbb{S}^{Y^n} where $\vec{y} \in Y^n$, i.e., the \vec{y} th free generator of $\langle Y^n \rangle_{\text{Frm}}$, we put

$$(\mathbb{S}_Z^{g_Z^n})^*(\{(z, S) \in Z \times \mathbb{S}^{Y^n} \mid S(\vec{y})\}) := \{(z, R) \in Z \times \mathbb{S}^{X^n} \mid \bigvee_{\vec{x} \in X^n} ((g_Z^n(z, \vec{x}) = (z, \vec{y})) \wedge R(\vec{x}))\}.$$

(Here we are using the internal logic in \mathbf{Loc} , where $R(\vec{x})$ is really an abbreviation for the atomic formula $[\vec{x}](R)$ where $[\vec{x}] \subseteq \mathbb{S}^{X^n}$ is the \vec{x} th subbasic open set. Note that the formula $g_Z^n(z, \vec{x}) = (z, \vec{y})$ defines an open set of z , since $\{\vec{y}\} \subseteq Y^n$ is open.) Similarly, let

$$\mathbb{S}_Z^{(g_Z^n)^{-1}} : Z \times \mathbb{S}^{Y^n} \longrightarrow Z \times \mathbb{S}^{X^n} \in \mathbf{Loc}$$

be the “fiberwise exponential of $(g_Z^n)^{-1}$ over Z ”. Using the internal logic, it is straightforward to check that these two maps are inverses:

$$\begin{aligned} & (\mathbb{S}_Z^{(g_Z^n)^{-1}})^*((\mathbb{S}_Z^{g_Z^n})^*({(z, S) \mid S(\vec{y})})) \\ &= \left\{ (z, S) \mid \bigvee_{\vec{x} \in X^n} \left((g_Z^n(z, \vec{x}) = (z, \vec{y})) \wedge ((z, S) \in (\mathbb{S}_Z^{(g_Z^n)^{-1}})^*({(z, R) \mid R(\vec{x})})) \right) \right\} \\ &= \left\{ (z, S) \mid \bigvee_{\vec{x} \in X^n} ((g_Z^n(z, \vec{x}) = (z, \vec{y})) \wedge \bigvee_{\vec{y}' \in Y^n} (((g_Z^n)^{-1}(z, \vec{y}') = (z, \vec{x})) \wedge S(\vec{y}')) \right\} \\ &= \{(z, S) \mid \bigvee_{\vec{x} \in X^n} ((g_Z^n(z, \vec{x}) = (z, \vec{y})) \wedge S(\vec{y}))\} = \{(z, S) \mid S(\vec{y})\} \end{aligned}$$

whence $\mathbb{S}_Z^{g_Z^n} \circ \mathbb{S}_Z^{(g_Z^n)^{-1}} = 1$, and similarly $\mathbb{S}_Z^{(g_Z^n)^{-1}} \circ \mathbb{S}_Z^{g_Z^n} = 1$. Now let

$$f_n := \mathbb{S}_Z^{g_Z^2} \times_Z g_Z^n : Z \times \mathbb{S}^{X^2} \times X^n \cong Z \times \mathbb{S}^{Y^2} \times Y^n.$$

It is again straightforward to check using the internal logic that these maps commute with the projections $p_n : Z \times \mathbb{S}^{X^2} \times (-)^n \rightarrow Z \times \mathbb{S}^{X^2} \times (-)^{n-1}$ as in the diagram (*), and that they take $Z \times \mathbf{DESC}(X, n)$ to $Z \times \mathbf{DESC}(Y, n)$:

$$\begin{aligned} & f_n^*(Z \times \mathbf{DESC}(Y, n)) \\ &= f_n^*({(z, S, y_0, \dots, y_{n-1}) \mid y_{n-1} S \cdots S y_0}) \\ &= \left\{ (z, R, \vec{x}) \mid \bigvee_{\vec{y} \in Y^n} \left((g_Z^n(z, \vec{x}) = (z, \vec{y})) \wedge ((z, R) \in (\mathbb{S}_Z^{g_Z^2})^*({(z, S) \mid y_{n-1} S \cdots S y_0})) \right) \right\} \\ &= \left\{ (z, R, \vec{x}) \mid \bigvee_{\vec{y} \in Y^n} ((g_Z^n(z, \vec{x}) = (z, \vec{y})) \wedge \bigvee_{\vec{x}' \in X^n} ((g_Z^n(z, \vec{x}') = (z, \vec{y})) \wedge (x'_{n-1} R \cdots R x'_0))) \right\} \\ &= \{(z, R, \vec{x}) \mid \bigvee_{\vec{y} \in Y^n} ((g_Z^n(z, \vec{x}) = (z, \vec{y})) \wedge (x_{n-1} R \cdots R x_0))\} \\ &= \{(z, R, \vec{x}) \mid x_{n-1} R \cdots R x_0\} = Z \times \mathbf{DESC}(X, n). \quad \square \end{aligned}$$

Theorem 4.4.2. *For any infinite set X , the ∞ -length canonical pruning sequence of $(\mathbf{DESC}(X, n))_n$ is strictly decreasing at each level n : for each n and ordinal $\alpha < \infty$, we have*

$$\mathbf{DESC}(X, n)^{(\alpha)} \supsetneq \mathbf{DESC}(X, n)^{(\alpha+1)}.$$

Proof. Let Y be an infinite set of cardinality $\geq |\alpha|$, and let Z, f_n be given by Lemma 4.4.1. By pullback-stability along $Z \rightarrow 1$, the canonical pruning sequence of $(Z \times \mathbf{DESC}(X, n))_n$ is the product of Z with the canonical pruning sequence of $(\mathbf{DESC}(X, n))_n$. By the natural isomorphism $(f_n)_n$, it is also isomorphic (in $\infty\mathbf{BorLoc}$) to the canonical pruning sequence of $(Z \times \mathbf{DESC}(Y, n))_n$, which again by pullback-stability, is the product of Z with that of $(\mathbf{DESC}(Y, n))_n$. So it suffices to show that $\mathbf{DESC}(Y, n)^{(\alpha)} \supsetneq \mathbf{DESC}(Y, n)^{(\alpha+1)}$ for each n, α , since that implies $Z \times \mathbf{DESC}(Y, n)^{(\alpha)} \supsetneq Z \times \mathbf{DESC}(Y, n)^{(\alpha+1)}$ because $Z \neq \emptyset$ (see Corollary 2.8.8), which in turn implies $\mathbf{DESC}(X, n)^{(\alpha)} \supsetneq \mathbf{DESC}(X, n)^{(\alpha+1)}$.

Since each p_n is ∞ -ary-to-one, by Remark 4.3.3, the construction of the canonical pruning sequence of $(\mathbf{DESC}(Y, n))_n$ is preserved by the spatialization functor $\mathbf{Sp} : \infty\mathbf{BorLoc} \rightarrow \mathbf{Set} \subseteq \infty\mathbf{Bor}$.

An easy induction shows that if $S \subseteq Y^2$ is the predecessor relation of a forest of height ω , and $y_{n-1} S \cdots S y_0$ is a finite path through S , then $(S, y_0, \dots, y_{n-1}) \in \text{Sp}(\text{DESC}(Y, n)^{(\alpha)})$ iff the subtree below y_{n-1} (or the entire forest, if $n = 0$) has rank $\geq \alpha$. Thus if we let S be a forest of rank α , together with a finite terminal segment $y_{n-1} S \cdots S y_0$ attached if $n \geq 1$, then (S, y_0, \dots, y_{n-1}) is a point in $\text{DESC}(Y, n)^{(\alpha)} \setminus \text{DESC}(Y, n)^{(\alpha+1)}$. \square

Let $\text{DESC}(X, \omega) := \varprojlim_n \text{DESC}(X, n)$, with projections $\pi_n : \text{DESC}(X, \omega) \rightarrow \text{DESC}(X, n)$. Clearly

$$\text{DESC}(X, \omega) \cong \{(R, x_0, x_1, \dots) \in \mathbb{S}^{X^2} \times X^{\mathbb{N}} \mid \bigwedge_n (x_{n+1} R x_n)\}_{\text{Loc}}$$

is the locale “of binary relations on X with an infinite descending sequence”.

Theorem 4.4.3. *There exists a continuous map $\pi_0 : \text{DESC}(\mathbb{N}, \omega) \rightarrow \text{DESC}(\mathbb{N}, 0)$ between quasi-Polish spaces which does not have a κ -Borel image for any $\omega_1 \leq \kappa \leq \infty$.*

In particular, for all $\omega_1 \leq \kappa \leq \infty$, we have $\mathcal{B}_\kappa(\mathbb{S}^{\mathbb{N}}) \subsetneq \kappa\Sigma_1^1(\mathbb{S}^{\mathbb{N}})$.

Proof. By Theorem 4.4.2, $(p_1(\text{DESC}(\mathbb{N}, 1)^{(\alpha)}))_\alpha = (\text{DESC}(\mathbb{N}, 0)^{(\alpha+1)})_\alpha$ does not stabilize below any κ , whence by Theorem 4.3.6(c), the $\kappa\Sigma_1^1$ -image $\text{im}^\kappa(\pi_0) \subseteq \text{DESC}(\mathbb{N}, 0)$ is not κ -Borel.

The last statement follows because $\text{DESC}(\mathbb{N}, 0) = \mathbb{S}^{\mathbb{N}^2} \cong \mathbb{S}^{\mathbb{N}}$. \square

Corollary 4.4.4 (Gaifman–Hales). *$\langle \mathbb{N} \rangle_{\text{CBOOL}}$ is a proper class.*

Proof. If $\langle \mathbb{N} \rangle_{\text{CBOOL}} = \mathcal{B}_\infty(\mathbb{S}^{\mathbb{N}})$ were a set, it would be a small complete lattice, whence every ∞ -Borel map $X \rightarrow \mathbb{S}^{\mathbb{N}}$ would have an ∞ -Borel image. \square

The following consequence was mentioned in Remark 3.4.15:

Corollary 4.4.5. *There exist $\kappa = \omega_1 < \lambda = (2^{\aleph_0})^+$ and a σ -Borel map $\pi_0 : \text{DESC}(\mathbb{N}, \omega) \rightarrow \text{DESC}(\mathbb{N}, 0) \in \kappa\text{BorLoc}_\kappa$ between standard κ -Borel locales, such that the (regular) epi-mono factorization of π_0 in κBorLoc is not preserved by the forgetful functor $\kappa\text{BorLoc} \rightarrow \lambda\text{BorLoc}$.*

In particular, the forgetful functor $\kappa\text{BorLoc} \rightarrow \lambda\text{BorLoc}$ does not preserve (regular) epimorphisms, and does not map the embedding $\kappa\Sigma_1^1\text{BorLoc} \subseteq \kappa\text{BorLoc}$ into $\lambda\Sigma_1^1\text{BorLoc} \subseteq \lambda\text{BorLoc}$.

Proof. Since $|\mathcal{B}_\sigma(\text{DESC}(\mathbb{N}, 0))| = 2^{\aleph_0}$ (because $\mathcal{B}_\sigma(\text{DESC}(\mathbb{N}, 0))$ is standard Borel), the $\infty(\mathcal{B}_\sigma)_\delta$ -image of π_0 is the λ -ary intersection of all $B \in \mathcal{B}_\sigma(\text{DESC}(\mathbb{N}, 0))$ such that $\pi_0^*(B) = \top$, hence is λ -Borel. Thus it cannot also be the $\infty(\mathcal{B}_\lambda)_\delta$ -image, or else it would be the λ -Borel image, which means the $\infty(\mathcal{B}_\lambda)_\delta$ -image must be strictly smaller. \square

We also state the purely Boolean-algebraic dual result, which is new as far as we know, and somewhat surprising given the good behavior of monomorphisms in κBool in other respects (e.g., Corollaries 2.12.7 and 2.12.9):

Corollary 4.4.6. *There exist $\kappa = \omega_1 < \lambda = (2^{\aleph_0})^+$ such that the free functor $\kappa\text{Bool} \rightarrow \lambda\text{Bool}$ does not preserve injective homomorphisms.* \square

Remark 4.4.7. The easiest classical proof of the existence of analytic, non-Borel sets is by diagonalizing a **universal analytic set** $A \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, i.e., an analytic set whose vertical fibers over each $x \in 2^{\mathbb{N}}$ yield all analytic sets in $2^{\mathbb{N}}$; see [Kec95, 14.2]. Such a technique cannot directly yield the full strength of Theorem 4.4.3, since every locale has only a small set of points, whereas $2^{\mathbb{N}}$ has

a proper class of ∞ -analytic sets (since it has a proper class of ∞ -Borel sets), so there cannot be a universal ∞ -analytic set. However, for each *fixed* non-inaccessible $\kappa \geq \omega_1$, say with $\lambda < \kappa$ such that $2^\lambda \geq \kappa$, a universal κ -analytic set in $2^\lambda \times 2^\lambda$ can probably be used to show the weaker statement that $\mathcal{B}_\kappa(X) \subsetneq \kappa\Sigma_1^1(X)$ for some κ -locale $X = 2^\lambda$ depending on κ .

A Appendix: nice categories of structures

In this appendix, we briefly review the basic theory of locally presentable and monadic categories, as well as their ordered analogs. These are well-behaved categories of (infinitary) first-order structures which are sufficiently “algebraic”, meaning they are defined by axioms of certain restricted forms, such that the usual universal categorical constructions are available: products, coproducts, quotients, presentations via generators and relations, etc.

See Section 2 for our general conventions on categories and cardinalities.

A.1 Limit theories and locally presentable categories

Let \mathcal{L} be an infinitary first-order signature, consisting of function and relation symbols, each with an associated arity which may be an arbitrary set (usually a cardinal).¹⁷ If every symbol in \mathcal{L} has arity of size $< \kappa$, then we say that \mathcal{L} is κ -ary. For now, we require \mathcal{L} to be a set, rather than a proper class (but see the end of this subsection).

Example A.1.1. The ω -ary signature of distributive lattices may be taken as $\{\perp, \vee, \top, \wedge, \leq\}$ where \perp, \top are nullary function symbols, \vee, \wedge are binary relation symbols, and \leq is a binary relation symbol. We may also omit \leq , since it can be defined from \vee or \wedge .

Example A.1.2. The ω_1 -ary signature of σ -frames may be taken as $\{\perp, \bigvee, \top, \wedge, \leq\}$, where \bigvee has arity ω and the other symbols are as before.

For $\kappa < \infty$ and a κ -ary signature \mathcal{L} , a **κ -limit axiom** over \mathcal{L} is an axiom (in the infinitary first-order logic $\mathcal{L}_{\kappa\kappa}$) of the form

$$(*) \quad \forall \vec{x} \left(\bigwedge_i \phi_i(\vec{x}) \rightarrow \exists ! \vec{y} \bigwedge_j \psi_j(\vec{x}, \vec{y}) \right)$$

where ϕ_i, ψ_j are atomic formulas (i.e., equations or atomic relations between terms), \bigwedge_i, \bigwedge_j are κ -ary conjunctions, and \vec{x}, \vec{y} are κ -ary families of variables. A **κ -limit theory** is a set of κ -limit axioms. The category of all (set-based) models of a κ -limit theory, together with all homomorphisms between them, is called a **locally κ -presentable category**. See [AR97] for a comprehensive reference.

Example A.1.3. An **algebraic theory** is a universal-equational theory, i.e., \mathcal{L} has only function symbols, and each axiom as in $(*)$ has empty left-hand side (no ϕ_i 's) and empty \vec{y} . Examples include the theories of κ -frames, κ - \bigvee -lattices, κ -Boolean algebras, or groups, rings, etc.

Example A.1.4. More generally, a **universal Horn theory** is a limit theory with no existentials (\vec{y} in $(*)$ is empty), e.g., the theory of posets.

¹⁷The general theory also allows multi-sorted signatures; see below.

Example A.1.5. More generally, we may allow structures with partial operations, where the domain of each operation must be defined by a conjunction of atomic formulas; such theories are sometimes called **essentially algebraic**. To axiomatize such structures via a limit theory, replace each partial operation with its graph relation, and use an axiom with $\exists!$ to say that it is a function. For example, \wedge -lattices with pairwise disjoint κ -ary joins (disjunctive κ -frames, as defined in Section 2.9; see Example A.3.2 below) may be axiomatized in this way.

Let \mathbf{C} be a locally κ -presentable category of structures, axiomatized by some κ -limit theory. Then \mathbf{C} has arbitrary small limits and colimits. Limits of structures are constructed in the usual way, by taking the limit of the underlying sets in \mathbf{Set} , equipped with the coordinatewise operations and relations. Colimits are generally not constructed as in \mathbf{Set} ; however, κ -directed colimits (meaning colimits of diagrams indexed by a κ -directed preordered set) are constructed as in \mathbf{Set} , with a relation holding in the colimit iff it already holds at some stage of the diagram.

Structures in \mathbf{C} may be presented via generators and relations: for any set G of generators, and any set R of relations between the elements of G (i.e., atomic formulas with variables from G),¹⁸ there is a **structure presented by G, R** , denoted

$$\langle G \mid R \rangle = \langle G \mid R \rangle_{\mathbf{C}} \in \mathbf{C},$$

which is the universal structure obeying the axioms and equipped with a map $\eta : G \rightarrow \langle G \mid R \rangle$ making all of the relations in R hold. That is, for any $A \in \mathbf{C}$, composition with η gives a bijection

$$\mathbf{C}(\langle G \mid R \rangle, A) \cong \{f : G \rightarrow A \mid \text{every relation in } R \text{ holds in } A \text{ after substituting } f\}.$$

The structure $\langle G \mid R \rangle$ may be constructed via a standard transfinite iteration of length $< \kappa^+$, starting with the algebra of terms generated from G by the function symbols in \mathcal{L} , and then at each stage, enforcing the relations in R as well as the κ -limit axioms in the theory relative to the previous stage, by adjoining new elements (to satisfy \exists), taking a quotient (to satisfy $\exists!$ and equations), and enlarging the relations (to satisfy other atomic formulas). Every $A \in \mathbf{C}$ has a presentation, i.e., $A \cong \langle G \mid R \rangle$ for some G, R , for example $G :=$ underlying set of A and $R :=$ all relations which hold in A .

Colimits in \mathbf{C} may be constructed via presentations: to construct a colimit of a diagram, in which each object is given by a presentation, take the disjoint union of the presentations (which presents the coproduct of the objects), then add equations identifying each $a \in A$ with $f(a) \in B$ for each morphism $f : A \rightarrow B$ in the diagram.

If $A \in \mathbf{C}$ is presented by some G, R which are λ -ary (respectively, just G is λ -ary), then A is called **λ -presented** (**λ -generated**). Let $\mathbf{C}_\lambda \subseteq \mathbf{C}$ denote the full **subcategory of λ -presented structures**. By the construction of colimits described above, \mathbf{C}_λ is closed under λ -ary colimits. The notion of λ -presentability in locally κ -presentable \mathbf{C} is most robust when $\lambda \geq \kappa$, in which case $\mathbf{C}_\lambda \subseteq \mathbf{C}$ can be intrinsically characterized as all objects A for which the representable functor $\mathbf{C}(A, -) : \mathbf{C} \rightarrow \mathbf{Set}$ preserves λ -directed colimits.

There is a **(special) adjoint functor theorem** for locally presentable categories: any functor $\mathbf{D} \rightarrow \mathbf{C}$ between locally presentable categories which preserves limits as well as λ -directed colimits for sufficiently large λ has a left adjoint $\mathbf{C} \rightarrow \mathbf{D}$. In particular, forgetful functors between \mathbf{D}, \mathbf{C}

¹⁸More generally, R can contain existentials which are already known to be unique; see Section 3.6, [Joh82, V 1.12].

axiomatized by κ -limit theories (which preserve limits and κ -directed colimits) have left adjoints, which construct the **free D-structure generated by a C-structure**:

$$\begin{aligned} \mathbf{C} &\longrightarrow \mathbf{D} \\ A &\longmapsto \langle A \mid \text{all C-structure relations which hold in } A \rangle_{\mathbf{D}} =: \langle A \text{ qua } \mathbf{C} \rangle_{\mathbf{D}} = \langle A \rangle_{\mathbf{D}}. \end{aligned}$$

We adopt this “**qua**” notation from [JV91], but omit it when context makes clear which category \mathbf{C} we are starting from. More generally, we may add further generators or impose further D-structure relations, via

$$\langle A \text{ qua } \mathbf{C}, G \mid R \rangle_{\mathbf{D}} := \langle A \sqcup G \mid \{\text{all C-relations which hold in } A\} \cup R \rangle_{\mathbf{D}}.$$

Any left adjoint $F : \mathbf{C} \rightarrow \mathbf{D}$ between locally κ -presentable categories whose right adjoint $G : \mathbf{D} \rightarrow \mathbf{C}$ preserves κ -directed colimits (in particular, any free functor F) preserves κ -presented objects, i.e., restricts to a functor $F : \mathbf{C}_{\kappa} \rightarrow \mathbf{D}_{\kappa}$. There is also an adjoint functor theorem for colimit-preserving functors between locally presentable categories: they all have right adjoints.

Two special types of adjunctions $F : \mathbf{C} \rightarrow \mathbf{D} \dashv G : \mathbf{D} \rightarrow \mathbf{C}$ are the **reflective** and **coreflective** adjunctions, where G , respectively F , is full and faithful, hence can be regarded (up to equivalence of categories) as a full subcategory inclusion. When \mathbf{C}, \mathbf{D} are locally κ -presentable and G preserves κ -directed colimits (e.g., G is forgetful), these interact with presentations as follows. In the coreflective case, as mentioned above, F preserves κ -presented objects; but since F is full and faithful, it also reflects them, i.e.,

$$\mathbf{C}_{\kappa} \simeq F^{-1}(\mathbf{D}_{\kappa}).$$

In the reflective case, we instead have

Proposition A.1.6. *Given a reflective adjunction $F \dashv G : \mathbf{D} \rightarrow \mathbf{C}$ between locally κ -presentable categories as above, where G preserves κ -directed colimits, every $B \in \mathbf{D}_{\kappa}$ is a retract of some $F(A)$ for $A \in \mathbf{C}_{\kappa}$. If furthermore κ is uncountable, then every B is isomorphic to some $F(A)$, i.e.,*

$$F(\mathbf{C}_{\kappa}) \simeq \mathbf{D}_{\kappa}.$$

Remark A.1.7. More generally, the conclusion above holds for the functor induced by F between coslice categories $A \downarrow \mathbf{C}_{\kappa} \rightarrow F(A) \downarrow \mathbf{D}_{\kappa}$ for every $A \in \mathbf{C}_{\kappa}$. In fact, this generalized statement completely characterizes when a κ -directed-colimit-preserving right adjoint between locally κ -presentable categories is full and faithful; see [MP87, 2.6].

Proposition A.1.6 is a consequence of

Lemma A.1.8. *Let \mathbf{D} be a locally κ -presentable category. If a κ -presented object $B \in \mathbf{D}_{\kappa}$ is a κ -directed colimit $B = \varinjlim_{i \in I} A_i$ of κ -presented objects $A_i \in \mathbf{D}_{\kappa}$ with morphisms $f_{ij} : A_i \rightarrow A_j$ for $i \leq j \in I$, for some κ -directed preorder I , then B is a retract of some A_{i_0} , as well as the colimit of some subsequence $A_{i_0} \rightarrow A_{i_1} \rightarrow \cdots$ of the A_i , where $i_0 \leq i_1 \leq \cdots \in I$.*

Proof. Since B is κ -presented, the identity $1_B : B \rightarrow \varinjlim_i A_i$ factors through the cocone map $\iota_{i_0} : A_{i_0} \rightarrow \varinjlim_i A_i$ for some $i_0 \in I$, say as $s : B \rightarrow A_{i_0}$, so that $\iota_{i_0} \circ s = 1_B$, whence B is a retract of A_{i_0} . Then the composite $A_{i_0} \xrightarrow{\iota_{i_0}} B \xrightarrow{s} A_{i_0} \xrightarrow{\iota_{i_0}} B = \varinjlim_i A_i$ is equal to ι_{i_0} , so since A_{i_0} is κ -presented,

the two maps $s \circ \iota_{i_0}, 1_{A_{i_0}} : A_{i_0} \rightarrow A_{i_0}$ are already equal when composed with the diagram map $f_{i_0 i_1} : A_{i_0} \rightarrow A_{i_1}$ for some $i_1 \geq i_0$. Then we have $\iota_{i_1} \circ f_{i_0 i_1} \circ s = \iota_{i_0} \circ s = 1_B$, so the composite $A_{i_1} \xrightarrow{\iota_{i_1}} B \xrightarrow{s} A_{i_0} \xrightarrow{f_{i_0 i_1}} A_{i_1} \xrightarrow{\iota_{i_1}} B = \varinjlim_i A_i$ is equal to ι_{i_1} , so as before, $f_{i_0 i_1} \circ s \circ \iota_{i_1}, 1_{A_{i_1}} : A_{i_1} \rightarrow A_{i_1}$ are already equal when composed with $f_{i_1 i_2} : A_{i_1} \rightarrow A_{i_2}$, i.e., we have $f_{i_0 i_2} \circ s \circ \iota_{i_1} = f_{i_1 i_2} : A_{i_1} \rightarrow A_{i_2}$, for some $i_2 \geq i_1$. Continue finding $i_2 \leq i_3 \leq \dots$ in this manner, such that for each j , we have

$$f_{i_0 i_{j+1}} \circ s \circ \iota_{i_j} = f_{i_j i_{j+1}} : A_{i_j} \rightarrow A_{i_{j+1}}.$$

Let $\iota'_j : A_{i_j} \rightarrow \varinjlim_j A_{i_j}$ be the cocone maps. We have a colimit comparison map $g : \varinjlim_j A_{i_j} \rightarrow \varinjlim_i A_i = B$, such that $g \circ \iota'_j = \iota_{i_j}$, as well as a map $h := \iota'_0 \circ s : B \rightarrow \varinjlim_j A_{i_j}$. We have $g \circ h = g \circ \iota'_0 \circ s = \iota_{i_0} \circ s = 1_B$, and conversely, $h \circ g = 1_{\varinjlim_j A_{i_j}}$ because for each j , we have $h \circ g \circ \iota'_j = \iota'_0 \circ s \circ \iota_{i_j} = \iota'_{j+1} \circ f_{i_0 i_{j+1}} \circ s \circ \iota_{i_j} = \iota'_{j+1} \circ f_{i_j i_{j+1}} = \iota'_j$. Thus $g^{-1} = h : B \cong \varinjlim_j A_{i_j}$. \square

Proof of Proposition A.1.6. Write $G(B) \in \mathcal{C}$ as a κ -directed colimit $\varinjlim_{i \in I} A_i$, for a κ -directed poset I , of κ -presented objects $A_i \in \mathcal{C}_\kappa$ and morphisms $f_{ij} : A_i \rightarrow A_j$ for $i \leq j$ (e.g., $A_i = \langle G_i \mid R_i \rangle$, for all κ -ary $G_i \subseteq G(A)$ and κ -ary sets R_i of relations which hold between them in $G(A)$). Then the adjunction counit $F(G(B)) \cong \varinjlim_{i \in I} F(A_i) \rightarrow B$ is an isomorphism. Since G preserves κ -directed colimits, each $F(A_i) \in \mathcal{D}_\kappa$. By Lemma A.1.8, B is a retract of some $F(A_{i_0})$, as well as isomorphic to $\varinjlim_j F(A_{i_j}) \cong F(\varinjlim_j A_{i_j})$ for some $i_0 \leq i_1 \leq \dots$; if κ is uncountable, then $\varinjlim_j A_{i_j} \in \mathcal{C}_\kappa$. \square

Many naturally occurring adjunctions between locally κ -presentable categories are not manifestly free/forgetful. For example, the usual forgetful functor $\mathbf{Bool} \rightarrow \mathbf{DLat}$ also has a *right* adjoint, taking each distributive lattice A to the Boolean subalgebra A_- of complemented elements. However, up to modifying the theories axiomatizing the categories involved, it is always possible to regard such adjunctions as free/forgetful. In this example, we may regard a distributive lattice A as a **two-sorted structure**, with its usual underlying set A as well as A_- as an additional “underlying set”, with the inclusion $A_- \hookrightarrow A$ regarded as a unary operation. The resulting class of structures is still ω -limit axiomatizable; the key point is that complements are uniquely defined by an ω -ary conjunction of atomic formulas $(x \wedge y = \perp) \wedge (x \vee y = \top)$. We can then regard $A \mapsto A_-$ as forgetting the usual underlying set A ; the usual forgetful functor $\mathbf{Bool} \rightarrow \mathbf{DLat}$ then takes $B \in \mathbf{Bool}$ to the “free distributive lattice A generated by $B \subseteq A_-$ ”. Any adjunction between locally presentable categories can be described as free/forgetful in a similar manner, so that the concrete constructions in terms of presentations described above can in fact be applied to arbitrary adjunctions.

For limit theories over a *large* signature, one can try to develop the above notions in the same way; the only issue is that the transfinite construction of presented structures may result in a proper class, hence small-presented structures need not be small. If it is known (by some other means) that small-presented structures are always small, e.g., in the case of \mathbf{VLat} or \mathbf{Frm} (by Corollary 2.3.1), then the above facts all go through.

On the other hand, for example, the result of Gaifman–Hales (Theorem 2.1.2) implies that the forgetful functor $\mathbf{CBool} \rightarrow \mathbf{Set}$ does not have a left adjoint. In this case, we denote the category of large models in all-caps, e.g., \mathbf{CBOOL} is the category of large complete Boolean algebras, and let $\mathbf{CBOOL}_\infty \subseteq \mathbf{CBOOL}$ denote the full subcategory of small-presented structures; these are all the large structures we will ever need in this paper. Note that \mathbf{CBOOL} , being a collection of possibly proper classes, is an “extra-large” category (hence would require extra foundational care to work with rigorously); however, $\mathbf{CBOOL}_\infty \subseteq \mathbf{CBOOL}$ is only a large category, since we may equivalently

define it to consist of small presentations, together with homomorphisms between the presented structures which are determined by their values on generators.

A.2 Algebraic theories and monadic categories

Categories of structures axiomatized by algebraic theories (in the sense of Example A.1.3) over a small signature are **monadic over Set**.¹⁹ See [Bor, II Ch. 2–4] for a detailed reference. In such categories, in addition to the notions from the preceding subsection, the following universal-algebraic notions can be categorically formulated.

Recall that a **monomorphism** in an arbitrary category \mathbf{C} is a morphism f such that $f \circ g = f \circ h \implies g = h$; the dual notion is an **epimorphism**. A **subobject** of an object $X \in \mathbf{C}$ is an equivalence class of monomorphisms $A \rightarrow X$ with respect to the preorder

$$(f : A \hookrightarrow X) \subseteq (g : B \hookrightarrow X) :\iff \exists h : A \rightarrow B (f = g \circ h)$$

(such an h is necessarily unique); this preorder descends to a partial order on the **poset of subobjects** $\text{Sub}(X) = \text{Sub}_{\mathbf{C}}(X)$. It is common to abuse notation regarding monomorphisms: we often treat single monomorphisms $A \rightarrow X$ as if they were subobjects, denoted $A \subseteq X$, and refer to the representing monomorphism as the **inclusion**. In \mathbf{C} monadic over **Set**, monomorphisms are precisely the embeddings of subalgebras; hence, $\text{Sub}(X) \cong \{\text{subalgebras of } X\}$.²⁰

For a morphism $f : X \rightarrow Y$ in an arbitrary category \mathbf{C} with finite limits, its **kernel** is the pullback $\ker(f) := X \times_Y X$ of f with itself. It is an instance of a **congruence** (also known as **equivalence relation**) on X , meaning a subobject $\sim \subseteq X^2$ which internally satisfies the axioms of an equivalence relation. In \mathbf{C} monadic over **Set**, these correspond to congruences in the usual universal-algebraic sense (i.e., equivalence relations which are also subalgebras).

Given a congruence $\sim \subseteq X^2$, we may take the coequalizer of the projections $\pi_1, \pi_2 : \sim \rightrightarrows X$. In \mathbf{C} monadic over **Set**, this is the usual **quotient algebra** X/\sim . The quotient map $q : X \twoheadrightarrow X/\sim$ has kernel \sim , and is a **regular epimorphism**, meaning that it is the coequalizer of its kernel (which implies that it is an epimorphism). So we have a bijection

$$\{\text{congruences on } X\} \xrightleftharpoons[\text{ker}]{\text{coeq}} \{\text{equiv. classes of regular epimorphisms from } X\} =: \text{RSub}_{\mathbf{C}^{\text{op}}}(X) \subseteq \text{Sub}_{\mathbf{C}^{\text{op}}}(X).$$

Any morphism $f : X \rightarrow Y \in \mathbf{C}$ factors uniquely-up-to-unique-isomorphism into a regular epimorphism $X \twoheadrightarrow X/\ker(f)$ followed by a monomorphism $X/\ker(f) \hookrightarrow Y$; the subobject of Y represented by this monomorphism is called the **image** of f . Moreover, regular epimorphisms are closed under arbitrary products, as well as under pullback across arbitrary morphisms; thus these same operations also preserve image factorizations of arbitrary morphisms.

Categories satisfying the above compatibility conditions between subobjects, congruences, and regular epimorphisms are called **(Barr-)exact completely regular** [CV98]. Thus, categories monadic over **Set** are exact completely regular. Moreover, in addition to limits, also images and coequalizers of congruences are computed as for the underlying sets (unlike in arbitrary locally presentable categories, e.g., the quotient of $\{0 < 1 < 2\}$ by $0 \sim 2$ in **Pos**).

¹⁹Multi-sorted algebraic theories are monadic over \mathbf{Set}^S where S is the set of sorts.

²⁰In categories of models of arbitrary limit theories, monomorphisms are injective homomorphisms, not necessarily embeddings (which are not an intrinsic categorical notion, since they depend on the choice of signature).

Presented algebras $\langle G \mid R \rangle$ may be computed in a single step, as the quotient of the free algebra $\langle G \rangle$ by the congruence generated by R (where each equation between terms in R is identified with a pair of elements in $\langle G \rangle$); there is no need for the transfinite iteration described in the preceding subsection (except in order to generate the congruence). It follows that an algebraic theory over a large signature defines a well-behaved category \mathbf{C} as soon as we know that free algebras $\langle G \rangle$ over small generating sets G are small, i.e., that the left adjoint to the forgetful functor $\mathbf{C} \rightarrow \mathbf{Set}$ exists; this then implies that small-presented algebras are small and \mathbf{C} is monadic over \mathbf{Set} , so all of the facts in this and the preceding section go through. For example, this applies to the categories $\mathbf{VLat}, \mathbf{Frm}$ (Corollary 2.3.1).

We will need the following standard technical facts:

Lemma A.2.1. *Let A be a κ -generated algebra for some algebraic theory, and let \sim be a congruence on A such that the quotient algebra A/\sim has a presentation with $< \kappa$ -many relations. Then \sim is a κ -generated congruence.*

Proof. Let $G \subseteq A$ be a κ -ary generating set, let $p : A \twoheadrightarrow A/\sim$ be the quotient map, and let $A/\sim = \langle H \mid S \rangle$ be some presentation with $|S| < \kappa$, so that $S \subseteq \langle H \rangle^2$ generates the kernel of the quotient map $q : \langle H \rangle \twoheadrightarrow A/\sim$. For each $g \in G$, pick some $u(g) \in A$ such that $p(g) = q(u(g)) \in A/\sim$, and for each $h \in H$, pick some $v(h) \in A$ such that $q(h) = p(v(h)) \in A/\sim$. Extend v to a homomorphism $v : \langle H \rangle \rightarrow A$; by definition of v on generators, v lifts q along p :

$$\begin{array}{ccc} \langle H \rangle & \xrightarrow{v} & A \\ & \searrow q & \downarrow p \\ & & A/\sim \end{array}$$

Then \sim is generated by

$$R := v(S) \cup \{(g, v(u(g))) \mid g \in G\}.$$

Indeed, R is contained in $\sim = \ker(p)$: $v(S) \subseteq v(\ker(q))$ is, because $q = p \circ v$; and for $g \in G$, we have $p(v(u(g))) = q(u(g)) = p(g)$. Conversely, it follows from $q = p \circ v$ that A/\sim is the quotient of the image $v(\langle H \rangle) \subseteq A$ by $v(\ker(q))$ which is the congruence generated by $v(S)$; while the extra pairs $(g, v(u(g)))$ for $g \in G$ ensure that every element of A is identified with some element of $v(\langle H \rangle)$. \square

Corollary A.2.2. *Let A be an algebra for some algebraic theory. Suppose A has $< \kappa$ -many generators $G \subseteq A$, as well as a presentation with $< \kappa$ -many relations. Then A has a κ -ary presentation using the generators G and $< \kappa$ -many relations.*

Proof. Apply Lemma A.2.1 to the kernel of $\langle G \rangle \twoheadrightarrow A$. \square

Lemma A.2.3. *Let \mathbf{C} be the category of algebras for some κ -algebraic theory. Then a morphism $f : X \rightarrow Y \in \mathbf{C}_\kappa$ between κ -presented algebras is a monomorphism, epimorphism, or regular epimorphism, respectively, in \mathbf{C}_κ iff it is so in \mathbf{C} .*

Proof. Since the inclusion $\mathbf{C}_\kappa \subseteq \mathbf{C}$ is faithful, it reflects monos and epis. Since $\mathbf{C}_\kappa \subseteq \mathbf{C}$ is closed under finite colimits, hence cokernels and coequalizers, epis and regular epis in \mathbf{C}_κ are also so in \mathbf{C} . If $f : X \rightarrow Y \in \mathbf{C}_\kappa$ is a mono, then since the free algebra $\langle 1 \rangle_{\mathbf{C}}$ on one generator is in \mathbf{C}_κ , we get that the underlying function of sets $\mathbf{C}_\kappa(\langle 1 \rangle_{\mathbf{C}}, f)$ is injective, whence f is a mono in \mathbf{C} . Finally, if

$f : X \rightarrow Y \in \mathbf{C}_\kappa$ is a regular epi in \mathbf{C} , i.e., a surjective homomorphism, then by Lemma A.2.1 its kernel is a κ -generated congruence $\sim \subseteq X^2$, whence letting $(g, h) : \langle Z \rangle_{\mathbf{C}} \rightarrow X^2$ for some κ -ary Z map generators to generators of \sim , we have that f is the coequalizer of $g, h : \langle Z \rangle_{\mathbf{C}} \rightrightarrows X \in \mathbf{C}_\kappa$. \square

Lemma A.2.4. *Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a finite-limit-preserving functor between categories with finite limits.*

- (a) *F is faithful iff for each $X \in \mathbf{C}$, F induces an embedding $\mathbf{RSub}_{\mathbf{C}}(X) \rightarrow \mathbf{RSub}_{\mathbf{D}}(F(X))$.*
- (b) *F is conservative iff for each $X \in \mathbf{C}$, F induces an embedding $\mathbf{Sub}_{\mathbf{C}}(X) \rightarrow \mathbf{Sub}_{\mathbf{D}}(F(X))$.*

Proof. (a) (\implies) Let $A, B \subseteq X$ be regular subobjects, the equalizers of parallel pairs $X \rightrightarrows Y$ and $X \rightrightarrows Z$ respectively, such that $F(A) \subseteq F(B) \subseteq F(X)$. Then $F(A)$ equalizes $F(X) \rightrightarrows F(Z)$, whence by faithfulness, A equalizes $X \rightrightarrows Z$, whence $A \subseteq B$.

(a) (\impliedby) Let $f, g : X \rightrightarrows Y \in \mathbf{C}$ with $F(f) = F(g) \in \mathbf{D}$. Then $F(\text{eq}(f, g)) = \text{eq}(F(f), F(g)) \subseteq F(X)$ is all of $F(X)$, whence since $F : \mathbf{RSub}_{\mathbf{C}}(X) \rightarrow \mathbf{RSub}_{\mathbf{D}}(F(X))$ is injective, $\text{eq}(f, g)$ is all of X , whence $f = g$.

(b) (\implies) Since F preserves finite limits, F induces a \wedge -lattice homomorphism $F : \mathbf{Sub}_{\mathbf{C}}(X) \rightarrow \mathbf{Sub}_{\mathbf{D}}(F(X))$ for each $X \in \mathbf{C}$. Thus by taking meets, it suffices to check that whenever $A \subseteq B \subseteq X$ with $F(A) = F(B) \subseteq F(X)$, then $A = B \subseteq X$. This follows from conservativity.

(b) (\impliedby) In any category with finite limits, morphisms $f : X \rightarrow Y$ are in bijection with their **graphs** $G \subseteq X \times Y$, which are arbitrary subobjects such that the first projection $G \rightarrow X$ is an isomorphism. So for any $f : X \rightarrow Y \in \mathbf{C}$ such that $F(f)$ is invertible, letting $G \subseteq X \times Y$ be the graph of f , the second projection $F(G) \rightarrow F(Y)$ must be an isomorphism. In particular, it is monic, i.e., its kernel is the diagonal $F(G) \hookrightarrow F(G) \times F(G) \cong F(G \times G)$, whence since $F : \mathbf{Sub}_{\mathbf{C}}(G \times G) \rightarrow \mathbf{Sub}_{\mathbf{D}}(F(G \times G))$ is injective, so must be the kernel of the second projection $G \rightarrow Y$, which is thus monic. Now regarding G as a subobject of Y , since $F : \mathbf{Sub}_{\mathbf{C}}(Y) \rightarrow \mathbf{Sub}_{\mathbf{D}}(F(Y))$ is injective and $F(G) \cong F(Y)$, we have $G \cong Y$, i.e., f is invertible. \square

A.3 Ordered theories and categories

A **locally ordered category** is a category \mathbf{C} , each of whose hom-sets is equipped with a partial order, such that composition is order-preserving on both sides. In other words, it is a category **enriched** over the category \mathbf{Pos} of posets. See [Kel82] for the standard reference on general enriched category theory. There are also enriched analogs of locally presentable and monadic categories; see [Kel82a], [Pow99]. In this subsection, we give an elementary description of the locally ordered case, in syntactic terms largely analogous to the previous two subsections. The basic idea will be to “replace $=$ with \leq throughout”.

Let \mathcal{L} be an infinitary first-order signature in the usual sense. An (infinitary first-order) **ordered \mathcal{L} -formula** will mean an $(\mathcal{L} \sqcup \{\leq\})$ -formula in the usual sense, where \leq is a binary relation symbol. Equivalently, we may think of \leq replacing $=$ as the only extra allowed relation symbol, by treating $x = y$ as an abbreviation for $(x \leq y) \wedge (y \leq x)$. An **ordered \mathcal{L} -structure** will mean an \mathcal{L} -structure in the usual sense, but with the underlying set replaced by an underlying poset, and with all function symbols required to be interpreted as order-preserving maps (from the product partial order). We define the interpretation of ordered \mathcal{L} -formulas in an ordered \mathcal{L} -structure in the expected manner, with \leq always interpreted as the partial order in the underlying poset.

To define ordered κ -limit axioms, we need to replace the $\exists!$ quantifier in ordinary κ -limit axioms, which involves a hidden $=$ to assert uniqueness, with an ordered analog. Note that a κ -limit axiom

$$\forall \vec{x} \left(\bigwedge_i \phi_i(\vec{x}) \rightarrow \exists! \vec{y} \bigwedge_j \psi_j(\vec{x}, \vec{y}) \right)$$

is equivalent to the combination of the two axioms

$$\begin{aligned} \forall \vec{x}, \vec{x}', \vec{y}, \vec{y}' \left(\bigwedge_i \phi_i(\vec{x}) \wedge \bigwedge_i \phi_i(\vec{x}') \wedge \bigwedge_j \psi_j(\vec{x}, \vec{y}) \wedge \bigwedge_j \psi_j(\vec{x}', \vec{y}') \wedge (\vec{x} = \vec{x}') \rightarrow (\vec{y} = \vec{y}') \right), \\ \forall \vec{x} \left(\bigwedge_i \phi_i(\vec{x}) \rightarrow \exists \vec{y} \bigwedge_j \psi_j(\vec{x}, \vec{y}) \right). \end{aligned}$$

By an **ordered κ -limit theory**, we mean a set of pairs of axioms of the form

$$\begin{aligned} \forall \vec{x}, \vec{x}', \vec{y}, \vec{y}' \left(\bigwedge_i \phi_i(\vec{x}) \wedge \bigwedge_i \phi_i(\vec{x}') \wedge \bigwedge_j \psi_j(\vec{x}, \vec{y}) \wedge \bigwedge_j \psi_j(\vec{x}', \vec{y}') \wedge (\vec{x} \leq \vec{x}') \rightarrow (\vec{y} \leq \vec{y}') \right), \\ \forall \vec{x} \left(\bigwedge_i \phi_i(\vec{x}) \rightarrow \exists \vec{y} \bigwedge_j \psi_j(\vec{x}, \vec{y}) \right) \end{aligned}$$

(where the ϕ_i and ψ_j are atomic and the variable tuples and \wedge 's are κ -ary). Clearly, these imply the previous two axioms with $=$. We will refer to the latter pair of axioms as a single **ordered κ -limit axiom**, and abbreviate it by

$$\forall \vec{x} \left(\bigwedge_i \phi_i(\vec{x}) \rightarrow \exists!_{\leq}^{\vec{x}} \vec{y} \bigwedge_j \psi_j(\vec{x}, \vec{y}) \right)$$

where $\exists!_{\leq}^{\vec{x}} \vec{y}$ is read “there exists unique \vec{y} , monotone in \vec{x} , s.t.”; but note that $\exists!_{\leq}^{\vec{x}}$ has no independent meaning in a formula. The category of all models of an ordered κ -limit theory, together with all homomorphisms (which are required to be monotone), is called a **locally κ -presentable locally ordered category**. These are in particular locally κ -presentable categories in the ordinary sense, since any ordered κ -limit theory over \mathcal{L} is an ordinary κ -limit theory over $\mathcal{L} \sqcup \{\leq\}$.

Example A.3.1. An ordered κ -limit axiom with empty \vec{y} is just an ordinary κ -Horn axiom.

Example A.3.2. The axiom $\forall \vec{x} \exists!_{\leq}^{\vec{x}} \vec{y} \bigwedge_j \psi_j(\vec{x}, \vec{y})$ says that $\bigwedge_j \psi_j$ is the graph of a monotone function. One can similarly axiomatize ordered structures with partial monotone operations, as in Example A.1.5. For example, the axioms

$$\begin{aligned} \forall x, y, z (J(x, y, z) \rightarrow (x \leq z) \wedge (y \leq z) \wedge (x \wedge y = \perp)), \\ \forall x, y, z, w (J(x, y, z) \wedge (x \leq w) \wedge (y \leq w) \rightarrow (z \leq w)), \\ \forall x, y ((x \wedge y = \perp) \rightarrow \exists!_{\leq}^{x, y} z J(x, y, z)) \end{aligned}$$

axiomatize disjunctive ω -frames, where J is the graph of the partial join operation.

Example A.3.3. Boolean algebras *cannot* be axiomatized by an ordered ω -limit theory. We cannot include \neg in the signature, since \neg is not a monotone function in Boolean algebras. If we were allowed ordinary limit axioms, we could say $\forall x \exists! y ((x \wedge y = \perp) \wedge (x \vee y = \top))$; but the corresponding ordered limit axiom is false in Boolean algebras, again because \neg is not monotone.

In locally ordered categories \mathbf{C} , one has **weighted limits and colimits** (called **indexed** in older literature), which are universal objects equipped with morphisms to/from a diagram satisfying specified inequalities rather than (or in addition to) equations; see [Kel82, Ch. 3], [Joh02, B1.1.3]. We will not need the general notion, but will use the following particular types of weighted limits:

- The **inserters** $\text{ins}(f, g)$ of a parallel pair of morphisms $f, g : A \rightrightarrows B$ is (if it exists) a universal object equipped with a morphism $\pi : \text{ins}(f, g) \rightarrow A$ such that $f \circ \pi \leq g \circ \pi$. Thus, inserters are ordered analogs of equalizers.

If inserters exist, then so do equalizers, via $\text{eq}(f, g) = \text{ins}(g \circ \pi, f \circ \pi : \text{ins}(f, g) \rightrightarrows B)$.

In **Pos**, inserters are given by $\text{ins}(f, g) = \{a \in A \mid f(a) \leq g(a)\}$.

- The **comma object** $A \downarrow_C B$ of two objects A, B equipped with morphisms $f : A \rightarrow C$ and $g : B \rightarrow C$ to a third object is (if it exists) a universal object equipped with morphisms $\pi_1 : A \downarrow_C B \rightarrow A$ and $\pi_2 : A \downarrow_C B \rightarrow B$ such that $f \circ \pi_1 \leq g \circ \pi_2$. These are ordered analogs of fiber products.

In **Pos**, comma objects are given by $A \downarrow_C B = \{(a, b) \in A \times B \mid f(a) \leq g(b)\}$.

- The **power** A^I of an object A by a poset I is (if it exists) a universal object equipped with a monotone I -indexed family of morphisms $(\pi_i)_{i \in I} : I \rightarrow \mathbf{C}(A^I, A)$. These are ordered analogs of ordinary powers (iterated products).

In **Pos**, powers are given by $A^I = \mathbf{Pos}(I, A)$ with the pointwise partial ordering.

In particular, when $I = \mathbb{S} = \{0 < 1\}$, we call $A^{\mathbb{S}} =: \leq_A$ the **(internal) (partial) order on** A ; it plays a role analogous to the diagonal $A \hookrightarrow A^2$ in ordinary categories.

Duals of these are called **coinserter**s, denoted $\text{coins}(f, g)$, **cocomma objects**, and **copowers** (or sometimes **tensors**). Much as ordinary limits may be constructed from products and equalizers, so too may weighted limits in locally ordered categories be constructed from products and inserters (or from products and comma objects, or from products, equalizers, and powers by \mathbb{S}); dually for weighted colimits. Locally presentable locally ordered categories have arbitrary small weighted limits and colimits; weighted limits of models of ordered κ -limit theories are constructed as in **Pos**.

Example A.3.4. If we regard **Bool** as locally discretely ordered, then it has inserters coinciding with equalizers; note that these are *not* computed as in **Pos**. This is another reflection of the fact (Example A.3.3) that **Bool** is not axiomatizable by an ordered limit theory.

Presentations of models of ordered κ -limit theories are the same as in the unordered context (of course, relations may now use \leq), and may be used to construct colimits as well as left adjoints of forgetful functors (which are locally order-preserving on morphisms). The notion of λ -presentability is also the same as before, and is preserved by coinserter, cocomma objects, and copowers by λ -ary posets. The same caveats as before regarding large signatures continue to apply.

By an **ordered algebraic theory**, we will mean an ordered limit theory whose signature has no relation symbols and axioms have empty left-hand side and no existentials, in analogy with Example A.1.3. Note that \leq may still be used; hence, these do not reduce to unordered algebraic theories (unless there are operations like \wedge, \vee which may be used to define \leq in terms of $=$). Locally ordered categories axiomatized by small ordered algebraic theories admit the following ordered universal-algebraic notions (see [Str82], where they are defined more generally for 2-categories).²¹

²¹These categories are strongly (i.e., **Pos**-enriched) monadic over **Pos**; see [Pow99]. However, unlike in the unordered setting, not every category strongly monadic over **Pos** is axiomatized by an ordered algebraic theory (in our sense): in general, one would need to allow operations whose “arity” is a poset, i.e., partial operations with domain defined by a system of inequalities. Categories axiomatized by such theories are less well-behaved: they do not admit a good notion of order-congruence which corresponds to surjective homomorphisms.

An **order-monomorphism** will mean a morphism f such that $f \circ g \leq f \circ h \implies g \leq h$. In \mathbf{Pos} , these are order-embeddings, whereas monomorphisms are injective monotone maps. An **order-embedded subobject** will mean a subobject which is represented by an order-monomorphism. Let $\mathbf{OSub}(X) = \mathbf{OSub}_{\mathbf{C}}(X) \subseteq \mathbf{Sub}(X)$ denote the poset of order-embedded subobjects. Then $\mathbf{OSub}_{\mathbf{C}}(X) \cong \{\text{subalgebras of } X\}$ (which is $\cong \mathbf{Sub}_{\mathbf{C}}(X)$ if \mathbf{C} is also monadic over \mathbf{Set} , e.g., $\mathbf{C} = \mathbf{VLat}$; but $\mathbf{Sub}_{\mathbf{Pos}}(X)$ consists of subsets of X equipped with a finer partial order, for example).

The **order-kernel** of $f : X \rightarrow Y$ will mean the comma object

$$\text{oker}(f) := (X \downarrow_Y X) = \{(x, x') \in X^2 \mid f(x) \leq f(x')\}.$$

Thus $\text{oker}(f)$ is an order-embedded subobject of X^2 containing $\ker(f)$, and is an **order-congruence** on X , meaning a subobject $\lesssim \subseteq X^2$ which is transitive and contains the partial order \leq_X on X (which implies reflexivity).

Conversely, given any order-congruence \lesssim , let $\sim := \lesssim \cap \lesssim^{-1}$ denote its symmetric part, which is a congruence (in the unordered sense) on X . Then \lesssim descends to a partial order on the quotient algebra X/\sim , which is characterized as the coinserter of the projections $\pi_1, \pi_2 : \lesssim \rightrightarrows X$, meaning that any homomorphism $f : X \rightarrow Y$ such that $x \lesssim x' \implies f(x) \leq f(x')$, i.e., $\lesssim \subseteq \text{oker}(f)$, descends to a monotone homomorphism $X/\sim \rightarrow Y$. The quotient map $q : X \twoheadrightarrow X/\sim$ is an **order-regular epimorphism**, meaning the coinserter of its order-kernel. So we have a bijection

$$\{\text{order-congruences on } X\} \begin{matrix} \xrightarrow{\text{coins}} \\ \xleftarrow{\text{oker}} \end{matrix} \{\text{equiv. classes of order-regular epimorphisms from } X\}.$$

Any morphism $f : X \rightarrow Y \in \mathbf{C}$ factors uniquely-up-to-unique-isomorphism into an order-regular epimorphism $X \twoheadrightarrow X/\ker(f)$ followed by an order-monomorphism $X/\ker(f) \hookrightarrow Y$; the order-embedded subobject of Y represented by this monomorphism is called the **image** of f . Moreover, order-regular epimorphisms are closed under products and pullback across arbitrary morphisms.

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