14 Galois cohomology

14.1 群の cohomology

定義 14.1. G : 群、 M : 加法 (Abel) 群で G は M に加群としての作用をしているとする。ここで以下のように G^n から M への写像全体の集合を $C^n(n \in \mathbb{Z}_{\geq 0})$ として定める。

$$C^n = C^n(G,M) := \{f : G^n \longrightarrow M\} = \mathrm{Map}(G^n, M)$$

ただし $G^0 = \{e\}$ と考えることで $C^0 := M$ と定める。この C^n の各元を n コチェイン (cochain)という。C^n 上へは $f, g \in C^n$ に対して $(f+g)(x) := f(x) + g(x)$ と演算を定めることで C^n は加法群となる。

定義 14.2. C^n から C^{n+1} への以下のように定まる写像 ∂ を考える。

$$\begin{aligned} \partial = \partial^n : C^n &\longrightarrow C^{n+1} \\ f &\longmapsto \partial f \end{aligned}$$

ここで $\partial f : G^{n+1} \longrightarrow M$ は G が M へ作用していることに注意して

$$\begin{aligned} \partial f(g_1, \ldots, g_{n+1}) = g_1 f(g_2, \ldots, g_{n+1}) & \\ + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) & \\ + (-1)^{n+1} f(g_1, \ldots, g_n) & \end{aligned}$$

と定める。このときこの $\partial (= \partial^n) : C^n(G,M) \longrightarrow C^{n+1}(G,M)$ は加法群の準同型になり、これを n 次のコバウンダリー (双対境界) 作用素 (coboundary operator)とよぶ。

命題 14.3. コバウンダリー作用素 ∂ に対して $\partial^{n+1} \circ \partial^n = 0$ が成り立つ。

Proof. $4 \leq n$ でまず考える。

$(\partial^{n+1} \circ \partial^n)(f)(g_1, \ldots, g_{n+2}) = \partial^{n+1}(\partial^n f)(g_1, \ldots, g_{n+2})$ なので $f' := \partial^n f$ として $\partial^{n+1} f'(g_1, \ldots, g_{n+2})$ は

$$\begin{aligned} \partial^{n+1} f'(g_1, \ldots, g_{n+2}) = g_1 f'(g_2, \ldots, g_{n+2}) & \\ + \sum_{i=1}^{n+1} (-1)^i f'(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+2}) & \\ + (-1)^{n+1} f'(g_1, \ldots, g_{n+1}) & \end{aligned}$$

である。$f'(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+2}) = \partial^n f(g_1, \ldots g_i g_{i+1}, \ldots, g_{n+2})$ を i の値によって計算する。

・ $i = 1$ のとき

$$\begin{aligned} \partial^n f(g_1 g_2, \ldots, g_{n+2}) = g_1 g_2 f(g_3, \ldots, g_{n+2}) & \\ + (-1)^1 f((g_1 g_2) g_3, g_4, \ldots, g_{n+2}) & \\ + \sum_{k=3}^{n+1} (-1)^{k-1} f(g_1 g_2, g_3, \ldots, g_i g_{i+1}, \ldots, g_{n+2}) & \\ + (-1)^{n+1} f(g_1 g_2, g_3, \ldots, g_n) & \end{aligned}$$

・ $i=2$ のとき

$$\begin{aligned}\partial^n f(g_1, g_2g_3, g_4, \dots, g_{n+2}) &= g_1 f(g_2g_3, g_4, \dots, g_{n+2}) \\ &\quad + (-1)^1 f(g_1(g_2g_3), g_4, \dots, g_{n+2}) \\ &\quad + (-1)^2 f(g_1, (g_2g_3)g_4, g_5, \dots, g_{n+2}) \\ &\quad + \sum_{k=4}^{n+1} (-1)^{k-1} f(g_1, g_2g_3, g_4, \dots, g_i g_{i+1}, \dots, g_{n+2}) \\ &\quad + (-1)^{n+1} f(g_1, g_2g_3, g_4, \dots, g_{n+1})\end{aligned}$$

・ $3 \leq i \leq n-1$ のとき

$$\begin{aligned}\partial^n f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+2}) &= g_1 f(g_2, \dots, g_i g_{i+1}, \dots, g_{n+2}) \\ &\quad + \sum_{k=1}^{i-2} (-1)^k f(g_1, \dots, g_k g_{k+1}, \dots, g_i g_{i+1}, \dots, g_{n+2}) \\ &\quad + (-1)^{i-1} f(g_1, \dots, g_{i-2}, g_{i-1}(g_i g_{i+1}), g_{i+2}, \dots, g_{n+2}) \\ &\quad + (-1)^i f(g_1, \dots, g_{i-1}, (g_i g_{i+1}) g_{i+2}, g_{i+3}, \dots, g_{n+2}) \\ &\quad + \sum_{k=i+2}^{n+1} (-1)^{k-1} f(g_1, \dots, g_i g_{i+1}, \dots, g_k g_{k+1}, \dots, g_{n+2}) \\ &\quad + (-1)^{n+1} f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})\end{aligned}$$

・ $i=n$ のとき

$$\begin{aligned}\partial^n f(g_1, \dots, g_n g_{n+1}, g_{n+2}) &= g_1 f(g_2, \dots, g_n g_{n+1}, g_{n+2}) \\ &\quad + \sum_{k=1}^{n-2} (-1)^k f(g_1, \dots, g_k g_{k+1}, \dots, g_n g_{n+1}, g_{n+2}) \\ &\quad + (-1)^{n-1} f(g_1, \dots, g_{n-1}(g_n g_{n+1}), g_{n+2}) \\ &\quad + (-1)^n f(g_1, \dots, g_{n-1}, (g_n g_{n+1}) g_{n+2}) \\ &\quad + (-1)^{n+1} f(g_1, \dots, g_{n-1}, g_n g_{n+1})\end{aligned}$$

・ $i=n+1$ のとき

$$\begin{aligned}\partial^n f(g_1, \dots, g_{n+1} g_{n+2}) &= g_1 f(g_2, \dots, g_{n+1} g_{n+2}) \\ &\quad + \sum_{k=1}^{n-1} (-1)^k f(g_1, \dots, g_k g_{k+1}, \dots, g_n, g_{n+1} g_{n+2}) \\ &\quad + (-1)^n f(g_1, \dots, g_{n-1}, g_n(g_{n+1} g_{n+2})) \\ &\quad + (-1)^{n+1} f(g_1, \dots, g_n)\end{aligned}$$

となる。

また、 $g_1 f'(g_2, \dots, g_{n+2})$ と $(-1)^{n+2} f'(g_1, \dots, g_{n+1})$ は以下のようになる。

$$
\begin{aligned}
g_1 \partial^n f(g_2, \dots, g_{n+2}) &= g_1 (g_2 f(g_3, \dots, g_{n+2}) \\
&\quad + \sum_{i=2}^{n+1} (-1)^{i-1} f(g_2, \dots, g_i g_{i+1}, \dots, g_{n+2}) \\
&\quad + (-1)^{n+1} f(g_2, \dots, g_{n+1})) \\
(-1)^{n+2} \partial^n f(g_1, \dots, g_{n+1}) &= (-1)^{n+2} (g_1 f(g_2, \dots, g_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^{i} f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\
&\quad + (-1)^{n+1} f(g_1, \dots, g_n))
\end{aligned}
$$

これを $\partial^{n+1}(\partial^n f)(g_1,\ldots,g_{n+2})$ の式に代入すると

$$
\begin{aligned}
\partial^{n+1}(\partial^n f)(g_1,\ldots,g_{n+2} &= \{g_1g_2f(g_3,\ldots,g_{n+2})\\
&+\sum_{i=2}^{n+1}(-1)^{i-1}g_1f(g_2,\ldots,g_ig_{i+1},\ldots,g_{n+2})\\
&+(-1)^{n+1}g_1f(g_2,\ldots,g_{n+1})\}\\
&+(-1)^1\{g_1g_2f(g_3,\ldots,g_{n+2})\\
&+(-1)^1f((g_1g_2)g_3,g_4,\ldots,g_{n+2})\\
&+\sum_{k=3}^{n+1}(-1)^{k-1}f(g_1g_2,g_3,\ldots,g_ig_{i+1},\ldots,g_{n+2})\\
&+(-1)^{n+1}f(g_1g_2,g_3,\ldots,g_n)\}\\
&+(-1)^2\{g_1f(g_2g_3,g_4,\ldots,g_{n+2})\\
&+(-1)^1f(g_1(g_2g_3),g_4,\ldots,g_{n+2})\\
&+(-1)^2f(g_1,(g_2g_3)g_4,g_5,\ldots,g_{n+2})\\
&+\sum_{k=4}^{n+1}(-1)^{k-1}f(g_1,g_2g_3,g_4,\ldots,g_ig_{i+1},\ldots,g_{n+2})\\
&+(-1)^{n+1}f(g_1,g_2g_3,g_4,\ldots,g_{n+1})\}\\
&+\sum_{i=3}^{n-1}(-1)^i\{g_1f(g_2,\ldots,g_ig_{i+1},\ldots,g_{n+2})\\
&+\sum_{k=1}^{i-2}(-1)^kf(g_1,\ldots,g_kg_{k+1},\ldots,g_ig_{i+1},\ldots,g_{n+2})\\
&+(-1)^{i-1}f(g_1,\ldots,g_{i-2},g_{i-1}(g_ig_{i+1}),g_{i+2},\ldots,g_{n+2})\\
&+(-1)^if(g_1,\ldots,g_{i-1},(g_ig_{i+1})g_{i+2},g_{i+3},\ldots,g_{n+2})\\
&+\sum_{k=i+2}^{n+1}(-1)^{k-1}f(g_1,\ldots,g_ig_{i+1},\ldots,g_kg_{k+1},\ldots,g_{n+2})\\
&+(-1)^{n+1}f(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1})\}\\
&+\{g_1f(g_2,\ldots,g_ng_{n+1},g_{n+2})\\
&+\sum_{k=1}^{n-2}(-1)^kf(g_1,\ldots,g_kg_{k+1},\ldots,g_ng_{n+1},g_{n+2})\\
&+(-1)^{n-1}f(g_1,\ldots,g_{n-1}(g_ng_{n+1}),g_{n+2})\\
&+(-1)^nf(g_1,\ldots,g_{n-1},(g_ng_{n+1})g_{n+2})\\
&+(-1)^{n+1}f(g_1,\ldots,g_{n-1},g_ng_{n+1})\}\\
&+\{g_1f(g_2,\ldots,g_{n+1}g_{n+2})\\
&+\sum_{k=1}^{n-1}(-1)^kf(g_1,\ldots,g_kg_{k+1},\ldots,g_n,g_{n+1}g_{n+2})\\
&+(-1)^nf(g_1,\ldots,g_{n-1},g_n(g_{n+1}g_{n+2}))\\
&+(-1)^{n+1}f(g_1,\ldots,g_n)\}\\
&+\{(-1)^{n+2}(g_1f(g_2,\ldots,g_{n+1})\\
&+\sum_{i=1}^{n}(-1)^if(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1})\\
&+(-1)^{n+1}f(g_1,\ldots,g_n))\}
\end{aligned}
$$

$\square$