Non-Bayesian Time-Varying Vector Autoregressive Model

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$$\boldsymbol{y}_t = \boldsymbol{\nu} + \Phi_t^{(1)} \boldsymbol{y}_{t-1} + \cdots + \boldsymbol{\Phi}_t^{(p)} \boldsymbol{y}_{t-p} + \boldsymbol{\epsilon}_t, \qquad t = p+1, \ldots, T,$$

where

$$\underbrace{\boldsymbol{y}_t}_{d\times 1} = \begin{pmatrix} y_{1,t} \\ \vdots \\ y_{d,t} \end{pmatrix}, \qquad \underbrace{\boldsymbol{\nu}}_{d\times 1} = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix}, \qquad \underbrace{\boldsymbol{\epsilon}_t}_{d\times 1} = \begin{pmatrix} \epsilon_{1,t} \\ \vdots \\ \epsilon_{d,t} \end{pmatrix}, \qquad \underbrace{\boldsymbol{\Phi}_t^{(l)}}_{d\times d} = \begin{pmatrix} \phi_{11,t}^{(l)} & \cdots & \phi_{1d,t}^{(l)} \\ \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(l)} & \cdots & \phi_{dd,t}^{(l)} \end{pmatrix}, \qquad l = 1, \ldots, p.$$

We can notate as

$$\begin{aligned}
\begin{pmatrix} y_{1,t} \\ \vdots \\ y_{d,t} \end{pmatrix} &= \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} + \begin{pmatrix} \phi_{11,t}^{(1)} & \cdots & \phi_{1d,t}^{(1)} \\ \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ \vdots \\ y_{d,t-1} \end{pmatrix} + \cdots + \begin{pmatrix} \phi_{11,t}^{(p)} & \cdots & \phi_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(p)} & \cdots & \phi_{dd,t}^{(p)} \end{pmatrix} \begin{pmatrix} y_{1,t-p} \\ \vdots \\ y_{d,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \vdots \\ \epsilon_{d,t} \end{pmatrix} \\
&= \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} + \begin{pmatrix} \phi_{11,t}^{(1)} & \cdots & \phi_{1d,t}^{(1)} & \cdots & \phi_{11,t}^{(p)} & \cdots & \phi_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} & \cdots & \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ \vdots \\ y_{d,t-p} \\ \vdots \\ y_{1,t-1} \\ \vdots \\ y_{d,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \vdots \\ \epsilon_{d,t} \end{pmatrix}.
\end{aligned}$$

Thus, we can write

$$\boldsymbol{y}_t = \boldsymbol{\nu} + \boldsymbol{\Phi}_t \boldsymbol{Z}_t + \boldsymbol{\epsilon}_t,$$

where

$$\underbrace{\boldsymbol{\Phi}_t}_{d\times dp} = \begin{pmatrix} \phi_{11,t}^{(1)} & \cdots & \phi_{1d,t}^{(1)} & \cdots & \phi_{11,t}^{(p)} & \cdots & \phi_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} & \cdots & \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} \end{pmatrix}, \qquad \underbrace{\boldsymbol{Z}_t}_{dp\times 1} = \begin{pmatrix} \boldsymbol{y}_{t-1} \\ \vdots \\ \boldsymbol{y}_{t-p} \end{pmatrix} = \begin{pmatrix} y_{1,t-1} \\ \vdots \\ y_{d,t-1} \\ \vdots \\ y_{1,t-p} \\ \vdots \\ y_{d,t-p} \end{pmatrix}.$$

we can calculate

$$\begin{pmatrix} y_{1,t} \\ \vdots \\ y_{d,t} \end{pmatrix} = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} + \begin{pmatrix} \phi_{11,t}^{(1)} & \cdots & \phi_{1d,t}^{(1)} & \cdots & \phi_{11,t}^{(p)} & \cdots & \phi_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} & \cdots & \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ \vdots \\ y_{d,t-p} \\ \vdots \\ y_{1,t-1} \\ \vdots \\ y_{d,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \vdots \\ \epsilon_{d,t} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} + \begin{pmatrix} y_{1,t-1} & & & \cdots & y_{d,t-1} & & \\ & \ddots & & \cdots & & \ddots & \\ & & y_{1,t-p} & \cdots & & & y_{d,t-p} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \\ \phi_{11,t}^{(1)} \\ \vdots \\ \phi_{d1,t}^{(1)} \\ \vdots \\ \phi_{d1,t}^{(p)} \\ \vdots \\ \phi_{dd,t}^{(p)} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \vdots \\ \epsilon_{d,t} \end{pmatrix}$$

Therefore we can notate

$$\boldsymbol{y}_t = \begin{pmatrix} \boldsymbol{I}_d & \boldsymbol{Z}_t' \otimes \boldsymbol{I}_d \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu} \\ \mathrm{vec}(\boldsymbol{\Phi}_t) \end{pmatrix}$$

We can write

$$\begin{pmatrix} \boldsymbol{y}_{p+1} \\ \vdots \\ \boldsymbol{y}_T \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_d & \boldsymbol{Z}_{p+1}' \otimes \boldsymbol{I}_d & & \\ \vdots & & \ddots & \\ \boldsymbol{I}_d & & & \boldsymbol{Z}_T' \otimes \boldsymbol{I}_d \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu} \\ \mathrm{vec}(\boldsymbol{\Phi}_{p+1}) \\ \vdots \\ \mathrm{vec}(\boldsymbol{\Phi}_T) \end{pmatrix} + \begin{pmatrix} \boldsymbol{\epsilon}_{p+1} \\ \vdots \\ \boldsymbol{\epsilon}_T \end{pmatrix}$$

Our model can be rewritten as

$$\boldsymbol{Y} = \boldsymbol{Z}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where

$$\underbrace{\boldsymbol{Y}}_{d(T-p)\times 1} = \begin{pmatrix} \boldsymbol{y}_{p+1} \\ \vdots \\ \boldsymbol{y}_T \end{pmatrix}, \qquad \underbrace{\boldsymbol{Z}}_{d(T-p)\times(d+d^2p(T-p))} = \begin{pmatrix} \boldsymbol{I}_d & \boldsymbol{Z}_{p+1}' \otimes \boldsymbol{I}_d & & \\ \vdots & & \ddots & \\ \boldsymbol{I}_d & & & \boldsymbol{Z}_T' \otimes \boldsymbol{I}_d \end{pmatrix},$$

and

$$\underbrace{\boldsymbol{\beta}}_{(d+d^2p(T-p))\times 1} = \begin{pmatrix} \boldsymbol{\nu} \\ \mathrm{vec}(\boldsymbol{\Phi}_{p+1}) \\ \vdots \\ \mathrm{vec}(\boldsymbol{\Phi}_T) \end{pmatrix}, \qquad \underbrace{\boldsymbol{\epsilon}}_{d(T-p)\times 1} = \begin{pmatrix} \boldsymbol{\epsilon}_{p+1} \\ \vdots \\ \boldsymbol{\epsilon}_T \end{pmatrix}.$$

Assume

$$\phi_{ij,t}^{(l)} = \phi_{ij,t-1}^{(l)} + h_{ij,t}^{(l)}, \qquad i = 1,\ldots,d, \qquad j = 1,\ldots,d, \qquad l = 1,\ldots,p, \qquad t = p+1,\ldots,T.$$

Note that

$$\boldsymbol{\Phi}_t = \boldsymbol{\Phi}_{t-1} + \boldsymbol{H}_t$$

where

$$\underbrace{\boldsymbol{H}_t}_{d\times dp} = \begin{pmatrix} h_{11,t}^{(1)} & \cdots & h_{1d,t}^{(1)} & \cdots & h_{11,t}^{(p)} & \cdots & h_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ h_{d1,t}^{(1)} & \cdots & h_{dd,t}^{(1)} & \cdots & h_{d1,t}^{(p)} & \cdots & h_{dd,t}^{(p)} \end{pmatrix}.$$

Define

$$\underbrace{\boldsymbol{\beta}_t}_{d^2p\times 1} = \mathrm{vec}(\boldsymbol{\Phi}_t), \qquad \underbrace{\boldsymbol{\eta}_t}_{d^2p\times 1} = \mathrm{vec}(\boldsymbol{H}_t),$$

then

$$\boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t.$$

We can calculate

$$\begin{aligned}
\begin{pmatrix} \boldsymbol{\beta}_{p+1} \\ \vdots \\ \boldsymbol{\beta}_T \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\beta}_p \\ \vdots \\ \boldsymbol{\beta}_{T-1} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\eta}_{p+1} \\ \vdots \\ \boldsymbol{\eta}_T \end{pmatrix} \\
\begin{pmatrix} \mathbf{0}_{d^2p} \\ \vdots \\ \mathbf{0}_{d^2p} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\beta}_p - \boldsymbol{\beta}_{p+1} \\ \vdots \\ \boldsymbol{\beta}_{T-1} - \boldsymbol{\beta}_T \end{pmatrix} + \begin{pmatrix} \boldsymbol{\eta}_{p+1} \\ \vdots \\ \boldsymbol{\eta}_T \end{pmatrix} \\
\begin{pmatrix} -\boldsymbol{\beta}_p \\ \mathbf{0}_{d^2p} \\ \vdots \\ \mathbf{0}_{d^2p} \end{pmatrix} &= \begin{pmatrix} -\boldsymbol{\beta}_{p+1} \\ \boldsymbol{\beta}_{p+1} - \boldsymbol{\beta}_{p+2} \\ \vdots \\ \boldsymbol{\beta}_{T-1} - \beta_T \end{pmatrix} + \begin{pmatrix} \boldsymbol{\eta}_{p+1} \\ \vdots \\ \boldsymbol{\eta}_T \end{pmatrix} \\
&= \begin{pmatrix} -\boldsymbol{I}_{d^2p} & & & \\ \boldsymbol{I}_{d^2p} & -\boldsymbol{I}_{d^2p} & & \\ & \ddots & \ddots & \\ & & \boldsymbol{I}_{d^2p} & -\boldsymbol{I}_{d^2p} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_{p+1} \\ \boldsymbol{\beta}_{p+2} \\ \vdots \\ \boldsymbol{\beta}_T \end{pmatrix} + \begin{pmatrix} \boldsymbol{\eta}_{p+1} \\ \vdots \\ \boldsymbol{\eta}_T \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0}_{d^2p} & \cdots & \mathbf{0}_{d^2p} & -\boldsymbol{I}_{d^2p} & & & \\ \mathbf{0}_{d^2p} & \cdots & \mathbf{0}_{d^2p} & \boldsymbol{I}_{d^2p} & -\boldsymbol{I}_{d^2p} & & \\ \vdots & \cdots & \vdots & & \ddots & \ddots & \\ \mathbf{0}_{d^2p} & \cdots & \mathbf{0}_{d^2p} & & & \boldsymbol{I}_{d^2p} & -\boldsymbol{I}_{d^2p} \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu} \\ \boldsymbol{\beta}_{p+1} \\ \boldsymbol{\beta}_{p+2} \\ \vdots \\ \boldsymbol{\beta}_T \end{pmatrix} + \begin{pmatrix} \boldsymbol{\eta}_{p+1} \\ \vdots \\ \boldsymbol{\eta}_T \end{pmatrix}.
\end{aligned}$$

Thus, we can write

$$\boldsymbol{\Gamma} = \boldsymbol{W}\boldsymbol{\beta} + \boldsymbol{\eta}$$

where

$$\underbrace{\boldsymbol{\Gamma}}_{d^2p(T-p))\times 1} = \begin{pmatrix} -\mathrm{vec}(\boldsymbol{\Phi}_p) \\ \mathbf{0}_{d^2p} \\ \vdots \\ \mathbf{0}_{d^2p} \end{pmatrix}, \qquad \underbrace{\boldsymbol{\eta}}_{d^2p(T-p)\times 1} = \begin{pmatrix} \boldsymbol{\eta}_{p+1} \\ \vdots \\ \boldsymbol{\eta}_T \end{pmatrix},$$

and

$$\underbrace{\boldsymbol{W}}_{d^2pT\times(d+d^2p(T-p))} = \begin{pmatrix} \mathbf{0}_{d^2p} & \cdots & \mathbf{0}_{d^2p} & -\boldsymbol{I}_{d^2p} & & & \\ \mathbf{0}_{d^2p} & \cdots & \mathbf{0}_{d^2p} & \boldsymbol{I}_{d^2p} & -\boldsymbol{I}_{d^2p} & & \\ \vdots & \cdots & \vdots & & \ddots & \ddots & \\ \mathbf{0}_{d^2p} & \cdots & \mathbf{0}_{d^2p} & & & \boldsymbol{I}_{d^2p} & -\boldsymbol{I}_{d^2p} \end{pmatrix}.$$

Therefore, with all things considered, our model can be notated as

$$\boldsymbol{\psi} = \boldsymbol{\zeta}\boldsymbol{\beta} + \boldsymbol{\xi},$$

where

$$\underbrace{\boldsymbol{\psi}}_{(d(T-p)+d^2p(T-p))\times 1} = \begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{\Gamma} \end{pmatrix}, \qquad \underbrace{\boldsymbol{\zeta}}_{(d(T-p)+d^2p)(T-p))\times(d+d^2p(T-p))} = \begin{pmatrix} \boldsymbol{Z} \\ \boldsymbol{W} \end{pmatrix}, \qquad \underbrace{\boldsymbol{\xi}}_{(d(T-p)+d^2p(T-p))\times 1} = \begin{pmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \end{pmatrix}.$$