Clones of Borel Boolean Functions

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Abstract

We study the lattice of all *Borel clones* on $2 = \{0, 1\}$: classes of Borel functions $f : 2^n \to 2$, $n \le \omega$, which are closed under composition and include all projections. This is a natural extension to countable arities of Post's 1941 classification of all clones of finitary Boolean functions. Every Borel clone restricts to a finitary clone, yielding a "projection" from the lattice of all Borel clones to Post's lattice. It is well-known that each finitary clone of affine mod 2 functions admits a unique extension to a Borel clone. We show that over each finitary clone containing either the 2-out-of-3 median operation, or both $\wedge$ and $\vee$, there lie at least 2 but only finitely many Borel clones. Over the remaining clones in Post's lattice, we give only a partial classification of the Borel extensions, and present some evidence that the full structure may be quite complicated.

Contents

Post's lattice diagram labels:

$\mathbb{O}_2^{<\omega} = \mathrm{Pol}\,\varnothing = \langle \wedge, \vee, 0, 1, \neg \rangle$

$\mathbb{T}_0 = \mathrm{Pol}\{0\} = \langle \vee, \nrightarrow \rangle$ · $\mathbb{M} = \mathrm{Pol}\{\leq\} = \langle \wedge, \vee, 0, 1 \rangle$ · $\mathbb{T}_1 = \mathrm{Pol}\{1\} = \langle \wedge, \rightarrow \rangle$

$\mathbb{MT}_0 = \langle \wedge, \vee, 0 \rangle$ · $\mathbb{T}_0\mathbb{T}_1 = \langle ?: \rangle = \langle \vee, \lceil \rightarrow \rceil \rangle = \langle \wedge, \lfloor \nrightarrow \rfloor \rangle$ · $\mathbb{MT}_1 = \langle \wedge, \vee, 1 \rangle$

$\mathbb{T}_{0,2} = \mathrm{Pol}\{\wedge = 0\}$ · $\mathbb{T}_{1,2} = \mathrm{Pol}\{\vee = 1\}$

$\mathbb{MT}_{0,2}$ · $\mathbb{T}_{0,2}\mathbb{T}_1$ · $\mathbb{MT}_0\mathbb{T}_1 = \langle \wedge, \vee \rangle$ · $\mathbb{T}_0\mathbb{T}_{1,2}$ · $\mathbb{MT}_{1,2}$

$\mathbb{T}_{0,3} = \mathrm{Pol}\{\wedge^3 = 0\}$ · $\mathbb{T}_{1,3} = \mathrm{Pol}\{\vee^3 = 1\}$

$\mathbb{MT}_{0,3}$ · $\mathbb{T}_{0,3}\mathbb{T}_1$ · $\mathbb{MT}_{0,2}\mathbb{T}_1 = \langle \forall_2^3, \wedge \rangle$ · $\mathbb{D} = \mathrm{Pol}\{\neg\} = \langle \exists_2^3, \neg \rangle$ · $\langle \exists_2^3, \vee \rangle = \mathbb{MT}_0\mathbb{T}_{1,2}$ · $\mathbb{T}_0\mathbb{T}_{1,3}$ · $\mathbb{MT}_{1,3}$

$\mathbb{MT}_{0,3}\mathbb{T}_1 = \langle \forall_2^4 \rangle$ · $\langle \exists_2^3, +^3 \rangle = \mathbb{DT}_0 = \mathbb{DT}_1$ · $\langle \exists_2^4 \rangle = \mathbb{MT}_0\mathbb{T}_{1,3}$

$\mathbb{T}_{0,<\omega} = \langle \nrightarrow \rangle = \langle \lceil \neg \rceil \rangle$ · $\langle \exists_2^3 \rangle = \mathbb{DM}$ · $\mathbb{A} = \mathrm{Pol}\{+^3\}$ · $\mathbb{T}_{1,<\omega} = \langle \rightarrow \rangle$

$\mathbb{T}_{0,<\omega}\mathbb{T}_1 = \langle \lceil \rightarrow \rceil \rangle = \langle \lceil \lfloor \neg \rfloor \rceil \rangle$ · $\langle \lfloor \nrightarrow \rfloor \rangle = \mathbb{T}_0\mathbb{T}_{1,<\omega}$

$\mathbb{MT}_{0,<\omega}$ · $\mathrm{Pol}\{+\} = \langle + \rangle = \mathbb{AT}_0$ · $\mathbb{AD}$ · $\langle \leftrightarrow \rangle = \mathbb{AT}_1$ · $\mathbb{MT}_{1,<\omega}$

$\mathbb{MT}_{0,<\omega}\mathbb{T}_1 = \langle \lceil \vee \rceil \rangle$ · $\mathbb{MT}_0\mathbb{T}_{1,<\omega}$

$\mathbb{ADT}_0 = \mathbb{ADT}_1 = \langle +^3 \rangle$

$\Lambda = \mathrm{Pol}\{\wedge\} = \langle \wedge, 0, 1 \rangle$ · $\langle \neg, 0 \rangle$ · $\mathbb{V} = \mathrm{Pol}\{\vee\} = \langle \vee, 0, 1 \rangle$

$\Lambda\mathbb{T}_0 = \langle \wedge, 0 \rangle$ · $\Lambda\mathbb{T}_1 = \langle \wedge, 1 \rangle$ · $\mathbb{VT}_0 = \langle \vee, 0 \rangle$ · $\mathbb{VT}_1 = \langle \vee, 1 \rangle$

$\Lambda\mathbb{T}_0\mathbb{T}_1 = \langle \wedge \rangle$ · $\langle \neg \rangle$ · $\langle \vee \rangle = \mathbb{VT}_0\mathbb{T}_1$

$\langle 0, 1 \rangle$

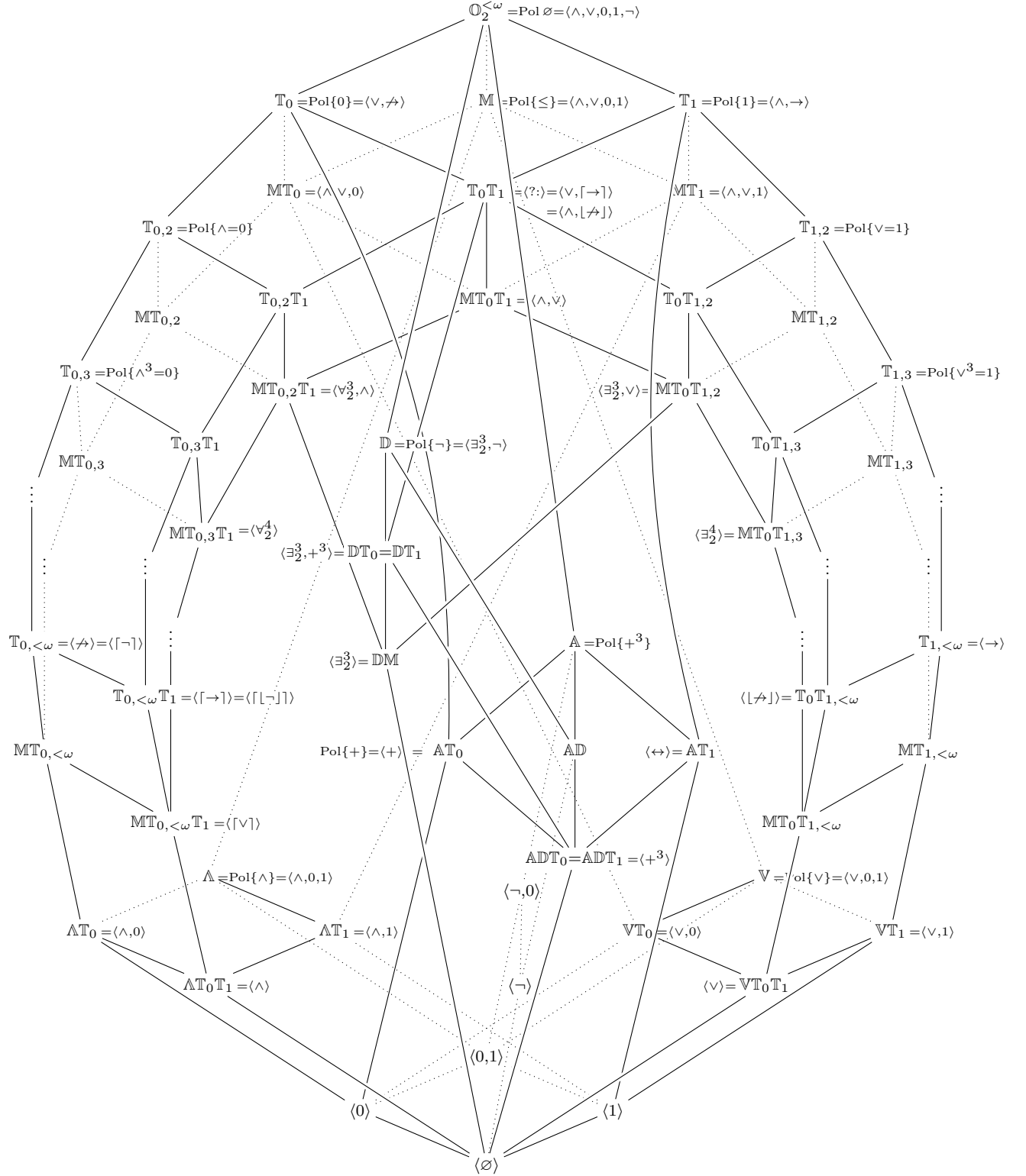$\langle 0 \rangle$ · $\langle 1 \rangle$

$\langle \varnothing \rangle$

Figure 1.1: Post's lattice $\mathcal{Clo}^{<\omega}(2)$ of all clones of finitary Boolean functions $2^n \to 2$. See Section 3 for the definitions of these clones, functions, and relations. (Here, all clones are implicitly restricted to finitary.)

1 Introduction

It is a standard exercise in propositional logic that the logical connectives $\wedge, \vee, \neg$ can express all Boolean functions $2^n \to 2$ of finite arity; in fact $\wedge, \neg$ already suffice. On the other hand, $\wedge, \vee$ do not suffice to generate all Boolean functions, for they are both "positive" connectives; and in fact together with the constant truth values $1, 0$, they generate precisely all monotone Boolean functions. Leaving out the constants $1, 0$, we obtain only the monotone functions fixing $1, 0$, etc. In 1941, Post [Pos41] gave a complete classification of all possible classes of finitary Boolean functions $2^n \to 2$ generated by a class of functions under composition, known as **clones** on 2; these may be thought of as all "sublogics" of classical finitary 2-valued propositional logic. The resulting countable lattice of all clones on 2, known as **Post's lattice**, is depicted in Figure 1.1. For background on Post's lattice and clone theory, see [FMMT22], [Lau06], [Sze86].

In this paper, we consider the analogous problem of classifying all clones of *countable* arity Boolean functions $2^n \to 2$, $n \le \omega$; these may be thought of as "sublogics" of countably infinitary propositional logic. A Boolean function $2^n \to 2$ may be obtained as a composition of the countable connectives $\bigwedge, \bigvee$ and $\neg$ iff it is a **Borel** function, i.e., the indicator function of a Borel set $A \subseteq 2^n$. Let $\mathbb{O}_2^{\mathsf{B}} \subseteq \bigsqcup_{n \le \omega} 2^{2^n}$ denote the clone of all Borel functions. We will mostly restrict attention to subclones of $\mathbb{O}_2^{\mathsf{B}}$, thereby ruling out pathological "connectives" such as nonprincipal ultrafilters which cannot be explicitly defined. Note that definability constraints such as being Borel can interact with the algebraic structure on 2 in intricate ways. For example, a Borel function $2^n \to 2$ of countable arity which is affine over $\mathbb{Z}/2\mathbb{Z}$ can in fact depend only on finitely many variables; but this is no longer true for any condition weaker (in a precise sense; see below) than affinity.

Even restricted to the Borel clones on 2, we do not obtain a complete classification in this paper. However, we obtain a classification of a large "region" of the lattice of all Borel clones, as well as a partial classification of the remaining "regions" along with some indications that they may be difficult to fully classify. In order to state our results more precisely, we now give an overview of our approach, which is based on Post's classification of the finitary clones.

1.A The bundle of Borel clones

Let $\mathbb{O}_2^{<\omega} := \bigsqcup_{n < \omega} 2^{2^n}$ denote the clone of all finitary functions. Thus Post's lattice 1.1 consists of all finitary subclones of $\mathbb{O}_2^{<\omega}$, while we seek to classify all countable-arity subclones of $\mathbb{O}_2^{\mathsf{B}}$. Given such a subclone $F \subseteq \mathbb{O}_2^{\mathsf{B}}$, we may restrict to the finitary functions $F \cap \mathbb{O}_2^{<\omega}$ within it. We get a map

$$(1.2) \qquad \mathcal{C}lo^{\mathsf{B}}(2) := \{\text{Borel clones on 2}\} \longrightarrow\!\!\!\!\!\rightarrow \{\text{finitary clones on 2}\} =: \mathcal{C}lo^{<\omega}(2)$$
$$F \longmapsto F \cap \mathbb{O}_2^{<\omega},$$

which turns out to be a complete lattice homomorphism (see Corollary 2.22 and (4.3)). Thus, we may regard the lattice $\mathcal{C}lo^{\mathsf{B}}(2)$ of Borel clones on 2 as a "fiber bundle" over Post's lattice $\mathcal{C}lo^{<\omega}(2)$, depicted in Figure 1.5; and the classification problem for Borel clones decomposes into, for each finitary clone G in Post's lattice, classifying the "fiber" over G, of all Borel clones F with finitary restriction G. When G is defined as all the finitary functions preserving certain finitary relations, then such F are precisely the Borel clones of functions preserving the same relations, and containing all the finitary such functions G.

To illustrate this, consider the following clones, which are the maximal nodes in Post's lattice (defined in more detail in Section 3):

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- $\mathbb{M}$:= all monotone (i.e., $\leq$-preserving) functions $2^n \to 2$.

- $\mathbb{T}_c$:= all functions $2^n \to 2$ preserving the constant $c \in \{0, 1\}$.

- $\mathbb{D}$:= all functions $2^n \to 2$ equal to their own de Morgan dual.

- $\mathbb{A}$:= all functions $2^n \to 2$ affine over $\mathbb{Z}/2\mathbb{Z}$.

We denote the finitary, respectively Borel, versions of these clones by a superscript $^{<\omega}$, resp., $^\mathsf{B}$. Thus, for example, $\mathbb{M}^{<\omega}$ is the clone of finitary monotone functions, generated by $\wedge, \vee, 0, 1$. Then the Borel clones F in the fiber of the bundle (1.2) over $\mathbb{M}^{<\omega}$ are those such that $\wedge, \vee, 0, 1 \in F \subseteq \mathbb{M}^\mathsf{B}$. There is a greatest such F, namely $\mathbb{M}^\mathsf{B}$, as well as a least, namely the Borel clone generated by $\wedge, \vee, 0, 1$; these are distinct since the former contains the countable disjunction (join/supremum) $\bigvee$. In contrast, in the fiber over $\mathbb{A}^{<\omega}$, there lies only a single Borel clone, by the aforementioned fact that Borel affine maps can depend on only finitely many variables. It follows that the same holds over each subclone of $\mathbb{A}^{<\omega}$ in Post's lattice, e.g., $\mathbb{AD}^{<\omega} := \mathbb{A}^{<\omega} \cap \mathbb{D}^{<\omega}$ (we use juxtaposition to abbreviate intersection of clones.).

1.B Main results

We are able to completely classify the Borel clones lying over a node near the top of Post's lattice:

Theorem 1.3 (see Section 4.A)**.**

(a) *Over each of the 8 finitary clones in the "cube" at the top of Post's lattice 1.1, consisting of all intersections of combinations of $\mathbb{M}, \mathbb{T}_0, \mathbb{T}_1$, there lie at least 2 but only finitely many Borel clones restricting to that finitary clone, namely: 2 Borel clones over $\mathbb{O}_2^{<\omega}$; 3 over each of $\mathbb{T}_0^{<\omega}, \mathbb{T}_1^{<\omega}$; 5 over $\mathbb{T}_0\mathbb{T}_1^{<\omega}$; 4 over $\mathbb{M}^{<\omega}$; 6 over each of $\mathbb{MT}_0^{<\omega}, \mathbb{MT}_1^{<\omega}$; and 9 over $\mathbb{MT}_0\mathbb{T}_1^{<\omega}$.*

(b) *Over the finitary clones $\mathbb{D}^{<\omega}, \mathbb{DT}_0^{<\omega}, \mathbb{DM}^{<\omega}$, there lie 2, 3, 3 Borel clones respectively.*

(c) *Over each finitary clone F in Post's lattice 1.1, there lies only 1 Borel clone iff $F \subseteq \mathbb{A}$.*

Figure 1.5 (shaded regions) depicts those fibers of the bundle $\mathcal{Clo}^\mathsf{B}(2) \twoheadrightarrow \mathcal{Clo}^{<\omega}(2)$ from (1.2) in which we get a complete classification of the Borel clones. The proofs are spread out over several results in Section 4.A; see there for more detailed pictures of each of the fibers individually.

Over the remaining finitary clones in Post's lattice 1.1, we do not get a complete classification of the Borel clones. Instead, we exhibit a wide variety of complex behaviors among the Borel clones. In order to state these, we recall some more clones from Post's lattice:

- $\mathbb{\Lambda}$:= all indicator functions $f : 2^n \to 2$ of filters $f^{-1}(1) \subseteq 2^n$, or $f = 0$.

- $\mathbb{T}_{0,k}$:= all indicator functions $f : 2^n \to 2$ of sets $f^{-1}(1) \subseteq 2^n$ with the k-ary intersection property (any k bit strings in $f^{-1}(1)$ have bitwise conjunction $\neq \vec{0}$); thus $\mathbb{T}_0 = \mathbb{T}_{0,1}$.

- $\mathbb{T}_{0,<\omega} := \bigcap_{k<\omega} \mathbb{T}_{0,k}$ = all indicator functions of sets with the finite intersection property.

- $\mathbb{V}, \mathbb{T}_{1,k}, \mathbb{T}_{1,<\omega}$ are the de Morgan duals of these, concerning ideals/disjunctions.

As before, e.g., $\mathbb{T}_{0,k}^{<\omega}$ denotes the clone of all finitary such functions, while $\mathbb{T}_{0,k}^\mathsf{B}$ denotes the corresponding Borel clone. Note that $\mathbb{T}_{0,<\omega}^{<\omega}$ thus consists of all finitary functions which are bounded above by some particular variable (by considering a conjunction of strings with a single 0); whereas $\mathbb{T}_{0,<\omega}^\mathsf{B} \supseteq \mathbb{\Lambda}^\mathsf{B}$, which includes all Borel filters $\subseteq 2^\omega$, is much more complicated.

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Theorem 1.4 (see Sections 4.B to 4.D, especially Theorems 4.94 and 4.61 and Example 4.62)**.**

(a) *Over each of the finitary clones* $\mathbb{T}_{0,k}, \mathbb{T}_{0,k}\mathbb{T}_1, \mathbb{MT}_{0,k}, \mathbb{MT}_{0,k}\mathbb{T}_1$ *in the left "side tube" of Post's lattice* 1.1*,* $k < \omega$*, there lie at least* $2^k+1, 2^k+k+2, 2^k+4, 2^k+k+6$ *Borel clones, respectively.*

(b) *Over each of the finitary clones* $\mathbb{T}_{0,<\omega}, \mathbb{T}_{0,<\omega}\mathbb{T}_1, \mathbb{MT}_{0,<\omega}, \mathbb{MT}_{0,<\omega}\mathbb{T}_1$ *(at the "base of the side tube"), there lie at least countably infinitely many Borel clones.*

(c) *Over each of the finitary clones* $\mathbb{\Lambda}, \mathbb{\Lambda}\mathbb{T}_1, \mathbb{\Lambda}\mathbb{T}_0, \mathbb{\Lambda}\mathbb{T}_0\mathbb{T}_1$ *(below the "side tube"), there lie precisely 3, 3, 4, 4 Borel clones* $F \subseteq 2^{2^{\leq\omega}}$ *respectively which are "countably closed" in* 2^{2^ω}*, i.e., contain all functions which agree at any countably many input strings* $\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots \in 2^\omega$ *with some function in* F*. However, there also exist other "non-countably closed" clones of Borel filters.*

Similarly for the de Morgan duals of these clones.

Figure 1.6 (shaded regions) depicts the fibers of the bundle (1.2) mentioned above, in which we get only a partial classification of the Borel clones (these clones are defined in Section 4.D); see also Figures 4.29, 4.53 and 4.71. As these pictures indicate, the lower bounds on the numbers of Borel clones in Theorem 1.4(a) and (b) are merely the numerical counts of detailed order-theoretic structures on these fibers. For instance, the constant terms in these lower bounds count the fully classified lower (solid-shaded in 1.6) portions of each fiber, with $2, 3, 4, 6$ elements respectively, which are isomorphic to four of the fibers in Theorem 1.3(a). On the other hand, the upper, hatch-shaded "cobweb" portions of 1.6 contribute 2^k distinct Borel clones in each fiber, with a recursively generated order-structure; but we cannot rule out the existence of other Borel clones between these.

Theorem 1.4(c) shows that in a precise sense, it is "difficult" to fully classify all Borel clones over $\mathbb{\Lambda}^{<\omega}$ and its variants. Recall that by standard clone theory, every clone $F \subseteq \bigsqcup_n 2^{2^n}$ can be defined as all Boolean functions which preserve some given set of k-ary relations $R \subseteq 2^k$ for various k, called **polymorphisms** of those relations; this includes $R = (\leq) \subseteq 2^2$ (yielding the monotone functions $\mathbb{M}$), $(\neq) \subseteq 2^2$ (yielding the self-dual functions $\mathbb{D}$), etc. The *countably closed* clones F as in 1.4(c) are precisely those which can be defined as the polymorphisms of *countable-arity* relations; for example, polymorphisms of the ω-ary convergence relation "$\lim_{i\to\infty} x_i = 0$" yield the functions $f : 2^n \to 2$ which fix and are continuous at $\vec{0}$, denoted $\mathbb{L}_0$ near the tops of Figures 1.5 and 1.6. Thus, Theorem 1.4(c) says that on the one hand, there are only a few Borel clones of filters which are definable by countable-arity relations; on the other hand, such relations cannot distinguish all distinct Borel clones.

We now briefly outline our proof strategy for the aforementioned positive classification results. The combinatorial core consists of a handful of simple "Wadge's lemma"-type dichotomies, showing that every Borel clone F must either be contained in a specific clone, or else contain a "minimally complex" function outside of that clone; see Lemmas 4.7, 4.45, 4.54 and 4.75. These dichotomies are used to bootstrap several abstract structural mappings, showing that large regions of the lattice of Borel clones $\mathcal{Clo}^{\mathsf{B}}(2)$ are isomorphic or embed into each other; see Figure 3.29 and Propositions 3.15, 3.22, 3.32, 4.34 and 4.87. By applying these isomorphisms repeatedly, we then transport the core dichotomies across various regions of the lattice $\mathcal{Clo}^{\mathsf{B}}(2)$, in order to resolve it into the pieces shown in Figures 1.5 and 1.6.
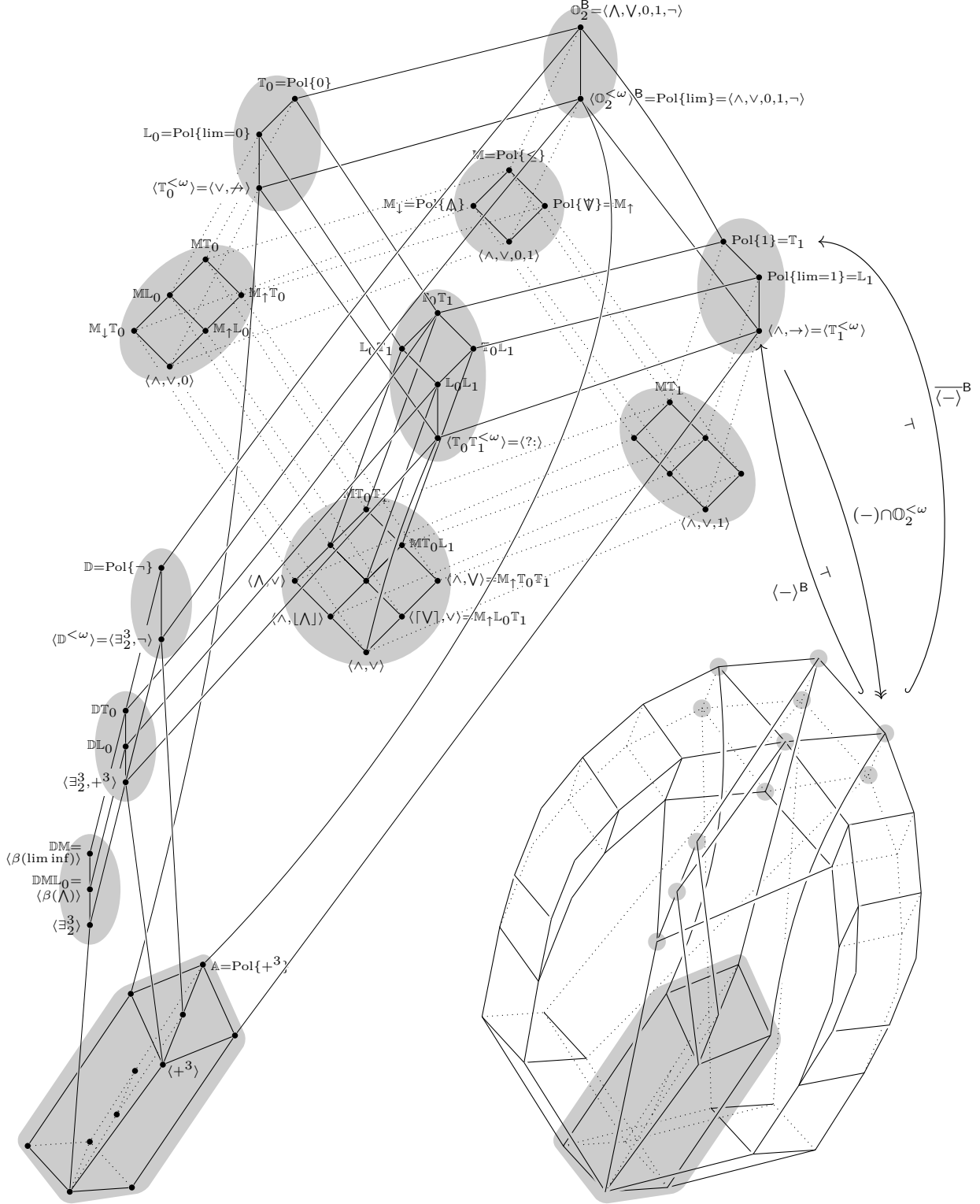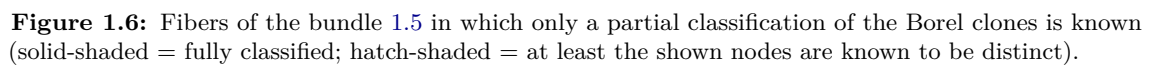
Figure 1.5: Bundle of Borel clones $\mathcal{C}lo^{\mathsf{B}}(2) \twoheadrightarrow \mathcal{C}lo^{<\omega}(2)$ projecting to Post's lattice 1.1 via finitary restriction, with fibers (shaded blobs) in which a complete classification of the Borel clones is known.

Figure 1.6: Fibers of the bundle 1.5 in which only a partial classification of the Borel clones is known (solid-shaded = fully classified; hatch-shaded = at least the shown nodes are known to be distinct).

1.C Related work and future directions

An unusual feature of our classification shown in Figure 1.6 is that among the Borel clones over $\mathbb{T}_{0,2}$ and its variants, we are able to fully characterize the bottommost as well as topmost clones, but not the intermediate clones $\mathbb{L}_{0,2,1}$ and its variants. This has to do with the special connection between the clone $\mathbb{MT}_{0,2}$ and the self-dual monotone functions $\mathbb{DM}$ (the function $\forall_2^3 = \exists_2^3 : 2^3 \to 2$ is the 2-out-of-3 median operation). It remains to be seen whether similar ideas may be used to characterize upper regions (the "cobwebs" in 1.6) of the $\mathbb{T}_{0,k}$ Borel clones for $k \geq 3$. In particular, the following basic problem is suggested by Figure 1.6:

Conjecture 1.7. *The number of Borel clones with finitary restriction $\mathbb{T}_{0,k}^{<\omega}$ is increasing in $k \leq \omega$.*

The Example 4.62 of a Borel clone not definable by countable-arity relations mentioned in Theorem 1.4(c) essentially follows from a result of Kanovei–Reeken [KR03] on Borel equivalence relations and ideals. This result uses deep set-theoretic tools such as forcing and Hjorth's turbulence theory [Hjo00], while showing something much stronger (namely, Borel non-reducibility of equivalence relations) than is needed for the application to Borel clones (namely, Rudin–Keisler non-reducibility between Borel filters; see Problem 4.63). We expect there to be many other examples of distinct Borel clones inseparable by countable-arity relations; a better understanding of the connections with the theory of Borel ideals and filters (see [Sol99], [Hru11], [Kan08]) may help with such a pursuit. In particular, in light of Example 4.62 and analogous non-reducibility results in the theory of Borel equivalence relations (see e.g., [AK00]), the following seems natural:

Conjecture 1.8. *There are $\geq 2^{\aleph_0}$ pairwise distinct Borel clones with finitary restriction $\mathbb{AT}_0 \mathbb{T}_1^{<\omega}$.*

More generally, we are pleasantly surprised that a large part of the lattice of Borel clones seems to be classifiable using only abstract lattice-theoretic reasoning plus a few simple combinatorial dichotomies, as described in the preceding subsection. In particular, we make no use in this paper of more powerful techniques such as determinacy, forcing, or effective descriptive set theory (see [Mos09], [Kan08]). This is especially surprising given the use of such techniques in a structure theory which is conceptually related to that of Borel clones: the Wadge hierarchy [Wad83]. Recall that a **Wadge class** (on 2^ω) is essentially a family F of functions $2^\omega \to 2$ which is closed under *right*-composition with arbitrary continuous functions. The lattice (essentially a well-order) of all Wadge classes of Borel functions has been completely described [Wad83], [Lou83].

We note that the classification of Borel clones in this paper is formally "orthogonal" to the Wadge hierarchy: every Wadge class contains all finitary (i.e., continuous) functions, whereas a Borel clone containing all finitary functions and a single discontinuous function must be the entirety of $\mathbb{O}_2^{\mathsf{B}}$. Rather, we expect that the machinery we have developed may be useful in future studies of infinitary clones of "definable" Boolean functions *beyond* Borel (under suitable determinacy hypotheses), with the Wadge hierarchy as a "backbone". For instance, as a natural next step beyond the Borel clones:

Problem 1.9 (assuming analytic determinacy)**.** Classify the subclones of the clone $\mathbb{O}_2^{\mathsf{C}}$ of indicator functions of **C-sets**, i.e., the clone generated by all Borel functions $2^n \to 2$ together with the Suslin operation $\mathcal{A} : 2^\omega \cong 2^{\omega^{<\omega}} \to 2$ where $\mathcal{A}((x_s)_{s \in \omega^{<\omega}}) := \bigvee_{S \in \omega^\omega} \bigwedge_{n < \omega} x_{S|n}$ (see [Kec95, 29.D]).

Partly for the sake of such generalizations, we have taken care to state key lemmas, such as the combinatorial dichotomies and structural lattice isomorphisms mentioned in the preceding subsection, as much as possible for arbitrary infinitary clones on 2, without assuming Borelness.

The main role of Borelness in our development is to enable an *exhaustive* classification in results such as Theorem 1.3, essentially by collapsing the Wadge hierarchy "backbone" so that a single discontinuous function $\bigvee : 2^\omega \to 2$ together with all finitary functions generates all Borel functions. (A more subtle role is played by reflection theorems for Borel sets that enable one to relate monotone and non-monotone Borel functions; see Corollary 4.15 and Proposition 4.40.)

Clones of polymorphisms have received much attention in recent years in connection with **constraint satisfaction problems (CSPs)**, a large class of computational/combinatorial problems, especially since the proof of the *CSP dichotomy theorem* [Bul17], [Zhu17] using universal-algebraic methods. See [Bod21] for a detailed survey of this area. Recently, clones and related concepts have also seen applications to CSPs in set theory [Tho22], [KTV23], [GV25]. We hope to investigate potential combinatorial applications of Borel clones on 2 in future work.

More generally, it would be of interest to study Borel clones on finite sets bigger than 2, which correspond to many naturally occurring CSPs (such a study was proposed in [GV25, 5.3]). While finitary such clones have not been completely classified (in contrast to Post's lattice), we expect that some of the machinery developed in this paper may be combined with the known theory of finitary clones on finite sets $X \geq 3$ (see [Lau06]) to derive useful consequences in the Borel context. We hope to investigate this also in future work.

Acknowledgments

2   Clone theory

We begin by reviewing the concepts from general clone theory which we will need. All of the ideas here are standard in universal algebra; see [FMMT22], [Lau06], [Sze86], [Bod21]. However, we will be needing the infinite-arity (even occasionally uncountable-arity) versions of the usual machinery, which may be less familiar. We will sketch the less routine proofs for the reader's convenience.

2.A   Lattices and adjunctions

Recall that a **lattice** is a poset in which any two elements a, b have a greatest lower bound or **meet** $a \wedge b$ as well as least upper bound or **join** $a \vee b$, while a **complete lattice** is a poset in which arbitrarily many elements a_i (possibly none) have a meet $\bigwedge_i a_i$ and join $\bigvee_i a_i$. A **meet-irreducible** element in a complete lattice is one which cannot be obtained as a meet of strictly smaller elements, i.e., has a unique immediate predecessor; **join-irreducible** is defined similarly.

Recall that an **adjunction** or **Galois connection** between two posets A, B is a pair of maps $f : A \to B$ (the **left adjoint**) and $g : B \to A$ (the **right adjoint**) satisfying

(2.1)
$$f(a) \leq b \iff a \leq g(b),$$

denoted $f \dashv g$. A left adjoint preserves all existing joins, while a right adjoint preserves all existing meets; the converses hold assuming A, B are complete lattices. If B is replaced with its order-reversal B^{op}, then (2.1) becomes
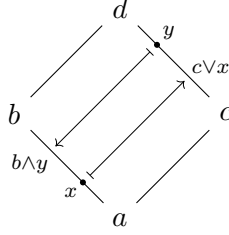
$$b \leq f(a) \iff a \leq g(b)$$

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Figure 2.3: The modularity adjunction $[a, b] \rightleftarrows [c, d]$ in a lattice.

(we call such f, g *dually adjoint on the right*), and each of f, g maps joins to meets. In either case, we have $g \circ f = \mathrm{id}_A$ iff f is injective iff g is surjective, and dually for $f \circ g = \mathrm{id}_B$.

The **interval** between elements $a, b \in A$ of a poset is

$$[a, b] := \{x \in A \mid a \le x \le b\}.$$

Given elements $a, b, c, d \in A$ of a lattice, such that $b, c \in [a, d]$, we have the **modularity adjunction**

(2.2)
$$[a, b] \xrightleftharpoons[b \wedge (-)]{\overset{c \vee (-)}{\perp}} [c, d].$$

If these form an isomorphism, then necessarily $a = b \wedge c$ and $d = b \vee c$. (Recall that the lattice A is called *modular* if we have such a modularity isomorphism for all $b, c \in A$.) See Figure 2.3.

For more information on general lattice theory, see [Grä03], [GHK$^+$03], [Joh82].

2.B   Clones

Convention 2.4. The **arity** of an operation or relation will always be a cardinal, by which we mean a (von Neumann) initial ordinal. We will use the letters k, l, m, n for cardinals treated as arities, including infinite ones. For a class N of cardinals, by N**-ary** we mean n-ary for some $n \in N$. We will use the letters K, L, M, N for classes of cardinals.

We abuse notation by writing n instead of $\{n\}$ when convenient. For a regular cardinal κ, we write $N = {<}\kappa$ for the set of all cardinals $n < \kappa$. This includes $\kappa = \omega$, where ${<}\omega$ is formally the same set as ω; however, "${<}\omega$-ary" means finitary, while "ω-ary" (i.e., $\{\omega\}$-ary) means countably infinitary. We will use the letters $\kappa, \lambda, \mu, \nu$ for regular cardinals. We also use $\le\kappa$ as an abbreviation for ${<}\kappa^+$.

We will occasionally use $N = {<}\infty$ to mean the class of all cardinals.

Notation 2.5. Let N be a class of cardinals, X be a set. We let

$$\mathbb{O}_X^N := \bigsqcup_{n \in N} X^{X^n}$$

denote the class of N-ary functions on X. We also let $\mathbb{O}_X := \mathbb{O}_X^{<\infty}$ denote *all* functions on X.

For a subclass $F \subseteq \mathbb{O}_X^N$ of N-ary functions and $n \in N$, we let

$$F^n := F \cap \mathbb{O}_X^n$$

denote all n-ary functions in F. For a subclass of cardinals $M \subseteq N$, we let

$$F^M := F \cap \mathbb{O}_X^M = \bigsqcup_{m \in M} F^m$$

denote the M-**ary restriction** of F. Note that the notation $\mathbb{O}_X^N$ can thus be consistently interpreted as the N-ary restriction of $\mathbb{O}_X = \mathbb{O}_X^{<\infty}$.

Definition 2.6. We say that $F \subseteq \mathbb{O}_X^N$ is an N-**ary clone** on X if it contains all **projections**

$$\pi_i = \pi_i^n : X^n \longrightarrow X$$
$$\vec{x} \longmapsto x_i$$

for $i < n \in N$, as well as all **compositions**

$$g(\vec{f}) = g \circ \vec{f} : X^m \longrightarrow X$$
$$\vec{x} \longmapsto g(f_0(\vec{x}), f_1(\vec{x}), \dots)$$

for $m, n \in N$, $g \in F^n$, $\vec{f} = (f_i)_{i<n} \in (F^m)^n$. Denote the complete lattice of N-ary clones on X by

$$\mathcal{C}lo^N(X) \subseteq \mathcal{P}(\mathbb{O}_X^N).$$

Notation 2.7. For an arbitrary class of N-ary functions $F \subseteq \mathbb{O}_X^N$, we let

$$\langle F \rangle^N \subseteq \mathbb{O}_X^N$$

denote the N-**ary clone generated by** F, i.e., the smallest N-ary clone containing F.

We also write $\langle F \rangle := \langle F \rangle^{<\infty}$.

Lemma 2.8. $\langle F \rangle^N \subseteq \mathbb{O}_X^N$ *is also the smallest class of N-ary functions containing all N-ary projections and closed under left composition with functions in F.*

Proof sketch. It suffices to verify that said smallest class, call it G, is closed under composition. Indeed, it is easily checked that $\{f \in \mathbb{O}_X^N \mid \forall \vec{g} \in G\,(f \circ \vec{g} \in G)\}$ contains all projections and is closed under left composition with functions in F, hence contains G. $\qquad\square$

Note that this description of $\langle F \rangle^N$ separately characterizes the set of n-ary functions in $\langle F \rangle^N$, for each $n \in N$ (independently of the other cardinals in N). Thus

Corollary 2.9. *For two classes of cardinals $M \subseteq N$ and $F \subseteq \mathbb{O}_X^M$, we have $\langle F \rangle^N \cap \mathbb{O}_X^M = \langle F \rangle^M$. Thus, if F is an M-ary clone, then $\langle F \rangle^N \cap \mathbb{O}_X^M = F$. In other words, the adjunction*

$$\mathcal{C}lo^M(X) \xrightleftharpoons[\;(-)\cap\mathbb{O}_X^M\;]{\;\langle - \rangle^N\;} \mathcal{C}lo^N(X)$$

exhibits the lattice $\mathcal{C}lo^M(X)$ as a retract of $\mathcal{C}lo^N(X)$. $\qquad\square$

When $M = {<}\mu$ and $N = {<}\nu$ for infinite regular cardinals $\mu \le \nu$, we have a simpler description of $\langle F \rangle^N$ for an M-ary clone F:

Lemma 2.10. *For a regular cardinal μ and $<\mu$-ary clone $F \subseteq \mathbb{O}_X^{<\mu}$, for any cardinal n,*

$$\langle F \rangle^n = \{ f \circ (\pi_{s(i)}^n)_{i<m} \mid m < \mu,\ f \in F^m,\ s : m \hookrightarrow n \}$$

consists of precisely the n-ary functions in F which depend on only $<\mu$-many variables.

Proof sketch. It is easily checked that said class contains all projections and is closed under left composition with functions in F (using regularity of μ to disjointify variables). $\qquad\square$

Corollary 2.11. *For regular cardinals $\mu \leq \nu$, the image of the map*

$$\langle - \rangle^{<\nu} : \mathcal{C}lo^{<\mu}(X) \longrightarrow \mathcal{C}lo^{<\nu}(X)$$

consists of precisely the $<\nu$-ary subclones of $\langle \mathbb{O}_X^{<\mu} \rangle^{<\nu}$. $\qquad\square$

Definition 2.12. We call a function $f : X^n \to X$ **essentially $<\mu$-ary** if it depends on only $<\mu$-many variables, i.e., it is in $\langle \mathbb{O}_X^{<\mu} \rangle$. We call a clone F **essentially $<\mu$-ary** if it consists entirely of essentially $<\mu$-ary functions, i.e., $F \subseteq \langle \mathbb{O}_X^{<\mu} \rangle$.

Remark 2.13. If X is finite, then an essentially finitary function $f : X^n \to X$ is one which is continuous with respect to the discrete topology on X and the product topology on X^n.

More generally, for a class of cardinals M, let us define a M-**topology** to mean a family of sets closed under arbitrary unions and M-ary intersections, and the **product M-topology** on X^n to mean the smallest M-topology containing preimages of arbitrary sets under the projections $\pi_i : X^n \to X$ (so interpolating between the usual product topology when $M = <\omega$ and the box topology when $M = <\infty$). Then, for $|X| < \mu$, an essentially $<\mu$-ary function $f : X^n \to X$ is one which is $<\mu$-**continuous**, i.e., the preimage of an arbitrary set belongs to the product $<\mu$-topology.

2.C   Polymorphisms

Notation 2.14. Let K be a class of cardinals, X be a set. We let

$$\mathcal{R}_X^K := \bigsqcup_{k \in K} \mathcal{P}(X^k)$$

denote the class of all K-ary relations on X. We will loosely refer to a subclass $\mathcal{M} \subseteq \mathcal{R}_X^K$ as a (relational) K-**ary structure** on X. We use similar notations as for functions (Notation 2.5) to denote arities: $\mathcal{M}^k := \mathcal{M} \cap \mathcal{R}_X^k$ means the k-ary relations in $\mathcal{M}$; $\mathcal{R}_X := \mathcal{R}_X^{<\infty}$; etc.

Definition 2.15. Let X be a set, $f : X^n \to X$ be an n-ary function, $R \subseteq X^k$ be a k-ary relation, for some cardinals n, k. The following phrases are synonymous:

- f **preserves** R, or is a **polymorphism** of R.

- f is a homomorphism $(X, R)^n \to (X, R)$, where $(X, R)^n := (X^n, R^n)$ is the **product structure** (in the category of structures with a single k-ary relation).

- R is f-**invariant**, or **closed** under f, or a **substructure** of the product structure $(X, f)^k$ where f acts coordinatewise.

- For any $k \times n$ matrix $(\vec{x}_i)_{i<k} = (x_{i,j})_{i<k, j<n} \in X^{k \times n}$, if each $(x_{i,j})_{i<k} \in R$ for fixed $j < n$, then also $(f(\vec{x}_i))_{i<k} = (f((x_{i,j})_{j<n}))_{i<k} \in R$.

$$
k \left\{ \overbrace{
\begin{bmatrix}
x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\
x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
}^{n}
\;\to\;
\begin{bmatrix}
f(\vec{x}_0) \\
f(\vec{x}_1) \\
\vdots
\end{bmatrix}
\right.
$$
$$
\cap \qquad \cap \qquad \cap \qquad\qquad\qquad \cap
$$
$$
R \qquad\; R \qquad\; R \qquad\qquad\qquad R
$$

If these hold, we write interchangeably

$$
f \in \mathrm{Pol}(R) \iff R \in \mathrm{Inv}(f)
$$

where $\mathrm{Pol}(R) \subseteq \mathbb{O}_X$ is the class of all polymorphisms of R, and $\mathrm{Inv}(f) \subseteq \mathcal{R}_X$ is the class of all f-invariant relations. More generally, for any $\mathcal{M} \subseteq \mathcal{R}_X$ and $F \subseteq \mathbb{O}_X$, we write

$$
\mathrm{Pol}(\mathcal{M}) := \bigcap_{R \in \mathcal{M}} \mathrm{Pol}(R), \qquad\qquad \mathrm{Inv}(F) := \bigcap_{f \in F} \mathrm{Inv}(f).
$$

We let also $\mathrm{Pol}^N(\mathcal{M}) := \mathrm{Pol}(\mathcal{M}) \cap \mathbb{O}_X^N$ and $\mathrm{Inv}^K(F) := \mathrm{Inv}(F) \cap \mathcal{R}_X^K$ for classes of cardinals N, K.

Thus for any classes of cardinals N, K, we have an (order-reversing) Galois connection

(2.16)
$$
\mathcal{P}(\mathbb{O}_X^N) \xrightleftharpoons[\mathrm{Pol}^N]{\mathrm{Inv}^K} \mathcal{P}(\mathcal{R}_X^K),
$$

i.e., $F \subseteq \mathrm{Pol}(\mathcal{M}) \iff \mathcal{M} \subseteq \mathrm{Inv}(F) \iff$ every $f \in F$ preserves every $R \in \mathcal{M}$.

Every class of functions of the form $\mathrm{Pol}(\mathcal{M})$ is a clone. Thus the fixed classes on the left side of the above Galois connection, i.e., those $F \subseteq \mathbb{O}_X^N$ for which $F = \mathrm{Pol}^N(\mathrm{Inv}^K(F))$, must in particular be clones. We may characterize them more precisely as follows:

Lemma 2.17. *For any class of functions $F \subseteq \mathbb{O}_X$, class of cardinals K, and n-ary function $g : X^n \to X$, we have $g \in \mathrm{Pol}^n(\mathrm{Inv}^K(F))$ iff for any K-ary family of n-tuples $(\vec{x}_i)_{i<k} \in (X^n)^k$, where $k \in K$, there is an $f \in \langle F \rangle^n$ such that $f(\vec{x}_i) = g(\vec{x}_i)$ for each $i < k$.*

In other words, this says that $\mathrm{Pol}^n(\mathrm{Inv}^K(F))$ is the K**-closure** of $\langle F \rangle^n \subseteq X^{X^n}$, with respect to the product K-topology from Remark 2.13. For instance:

- When $K = {<}\omega$, we get that $\mathrm{Pol}^n(\mathrm{Inv}^{<\omega}(F)) = \overline{\langle F \rangle^n}$ is the usual closure of $\langle F \rangle^n \subseteq X^{X^n}$.

- When $K = {<}\omega_1$, we get that $\mathrm{Pol}^n(\mathrm{Inv}^{<\omega_1}(F))$ consists of all functions agreeing on any countably many n-tuples with a function in $\langle F \rangle^n$; we call this the **countable closure** of $\langle F \rangle^n$.

- When $K \supseteq {\leq}|X|^n$, e.g., X is finite, n countable, and $K = {\leq}2^{\aleph_0}$, we get $\mathrm{Pol}^n(\mathrm{Inv}^K(F)) = \langle F \rangle^n$.

Proof. Any $\mathrm{Pol}^n(R)$ for a K-ary relation R is K-closed, since for a function to preserve R requires checking its values only on each K-ary family of n-tuples at a time; thus $\mathrm{Pol}^n(\mathrm{Inv}^K(F))$ contains the K-closure of F. Conversely, for $g \in \mathrm{Pol}^n(\mathrm{Inv}^K(F))$ and $(\vec{x}_i)_{i<k} \in (X^n)^k$, the forward $\langle F \rangle$-orbit $R := \{(f(\vec{x}_i))_{i<k} \mid f \in \langle F \rangle^n\}$ is F-invariant, hence preserved by g; we have $(x_{i,j})_{i<k} = (\pi_j^n(\vec{x}_i))_{i<k} \in R$ for each j, whence $(g(\vec{x}_i))_{i<k} \in R$, i.e., $(g(\vec{x}_i))_{i<k} = (f(\vec{x}_i))_{i<k}$ for some $f \in \langle F \rangle^n$. $\qquad\square$

On the right side of the adjunction (2.16), the fixed classes are characterized as follows:

Lemma 2.18. *For any class of relations $\mathcal{M} \subseteq \mathcal{R}_X$, regular cardinal ν, and k-ary relation $S \subseteq X^k$, we have $S \in \mathrm{Inv}^k(\mathrm{Pol}^{<\nu}(\mathcal{M}))$ iff S is a $<\nu$-directed union of relations **positive-primitively definable** from $\mathcal{M}$, i.e., (infinitary) first-order definable from the relations in $\mathcal{M}$ using $=, \exists$ over arbitrarily many variables, and $\bigwedge$ of arbitrary arity.*

Here by a $<\nu$-*directed union* we mean a union of a family of sets in which any $<\nu$-sized subfamily has an upper bound. Let

$$\mathcal{PPStr}^K(X) \subseteq \mathcal{P}(\mathcal{R}_X^K)$$

denote the class of **positive-primitive K-ary structures**, by which we mean structures closed under positive-primitive definability (sometimes called *coclones*; see [Lau06]). The above then says that $\mathrm{Inv}^K(\mathrm{Pol}^{<\nu}(\mathcal{M})) = \mathcal{M}$ is the closure under $<\nu$-directed union of the positive-primitive K-ary structure generated by $\mathcal{M}$. Note that for sufficiently large ν, namely $\nu > 2^{|X|^k}$, $<\nu$-directed unions in X^k are trivial; thus we simply get the positive-primitively definable relations in that case.

Proof. Any $\mathrm{Inv}^k(f)$ for a $<\nu$-ary function f is easily seen to be closed under $<\nu$-directed union and positive-primitive definability; thus $\mathrm{Inv}^k(\mathrm{Pol}^{<\nu}(\mathcal{M}))$ contains all such R. Conversely, let $S \in \mathrm{Inv}^k(\mathrm{Pol}^{<\nu}(\mathcal{M}))$. Then any family of tuples $(\vec{x}_j)_{j<n} \in S^n$ where $n < \nu$ generates a smallest $\mathrm{Pol}^{<\nu}(\mathcal{M})$-invariant subset $R((\vec{x}_j)_{j<n}) \subseteq X^k$ contained in S; and S is the $<\nu$-directed union of all of these $R((\vec{x}_j)_{j<n})$. But each $R((\vec{x}_j)_{j<n})$ is the saturation of $(\vec{x}_j)_{j<n}$ under all $f \in \mathrm{Pol}^n(\mathcal{M})$ (as in the proof of Lemma 2.17, with the "transpose matrix" of $(\vec{x}_j)_j$); and "$\exists f \in X^{X^n}$ (f is a polymorphism and $f((\vec{x}_j)_j) = \vec{y}$)" is a positive-primitive formula (where the existential is over $|X^n|$-many variables). $\qquad \square$

The preceding two lemmas are the natural generalizations of classical results of Geiger [Gei68] and Bodnarčuk–Kalužnin–Kotov–Romov [BKKR69] to arbitrary arities.

2.D Comparing arities

The Pol–Inv adjunction (2.16) interacts with the change of arity adjunction from Corollary 2.9 as follows. For classes of cardinals $M \subseteq N$ and K, we have the diagram

(2.19)

in which the two diagrams of left/right adjoints each commute, and the unmarked arrows are inclusions. For sufficiently large K, Inv^K is an embedding of both $\mathcal{Clo}^N(X)$ and $\mathcal{Clo}^M(X)$ into $\mathcal{PPStr}^K(X)^{\mathsf{op}}$ by Lemma 2.17. Thus for instance, commutativity of the triangle yields

Corollary 2.20. *For a set of finitary functions $F \subseteq \mathbb{O}_X^{<\omega}$, the countable clone it generates is $\langle F \rangle^{<\omega_1} = \mathrm{Pol}^{<\omega_1}(\mathrm{Inv}(F))$. If X is finite, it suffices to take $\mathrm{Inv}^{\leq 2^{\aleph_0}}$.* $\qquad \square$

On the other hand, if M, K are such that $\mathrm{Pol}^M, \mathrm{Inv}^K$ form inverse order-isomorphisms $\mathcal{C}lo^M(X) \cong \mathcal{PPS}tr^K(X)^{\mathsf{op}}$ by Lemmas 2.17 and 2.18, e.g., X is finite and $M = K = {<}\omega$, then the triangle in (2.19) exhibits this common lattice as a retract of $\mathcal{C}lo^N(X)$ in two different ways.

Lemma 2.21. *For a finite X, regular cardinal ν, and ${<}\nu$-ary clone F, $\mathrm{Pol}^{<\omega}(\mathrm{Inv}^{<\omega}(F)) = F \cap \mathbb{O}_X^{<\omega}$.*

Proof. For $n < \omega$, $\mathrm{Pol}^n(\mathrm{Inv}^{<\omega}(F))$ is by Lemma 2.17 the closure of F^n; but F^n is finite. $\qquad\square$

Corollary 2.22. *For a finite X and regular cardinal ν, the adjunction of Corollary 2.9 admits a further right adjoint:*

$$
\mathcal{C}lo^{<\nu}(X)
$$

$$
\langle -\rangle^{<\nu} \dashv \bigg| (-)\cap\mathbb{O}_X^{\leq\omega} \dashv \bigg) \mathrm{Pol}^{<\nu}\circ\mathrm{Inv}^{<\omega}=\overline{\langle -\rangle^{<\nu}}
$$

$$
\mathcal{C}lo^{<\omega}(X)
$$

In other words, the complete lattice $\mathcal{C}lo^{<\nu}(X)$ of ${<}\nu$-ary clones admits the complete lattice $\mathcal{C}lo^{<\omega}(X)$ of finitary clones as a quotient (via restriction to finitary functions). Thus we may regard the former lattice as a "bundle" over the latter. Each finitary clone $F \in \mathcal{C}lo^{<\omega}(X)$ admits a least extension to a ${<}\nu$-ary clone, namely $\langle F\rangle^{<\nu} = \{$essentially finite ν-ary versions of functions in $F\}$, as well as a greatest extension to a ${<}\nu$-ary clone, namely $\overline{\langle F\rangle^{<\nu}} = \{$pointwise limits of functions in $\langle F\rangle^{<\nu}\}$. $\square$

Notation 2.23. In the situation of Corollary 2.22, for a finitary clone $F \in \mathcal{C}lo^{<\omega}(X)$, we write

$$
\mathcal{C}lo_F^{<\nu}(X) := [\langle F\rangle^{<\nu}, \overline{\langle F\rangle^{<\nu}}] \subseteq \mathcal{C}lo^{<\nu}(X)
$$

for the interval of ${<}\nu$-ary clones with finitary restriction F, i.e., the fiber of the bundle 2.22 over F.

As is typical in discussions of Post's lattice, we will also adopt the following

Convention 2.24. We will henceforth assume that clones do not contain nullary functions (constants). Thus, when we say e.g., a "${<}\omega$-ary clone", we really mean N-ary for $N = \{n \mid 0 < n < \omega\}$. When we refer to a constant function, e.g., $0 \in 2$, we will always mean $0 : 2^n \to 2$ for some $n > 0$.

This convention is used to eliminate uninteresting duplication when classifying clones on a given set such as $X = 2$. Without it, for each clone F containing a positive-arity constant function $c = 0$ or 1, we may arbitrarily choose whether or not to include the nullary versions of all constants in F. Thus, the classification of clones on 2 possibly containing nullary functions is essentially the same as for clones without, except that clones with constants are duplicated.

3 Clones on 2

In this section, we discuss general aspects of clones of arbitrary arity on $2 = \{0, 1\}$. In Section 3.A, we define the standard operations (logical connectives) and invariant relations used to specify the clones in Post's lattice $\mathcal{C}lo^{<\omega}(2)$, as well as their infinitary generalizations. We then define the standard clones in Post's lattice and state Post's theorem in Section 3.B. In Sections 3.C to 3.E, we introduce some operators on functions that map between different regions of Post's lattice.

3.A   Basic functions and relations on 2

Notation 3.1. For two functions $f, g : 2^n \to 2$, we write $f \leq g$ for the pointwise ordering, i.e.,

$$f \leq g \; :\Longleftrightarrow \; \forall \vec{x} \, (f(\vec{x}) \leq g(\vec{x})).$$

Definition 3.2. We define the following functions on 2 of various arities:

- $\neg : 2 \to 2$ is the bit flip (logical negation).

Given now any other function $f : 2^n \to 2$, its **de Morgan dual** is its $\neg$-conjugate

$$\delta(f) := \neg \circ f \circ \neg.$$

This is an automorphism of the clone $\mathbb{O}_2$; thus below, anything said about a function or a clone automatically transfers to its de Morgan dual.

- $0^n, 1^n : 2^n \to 2$ are the constant functions.

- $\wedge = \wedge^2 : 2^2 \to 2$ is binary conjunction (minimum); $\vee = \delta(\wedge)$ is binary disjunction.

- $\wedge^n$ or $\bigwedge^n : 2^n \to 2$ is n-ary conjunction; $\bigwedge$ by default means $\bigwedge^\omega$. Similarly for $\bigvee^n := \delta(\bigwedge^n)$.

- $\forall_k^n : 2^n \to 2$ and its dual $\exists_k^n : 2^n \to 2$, where $0 \leq k \leq n^+$, are given by:

$$\forall_k^n(\vec{x}) := \bigwedge_{\substack{I \subseteq n \\ |I| \geq k}} \bigvee_{i \in I} x_i = \bigvee_{\substack{J \subseteq n \\ |n \setminus J| < k}} \bigwedge_{j \in J} x_j = \text{``all but} < k \text{ inputs are true''},$$

$$\exists_k^n(\vec{x}) := \bigvee_{\substack{I \subseteq n \\ |I| \geq k}} \bigwedge_{i \in I} x_i = \bigwedge_{\substack{J \subseteq n \\ |n \setminus J| < k}} \bigvee_{j \in J} x_j = \text{``at least } k \text{ inputs are true''}.$$

Note that $\forall_1^n = \wedge^n$, and $\exists_1^n = \vee^n$. Note also that for $k, n < \omega$, we have $\forall_k^n = \exists_{n-k+1}^n$. In particular, $\forall_2^3 = \exists_2^3$ is the 2-out-of-3 median function

$$\forall_2^3(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = \exists_2^3(x, y, z).$$

By default, $\forall_k$ means $\forall_k^\omega : 2^\omega \to 2$, and similarly $\exists_k := \exists_k^\omega$.

- $\liminf := \forall_\omega : 2^\omega \to 2$ and its dual $\limsup := \exists_\omega$, which may be written more simply as:

$$\liminf \vec{x} = \bigvee_{i \in \omega} \bigwedge_{j \geq i} x_j, \qquad\qquad \limsup \vec{x} = \bigwedge_{i \in \omega} \bigvee_{j \geq i} x_j.$$

We have the following pointwise ordering among all aforementioned functions of arity ω:

$$\forall_0 = 0 \leq \forall_1 = \bigwedge \leq \forall_2 \leq \forall_3 \leq \cdots \leq \forall_\omega = \liminf \leq \limsup = \exists_\omega \leq \cdots \leq \exists_2 \leq \exists_1 \leq \exists_0.$$

- $+^n : 2^n \to 2$ is the n-ary addition mod 2 (i.e., "XOR"). By default, $+$ means $+^2$.

- $(?:) : 2^3 \to 2$ is the ternary conditional ("if-then-else")

$$(x \; ? \; y : z) := (x \wedge y) \vee (\neg x \wedge z).$$

- $\to : 2^2 \to 2$ is the Boolean implication $x \to y = \neg x \vee y$. Its de Morgan dual is $\delta(\to) = \not\leftarrow$, the variable transposition of $x \not\leftarrow y = x \wedge \neg y$.

Definition 3.3. For $f : 2^n \to 2$, define its **upper- and lower-bounded versions**

$$\lceil f \rceil : 2^{1+n} \to 2 \qquad\qquad\qquad \lfloor f \rfloor : 2^{1+n} \to 2$$
$$(x_0, x_1, \dots) \mapsto x_0 \wedge f(x_1, \dots), \qquad\qquad (x_0, x_1, \dots) \mapsto x_0 \vee f(x_1, \dots).$$

Examples include:

- $\leftarrow = \lfloor \neg \rfloor$ (where $x \leftarrow y := y \to x$), and similarly $\not\to = \lceil \neg \rceil$.
- $\lceil \to \rceil : 2^3 \to 2$ is thus given by $\lceil \to \rceil(x, y, z) = \lceil \lfloor \neg \rfloor \rceil(x, z, y) = x \wedge (y \to z)$. Its de Morgan dual is $\lfloor \not\to \rfloor(x, y, z) = \lfloor \lceil \neg \rceil \rfloor(x, y, z) = x \vee (y \wedge \neg z)$.
- $\lceil \vee \rceil : 2^3 \to 2$ is similarly given by $\lceil \vee \rceil(x, y, z) = x \wedge (y \vee z)$; dually, $\lfloor \wedge \rfloor(x, y, z) = x \vee (y \wedge z)$.
- $\lceil \bigvee \rceil : 2^\omega \to 2$ is similarly given by $\lceil \bigvee \rceil(x_0, x_1, x_2, \dots) = x_0 \wedge (x_1 \vee x_2 \vee \cdots)$.
- $\lceil \wedge \rceil = \wedge^3$ and $\lceil \bigwedge \rceil = \bigwedge$.

Notation 3.4. For a family of functions $(f^n : 2^n \to 2)_n$ of various arities $n \geq 1$, when we write $f^{<\nu}$ as part of a set of functions, we mean that f^n for each $1 \leq n < \nu$ is included. For instance,

$$\langle \vee, \bigwedge^{<\omega_1} \rangle = \langle \vee, \mathrm{id}, \wedge, \wedge^3, \wedge^4, \dots, \bigwedge \rangle$$

consists of all functions built from binary joins and countable meets (which is also just $\langle \vee, \bigwedge \rangle$).

We use similar notation for sets of relations indexed over a cardinal (see below).

Definition 3.5. We define the following relations on 2 of various arities:

- $(\leq) \subseteq 2^2$ is the usual linear order where $0 < 1$. A function preserves $\leq$ iff it is monotone.

If $f : 2^k \to 2$ is a function of arity k, or more generally a partial function, then we may also treat f as the $(1 + k)$-ary relation given by its graph. Examples include:

- The constant 0 may be treated as the unary relation $\{0\}$. A function g preserves 0 iff $g(\vec{0}) = 0$.
- $\neg : 2^1 \to 2$ is identified with the binary relation $\neq$. A function g preserves $\neg$ iff it is **self-dual**, i.e., $\delta(g) = g$, i.e., whenever $\vec{x}, \vec{y}$ differ in *every* coordinate (so $\vec{y} = \neg \vec{x}$), then $g(\vec{x}) \neq g(\vec{y})$.
- $+ : 2^2 \to 2$ is identified with the ternary relation $\{(x, y, z) \in 2^3 \mid x + y + z = 0\}$. A function preserves $+$ iff it is a linear transformation (over $\mathbb{Z}/2\mathbb{Z}$).
- $+^3 : 2^3 \to 2$ is identified with the quaternary relation $\{(x, y, z, w) \in 2^4 \mid x + y + z + w = 0\}$. A function preserves $+^3$ iff it is an affine transformation.
- $\wedge : 2^2 \to 2$ as a ternary relation is preserved by $g : 2^n \to 2$ iff g is a meet-semilattice homomorphism, which means that $g^{-1}(1) \subseteq 2^n$ is empty or a filter.
- The partial increasing join operation $\bigvee\!\!\!\!/ : 2^\omega \to 2$, defined by

$$\bigvee\!\!\!\!/ \,\vec{x} = y \;:\Longleftrightarrow\; (x_0 \leq x_1 \leq \cdots) \wedge \left(\bigvee_i x_i = y \right),$$

as a $1 + \omega = \omega$-ary relation is preserved by $g : 2^n \to 2$, $n \leq \omega$, iff g is *Scott-continuous*, i.e., monotone and $g^{-1}(0) \subseteq 2^n$ is closed under increasing joins, or equivalently closed in the Cantor topology (see e.g., [GHK+03, III-1.6]). Similarly for decreasing meet $\bigwedge\!\!\!\!/ : 2^\omega \to 2$.

- Similarly, the partial limit operation $\lim : 2^\omega \to 2$ is preserved by $g : 2^\omega \to 2$ iff g is continuous.

Definition 3.6. For $f : 2^k \to 2$, we also have two k-ary relations, distinct from the graph of f, which we denote using the "probabilist's event notation"

$$(f{=}0) := f^{-1}(0) = \{\vec{x} \in 2^k \mid f(\vec{x}) = 0\}, \qquad (f{=}1) := f^{-1}(1) = \{\vec{x} \in 2^k \mid f(\vec{x}) = 1\}.$$

Examples include:

- $(\bigwedge^k{=}0) \subseteq 2^k$ is the k-ary disjointness relation ($k \geq 1$). A function $g : 2^n \to 2$ preserves $\bigwedge^k{=}0$ iff $g^{-1}(1) \subseteq 2^n$ does not contain k strings with bitwise meet $\vec{0}$. If $n \leq k$, then (by considering the strings with exactly one 1) this is equivalent to $g \leq \pi_i$ for some $i < n$.

- In particular, $\bigwedge^1{=}0$ yields the same unary relation as the constant 0.

- $(\lim{=}0) \subseteq 2^\omega$ is the set of eventually 0 sequences. A function $g : 2^n \to 2$ for $n \leq \omega$ preserves $\lim{=}0$ iff g vanishes on a neighborhood of $\vec{0} \in 2^n$, i.e., $g(\vec{0}) = 0$ and g is continuous at $\vec{0} \in 2^n$.

3.B Post's lattice

Definition 3.7. We use the following names for certain clones on 2:

- $\mathbb{A} := \mathrm{Pol}\{+^3\}$ consists of the affine functions.

- $\mathbb{D} := \mathrm{Pol}\{\neg\}$ consists of the self-dual functions.

- $\mathbb{M} := \mathrm{Pol}\{\leq\}$ consists of the monotone functions.

- $\mathbb{M}_\downarrow := \mathrm{Pol}\{\bigwedge\}$ and $\mathbb{M}_\uparrow := \mathrm{Pol}\{\bigvee\}$ (note that these are contained in $\mathbb{M}$, by definition of the domains of the partial operations $\bigwedge, \bigvee$: for example, $x \leq y \iff \bigwedge(y, x, x, \dots) = x$).

- $\mathbb{T}_{0,k} := \mathrm{Pol}\{\bigwedge^k{=}0\}$ and $\mathbb{T}_{1,k} := \mathrm{Pol}\{\bigvee^k{=}1\}$, where k is a positive cardinal.

- $\mathbb{T}_c := \mathbb{T}_{c,1} = \mathrm{Pol}\{c\}$ consists of the functions preserving the constant $c \in 2$.

- $\mathbb{T}_{c,<\kappa} := \bigcap_{1 \leq k < \kappa} \mathbb{T}_{c,k}$ for a regular cardinal κ. Thus

$$\mathbb{T}_c = \mathbb{T}_{c,1} \supseteq \mathbb{T}_{c,2} \supseteq \mathbb{T}_{c,3} \supseteq \cdots \supseteq \mathbb{T}_{c,<\omega} \supseteq \mathbb{T}_{c,\omega} = \mathbb{T}_{c,<\omega_1} \supseteq \mathbb{T}_{c,\omega_1} \supseteq \cdots.$$

  (Recall from 3.6 that e.g., $\mathbb{T}_{0,\omega}^{<\omega_1}$ consists of functions $f : 2^n \to 2$ such that $f \leq \pi_i$ for some i.)

- $\mathbb{L}_c := \mathrm{Pol}\{\lim{=}c\}$ consists of functions in $\mathbb{T}_c$ which are continuous at $\vec{c} \in 2^n$ (recall again 3.6). (Generalizations called $\mathbb{L}_{c,k}$ and $\mathbb{L}_{c,k,t}$ will be defined later in Section 4.D.)

- $\mathbb{A} := \mathrm{Pol}\{\wedge\}$ and $\mathbb{V} := \mathrm{Pol}\{\vee\}$; more generally, $\mathbb{A}_k := \mathrm{Pol}\{\bigwedge^k\}$ and $\mathbb{V}_k := \mathrm{Pol}\{\bigvee^k\}$.

For two (or more) clones $\mathbb{X}, \mathbb{Y}$ in this list, we use the abbreviation

$$\mathbb{XY} := \mathbb{X} \cap \mathbb{Y}.$$

Thus for instance, $\mathbb{AT}_0 = \mathrm{Pol}\{+^3, 0\} = \mathrm{Pol}\{+\}$ consists of the linear functions.

Note that we use these symbols for the $<\infty$-ary clones (see Notation 2.5), consisting of *all* functions preserving said relations. We may restrict to functions of a certain arity by writing e.g., $\mathbb{M}^{<\omega}$ for the finitary monotone functions.

Theorem 3.8 (Post [Pos41])**.** *The lattice $\mathcal{C}lo^{<\omega}(2)$ of finitary clones on 2 is countably infinite, and consists of precisely the finitary restrictions of the following clones*:

- *The "cube" of 8 clones formed from intersecting all combinations of the 3 clones $\mathbb{M}, \mathbb{T}_0, \mathbb{T}_1$.*

- *The 8 infinite series, consisting of the $\omega + 1$ clones $\mathbb{T}_{0,k}$, where k is one of $2, 3, \ldots, <\omega$; the intersections of these with $\mathbb{M}, \mathbb{T}_1$, or both; and the de Morgan duals of these (consisting of $\mathbb{T}_{1,k}$ and their intersections with $\mathbb{M}$ and/or $\mathbb{T}_0$).*

- *The 8 clones consisting of Λ; its intersections with $\mathbb{T}_0$ and/or $\mathbb{T}_1$; and their de Morgan duals.*

- *The 11 subclones of $\mathbb{A}$: $\mathbb{A}, \mathbb{AD}, \mathbb{AT}_0, \mathbb{AT}_1, \mathbb{ADT}_0 = \mathbb{ADT}_1$; and the clone $\langle \neg, 0 \rangle = \langle \neg, 1 \rangle$ of essentially unary functions and its subclones $\langle \neg \rangle, \langle 0, 1 \rangle, \langle 0 \rangle, \langle 1 \rangle$, and the trivial clone $\langle \varnothing \rangle$.*

- *The 3 subclones of $\mathbb{D}$ which are not subclones of $\mathbb{A}$: $\mathbb{D}, \mathbb{DT}_0 = \mathbb{DT}_1$, and $\mathbb{DM}$.*

The ordering between these clones, as well as generators for them, are depicted in Figure 1.1.

It follows from Corollary 2.22 that the task of analyzing $\mathcal{C}lo^{<\nu}(2)$ for higher cardinals $\nu > \omega$ largely reduces to analyzing the lattice $\mathcal{C}lo_F^{\leq\nu}(2)$ of $<\nu$-ary clones restricting to F (Notation 2.23), separately for each finitary clone $F \in \mathcal{C}lo^{<\omega}(2)$ given in Theorem 3.8, i.e., in Figure 1.1.

3.C The bounding and cross-sectioning operators

In this and the following subsections, we consider several operators $\mathbb{O}_2 \to \mathbb{O}_2$, turning functions on 2 into new functions. We will use these to show some global structural relationships between distinct "regions" of (infinitary) Post's lattice; see Figure 3.29.

First, recall the bounding operators $f \mapsto \lceil f \rceil, \lfloor f \rfloor$ from Definition 3.3.

Remark 3.9. We have the following simple identities:

$$\lceil f \circ \vec{g} \rceil = \lceil f \rceil \circ (\pi_0, \lceil \vec{g} \rceil), \qquad\qquad \lfloor f \circ \vec{g} \rfloor = \lfloor f \rfloor \circ (\pi_0, \lfloor \vec{g} \rfloor)$$

(where $\lceil (g_0, g_1, \ldots) \rceil := (\lceil g_0 \rceil, \lceil g_1 \rceil, \ldots)$ and similarly for $\lfloor \vec{g} \rfloor$). Also, clearly

$$\lceil f \rceil \in \langle f, \wedge \rangle, \qquad\qquad \lfloor f \rfloor \in \langle f, \vee \rangle.$$

Lemma 3.10. *For any regular cardinal ν and set of functions $F \subseteq \mathbb{O}_2^{\leq\nu}$, we have*

$$\langle \lceil \langle F \rangle^{<\nu} \rceil \rangle^{<\nu} = \langle \lceil F \rceil \cup \{\wedge\} \rangle^{<\nu}, \qquad\qquad \langle \lfloor \langle F \rangle^{<\nu} \rfloor \rangle^{<\nu} = \langle \lfloor F \rfloor \cup \{\vee\} \rangle^{<\nu}$$

(where $\lceil F \rceil := \{\lceil f \rceil \mid f \in F\}$).

Proof. We only prove the first identity; the second is dual.

$\supseteq$ follows easily from $\wedge = \lceil \mathrm{id} \rceil \in \lceil \langle F \rangle^{<\nu} \rceil$.

$\subseteq$ follows from $\lceil \langle F \rangle^{<\nu} \rceil \subseteq \langle \lceil F \rceil \cup \{\wedge\} \rangle^{<\nu}$, which follows from the fact that the preimage $\lceil - \rceil^{-1}(\langle \lceil F \rceil \cup \{\wedge\} \rangle^{<\nu})$ is closed under $\circ$ and contains all projections, by the above remark. $\qquad\square$

Lemma 3.11. *For any regular cardinal ν and $<\nu$-ary clone $\wedge \in F \subseteq \mathbb{O}_2^{\leq\nu}$, we have*

$$F \cap \mathbb{T}_{0,<\nu} = \langle \lceil F \rceil \rangle^{<\nu}.$$

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Proof. $\supseteq$ follows from Remark 3.9. Conversely, for $f \in F \cap \mathbb{T}_{0,<\nu}$, we have $f \leq \pi_i$ for some i (recall Definition 3.7), whence $f(\vec{x}) = \lceil f \rceil(x_i, \vec{x})$ is in $\langle \lceil F \rceil \rangle$. $\qquad\square$

We now consider an operator that forms a partial inverse to $\lceil - \rceil, \lfloor - \rfloor$.

Definition 3.12. For $f : 2^{1+n} \to 2$ and a constant $c \in 2$, define the **cross-section**[1] $f_c : 2^n \to 2$ by

$$f_c(\vec{x}) := f(c, \vec{x}).$$

These satisfy the identities

$$c^n = (\pi_0^{1+n})_c, \qquad \pi_i^n = (\pi_{1+i}^{1+n})_c, \qquad f_c \circ \vec{g}_c = (f \circ (\pi_0, \vec{g}))_c, \qquad f = (f \circ (\pi_{1+i})_i)_c.$$

Lemma 3.13. *For a regular cardinal ν, $<\nu$-ary clone $F \subseteq \mathbb{O}_2^{\leq \nu}$, and $c \in 2$,*

$$\langle F \cup \{c\} \rangle^{<\nu} = \{f_c \mid f \in F^{\geq 2}\}.$$

(Here, following Convention 2.24, c on the left denotes a positive-arity constant function.)

Proof. $\supseteq$ is obvious; $\subseteq$ follows from the above identities which ensure that the right-hand side is a clone containing $F \cup \{c\}$. $\qquad\square$

Remark 3.14. The above operators are related via

$$\lfloor f \rfloor_0 = f, \qquad \lfloor f_0 \rfloor(x_0, x_1, \dots) = x_0 \vee f(0, x_1, \dots) = x_0 \vee f(x_0, x_1, \dots),$$
$$\lceil f \rceil_1 = f, \qquad \lceil f_1 \rceil(x_0, x_1, \dots) = x_0 \wedge f(1, x_1, \dots) = x_0 \wedge f(x_0, x_1, \dots).$$

Proposition 3.15. *Let ν be a regular cardinal. The composite of the modularity adjunctions (2.2) between the three intervals*

$$F \mapsto \langle F \cup \{1\} \rangle^{<\nu} = \{f_1 \mid f \in F^{\geq 2}\}$$

$$[\langle \wedge \rangle^{<\nu}, \mathbb{O}_2^{\leq \nu}] \underset{\longleftarrow}{\overset{\bot}{\rightrightarrows}} [\langle \wedge, 1 \rangle^{<\nu}, \mathbb{O}_2^{\leq \nu}]$$

$$\Big\uparrow \dashv \Big\downarrow \underset{G \cap \mathbb{T}_{0,<\nu} = \langle \lceil G \rceil \rangle^{<\nu}}{\overset{G}{\downarrow}}$$

$$[\langle \wedge \rangle^{<\nu}, \mathbb{T}_{0,<\nu}^{<\nu}] \xleftarrow{\quad\cong\quad}$$

is an isomorphism. (Here the $\hookrightarrow$ arrows are inclusions.)

Moreover, the composite $F \mapsto \langle F \cup \{1\} \rangle^{<\nu} \mapsto \langle F \cup \{1\} \rangle^{<\nu} \cap \mathbb{T}_{0,<\nu} : [\langle \wedge \rangle^{<\nu}, \mathbb{O}_2^{\leq \nu}] \to [\langle \wedge \rangle^{<\nu}, \mathbb{T}_{0,<\nu}^{<\nu}]$ is equal to $G \mapsto G \cap \mathbb{T}_{0,<\nu}$, which thus has a further right adjoint $F \mapsto \langle F \cup \{1\} \rangle^{<\nu}$.

Proof. We prove the last statement first. For a clone $\wedge \in F \subseteq \mathbb{O}_2^{\leq \nu}$, we have $\langle F \cup \{1\} \rangle^{<\nu} \cap \mathbb{T}_{0,<\nu} = \langle \lceil \{f_1 \mid f \in F^{\geq 2}\} \rceil \rangle^{<\nu}$ (by Lemmas 3.11 and 3.13), which by the preceding remark is contained in $F \cap \mathbb{T}_{0,<\nu}$ (since $\lceil f_1 \rceil \in \langle f, \wedge \rangle \subseteq F$); the other inclusion is obvious.

Now to finish proving that $[\langle \wedge \rangle^{<\nu}, \mathbb{T}_{0,<\nu}^{<\nu}] \cong [\langle \wedge, 1 \rangle^{<\nu}, \mathbb{O}_2^{\leq \nu}]$, it remains to show that for G in the latter interval, we have $G \subseteq \langle \lceil G \rceil \cup \{1\} \rangle^{<\nu}$, which again follows from the preceding remark. $\qquad\square$

Proposition 3.15 shows that, analogously to Corollary 2.22, the entire interval $[\langle \wedge \rangle^{<\nu}, \mathbb{O}_2^{\leq \nu}]$ may be regarded as a "bundle" over the intervals $[\langle \wedge \rangle^{<\nu}, \mathbb{T}_{0,<\nu}^{<\nu}] \cong [\langle \wedge, 1 \rangle^{<\nu}, \mathbb{O}_2^{\leq \nu}]$, which contain respectively the least and greatest elements in each fiber. This is depicted in Figure 3.29(a) for $\nu = \omega$: the bottommost and topmost shaded intervals are isomorphic, and everything between them is between two clones corresponding to each other in the bottom interval and the top interval.

---

[1]These are sometimes called *translations* or *freezifications* in universal algebra; see [FMMT22, §6.3].

3.D The downward-closure of a clone

We now extend Proposition 3.15 to show that in fact, "most" of $[\langle \wedge \rangle^{<\nu}, \mathbb{O}_2^{\leq \nu}]$ decomposes as a product of two intervals, i.e., "most" clones containing $\wedge$ are determined by two "orthogonal projections", one onto $[\langle \wedge \rangle^{<\nu}, \mathbb{T}_{0,<\nu}^{<\nu}] \cong [\langle \wedge, 1 \rangle^{<\nu}, \mathbb{O}_2^{\leq \nu}]$ as in 3.15, and the other given as follows.

Definition 3.16. For a class of functions $G \subseteq \mathbb{O}_2$, let $\downarrow G$ denote its **pointwise downward-closure**:

$$\downarrow G := \{f : 2^n \to 2 \mid \exists g \in G^n \, (f \leq g)\}.$$

For example:

- $\downarrow \langle \varnothing \rangle^{<\nu} = \{f : 2^n \to 2 \mid n < \nu, \, \exists i < n \, (f \leq \pi_i)\} = \mathbb{T}_{0,<\nu}^{<\nu}$.

- $\downarrow \langle \mathbb{MT}_0 \mathbb{T}_1^{<\omega} \rangle^{<\omega_1} = \mathbb{L}_0^{<\omega_1}$. (Recall Definition 3.7. Given $f \in \mathbb{L}_0^\omega$, i.e., $f : 2^\omega \to 2$ which is 0 on a neighborhood of $\vec{0} \in 2^\omega$, which we may assume to be clopen, downward-closed and $\subsetneq 2^\omega$, the indicator function g of the complement of that neighborhood is in $\langle \mathbb{MT}_0 \mathbb{T}_1^{<\omega} \rangle$.)

Lemma 3.17. *If $G \subseteq \mathbb{M}$ is a clone, then so is $\downarrow G$.*

Proof. Let $f \leq g \in G^n$, and $f_i \leq g_i \in G^m$ for each $i < n$. Then $f \circ \vec{f} \leq g \circ \vec{f} \leq g \circ \vec{g} \in G^m$. $\square$

Remark 3.18. Clearly $G \subseteq F \subseteq \downarrow G \implies \downarrow F = \downarrow G$. Thus if $G \subseteq \mathbb{O}_2$ is such that $\downarrow G$ is a clone, then so is $\downarrow F$ for every $F \in [G, \downarrow G]$. In particular, if $F \subseteq \downarrow(F \cap \mathbb{M})$, then $\downarrow F$ is a clone.

The following generalizes Lemma 3.11, which is the case $G = \langle \varnothing \rangle^{<\nu}$:

Lemma 3.19. *For any regular cardinal ν, $G \subseteq \mathbb{O}_2^{\leq \nu}$, and $<\nu$-ary clone $G \cup \{\wedge\} \subseteq F \subseteq \mathbb{O}_2^{\leq \nu}$,*

$$\langle F \cap \downarrow G \rangle^{<\nu} = \langle \lceil F \rceil \cup G \rangle^{<\nu}.$$

Proof. $\supseteq$ follows from Remark 3.9. Conversely, for $f \in F \cap \downarrow G$, we have $f \leq g$ for some $g \in G$, whence $f(\vec{x}) = g(\vec{x}) \wedge f(\vec{x}) = \lceil f \rceil(g(\vec{x}), \vec{x})$ is in $\langle \lceil F \rceil \rangle$. $\square$

Corollary 3.20. *For any regular cardinal ν and $G \subseteq \mathbb{O}_2^{\leq \nu}$,*

$$\langle \downarrow G \rangle^{<\nu} = \langle \mathbb{T}_{0,<\nu}^{<\nu} \cup G \rangle^{<\nu}.$$

Proof. Take $F = \mathbb{O}_2^{\leq \nu}$ above. $\square$

Corollary 3.21. *For any regular cardinal ν and $<\nu$-ary clone $\wedge \in G \subseteq \mathbb{O}_2^{\leq \nu}$,*

$$\langle G \cup \{1\} \rangle^{<\nu} \cap \downarrow G = G.$$

This says that G is "downward-closed within its fiber from Proposition 3.15".

Proof. Take $F = \langle G \cup \{1\} \rangle^{<\nu}$ above, noting that $\langle \lceil \langle G \cup \{1\} \rangle^{<\nu} \rceil \rangle^{<\nu} = G \cap \mathbb{T}_{0,<\nu}$ by 3.15. $\square$

Summarizing the preceding results, we have:

Proposition 3.22. *Let ν be a regular cardinal. We have an adjunction*

$$
\begin{array}{ccc}
& & (\langle F \cup \{1\}\rangle^{<\nu}, \langle {\downarrow} F\rangle^{<\nu}) \\
& & \rotatebox{90}{$\in$} \\
(F \cap \mathbb{T}_{0,<\nu}, \langle {\downarrow} F\rangle^{<\nu}) \ \in\ [\langle\wedge\rangle^{<\nu}, \mathbb{T}_{0,<\nu}^{<\nu}] \times [\mathbb{T}_{0,<\nu}^{<\nu}, \mathbb{O}_2^{<\nu}] \ \cong\ [\langle\wedge,1\rangle^{<\nu}, \mathbb{O}_2^{<\nu}] \times [\mathbb{T}_{0,<\nu}^{<\nu}, \mathbb{O}_2^{<\nu}] \\
\Big\uparrow \qquad\qquad\quad \Big\uparrow{\dashv}\Big\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \rotatebox{90}{$\in$}\qquad (G,H) \\
F \qquad\quad \in \qquad\quad [\langle\wedge\rangle^{<\nu}, \mathbb{O}_2^{<\nu}] \qquad \ni \qquad G \cap H
\end{array}
$$

which restricts to an order-embedding on those $F \in [\langle\wedge\rangle^{<\nu}, \mathbb{O}_2^{<\nu}]$ such that ${\downarrow}F$ is a clone. This includes all monotone F, and more generally, all $F \in [G, {\downarrow}G]$ for some subclone $G \subseteq \mathbb{M}^{<\nu}$. $\qquad\square$

In other words, every such clone F is determined by its two "projections" or "coordinates" $F \cap \mathbb{T}_{0,<\nu}$ and ${\downarrow}F$, in the intervals $[\langle\wedge\rangle^{<\nu}, \mathbb{T}_{0,<\nu}^{<\nu}]$ and $[\mathbb{T}_{0,<\nu}^{<\nu}, \mathbb{O}_2^{<\nu}]$. This gives a detailed analysis of the "tube" on the left side of Post's lattice 1.1 and its $<\nu$-ary version; see Figure 3.29(b).

Some particular consequences of Proposition 3.22 are:

Corollary 3.23. *For any regular cardinal ν and $<\nu$-ary clones $\wedge \in G \subseteq \mathbb{M}\mathbb{T}_{0,<\nu}^{<\nu}$ and $G \subseteq H \subseteq \mathbb{T}_{0,<\nu}^{<\nu}$, the modularity adjunction*

$$[G, \langle G \cup \{1\}\rangle^{<\nu}] \rightleftarrows [H, \langle H \cup \{1\}\rangle^{<\nu}]$$
$$F \mapsto \langle F \cup H\rangle^{<\nu}$$

exhibits the left interval as a retract of the right. $\qquad\square$

This says that the "vertical fibers" of the image of the embedding in 3.22 (i.e., the fibers of the bundle 3.15) are "increasing as the x-coordinate (in $[\langle\wedge\rangle^{<\nu}, \mathbb{T}_{0,<\nu}^{<\nu}]$) increases".

Corollary 3.24. *Let ν be a regular cardinal, $\wedge \in H \subseteq \mathbb{M}^{<\nu}$ be a $<\nu$-ary clone. Then we have modularity isomorphisms between the intervals*

$$[H \cap \mathbb{T}_{0,<\nu}, \mathbb{T}_{0,<\nu}^{<\nu}] \cong [H, {\downarrow}H] \quad \cong [\langle H \cup \{1\}\rangle^{<\nu}, \mathbb{O}_2^{<\nu}]$$
$$F \mapsto \langle F \cup H\rangle^{<\nu} \mapsto \langle F \cup H \cup \{1\}\rangle^{<\nu} = \langle F \cup \{1\}\rangle^{<\nu}$$
$$G \cap \mathbb{T}_{0,<\nu} = \langle\lceil G\rceil\rangle^{<\nu} \leftarrow\!\shortmid \quad G \cap {\downarrow}H \leftarrow\!\shortmid G.$$
$\qquad\square$

This says that the "horizontal slices" of 3.22, with fixed "y-coordinate ${\downarrow}H \in [\mathbb{T}_{0,<\nu}^{<\nu}, \mathbb{O}_2^{<\nu}]$", are isomorphic to the top and bottom slices between which they are sandwiched. For example, taking $H = \langle\mathbb{M}\mathbb{T}_0\mathbb{T}_1^{<\omega_1}\rangle = \langle\wedge,\vee\rangle^{\omega_1}$, so that ${\downarrow}H = \mathbb{L}_0^{<\omega_1}$ (see Definition 3.16), we get

(3.25) $\qquad [\langle\lceil\vee\rceil\rangle^{<\omega_1}, \mathbb{T}_{0,\omega}^{<\omega_1}] \cong [\langle\wedge,\vee\rangle^{<\omega_1}, \mathbb{L}_0^{<\omega_1}] \cong [\langle\wedge,\vee,1\rangle^{<\omega_1}, \mathbb{O}_2^{<\omega_1}]$

$$F \mapsto \langle F \cup \{\vee\}\rangle^{<\omega_1} \quad \mapsto \langle F \cup \{\vee,1\}\rangle^{<\omega_1} = \langle F \cup \{1\}\rangle^{<\omega_1}$$
$$G \cap \mathbb{T}_{0,\omega} = \langle\lceil G\rceil\rangle^{<\omega_1} \leftarrow\!\shortmid \quad G \cap \mathbb{L}_0 \leftarrow\!\shortmid G.$$

When instead $H = \mathbb{M}\mathbb{T}_{0,2}\mathbb{T}_1^{<\omega}$, this yields the slice depicted in Figure 3.29(a) and (b).

Remark 3.26. When $\nu = \omega$, by an inspection of finitary Post's lattice 1.1, we see that Proposition 3.22 in fact applies to all finitary clones $F \ni \wedge$, i.e., ${\downarrow}F = \langle F \cup \mathbb{T}_{0,<\omega}^{<\omega}\rangle^{<\omega}$ is always a clone. It does not seem possible to prove this in full generality for $\nu > \omega$. However, in the context of $<\omega_1$-ary Borel clones, it turns out that Proposition 3.22 does always apply; see Corollary 4.37.
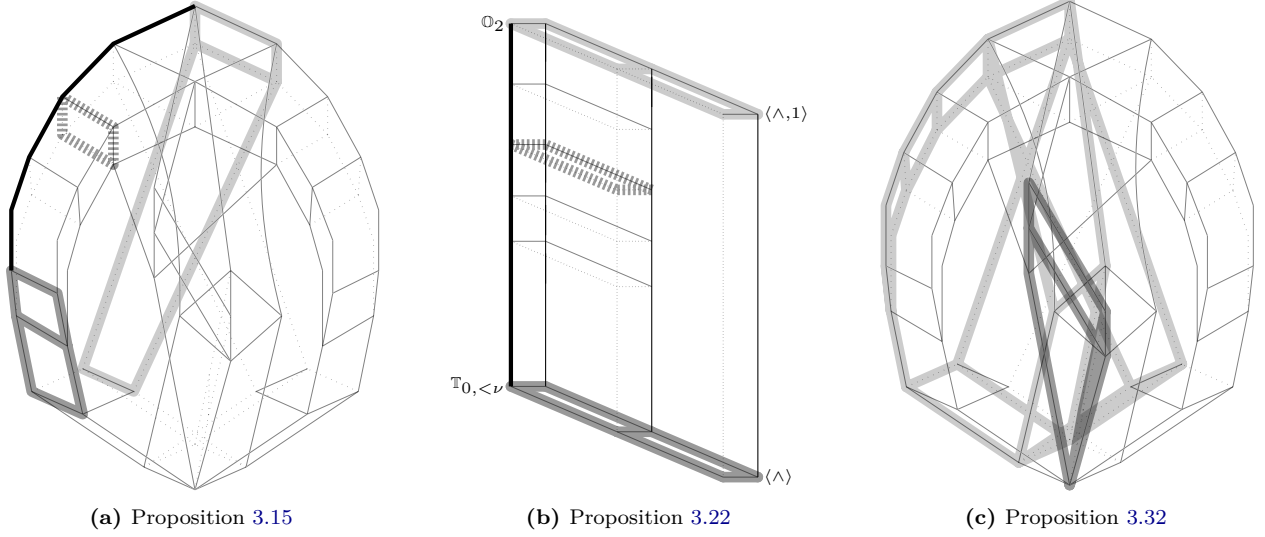
(a) Proposition 3.15   **(b)** Proposition 3.22   **(c)** Proposition 3.32

Figure 3.29: Modularity isomorphisms/retractions between various subintervals of Post's lattice.

For a clone $F \ni \wedge$, the following notion can be useful in determining whether $\downarrow F$ is also a clone:

Definition 3.27. For a function $f : 2^n \to 2$, let $\uparrow f : 2^n \to 2$ denote the indicator function of the **upward-closure** $\uparrow f^{-1}(1)$ of $f^{-1}(1) \subseteq 2^n$, namely

$$(\uparrow f)(\vec{x}) := \bigvee_{\vec{y} \leq \vec{x}} f(\vec{y}).$$

For a class of functions $F \subseteq \mathbb{O}_2$, let $\uparrow[F] := \{\uparrow f \mid f \in F\}$. (Note that this is *not* dual to the pointwise downward-closure $\downarrow G$ of a set of functions from Definition 3.16, which refers to the pointwise ordering on functions $2^n \to 2$, whereas this refers to the bitwise ordering on 2^n.)

The upward-closure $\uparrow f$ is the least monotone function $\geq f$. It follows that for any $F \subseteq \mathbb{O}_2$,

(3.28) $F \cap \mathbb{M} \subseteq \uparrow[F] \subseteq \mathbb{M}$, $F \cap \mathbb{M} = F \cap \uparrow[F]$, $F \subseteq \downarrow\uparrow[F]$.

Thus if $\uparrow[F] \subseteq F$, then $G := \uparrow[F] \subseteq \mathbb{M}$ obeys $G \subseteq F \subseteq \downarrow G$, and so $\downarrow F = \downarrow G$ is a clone. (It is not always true that $\uparrow[F] \subseteq F$ for a clone F; see Remark 4.31.)

3.E   The self-dualizing operator

Next, we consider a different region of Post's lattice on which the cross-sectioning operators yield an isomorphism: the self-dual functions. (This is the same approach used to classify the self-dual clones in the original finitary Post's lattice in e.g., [Lau06, §3.2.3].)

Definition 3.30. For $f : 2^n \to 2$, define $\beta(f) : 2^{1+n} \to 2$ by

$$\beta(f)(x_0, \vec{x}) := (x_0 \ ? \ \delta(f)(\vec{x}) : f(\vec{x})).$$

(Recall from Definition 3.2 that ?: denotes the ternary conditional.)

Remark 3.31. It is easily seen that for any cardinal n, including $n = 0$, β is a bijection

$$2^{2^n} \overset{\beta}{\cong} \mathbb{D}^{1+n} \subseteq 2^{2^{1+n}}$$
$$\underbrace{\phantom{2^{2^n} \cong \mathbb{D}^{1+n} \subseteq 2^{2^{1+n}}}}_{f_0 \leftmapsto f}$$

with retraction $(-)_0 : 2^{2^{1+n}} \twoheadrightarrow 2^{2^n}$ (the cross-section from Definition 3.12), i.e., for any $f : 2^{1+n} \to 2$,

$$f \in \mathbb{D}^{1+n} \iff f \in \operatorname{im}(\beta) \iff f = \beta(f_0).$$

Proposition 3.32. *For any regular cardinal ν, we have an order-isomorphism*

$$[\langle \varnothing \rangle^{<\nu}, \mathbb{D}^{<\nu}] \cong \{G \in \mathcal{C}lo^{<\nu}(2) \mid 0 \in G^1,\ \beta(G) \subseteq G\} \subseteq [\langle 0 \rangle^{<\nu}, \mathbb{O}_2^{\leq \nu}]$$
$$F \mapsto \langle F \cup \{0\} \rangle^{<\nu} = \{f_0 \mid f \in F^{\geq 2}\}$$
$$G \cap \mathbb{D} = \langle \beta(G) \rangle^{<\nu} \leftmapsto G.$$

Proof. By Remark 3.31, $F \mapsto F_0 := \{f_0 \mid f \in F\}$ is a bijection between all subsets of $\mathbb{D}$ and arbitrary sets of functions on 2, including nullary functions f_0 which occur when $f \in F$ is unary. But for a clone F and unary $f \in F$, the unary constant function $x \mapsto f(0) = f_0$ also appears as $(f \circ \pi_0^2)_0$; thus F_0 is completely determined by the positive-arity functions $F_0^{\geq 1} \subseteq F_0$, namely a nullary function is in F_0 iff the corresponding unary constant function is in $F_0^{\geq 1}$. Hence, $F \mapsto \langle F \cup \{0\} \rangle^{<\nu} = F_0^{\geq 1}$ (by Lemma 3.13) remains injective on clones F, with inverse (by Remark 3.31) taking $G = F_0^{\geq 1}$ to $\beta(F_0)$, which can be obtained from $\beta(G) = \beta(F_0^{\geq 1})$ by adding a unary function f whenever the corresponding binary function $f'(x, y) := f(x)$ is in $\beta(G)$; since $f(x) = f'(x, x)$, this means $\beta(F_0) = \langle \beta(G) \rangle^{<\nu}$. This proves that $F \mapsto \langle F \cup \{0\} \rangle^{<\nu}$ is injective, with inverse on its image given by $G \mapsto \langle \beta(G) \rangle^{<\nu}$, which must hence equal the right adjoint $G \mapsto G \cap \mathbb{D}$ (recall again (2.2)).

For a clone of the form $G = \langle F \cup \{0\} \rangle^{<\nu}$, we clearly have $0 \in G^1$ and $\beta(G) \subseteq \langle \beta(G) \rangle^{<\nu} = F \subseteq G$. Conversely, if $G \in \mathcal{C}lo^{<\nu}(2)$ satisfies these two conditions, then clearly $\langle \langle \beta(G) \rangle^{<\nu} \cup \{0\} \rangle^{<\nu} \subseteq G$, and again by Remark 3.31 and Lemma 3.13, $G = G^{\geq 1} = \beta(G)_0^{\geq 1} \subseteq (\langle \beta(G) \rangle^{<\nu})_0^{\geq 1} = \langle \langle \beta(G) \rangle^{<\nu} \cup \{0\} \rangle^{<\nu}$. This shows that the image of the isomorphism is as claimed. $\square$

Thus, the classification of self-dual clones reduces to the classification of clones containing 0 and closed under the β operator. See Figure 3.29(c), which should probably be viewed only impressionistically as it is rather difficult to tell at a glance which clones are closed under β. In the finitary Post's lattice 1.1, the three non-affine clones containing 0 and closed under β are

$$(3.33) \qquad \langle \mathbb{D}^{<\omega} \cup \{0\} \rangle^{<\omega} = \mathbb{O}_2^{<\omega}, \qquad \langle \mathbb{DT}_0^{<\omega} \cup \{0\} \rangle^{<\omega} = \mathbb{T}_0^{<\omega}, \qquad \langle \mathbb{DM}^{<\omega} \cup \{0\} \rangle^{<\omega} = \mathbb{MT}_{0,2}^{<\omega}.$$

The following lemma can simplify checking closure of a clone under β in general:

Lemma 3.34. *For any set of functions $G \subseteq \mathbb{O}_2^{\leq \nu}$, we have*

$$\beta(\langle G \rangle^{<\nu}) \subseteq \langle \beta(G) \rangle^{<\nu}.$$

Thus, $\beta(G) \subseteq \langle G \rangle^{<\nu}$ iff $\beta(\langle G \rangle^{<\nu}) \subseteq \langle G \rangle^{<\nu}$.

Proof. By Remark 3.31, $\beta^{-1}(\langle \beta(G) \rangle^{<\nu}) = (\langle \beta(G) \rangle^{<\nu})_0 \supseteq \beta(G)_0 = G$, whence by Lemma 3.13, $\beta^{-1}(\langle \beta(G) \rangle^{<\nu})^{\geq 1} = (\langle \beta(G) \rangle^{<\nu})_0^{\geq 1} = \langle \langle \beta(G) \rangle^{<\nu} \cup \{0\} \rangle^{<\nu}$ is a clone containing G. $\square$

4   Borel clones on 2

Definition 4.1. Recall that a **Borel set** $A \subseteq 2^n$, where $n \leq \omega$, is a set obtained as a countable Boolean combination of the subbasic clopen sets $\pi_i^{-1}(1)$ for each $i < n$. A **Borel function** $f : 2^n \to 2$ is one such that $f^{-1}(0), f^{-1}(1)$ are Borel sets, i.e., f is the indicator function of a Borel set, i.e., f is obtained by applying $\bigwedge^\omega, \bigvee^\omega, \neg$ pointwise to the π_i. Thus

$$\mathbb{O}_2^{\mathsf{B}} := \langle \bigwedge, \bigvee, \neg \rangle^{<\omega_1} \subseteq \mathbb{O}_2^{<\omega_1}$$

is the $<\omega_1$-ary clone of all Borel functions of countable arity on 2.

We are interested in classifying subclones of $\mathbb{O}_2^{\mathsf{B}}$, which we call **Borel clones** on 2. As such, we adopt the following

Notation 4.2. We will treat the superscript $^{\mathsf{B}}$ for "Borel" as if it were a class of arities, intermediate between finite $(<\omega)$ and countable $(<\omega_1)$. Thus, in analogy with Notation 2.5, for a class of functions $F \subseteq \mathbb{O}_2$, we write

$$F^{\mathsf{B}} := F \cap \mathbb{O}_2^{\mathsf{B}}$$

for the Borel functions in F. The following are some example uses of this notation:

- $\mathbb{M}^{\mathsf{B}} \subseteq \mathbb{O}_2^{\mathsf{B}}$ denotes the clone of monotone Borel functions $2^n \to 2$, $n \leq \omega$ (recall Definition 3.7).

- If $\mathcal{M} \subseteq \mathcal{R}_2$ is a class of relations on 2, then $\mathrm{Pol}^{\mathsf{B}}(\mathcal{M}) = \mathrm{Pol}(\mathcal{M}) \cap \mathbb{O}_2^{\mathsf{B}} = \mathrm{Pol}^{<\omega_1}(\mathcal{M}) \cap \mathbb{O}_2^{\mathsf{B}}$ consists of all Borel polymorphisms of $\mathcal{M}$.

- If $F \subseteq \mathbb{O}_2^{\mathsf{B}}$ is a class of Borel functions, then $\langle F \rangle^{\mathsf{B}} = \langle F \rangle \cap \mathbb{O}_2^{\mathsf{B}} = \langle F \rangle^{<\omega_1}$, while $\overline{\langle F \rangle}^{\mathsf{B}} = \overline{\langle F \rangle} \cap \mathbb{O}_2^{\mathsf{B}}$ consists of the Borel functions which are pointwise limits of functions in $\langle F \rangle^{\mathsf{B}}$. Note that if $F = \mathrm{Pol}^{<\omega}(\mathcal{M})$ consists of all finitary polymorphisms of a class of relations, then $\overline{\langle F \rangle}^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}(\mathrm{Inv}^{<\omega}(\mathrm{Pol}^{<\omega}(\mathcal{M}))) = \mathrm{Pol}^{\mathsf{B}}(\mathcal{M})$ (by Corollary 2.22); e.g., $\overline{\langle \mathbb{M}^{<\omega} \rangle}^{\mathsf{B}} = \mathbb{M}^{\mathsf{B}}$.

We also write

$$\mathcal{C}lo^{\mathsf{B}}(2) := [\langle \varnothing \rangle^{\mathsf{B}}, \mathbb{O}_2^{\mathsf{B}}] \subseteq \mathcal{C}lo^{<\omega_1}(2)$$

for the sublattice of all clones of Borel functions. Thus, the "bundle" of Corollary 2.22 specializes to

(4.3)
$$
\begin{array}{c}
\mathcal{C}lo^{\mathsf{B}}(2) \\
{\scriptstyle \langle - \rangle^{\mathsf{B}}} \uparrow \Big\downarrow {\scriptstyle \dashv} \Big| {\scriptstyle (-) \cap \mathbb{O}_2^{<\omega}} \quad {\scriptstyle \dashv} \Big) {\scriptstyle \mathrm{Pol}^{\mathsf{B}} \circ \mathrm{Inv}^{<\omega} = \overline{\langle - \rangle}^{\mathsf{B}}} \\
\mathcal{C}lo^{<\omega}(2)
\end{array}
$$

As in Notation 2.23, put

$$\mathcal{C}lo_F^{\mathsf{B}}(2) := [\langle F \rangle^{\mathsf{B}}, \overline{\langle F \rangle}^{\mathsf{B}}] \subseteq \mathcal{C}lo^{\mathsf{B}}(2)$$

for the fiber of this bundle over each finitary clone $F \in \mathcal{C}lo^{<\omega}(2)$ in Post's lattice.

In the rest of this section, we will describe the structure of $\mathcal{C}lo_F^{\mathsf{B}}(2)$ for F in various "regions" of Post's lattice $\mathcal{C}lo^{<\omega}(2)$ (see Figure 1.5).

We begin by dispensing with the simplest case: the affine (over $\mathbb{Z}/2\mathbb{Z}$) functions. The following classical result is a special case of general "automatic continuity" results for well-behaved topological groups, and can be proved using either Haar measure (as the Steinhaus–Weil theorem) or Baire category (as the Pettis's theorem). See e.g., [Kec95, 9.9], [Ros09].

Theorem 4.4. *Let $A \subseteq 2^\omega$ be a Borel subgroup under addition mod 2. If A has countable index, then A is clopen.*

Corollary 4.5. $\mathbb{A}^{\mathsf{B}} = \langle \mathbb{A}^{<\omega} \rangle^{\mathsf{B}}$, *i.e., every affine Borel map $f : 2^n \to 2$ for $n \leq \omega$ is continuous.*

Proof. Either $f^{-1}(0)$ or $f^{-1}(1)$ is an index ≤ 2 Borel subgroup of 2^n. $\qquad\square$

Corollary 4.6. *For every finitary clone $F \subseteq \mathbb{A}^{<\omega}$, there is a unique Borel clone restricting to F, namely $\langle F \rangle^{\mathsf{B}} = \overline{\langle F \rangle}^{\mathsf{B}} \in \mathcal{C}lo_F^{\mathsf{B}}(2)$.* $\qquad\square$

This is illustrated in the large shaded region near the bottom of Figure 1.5.

4.A The top cube

We now turn to the Borel clones lying over one of the 8 finitary cones in the "top cube" of Post's lattice 1.1, between $\mathbb{MT}_0\mathbb{T}_1^{<\omega}$ and $\mathbb{O}_2^{<\omega}$. We will give a complete classification of the corresponding Borel clones, of which there are only finitely many (see Figure 1.5). Our main tool will be the modularity isomorphisms from Corollary 3.24.

The following lemma consists of variations of the well-known fact that countable supremum is the "simplest" discontinuous function (the base case of Wadge's lemma; see e.g., [Kec95, 21.16]).

Lemma 4.7. *Let $f : 2^\omega \to 2$.*

(a) *If f is discontinuous, then $\bigvee \in \langle \{f\} \cup \mathbb{T}_{1,<\omega}^{<\omega} \rangle$.*

(b) *If $f \in \mathbb{T}_0 \setminus \mathbb{L}_0$, then $\bigvee \in \langle \{f\} \cup \mathbb{T}_0\mathbb{T}_{1,<\omega}^{<\omega} \rangle$.*

(c) *If $f \in \mathbb{M} \setminus \mathbb{M}_\downarrow$, then $\bigvee \in \langle \{f\} \cup \mathbb{MT}_{1,<\omega}^{<\omega} \rangle$.*

(d) *If $f \in \mathbb{MT}_0 \setminus \mathbb{L}_0$, then $\bigvee \in \langle \{f\} \cup \mathbb{MT}_0\mathbb{T}_{1,<\omega}^{<\omega} \rangle$.*

Proof. (a) Suppose f is discontinuous at $\vec{x}_\infty \in 2^\omega$. Then there is a sequence of strings $\vec{x}_0, \vec{x}_1, \ldots \in 2^\omega$ converging to $\vec{x}_\infty$ such that $f(\vec{x}_0) = f(\vec{x}_1) = \cdots \neq f(\vec{x}_\infty)$. Define

$$g : 2^\omega \longrightarrow 2^\omega$$
$$1\cdots \longmapsto \vec{1}$$
$$01\cdots \longmapsto \vec{x}_0$$
$$001\cdots \longmapsto \vec{x}_1$$
$$\vdots$$
$$\vec{0} \longmapsto \vec{x}_\infty$$

where each tail "$\cdots$" may be an arbitrary string. This is evidently a continuous function, such that each coordinate $g_i := \pi_i \circ g \geq \pi_0 : 2^\omega \to 2$, hence is in $\langle \mathbb{T}_{1,<\omega}^{<\omega} \rangle$; thus $f \circ g = f \circ (g_i)_i \in \langle \{f\} \cup \mathbb{T}_{1,<\omega}^{<\omega} \rangle$. If $f(\vec{x}_\infty) = 0$, then $f \circ g$ agrees with $\bigvee : 2^\omega \to 2$ on all strings of the form $0\cdots$, whence $\bigvee = \pi_0 \vee (f \circ g) \in \langle \{f\} \cup \mathbb{T}_{1,<\omega}^{<\omega} \rangle$. Otherwise, $\bigvee = \pi_0 \vee \neg(f \circ g) = (f \circ g) \to \pi_0 \in \langle \{f\} \cup \mathbb{T}_{1,<\omega}^{<\omega} \rangle$.

(b) Let $\vec{x}_0, \vec{x}_1, \ldots \in f^{-1}(1)$ converge to $\vec{x}_\infty := \vec{0} \in f^{-1}(0)$. Then the function g defined above preserves 0 coordinatewise, and we know $f(\vec{x}_\infty) = 0$, whence $\bigvee = \pi_0 \vee (f \circ g) \in \langle \{f\} \cup \mathbb{T}_0\mathbb{T}_{1,<\omega}^{<\omega} \rangle$.

(c) Let $\vec{x}_0 \geq \vec{x}_1 \geq \cdots$ converge to $\vec{x}_\infty \in 2^\omega$ with $f(\vec{x}_0) = f(\vec{x}_1) = \cdots > f(\vec{x}_\infty)$. Then g defined above is monotone, and $f(\vec{x}_\infty) = 0$, whence $\bigvee = \pi_0 \vee (f \circ g) \in \langle \{f\} \cup \mathbb{MT}_{1,<\omega}^{<\omega} \rangle$.

(d) Let $\vec{x}_0, \vec{x}_1, \ldots \in f^{-1}(1)$ converge to $\vec{x}_\infty := \vec{0} \in f^{-1}(0)$. Then $\vec{x}_i' := \bigvee_{j \geq i} \vec{x}_j$ converge monotonically to $\vec{0}$, and $f(\vec{x}_i') \geq f(\vec{x}_i) = 1$. Combine the arguments of (b) and (c). $\qquad\square$
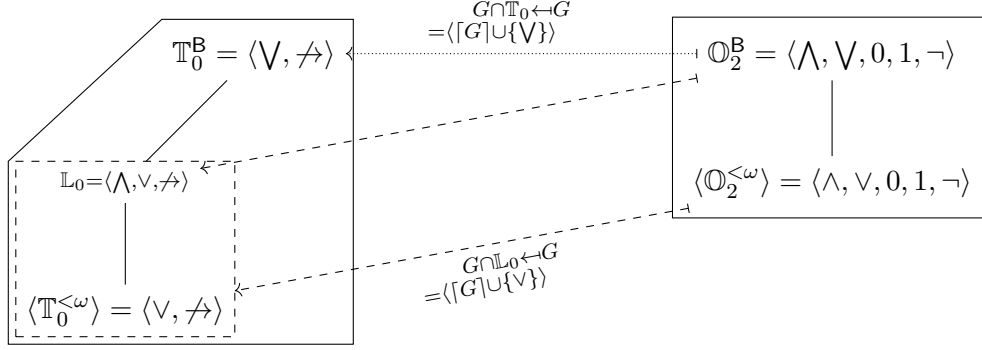
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Figure 4.9: Borel clones with finitary restrictions $\mathbb{T}_0^{<\omega}$ and $\mathbb{O}_2^{<\omega}$, respectively, along with modularity isomorphisms from Corollary 3.24 used to deduce the former from the latter. See also Figure 1.5. (Here, and in the below figures, all clones are $<\omega_1$-ary and restricted to Borel functions, unless otherwise indicated.)

Corollary 4.8. *There are precisely 2 Borel clones on* 2 *restricting to* $\mathbb{O}_2^{<\omega}$ *(see Figure* 4.9*):*

(i) $\mathbb{O}_2^{\mathsf{B}} = \langle \bigvee, \neg \rangle^{\mathsf{B}} = \langle \bigwedge, \bigvee, 0, 1, \neg \rangle^{\mathsf{B}}.$

(ii) $\langle \mathbb{O}_2^{<\omega} \rangle^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\lim\} = \langle \vee, \neg \rangle^{\mathsf{B}} = \langle \wedge, \vee, 0, 1, \neg \rangle^{\mathsf{B}}.$

Proof. If $\langle \mathbb{O}_2^{<\omega} \rangle^{\mathsf{B}} \neq F \in \mathcal{C}lo_{\mathbb{O}_2^{<\omega}}^{\mathsf{B}}(2)$, then F contains a discontinuous function, hence contains $\bigvee$ by Lemma 4.7(a), hence contains all Borel functions since it contains $\neg \in \mathbb{O}_2^{<\omega}$. $\square$

Theorem 4.10. *There are precisely 3 Borel clones on* 2 *restricting to* $\mathbb{T}_0^{<\omega}$ *(see Figure* 4.9*):*

(i) $\mathbb{T}_0^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{0\} = \langle \bigvee, \not\rightarrow \rangle^{\mathsf{B}} = \langle \bigwedge, \bigvee, 0, \not\rightarrow \rangle^{\mathsf{B}}.$

(ii) $\mathbb{L}_0^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\lim{=}0\} = \langle \bigwedge, \vee, \not\rightarrow \rangle^{\mathsf{B}} = \langle \lceil \bigvee \rceil, \vee, \not\rightarrow \rangle^{\mathsf{B}} = \langle \bigwedge, \lceil \bigvee \rceil, \vee, 0, \not\rightarrow \rangle^{\mathsf{B}}.$

(iii) $\langle \mathbb{T}_0^{<\omega} \rangle^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\lim, 0\} = \langle \vee, \not\rightarrow \rangle^{\mathsf{B}} = \langle \wedge, \vee, 0, \not\rightarrow \rangle^{\mathsf{B}}.$

Proof. By Lemma 4.7(b), the greatest clone in $\mathcal{C}lo_{\mathbb{T}_0^{<\omega}}^{\mathsf{B}}(2)$ below the maximum $\mathbb{T}_0^{\mathsf{B}}$ is $\mathbb{L}_0^{\mathsf{B}}$ (which is indeed below the maximum since $\bigvee : 2^\omega \to 2$ preserves 0 but is discontinuous at $\vec{0}$).

By Corollary 3.24 with $H = \langle \mathbb{M}\mathbb{T}_0 \mathbb{T}_1^{<\omega} \rangle^{<\omega_1} = \langle \wedge, \vee \rangle^{\mathsf{B}}$ and $\downarrow H = \mathbb{L}_0^{<\omega_1}$ as in (3.25), the clones between $\langle \mathbb{T}_0^{<\omega} \rangle^{\mathsf{B}}$ and $\mathbb{L}_0^{\mathsf{B}}$ are in order-preserving bijection, via $F \mapsto \langle F \cup \{1\} \rangle^{\mathsf{B}}$, with the clones in $\mathcal{C}lo_{\mathbb{O}_2^{<\omega}}^{\mathsf{B}}(2)$ (since $\langle \mathbb{T}_0^{<\omega} \cup \{1\} \rangle^{\mathsf{B}} = \langle \vee, \not\rightarrow, 1 \rangle^{\mathsf{B}} = \langle \mathbb{O}_2^{<\omega} \rangle^{\mathsf{B}}$ and $\langle \mathbb{L}_0^{\mathsf{B}} \cup \{1\} \rangle^{\mathsf{B}} \supseteq \langle \wedge, \not\rightarrow, 1 \rangle^{\mathsf{B}} = \mathbb{O}_2^{\mathsf{B}}$). The inverse of this bijection takes $G \in \mathcal{C}lo_{\mathbb{O}_2^{\leq\omega}}^{\mathsf{B}}(2)$ to $G \cap \mathbb{L}_0$, which can also be obtained (see again (3.25)) by first restricting all the way to $G \cap \mathbb{T}_{0,\omega} = \langle \lceil G \rceil \rangle^{\mathsf{B}} \ni \wedge$, then adding $\vee$ to yield $\langle \lceil G \rceil \cup \{\vee\} \rangle^{\mathsf{B}}$. Applying this to the two clones in Corollary 4.8 yields

$$\mathbb{O}_2^{\mathsf{B}} \cap \mathbb{L}_0 = \mathrm{Pol}^{\mathsf{B}}\{\lim{=}0\} = \langle \lceil \bigvee \rceil, \lceil \neg \rceil, \vee \rangle^{\mathsf{B}} = \langle \lceil \bigvee \rceil, \not\rightarrow, \vee \rangle^{\mathsf{B}} \qquad \text{(by Lemma 3.10)}$$
$$= \langle \lceil \bigwedge \rceil, \lceil \neg \rceil, \vee \rangle^{\mathsf{B}} = \langle \bigwedge, \not\rightarrow, \vee \rangle^{\mathsf{B}} \qquad \text{(by Lemma 3.10)},$$
$$\langle \mathbb{O}_2^{<\omega} \rangle^{\mathsf{B}} \cap \mathbb{L}_0 = \mathrm{Pol}^{\mathsf{B}}\{\lim, 0\} = \langle \mathbb{T}_0^{<\omega} \rangle^{\mathsf{B}} \qquad \text{(by Remark 2.13)}.$$

Thus, these are the two clones in $\mathcal{C}lo_{\mathbb{T}_0^{<\omega}}^{\mathsf{B}}(2)$ below the maximum.

Finally, to see that the maximum $\mathbb{T}_0^{\mathsf{B}}$ is generated by $\bigvee, \not\rightarrow$, we can again apply Corollary 3.24 but with $H = \langle \wedge, \bigvee \rangle^{\mathsf{B}}$, which clearly bounds all $\mathbb{T}_0^{<\omega_1}$ functions, to get that the clones between $\langle \bigvee, \not\rightarrow \rangle^{\mathsf{B}}$ and $\mathbb{T}_0^{\mathsf{B}}$ correspond to the clones above $\langle \bigvee, \not\rightarrow, 1 \rangle^{\mathsf{B}} = \mathbb{O}_2^{\mathsf{B}}$, whence $\langle \bigvee, \not\rightarrow \rangle^{\mathsf{B}} = \mathbb{T}_0^{\mathsf{B}}$. $\square$
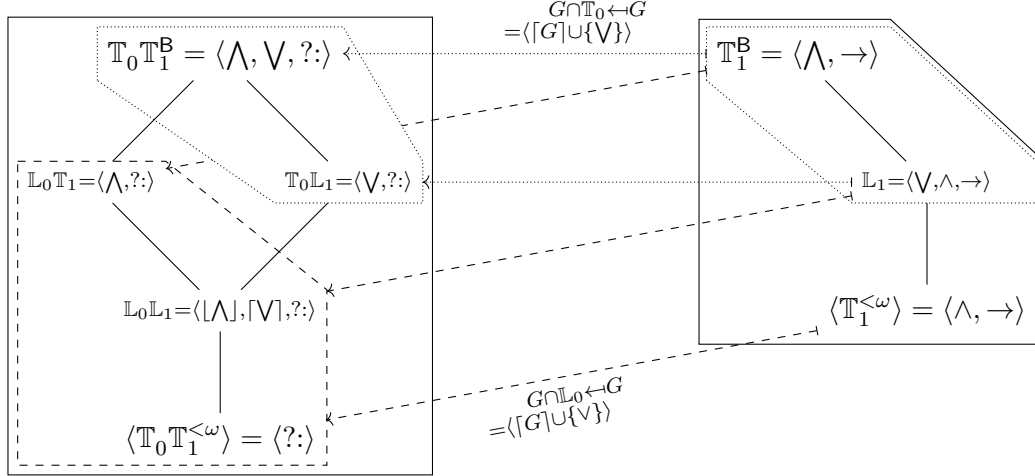
Figure 4.12: Borel clones restricting to $\mathbb{T}_0\mathbb{T}_1^{<\omega}$ (left), obtained by applying modularity isomorphisms from 3.24 to the Borel clones restricting to $\mathbb{T}_1^{<\omega}$ (right), which are isomorphic to $\mathcal{C}lo_{\mathbb{T}_0}^{\mathsf{B}}(2)$ from Figure 4.9.

This proof technique is illustrated in Figure 4.9: we partition the lattice $\mathcal{C}lo_{\mathbb{T}_0}^{\mathsf{B}<\omega}(2)$ into two pieces, each of which is isomorphic by Corollary 3.24 to a part of the previously known lattice $\mathcal{C}lo_{\mathbb{O}_2^{<\omega}}^{\mathsf{B}}(2)$. We now repeatedly apply this technique to the remaining clones in the top cube.

Theorem 4.11. *There are precisely 5 Borel clones on 2 restricting to $\mathbb{T}_0\mathbb{T}_1^{<\omega}$ (see Figure 4.12):*

(i) $\mathbb{T}_0\mathbb{T}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{0,1\} = \langle \bigwedge, \bigvee, ?: \rangle^{\mathsf{B}}$.

(ii) $\mathbb{T}_0\mathbb{L}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{0, \lim{=}1\} = \langle \bigvee, ?: \rangle^{\mathsf{B}}$.

(iii) $\mathbb{L}_0\mathbb{T}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\lim{=}0, 1\} = \langle \bigwedge, ?: \rangle^{\mathsf{B}}$.

(iv) $\mathbb{L}_0\mathbb{L}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\lim{=}0, \lim{=}1\} = \langle \lfloor\bigwedge\rfloor, ?: \rangle^{\mathsf{B}} = \langle \lceil\bigvee\rceil, ?: \rangle^{\mathsf{B}}$.

(v) $\langle \mathbb{T}_0\mathbb{T}_1^{<\omega}\rangle^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\lim, 0, 1\} = \langle ?: \rangle^{\mathsf{B}}$.

Proof. By Lemma 4.7(b), each clone in $\mathcal{C}lo_{\mathbb{T}_0\mathbb{T}_1^{<\omega}}^{\mathsf{B}}(2)$ either is contained in $\mathbb{L}_0\mathbb{T}_1$ or else contains $\langle\{\bigvee\} \cup \mathbb{T}_0\mathbb{T}_1^{<\omega}\rangle^{\mathsf{B}} = \langle \bigvee, ?: \rangle^{\mathsf{B}}$ (and these are mutually exclusive, again since $\bigvee$ is discontinuous at $\vec{0}$).

As in the proof of Theorem 4.10, by Corollary 3.24, the clones below $\mathbb{L}_0\mathbb{T}_1$ may be obtained by applying $G \mapsto G \cap \mathbb{L}_0 = \langle\lceil G\rceil \cup \{\vee\}\rangle^{\mathsf{B}}$ to each of the clones in $\mathcal{C}lo_{\mathbb{T}_1}^{\mathsf{B}<\omega}(2)$, which are the de Morgan duals of the clones in Theorem 4.10; these are easily seen to map to the above-listed clones (iii) to (v) (see Figure 4.12; recall from Post's lattice 1.1 that $\langle ?: \rangle = \langle \vee, \lceil\to\rceil\rangle = \langle \wedge, \lfloor\not\to\rfloor\rangle$).

Again by Corollary 3.24 but with $H = \langle \wedge, \bigvee\rangle^{\mathsf{B}}$, clones above $\langle\bigvee, ?: \rangle^{\mathsf{B}}$ and below $\mathbb{T}_0\mathbb{T}_1^{\mathsf{B}}$ (which are all bounded by $\bigvee$) are obtained by applying $G \mapsto G \cap \mathbb{T}_0 = \langle\lceil G\rceil \cup \{\bigvee\}\rangle^{\mathsf{B}}$ to each of the clones in $\mathcal{C}lo_{\mathbb{T}_1}^{\mathsf{B}<\omega}(2)$ containing $\bigvee$; this yields the clones (i) and (ii). $\quad\square$

In order to apply the same technique to subclones of $\mathbb{M}^{\mathsf{B}}$, we need the following standard facts. The next lemma is well-known in topological lattice theory; see e.g., [Joh82, VII 1.7].

Lemma 4.13. *A monotone function $f : 2^\omega \to 2$ is continuous iff it preserves both increasing limits (=joins) and decreasing limits (=meets). In other words, $\langle \mathbb{M}^{<\omega}\rangle^{<\omega_1} = \mathrm{Pol}^{<\omega_1}\{\leq, \lim\} = \mathbb{M}_\downarrow\mathbb{M}_\uparrow^{<\omega_1}$.*

Proof. For a sequence of bits $x_0, x_1, \ldots, x_\infty \in 2$, we have

$$\lim_{n \to \infty} x_n = x_\infty \iff \exists (y_n)_{n \in \mathbb{N}}, (z_n)_{n \in \mathbb{N}} \in 2^\omega \left(\bigwedge_n (y_n \le x_n \le z_n) \wedge \left(\bigvee_n y_n = x_\infty = \bigwedge_n z_n \right) \right)$$

which is a positive-primitive definition (recall Lemma 2.18) of lim from $\le, \bigvee, \bigwedge$. $\qquad \square$

Recall from Definition 3.5 that $g : 2^n \to 2$, where $n \le \omega$, preserves $\bigvee$ iff it is Scott-continuous, i.e., g is the indicator function of a Scott-open set $g^{-1}(1) \subseteq 2^n$, meaning $g^{-1}(1)$ is upward-closed and its complement is closed under directed joins. The following is again well-known in the theory of topological posets; see e.g., [Joh82, VII 4.8].

Lemma 4.14. $\mathbb{M}_\uparrow^{<\omega_1} = \langle \wedge, \bigvee, 0, 1 \rangle^{<\omega_1}$.

Proof. Let $n \le \omega$. We are to show that if $U \subseteq 2^n$ is Scott-open, then its indicator function is a join of finite meets of projections, i.e., it is open in the product topology on 2^n where 2 has the *Sierpiński topology* $\{\varnothing, \{1\}, \{0, 1\}\}$. We identify 2^n with the powerset $\mathcal{P}(n)$. For every $a \in U$, since a is a directed union of finite subsets, there is a finite $b \subseteq a$ such that $b \in U$, whence $a \in [b, n] := \{c \in \mathcal{P}(n) \mid b \subseteq c\} \subseteq U$. Thus U is a union of such $[b, n]$, each of which is a finite intersection of subbasic opens $[b, n] = \bigcap_{i \in b} \{c \in \mathcal{P}(n) \mid i \in c\}$ in the product topology. $\qquad \square$

Corollary 4.15. *There are precisely 4 Borel clones on 2 restricting to $\mathbb{M}^{<\omega}$ (see Figure 4.16):*

(i) $\mathbb{M}^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\le\} = \langle \bigwedge, \bigvee, 0, 1 \rangle^\mathsf{B}$.

(ii) $\mathbb{M}_\uparrow^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\bigvee\} = \langle \wedge, \bigvee, 0, 1 \rangle^\mathsf{B}$.

(iii) $\mathbb{M}_\downarrow^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\bigwedge\} = \langle \bigwedge, \vee, 0, 1 \rangle^\mathsf{B}$.

(iv) $\langle \mathbb{M}^{<\omega} \rangle^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\lim, \le\} = \langle \wedge, \vee, 0, 1 \rangle^\mathsf{B}$.

Proof. $\mathbb{M}^\mathsf{B} = \langle \bigwedge, \bigvee, 0, 1 \rangle^\mathsf{B}$ by Dyck's monotone version of the Lusin separation theorem for analytic sets; see [Kec95, 28.12], [Dou88, §5]. The generators for $\mathbb{M}_\uparrow^\mathsf{B}$ and $\mathbb{M}_\downarrow^\mathsf{B}$ are by Lemma 4.14 and its dual; and $\langle \mathbb{M}^{<\omega} \rangle^\mathsf{B} = \langle \wedge, \vee, 0, 1 \rangle^\mathsf{B}$ from Post's lattice 1.1. By Lemma 4.7(c), each clone in $\mathcal{C}lo_{\mathbb{M}^{<\omega}}^\mathsf{B}(2)$ either is contained in $\mathbb{M}_\downarrow^\mathsf{B}$, or else contains $\bigvee$ and hence all of $\langle \wedge, \bigvee, 0, 1 \rangle^\mathsf{B} = \mathbb{M}_\uparrow^\mathsf{B}$; dually, it either is contained in $\mathbb{M}_\uparrow^\mathsf{B}$, or else contains $\mathbb{M}_\downarrow^\mathsf{B}$. If it contains both $\mathbb{M}_\uparrow^\mathsf{B}$ and $\mathbb{M}_\downarrow^\mathsf{B}$, then it contains $\langle \bigwedge, \bigvee, 0, 1 \rangle^\mathsf{B} = \mathbb{M}^\mathsf{B}$. If it is contained in both, then it is contained in $\langle \mathbb{M}^{<\omega} \rangle^\mathsf{B}$ by Lemma 4.13. $\qquad \square$

Lemma 4.17. $\mathbb{M}_\downarrow \mathbb{T}_0^{<\omega_1} \subseteq \mathbb{L}_0$.

Proof. This is easily seen directly, or via the dual of Lemma 4.14, or the positive-primitive definition

$$\lim_{n \to \infty} x_n = 0 \iff \exists (y_n)_n \left(\bigwedge_n (x_n \le y_n) \wedge \left(\bigwedge_n y_n = 0 \right) \right). \qquad \square$$

Theorem 4.18. *There are precisely 6 Borel clones on 2 restricting to $\mathbb{M}\mathbb{T}_0^{<\omega}$ (see Figure 4.16):*

(i) $\mathbb{M}\mathbb{T}_0^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\le, 0\} = \langle \bigwedge, \bigvee, 0 \rangle^\mathsf{B}$.

(ii) $\mathbb{M}_\uparrow \mathbb{T}_0^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\bigvee, 0\} = \langle \wedge, \bigvee, 0 \rangle^\mathsf{B}$.

(iii) $\mathbb{M}\mathbb{L}_0^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\le, \lim=0\} = \langle \bigwedge, \lceil \bigvee \rceil, \vee, 0 \rangle^\mathsf{B}$.
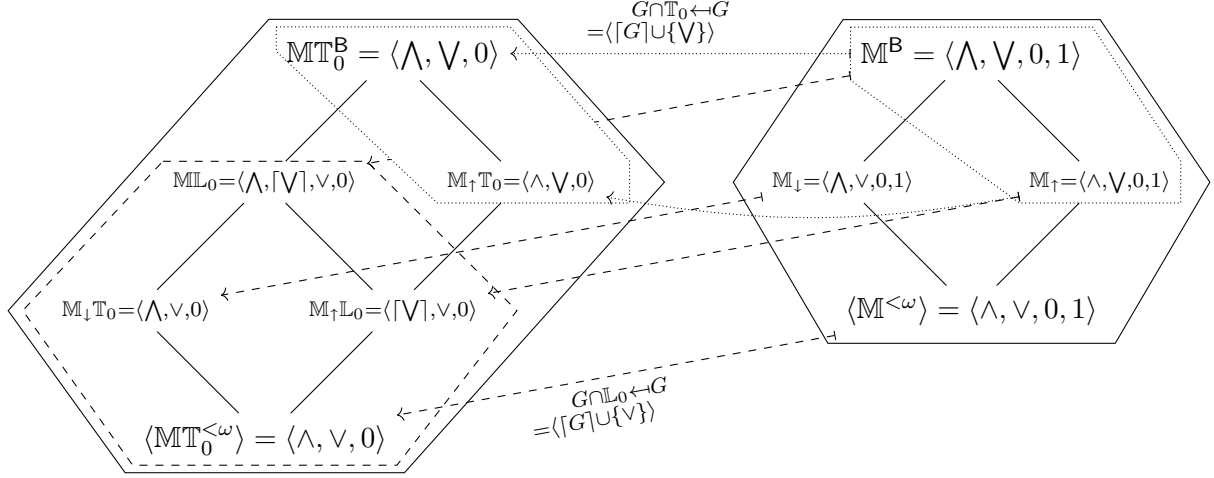
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Figure 4.16: Borel clones restricting to $\mathbb{M}\mathbb{T}_0^{<\omega}$ and $\mathbb{M}^{<\omega}$, along with modularity isomorphisms from 3.24 used to deduce the former from the latter.

  (iv) $\mathbb{M}_\uparrow\mathbb{L}_0^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\mathbb{W}, \lim{=}0\} = \langle\lceil\bigvee\rceil, \vee, 0\rangle^{\mathsf{B}}$.

  (v) $\mathbb{M}_\downarrow\mathbb{T}_0^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\mathbb{A}, 0\} = \mathrm{Pol}^{\mathsf{B}}\{\mathbb{A}, \lim{=}0\} = \langle\bigwedge, \vee, 0\rangle^{\mathsf{B}}$.

  (vi) $\langle\mathbb{M}\mathbb{T}_0^{<\omega}\rangle^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\leq, \lim, 0\} = \langle\wedge, \vee, 0\rangle^{\mathsf{B}}$.

Proof. By Lemma 4.7(d), each clone in $\mathcal{C}lo^{\mathsf{B}}_{\mathbb{M}\mathbb{T}_0^{<\omega}}(2)$ either is contained in $\mathbb{M}\mathbb{L}_0^{\mathsf{B}}$ or contains $\bigvee$. As in the proof of Theorem 4.11, the former clones, namely (iii) to (vi) above, are obtained by applying $G \mapsto G \cap \mathbb{L}_0 = \langle\lceil G\rceil \cup \{\vee\}\rangle^{\mathsf{B}}$ to the clones in $\mathcal{C}lo^{\mathsf{B}}_{\mathbb{M}}(2)$ from Corollary 4.15, while the latter clones (i) and (ii) are obtained by applying $G \mapsto G \cap \mathbb{T}_0 = \langle\lceil G\rceil \cup \{\bigvee\}\rangle^{\mathsf{B}}$ to the clones in $\mathcal{C}lo^{\mathsf{B}}_{\mathbb{M}}(2)$ containing $\bigvee$ (see Figure 4.16). $\qquad\square$

Theorem 4.19. *There are precisely 9 Borel clones on 2 restricting to $\mathbb{M}\mathbb{T}_0\mathbb{T}_1^{<\omega}$ (see Figure 4.20):*

  (i) $\mathbb{M}\mathbb{T}_0\mathbb{T}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\leq, 0, 1\} = \langle\bigwedge, \bigvee\rangle^{\mathsf{B}}$.

  (ii) $\mathbb{M}\mathbb{T}_0\mathbb{L}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\leq, 0, \lim{=}1\} = \langle\lfloor\bigwedge\rfloor, \wedge, \bigvee\rangle^{\mathsf{B}}$.

  (iii) $\mathbb{M}_\uparrow\mathbb{T}_0\mathbb{T}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\mathbb{W}, 0, 1\} = \langle\wedge, \bigvee\rangle^{\mathsf{B}}$.

  (iv) $\mathbb{M}\mathbb{L}_0\mathbb{T}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\leq, \lim{=}0, 1\} = \langle\bigwedge, \lceil\bigvee\rceil, \vee\rangle^{\mathsf{B}}$.

  (v) $\mathbb{M}_\downarrow\mathbb{T}_0\mathbb{T}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\mathbb{A}, 0, 1\} = \langle\bigwedge, \vee\rangle^{\mathsf{B}}$.

  (vi) $\mathbb{M}\mathbb{L}_0\mathbb{L}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\leq, \lim{=}0, \lim{=}1\} = \langle\lfloor\bigwedge\rfloor, \lceil\bigvee\rceil\rangle^{\mathsf{B}}$.

  (vii) $\mathbb{M}_\uparrow\mathbb{L}_0\mathbb{T}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\mathbb{W}, \lim{=}0, 1\} = \langle\lceil\bigvee\rceil, \vee\rangle^{\mathsf{B}}$.

 (viii) $\mathbb{M}_\downarrow\mathbb{T}_0\mathbb{L}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\mathbb{A}, 0, \lim{=}1\} = \langle\lfloor\bigwedge\rfloor, \wedge\rangle^{\mathsf{B}}$.

  (ix) $\langle\mathbb{M}\mathbb{T}_0\mathbb{T}_1^{<\omega}\rangle^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\leq, \lim, 0, 1\} = \langle\wedge, \vee\rangle^{\mathsf{B}}$.

Proof. By Lemma 4.7(d), every clone in $\mathcal{C}lo^{\mathsf{B}}_{\mathbb{M}\mathbb{T}_0\mathbb{T}_1^{<\omega}}(2)$ either is contained in $\mathbb{M}\mathbb{L}_0\mathbb{T}_1$ or contains $\bigvee$. As in the proof of Theorem 4.11, the former clones (iv) to (ix) are obtained by applying $G \mapsto G \cap \mathbb{L}_0 = \langle\lceil G\rceil \cup \{\vee\}\rangle^{\mathsf{B}}$ from Corollary 3.24 to the clones in $\mathcal{C}lo^{\mathsf{B}}_{\mathbb{M}\mathbb{T}_1^{<\omega}}(2)$, which are isomorphic
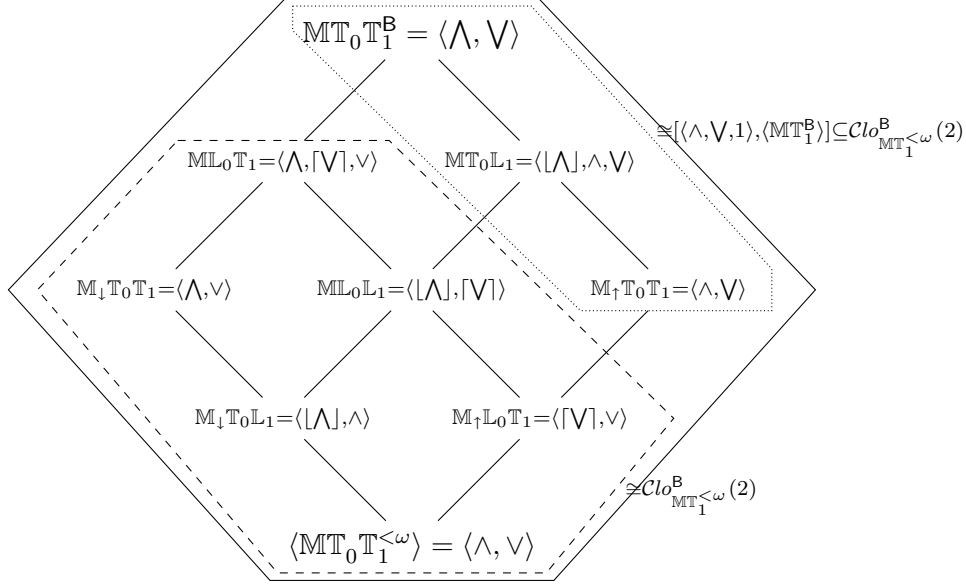
30

Figure 4.20: Borel clones restricting to $\mathbb{MT}_0\mathbb{T}_1^{<\omega}$, with the two indicated sublattices obtained by applying the modularity isomorphisms from Corollary 3.24 to the Borel clones restricting to $\mathbb{MT}_1^{<\omega}$ (not shown).

via de Morgan duality to $Clo^{\mathsf{B}}_{\mathbb{MT}_0^{<\omega}}(2)$ from Theorem 4.18. For example, the de Morgan dual of the clone (iii) in Theorem 4.18 is

$$\mathbb{ML}_1^{\mathsf{B}} = \langle\lfloor\textstyle\bigwedge\rfloor, \wedge, \textstyle\bigvee, 1\rangle^{\mathsf{B}} \in Clo^{\mathsf{B}}_{\mathbb{MT}_1^{<\omega}}(2),$$

which gets mapped to

$$\begin{aligned}
\mathbb{ML}_0\mathbb{L}_1^{\mathsf{B}} &= \langle\lceil\lfloor\textstyle\bigwedge\rfloor\rceil, \lceil\wedge\rceil, \lceil\textstyle\bigvee\rceil, \lceil 1\rceil, \vee\rangle^{\mathsf{B}} \\
&= \langle\lceil\lfloor\textstyle\bigwedge\rfloor\rceil, \wedge^3, \lceil\textstyle\bigvee\rceil, \pi_0, \vee\rangle^{\mathsf{B}} \\
&= \langle\lfloor\textstyle\bigwedge\rfloor, \lceil\textstyle\bigvee\rceil\rangle^{\mathsf{B}}
\end{aligned}$$

since $\wedge^3 \in \langle\wedge\rangle \subseteq \langle\lceil\bigvee\rceil\rangle$ (Lemma 3.10), $\vee \in \langle\lfloor\bigwedge\rfloor\rangle$ similarly, and $\lfloor\bigwedge\rfloor \in \mathbb{ML}_0\mathbb{L}_1$. The above clones (i) to (iii) are similarly obtained by applying $G \mapsto G \cap \mathbb{T}_0 = \langle\lceil G\rceil \cup \{\textstyle\bigvee\}\rangle^{\mathsf{B}}$ to the 3 clones in $Clo^{\mathsf{B}}_{\mathbb{MT}_1^{<\omega}}(2)$ containing $\bigvee$ (or alternatively, by applying de Morgan duality within $Clo^{\mathsf{B}}_{\mathbb{MT}_0\mathbb{T}_1^{<\omega}}(2)$). □

This completes the classification of the Borel clones restricting to the top cube of Post's lattice, i.e., the interval $[\langle\mathbb{MT}_0\mathbb{T}_1^{<\omega}\rangle^{\mathsf{B}}, \mathbb{O}_2^{\mathsf{B}}] \subseteq Clo^{\mathsf{B}}(2)$, which is depicted altogether in Figure 1.5.

Remark 4.21. From this classification, we may read off that every Borel clone in $[\langle\mathbb{MT}_0\mathbb{T}_1^{<\omega}\rangle^{\mathsf{B}}, \mathbb{O}_2^{\mathsf{B}}]$ is some intersection of the following, which form the meet-irreducible elements of $[\langle\mathbb{MT}_0\mathbb{T}_1^{<\omega}\rangle^{\mathsf{B}}, \mathbb{O}_2^{\mathsf{B}}]$.

$$\langle\mathbb{O}_2^{<\omega}\rangle, \qquad \mathbb{T}_0, \mathbb{T}_1, \qquad \mathbb{L}_0, \mathbb{L}_1, \qquad \mathbb{M}, \qquad \mathbb{M}_\downarrow, \mathbb{M}_\uparrow.$$

Dually, every Borel clone in $[\langle\mathbb{MT}_0\mathbb{T}_1^{<\omega}\rangle^{\mathsf{B}}, \mathbb{O}_2^{\mathsf{B}}]$ is generated by some subset of the following functions, together with the generators $\wedge, \vee$ of $\mathbb{MT}_0\mathbb{T}_1^{<\omega}$. In other words, the clones generated by each of the following functions, together with $\wedge, \vee$, yield the join-irreducible elements of $[\langle\mathbb{MT}_0\mathbb{T}_1^{<\omega}\rangle^{\mathsf{B}}, \mathbb{O}_2^{\mathsf{B}}]$.

$$0, 1, \qquad \textstyle\bigwedge, \bigvee, \qquad \lfloor\textstyle\bigwedge\rfloor, \lceil\bigvee\rceil, \qquad ? : (\text{or } \lceil\rightarrow\rceil, \text{ or } \lfloor\nrightarrow\rfloor).$$
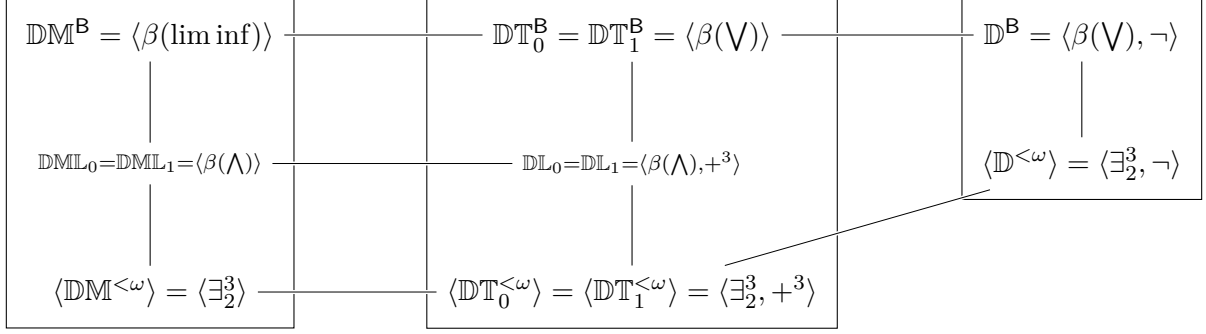
$$\mathbb{DM}^\mathsf{B} = \langle\beta(\liminf)\rangle \quad\quad \mathbb{DT}_0^\mathsf{B} = \mathbb{DT}_1^\mathsf{B} = \langle\beta(\bigvee)\rangle \quad\quad \mathbb{D}^\mathsf{B} = \langle\beta(\bigvee),\neg\rangle$$

$$\mathbb{DML}_0 = \mathbb{DML}_1 = \langle\beta(\bigwedge)\rangle \quad\quad \mathbb{DL}_0 = \mathbb{DL}_1 = \langle\beta(\bigwedge),+^3\rangle \quad\quad \langle\mathbb{D}^{<\omega}\rangle = \langle\exists_2^3,\neg\rangle$$

$$\langle\mathbb{DM}^{<\omega}\rangle = \langle\exists_2^3\rangle \quad\quad \langle\mathbb{DT}_0^{<\omega}\rangle = \langle\mathbb{DT}_1^{<\omega}\rangle = \langle\exists_2^3,+^3\rangle$$

Figure 4.23: Borel clones with finitary restrictions $\mathbb{DM}^{<\omega}, \mathbb{DT}_0^{<\omega}, \mathbb{D}^{<\omega}$ respectively.

Using the classification of the Borel clones over $\mathbb{O}_2^{<\omega}, \mathbb{T}_0^{<\omega}$, and the correspondence between self-dual clones and clones closed under $0, \beta$ given by Proposition 3.32, we easily also obtain a classification of the Borel clones lying over 2 out of the 3 finitary self-dual, non-affine clones (3.33):

Corollary 4.22. *There are precisely 2 Borel clones restricting to $\mathbb{D}^{<\omega}$:*

(i) $\mathbb{D}^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\neg\} = \langle\beta(\bigvee),\neg\rangle^\mathsf{B} = \langle\beta(\bigwedge),\neg\rangle^\mathsf{B}$.

(ii) $\langle\mathbb{D}^{<\omega}\rangle^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\lim,\neg\} = \langle\exists_2^3,\neg\rangle^\mathsf{B}$.

And there are precisely 3 Borel clones restricting to $\mathbb{DT}_0^{<\omega} = \mathbb{DT}_1^{<\omega}$:

(iii) $\mathbb{DT}_0^\mathsf{B} = \mathbb{DT}_1^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\neg,0\} = \langle\beta(\bigvee),+^3\rangle^\mathsf{B} = \langle\beta(\bigvee)\rangle^\mathsf{B}$.

(iv) $\mathbb{DL}_0^\mathsf{B} = \mathbb{DL}_1^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\neg,\lim=0\} = \langle\beta(\bigwedge),+^3\rangle^\mathsf{B}$.

(v) $\langle\mathbb{DT}_0^{<\omega}\rangle^\mathsf{B} = \langle\mathbb{DT}_1^{<\omega}\rangle^\mathsf{B} = \mathrm{Pol}^\mathsf{B}\{\neg,\lim,0\} = \langle\exists_2^3,+^3\rangle^\mathsf{B}$.

(See Figure 4.23.) The functions $\beta(\bigvee),\beta(\bigwedge) : 2^\omega \to 2$ are given by

$$\beta(\bigwedge)(x_0,x_1,\dots) = \Big(x_0 \wedge \bigvee_{i\geq1} x_i\Big) \vee \bigwedge_{i\geq1} x_i, \qquad \beta(\bigvee)(x_0,x_1,\dots) = \Big(\neg x_0 \wedge \bigvee_{i\geq1} x_i\Big) \vee \bigwedge_{i\geq1} x_i.$$

Proof. By applying Proposition 3.32 to the Borel clones restricting to $\mathbb{O}_2^{<\omega}$ (Corollary 4.8) and $\mathbb{T}_0^{<\omega}$ (Theorem 4.10), which are all easily seen to be closed under β. The generating sets above are obtained using Lemma 3.34, by applying β to generating sets for the respective earlier clones (namely $\{\bigvee,\neg\}, \{\bigwedge,\neg\}, \{\bigvee,+\}$, and $\{\bigwedge,+\}$ for the non-essentially finite clones). $\qquad\square$

The classification of the Borel clones lying over $\mathbb{DM}^{<\omega}$, which correspond via 3.32 to the Borel clones over $\mathbb{MT}_{0,2}^{<\omega}$, will be given in Corollary 4.98 after we have treated those latter clones.

4.B   Bounded functions

We now turn our attention to the remaining regions in Post's lattice 1.1: the "side tubes", between $\langle\wedge\rangle$ and $\mathbb{T}_{0,2}$, or dually between $\langle\vee\rangle$ and $\mathbb{T}_{1,2}$; we find it more convenient to focus on the former (left side of the diagram 1.1). Unlike above, here we are unable to give a full classification of the corresponding Borel clones. Nonetheless, we can show some interesting partial structure, as well as give some indications that the remaining structure may be quite complicated (see Figure 1.6).

In this subsection, we focus on the "base" of the "side tube", below $\mathbb{T}_{0,<\omega} = \mathrm{Pol}\{\wedge^{<\omega}{=}0\}$, which consists of the indicator functions $f : 2^n \to 2$ of subsets $f^{-1}(1) \subseteq 2^n$ with the *finite intersection property* (the meet of finitely many strings in $f^{-1}(1)$ is not $\vec{0}$). Contained within these is $\mathbb{T}_{0,\omega} = \mathrm{Pol}\{\wedge{=}0\}$, the indicator functions of subsets with the *countable intersection property*, or equivalently functions $\leq \pi_i$ for some i (assuming countable arities; recall Definition 3.6). Clearly,

$$(4.24) \qquad\qquad \mathbb{T}_{0,\omega}^{<\omega_1} \subseteq \mathbb{L}_0.$$

We also have the following interactions with the clones from the top cube (from Remark 4.21):

Lemma 4.25. $\mathbb{T}_{0,<\omega}\mathbb{L}_1^{<\omega_1} \subseteq \mathbb{T}_{0,\omega}$.

Proof. We have the following positive-primitive definition of $(\bigwedge{=}0) \subseteq 2^\omega$:

$$\bigwedge \vec{x} = 0 \iff \exists \vec{y} \in 2^\omega \left((\lim \vec{y} = 1) \wedge \bigwedge_{i<\omega} (x_0 \wedge \cdots \wedge x_{i-1} \wedge y_i = 0) \right). \qquad \square$$

Corollary 4.26. $\mathbb{M}_\uparrow \mathbb{T}_{0,<\omega}^{<\omega_1} \subseteq \mathbb{T}_{0,\omega}$.

Proof. By the dual of Lemma 4.17, a Scott-continuous function $f \in \mathbb{M}_\uparrow$ is either the constant 0 function, which is clearly in $\mathbb{T}_{0,\omega}$, or in $\mathbb{L}_1$, hence in $\mathbb{T}_{0,\omega}$ by the above. $\qquad \square$

Remark 4.27. It is not true that $\mathbb{M}_\downarrow \mathbb{T}_{0,<\omega}^{<\omega_1} \subseteq \mathbb{T}_{0,\omega}$; see Example 4.44.

We have already worked out the structure of the Borel clones in $[\langle \mathbb{M}\mathbb{T}_{0,<\omega}\mathbb{T}_1^{<\omega} \rangle^{\mathsf{B}}, \mathbb{T}_{0,\omega}^{\mathsf{B}}]$, as part of the classification of the top cube in the preceding subsection:

Corollary 4.28. *There are 2, 3, 4, 6 Borel clones contained in $\mathbb{T}_{0,\omega}$ restricting to the finitary clones $\mathbb{T}_{0,<\omega}^{<\omega}, \mathbb{T}_{0,<\omega}\mathbb{T}_1^{<\omega}, \mathbb{M}\mathbb{T}_{0,<\omega}^{<\omega}, \mathbb{M}\mathbb{T}_{0,<\omega}\mathbb{T}_1^{<\omega}$ respectively, namely $\langle \lceil - \rceil \rangle^{\mathsf{B}}$ applied to each of the Borel clones in $\mathcal{C}lo_{\mathbb{O}_2^{<\omega}}^{\mathsf{B}}(2)$ (Corollary 4.8), $\mathcal{C}lo_{\mathbb{T}_1^{<\omega}}^{\mathsf{B}}(2)$ (dual of Theorem 4.10), $\mathcal{C}lo_{\mathbb{M}^{<\omega}}^{\mathsf{B}}(2)$ (Corollary 4.15), and $\mathcal{C}lo_{\mathbb{M}\mathbb{T}_1^{<\omega}}^{\mathsf{B}}(2)$ (dual of Theorem 4.18); see Figure 4.29.*

Proof. By Proposition 3.15. (We computed $\langle \lceil - \rceil \rangle^{\mathsf{B}}$ of these clones in proving Theorems 4.10, 4.11, 4.18 and 4.19; here we need only omit the last step of adding back $\vee$.) $\qquad \square$

Recall that by Proposition 3.15, every Borel clone above $\langle \mathbb{M}\mathbb{T}_{0,<\omega}\mathbb{T}_1^{<\omega} \rangle^{\mathsf{B}}$ is sandwiched between exactly one of the above 15 clones in $[\langle \mathbb{M}\mathbb{T}_{0,<\omega}\mathbb{T}_1^{<\omega} \rangle^{\mathsf{B}}, \mathbb{T}_{0,\omega}^{\mathsf{B}}]$ (the dashed blocks in Figure 4.29) and the corresponding clone above $\langle \mathbb{M}\mathbb{T}_1^{<\omega} \rangle^{\mathsf{B}} = \langle \wedge, \vee, 1 \rangle^{\mathsf{B}}$ (Figures 4.9, 4.12 and 4.16). More generally, the same holds for $<\omega_1$-ary clones; the only difference is that there are more than just 15 clones below $\mathbb{T}_{0,\omega}^{<\omega_1}$, and we do not have a classification of all of them. Nonetheless, we may deduce the following "dichotomies" for arbitrary $<\omega_1$-ary clones above $\langle \mathbb{M}\mathbb{T}_{0,<\omega}\mathbb{T}_1^{<\omega} \rangle^{<\omega_1}$:

Corollary 4.30 (of preceding subsection). *Let $\mathbb{M}\mathbb{T}_{0,<\omega}\mathbb{T}_1^{<\omega} \subseteq F \subseteq \mathbb{O}_2^{<\omega_1}$ be a $<\omega_1$-ary clone.*

(a) *Either $F \subseteq \langle \mathbb{O}_2^{<\omega} \rangle$, or $F \subseteq \mathbb{L}_1$, or $F \subseteq \mathbb{M}_\uparrow$, or $\bigwedge \in F$.*

(b) *Either $F \subseteq \langle \mathbb{O}_2^{<\omega} \rangle$, or $F \subseteq \mathbb{M}_\uparrow$, or $\lceil \lfloor \bigwedge \rfloor \rceil \in F$.*

(c) *Either $F \subseteq \langle \mathbb{O}_2^{<\omega} \rangle$, or $F \subseteq \mathbb{M}_\downarrow$, or $\lceil \bigvee \rceil \in F$.*

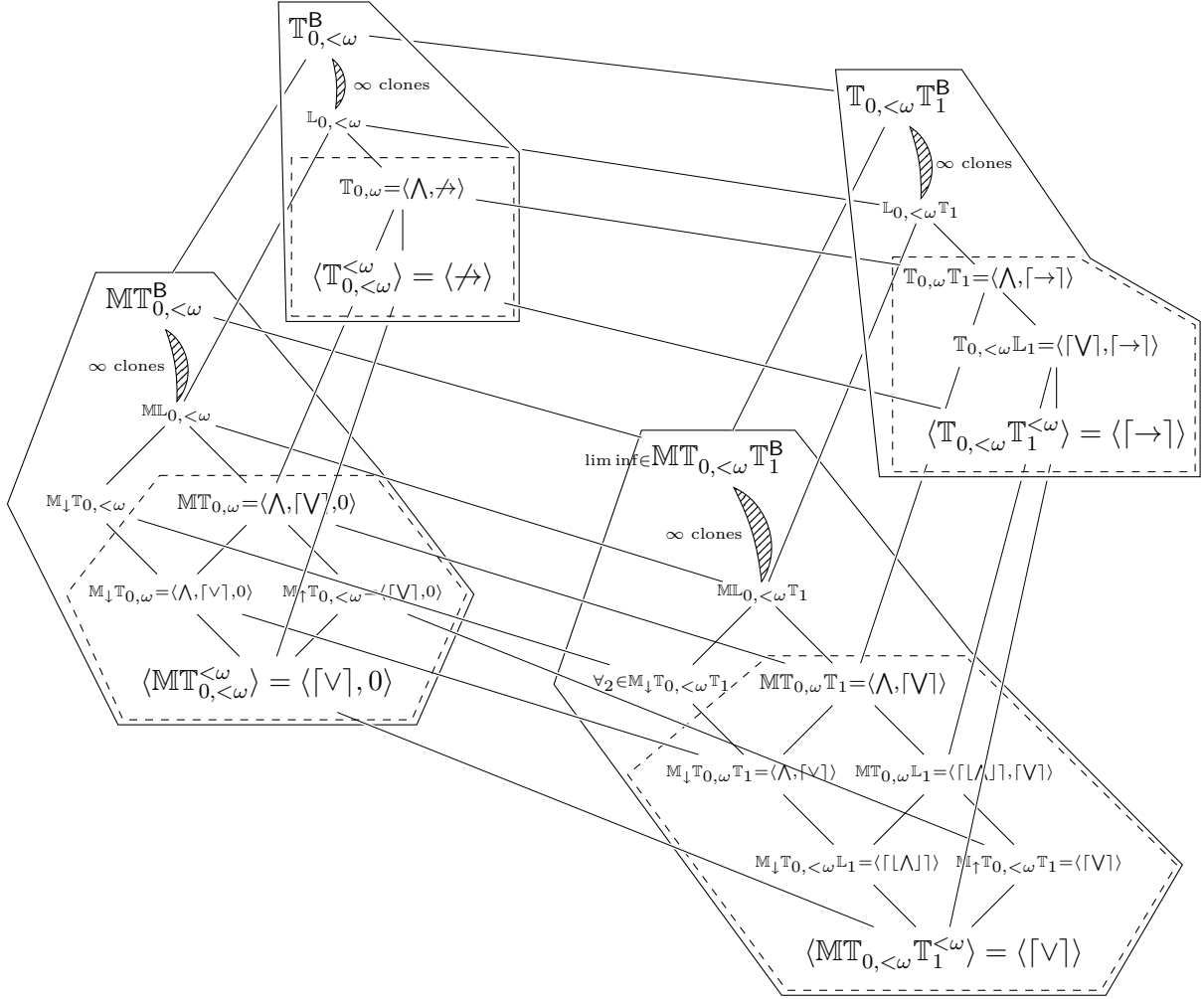Note that in each case, the last alternative is mutually exclusive with the others.

Figure 4.29: All Borel clones restricting to $\mathbb{T}^{<\omega}_{0,<\omega}$, $\mathbb{T}_{0,<\omega}\mathbb{T}^{<\omega}_1$, $\mathbb{MT}^{<\omega}_{0,<\omega}$, or $\mathbb{MT}_{0,<\omega}\mathbb{T}^{<\omega}_1$ that are contained within $\mathbb{T}_{0,\omega}$ (dashed blocks), plus some examples of Borel clones restricting to these finitary clones that are not contained within $\mathbb{T}_{0,\omega}$ (see Figure 1.6 and Section 4.D for structure of the infinite shaded regions).

Proof. (a) is a restatement of the dual of Lemma 4.7 (using the case there corresponding to the finitary restriction of $\langle F \cup \{1\}\rangle$).

(c) follows from applying Lemma 4.7 to $\langle F \cup \{1\}\rangle^{<\omega_1} \in [\langle\mathbb{MT}^{<\omega}_1\rangle^{<\omega_1}, \mathbb{O}^{<\omega_1}_2]$, which shows that if $F \not\subseteq \langle\mathbb{O}^{<\omega}_2\rangle, F \not\subseteq \mathbb{M}_\downarrow$, then $\bigvee \in \langle F \cup \{1\}\rangle^{<\omega_1}$, so by Corollary 3.24, $\lceil\bigvee\rceil \in \lceil\langle F \cup \{1\}\rangle^{<\omega_1}\rceil \subseteq F \cap \mathbb{T}_{0,\omega}$.

(b) Suppose $F \not\subseteq \langle\mathbb{O}^{<\omega}_2\rangle, \mathbb{M}_\uparrow$. If $F \not\subseteq \mathbb{L}_1$, then $\bigwedge \in F$ by (a), and so clearly $\lceil\lfloor\bigwedge\rfloor\rceil \in F$ (since $F \supseteq \mathbb{MT}_{0,<\omega}\mathbb{T}^{<\omega}_1 \ni \wedge, \lceil\vee\rceil$). Otherwise, we have $\langle F \cup \{1\}\rangle^{<\omega_1} \in [\langle\mathbb{MT}^{<\omega}_1\rangle^{<\omega_1}, \mathbb{L}^{<\omega_1}_1]$. By the dual of Corollary 3.24 with $H = \langle\mathbb{MT}_0\mathbb{M}11^{<\omega}\rangle^{<\omega_1}$ and $\uparrow H = \mathbb{L}^{<\omega_1}_1$ as in (3.25), $\langle F \cup \{1\}\rangle^{<\omega_1}$ corresponds to $\langle F \cup \{0,1\}\rangle^{<\omega_1} \in [\langle\mathbb{M}^{<\omega}\rangle^{<\omega_1}, \mathbb{O}^{<\omega_1}_2]$, which is not contained in $\langle\mathbb{O}^{<\omega}_2\rangle$ or $\mathbb{M}_\uparrow$, hence contains $\bigwedge$ by the dual of Lemma 4.7, whence $\lfloor\bigwedge\rfloor \in \lfloor\langle F \cup \{0,1\}\rangle^{<\omega_1}\rfloor \subseteq \langle F \cup \{1\}\rangle^{<\omega_1} \cap \mathbb{T}_{1,\omega}$. Applying Corollary 3.24 again yields $\lceil\lfloor\bigwedge\rfloor\rceil \in \lceil\langle F \cup \{1\}\rangle^{<\omega_1}\rceil \subseteq F \cap \mathbb{T}_{0,\omega}$. (This is essentially a more explicit version of the argument used in Theorems 4.10 and 4.18 to determine $[\langle\mathbb{MT}^{<\omega}_0\rangle^{\mathsf{B}}, \mathbb{L}^{\mathsf{B}}_0]$.) $\square$

Using the preceding dichotomies, we now simplify some of the machinery from Section 3.D, on

the global structure of the "side tubes" of Post's lattice, for Borel and/or $<\omega_1$-ary clones.

Recall Definition 3.16 of the downward-closure $\downarrow G$ of a clone, as well as Definition 3.27 of the upward-closure $\uparrow f$ of a function, which is the canonical candidate for a monotone function $\geq f$.

Remark 4.31. For a Borel function $f : 2^\omega \to 2$, $\uparrow f$ need not be Borel. Conceptually, this is because the disjunction in Definition 3.27 is over 2^ω, hence yields an analytic rather than Borel set.

For a concrete example, let $A \subseteq 2^\omega \times 2^\omega$ be a Borel set whose projection onto the first coordinate is not Borel (see e.g., [Kec95, 14.2]), and let $e : 2^\omega \to 2^\omega$ be a continuous embedding whose image is an antichain, e.g., $e(\vec{x}) := (x_0, \neg x_0, x_1, \neg x_1, \dots)$. Then the upward-closure of $(e \times e)(A) \subseteq 2^\omega \times 2^\omega \cong 2^\omega$ is not Borel, since the first projection of A is $\{\vec{x} \in 2^\omega \mid (e(\vec{x}), \vec{1}) \in (e \times e)(A)\}$.

Thus for a Borel clone $\wedge \in F \subseteq \mathbb{O}_2^\mathsf{B}$, the canonical candidate for a monotone subclone $G \subseteq \mathbb{M}$ with $G \subseteq F \subseteq \downarrow G$, namely $G = \uparrow[F]$, need not be Borel (hence need not be contained in F). The following shows that nonetheless, $\uparrow[F]$ is always "approximately" contained in F:

Lemma 4.32. Let $\wedge \in F \subseteq \mathbb{O}_2^{<\omega_1}$ be a $<\omega_1$-ary clone such that $F \not\subseteq \mathbb{M}, \langle \mathbb{O}_2^{<\omega} \rangle$, let $f \in F^n$, and let $h : 2^n \to 2^n$ be Borel (meaning that each $h_i := \pi_i \circ h : 2^n \to 2$ is Borel) with $h \leq \mathrm{id}$. Then $f \circ h \in F$.

Note that the assumption $h \leq \mathrm{id}$ implies that $f \circ h \leq \uparrow f$. Thus, h may be regarded as a Borel "choice function" witnessing the disjunction in the definition of $\uparrow f$.

Proof. Since $F \not\subseteq \mathrm{Pol}^{<\omega_1}\{\leq\}$, we have $F \cap \mathbb{O}_2^{<\omega} \not\subseteq \mathbb{M}^{<\omega}$ (by the right adjunction in Corollary 2.22), and so (from Post's lattice 1.1) $\lceil \to \rceil \in \mathbb{T}_0 \mathbb{T}_1^{<\omega} \subseteq F$; and if $F \not\subseteq \mathbb{T}_1$, then $\not\to \in \mathbb{T}_0^{<\omega} \subseteq F$. Also since $F \not\subseteq \mathbb{M}, \langle \mathbb{O}_2^{<\omega} \rangle$, from Corollary 4.30, we have $\lceil \lfloor \wedge \rfloor \rceil \in F$; and if $F \not\subseteq \mathbb{L}_1$, then $\wedge \in F$.

First suppose $F \not\subseteq \mathbb{T}_1$. Since $h \leq \mathrm{id}$, each $h_i \leq \pi_i$, so $h_i \in \mathbb{T}_{0,\omega}^\mathsf{B} = \langle \wedge, \not\to \rangle^\mathsf{B} \subseteq F$, and so $f \circ h \in F$.

Now suppose $F \subseteq \mathbb{T}_1$. Then either $F \subseteq \mathbb{L}_1$ or not; in either case, we have $\bigwedge^{<\kappa} \in F$, and there is $J \subseteq n$ of size $|J| < \kappa$ such that $f(\vec{x}) = 1$ whenever $\bigwedge_{j \in J} x_j = 1$, for $\kappa = \omega$ or ω_1 respectively. We may assume that $h(\vec{x}) = \vec{x}$ for all such $\vec{x}$, by replacing h with $h(\vec{x}) \vee (\bigwedge_{j \in J} x_j \wedge \vec{x})$ which does not affect $f \circ h$. From above, each $h_i \in \langle \wedge, \not\to \rangle$, whence for any $j < n$, we have $\pi_j \vee h_i \in \langle \lfloor \wedge \rfloor, \lfloor \not\to \rfloor \rangle$ (dual of Lemma 3.11), whence

$$h_i = \bigwedge_{j \in J} (\pi_i \wedge (\pi_j \vee h_i)) \in \langle \bigwedge^{<\kappa}, \lceil \lfloor \wedge \rfloor \rceil, \lceil \lfloor \not\to \rfloor \rceil \rangle^{<\omega_1} = \langle \bigwedge^{<\kappa}, \lceil \lfloor \wedge \rfloor \rceil, \lceil \to \rceil \rangle^{<\omega_1} \subseteq F. \qquad \square$$

Recall from Lemma 2.17 that the *countable closure* (or $<\omega_1$-*closure*) of a clone $F \subseteq \mathbb{O}_2^{<\omega_1}$ consists of all functions which agree on any countably many inputs with a function in F, or equivalently, all functions which preserve every $<\omega_1$-ary relation preserved by every function in F.

Corollary 4.33. Let $\wedge \in F \subseteq \mathbb{O}_2^{<\omega_1}$ be a $<\omega_1$-ary clone. Then for every $f \in F^n$, $\uparrow f$ is in the countable closure of F.

Thus if F is countably closed, i.e., defined as $\mathrm{Pol}^{<\omega_1}$ of a set of $<\omega_1$-ary relations, then $\uparrow[F] = F \cap \mathbb{M}$, so $F \subseteq \downarrow(F \cap \mathbb{M})$, and so F belongs to the domain of the embedding in Proposition 3.22.

Proof. If f is monotone, then $\uparrow f = f$. If f depends on finitely many variables, then $\uparrow f$ depends on those same variables, and the result follows from inspecting finitary Post's lattice 1.1. Otherwise, in order to find $g \in F^n$ agreeing with $\uparrow f$ on countably many $\vec{a}_i \in 2^n$, apply Lemma 4.32 with the "choice function" $h : 2^n \to 2^n$ which is the identity at all $\vec{x}$ except the $\vec{a}_i \in \uparrow f^{-1}(1)$, which it maps to any $\vec{a}_i \geq h(\vec{a}_i) \in f^{-1}(1)$. The last statement follows from (3.28). $\qquad \square$

We also have the following, which concerns the "left-versus-right" direction ($\mathbb{T}_1$ or not) of the "side tubes" of Post's lattice 1.1, rather than the "front-versus-back" direction ($\mathbb{M}$ or not):

Proposition 4.34. *We have isomorphisms*

$$[\langle\lceil\vee\rceil\rangle^{<\omega_1}, \mathbb{L}_1^{<\omega_1}] \cong \{G \in [\langle\lceil\vee\rceil, 0\rangle^{<\omega_1}, \mathbb{O}_2^{<\omega_1}] \mid G \subseteq \downarrow(G \cap \mathbb{L}_1)\}$$

$$F \mapsto \langle F \cup \{0\}\rangle^{<\omega_1} = \{f_0 \mid f \in F^{\geq 2}\}$$

$$G \cap \mathbb{L}_1 \leftarrow\!\shortmid G,$$

$$[\langle\bigwedge\rangle^{<\omega_1}, \mathbb{T}_1^{<\omega_1}] \cong [\langle\bigwedge, 0\rangle^{<\omega_1}, \mathbb{O}_2^{<\omega_1}]$$

$$F \mapsto \langle F \cup \{0\}\rangle^{<\omega_1}$$

$$G \cap \mathbb{T}_1 \leftarrow\!\shortmid G.$$

In each case, for every clone F on the left, we have $\langle F \cup \{0\}\rangle^{<\omega_1} \subseteq \downarrow F$.

Proof. To show that these maps compose to the identity on the left, in each case, let $\bigwedge^{<\kappa} \in F \subseteq \mathbb{T}_1^{<\omega_1}$ be a clone such that each $f \in F$ is $<\kappa$-continuous at $\vec{1}$, for $\kappa = \omega, \omega_1$ respectively. So for $f \in F^{1+n}$, there is $J \subseteq n$ of size $< \kappa$ such that $f(1, \vec{x}) = 1$ whenever $\bigwedge_{j \in J} x_j = 1$. Thus

$$f_0(\vec{x}) = f(0, \vec{x}) \leq f\left(\bigwedge_{j \in J} x_j, \vec{x}\right).$$

This shows that $\langle F \cup \{0\}\rangle^{<\omega_1} \subseteq \downarrow F$. If f_0 is also $<\kappa$-continuous at $\vec{1}$, then we may pick J so that also $f(0, \vec{x}) = 1$ whenever $\bigwedge_{j \in J} x_j = 1$, whence the above $\leq$ becomes $=$, showing that $f_0 \in F$. Thus $\langle F \cup \{0\}\rangle^{<\omega_1} \cap \mathbb{L}_1$, respectively $\langle F \cup \{0\}\rangle^{<\omega_1} \cap \mathbb{T}_1$, is contained in F; the other inclusion is trivial.

Now let $\lceil\vee\rceil, 0 \in G \subseteq \mathbb{O}_2^{<\omega_1}$ be a clone such that $G \subseteq \downarrow F$ for $F := G \cap \mathbb{L}_1$; we must show $G \subseteq \langle F \cup \{0\}\rangle$. Let $g \in G^n$, and let $g \leq f \in F^n = G^n \cap \mathbb{L}_1$. Then $h : 2^{1+n} \to 2$ with

$$h(y, \vec{x}) := f(\vec{x}) \wedge (y \vee g(\vec{x})) = \lceil\vee\rceil(f(\vec{x}), y, g(\vec{x}))$$

has $h_0 = g$ and $h_1 = f$, thus $h \in \mathbb{L}_1$, and so $h \in G \cap \mathbb{L}_1 = F$ with $g = h_0 \in \langle F \cup \{0\}\rangle$.

Finally, let $\bigwedge, 0 \in G \subseteq \mathbb{O}_2^{<\omega_1}$ be a clone, and let $F := G \cap \mathbb{T}_1$; we must show $G \subseteq \langle F \cup \{0\}\rangle$. If $G \subseteq \mathbb{M}$, then clearly the only monotone function not in $\mathbb{T}_1$ is 0, so $F = G \setminus \{0\}$ and $G = F \cup \{0\}$. If $G \subseteq \langle\mathbb{O}_2^{<\omega}\rangle$, the result follows by inspection of Post's lattice 1.1. Otherwise, for every $g \in G \setminus \mathbb{T}_1 \setminus \{0\}$, we have $f := g \vee \bigwedge \in G \cap \mathbb{T}_1 = F$ by Lemma 4.32 with $h : 2^n \to 2^n$ which is the identity at all $\vec{x}$ except for $\vec{1}$ which gets mapped to any element of $g^{-1}(1)$. Since $G \not\subseteq \mathbb{M}$, from Post's lattice 1.1 we also have $\lceil\to\rceil \in G \cap \mathbb{T}_1 = F$, so $g = f \not\to \bigwedge \in \langle f, \not\to, \bigwedge\rangle \subseteq \langle f, \lceil\to\rceil, 0, \bigwedge\rangle \subseteq \langle F \cup \{0\}\rangle$. $\square$

Similarly to Remark 4.31, note that for a Borel clone $G \subseteq \mathbb{O}_2^{\mathsf{B}}$, $\downarrow G$ may contain non-Borel functions (e.g., if $1 \in G$, then $\downarrow G$ contains all functions). Following Notation 4.2, we adopt

Notation 4.35. For a Borel clone $G \subseteq \mathbb{O}_2^{\mathsf{B}}$, its **Borel downward-closure** is

$$\downarrow^{\mathsf{B}} G := \downarrow G \cap \mathbb{O}_2^{\mathsf{B}}.$$

Proposition 4.36. *For every Borel clone $\wedge \in G \subseteq \mathbb{O}_2^{\mathsf{B}}$, $\downarrow^{\mathsf{B}} G$ is a clone.*

Proof. If $G \subseteq \mathbb{M}$, this follows from Lemma 3.17; if $G \subseteq \langle \mathbb{O}_2^{<\omega} \rangle$, this follows from inspecting finitary Post's lattice 1.1. Suppose neither of these holds. Then from Post's lattice 1.1, $\mathbb{MT}_{0,<\omega}\mathbb{T}_1^{<\omega} \subseteq G$, whence by Corollary 4.30, $\langle G \cup \{0\} \rangle^{\mathsf{B}} = \langle G \cup \{0, \bigwedge\} \rangle^{\mathsf{B}} = \langle G \cup \mathbb{T}_{0,\omega}^{\mathsf{B}} \rangle^{\mathsf{B}}$ (since $\bigwedge \in \langle 0, \lceil \lfloor \bigwedge \rfloor \rceil \rangle$), which by Lemma 3.19 with $F = \mathbb{O}_2^{\mathsf{B}}$ is equal to $\langle \downarrow^{\mathsf{B}} G \rangle^{\mathsf{B}}$. But by Lemma 3.13, every function in $\langle G \cup \{0\} \rangle^{\mathsf{B}}$ is g_0 for some $g \in G$, which by Proposition 4.34 is in $\downarrow G$. So $\langle \downarrow^{\mathsf{B}} G \rangle^{\mathsf{B}} \subseteq \downarrow^{\mathsf{B}} G$. $\qquad\square$

Corollary 4.37 (cf. Proposition 3.22). *We have an order-embedding with right adjoint retraction*

Proof. By Corollary 3.21, $\langle F \cup \{1\} \rangle^{\mathsf{B}} \cap \downarrow^{\mathsf{B}} F = \langle F \cup \{1\} \rangle^{\mathsf{B}} \cap \downarrow F = F$. $\qquad\square$

Thus, Figure 3.29(b) describes the entire left side (above $\langle \wedge \rangle^{\mathsf{B}}$) of the Borel version of Post's lattice. We may also drop the monotonicity assumption from Corollaries 3.23 and 3.24:

Corollary 4.38. *For any Borel clones $\wedge \in G \subseteq \mathbb{T}_{0,\omega}^{\mathsf{B}}$ and $G \subseteq H \subseteq \mathbb{T}_{0,\omega}^{\mathsf{B}}$, the modularity adjunction*

$$[G, \langle G \cup \{1\} \rangle^{\mathsf{B}}] \rightleftarrows [H, \langle H \cup \{1\} \rangle^{\mathsf{B}}]$$

exhibits the left interval as a retract of the right. $\qquad\square$

Corollary 4.39. *For any Borel clone $\wedge \in H \subseteq \mathbb{O}_2^{\mathsf{B}}$, we have modularity isomorphisms*

$$[H \cap \mathbb{T}_{0,\omega}, \mathbb{T}_{0,\omega}^{\mathsf{B}}] \cong [H, \downarrow H] \cong [\langle H \cup \{1\} \rangle^{\mathsf{B}}, \mathbb{O}_2^{\mathsf{B}}]. \qquad\square$$

Corollary 4.33 continues to apply to $<\omega_1$-ary clones, rather than Borel clones; there does not seem to be a direct Borel analogue, since (as noted in Remark 4.31) even for a Borel clone F, $\uparrow[F]$ need not be Borel. (Note that the last conclusion of Corollary 4.33 is subsumed in the Borel setting by Proposition 4.36.) Nonetheless, we do have the following Borel variant of Corollary 4.33:

Proposition 4.40. *Let $\wedge \in F \subseteq \mathbb{O}_2^{\mathsf{B}}$ be a Borel clone. Suppose $F = \mathrm{Pol}^{\mathsf{B}}(\mathcal{M})$ for a countable set $\mathcal{M}$ of Borel downward-closed relations $R \subseteq 2^k$, $k \leq \omega$. Then $\uparrow[F] \subseteq \downarrow(F \cap \mathbb{M})$, and so $F \subseteq \downarrow(F \cap \mathbb{M})$.*

Proof. For each $f \in F^n$, we have $\uparrow f \in \mathrm{Pol}(\mathcal{M}) \cap \mathbb{M}$ by Corollary 4.33. Apply now the reflection theorem [Kec95, 35.10] to the analytic set $\uparrow f^{-1}(1) \subseteq 2^n$, to get a Borel superset $g^{-1}(1) \supseteq \uparrow f^{-1}(1)$ whose indicator function g is in $\mathrm{Pol}^n(\mathcal{M})$. To do so, we need to verify the hypothesis that the class of sets $g^{-1}(1) \subseteq 2^n$ whose indicator function g is in $\mathrm{Pol}(\mathcal{M})$ is $\mathbf{\Pi}_1^1$ on $\mathbf{\Sigma}_1^1$, which means (see again [Kec95, 35.10]), in terms of the indicator functions, that for any family of functions $(g_\alpha : 2^n \to 2)_{\alpha \in 2^\omega}$ which is the indicator function in two variables of an analytic set $\{(\alpha, \vec{x}) \mid g_\alpha(\vec{x}) = 1\} \subseteq 2^\omega \times 2^n$, the set of indices α such that $g_\alpha \in \mathrm{Pol}(\mathcal{M})$ is $\mathbf{\Pi}_1^1$. Indeed,

$$g_\alpha \in \mathrm{Pol}(\mathcal{M}) \iff \forall R \in \mathcal{M}^k \, \forall (x_{i,j})_{i<k,j<n} \in 2^{k \times n} \left(\forall j < n \, \big((x_{i,j})_i \in R \big) \implies \big(g_\alpha((x_{i,j})_j) \big)_i \in R \right);$$

and by downward-closure of R,

$$\left(g_\alpha((x_{i,j})_j)\right)_i \in R \iff \forall \vec{y} \in 2^k \left(\vec{y} \le \left(g_\alpha((x_{i,j})_j)\right)_i \implies \vec{y} \in R\right)$$
$$\iff \forall \vec{y} \in 2^k \left(\forall i < k \left(y_i = 1 \implies g_\alpha((x_{i,j})_j) = 1\right) \implies \vec{y} \in R\right)$$

which are clearly $\mathbf{\Pi}_1^1$. So we have found Borel $\uparrow\! f \le g \in \mathrm{Pol}(\mathcal{M})$. By Dyck's monotone separation theorem [Kec95, 28.12], there is a monotone Borel $\uparrow\! f \le h \le g$. Since each $R \in \mathcal{M}$ is downward-closed, we still have $h \in \mathrm{Pol}(\mathcal{M})$, so $h \in F \cap \mathbb{M}$ with $\uparrow\! f \le h$, as desired. $\qquad\square$

Example 4.41. Each $\mathbb{T}_{0,k}^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\bigwedge^k\!=\!0\}$ is defined by a countable-arity downward-closed Borel relation $(\bigwedge^k\!=\!0) \subseteq 2^k$. Thus, the above shows that for every Borel $f \in \mathbb{T}_{0,k}$, there is a monotone Borel $g \in \mathbb{T}_{0,k}$ such that $f \le g$. (See Section 4.D for similar but more involved examples.)

Remark 4.42. The above proof applies more generally if $\mathcal{M}$ is a possibly uncountable family $\{R_i\}_{i \in I}$ of relations, indexed over some standard Borel space I, such that $\{(i, \vec{x}) \mid \vec{x} \in R_i\}$ is $\mathbf{\Pi}_1^1$.

We close this subsection with some basic remarks on the Borel clones below $\mathbb{T}_{0,<\omega}$ which are *not* contained in $\mathbb{T}_{0,\omega}$; we will study these clones in more detail in Section 4.D (producing in particular the countably infinitely many examples in the crescent-shaped shaded regions in Figure 4.29).

Example 4.43. $\liminf : 2^\omega \to 2$ is in $\mathbb{MT}_{0,<\omega}\mathbb{T}_1^{\mathsf{B}}$, indeed in $\mathbb{AT}_0\mathbb{T}_1^{\mathsf{B}}$ (it is the indicator function of the Fréchet filter of cofinite sets). But it is not in $\mathbb{T}_{0,\omega}$, or even (4.24) in $\mathbb{L}_0$, since the strings of the form $0\cdots0111\cdots \in \liminf^{-1}(1)$ converge to $\vec{0}$.

Example 4.44. The quantifier $\forall_2 : 2^\omega \to 2$, "for all but at most one input bits", namely

$$\forall_2(\vec{x}) = \bigwedge_{i<j<\omega} (x_i \vee x_j) = \bigvee_{i<\omega} \bigwedge_{j \ne i} x_j$$

(recall Definition 3.2), is also in $\mathbb{MT}_{0,<\omega}\mathbb{T}_1^{\mathsf{B}} \setminus \mathbb{T}_{0,\omega}$. Unlike $\liminf$, it is in $\langle \bigwedge, \vee \rangle^{\mathsf{B}} = \mathbb{M}_\downarrow \mathbb{T}_0 \mathbb{T}_1^{\mathsf{B}} \subseteq \mathbb{L}_0^{\mathsf{B}}$.

In fact, $\forall_2$ is a "minimal" example of a non-$\mathbb{T}_{0,\omega}$ function:

Lemma 4.45. If $f : 2^\omega \to 2 \notin \mathbb{T}_{0,\omega}$, then $\forall_2 \in \langle \{f\} \cup \mathbb{M}_\downarrow \mathbb{T}_{0,\omega} \mathbb{T}_1^{\mathsf{B}} \rangle = \langle f, \bigwedge, \lceil\vee\rceil \rangle$.

Proof. Since $f \notin \mathbb{T}_{0,\omega}$, there are $\vec{x}_0, \vec{x}_1, \ldots \in f^{-1}(1)$ with $x_{i,i} = \pi_i(\vec{x}_i) = 0$. Define

$$g_i : 2^\omega \longrightarrow 2^\omega$$
$$\vec{y} \longmapsto (y_i \vee \vec{x}_i) \wedge \vec{y}$$
$$= \begin{cases} \vec{x}_i \wedge \vec{y} & \text{if } y_i = 0, \\ \vec{y} & \text{if } y_i = 1. \end{cases}$$

Then each coordinate $g_{i,j} := \pi_j \circ g_i$ of each g_i is either π_j (if $x_{i,j} = 1$) or $\pi_i \wedge \pi_j$ (if $x_{i,j} = 0$), so $g_{i,j} \in \langle \wedge \rangle$. Thus

$$\bigwedge_i g_i : 2^\omega \longrightarrow 2^\omega$$
$$0111\cdots \longmapsto \vec{x}_0$$
$$1011\cdots \longmapsto \vec{x}_1$$
$$1101\cdots \longmapsto \vec{x}_2$$
$$\vdots$$

has each coordinate in $\langle \bigwedge \rangle$, and so we have $f \circ \bigwedge_i g_i \in \langle f, \bigwedge \rangle$, mapping each string with exactly one 0 in the ith coordinate to $f(\vec{x}_i) = 1$. By Corollary 4.33, we then in fact have some $\forall_2 \leq h \in \langle f, \bigwedge \rangle$. Since $\forall_2 \in \mathbb{M}_\downarrow \mathbb{T}_{0,<\omega} \mathbb{T}_1^{\mathsf{B}} \subseteq \mathbb{M}_\downarrow \mathbb{T}_1^{\mathsf{B}} = \langle \mathbb{M}_\downarrow \mathbb{T}_{0,\omega} \mathbb{T}_1^{\mathsf{B}} \cup \{1\} \rangle^{<\omega_1} \subseteq \langle \{f\} \cup \mathbb{M}_\downarrow \mathbb{T}_{0,\omega} \mathbb{T}_1^{\mathsf{B}} \cup \{1\} \rangle^{<\omega_1}$, by Corollary 3.21 it follows that $\forall_2 \in \langle \{f\} \cup \mathbb{M}_\downarrow \mathbb{T}_{0,\omega} \mathbb{T}_1^{\mathsf{B}} \rangle^{<\omega_1}$. $\qquad \square$

Corollary 4.46. *Let* $\mathbb{M}\mathbb{T}_{0,<\omega} \mathbb{T}_1^{<\omega} \subseteq F \subseteq \mathbb{T}_{0,<\omega}^{<\omega_1}$ *be a* $<\omega_1$*-ary clone. Then* $F \subseteq \mathbb{T}_{0,\omega}$ *or* $\forall_2 \in F$.

Note that as in Corollary 4.30, Borelness of F is not required.

Proof. If $F \not\subseteq \mathbb{T}_{0,\omega}$, then by Lemma 4.25 and Corollary 4.26, $F \not\subseteq \mathbb{L}_1, \mathbb{M}_\uparrow$, and clearly also $F \not\subseteq \langle \mathbb{O}_2^{<\omega} \rangle$ (since $F \not\subseteq \mathbb{T}_{0,\omega}$), whence by Corollary 4.30, $\bigwedge \in F$; now apply the preceding lemma. $\qquad \square$

Remark 4.47. We do not know whether $\mathbb{M}_\downarrow \mathbb{T}_{0,<\omega} \mathbb{T}_1^{\mathsf{B}} = \langle \forall_2 \rangle^{\mathsf{B}}$, but suspect this to be the case. For instance, we have $\forall_3 \in \langle \forall_2 \rangle$ (recall again Definition 3.2), since it is easily seen that

$$\forall_3(\vec{x}) \leq \forall_2(\forall_2(x_{i_{0,0}}, x_{i_{0,1}}, \dots), \forall_2(x_{i_{1,0}}, x_{i_{1,1}}, \dots), \dots),$$

where $(i_{j,k})_{j,k \in \omega}$ is any bijection $\omega \times \omega \cong \omega$; now apply Corollary 3.21. More generally, $\langle \forall_2 \rangle = \langle \forall_k \rangle$ for any $2 \leq k < \omega$.

Note that if $\mathbb{M}_\downarrow \mathbb{T}_{0,<\omega} \mathbb{T}_1^{\mathsf{B}} = \langle \forall_2 \rangle^{\mathsf{B}}$, then by Corollary 4.39, we would have a complete classification of all Borel clones "just above" $\mathbb{T}_{0,\omega}$ in Figure 4.29, i.e., between $\mathbb{M}_\downarrow \mathbb{T}_{0,<\omega} \mathbb{T}_1$ and $\mathbb{L}_{0,<\omega}$ which is the downward-closure $\downarrow \mathbb{M}_\downarrow \mathbb{T}_{0,<\omega} \mathbb{T}_1$ (see Remark 4.84).

4.C Filters and ideals

Next, we consider the Borel clones below $\mathbb{A} = \mathrm{Pol}\{\wedge\}$, which consists of indicator functions $f : 2^n \to 2$ of filters $f^{-1}(1) \subseteq 2^n$ or the empty set; the latter is excluded in $\mathbb{A}\mathbb{T}_1 = \mathrm{Pol}\{\wedge, 1\}$, while the improper filter 2^n is excluded in $\mathbb{A}\mathbb{T}_0$. Similarly, $\mathbb{A}_\omega = \mathrm{Pol}\{\bigwedge\}$ consists of indicator functions of σ-filters. But since 2^ω is countably generated under meets, a σ-filter in it is just the principal filter above some element $\vec{x}$, namely $\{\vec{y} \in 2^\omega \mid \vec{x} \leq \vec{y}\}$, whose indicator function is $\bigwedge_{\substack{i < \omega \\ x_i = 1}} \pi_i$. This yields

$$(4.48) \qquad \mathbb{A}_\omega^{<\omega_1} = \langle \bigwedge, 0, 1 \rangle^{<\omega_1}, \qquad\qquad \mathbb{A}_\omega \mathbb{T}_1^{<\omega_1} = \langle \bigwedge, 1 \rangle^{<\omega_1},$$
$$\mathbb{A}_\omega \mathbb{T}_0^{<\omega_1} = \langle \bigwedge, 0 \rangle^{<\omega_1}, \qquad\qquad \mathbb{A}_\omega \mathbb{T}_0 \mathbb{T}_1^{<\omega_1} = \langle \bigwedge \rangle^{<\omega_1}.$$

We also clearly have

$$(4.49) \qquad\qquad \mathbb{A}\mathbb{M}_\downarrow = \mathrm{Pol}\{\wedge, \mathbb{A}\} = \mathrm{Pol}\{\bigwedge\} = \mathbb{A}_\omega,$$
$$(4.50) \qquad\qquad \mathbb{A}_\omega \mathbb{T}_0 = \mathrm{Pol}\{\bigwedge, 0\} \subseteq \mathrm{Pol}\{\bigwedge = 0\} = \mathbb{T}_{0,\omega}.$$

Lemma 4.51. $\mathbb{A}\mathbb{L}_0^{<\omega_1} = \mathbb{A}\mathbb{T}_{0,\omega}^{<\omega_1}$.

Proof. $\supseteq$ holds by (4.24). Conversely, if $f : 2^\omega \to 2$ is the indicator function of a filter $f^{-1}(1)$ but $f \not\leq \pi_i$ for each $i < \omega$, then each of the strings $\vec{x}_i := 1 \cdots 101 \cdots$ with a single 0 in the ith bit would be in $f^{-1}(1)$, whence $\vec{x}_0, \vec{x}_0 \wedge \vec{x}_1, \cdots \in f^{-1}(1)$ would converge to $\vec{0}$, whence $f \notin \mathbb{L}_0$. $\qquad \square$

Lemma 4.52. $\mathbb{A}\mathbb{L}_1^{<\omega_1} = \langle \mathbb{A}\mathbb{T}_1^{<\omega} \rangle^{<\omega_1}$.

Proof. Suppose, dually, that $f : 2^\omega \to 2 \in \mathbb{V}\mathbb{L}_0^{<\omega_1}$ is the indicator function of the complement of an ideal $f^{-1}(0) \subseteq 2^\omega$, such that f is continuous at $\vec{0}$, i.e., $\vec{0}$ is in the interior of $f^{-1}(0)$. Then $f^{-1}(0)$ is a clopen subgroup of $(\mathbb{Z}/2\mathbb{Z})^\omega$, hence f is continuous and so generated by finitary functions. $\qquad \square$
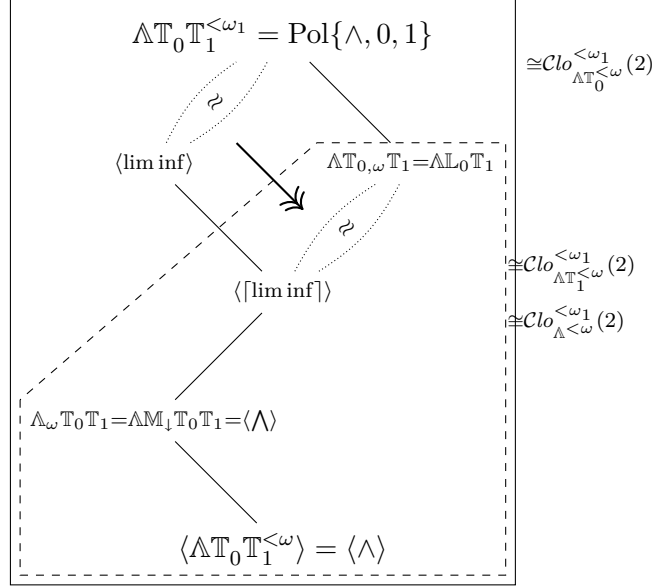
Figure 4.53: All countable arity clones restricting to $\mathbb{A}\mathbb{T}_0\mathbb{T}_1^{<\omega}$. Aside from the 6 labelled, all other clones must lie in one of the two '$\approx$' regions, inside of which no clones can be distinguished by countable arity relations. The clones restricting to $\mathbb{A}\mathbb{T}_0^{<\omega}$ are isomorphic to these, while the clones restricting to each of $\mathbb{A}\mathbb{T}_1^{<\omega}, \mathbb{A}^{<\omega}$ are isomorphic to the sublattice within the dashed block.

From these observations, we may compute all Borel clones that are the intersection of $\mathbb{A}$ with a clone from the top cube (Remark 4.21); see Figure 4.53.

We now show that these are "approximately" all of the Borel clones, indeed all of the $<\omega_1$-ary clones, with finitary restriction $\mathbb{A}^{<\omega}$ or one of its relatives. To do so, we need "dichotomies" along the lines of Corollary 4.30 and Lemma 4.45. Note however that the proof technique used there does not apply, since it depended on a prior understanding of $[\langle \mathbb{M}\mathbb{T}_{0,<\omega}\mathbb{T}_1 \rangle, \mathbb{T}_{0,\omega}]$, whereas we lack a similar understanding of $\mathbb{A}\mathbb{T}_{0,\omega}$. Instead, we use the following, which is implicit in [Kah92]:

Lemma 4.54 (Kahane). *If* $f : 2^\omega \to 2 \in \mathbb{M}\mathbb{T}_{0,2} \setminus \mathbb{L}_0$, *then* $\liminf \in \langle f, \wedge \rangle$.

(Monotonicity of f is automatic if $f \in \mathbb{A}$, but is in fact not required otherwise; see Lemma 4.75.)

Proof. As in Lemma 4.7(d), there are $\vec{x}_0 \geq \vec{x}_1 \geq \cdots$ in $f^{-1}(1)$ converging to $\vec{0} \in f^{-1}(0)$. By monotonicity, it follows that $f(\vec{y}) = 1$ whenever $\vec{y}$ has cofinally many 1's, i.e.,

$$\liminf \leq f \qquad \text{which implies by de Morgan duality} \qquad \delta(f) = \neg f \neg \leq \limsup.$$

But since $f \in \mathbb{T}_{0,2}$, we have $f \wedge f\neg = 0$, i.e., $f \leq \neg f\neg$; thus

$$f(\vec{y}) = \lim \vec{y} \quad \text{whenever } \vec{y} \in 2^\omega \text{ converges.}$$

Hence

$$\liminf \vec{y} = \lim_{n\to\infty} \inf_{i\geq n} y_i = \lim_{n\to\infty} \lim_{m\to\infty} (y_i \wedge y_{i+1} \wedge \cdots \wedge y_m)$$

$$= f \begin{pmatrix} f(y_0, y_0 \wedge y_1, y_0 \wedge y_1 \wedge y_2, \dots), \\ f(y_1, y_1 \wedge y_2, y_1 \wedge y_2 \wedge y_3, \dots), \\ f(y_2, y_2 \wedge y_3, y_2 \wedge y_3 \wedge y_4, \dots), \\ \dots \end{pmatrix}. \qquad \square$$

Corollary 4.55. *If* $f : 2^\omega \to 2 \in \mathbb{A} \setminus \mathbb{M}_\downarrow$*, then* $\lceil \liminf \rceil \in \langle f, \wedge \rangle$*.*

Proof. Since $f \notin \mathbb{M}_\downarrow$, it is clearly not a constant function; thus $f \in \mathbb{A}\mathbb{T}_0\mathbb{T}_1$. If $f \notin \mathbb{L}_0$, then Lemma 4.54 applies; so suppose $f \in \mathbb{A}\mathbb{L}_0^{<\omega_1} = \mathbb{A}\mathbb{T}_{0,\omega}^{<\omega_1}$ (Lemma 4.51). Let $\vec{x} := \bigwedge f^{-1}(1)$ and

$$g(\vec{y}) := f(\vec{x} \vee \vec{y}) = f\big((x_i ? 1 : y_i)_{i<\omega}\big);$$

then $g \in \langle f, 1 \rangle \subseteq \mathbb{A}$. Since f does not preserve $\mathbb{\Lambda}$, and $f^{-1}(1) \subseteq 2^\omega$ is a filter, there is a decreasing sequence in $f^{-1}(1)$ converging to $\vec{x} \in f^{-1}(0)$; the meet of each term in this sequence with $\neg\vec{x}$ is then a sequence in $g^{-1}(1)$ converging to $\vec{0} \in g^{-1}(0)$, and so $g \in \mathbb{A}\mathbb{T}_0 \setminus \mathbb{L}_0$. By Lemma 4.54, $\liminf \in \langle g, \wedge \rangle$, whence by Proposition 3.15, $\langle f, \wedge \rangle = \langle \lceil \langle f, \wedge, 1 \rangle \rceil \rangle \supseteq \langle \lceil \langle g, \wedge \rangle \rceil \rangle \ni \lceil \liminf \rceil$. $\qquad \square$

Corollary 4.56. *If* $f : 2^\omega \to 2 \in \mathbb{A}$ *is discontinuous, then* $\bigwedge \in \langle f, \wedge \rangle$*.*

Proof. If $f \notin \mathbb{M}_\downarrow$, then by the preceding corollary, we get $\lceil \liminf \rceil \in \langle f, \wedge \rangle$, from which we get $\bigwedge \vec{x} = \lceil \liminf \rceil (x_0; x_1; x_1, x_2; x_1, x_2, x_3; x_1, x_2, x_3, x_4; \dots)$. Otherwise, $f \in \mathbb{A}\mathbb{M}_\downarrow^{<\omega_1} = \mathbb{A}_\omega^{<\omega_1} = \langle \bigwedge, 0, 1 \rangle^{\omega_1}$ ((4.49) and (4.48)); since f is discontinuous, it must be a meet of infinitely many variables, from which we easily get $\bigwedge$ via a variable substitution. $\qquad \square$

These results show that the $<\omega_1$-ary clones $\mathcal{C}lo_{\mathbb{A}\mathbb{T}_0\mathbb{T}_1^{<\omega}}^{<\omega_1}(2)$ restricting to $\mathbb{A}\mathbb{T}_0\mathbb{T}_1^{<\omega}$, say, are as depicted in Figure 4.53, with all other potential clones in one of the intervals $[\liminf, \mathbb{A}\mathbb{T}_0\mathbb{T}_1]$ or $[\lceil \liminf \rceil, \mathbb{A}\mathbb{T}_{0,\omega}\mathbb{T}_1]$. Within these intervals, we have a mismatch between the minimal functions provided by Lemma 4.54 and Corollary 4.55 and the invariant relations: for example, not every Borel filter $f \in \mathbb{A}\mathbb{T}_0\mathbb{T}_1^{\mathsf{B}}$ can be constructed from the Fréchet filter $\liminf$. (See Example 4.62.) We point out however that this is true "approximately". Recall again from Lemma 2.17 that the *countable closure* of a clone consists of all functions agreeing on any countably many input strings.

Proposition 4.57. *For any finitary clone* $\wedge \in F \subseteq \mathbb{O}_2^{<\omega}$ *in Post's lattice, the infinitary clone* $\overline{\langle F \rangle}$ *(of arbitrary arity) is the countable closure of* $\langle F \cup \{\liminf\} \rangle$*.*

(Here, as usual, $\liminf$ means $\liminf^\omega : 2^\omega \to 2$.)

Proof. Since $\wedge \in F$, we have $\bigwedge \in \overline{\langle F \rangle}$, and so $\liminf \in \overline{\langle F \rangle}$; this shows $\supseteq$. To show $\subseteq$: let $g \in \overline{\langle F \rangle}^n$ and $\vec{x}_0, \vec{x}_1, \dots \in 2^n$; we must find $f \in \langle F \cup \{\liminf\} \rangle^n$ agreeing with g on each $\vec{x}_i$. Since $g \in \overline{\langle F \rangle}$, we may find $f_0, f_1, \dots \in \langle F \rangle^n$ such that f_i agrees with g on $\vec{x}_0, \dots, \vec{x}_{i-1}$. Let $f := \liminf_i f_i$. $\qquad \square$

Corollary 4.58. $\mathbb{A}\mathbb{T}_0\mathbb{T}_1^\omega \subseteq 2^{2^\omega}$ *is the countable closure of* $\langle \liminf \rangle^\omega$*.* $\qquad \square$

Corollary 4.59. $\mathbb{A}\mathbb{T}_0^\omega, \mathbb{A}\mathbb{T}_1^\omega, \mathbb{A}^\omega$ *are the countable closures of* $\langle \liminf, 0 \rangle^\omega, \langle \liminf, 1 \rangle^\omega, \langle \liminf, 0, 1 \rangle^\omega$*.*

Proof. A function $f \in \mathbb{M}$ either is constant or preserves both $0, 1$. $\qquad \square$

Corollary 4.60. $\mathbb{A}\mathbb{T}_{0,\omega}\mathbb{T}_1^{\omega}, \mathbb{A}\mathbb{T}_{0,\omega}^{\omega}$ *are the countable closures of* $\langle\lceil\liminf\rceil\rangle^{\omega}, \langle\lceil\liminf\rceil, 0\rangle^{\omega}$.

Proof. This may be seen using the modularity isomorphisms (similarly to Corollary 4.55), or by slightly modifying the proof of Proposition 4.57: if $g \in \mathbb{M}\mathbb{T}_{0,\omega}$, say (up to a variable permutation) $g \leq \pi_0$, then we may pick the f_i in the proof of Proposition 4.57 to agree with g on $0111\cdots$, hence also $f_i \leq \pi_0$; then $f = \lceil\liminf\rceil(\pi_0, f_0, f_1, \dots)$. $\qquad\square$

The following summarizes our positive classification results about $\mathbb{A}^{<\omega}$ and its variants:

Theorem 4.61. *There are precisely 4 countably closed* $<\omega_1$*-ary clones on* 2 *restricting to* $\mathbb{A}\mathbb{T}_0\mathbb{T}_1^{<\omega}$ (*see Figure* 4.53):

   (i) $\mathbb{A}\mathbb{T}_0\mathbb{T}_1^{<\omega_1} = \mathrm{Pol}^{<\omega_1}\{\wedge, 0, 1\} =$ *countable closure of* $\langle\liminf\rangle^{\mathsf{B}}$.

   (ii) $\mathbb{A}\mathbb{T}_{0,\omega}\mathbb{T}_1^{<\omega_1} = \mathbb{A}\mathbb{L}_0\mathbb{T}_1^{<\omega_1} = \mathrm{Pol}^{<\omega_1}\{\wedge, \lim{=}0, 1\} =$ *countable closure of* $\langle\lceil\liminf\rceil\rangle^{\mathsf{B}}$.

   (iii) $\mathbb{A}_{\omega}\mathbb{T}_0\mathbb{T}_1^{<\omega_1} = \mathbb{A}\mathbb{M}_{\downarrow}\mathbb{T}_0\mathbb{T}_1^{<\omega_1} = \mathrm{Pol}^{<\omega_1}\{\bigwedge, 0, 1\} = \langle\bigwedge\rangle^{\mathsf{B}}$.

   (iv) $\langle\mathbb{A}\mathbb{T}_0\mathbb{T}_1^{<\omega}\rangle^{<\omega_1} = \mathbb{A}\mathbb{M}_{\uparrow}\mathbb{T}_0\mathbb{T}_1^{<\omega_1} = \mathbb{A}\mathbb{T}_0\mathbb{L}_1^{<\omega_1} = \mathrm{Pol}^{<\omega_1}\{\wedge, \lim, 0, 1\} = \langle\wedge\rangle^{\mathsf{B}}$.

Every other (*non-countably-closed*) $<\omega_1$*-ary clone restricting to* $\mathbb{A}\mathbb{T}_0\mathbb{T}_1^{<\omega}$ *lies in one of the intervals in* (i) *and* (ii), *between the generated clone and its countable closure. Moreover*:

   (a) *The* $<\omega_1$*-ary clones restricting to* $\mathbb{A}\mathbb{T}_0^{<\omega}$ *are isomorphic to these, via* $F \mapsto \langle F \cup \{0\}\rangle^{<\omega_1}$.

   (b) *The* $<\omega_1$*-ary clones restricting to* $\mathbb{A}\mathbb{T}_1^{<\omega}$ *are isomorphic to those in* $[\langle\mathbb{A}\mathbb{T}_0\mathbb{T}_1^{<\omega}\rangle^{\mathsf{B}}, \mathbb{A}\mathbb{T}_{0,\omega}\mathbb{T}_1^{<\omega_1}]$, *via* $F \mapsto \langle F \cup \{1\}\rangle^{<\omega_1}$, *as are the* $<\omega_1$*-ary clones restricting to* $\mathbb{A}^{<\omega}$, *via* $F \mapsto \langle F \cup \{0, 1\}\rangle^{<\omega_1}$.

   (c) *Every* $F \in [\langle\lceil\liminf\rceil\rangle^{\mathsf{B}}, \mathbb{A}\mathbb{T}_{0,\omega}\mathbb{T}_1^{<\omega_1}]$ *is* $\mathbb{T}_{0,\omega}$ *intersected with some* $G \in [\langle\liminf\rangle^{\mathsf{B}}, \mathbb{A}\mathbb{T}_0\mathbb{T}_1^{<\omega_1}]$. (*Similarly for the clones in* $[\langle\lceil\liminf\rceil, 0\rangle^{\mathsf{B}}, \mathbb{A}\mathbb{T}_{0,\omega}^{<\omega_1}]$ *versus* $[\langle\liminf, 0\rangle^{\mathsf{B}}, \mathbb{A}\mathbb{T}_0^{<\omega_1}]$.)

Proof. The equivalent defining invariant relations are by Lemmas 4.51 and 4.52, the countable closure definitions are by the preceding corollaries, and that these options exhaust $\mathcal{C}lo^{<\omega_1}_{\mathbb{A}\mathbb{T}_0\mathbb{T}_1^{<\omega}}(2)$ is by Lemma 4.54 and Corollaries 4.55 and 4.56.

(a) follows from Proposition 4.34, which shows that the clones containing $\bigwedge$ in both fibers are isomorphic, and Corollary 4.56, which shows that the only clones not containing $\bigwedge$ in both fibers are the essentially finitary ones.

(b) follows from Proposition 3.15 and (a), as does (c) since $[\langle\liminf\rangle^{\mathsf{B}}, \mathbb{A}\mathbb{T}_0\mathbb{T}_1^{<\omega_1}]$ is sandwiched between the two intervals in (b). $\qquad\square$

We close this subsection by showing that indeed, $\langle\liminf\rangle^{\mathsf{B}} \subsetneq \mathbb{A}\mathbb{T}_0\mathbb{T}_1^{\mathsf{B}}$:

Example 4.62. The indicator function $f : 2^{\omega} \to 2$ of the **cosummable filter**,

$$f(\vec{x}) = 1 \;:\Longleftrightarrow\; \sum_{\substack{i<\omega \\ x_i=0}} \frac{1}{i+1} < \infty,$$

is not in $\langle\liminf\rangle$. This follows from a much more general result of Kanovei–Reeken [KR03].

Indeed, note, first, that every function $g \in \langle\liminf\rangle^{\omega}$, which is *a priori* a composite of copies of $\liminf$, may be written as such a composite in "normal form" as follows: consider first the smallest subclass $L \subseteq \langle\liminf\rangle^{\omega}$ of functions containing all projections and closed under left composition

$(h_0, h_1, \dots) \mapsto \liminf_i h_i$ with $\liminf$, *subject to the constraint that the h_i depend on disjoint sets of variables*; then g can be obtained from some $h \in L$ via a variable substitution

$$g(x_0, x_1, \dots) = h(x_{s(0)}, x_{s(1)}, \dots) = h(s^*(\vec{x})) \qquad \text{where} \qquad s^*(\vec{x}) := (x_{s(0)}, x_{s(1)}, \dots)$$

for some $s : \omega \to \omega$. (This is easily seen via a minor variation of Lemma 2.8.) This means that $g^{-1}(1) \subseteq 2^\omega$ is the preimage of $h^{-1}(1) \subseteq 2^\omega$ under $s^* : 2^\omega \to 2^\omega$; such an s is called a **Rudin–Keisler reduction** of the filter $g^{-1}(1)$ to the filter $h^{-1}(1)$. Now the construction of h via left composition of $\liminf$ with functions h_i with disjoint variable sets means in turn that $h^{-1}(1)$ is a **Fubini product**, mod the Fréchet filter $\liminf^{-1}(1)$, of the filters $h_i^{-1}(1)$. Thus, to say that $g \in \langle \liminf \rangle$ means that the filter $g^{-1}(1) \subseteq 2^\omega$ Rudin–Keisler reduces to a filter $h^{-1}(1)$ which is a transfinitely iterated Fubini product, mod the Fréchet filter, of principal filters $\pi_i^{-1}(1)$. Equivalently, the dual ideal $\neg[g^{-1}(1)] \subseteq 2^\omega$ (i.e., subgroup mod 2) Rudin–Keisler reduces to an ideal $\neg[h^{-1}(1)]$ which is a transfinitely iterated Fubini product, mod the Fréchet ideal of finite sets, of principal ideals (or trivial subgroups). This implies in particular that the coset equivalence relation of the subgroup $\neg[g^{-1}(1)]$ is a continuous preimage of an iterated Fubini product mod Fréchet of equality relations. But Kanovei–Reeken [KR03] showed that the coset equivalence relations of many Borel ideals $\subseteq 2^\omega$, including the summable ideal $\neg[f^{-1}(1)]$, do not admit a continuous (or even Borel) reduction to an iterated Fubini product mod Fréchet of equalities.

Thus the topmost $\approx$ interval in Figure 4.53 contains at least two distinct clones. On the other hand, by Proposition 4.57 there is no countable-arity relation $R \subseteq 2^\omega$ that can distinguish between such clones (i.e., that is preserved by $\liminf$ but not by the indicator function of the cosummable filter). So in some sense, while Theorem 4.61 does not completely classify all Borel clones with finitary restriction $\mathbb{AT}_0 \mathbb{T}_1^{<\omega}$, the remaining classification problem is "strictly harder" than all of the positive classifications we have obtained in this paper.

We expect there to be many other Borel clones between $\langle \liminf \rangle^{\mathsf{B}}$ and $\mathbb{AT}_0 \mathbb{T}_1^{\mathsf{B}}$, i.e., Borel filters (or ideals) which are indistinguishable by countable-arity relations but not equivalent up to Rudin–Keisler reducibility and Fubini product (see Conjecture 1.8). Note that the proofs in [KR03] use deep tools from set-theoretic forcing and the theory of Borel equivalence relations (including ideas from Hjorth's turbulence theory [Hjo00]), but show a much stronger Borel non-reducibility result. It would be desirable to find a more elementary proof of Rudin–Keisler non-reducibility, which is all that is needed to yield distinct Borel clones; a natural approach is

Problem 4.63. Give an "explicit" example of a relation $R \subseteq 2^k$ (necessarily of uncountable arity $k \geq \omega_1$) which is preserved by $\liminf$ but not by the indicator function of the cosummable filter.

Note that by Lemma 2.17, such a relation of arity $\leq 2^{\aleph_0}$ exists, and can be described abstractly as a forward orbit of $\langle \liminf \rangle$. The hope is to find a simpler example that does not depend on the techniques used in [KR03], which may lead to new examples of Borel clones between $\langle \liminf \rangle^{\mathsf{B}}$ and $\mathbb{AT}_0 \mathbb{T}_1^{\mathsf{B}}$. See [Hru11], [Kan08], [Sol99] for more information on Borel filters and ideals.

4.D The k-ary intersection property

In this final section, we give some results on the structure of the left "side tube" $[\langle \mathbb{MT}_{0,<\omega} \mathbb{T}_1^{<\omega} \rangle^{\mathsf{B}}, \mathbb{T}_{0,2}^{\mathsf{B}}]$. We will produce a large yet richly structured family of distinct Borel clones lying over each $\mathbb{T}_{0,k}^{<\omega}$ and its variants, depicted in Figure 1.6 and in more detail in Figure 4.71(b).

Recall from Definition 3.7 the clone $\mathbb{L}_0 = \mathrm{Pol}\{\lim=0\} \subseteq \mathbb{T}_0$; note that for countable n,

$$\mathbb{L}_0^n = \{f : 2^n \to 2 \mid \vec{0} \notin \overline{f^{-1}(1)}\},$$

where $\overline{f^{-1}(1)}$ denotes the topological closure of $f^{-1}(1) \subseteq 2^n$. Recall also the clones $\mathbb{T}_{0,k} = \mathrm{Pol}\{\wedge^k=0\}$ $(0 < k < \omega)$, of functions f such that $f^{-1}(1)$ has the **k-ary intersection property**:

$$\vec{0} \notin \underbrace{f^{-1}(1) \wedge \cdots \wedge f^{-1}(1)}_{k} =: f^{-1}(1)^{\wedge k}.$$

Here by $A \wedge B$ for two sets $A, B \subseteq 2^n$, we mean $\{\vec{a} \wedge \vec{b} \mid \vec{a} \in A,\, \vec{b} \in B\}$.

Definition 4.64. Generalizing $\mathbb{L}_0$, for each $0 < k < \omega$ and $n \le \omega$, let

$$\mathbb{L}_{0,k}^n := \{f : 2^n \to 2 \mid \text{the indicator function of } \overline{f^{-1}(1)} \subseteq 2^n \text{ is in } \mathbb{T}_{0,k}\}$$
$$= \{f : 2^n \to 2 \mid \vec{0} \notin \overline{f^{-1}(1)}^{\wedge k}\}.$$

It is easily seen (using compactness of 2^n) that this is

$$= \{f : 2^n \to 2 \mid \nexists(\vec{x}_{r,q} \in f^{-1}(1))_{r<k,q<\omega} \text{ s.t. } \lim_{q \to \infty}(\vec{x}_{0,q} \wedge \cdots \wedge \vec{x}_{k-1,q}) = \vec{0}\}.$$

Thus, we define the partial operation

$$\lim \wedge^k : 2^\omega \cong 2^{k \times \omega} \longrightarrow 2$$
$$\vec{x} = (x_i)_{i<\omega} = (x_{qk+r})_{r<k,q<\omega} \longmapsto \lim_{q \to \infty}(x_{qk} \wedge x_{qk+1} \wedge \cdots \wedge x_{qk+k-1}),$$

and put

$$\mathbb{L}_{0,k} := \mathrm{Pol}\{\lim \wedge^k=0\}.$$

This recovers the above definition of $\mathbb{L}_{0,k}^n$ for n countable, and shows that $\mathbb{L}_{0,k}^{<\omega_1}$ forms a clone.

Example 4.65. To illustrate, consider $\mathbb{L}_{0,3}$. Recalling Definition 2.15, for a function $f : 2^\omega \to \omega$ to preserve the relation $(\lim \wedge^3=0)$ means that for every $\omega \times \omega$ matrix, such as

$$
\begin{array}{ccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & \to \quad f(011111111\cdots) \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & \to \quad f(101111111\cdots) \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & \to \quad f(110111111\cdots) \\
\hline
0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & \cdots & \to \quad f(011011111\cdots) \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & \cdots & \to \quad f(101101111\cdots) \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & \cdots & \to \quad f(110110111\cdots) \\
\hline
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \cdots & \to \quad f(011011011\cdots) \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & \cdots & \to \quad f(101101101\cdots) \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & \cdots & \to \quad f(110110110\cdots) \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \qquad \vdots
\end{array}
$$

if in each original column, cofinally many blocks of 3 entries have at least one 0, then the same must hold for the resulting column obtained by applying f to each row. (See Lemma 4.69 for an application of the particular matrix shown above.)

Note that if this does *not* hold, then after passing to a subsequence of blocks of 3 rows, we may assume that the resulting column of $f(\cdots)$'s is $\vec{1}$; then the rows will yield 3 sequences in $f^{-1}(1)$ whose termwise meet converges to $\vec{0}$. By passing to further subsequences, we may assume each of these 3 sequences individually converges, and that the meet of the 3 limit strings is $\vec{0}$.

Definition 4.66. More generally still, for $0 < k < \omega$, $0 \leq t \leq k$, and $n \leq \omega$, let

$$\mathbb{L}^n_{0,k,t} := \left\{ f : 2^n \to 2 \ \middle|\ \vec{0} \notin f^{-1}(1)^{\wedge(k-t)} \wedge \overline{f^{-1}(1)}^{\wedge t} = \underbrace{f^{-1}(1) \wedge \cdots}_{k-t} \wedge \underbrace{\overline{f^{-1}(1)} \wedge \cdots}_{t} \right\}.$$

It is easily seen that this is

$$= \left\{ f : 2^n \to 2 \ \middle|\ \begin{array}{c} \nexists (\vec{x}_r \in f^{-1}(1))_{r<k-t}, (\vec{x}_{r,q} \in f^{-1}(1))_{k-t \leq r < k, q < \omega} \text{ s.t.} \\ \lim_{q \to \infty} (\vec{x}_0 \wedge \cdots \wedge \vec{x}_{k-t-1} \wedge \vec{x}_{k-t,q} \wedge \cdots \wedge \vec{x}_{k-1,q}) = \vec{0} \end{array} \right\}.$$

Thus, we define the partial operation

$$\wedge^{k-t} \wedge \lim \wedge^t : 2^\omega \cong 2^{(k-t)+t\times\omega} \longrightarrow 2$$
$$\vec{x} = (x_0, \ldots, x_{k-t-1}, x_{qt+r})_{k-t \leq r < k, q < \omega} \longmapsto x_0 \wedge \cdots \wedge x_{k-t-1} \wedge \lim_{i \to \infty} (x_{qt+k-t} \wedge \cdots \wedge x_{qt+k-1}),$$

and put
$$\mathbb{L}_{0,k,t} := \mathrm{Pol}\{\wedge^{k-t} \wedge \lim \wedge^t = 0\}.$$

The matrix involved is as in Example 4.65, except that there are $k-t$ distinguished rows at the top, then repeating blocks of t rows; a column obeys $\wedge^{k-t} \wedge \lim \wedge^t = 0$ iff either one of the first $k-t$ entries is 0, or cofinally many of the repeating blocks contain at least one 0.

When $t = 0$, $\wedge^{k-t} \wedge \lim \wedge^t$ becomes $\wedge^k$ (with domain 2^k, rather than 2^ω); thus

$$\mathbb{L}_{0,k,0} = \mathbb{T}_{0,k} \quad (= \mathbb{T}_0 \text{ if } k = 1).$$

On the other hand,
$$\mathbb{L}_{0,k,k} = \mathbb{L}_{0,k} \quad (= \mathbb{L}_0 \text{ if } k = 1).$$

We have the obvious relations

$$(4.67) \quad \mathbb{T}_{0,\omega} \subseteq \cdots \subseteq \mathbb{L}_{0,k+2,k} \subseteq \mathbb{L}_{0,k+1,k} \subseteq \mathbb{L}_{0,k} = \mathbb{L}_{0,k,k} \subseteq \mathbb{L}_{0,k,k-1} \subseteq \cdots \subseteq \mathbb{L}_{0,k,1} \subseteq \mathbb{L}_{0,k,0} = \mathbb{T}_{0,k}$$

(see Figure 4.71(a)). Based on these inclusions, it is natural to also define

$$\mathbb{L}_{0,0,0} := \mathbb{L}_{0,0} := \mathbb{T}_{0,0} := \mathbb{O}_2,$$
$$\mathbb{L}_{0,<\omega,t} := \bigcap_{k<\omega} \mathbb{L}_{0,k,t},$$
$$\mathbb{L}_{0,<\omega,<\omega} := \mathbb{L}_{0,<\omega} := \bigcap_{k<\omega} \mathbb{L}_{0,k} = \bigcap_{t \leq k < \omega} \mathbb{L}_{0,k,t}.$$

Let also $\mathbb{L}_{1,k,t}$ be the de Morgan dual of $\mathbb{L}_{0,k,t}$, consisting of all functions f preserving the relation $(\vee^{k-t} \vee \lim \vee^t = 1)$ defined analogously, or (in countable arities) such that $f^{-1}(0)$ does not contain $k-t$ elements together with t elements in its closure with join $\vec{1}$.

The poset of all $\mathbb{L}_{0,k,t}$ clones is depicted in Figure 4.71(a). *However, the $\mathbb{L}_{0,k,t}$'s do not form a sublattice of $\mathcal{Clo}^{\mathsf{B}}(2)$. For example, $\mathbb{L}_{0,3,1} \cap \mathbb{L}_{0,2} \supsetneq \mathbb{L}_{0,3,2}$; see Figure 4.71(b). In fact, there are no nontrivial meet relations* (that are not implied by the ordering) among the $\mathbb{L}_{0,k,t}$'s, as we now show.

Definition 4.68. Given functions $f_i : 2^{n_i} \to 2 \in \mathbb{T}_1$, $i < k$, their **orthogonal disjunction** is[2]

$$\bigsqcup_{i<k} f_i : 2^{\bigsqcup_{i<k} n_i = \{(i,j)|i<k, j<n_i\}} \longrightarrow 2$$

$$(x_{i,j})_{i,j} \longmapsto \bigvee_{i<k} \left(f_i((x_{i,j})_j) \wedge \bigwedge_{i' \neq i} \bigwedge_{j<n_{i'}} x_{i',j} \right)$$

$$= \forall_2^k \left(\left(\bigwedge_{j<i} x_{i,j} \right)_{i<k} \wedge \bigvee_{i<k} f_i((x_{i,j})_j) \right).$$

(The binary operation $\sqcup$ is associative up to a canonical permutation of the input bits.)

For example, when $f_i = 1 : 2 \to 2$ for all i, we get $\bigsqcup_{i<k} f_i = \forall_2^k : 2^k \to 2$ (recall Definition 3.2).

More generally, when $k < \omega$, $f_0 = \cdots = f_{k-t-1} = 1 : 2 \to 2$, and $f_{k-t} = \cdots = f_{k-1} = \liminf :$ $2^\omega \to 2$, we may identify $2^{(k-t)+t\times\omega} \cong 2^\omega$ (assuming $t > 0$), so that $\bigsqcup_{i<k} f_i$ becomes

$$\forall_2^{k-t} \sqcup \liminf^{\sqcup t} : 2^\omega \cong 2^{(k-t)+t\times\omega} \longrightarrow 2$$

$$\vec{x} = (x_0, \ldots, x_{k-t-1}, x_{qt+r})_{k-t \leq r < k, q < \omega} \longmapsto \forall_2^k(x_0, \ldots, x_{k-t-1}, \textstyle\bigwedge_q x_{qt+k-t}, \ldots, \bigwedge_q x_{qt+k-1}) \wedge \liminf \vec{x}.$$

When $t = k$, this becomes

$$\liminf^{\sqcup t} : 2^\omega \cong 2^{k\times\omega} \longrightarrow 2$$

$$\vec{x} = (x_{qk+r})_{r<k,q<\omega} \longmapsto \forall_2^k(\textstyle\bigwedge_q x_{qk}, \ldots, \bigwedge_q x_{qk+k-1}) \wedge \liminf \vec{x}.$$

For example, when $t = k = 2$, we get

$$\liminf \sqcup \liminf : 2^\omega \cong 2^{2\times\omega} \longrightarrow 2$$

$$\vec{x} \longmapsto \left(\liminf_{q\to\infty} x_{2q} \wedge \bigwedge_{q<\omega} x_{2q+1} \right) \vee \left(\bigwedge_{q<\omega} x_{2q} \wedge \liminf_{q\to\infty} x_{2q+1} \right).$$

Lemma 4.69. *For any $t \leq k < \omega$, we have $\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \in \mathbb{MT}_1 \cap \bigcap_{\substack{t' \leq k' < \omega \\ k' < k \text{ or } t' < t}} \mathbb{L}_{0,k',t'} \setminus \mathbb{L}_{0,k,t}$.*

Proof. To show that $\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \notin \mathbb{L}_{0,k,t}$, consider a matrix like the one in Example 4.65, which is the case $t = k = 3$; each column obeys $\lim \wedge^3 = 0$, while each row obeys $\liminf^{\sqcup 3} = 1$. When $t < k$, we consider instead such a matrix split into blocks of t, and then prepend $(k - t)$ rows and columns which are all 1's except for $(k - t)$ 0's down the diagonal; then we again get that each column obeys $\wedge^{k-t} \wedge \lim \wedge^t = 0$ while each row obeys $\forall_2^{k-t} \sqcup \liminf^{\sqcup t} = 1$.

It is clear that $\forall_2^{k-t} \sqcup \liminf^{\sqcup t}$ is in $\mathbb{MT}_1$. To show that it is in $\mathbb{L}_{0,k',t'}$, for $k' < k$: it is easily seen that $\overline{(\forall_2^{k-t} \sqcup \liminf^{\sqcup t})^{-1}(1)}$ has indicator function

$$(*) \qquad (\forall_2^{k-t} \sqcup (1^\omega)^{\sqcup t})(\vec{x}) = \forall_2^k(x_0, \ldots, x_{k-t-1}, \textstyle\bigwedge_q x_{qt+k-t}, \ldots, \bigwedge_q x_{qt+k-1}),$$

which is in $\mathbb{T}_{0,k'}$, whence the former function is in $\mathbb{L}_{0,k'} \subseteq \mathbb{L}_{0,k',t'}$.

To show that it is in $\mathbb{L}_{0,k',t'}$, for $t' < t$: let $\vec{y}_0, \ldots, \vec{y}_{k'-t'-1} \in \overline{(\forall_2^{k-t} \sqcup \liminf^{\sqcup t})^{-1}(1)}$ and $\vec{y}_{k'-t'}, \ldots, \vec{y}_{k'-1} \in (\forall_2^{k-t} \sqcup \liminf^{\sqcup t})^{-1}(1)$, i.e., when $\vec{x} = \vec{y}_{k'-t'}, \ldots, \vec{y}_{k'-1}$, all but 1 of the arguments of the $\forall_2^k$ on the right-hand side of $(*)$ above evaluate to 1. Since $t' < t$, there must be some $k - t \leq r < k$ such that $\bigwedge_q x_{qt+r} = 1$ whenever $\vec{x} = \vec{y}_{k'-t'}, \ldots, \vec{y}_{k'-1}$. Since also $\liminf_q x_{qt+r} = 1$ when $\vec{x} = \vec{y}_0, \ldots, \vec{y}_{k'-t'-1}$, it follows that $\vec{y}_0 \wedge \cdots \wedge \vec{y}_{k'-1}$ has infinitely many 1's. This shows $\vec{0} \notin (\forall_2^{k-t} \sqcup \liminf^{\sqcup t})^{-1}(1)^{\wedge(k'-t')} \wedge \overline{(\forall_2^{k-t} \sqcup \liminf^{\sqcup t})^{-1}(1)}^{\wedge t'}$, hence $\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \in \mathbb{L}_{0,k',t'}$. $\square$

---

[2]Identifying $f : 2^n \to 2$ with the set $f^{-1}(1)$, a standard name from topology for $\sqcup$ would be the *wedge sum* $\vee$; but that notation conflicts with ordinary disjunction.

Recall that for a poset P, the **free complete meet-semilattice** $\widehat{P}$ generated by P is a complete lattice equipped with an order-embedding $P \to \widehat{P}$ whose image generates P under meets, and such that every equation between meets of elements of P which holds in $\widehat{P}$ is implied by the ordering, in the precise sense that any monotone map $P \to Q$ to another complete lattice Q admits a unique meet-preserving extension $\widehat{P} \to Q$. Concretely, $\widehat{P}$ may be constructed as the poset of upward-closed subsets $U \subseteq P$ under *reverse* inclusion; we think of such U as the formal meet of its elements. If we exclude $U = \varnothing$, then we obtain instead the free poset with *nonempty* meets $\bigwedge_{i \in I}$, $I \neq \varnothing$.

We claim that there are no nontrivial meet relations between the Borel clones $\mathbb{L}^{\mathsf{B}}_{0,k,t}$, other than those implied by the order relations (4.67) (depicted in Figure 4.71(a)). Formally, this means

Corollary 4.70. *The closure of the Borel clones $\mathbb{L}^{\mathsf{B}}_{0,k,t} \in \mathcal{Clo}^{\mathsf{B}}(2)$, $t \leq k < \omega$, under nonempty intersections yields the free complete nonempty-meet-semilattice generated by the poset depicted in Figure 4.71(a), which is the poset of indices $\{(k,t) \mid t \leq k < \omega\}$ under the coordinatewise reverse ordering $\geq$. In other words, we have an order-embedding*

$$(\{\varnothing \neq D \subseteq \{(k,t) \mid t \leq k < \omega\} \mid D \text{ is } \leq\text{-downward-closed}\}, \supseteq) \hookrightarrow (\mathcal{Clo}^{\mathsf{B}}(2), \subseteq)$$

$$D \longmapsto \bigcap_{(k,t) \in D} \mathbb{L}^{\mathsf{B}}_{0,k,t}.$$

This remains an embedding upon intersecting with $\mathbb{M}$ and/or $\mathbb{T}_1$.

Proof. If $D' \not\supseteq D$ are two such downward-closed sets of indices, then there is $(k,t) \in D \setminus D'$, whence for all $(k',t') \in D'$, we have $(k,t) \not\leq (k',t')$; by Lemma 4.69, we have $\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \in \mathbb{M}\mathbb{T}_1 \cap \bigcap_{(k',t') \in D'} \mathbb{L}_{0,k',t'} \setminus \mathbb{L}_{0,k,t}$. $\square$

The "cobweb" region of Figure 1.6, above $\mathbb{L}_{0,<\omega}$, depicts all clones F obtained by intersecting the $\mathbb{L}_{0,k,t}$'s (except for $\mathbb{L}_{0,0,0} = \mathbb{O}_2$), which are isomorphic to the poset of all downward-closed $\varnothing \neq D \subseteq \{(k,t) \mid t \leq k < \omega\}$ under reverse inclusion. Figure 4.71(a) depicts a typical such D (the region above the thick dotted line), which determines the intersection clone $F = \mathbb{L}_{0,3,1}\mathbb{L}_{0,2}$ (since these are the minimal clones above the dotted line). We may categorize such F as follows:

- If D contains a greatest $(k,0)$, then the intersection $F = \bigcap_{(k',t') \in D} \mathbb{L}_{0,k',t'}$ is contained in $\mathbb{L}_{0,k,0} = \mathbb{T}_{0,k}$ but not $\mathbb{L}_{0,k+1,0} = \mathbb{T}_{0,k+1}$, and contains $\mathbb{L}_{0,k,k} = \mathbb{L}_{0,k} \supseteq \langle \mathbb{T}^{<\omega}_{0,k} \rangle$ (since every $(k',t') \in D$ must be $\leq (k,k)$). Thus F has finitary restriction $\mathbb{T}^{<\omega}_{0,k}$.

  Note that there are precisely 2^k such $(k,0) \in D \not\supseteq (k+1,0)$. Indeed, D either contains or does not contain $(k,1)$; the latter D are isomorphic to $\{D' \mid (k-1,0) \in D' \not\supseteq (k,0)\}$, via $D \mapsto D \setminus \{(k,0)\}$, as are the former D, via $D \mapsto \{(k'-1,t'-1) \mid (k',t') \in D\}$. (In Figure 4.71(a), there are 2^3 possible thick dotted lines starting between $\mathbb{L}_{0,3,0}$ and $\mathbb{L}_{0,4,0}$ and moving right for 3 steps, since at each step it may move down or up.) Thus the poset of all D such that $(k,0) \in D \not\supseteq (k+1,0)$ may be partitioned into two isomorphic copies of the poset of all D' such that $(k-1,0) \in D' \not\supseteq (k,0)$. By induction, there are 2^k such D.

  For $k = 3$, the $2^3 = 8$ such clones are depicted in Figure 4.71(b) (above $\mathbb{L}_{0,3}$). Note that the "lower and upper halves", below $\mathbb{L}_{0,3,1}$ or not, are each order-isomorphic to the 4 intersection clones obtained for $k = 2$.

Corollary 4.72. *For each $k < \omega$, there are at least 2^k Borel clones between $\mathbb{L}^{\mathsf{B}}_{0,k}$ and $\mathbb{T}^{\mathsf{B}}_{0,k}$ with finitary restriction $\mathbb{T}^{<\omega}_{0,k}$. Similarly for $\mathbb{M}\mathbb{T}^{<\omega}_{0,k}, \mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1, \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1$.* $\square$

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(a) Inclusion ordering among the clones $\mathbb{L}_{0,k,t}$, together with an upward-closed set (above thick dotted line), whose intersection $\mathbb{L}_{0,3,1}\mathbb{L}_{0,2}$ is not contained in any clones below.

(b) Some Borel clones with finitary restrictions $\mathbb{T}_{0,3}^{<\omega}, \mathbb{MT}_{0,3}\mathbb{T}_1^{<\omega}$ respectively, obtained by intersecting $\mathbb{L}_{0,k,t}$'s with clones from top cube (Remark 4.21). The dashed blocks are exhaustive below $\mathbb{L}_{0,3}$ and are isomorphic to corresponding intervals below $\mathbb{T}_{0,\omega}$ (Figure 4.29). (Similar blocks over $\mathbb{T}_{0,3}\mathbb{T}_1^{<\omega}, \mathbb{MT}_{0,3}^{<\omega}$ not shown.)
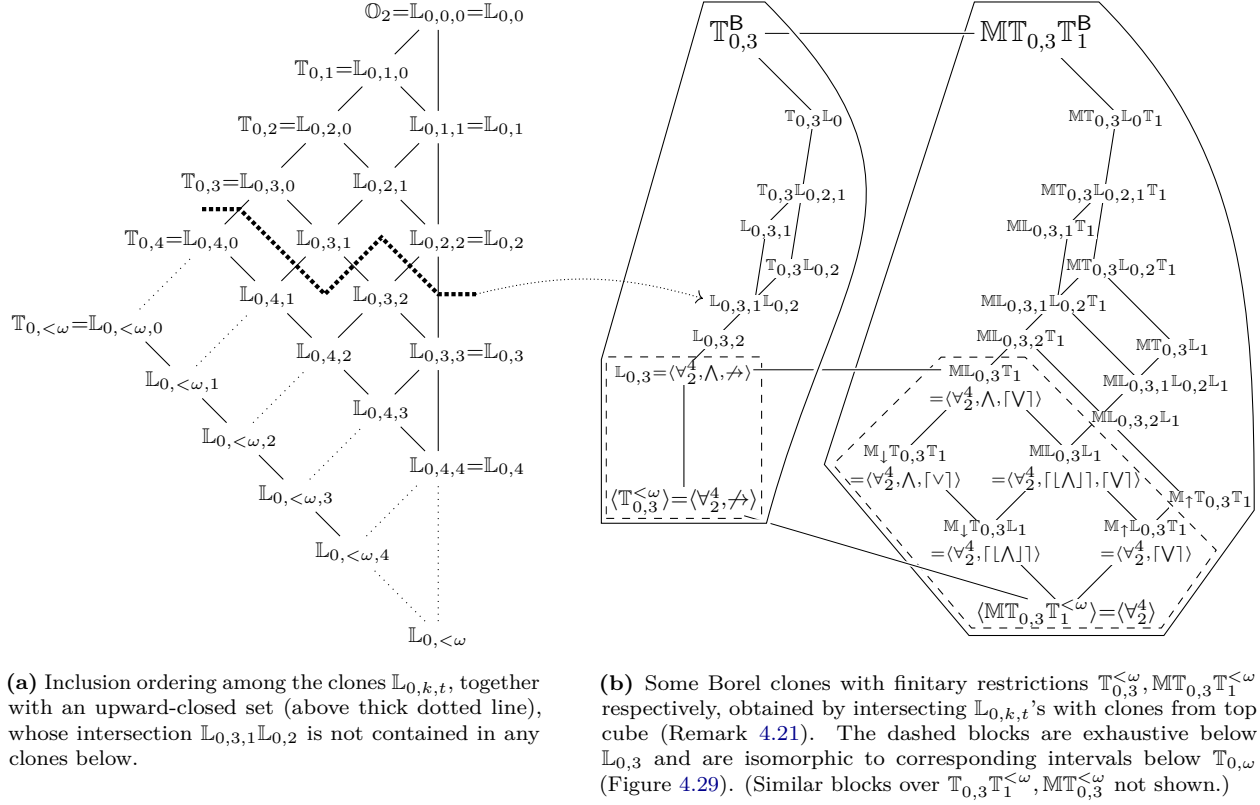
Figure 4.71: Examples of Borel clones over left "side tube" of Post's lattice.

- On the other hand, if D contains $(k,0)$ for all $k < \omega$, then the intersection F is contained in $\bigcap_{k<\omega} \mathbb{T}_{0,k} = \mathbb{T}_{0,<\omega}$, and contains $\mathbb{L}_{0,<\omega} \supseteq \langle \mathbb{T}_{0,<\omega}^{<\omega} \rangle$. Thus F has finitary restriction $\mathbb{T}_{0,<\omega}^{<\omega}$.

  We may further categorize such F, by considering whether or not there is a largest $t < \omega$ such that $(k,t) \in D$ for all $k < \omega$. If not, then $F = \mathbb{L}_{0,<\omega}$. If so, then F is contained in $\mathbb{L}_{0,<\omega,t}$ but not $\mathbb{L}_{0,<\omega,t+1}$. Then there is a least $k \geq t$ such that $(k+1,t+1) \notin D$, which means $F \subseteq \mathbb{L}_{0,k,t+1}$ but $F \not\subseteq \mathbb{L}_{0,k+1,t+1}$. For fixed t,k, the poset of all such D is order-isomorphic to the poset of D' with $(k-t-1,0) \in D \not\ni (k-t,0)$. (This means the thick dotted line in Figure 4.71(a) starts between $\mathbb{L}_{0,<\omega,t}$ and $\mathbb{L}_{0,<\omega,t+1}$, and moves up until just before it crosses $\mathbb{L}_{0,k,t+1}$, then bends down.) So the poset of all D with $\{(k,0) \mid k \in \omega\} \subseteq D$, except for the maximum $D = \{(k,t) \mid t \leq k < \omega\}$, may be partitioned into countably many mutually isomorphic subsets, each of which may be partitioned into copies of $\{D' \mid (k,0) \in D' \not\ni (k+1,0)\}$ for all k. In particular, there are countably infinitely many such D.

Corollary 4.73. *There are at least countably infinitely many distinct Borel clones above $\mathbb{L}_{0,<\omega}^{\mathsf{B}}$ with finitary restriction $\mathbb{T}_{0,<\omega}^{<\omega}$. Similarly for $\mathbb{MT}_{0,<\omega}^{<\omega}, \mathbb{T}_{0,<\omega}\mathbb{T}_1^{<\omega}, \mathbb{MT}_{0,<\omega}\mathbb{T}_1^{<\omega}$.* ☐

Remark 4.74. It would be desirable if the recursive order-isomorphisms of the various kinds of index sets $D \subseteq \{(k,t) \mid t \leq k < \omega\}$ mentioned in the above discussion could be enhanced to isomorphisms between the respective intervals of clones, perhaps by using certain "operators" on functions along the lines of Sections 3.C to 3.E. For instance, perhaps the (Borel) clones between $\mathbb{L}_{0,3,1}, \mathbb{T}_{0,3}^{<\omega}$ (see Figure 4.71(b)) could be shown to be isomorphic to those between $\mathbb{T}_{0,2}, \mathbb{T}_{0,2}^{<\omega}$.

This would be beneficial to the remaining classification problem for the Borel clones in the "side tubes". We currently only know of the finite lower bound of 2^k (plus a little bit; see below). For instance, in stark contrast to what Figure 1.6 suggests, we currently cannot even rule out the possibility of there being strictly more Borel clones over $\mathbb{T}_{0,2}^{<\omega}$ than over $\mathbb{T}_{0,3}^{<\omega}$ (see Conjecture 1.7).

We now show a converse to Lemma 4.69: the functions $\forall_2^{k-t} \sqcup \liminf^{\sqcup t}$ are "minimal" $\notin \mathbb{L}_{0,k,t}$. This generalizes Lemma 4.54 (when $k = t = 1$), and is also analogous to Lemma 4.45.

Lemma 4.75. *If* $f : 2^n \to 2 \in \mathbb{O}_2^{<\omega_1} \setminus \mathbb{L}_{0,k,t}$, *then* $\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \in \langle \{f\} \cup \mathbb{MT}_{0,\omega}\mathbb{T}_1^{\mathsf{B}} \rangle = \langle f, \bigwedge, \lceil \bigvee \rceil \rangle$.

Proof. Since $f \notin \mathbb{L}_{0,k,t}$, there are $\vec{x}_0, \dots, \vec{x}_{k-t-1}, (\vec{x}_{k-t,q})_{q<\omega}, \dots, (\vec{x}_{k-1,q})_{q<\omega}$ in $f^{-1}(1)$ such that

$$\lim_{q \to \infty} (\vec{x}_0 \wedge \cdots \wedge \vec{x}_{k-t-1} \wedge \vec{x}_{k-t,q} \wedge \cdots \wedge \vec{x}_{k-1,q}) = \vec{0}.$$

We now modify these $\vec{x}_r, \vec{x}_{r,q}$'s to make them resemble points in $(\forall_2^{k-t} \sqcup \liminf^{\sqcup t})^{-1}(1)$.

By passing to subsequences, we may assume that for each $k - t \le r < k$, the sequence $(\vec{x}_{r,q})_q$ converges to some $\vec{x}_r \in 2^n$. Thus $\bigwedge_{0 \le r < k} \vec{x}_r = \vec{0}$.

In fact, we may assume that $\vec{x}_{r,0} \ge \vec{x}_{r,1} \ge \vec{x}_{r,2} \ge \cdots$ for each $k - t \le r < k$. Indeed, this becomes true if we replace each $\vec{x}_{r,q}$ with $\bigvee_{p \ge q} \vec{x}_{r,p}$ (as in Lemma 4.7(d)), and the above limit remains $\vec{0}$, provided that each $(\vec{x}_{r,q})_q$ converges. Now these new points $\vec{x}_{r,q}$ may no longer be in $f^{-1}(1)$; but since they are $\ge$ the original points, and there are only countably many, by Corollary 4.33 we may replace f with another function in $\langle \{f\} \cup \mathbb{MT}_{0,\omega}\mathbb{T}_1^{\mathsf{B}} \rangle$ which is 1 at these new points.

Finally, we may assume that the $\vec{x}_r$'s ($= \lim_q \vec{x}_{r,q}$ for $r \ge k - t$) have pairwise joins $\vec{1}$. This is by replacing each $\vec{x}_r$ (or $\vec{x}_{r,q}$) with its join with $\bigvee_{s<r} \neg \vec{x}_s$. Again, these new points are $\ge$ the original ones, so by Corollary 4.33 we may replace f to make it 1 at these new points. We still have $\bigwedge_{r<k} \vec{x}_r = \vec{0}$; thus for each $j < n$, there is a unique $r < k$ such that $x_{r,j} = 0$.

Now define the variable substitution

$$g : 2^{(k-t)+t\times\omega} \longrightarrow 2^n$$

$$(y_0, \dots, y_{k-t-1}, y_{qt+r})_{k-t\le r<k, q<\omega} \longmapsto \left(\begin{cases} y_r & \text{if } x_{r,j} = 0 \text{ for } r < k - t, \\ y_{\min\{q<\omega \mid x_{r,q,j}=0\}\cdot t+r} & \text{if } x_{r,j} = 0 \text{ for } k - t \le r < k \end{cases} \right)_{j<n}.$$

Then $g(\vec{y}) \in f^{-1}(1)$ whenever $\vec{y} \in (\forall_2^{k-t} \sqcup \liminf^{\sqcup t})^{-1}(1)$, so $\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \le f \circ g \in \langle f \rangle$. Since clearly $\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \in \langle \bigwedge, \bigvee \rangle \subseteq \langle \mathbb{MT}_{0,\omega}\mathbb{T}_1^{\mathsf{B}} \cup \{1\} \rangle$, by Corollary 3.21 it follows that $\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \in \langle \{f\} \cup \mathbb{MT}_{0,\omega}\mathbb{T}_1^{\mathsf{B}} \rangle$. $\square$

Corollary 4.76. *For a* $<\omega_1$*-ary clone* $F \supseteq \mathbb{MT}_{0,\omega}\mathbb{T}_1^{\mathsf{B}}$, *either* $F \subseteq \mathbb{L}_{0,k,t}$ *or* $\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \in F$. $\square$

Corollary 4.77. *Every* $<\omega_1$*-ary clone* $F \supseteq \mathbb{MT}_{0,\omega}\mathbb{T}_1^{\mathsf{B}}$ *belongs to exactly one of the intervals*

$$[\langle \{\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \mid (k,t) \notin D\} \rangle^{\mathsf{B}}, \bigcap_{(k,t)\in D} \mathbb{L}_{0,k,t}^{<\omega_1}]$$

for a downward-closed $\varnothing \ne D \subseteq \{(k,t) \mid t \le k < \omega\}$ *(namely for* $D = \{(k,t) \mid F \subseteq \mathbb{L}_{0,k,t}\}$*).*

Proof. For said D, for every $(k,t) \notin D$, we have $\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \in F$ by the preceding corollary. And these intervals are disjoint for $D \ne D'$: if without loss of generality $D \not\subseteq D'$, then for $(k,t) \in D$, $\forall_2^{k-t} \sqcup \liminf^{\sqcup t}$ is in every clone in the latter interval but none in the former by Lemma 4.69. $\square$

Remark 4.78. Each of the above intervals is nonempty, i.e., the lower bound is contained in the upper bound, by Lemma 4.69; in fact there are at least 4 Borel clones in the interval, namely $\bigcap_{(k,t)\in D} \mathbb{L}^{\mathsf{B}}_{0,k,t}$ and its intersections with $\mathbb{M}$ and/or $\mathbb{T}_1$, or alternatively the clones generated by $\{\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \mid (k,t) \notin D\}$ together with $\mathbb{T}^{\mathsf{B}}_{0,\omega}, \mathbb{T}_{0,\omega}\mathbb{T}^{\mathsf{B}}_1, \mathbb{M}\mathbb{T}^{\mathsf{B}}_{0,\omega}, \mathbb{M}\mathbb{T}_{0,\omega}\mathbb{T}^{\mathsf{B}}_1$.

We do not know if, within each combination of $\mathbb{M}$ and/or $\mathbb{T}_1$, the generators match the relations. For example, we do not know if each inclusion

$$\langle \mathbb{T}^{\mathsf{B}}_{0,\omega} \cup \{\forall_2^{k-t} \sqcup \liminf^{\sqcup t} \mid (k,t) \notin D\}\rangle^{\mathsf{B}} \subseteq \bigcap_{(k,t)\in D} \mathbb{L}^{\mathsf{B}}_{0,k,t}$$

is in fact an equality. The only known cases are when $D \subseteq \{(0,0),(1,0),(1,1)\}$, which yield the clones over $\mathbb{T}_0^{<\omega}, \mathbb{O}_2^{<\omega}$ from Section 4.A; $D = \{(k',t) \mid t \le k' \le k\}$ for fixed $k < \omega$, which yield the clones $\mathbb{L}^{\mathsf{B}}_{0,k}$ whose generators are determined by Example 4.81 below; and $(2,0) \in D \not\ni (2,1)$, which yield a few sporadic clones over $(\mathbb{M})\mathbb{T}_{0,2}(\mathbb{T}_1)^{<\omega}$ related to self-dual functions (see Corollary 4.97).

In contrast to the analogous situation with the Borel clones over $\mathbb{A}^{<\omega}$ (Theorem 4.61), here we do not even know if such generation occurs up to countable closure, i.e., up to approximation at any given countably many input strings in 2^ω.

The above dichotomies only apply to Borel clones F containing $\mathbb{M}\mathbb{T}_{0,\omega}\mathbb{T}^{\mathsf{B}}_1$, i.e., whose "projection" $F \cap \mathbb{T}_{0,\omega}$ to the "base of the side tube" as in Proposition 3.15 and Figure 3.29(b) is one of the top 4 nodes given by Corollary 4.28 (Figure 4.29). In order to extend these results to F for which $F \cap \mathbb{T}_{0,\omega}$ is one of the other 11 clones in Corollary 4.28, we need to consider the interactions between the $\mathbb{L}_{0,k,t}$ and the clones from the top cube (Remark 4.21).

Proposition 4.79. *For each $k < \omega$, we have $\mathbb{L}^{<\omega_1}_{0,k} = \downarrow\langle \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1\rangle^{\mathsf{B}}$.*

(Recall Definition 3.16 of the downward-closure $\downarrow$, as well as Notation 4.35 for the Borel downward-closure $\downarrow^{\mathsf{B}}$. Note that it follows that $\mathbb{L}^{\mathsf{B}}_{0,k} = \downarrow^{\mathsf{B}}\langle \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1\rangle^{\mathsf{B}}$.)

Proof. Clearly from its definition, $\mathbb{L}_{0,k}$ is downward-closed, which shows $\supseteq$. To show $\subseteq$: let $f : 2^n \to 2 \in \mathbb{L}^{<\omega_1}_{0,k}$. If $f = 0$, then $f \le \pi_0 \in \langle \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1\rangle$; so suppose not. Then $\overline{f^{-1}(1)} \subseteq 2^n$ has the k-ary intersection property, i.e., $\overline{f^{-1}(1)}^k \subseteq (2^n)^k$ is disjoint from $(\wedge^k : (2^n)^k \to 2^n)^{-1}(\vec{0})$. Since $\overline{f^{-1}(1)} \subseteq 2^n$ is a downward-directed intersection of clopen sets, by compactness of $(2^n)^k$, for some such clopen $f^{-1}(1)(1) \subseteq A \subseteq 2^n$, A^k must also be disjoint from $(\wedge^k)^{-1}(\vec{0})$, i.e., A also has the k-ary intersection property. Then the upward-closure $\uparrow A \subseteq 2^n$ is clearly a clopen upward-closed set still with the k-ary intersection property, hence its indicator function g is in $\langle \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1\rangle$ with $f \le g$. $\square$

Corollary 4.80. *For each $k < \omega$, there are 2, 3, 4, 6 Borel clones contained in $\mathbb{L}_{0,k}$ restricting to the finitary clones $\mathbb{T}^{<\omega}_{0,k}, \mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1, \mathbb{M}\mathbb{T}^{<\omega}_{0,k}, \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1$ respectively, namely generated by the respective clones below $\mathbb{T}_{0,\omega}$ from Corollary 4.28 (Figure 4.29) together with the generator $\forall_2^{k+1} \in \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1$.*

The simplest (2) and most complicated (6) of these are depicted in Figure 4.71(b) (dashed blocks), for $k = 3$; for the others, see Figure 1.6 (solid-shaded regions).

Proof. By Corollary 4.39 applied to $H = \langle \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1\rangle^{\mathsf{B}}$. $\square$

Example 4.81. We get

$$\mathbb{M}\mathbb{L}_{0,k}\mathbb{T}^{\mathsf{B}}_1 = \langle \mathbb{M}\mathbb{T}_{0,\omega}\mathbb{T}^{\mathsf{B}}_1 \cup \{\forall_2^{k+1}\}\rangle^{\mathsf{B}} = \langle \wedge, \lceil\vee\rceil, \forall_2^{k+1}\rangle^{\mathsf{B}},$$
$$\mathbb{L}^{\mathsf{B}}_{0,k} = \langle \mathbb{T}^{\mathsf{B}}_{0,\omega} \cup \{\forall_2^{k+1}\}\rangle^{\mathsf{B}} = \langle \wedge, \nrightarrow, \forall_2^{k+1}\rangle^{\mathsf{B}}$$

(as shown in Figure 4.71(b) for $k = 3$). Note that $\forall_2^{k+1} = \forall_2^{k+1-0} \sqcup \liminf^{\sqcup 0}$ (Definition 4.68), which matches the candidate generating set for $\mathbb{L}_{0,k}^{\mathsf{B}}$ from Remark 4.78.

Remark 4.82. For $k < \omega$, we have $\mathbb{M}_\downarrow \mathbb{T}_{0,k}^{<\omega_1} \subseteq \mathbb{L}_{0,k}$, since for $f \in \mathbb{M}_\downarrow^{<\omega_1}$, $f^{-1}(1)$ is already closed (by the dual of Lemma 4.14; cf. Lemma 4.17).

Corollary 4.83. $\mathbb{L}_{0,<\omega}^{<\omega_1} = {\downarrow}\mathbb{M}_\downarrow \mathbb{T}_{0,<\omega} \mathbb{T}_1^{\mathsf{B}}$.

Proof. $\supseteq$ by intersecting the preceding remark over k; $\subseteq$ because for $f \in \mathbb{L}_{0,<\omega}^{<\omega_1}$, by Proposition 4.79 we may write $f \le g_k \in \langle \mathbb{M}\mathbb{T}_{0,k} \mathbb{T}_1^{<\omega} \rangle^{\mathsf{B}}$ for each k, whence $f \le \bigwedge_k g_k \in \mathbb{M}_\downarrow \mathbb{T}_{0,<\omega} \mathbb{T}_1^{\mathsf{B}}$. $\square$

Remark 4.84. It follows from Corollary 4.39 that $[\mathbb{M}_\downarrow \mathbb{T}_{0,<\omega} \mathbb{T}_1^{\mathsf{B}}, \mathbb{L}_{0,<\omega}^{\mathsf{B}}] \cong [\mathbb{M}_\downarrow \mathbb{T}_{0,\omega} \mathbb{T}_1^{\mathsf{B}}, \mathbb{T}_{0,\omega}^{\mathsf{B}}]$, which consists of 6 clones (see Figure 4.29).

If, as suggested in Remark 4.47, we had $\mathbb{M}_\downarrow \mathbb{T}_{0,<\omega} \mathbb{T}_1^{\mathsf{B}} = \langle \forall_2 \rangle^{\mathsf{B}}$, then together with Corollary 4.28 this would give a complete classification of all Borel clones in $[\langle \mathbb{M}\mathbb{T}_{0,<\omega} \mathbb{T}_1^{<\omega} \rangle^{\mathsf{B}}, \mathbb{L}_{0,<\omega}^{\mathsf{B}}]$.

The following is analogous to Lemma 4.25:

Lemma 4.85. *For each $0 < k < \omega$, we have $\mathbb{T}_{0,k} \mathbb{L}_1^{<\omega_1} \subseteq \mathbb{L}_{0,k-1}$.*

Proof. We have the following positive-primitive definition of the relation $(\lim \wedge^{k-1} {=} 0)$ (Definition 4.64), which we treat here as a subset of $(2^\omega)^{k-1}$ rather than $2^{(k-1)\times\omega} \cong 2^\omega$ for clarity:

$$\lim(\vec{x}_0 \wedge \cdots \wedge \vec{x}_{k-2}) = 0 \iff \exists \vec{y} \in 2^\omega \left((\lim \vec{y} = 1) \wedge \bigwedge_{q<\omega} (x_{0,q} \wedge \cdots \wedge x_{k-2,q} \wedge y_q = 0) \right). \quad \square$$

Analogously to Corollary 4.26, we may deduce from this that $\mathbb{M}_\uparrow \mathbb{T}_{0,k}^{<\omega_1} \subseteq \mathbb{L}_{0,k-1}$. But in fact:

Lemma 4.86. *For each $k < \omega$, we have $\mathbb{M}_\uparrow \mathbb{T}_{0,k}^{<\omega_1} \subseteq \mathbb{L}_{0,k,k-1}$.*

Proof. We have the following positive-primitive definition of the relation $(\wedge^1 \wedge \lim \wedge^{k-1} {=} 0)$ (Definition 4.66), which we treat here as a subset of $2 \times (2^\omega)^{k-1}$:

$$x_0 \wedge \lim(\vec{x}_1 \wedge \cdots \wedge \vec{x}_{k-1}) = 0 \iff \exists \vec{y} \in 2^\omega \left((\bigvee \vec{y} = x_0) \wedge \bigwedge_{q<\omega} (x_{1,q} \wedge \cdots \wedge x_{k-1,q} \wedge y_q = 0) \right). \square$$

These lemmas explain why in the right $\mathbb{M}\mathbb{T}_{0,3}\mathbb{T}_1$ part of Figure 4.71(b), below the primary "cobweb" of 8 clones $[\mathbb{M}\mathbb{L}_{0,3}\mathbb{T}_1, \mathbb{M}\mathbb{T}_{0,3}\mathbb{T}_1]$, there is a "secondary row" of only the clones below $\mathbb{L}_{0,2}$ intersected with $\mathbb{L}_1$, and why below only $\mathbb{L}_{0,3,2}$, there is a "tertiary" intersection with $\mathbb{M}_\uparrow$. (Over $\mathbb{T}_{0,k}\mathbb{T}_1$, there would only be the "secondary row", whereas over $\mathbb{M}\mathbb{T}_{0,k}$, there would only be the "tertiary" intersection with $\mathbb{M}_\uparrow$; see Figure 1.6.)

We now show that such "secondary" and "tertiary" Borel clones indeed correspond to the obvious "primary" clones in the "cobweb" suggested by Figures 4.71(b) and 1.6.

Proposition 4.87. *For each $0 < k < \omega$, we have modularity isomorphisms*

$$[\langle \mathbb{M}\mathbb{T}_{0,k} \mathbb{T}_1^{<\omega} \rangle^{\mathsf{B}}, \mathbb{T}_{0,k} \mathbb{L}_1^{<\omega_1}] \cong [\langle \mathbb{M}\mathbb{T}_{0,k}^{<\omega} \rangle^{\mathsf{B}}, \mathbb{T}_{0,k} \mathbb{L}_{0,k-1}^{<\omega_1}]$$

$$F \mapsto \langle F \cup \{0\} \rangle^{<\omega_1}$$

$$G \cap \mathbb{L}_1 \mapsfrom G,$$

$$[\mathbb{M}_\downarrow \mathbb{T}_{0,k} \mathbb{L}_1^{\mathsf{B}}, \mathbb{T}_{0,k} \mathbb{L}_1^{<\omega_1}] \cong [\mathbb{M}_\downarrow \mathbb{T}_{0,k} \mathbb{T}_1^{\mathsf{B}}, \mathbb{T}_{0,k} \mathbb{L}_{0,k-1} \mathbb{T}_1^{<\omega_1}] \cong [\mathbb{M}_\downarrow \mathbb{T}_{0,k}^{\mathsf{B}}, \mathbb{T}_{0,k} \mathbb{L}_{0,k-1}^{<\omega_1}]$$

$$F \mapsto \langle F \cup \{\wedge\} \rangle^{<\omega_1} \qquad\qquad \mapsto \langle F \cup \{0\} \rangle^{<\omega_1}.$$

Proof. The latter two isomorphisms above follow from the first isomorphism above together with the second isomorphism in Proposition 4.34.

For the first isomorphism above, note first that the map is well-defined, since the upper bound of the left-hand interval is equal to $\mathbb{T}_{0,k}\mathbb{L}_{0,k-1}\mathbb{L}_1^{<\omega_1}$ by Lemma 4.85. And it is an embedding by the first isomorphism in Proposition 4.34, with image consisting of all clones G in the right-hand interval such that $G \subseteq \downarrow(G \cap \mathbb{L}_1)$; thus it suffices to prove this for all G in the right-hand interval. For that, it suffices to prove that

$$\forall g \in \mathbb{L}_{0,k-1}^{<\omega_1},\ g \in \downarrow\big(\langle\{g\} \cup \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}_1^{<\omega}\rangle \cap \mathbb{L}_1\big).$$

By Proposition 4.79, $\mathbb{L}_{0,k-1}^{<\omega_1} = \downarrow\langle\mathbb{M}\mathbb{T}_{0,k-1}\mathbb{T}_1^{<\omega}\rangle^\mathsf{B} = \downarrow\langle\forall_2^k\rangle^\mathsf{B}$; thus it suffices to prove

$$\langle\forall_2^k\rangle^\mathsf{B} \subseteq \big\{f \in \langle\mathbb{M}\mathbb{T}_{0,k-1}\mathbb{T}_1^{<\omega}\rangle^\mathsf{B} \;\big|\; \forall g \leq f,\ g \in \downarrow\big(\langle\{g\} \cup \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}_1^{<\omega}\rangle \cap \mathbb{L}_1\big)\big\}.$$

Clearly the right-hand side contains each π_i ($g \leq \pi_i \in \langle\varnothing\rangle \cap \mathbb{L}_1$); thus it suffices (Lemma 2.8) to prove that it is closed under left-composition with $\forall_2^k$. Let $f_0, \ldots, f_{k-1} : 2^n \to 2$ belong to the right-hand side, and let $g \leq \forall_2^k(f_0, \ldots, f_{k-1})$. Then by the induction hypothesis, there are h_i with

$$f_i \wedge g \leq h_i \in \langle\{f_i \wedge g\} \cup \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}_1^{<\omega}\rangle \cap \mathbb{L}_1$$
$$\subseteq \langle\{g\} \cup \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}_1^{<\omega}\rangle \cap \mathbb{L}_1$$

by Corollary 3.21 since $f_i \wedge g \in \downarrow\langle\{g\} \cup \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}_1^{<\omega}\rangle \cap \langle\{g\} \cup \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}_1^{<\omega} \cup \{1\}\rangle$. Then

$$g \leq \forall_2^{k+1}(g, h_0, \ldots, h_{k-1}) \in \langle\{g\} \cup \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}_1^{<\omega}\rangle \cap \mathbb{L}_1,$$

since $g = g \wedge \forall_2^k(f_0, \ldots, f_{k-1}) \leq \forall_2^{k+1}(g, h_0, \ldots, h_{k-1})$, and $h_0 \wedge \cdots \wedge h_{k-1} \leq \forall_2^{k+1}(g, h_0, \ldots, h_{k-1})$ whence $\forall_2^{k+1}(g, h_0, \ldots, h_{k-1})$ is continuous at $\vec{1}$. $\square$

Corollary 4.88. *For any $t < k < \omega$, we have $\mathbb{L}_{0,k,t}\mathbb{L}_{0,k-1}^{<\omega_1} = \downarrow\mathbb{M}\mathbb{L}_{0,k,t}\mathbb{L}_{0,k-1}\mathbb{L}_1^{<\omega_1}$ and $\mathbb{L}_{0,k,t}\mathbb{L}_{0,k-1}^\mathsf{B} = \downarrow^\mathsf{B}\mathbb{M}\mathbb{L}_{0,k,t}\mathbb{L}_{0,k-1}\mathbb{L}_1^\mathsf{B}$.*

Proof. $\supseteq$ is obvious; $\subseteq$ follows from the above and that $\langle F \cup \{0\}\rangle^{<\omega_1} \subseteq \downarrow F$ for a $<\omega_1$-ary clone $F \subseteq \mathbb{M}$ (Corollary 3.20), as well as Proposition 4.40 in the Borel case to get $\mathbb{L}_{0,k,t}\mathbb{L}_{0,k-1}^\mathsf{B} \subseteq \downarrow^\mathsf{B}\mathbb{M}\mathbb{L}_{0,k,t}\mathbb{L}_{0,k-1}^\mathsf{B} \subseteq \downarrow^\mathsf{B}\langle\mathbb{M}\mathbb{L}_{0,k,t}\mathbb{L}_{0,k-1}\mathbb{L}_1^\mathsf{B} \cup \{0\}\rangle^\mathsf{B} \subseteq \downarrow^\mathsf{B}\mathbb{M}\mathbb{L}_{0,k,t}\mathbb{L}_{0,k-1}\mathbb{L}_1^\mathsf{B}$. $\square$

Thus for instance, in Figure 4.71(b), the interval $[\mathbb{L}_{0,3}, \mathbb{T}_{0,3}\mathbb{L}_{0,2}]$ is isomorphic to the two parallel intervals on the right. Note that the proof of Proposition 4.87 applies to *all $<\omega_1$-ary clones in these intervals, not just the 4 shown*. This includes non-Borel $<\omega_1$-ary clones, as well as any potential unknown Borel clones between the upper 3 shown and their proposed generators from Remark 4.78.

Remark 4.89. From the proof of Proposition 4.87, we may read off a procedure for obtaining an upper bound $h \in \langle\{g\} \cup \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}_1^{<\omega}\rangle \cap \mathbb{L}_1$ for $g \in \mathbb{T}_{0,k}\mathbb{L}_{0,k-1}$, given an upper bound $g \leq f \in \langle\forall_2^k\rangle$. Namely, we induct on the construction of f; whenever $f = \forall_2^k(f_0, \ldots, f_{k-1})$, then we recursively find upper bounds h_i for $f_i \wedge g$, and then take

$$h := \forall_2^{k+1}(g, h_0, \ldots, h_{k-1}) = g \vee (h_0 \wedge \cdots \wedge h_{k-1}).$$

By Corollary 3.21, it follows that the preimage of $\langle\{g\} \cup \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}_1^{<\omega}\rangle$ under the second isomorphism in Proposition 4.87 is $\langle\{h\} \cup \mathbb{M}\mathbb{T}_{0,k}\mathbb{T}_1^{<\omega}\rangle$ (since g is generated by the latter together with 0).

We may apply this to $g = \vee_2^{k-t} \sqcup \liminf^{\sqcup t} \in \mathbb{ML}_{0,k,t-1}\mathbb{L}_{0,k-1}\mathbb{T}_1 \setminus \mathbb{L}_{0,k,t}$ from Lemma 4.69, for each $1 \leq t \leq k$. An easy upper bound is $f = \vee_2^k(\pi_0, \ldots, \pi_{k-1})$, which gives

$$h = (\vee_2^{k-t} \sqcup \liminf^{\sqcup t}) \vee (\pi_0 \wedge \cdots \wedge \pi_{k-1}).$$

For example, when $k = t = 2$ (cf. Definition 4.68), we get

$$h(\vec{x}) = \left(\liminf_{q \to \infty} x_{2q} \wedge \bigwedge_{q < \omega} x_{2q+1}\right) \vee \left(\bigwedge_{q < \omega} x_{2q} \wedge \liminf_{q \to \infty} x_{2q+1}\right) \vee (x_0 \wedge x_1).$$

Using the above, we may convert Lemma 4.75 into a "dichotomy" for such h:

Corollary 4.90. *For any $0 < t \leq k < \omega$ and function $f : 2^n \to 2 \in \mathbb{T}_{0,k}\mathbb{L}_1^{<\omega_1} \setminus \mathbb{L}_{0,k,t}$, we have* $(\vee_2^{k-t} \sqcup \liminf^{\sqcup t}) \vee (\pi_0 \wedge \cdots \wedge \pi_{k-1}) \in \langle \{f\} \cup \mathbb{ML}_{0,k}\mathbb{L}_1^{\mathsf{B}} \rangle = \langle f, \vee_2^{k+1}, \lceil \lfloor \bigwedge \rfloor \rceil, \lceil \bigvee \rceil \rangle.$

Proof. The clone $F := \langle \{f\} \cup \mathbb{ML}_{0,k}\mathbb{L}_1^{\mathsf{B}} \rangle^{<\omega_1} \not\subseteq \mathbb{L}_{0,k,t}\mathbb{L}_1$ corresponds via the first isomorphism in Proposition 4.87 to $\langle F \cup \{0\} \rangle^{<\omega_1} \not\subseteq \mathbb{L}_{0,k,t}$, which hence contains $\langle \{\vee_2^{k-t} \sqcup \liminf^{\sqcup t}\} \cup \mathbb{ML}_{0,k}^{\mathsf{B}} \rangle^{\mathsf{B}}$ by Lemma 4.75, whence the original clone F contains h from the preceding remark.

(The generators $\vee_2^{k+1}, \lceil \lfloor \bigwedge \rfloor \rceil, \lceil \bigvee \rceil$ for $\mathbb{ML}_{0,k}\mathbb{L}_1^{\mathsf{B}}$ are by Corollary 4.80.) $\qquad\square$

This completes the description of the "secondary row" $[\mathbb{ML}_{0,3}\mathbb{L}_1, \mathbb{MT}_{0,3}\mathbb{L}_1]$ in Figure 4.71(b), again modulo the caveat (as in Remark 4.78) that we do not know if the candidate generators for $(\mathbb{M})\mathbb{L}_{0,k,t}\mathbb{L}_1^{\mathsf{B}}$ match the relations once $t < k$.

It remains to explain the single "tertiary" node $\mathbb{M}_{\uparrow}\mathbb{T}_{0,3}\mathbb{T}_1 = \mathbb{M}_{\uparrow}\mathbb{L}_{0,3,2}\mathbb{L}_1$ in Figure 4.71(b):

Proposition 4.91. *For each $1 \leq k < \omega$, we have $\mathbb{L}_{0,k,k-1}^{<\omega_1} = {\downarrow}\mathbb{M}_{\uparrow}\mathbb{T}_{0,k}\mathbb{T}_1^{\mathsf{B}}$.*

Proof. $\supseteq$ by Lemma 4.86. Now let $f : 2^n \to 2 \in \mathbb{L}_{0,k,k-1}^{<\omega_1}$; thus

$$\vec{0} \notin f^{-1}(1) \wedge \overline{f^{-1}(1)}^{\wedge(k-1)}.$$

Using metrizability of 2^n for $n \leq \omega$, write $\overline{f^{-1}(1)} \subseteq 2^n$ as a countable decreasing intersection of open sets $U_0 \supseteq \overline{U_1} \supseteq U_1 \supseteq \overline{U_2} \supseteq U_2 \supseteq \cdots$. Then for each $\vec{x} \in f^{-1}(1)$, there is some i such that

$$\vec{0} \notin \vec{x} \wedge \overline{U_i}^{\wedge(k-1)}, \text{ i.e., } \vec{x} \notin {\downarrow}\neg[\overline{U_i}^{\wedge(k-1)}].$$

Let $U := \bigcup_i (U_i \setminus {\downarrow}\neg[\overline{U_i}^{\wedge(k-1)}])$. Then $f^{-1}(1) \subseteq U$, and $\vec{0} \notin U^{\wedge k}$, since for $\vec{x}_0, \ldots, \vec{x}_{k-1} \in U$ where each $\vec{x}_j \in U_{i_j} \setminus {\downarrow}\neg[\overline{U_{i_j}}^{\wedge(k-1)}]$, with $i_0 \leq i_1, \ldots, i_{k-1}$, we have $\vec{x}_0 \wedge \cdots \wedge \vec{x}_{k-1} \in \vec{x}_0 \wedge U_{i_1} \wedge \cdots \wedge U_{i_{k-1}} \subseteq \vec{x}_0 \wedge U_{i_0}^{\wedge(k-1)} \not\ni \vec{0}$. So the indicator function g of the upward-closure of U obeys $f \leq g \in \mathbb{M}_{\uparrow}\mathbb{T}_{0,k}\mathbb{T}_1$. $\qquad\square$

Remark 4.92. The above argument is reminiscent of the T_5 separation axiom in metrizable spaces. As such, it fails for uncountable arities $n > \omega$. Indeed, if 2^n is not T_5 (see e.g., [Kel75, 4.F]), then taking $A, B \subseteq 2^n$ disjoint from each other's closures but inseparable by open sets, and then embedding $2^n \hookrightarrow 2^{2+2n}$ as an antichain as in Remark 4.31, with meet $\neq \vec{0}$ and join $\neq \vec{1}$ (e.g., take only strings beginning with 01), the indicator function f of $A \cup \neg[B] \subseteq 2^{2+2n}$ will be in the n-ary analogue of $\mathbb{L}_{0,2,1}$ (meaning $\vec{0} \notin f^{-1}(1) \wedge \overline{f^{-1}(1)}$) but not $\leq$ any Scott-continuous $\mathbb{T}_{0,2}$ function.

It follows from the above (and the fact that $\mathbb{L}_{0,k,k-1} \not\subseteq \mathbb{L}_{0,k}$ by Lemma 4.69) that the clones $\mathbb{M}_\uparrow \mathbb{T}_{0,k}(\mathbb{T}_1)^{\mathsf{B}} = \mathbb{M}_\uparrow \mathbb{L}_{0,k,k-1}(\mathbb{T}_1)^{\mathsf{B}}$ are not contained in $\mathbb{L}_{0,k}$, as shown in Figure 4.71(b).

However, in contrast to the situation with $\mathbb{L}_1$ (Proposition 4.87), here we do not know if *every* subclone of $\mathbb{M}\mathbb{L}^{\mathsf{B}}_{0,k,k-1}$ outside $\mathbb{L}_{0,k}$ corresponds to a subclone of $\mathbb{M}_\uparrow \mathbb{L}^{\mathsf{B}}_{0,k,k-1}$. Nor do we have a candidate generating set for $\mathbb{M}_\uparrow \mathbb{L}^{\mathsf{B}}_{0,k,k-1}$, or know if there is a single function generating a minimal subclone of $\mathbb{M}_\uparrow \mathbb{L}_{0,k,k-1}$ outside $\mathbb{L}_{0,k}$ (analogous to $\liminf^{\sqcup k} \vee (\pi_0 \wedge \cdots \wedge \pi_{k-1})$ which is minimal in $\mathbb{M}\mathbb{L}_{0,k,k-1} \setminus \mathbb{L}_{0,k}$ from Corollary 4.90). An example of such a function is given by

Example 4.93. The indicator function $g : 2^\omega \to 2$ of the set of all strings $\geq$ one of

$$11\ 00\ 00\ 00\ \cdots,$$
$$10\ 11\ 00\ 00\ \cdots,$$
$$01\ 11\ 00\ 00\ \cdots,$$
$$10\ 10\ 11\ 00\ \cdots,$$
$$01\ 01\ 11\ 00\ \cdots,$$
$$\vdots$$

is in $\mathbb{M}_\uparrow \mathbb{L}_{0,2,1} \mathbb{T}^{\mathsf{B}}_1 \setminus \mathbb{L}_{0,2}$ (the strings $101010\cdots, 010101\cdots \in \overline{g^{-1}(1)}$ are disjoint). Note that this is an example of an upper bound for $\liminf \sqcup \liminf \in \mathbb{L}_{0,2,1}$, as provided by Proposition 4.91.

The following summarizes all of the positive classification results we have obtained for Borel clones lying over the left "side tube" of Post's lattice 1.1, from this subsection and Section 4.B:

Theorem 4.94. *For a Borel clone* $\mathbb{M}\mathbb{T}_{0,<\omega}\mathbb{T}^{<\omega}_1 = \langle \lceil \vee \rceil \rangle^{<\omega} \subseteq F \subseteq \mathbb{O}^{\mathsf{B}}_2$, *one of the following holds. These cases can all occur, and are mutually exclusive except for the boundary cases indicated in* (e).

(a) *For some coordinatewise downward-closed* $\varnothing \neq D \subseteq \{(k,t) \mid t \leq k < \omega\}$, *we have one of*

$$\mathbb{T}^{\mathsf{B}}_{0,\omega} \subseteq \langle \{\wedge, \nrightarrow\} \cup \{\forall^{k-t}_2 \sqcup \liminf^{\sqcup t} \mid (k,t) \notin D\}\rangle^{\mathsf{B}} \subseteq F \subseteq \bigcap_{(k,t)\in D} \mathbb{L}_{0,k,t},$$

$$\mathbb{T}_{0,\omega}\mathbb{T}^{\mathsf{B}}_1 \subseteq \langle \{\wedge, \lceil \to \rceil\} \cup \{\forall^{k-t}_2 \sqcup \liminf^{\sqcup t} \mid (k,t) \notin D\}\rangle^{\mathsf{B}} \subseteq F \subseteq \mathbb{T}_1 \cap \bigcap_{(k,t)\in D} \mathbb{L}_{0,k,t},$$

$$\mathbb{M}\mathbb{T}^{\mathsf{B}}_{0,\omega} \subseteq \langle \{\wedge, \lceil \vee \rceil, 0\} \cup \{\forall^{k-t}_2 \sqcup \liminf^{\sqcup t} \mid (k,t) \notin D\}\rangle^{\mathsf{B}} \subseteq F \subseteq \mathbb{M} \cap \bigcap_{(k,t)\in D} \mathbb{L}_{0,k,t}, \ \text{or}$$

$$\mathbb{M}\mathbb{T}_{0,\omega}\mathbb{T}^{\mathsf{B}}_1 \subseteq \langle \{\wedge, \lceil \vee \rceil\} \cup \{\forall^{k-t}_2 \sqcup \liminf^{\sqcup t} \mid (k,t) \notin D\}\rangle^{\mathsf{B}} \subseteq F \subseteq \mathbb{M}\mathbb{T}_1 \cap \bigcap_{(k,t)\in D} \mathbb{L}_{0,k,t}$$

(Corollaries 4.77 and 4.28). The first two of these intervals of F are isomorphic, as are the latter two (Proposition 4.34), which order-embed into the first two (Corollary 4.38); the image includes all Borel clones defined as Pol *of countably many Borel downward-closed relations $R \subseteq 2^\omega$ (Proposition 4.40). Moreover, we have exactly one of the following subcases.*

   (i) *There is a greatest $k < \omega$ for which $(k,0) \in D$, in which case $\forall^{k+1}_2 \in F \subseteq \mathbb{T}_{0,k}$, so F has finitary restriction $\mathbb{T}^{<\omega}_{0,k}, \mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1, \mathbb{M}\mathbb{T}^{<\omega}_{0,k}$ or $\mathbb{M}\mathbb{T}_{0,k}\mathbb{T}^{<\omega}_1$ respectively. There are 2^k of each of the four types of intervals above in this case (Corollary 4.72).*

   (ii) *We have $(k,0) \in D$ for all $k < \omega$ (so $F \subseteq \mathbb{L}_{0,<\omega,0} = \mathbb{T}_{0,<\omega}$), and $\forall_2 = \forall^\omega_2 \in F$. There are countably infinitely many of each of the four intervals above in this case (Corollary 4.73), the lowest one being $[\langle \forall_2, \wedge, \lceil \vee \rceil \rangle^{\mathsf{B}}, \mathbb{M}\mathbb{L}_{0,<\omega}\mathbb{L}^{\mathsf{B}}_1]$ when $D = \{(k,t) \mid t \leq k < \omega\}$.*

(iii) *We have* $\forall_2 \notin F$, *in which case* $F = \mathbb{T}_{0,\omega}^\mathsf{B}, \mathbb{T}_{0,\omega}\mathbb{T}_1^\mathsf{B}, \mathbb{MT}_{0,\omega}^\mathsf{B}$ *or* $\mathbb{MT}_{0,\omega}\mathbb{T}_1^\mathsf{B}$ (*Corollary* 4.46).

(*These are the "primary cobwebs" in Figures* 4.71(b) *and* 1.6.)

(b) *For some* $t < k < \omega$, F *has finitary restriction* $\mathbb{T}_{0,k}\mathbb{T}_1^{<\omega}$ *or* $\mathbb{MT}_{0,k}\mathbb{T}_1^{<\omega}$, *and obeys respectively*

$$\mathbb{L}_{0,k}\mathbb{L}_1^\mathsf{B} \subseteq \langle \forall_2^{k+1}, \lceil \bigvee \rceil, \lceil \rightarrow \rceil, (\forall_2^{k-t-1} \sqcup \liminf{}^{\sqcup(t+1)}) \vee (\pi_0 \wedge \cdots \wedge \pi_{k-1}) \rangle^\mathsf{B} \subseteq F \subseteq \mathbb{L}_{0,k,t}\mathbb{L}_{0,k-1}\mathbb{L}_1 \text{ or}$$
$$\mathbb{ML}_{0,k}\mathbb{L}_1^\mathsf{B} \subseteq \langle \forall_2^{k+1}, \lceil \lfloor \bigwedge \rfloor \rceil, \lceil \bigvee \rceil, (\forall_2^{k-t-1} \sqcup \liminf{}^{\sqcup(t+1)}) \vee (\pi_0 \wedge \cdots \wedge \pi_{k-1}) \rangle^\mathsf{B} \subseteq F \subseteq \mathbb{ML}_{0,k,t}\mathbb{L}_{0,k-1}\mathbb{L}_1$$

(*Corollary* 4.90). *These intervals are isomorphic to the second and fourth intervals in* (a)(i), *for* $(k,0), (k-1,k-1) \in D \not\ni (k+1,0), (k,k)$ (*Proposition* 4.87). (*These are the "secondary rows" below the "cobwebs" in Figures* 4.71(b) *and* 1.6.)

(c) *For some* $0 < k < \omega$, F *has finitary restriction* $\mathbb{MT}_{0,k}^{<\omega}$ *or* $\mathbb{MT}_{0,k}\mathbb{T}_1^{<\omega}$, *and obeys respectively*

$$\mathbb{M}_\uparrow\mathbb{L}_{0,k}^\mathsf{B} = \langle \forall_2^{k+1}, \lceil \bigvee \rceil, 0 \rangle^\mathsf{B} \subsetneqq F \subseteq \mathbb{M}_\uparrow\mathbb{T}_{0,k}^\mathsf{B} = \mathbb{M}_\uparrow\mathbb{L}_{0,k,k-1}^\mathsf{B} \text{ or}$$
$$\mathbb{M}_\uparrow\mathbb{L}_{0,k}\mathbb{T}_1^\mathsf{B} = \langle \forall_2^{k+1}, \lceil \bigvee \rceil \rangle^\mathsf{B} \subsetneqq F \subseteq \mathbb{M}_\uparrow\mathbb{T}_{0,k}\mathbb{T}_1^\mathsf{B} = \mathbb{M}_\uparrow\mathbb{L}_{0,k,k-1}\mathbb{L}_1^\mathsf{B}.$$

These two sets of F *are isomorphic to each other* (*Proposition* 4.87), *and embed into the third and fourth intervals in* (a)(i) *for* $D = \{(k',t') \mid (k',t') \leq (k,k-1)\}$ (*Corollary* 4.38) *with cofinal image* (*Proposition* 4.91). (*These are the "tertiary" nodes in Figures* 4.71(b) *and* 1.6.)

(d) F *has finitary restriction* $\mathbb{MT}_{0,<\omega}^{<\omega}$ *or* $\mathbb{MT}_{0,<\omega}\mathbb{T}_1^{<\omega}$, *and obeys respectively*

$$\langle \forall_2, 0 \rangle^\mathsf{B} \subseteq F \subseteq \mathbb{M}_\downarrow\mathbb{T}_{0,<\omega}^\mathsf{B} \text{ or}$$
$$\langle \forall_2 \rangle^\mathsf{B} \subseteq F \subseteq \mathbb{M}_\downarrow\mathbb{T}_{0,<\omega}\mathbb{T}_1^\mathsf{B}.$$

These two intervals are isomorphic (*Proposition* 4.34), *and they embed into the respective intervals below* $\mathbb{L}_{0,<\omega}$ *in* (a)(ii) (*Corollary* 4.38) *with cofinal image* (*Corollary* 4.83).

(e) F *is equal to one of the precisely* 2, 3, 4, 6 *Borel clones contained in* $\mathbb{L}_{0,k}$ *restricting to the finitary clones* $\mathbb{T}_{0,k}^{<\omega}, \mathbb{T}_{0,k}\mathbb{T}_1^{<\omega}, \mathbb{MT}_{0,k}^{<\omega}, \mathbb{MT}_{0,k}\mathbb{T}_1^{<\omega}$ *respectively, for some* $0 \leq k < \omega$ (*Corollary* 4.80) *or* $k = \omega$ (*Corollary* 4.28; *by convention* $\mathbb{L}_{0,\omega} := \mathbb{T}_{0,\omega}$). *These are all isomorphic as* k *varies, and are obtained by adjoining the generator* $\forall_2^{k+1}$ *to the respective clone below* $\mathbb{T}_{0,\omega}$ *from Figure* 4.29 *for* $k < \omega$, *or alternately by intersecting* $\mathbb{L}_{0,k}$ *with the respective clone above* $\langle \mathbb{MT}_1^{<\omega} \rangle^\mathsf{B}$ *in the top cube from Section* 4.A (*Proposition* 3.15).

For each k, *and in each of the four cases* ($\mathbb{M}$ *and/or* $\mathbb{T}_1$), *the top clone* $(\mathbb{M})\mathbb{L}_{0,k}(\mathbb{T}_1)^\mathsf{B}$ *here coincides with one of the extreme cases from* (a)(i) *or* (iii), *where* $(k,k) \in D \not\ni (k+1,0)$ (*or* $D = \{(k,t) \mid t \leq k < \omega\}$, *when* $k = \omega$). $\qquad\square$

Note that in each of the "imprecise" cases (a)–(d) above, *except for* (c), we have examples of "minimal" functions contained in all clones in the interval. If these can be shown to generate the upper bound of the interval, then the entire interval would collapse. We know this to happen only in the "bottommost" cases covered by (e), in the top cube cases $k \leq 1$ from Section 4.A, and in a few sporadic $k = 2$ cases; see below. Also, in each of the "topmost" cases $(1,1) \notin D$ in (a), yielding an interval $\forall_2^{1-1} \sqcup \liminf{}^{\sqcup 1} = \liminf \in F \subseteq \mathbb{T}_{0,k}$ (or $\subseteq \mathbb{T}_{0,<\omega}$ when $k = \omega$), we know by Proposition 4.57 that the interval collapses *up to taking countable closure*, i.e., every $<\omega_1$-ary

function in $\mathbb{T}_{0,k}$ can be approximated at any countably many inputs by a Borel function built from the candidate generators.

We conclude by relating the partial classification of Borel clones over $\mathbb{MT}_{0,2}^{<\omega}$ given by the above, to the Borel self-dual clones restricting to $\mathbb{DM}^{<\omega}$, the last of the 3 finitary self-dual, non-affine clones (3.33) which corresponds to $\mathbb{MT}_{0,2}^{<\omega}$ via the self-dualizing operator β (Proposition 3.32).

Recall that for the other 2 such finitary self-dual clones, $\mathbb{D}^{<\omega}$ and $\mathbb{DT}_0^{<\omega}$, the Borel clones were fully classified using β in Corollary 4.22; recall from there the definition of $\beta(\bigwedge) \in \mathbb{DL}_0$. Note that, in terms of Definition 4.68, $\beta(\bigwedge)$ may be written as

$$\beta(\textstyle\bigwedge) = 1 \sqcup \bigvee \geq 1 \sqcup \liminf .$$

It follows by Corollary 3.21 that $1 \sqcup \liminf \in \langle \beta(\bigwedge), 0 \rangle$. It is also easily seen that

$$(4.95) \qquad \beta(\textstyle\bigwedge)(\vec{x}) = (1 \sqcup \liminf)(x_0, x_1, \exists_2^3(x_0, x_1, x_2), \exists_2^3(x_0, x_1, \exists_2^3(x_0, x_2, x_3)), \dots),$$
$$\exists_2^3(x, y, z) = \beta(\textstyle\bigwedge)(x, y, z, y, z, y, z, \dots).$$

Thus

$$\langle \beta(\textstyle\bigwedge), 0 \rangle^{\mathsf{B}} = \langle 1 \sqcup \liminf, \exists_2^3, 0 \rangle^{\mathsf{B}}$$

are two equivalent versions of the candidate generators for $\mathbb{MT}_{0,2}\mathbb{L}_0^{\mathsf{B}}$ given by Corollary 4.77 (for the downward-closed set of indices $D = \{(2,0), (1,0), (1,1), (0,0)\}$).

Proposition 4.96. $\mathbb{MT}_{0,2}\mathbb{L}_0^{\mathsf{B}} = \langle \beta(\bigwedge), 0 \rangle^{\mathsf{B}}$.

Proof. Let $f \in \mathbb{MT}_{0,2}\mathbb{L}_0^{\mathsf{B}}$; then $f \leq \pi_0 \vee \cdots \vee \pi_{m-1}$ for some $m < \omega$, and we must show $f \in \langle \beta(\bigwedge), 0 \rangle$. We induct on m. If $m = 0$, then $f = 0$. Now suppose the claim holds for m, and $f \leq \pi_0 \vee \cdots \vee \pi_m$. Then (the cross-section) $f_0 \leq \pi_0 \vee \cdots \vee \pi_{m-1}$, so $f_0 \in \langle \beta(\bigwedge), 0 \rangle^{\mathsf{B}}$, and so $\beta(f_0) \in \langle \beta(\bigwedge), 0 \rangle$, since $\langle \beta(\bigwedge), 0 \rangle$ is the clone containing 0 corresponding via Proposition 3.32 to $\langle \beta(\wedge) \rangle \subseteq \mathbb{D}$, hence is closed under β. But since $f \in \mathbb{MT}_{0,2}$, we have $f \leq \beta(f_0)$, thus $f \in \langle \beta(\bigwedge), 0 \rangle$ by Corollary 3.21. $\qquad \square$

Corollary 4.97. *We have*:

(a) $\mathbb{T}_{0,2}\mathbb{L}_0^{\mathsf{B}} = \langle \beta(\bigwedge), \nrightarrow \rangle^{\mathsf{B}}$, $\mathbb{T}_{0,2}\mathbb{L}_0\mathbb{T}_1^{\mathsf{B}} = \langle \beta(\bigwedge), \bigwedge, \lceil\rightarrow\rceil \rangle^{\mathsf{B}}$, *and* $\mathbb{MT}_{0,2}\mathbb{L}_0\mathbb{T}_1^{\mathsf{B}} = \langle \beta(\bigwedge), \bigwedge \rangle^{\mathsf{B}}$.

(b) $\mathbb{T}_{0,2}\mathbb{L}_0\mathbb{L}_1^{\mathsf{B}} = \langle \beta(\bigwedge), \lceil\rightarrow\rceil \rangle^{\mathsf{B}}$ *and* $\mathbb{MT}_{0,2}\mathbb{L}_1^{\mathsf{B}} = \langle \beta(\bigwedge), \wedge \rangle^{\mathsf{B}}$.

(c) $\mathbb{T}_{0,2}^{\mathsf{B}} = \langle \liminf, \exists_2^3, \nrightarrow \rangle^{\mathsf{B}}$ *and* $\mathbb{MT}_{0,2}^{\mathsf{B}} = \langle \liminf, \exists_2^3, 0 \rangle^{\mathsf{B}}$.

Proof. The first equality in (a) follows from Proposition 4.40, Corollary 3.21, and Proposition 4.96; the rest then follows from Proposition 4.34.

(b) follows from Proposition 4.87.

(c): Note first that from $\liminf, \exists_2^3$, we easily get $1 \sqcup \liminf$, via a formula similar to (4.95), which then yields $\beta(\bigwedge)$; so the two generated clones in question contain $\beta(\bigwedge)$ and $\nrightarrow, 0$ respectively, hence contain all of $(\mathbb{M})\mathbb{T}_{0,2}\mathbb{L}_0^{\mathsf{B}}$ by Proposition 4.96 and (a). Now for $f \in (\mathbb{M})\mathbb{T}_{0,2}^{\mathsf{B}}$, we have $f = \liminf_{m \to \infty}(f \wedge (\pi_0 \vee \cdots \vee \pi_{m-1}))$ with each $f \wedge (\pi_0 \vee \cdots \vee \pi_{m-1}) \in (\mathbb{M})\mathbb{T}_{0,2}\mathbb{L}_0^{\mathsf{B}}$. $\qquad \square$

Thus among the Borel clones with finitary restriction $\mathbb{T}_{0,2}^{<\omega}$ and its variants, we have the curious situation (see Figure 1.6) that sufficiently small (below $\mathbb{L}_{0,2}$) or large (above $\mathbb{MT}_{0,2}\mathbb{L}_0\mathbb{L}_1$) such clones are fully classified (solid-shaded regions in 1.6), whereas the intermediate clones, $\mathbb{L}_{0,2,1}$ and its variants, remain open (hatch-shaded regions in 1.6). Nonetheless, this is enough to yield

Corollary 4.98. *There are precisely 3 Borel clones on* 2 *restricting to* $\mathbb{DM}^{<\omega}$ *(see Figure* 4.23*):*

  (i) $\mathbb{DM}^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\neg, \leq\} = \langle \beta(\liminf) \rangle^{\mathsf{B}}$,

  (ii) $\mathbb{DML}_0^{\mathsf{B}} = \mathbb{DML}_1^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\neg, \leq, \lim{=}0\} = \langle \beta(\bigwedge) \rangle^{\mathsf{B}}$,

  (iii) $\langle \mathbb{DM}^{<\omega} \rangle^{\mathsf{B}} = \mathrm{Pol}^{\mathsf{B}}\{\neg, \leq, \lim\} = \langle \exists_2^3 \rangle^{\mathsf{B}}$,

where

$$\beta(\liminf)(x_0, x_1, \dots) = \big(x_0 \wedge \limsup_{i \to \infty} x_i\big) \vee \liminf_{i \to \infty} x_i.$$

Proof. By Proposition 3.32 (and the fact that $\mathbb{DM}^{<\omega} = \langle \beta(\mathbb{MT}_{0,2}^{<\omega}) \rangle^{<\omega}$), such Borel clones are $\langle \beta(G) \rangle^{\mathsf{B}}$ for all Borel clones G restricting to $\mathbb{MT}_{0,2}^{<\omega}$ which are closed under β. If G contains a discontinuous function g, then by Lemma 4.13, either $g \notin \mathbb{M}_\uparrow$, or $g \notin \mathbb{M}_\downarrow$ in which case clearly from Definition 3.30, $\beta(g) \notin \mathbb{M}_\uparrow$; thus by Corollary 4.30, $\bigwedge \in G$, so $\beta(\bigwedge) \in G$, so by Proposition 4.96, $\mathbb{MT}_{0,2}\mathbb{L}_0^{\mathsf{B}} \subseteq G$, and so by Lemma 4.54 and Corollary 4.97(c), either $G = \mathbb{MT}_{0,2}\mathbb{L}_0^{\mathsf{B}}$ or $G = \mathbb{MT}_{0,2}^{\mathsf{B}}$. So

$$\mathbb{MT}_{0,2}^{\mathsf{B}} = \langle \liminf, \exists_2^3, 0 \rangle^{\mathsf{B}}, \qquad \mathbb{MT}_{0,2}\mathbb{L}_0^{\mathsf{B}} = \langle \beta(\bigwedge), 0 \rangle^{\mathsf{B}}, \qquad \langle \mathbb{MT}_{0,2}^{<\omega} \rangle^{\mathsf{B}} = \langle \exists_2^3, 0 \rangle^{\mathsf{B}}$$

are the 3 Borel clones lying over $\mathbb{MT}_{0,2}^{<\omega}$ which are closed under β; these are easily seen to correspond via Proposition 3.32 to the above clones lying over $\mathbb{DM}^{<\omega}$. $\qquad \square$

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