

Prismatic Approaches to Regularity and Perfectoid Towers

プリズムを用いた正則性とパーフェクトトイド塔の研究

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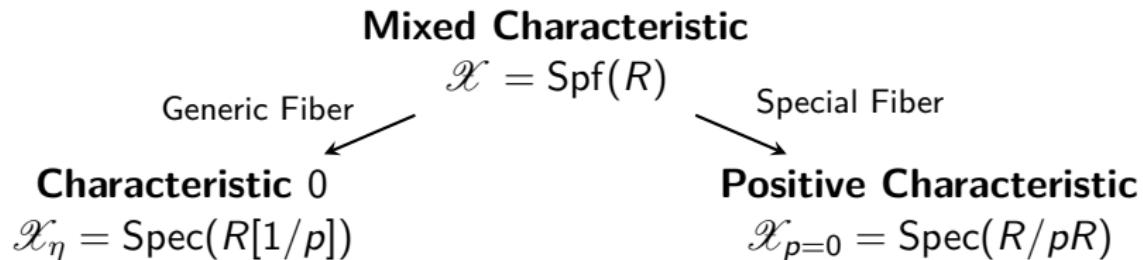
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Introduction: Mixed Characteristic

Background

- ▶ **Characteristic 0 (e.g., complex varieties):** Analytic methods (e.g., resolution of singularities, vanishing theorems).
- ▶ **Positive characteristic ($p > 0$) (e.g., varieties over finite fields):** The Frobenius map $F(x) = x^p$ plays a central role (e.g., Kunz's theorem, F -singularities).
- ▶ **Mixed characteristic (e.g., p -adic formal schemes):** There are neither analytic methods nor a Frobenius map, but it connects both worlds.



Introduction: Recent developments in perfectoid methods

- ▶ **Perfectoid spaces (Scholze, 2012):** Introduced perfectoid rings, which generalize perfect rings in characteristic p , and relate mixed characteristic to characteristic p via tilting.
- ▶ **Direct Summand Conjecture (André, 2018):** Proved using perfectoid techniques; this got modern mixed characteristic commutative algebra started.
- ▶ **Prisms (Bhatt–Scholze, 2022)¹:** Generalize perfectoid rings and introduce prismatic cohomology, a new integral p -adic cohomology theory.
- ▶ **Perfectoid singularities (BIM, BMPSTWW):** Introduced mixed characteristic analogues of “ F -singularities” using perfectoid techniques.
- ▶ **Perfectoid towers (Ishiro–Nakazato–Shimomoto, 2025)²:** Approximate perfectoid rings by sequences of (Noetherian) rings.

¹Annals of Mathematics, 196(3) (2022) 1135–1275.

²Algebra & Number Theory, 19(12) (2025) 2307–2358

Introduction: Summary of Results

Main Theme of this Thesis

Study commutative algebra in mixed characteristic using prismatic methods.

This thesis has two main parts:

1. Prismatic Kunz's Theorem

- ▶ Characterizes regularity of Noetherian local rings via the faithful flatness of the Frobenius lift φ on a prismatic complex $\Delta_{R/A}$.

2. Perfectoid towers from prisms

- ▶ Constructs perfectoid towers from a large class of prisms.
- ▶ Provides systematic examples of perfectoid towers and their tilts beyond the (log-)regular setting.

Prismatic Kunz's Theorem: Kunz and p -adic Kunz

Kunz's Theorem (Kunz, 1969)³

Let R be a Noetherian ring of $\text{char } p > 0$. TFAE:

1. R is regular.
2. $F: R \rightarrow R$ is (faithfully) flat.

p -adic Kunz's Theorem (Bhatt–Iyengar–Ma, 2019)⁴

Let R be a Noetherian ring with $p \in \text{Jac}(R)$. TFAE:

1. R is regular.
2. There exists a perfectoid ring S and a faithfully flat map $R \rightarrow S$.

In mixed characteristic, singularities can be detected via morphisms to perfectoid rings rather than via Frobenius.

³American Journal of Mathematics, 91 (1969) 772–784.

⁴Communications in Algebra, 47(6) (2019) 2367–2383.

Prismatic Kunz's Theorem: Prismatic Approach

- ▶ p -adic Kunz's theorem uses perfectoid rings but does not explicitly involve an endomorphism like Frobenius.
- ▶ **Heuristic (following Bhatt):** A “mixed characteristic Frobenius” for a ring R is the Frobenius lift φ on a prismatic cohomology $\Delta_{R/A}$.

Definition (Prisms: p -torsion-free case)

A pair (A, I) is a (*bounded*) *prism* if:

1. A has a Frobenius lift $\varphi_A: A \rightarrow A$, i.e., a ring endomorphism such that $\varphi_A(a) \equiv a^p \pmod{pA}$ for all $a \in A$.
2. $I \subset A$ is an invertible ideal of A .
3. A is (p, I) -adically complete and A/I has bounded p^∞ -torsion.
4. $p \in I + \varphi_A(I)A$.

- ▶ For example, $(A, I) = (\mathbb{Z}_p[[T_1, \dots, T_n]], (p - f))$ for some $f \in (T_1, \dots, T_n)$ and $\varphi_A(T_i) := T_i^p$.

Prismatic Kunz's Theorem: Prismatic Cohomology

Prismatic Cohomology (Bhatt–Scholze)

Let (A, I) be a bounded prism (e.g., $(\mathbb{Z}_p[[T_1, \dots, T_n]], (p - f))$). For an A/I -algebra $A/I \rightarrow R$, we define the *prismatic cohomology* $\Delta_{R/A}$ of $A/I \rightarrow R$. Naively, this is a commutative algebra object in the derived category $D(A)$ of A -modules:

$$\Delta_{R/A} \in D(A).$$

Moreover, $\Delta_{R/A}$ is equipped with an endomorphism $\varphi: \Delta_{R/A} \rightarrow \Delta_{R/A}$, called the *Frobenius lift*.

- ▶ Prismatic cohomology is defined as the non-additive derived functor of the cohomology of the prismatic site $(R/A)_{\Delta}$ for smooth A/I -algebras R .

Prismatic Kunz's Theorem: Main Theorem 1

Question: Can we characterize the regularity of R in terms of the Frobenius lift φ on $\mathbb{A}_{R/A}$ for some A ?

Theorem (Prismatic Kunz's Theorem (LCI case), I.-Nakazato⁵)

Let (R, \mathfrak{m}, k) be a complete Noetherian local ring of residue characteristic p . Suppose that R is a complete intersection. Then there exist a bounded prism (A, I) and a surjection $A/I \twoheadrightarrow R = A/(I, f_1, \dots, f_r)$ such that TFAE:

1. R is regular.
2. The Frobenius lift $\varphi: \mathbb{A}_{R/A} \rightarrow \mathbb{A}_{R/A}$ is (p -completely) faithfully flat.

- ▶ The proof needs “regular prisms” and the faithful flatness of prismatic complexes.
- ▶ This extends to general complete Noetherian local rings of residue characteristic p by using the derived quotient $A/{}^L(I, f_1, \dots, f_r)$.

⁵Journal of Algebra, 693 (2026) 732–759.

Prismatic Kunz's Theorem: Key Ingredients

Lemma

Let (R, \mathfrak{m}, k) be a complete Noetherian local ring of residue characteristic p . By Cohen structure theorem, we take a surjection

$$A := C(k)[|T_1, \dots, T_e|] \twoheadrightarrow R,$$

where $e := \dim_k \mathfrak{m}/\mathfrak{m}^2$ and $C(k)$ is the Cohen ring of k . Then there exists $f \in (T_1, \dots, T_e)$ such that $(A, (p - f))$ is a bounded prism and $p - f \mapsto 0$.

Lemma

Let (A, I) be a bounded prism and let f_1, \dots, f_r be a regular sequence in A/I . Set $R := A/(I, f_1, \dots, f_r)$. Then $\Delta_{R/A}$ is concentrated in degree 0, and the structure morphism

$$R \rightarrow \overline{\Delta}_{R/A} := \Delta_{R/A} \otimes_A A/I$$

is p -completely faithfully flat.

Prismatic Kunz's Theorem: Proof Sketch

Regularity \Rightarrow faithful flatness

If R is regular, then $A/I \cong R$. Thus, $\Delta_{R/A} \cong A$ and the Frobenius lift φ_A is faithfully flat.

Faithful flatness \Rightarrow regularity

If φ is faithfully flat, then so is the canonical morphism

$$\Delta_{R/A} \rightarrow \operatorname{colim}_{e \geq 0} \varphi_*^e \Delta_{R/A}.$$

Reducing modulo I and using the lemma, we obtain a faithfully flat morphism

$$R \rightarrow \left(\operatorname{colim}_{e \geq 0} \varphi_*^e \Delta_{R/A} \right) /I.$$

The target is a perfectoid ring, so R is regular by p -adic Kunz's theorem.



Perfectoid Towers: Motivation

Observation

- ▶ Perfectoid rings are powerful but non-Noetherian.
- ▶ Many perfectoid rings over Noetherian bases arise as colimits of sequences of Noetherian rings, e.g., $\mathbb{Z}[p^{1/p^\infty}]^{\wedge_p}$.
- ▶ We want to use tilting techniques for Noetherian rings directly.

Perfectoid tower (Ishiro–Nakazato–Shimomoto)

A *perfectoid tower* is a sequence of p -adically complete rings

$R_0 \rightarrow R_1 \rightarrow R_2 \rightarrow \dots$ satisfying certain axioms. For example, R_1 has an ideal I_1 satisfying $I_1^p = pR_1$ and, for each $i \geq 0$, a commutative diagram:

$$\begin{array}{ccc} R_{i+1}/I_1 R_{i+1} & \xrightarrow{a \mapsto a^p} & R_{i+1}/pR_{i+1} \\ & \searrow \exists \cong & \uparrow \\ & & R_i/pR_i. \end{array}$$

Perfectoid Towers: Consequences of Perfectoid Towers

Example

- ▶ $\mathbb{Z}_p \rightarrow \mathbb{Z}_p[p^{1/p}] \rightarrow \mathbb{Z}_p[p^{1/p^2}] \rightarrow \dots$.
- ▶ Any complete (log-)regular local ring R of mixed characteristic, e.g.,

$$R \cong W[|\underline{T}|]/(p - f) \rightarrow W[|p^{1/p}, \underline{T}^{1/p}|]/(p - f) \rightarrow \dots .$$

- ▶ The colimit $R_\infty := (\operatorname{colim}_i R_i)^{\wedge_p}$ is a perfectoid ring.
- ▶ We can take the *tilt* of the tower:

$$R_0^{s.\flat} \rightarrow R_1^{s.\flat} \rightarrow R_2^{s.\flat} \rightarrow \dots ,$$

which is a tower of \mathbb{F}_p -algebras with an element $p^{s.\flat} \in R_0^{s.\flat}$.

- ▶ An isomorphism $R_i^{s.\flat}/p^{s.\flat} \cong R_i/p$ holds for each i .
- ▶ $\{R_i\}_{i \geq 0}$ is a *lim Cohen–Macaulay sequence* if R_0 is a complete Noetherian local domain of mixed characteristic with perfect residue field (Bhatt–Hochster–Ma, Ishiro–Shimomoto).

Perfectoid Towers: Main Theorem 2

Issue: Previous constructions were restricted to (log-)regular local rings, and the proofs were often ad hoc.

We construct perfectoid towers from *prisms* systematically.

Theorem (Perfectoid Towers from Prisms, I.⁶)

Let (A, dA) be a prism such that p, d is a regular sequence on A and A/pA is p -root closed in $(A/pA)[1/d]$. Then the sequence

$$A/dA \xrightarrow{\varphi} A/\varphi(d)A \xrightarrow{\varphi} A/\varphi^2(d)A \rightarrow \dots$$

forms a perfectoid tower over A/dA . Its tilt is given by

$$(A/pA)^{\wedge_d} \xrightarrow{F} (A/pA)^{\wedge_d} \xrightarrow{F} (A/pA)^{\wedge_d} \rightarrow \dots,$$

where $(A/pA)^{\wedge_d}$ is the d -adic completion of A/pA and F is the Frobenius map.

⁶to appear in Nagoya Mathematical Journal.

Perfectoid Towers: Concrete Examples

As a special case, we consider prisms of the form $(A, I) = (R_0[|T|], (p - T))$, where R_0 is a ring equipped with a Frobenius lift φ satisfying mild conditions.

Corollary

Let (R_0, \mathfrak{m}) be a p -torsion-free complete Noetherian local ring of residue characteristic p such that R_0/pR_0 is reduced. If it admits a Frobenius lift φ , then there exists a perfectoid tower over R_0 of the form

$$R_0 \xrightarrow{\varphi \otimes \text{inclusion}} R_0 \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p}] \xrightarrow{\varphi \otimes \text{inclusion}} R_0 \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p^2}] \rightarrow \dots$$

whose tilt is given by

$$R_0/pR_0[|T|] \xrightarrow{F} R_0/pR_0[|T|] \xrightarrow{F} R_0/pR_0[|T|] \rightarrow \dots$$

Our method recovers the (log-)regular computations of perfectoid towers due to Ishiro–Nakazato–Shimomoto.

Perfectoid Towers: Examples

Explicitly, our method produces perfectoid towers over various singular rings.

Example

- ▶ $\mathbb{Z}_p[[X, Y, Z, W]]/(XY, p - ZW)$: a ramified Ici non-domain.
- ▶ $\mathbb{Z}_p[[S, ST, ST^3, ST^4]]$: a non-CM, non-normal domain.
- ▶ $\mathbb{Z}_p[[S^2, S^3]]$: an Ici domain that is not normal.
- ▶ $\mathbb{Z}_p[[X, Y, Z]]/(XY, YZ)$: a reduced, non-CM, and non-domain.
- ▶ $\mathbb{Z}_2[[X, Y, Z]]/(X^3 + Y^4 + Z^5, Y^8 + X^3Y^4 + X^6)$: an unramified Ici domain that is not log-regular.
- ▶ $R(\mathcal{A}, \mathcal{L})$, where \mathcal{A} is a canonical lift of an ordinary abelian variety A over a perfect field of characteristic p and \mathcal{L} is an ample line bundle on \mathcal{A} (the section ring of \mathcal{L}): a non-CM normal domain if $\dim A > 1$.

Summary and Future Outlook

Summary

1. **Prismatic Kunz's Theorem:** Characterized regularity using the Frobenius lift on prismatic cohomology.
2. **Perfectoid towers:** Developed a general construction from prisms, applicable even to singular rings.

Future Directions

- ▶ Study mixed characteristic singularities defined via the Frobenius lift on prismatic cohomology.
- ▶ Apply perfectoid towers to mixed characteristic singularities.
- ▶ More broadly, refine and extend mixed characteristic singularity theory using prismatic and perfectoid methods.