「REGULAR RINGS AND PERFECT(OID) ALGEBRAS」の紹介

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This is a note for the 18-th summer school on Commutative algebra at the Tokyo Institute of Technology to introduce the paper [BIM].

Convention

- All rings are assumed to be commutative and contains unity.
- Let p denote a fixed prime number.

Contents

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Introduction

The paper [BIM] explores some homological properties of perfect(oid) algebras over commutative noetherian rings. One of results is Kunz's theorem in mixed characteristic case.

Recall that Kunz's theorem asserts that a noetherian $\mathbb{F}_p$-algebra R is regular if and only if the Frobenius map $R \to R$ is flat. One can reformulate this result as the following assertion: such an R is regular exactly when there exists a faithfully flat map $R \to A$ with A perfect (Proposition 1.6). Our p-adic generalization is the following:

Theorem A (see Theorem 1.10)**.** *Let R be a noetherian ring such that p lies in the Jacobson radical of R (for example, R could be p-adically complete). Then R is regular if and only if there exists a faithfully flat map $R \to A$ with A perfectoid.*

§1 is devoted to Kunz's theorem in both positive characteristic case and mixed characteristic case.

In §2, we introduce the results about finiteness of flat dimension. In fact, the original proof of Theorem A in [BIM] is based on results in this section.

In §3, we focus on two algebras: the absolute integral closure R^+ and the perfect closure R_{perf}. In addition, we will introduce the notions of *proregular sequences* and *weakly proregular sequences*. It will turn out that systems of parameters of R are weakly proregular on R^+ and on R_{perf} under some conditions (**??**). Because of this, we obtain a criterion of regulaness using the vanishing of $\mathrm{Tor}_i^R(R^+, k)$ or $\mathrm{Tor}_i^R(R_{\mathrm{perf}}, k)$ (**??**)

1. Kunz's theorem

In this section we give a quick but reasonably detailed overview of the proof of Kunz's theorem in mixed characteristic case.

First in §1.1 we recall classical Kunz's theorem and some applications. This subsection ends with reformulation of Kunz's theorem

After understanding the proof of reformulation of Kunz's theorem in positive characteristic, we proceed in §1.2 to generalizing to mixed characteristic, using the notion of perfectoid rings.

1.1. Positive characteristic.

Throughout this subsection, let R denote a noetherian $\mathbb{F}_p$-algebra.

Let us first briefly recall Kunz's theorem.

Theorem 1.1 ([Kunz69, Theorem 2.1, Corollary 2.7])**.** *The following conditions are equivalent.*

(1) *R is regular.*
(2) *The absolute Frobenius $\varphi\colon R \to R$ is flat.*

Sketch. We may assume that R is complete local (here we use the fact that a local homomorphism $A \to B$ of noetherian local rings is (faithfully) flat if and only if so is $\widehat{A} \to \widehat{B}$, which is a consequence of local criterion of flatness). Let $k := R/\mathfrak{m}_R$.

(1) $\Rightarrow$ (2): By Cohen's structure theorem, we may assume that $R = k[[X_1, \dots, X_n]]$, where $n = \dim R$. Then the canonical injection $R^p \hookrightarrow R$ can be decomposed as

$$R^p = k^p[[X_1^p, \dots, X_n^p]] \hookrightarrow k[[X_1^p, \dots, X_n^p]] \hookrightarrow k[[X_1, \dots, X_n]] = R.$$

Since $k[[X_1, \dots, X_n]]$ is free over $k[[X_1^p, \dots, X_n^p]]$ on the basis $\{X_1^{\alpha_1} \cdots X_n^{\alpha_n} \mid 0 \le \alpha_i \le p-1\}$ and since the flatness of $k^p \hookrightarrow k$ implies that of $k^p[[X_1^p, \dots, X_n^p]] \hookrightarrow k[[X_1^p, \dots, X_n^p]]$, it follows that $R^p \hookrightarrow R$ is flat.

(2) $\Rightarrow$ (1): Let $x_1, \dots, x_r$ be a minimal basis of $\mathfrak{m}_R$. Then by Cohen's structure theorem, we have a surjection

$$S := k[[X_1, \dots, X_r]] \twoheadrightarrow R, \quad X_i \mapsto x_i.$$

Let $\mathfrak{a}$ be the kernel. Then, for each p-power $q = p^\nu$, the surjection induces the short exact sequence

$$0 \longrightarrow (\mathfrak{a} + \mathfrak{m}_S^q)/\mathfrak{m}_S^q \longrightarrow S/\mathfrak{m}_S^q \longrightarrow R/\mathfrak{m}_R^q \longrightarrow 0.$$

Using the notion of independendence in the sense of Lech, we can prove that $l_S(S/\mathfrak{m}_S^q) = q^r = l_R(R/\mathfrak{m}_R^q) = l_S(R/\mathfrak{m}_R^q)$, so that $(\mathfrak{a} + \mathfrak{m}_S^q)/\mathfrak{m}_S^q = 0$, i.e., $\mathfrak{a} \subset \mathfrak{m}_S^q$. Since this holds true for any p-power $q = p^\nu$, one has

$$\mathfrak{a} \subseteq \bigcap_{\nu > 0} \mathfrak{m}_S^q = (0).$$

Thus $R \cong S = k[[X_1, \dots, X_r]]$, which is a regular local ring. □

Let us mention applications of Kunz's theorem.

The fact that a localization of a regular local ring is again regular is proved by (Auslander-Buchbaum-)Serre's theorem for regular local rings. But in positive characteristic case, this fact is an immediate consequence of Kunz's theorem:

Corollary 1.2 ([Kunz69, Corollary 2.2])**.** *If R is a regular local ring, then so is $R_\mathfrak{p}$ of R for each $\mathfrak{p} \in \operatorname{Spec} R$.*

In addition, Kunz's theorem yields the following result about excellence of $\mathbb{F}_p$-algebras.

Theorem 1.3 ([Kunz76, Theorem 2.5])**.** *Let R be a noetherian $\mathbb{F}_p$-algebra. If the Frobenius endomorphism $\varphi\colon R \to R$ is finite, then R is excellent.*

The following corollay leads to our reformulation of Kunz's theorem.

Corollary 1.4. *If R is a regular $\mathbb{F}_p$-algebra, then $R \to R_{\mathrm{perf}}$ is faithfully flat.*

Proof. It suffices to show that each $\varphi^n\colon R \to R$ is faithfully flat (cf. [SP, Tag 090N]). Since R is a regular $\mathbb{F}_p$-algebra, φ (hence φ^n) is flat by Kunz's theorem. Moreover, given a maximal ideal $\mathfrak{m} \subset R$, we have $\varphi^n(m)R \subset \mathfrak{m}^{p^n} R \subset \mathfrak{m} \neq R$. Thus we conclude that $\varphi^n\colon R \to R$ is faithfully flat. □

We will show that the converse holds, and at the same time reformulate Kunz's theorem.

Let us start with the following lemma.

Lemma 1.5 (cf. [BIM, Lemma 3.2])**.** *Let A be a perfect $\mathbb{F}_p$-algebra, and $\boldsymbol{x} = x_1, \dots, x_n$ a sequence of elements in A.*

(1) $\sqrt{\boldsymbol{x}A} = (x_1^{1/p^\infty}, \dots, x_n^{1/p^\infty})$.
(2) $\mathsf{fd}_A(A/\sqrt{\boldsymbol{x}A}) \le n$.

Proof. (1) Straightforward.

(2) We proceed by induction on n.

$\underline{n=1}$ Relabel $x = x_1$ for visual convenience. It suffices to check that the ideal $I := (x^{1/p^\infty}) \subset A$ is flat as an S-module (then $0 \to I \to A \to A/I \to 0$ is a flat resolution of A/I). Observe that the morphism of direct systems[1]

$$\begin{tikzcd} A \arrow[r, "x^{1-\frac{1}{p}}"] \arrow[d, "x"] & A \arrow[r, "x^{\frac{1}{p}-\frac{1}{p^2}}"] \arrow[d, "x^{1/p}"] & A \arrow[r, "x^{\frac{1}{p^2}-\frac{1}{p^3}}"] \arrow[d, "x^{1/p^2}"] & \cdots \\ (x) & \subset \; (x^{1/p}) & \subset \; (x^{1/p^2}) & \subset \; \cdots \end{tikzcd}$$

induces the morphism of direct limits

$$\varinjlim \left(A \xrightarrow{x^{1-\frac{1}{p}}} A \xrightarrow{x^{\frac{1}{p}-\frac{1}{p^2}}} A \to \cdots \right) \to (x^{1/p^\infty}) = I.$$

The surjectivity is clear, and we can check the injectivity using the fact that A is reduced.

$\underline{n>1}$ Set $A' := A/\sqrt{x_1 A})$, and let $\boldsymbol{x}'$ be the image of the sequence $x_2, \dots, x_n$ in A'. Then A' is also perfect, and thus by the induction hypothesis, we obtain

$$\mathsf{fd}_A(A/\sqrt{\boldsymbol{x}A}) \le \mathsf{fd}_{A'}(A'/\sqrt{\boldsymbol{x}'A'}) + 1 \le (n-1)+1 = n,$$

(see [AF, 4.2 Corollary (b) (F)] or [SP, Tag 066K], for the first inequality). $\square$

Now one can reformulate Kunz's theorem as the following assertion:

Proposition 1.6. *Let R be a noetherian $\mathbb{F}_p$-algebra. Then the following conditions are equivalent.*
(1) *R is regular.*
(2) *$R \to R_{\mathrm{perf}}$ is faithfully flat.*
(3) *There exists a faithfully flat ring homomorphism $R \to A$ with A perfect.*

Proof. (1) $\Longrightarrow$ (2): Corollary 1.4.

(2) $\Longrightarrow$ (3): Trivial.

(3) $\Longrightarrow$ (1): Pick $\mathfrak{p} \in \operatorname{Spec} R$. Since $R \to A$ is faithfully flat, there exists $P \in \operatorname{Spec} A$ such that $P \cap R = \mathfrak{p}$. Then the induced local homomorphism $R_\mathfrak{p} \to A_P$ is (faithfully) flat. Thus we may assume that R, A are local and that the flat ring homomorphism $R \to A$ is local. Set $k := R/\mathfrak{m}_R$. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a s.o.p. of R ($n = \dim R$). Then lemma 1.9 yields

$$\mathsf{fd}_A(A/\sqrt{\boldsymbol{x}A}) \le n. \tag{1.1}$$

Hence, if $i > n$,

$$\begin{aligned} 0 &= \operatorname{Tor}_i^A(k \otimes_R A, A/\sqrt{\mathfrak{m}_R A}) \\ &\overset{\text{flat}}{=} \operatorname{Tor}_i^R(k, A/\sqrt{\mathfrak{m}_R A}) \\ &= \operatorname{Tor}_i^R(k,k)^{\oplus I} \end{aligned}$$

where $A/\sqrt{\boldsymbol{x}A} \cong k^{\oplus I}$ (since $A/\sqrt{\boldsymbol{x}A} = A/\sqrt{\sqrt{\boldsymbol{x}R}A} = A/\sqrt{\mathfrak{m}A}$ is a k-vector space). Since $R \to A$ is a local homomorphism, it follows that $I \ne \emptyset$, so that $\operatorname{Tor}_i^R(k,k) = 0$ for $i > n$. This means that $\operatorname{gl.dim} R = \mathsf{pd}_R k \le n < \infty$. $\square$

Note that the essential part of the proof the finiteness of flat dimension (1.1).

[1]The notation $x^{\frac{1}{p^e} - \frac{1}{p^{e+1}}}$ makes sense because $\frac{1}{p^e} - \frac{1}{p^{e+1}} = \frac{p-1}{p^{e+1}}$.

1.2. **Mixed characteristic.** We insert here a brief review of the definition and some properties of perfectoid rings (cf. [BMS1]).

Definition 1.7 ([BIM, Definition 3.5])**.** We say that a ring A is *perfectoid* if it satisfies the following conditions.

(1) A is p-adically complete.
(2) The $\mathbb{F}_p$-algebra A/pA is semiperfect.
(3) The kernel of Fontaine's map $\theta\colon W(A^\flat) \to A$ is principal.
(4) There exist $\pi \in A$ and $u \in A^\times$ such that $\pi^p = pu$.

For the equivalence of this definition and other characterizations, see [BMS1, Lemma 3.9, Proposition 3.10]. If A is perfectoid, then the following hold:

- Fontaine's map $\theta\colon W(A^\flat) \to A$ is surjective (see [BMS1, Lemma 3.9]).
- An element $\xi = (\xi_0, \xi_1, \dots) \in \ker\theta$ generates $\ker\theta$ if and only if ξ is distinguished, i.e., $\xi_1 \in (A^\flat)^\times$ (see [BMS1, Remark 3.11]).

Note also that an arbitrary product of perfectoid rings is perfectoid ([BIM, Example 3.8 (8)]). Indeed, the condition (3) in Definition 1.7 is satisfied because the functor $A \mapsto W(A^\flat)$ commutes with products. The other conditions are obvious.

We now turn to generalizing Kunz's theorem to mixed characteristic. The most important features of perfectoid rings are the following.

Lemma 1.8 ([BIM, Lemma 3.7])**.** *Let A be a perfectoid ring.*

(1) *The $\mathbb{F}_p$-algebra $\overline{A} := A/\sqrt{pA}$ is perfect.*
(2) *The ideal $\sqrt{pA} \subset A$ is a flat A-module.*

Proof. (1) We first show that the element π appearing in Definition 1.7 can be assumed to admit a compatible system of p-power roots $\{\pi^{1/p^n}\}_{n\geq 1}$. Let $\xi = (\xi_0, \xi_1, \dots) \in W(A^\flat)$ be a generator of $\ker\theta = (\xi)$, and set $\pi \in A$ to be the image of $[\xi_0]$. Then π satisfies the condition (4) in Definition 1.7 (here we use that ξ is distinguished) and admits a compatible system of p-power roots, namely the images of $[\xi_0^{1/p^{n+1}}]$.

Since $(p) = (\pi^p)$, one has $\sqrt{pA} = (p^{1/p^\infty}) = (\pi^{1/p^\infty})$, and so it suffices to show that $A/(\pi^{1/p^\infty})$ is perfect. The isomorphism $W(A^\flat)/(\xi) \xrightarrow{\sim} A$ and the definition of our π yield

$$A/(\pi^{1/p^\infty}) \cong W(A^\flat)/(\xi, [\xi_0^{1/p^\infty}]) \cong \frac{W(A^\flat)/(p)}{(\xi, [\xi_0^{1/p^\infty}])/(p)} \cong A^\flat/\left(\xi_0^{1/p^\infty}\right) \overset{\text{Lemma 1.5 (1)}}{=} A^\flat/\sqrt{(\xi_0)}.$$

This ring is perfect since $A^\flat$ is perfect.

(2) We prove only in the case where A is p-torsion free. (The general case is difficult.) We check that $\mathsf{fd}_A(\overline{A}) \leq 1$; this is equivalent to showing that $\sqrt{pA}$ is flat. Since A is p-torsion free and $\pi^p = pu$ for some $u \in A^\times$, we can see that each π^{1/p^e} is a non-zero-divisor of A. Thus each $\varpi^{1/p^e}A$ is isomorphic to A, hence is free. The directed union (π^{1/p^∞}) is also flat. ☐

By this lemma, we deduce the desired finiteness of flat dimension. For the sake of the later applications, we give this result in more general setting. We say that a ring A is *(positive characteristic p and) perfect modulo a flat ideal* if there exists an ideal $I \subset A$ containing p such that A/I is perfect. Lemma 1.8 shows that any perfectoid is perfect modulo a flat ideal.

Lemma 1.9. *Let A be a ring that is perfect modulo a flat ideal I, and set $\overline{A} := A/I$. If $J \subset A$ is an ideal containing I such that $J\overline{A} = \sqrt{\boldsymbol{x}\overline{A}}$ for some sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\overline{A}$. Then $\mathsf{fd}_A(A/J) \leq n+1$.*

Proof. $\mathsf{fd}_A(A/J) \leq \mathsf{fd}_{\overline{A}}(\overline{A}/J\overline{A}) + \mathsf{fd}_A \overline{A} \overset{\text{Lemma 1.8}}{\leq} n+1.$ ☐

Now we can prove a mixed characteristic generalization of Kunz's theorem.

Theorem 1.10 ([BIM, Theorem 4.7])**.** *Let R be a noetherian ring with $p \in \operatorname{rad} R$. Then the following conditions are equivalent.*

(1) *R is regular.*
(2) *There exists a faithfully flat ring homomorphism $R \to A$ with A perfectoid.*

Proof. (2) ⇒ (1): Because of Lemma 1.8 and Lemma 1.9, the argument similar to that in Proposition 1.6 works.

(1) $\Rightarrow$ (2): Assume that R is regular with $p \in \operatorname{rad} R$. We must construct a faithfully flat ring homomorphism $R \to A$ with perfectoid.

(Step 1): Reduction to the case where R is complete local. Assume that, for each $\mathfrak{m} \in \operatorname{Max} R$, we obtain a faithfully flat ring homomorphism $\widehat{R}_\mathfrak{m} \to A(\mathfrak{m})$ with $A(\mathfrak{m})$ perfectoid. Consider the resulting ring homomorphism

$$R \to \prod_{\mathfrak{m} \in \operatorname{Max} R} \widehat{R}_\mathfrak{m} \to \prod_{\mathfrak{m} \in \operatorname{Max} R} A(\mathfrak{m}).$$

As R is noetherian, an arbitrary product of flat R-modules is flat ([岩永-佐藤, 命題 8-2-7]), so the above map is flat. Moreover, it is also faithfully flat: the induced map $\operatorname{Spec}\left(\prod_{\mathfrak{m} \in \operatorname{Max} R} A(\mathfrak{m})\right) \to \operatorname{Spec} R$ is open (by flatness), and thus its image is generization-closed. Moreover, the image contains all closed points by construction. As a product of perfectoid rings is perfectoid, we have constructed the desired covers.

(Step 2): Reduction to the case where R is a domain. Since R is regular, we can write $R = \prod_{i \in I} R_i$ with R_i regular domain and I finite ([BH, Corollary 2.2.20]). If we obtain a faithfully flat ring homomorphism $R_i \to A_i$ with A_i perfectoid, for each $i \in I$, then the product $R = \prod_{i \in I} R_i \to \prod_{i \in I} A_i =: A$ is a faithfully flat ring homomorphism with A perfectoid (faithfully flatness follows since I is finite).

(Step 3): Finish. Since we have seen the positive characteristic case in Proposition 1.6, it remains the case of mixed characteristic $(0, p)$ (we note that $p \in \operatorname{rad} R$). By [BouAC2, IX, App., Theorem 1, Corolally], there exists a *gonflement* $R \to S$ such that the residue field of S is an algebraically closure of $R/\mathfrak{m}_R$. Then $R \to S$ is faithfully flat ([BouAC2, IX, App., Proposition 2, b)]) and S is also regular ([BouAC2, IX, App., Proposition 2, Corollary)]). Thus we may replace R by S, hence may assume that $R/\mathfrak{m}_R$ is perfect. Then, by Cohen's structure theorem,

$$R = \begin{cases} W(k)[[X_2, \dots, X_d]] & \text{if } R \text{ is unramified,} \\ W(k)[[X_1, \dots, X_d]]/(p - f) & \text{if } R \text{ is ramified.} \end{cases}$$

where $f \in (x_1, \dots, x_d)^2 \setminus (p)$. We take

$$A := \begin{cases} \left(W(k)[p^{1/p^\infty}][[X_2^{1/p^\infty}, \dots, X_d^{1/p^\infty}]]\right)_p^\wedge & \text{if } R \text{ is unramified,} \\ \left(W(k)[[X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]]/(p - f)\right)_p^\wedge & \text{if } R \text{ is ramified.} \end{cases}$$

Indeed, A is perfectoid and $R \to A$ is faithfully flat:

(perfectoid): The unramified case follows as in the case $k = \mathbb{F}_p$ (hence $W(k) = \mathbb{Z}_p$). The ramified case is due to Shimomoto [Shi16, Proposition 4.9].

(faithfully flat): Observe that

$$S := \begin{cases} W(k)[p^{1/p^\infty}][[X_2^{1/p^\infty}, \dots, X_d^{1/p^\infty}]] = \bigcup_{n > 0} W(k)[p^{1/p^n}][[X_2^{1/p^n}, \dots, X_d^{1/p^n}]], \\ W(k)[[X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]]/(p - f) = \bigcup_{n > 0} W(k)[[X_1^{1/p^n}, \dots, X_d^{1/p^n}]]/(p - f). \end{cases}$$

In the ramified case, $R \to A$ is faithfully flat by [Bha18, Proposition 5.12] and the fact that $R/pR \to S/pS \xrightarrow{\sim} A/pA$ is faithfully flat. □

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