

Non-Bayesian Time-Varying Vector Autoregressive Model

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1 Model

$$\mathbf{y}_t = \boldsymbol{\nu} + \boldsymbol{\Phi}_t^{(1)} \mathbf{y}_{t-1} + \cdots + \boldsymbol{\Phi}_t^{(p)} \mathbf{y}_{t-p} + \boldsymbol{\epsilon}_t, \quad t = p+1, \dots, T,$$

where

$$\underbrace{\mathbf{y}_t}_{d \times 1} = \begin{pmatrix} y_{1,t} \\ \vdots \\ y_{d,t} \end{pmatrix}, \quad \underbrace{\boldsymbol{\nu}}_{d \times 1} = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix}, \quad \underbrace{\boldsymbol{\epsilon}_t}_{d \times 1} = \begin{pmatrix} \epsilon_{1,t} \\ \vdots \\ \epsilon_{d,t} \end{pmatrix}, \quad \underbrace{\boldsymbol{\Phi}_t^{(l)}}_{d \times d} = \begin{pmatrix} \phi_{11,t}^{(l)} & \cdots & \phi_{1d,t}^{(l)} \\ \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(l)} & \cdots & \phi_{dd,t}^{(l)} \end{pmatrix}, \quad l = 1, \dots, p.$$

We can notate as

$$\begin{aligned} \begin{pmatrix} y_{1,t} \\ \vdots \\ y_{d,t} \end{pmatrix} &= \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} + \begin{pmatrix} \phi_{11,t}^{(1)} & \cdots & \phi_{1d,t}^{(1)} \\ \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ \vdots \\ y_{d,t-1} \end{pmatrix} + \cdots + \begin{pmatrix} \phi_{11,t}^{(p)} & \cdots & \phi_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(p)} & \cdots & \phi_{dd,t}^{(p)} \end{pmatrix} \begin{pmatrix} y_{1,t-p} \\ \vdots \\ y_{d,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \vdots \\ \epsilon_{d,t} \end{pmatrix} \\ &= \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} + \begin{pmatrix} \phi_{11,t}^{(1)} & \cdots & \phi_{1d,t}^{(1)} & \cdots & \phi_{11,t}^{(p)} & \cdots & \phi_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} & \cdots & \phi_{d1,t}^{(p)} & \cdots & \phi_{dd,t}^{(p)} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ \vdots \\ y_{d,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \vdots \\ \epsilon_{d,t} \end{pmatrix}. \end{aligned}$$

Thus, we can write

$$\mathbf{y}_t = \boldsymbol{\nu} + \boldsymbol{\Phi}_t \mathbf{Z}_t + \boldsymbol{\epsilon}_t,$$

where

$$\underbrace{\boldsymbol{\Phi}_t}_{d \times dp} = \begin{pmatrix} \phi_{11,t}^{(1)} & \cdots & \phi_{1d,t}^{(1)} & \cdots & \phi_{11,t}^{(p)} & \cdots & \phi_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} & \cdots & \phi_{d1,t}^{(p)} & \cdots & \phi_{dd,t}^{(p)} \end{pmatrix}, \quad \underbrace{\mathbf{Z}_t}_{dp \times 1} = \begin{pmatrix} \mathbf{y}_{t-1} \\ \vdots \\ \mathbf{y}_{t-p} \end{pmatrix} = \begin{pmatrix} y_{1,t-1} \\ \vdots \\ y_{d,t-1} \\ \vdots \\ y_{1,t-p} \\ \vdots \\ y_{d,t-p} \end{pmatrix}.$$

we can calculate

$$\begin{aligned}
\begin{pmatrix} y_{1,t} \\ \vdots \\ y_{d,t} \end{pmatrix} &= \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} + \begin{pmatrix} \phi_{11,t}^{(1)} & \cdots & \phi_{1d,t}^{(1)} & \cdots & \phi_{11,t}^{(p)} & \cdots & \phi_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} & \cdots & \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ \vdots \\ y_{d,t-p} \\ \vdots \\ y_{1,t-1} \\ \vdots \\ y_{d,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \vdots \\ \epsilon_{d,t} \end{pmatrix} \\
&= \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} + \begin{pmatrix} y_{1,t-1} & & \cdots & y_{d,t-1} \\ & \ddots & & \\ & & y_{1,t-p} & \cdots & y_{d,t-p} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \\ \phi_{11,t}^{(1)} \\ \vdots \\ \phi_{d1,t}^{(1)} \\ \vdots \\ \phi_{d1,t}^{(p)} \\ \vdots \\ \phi_{dd,t}^{(p)} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \vdots \\ \epsilon_{d,t} \end{pmatrix}
\end{aligned}$$

Therefore we can notate

$$\mathbf{y}_t = (\mathbf{I}_d \quad \mathbf{Z}'_t \otimes \mathbf{I}_d) \begin{pmatrix} \boldsymbol{\nu} \\ \text{vec}(\boldsymbol{\Phi}_t) \end{pmatrix}$$

We can write

$$\begin{pmatrix} \mathbf{y}_{p+1} \\ \vdots \\ \mathbf{y}_T \end{pmatrix} = \begin{pmatrix} \mathbf{I}_d & \mathbf{Z}'_{p+1} \otimes \mathbf{I}_d & & \\ \vdots & & \ddots & \\ \mathbf{I}_d & & & \mathbf{Z}'_T \otimes \mathbf{I}_d \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu} \\ \text{vec}(\boldsymbol{\Phi}_{p+1}) \\ \vdots \\ \text{vec}(\boldsymbol{\Phi}_T) \end{pmatrix} + \begin{pmatrix} \boldsymbol{\epsilon}_{p+1} \\ \vdots \\ \boldsymbol{\epsilon}_T \end{pmatrix}$$

Our model can be rewritten as

$$\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where

$$\underbrace{\mathbf{Y}}_{d(T-p) \times 1} = \begin{pmatrix} \mathbf{y}_{p+1} \\ \vdots \\ \mathbf{y}_T \end{pmatrix}, \quad \underbrace{\mathbf{Z}}_{d(T-p) \times (d+d^2p(T-p))} = \begin{pmatrix} \mathbf{I}_d & \mathbf{Z}'_{p+1} \otimes \mathbf{I}_d & & \\ \vdots & & \ddots & \\ \mathbf{I}_d & & & \mathbf{Z}'_T \otimes \mathbf{I}_d \end{pmatrix},$$

and

$$\underbrace{\boldsymbol{\beta}}_{(d+d^2p(T-p)) \times 1} = \begin{pmatrix} \boldsymbol{\nu} \\ \text{vec}(\boldsymbol{\Phi}_{p+1}) \\ \vdots \\ \text{vec}(\boldsymbol{\Phi}_T) \end{pmatrix}, \quad \underbrace{\boldsymbol{\epsilon}}_{d(T-p) \times 1} = \begin{pmatrix} \boldsymbol{\epsilon}_{p+1} \\ \vdots \\ \boldsymbol{\epsilon}_T \end{pmatrix}.$$

Assume

$$\phi_{ij,t}^{(l)} = \phi_{ij,t-1}^{(l)} + h_{ij,t}^{(l)}, \quad i = 1, \dots, d, \quad j = 1, \dots, d, \quad l = 1, \dots, p, \quad t = p+1, \dots, T.$$

Note that

$$\Phi_t = \Phi_{t-1} + H_t$$

where

$$\underbrace{H_t}_{d \times dp} = \begin{pmatrix} h_{11,t}^{(1)} & \cdots & h_{1d,t}^{(1)} & \cdots & h_{11,t}^{(p)} & \cdots & h_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ h_{d1,t}^{(1)} & \cdots & h_{dd,t}^{(1)} & \cdots & h_{d1,t}^{(p)} & \cdots & h_{dd,t}^{(p)} \end{pmatrix}.$$

Define

$$\underbrace{\beta_t}_{d^2 p \times 1} = \text{vec}(\Phi_t), \quad \underbrace{\eta_t}_{d^2 p \times 1} = \text{vec}(H_t),$$

then

$$\beta_t = \beta_{t-1} + \eta_t.$$

We can calculate

$$\begin{aligned} \begin{pmatrix} \beta_{p+1} \\ \vdots \\ \beta_T \end{pmatrix} &= \begin{pmatrix} \beta_p \\ \vdots \\ \beta_{T-1} \end{pmatrix} + \begin{pmatrix} \eta_{p+1} \\ \vdots \\ \eta_T \end{pmatrix} \\ \begin{pmatrix} \mathbf{0}_{d^2 p} \\ \vdots \\ \mathbf{0}_{d^2 p} \end{pmatrix} &= \begin{pmatrix} \beta_p - \beta_{p+1} \\ \vdots \\ \beta_{T-1} - \beta_T \end{pmatrix} + \begin{pmatrix} \eta_{p+1} \\ \vdots \\ \eta_T \end{pmatrix} \\ \begin{pmatrix} -\beta_p \\ \mathbf{0}_{d^2 p} \\ \vdots \\ \mathbf{0}_{d^2 p} \end{pmatrix} &= \begin{pmatrix} -\beta_{p+1} \\ \beta_{p+1} - \beta_{p+2} \\ \vdots \\ \beta_{T-1} - \beta_T \end{pmatrix} + \begin{pmatrix} \eta_{p+1} \\ \vdots \\ \eta_T \end{pmatrix} \\ &= \begin{pmatrix} -\mathbf{I}_{d^2 p} & & & \\ \mathbf{I}_{d^2 p} & -\mathbf{I}_{d^2 p} & & \\ & \ddots & \ddots & \\ & & \mathbf{I}_{d^2 p} & -\mathbf{I}_{d^2 p} \end{pmatrix} \begin{pmatrix} \beta_{p+1} \\ \beta_{p+2} \\ \vdots \\ \beta_T \end{pmatrix} + \begin{pmatrix} \eta_{p+1} \\ \vdots \\ \eta_T \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0}_{d^2 p} & \cdots & \mathbf{0}_{d^2 p} & -\mathbf{I}_{d^2 p} \\ \mathbf{0}_{d^2 p} & \cdots & \mathbf{0}_{d^2 p} & \mathbf{I}_{d^2 p} & -\mathbf{I}_{d^2 p} \\ \vdots & \cdots & \vdots & & \ddots & \ddots \\ \mathbf{0}_{d^2 p} & \cdots & \mathbf{0}_{d^2 p} & & \mathbf{I}_{d^2 p} & -\mathbf{I}_{d^2 p} \end{pmatrix} \begin{pmatrix} \nu \\ \beta_{p+1} \\ \beta_{p+2} \\ \vdots \\ \beta_T \end{pmatrix} + \begin{pmatrix} \eta_{p+1} \\ \vdots \\ \eta_T \end{pmatrix}. \end{aligned}$$

Thus, we can write

$$\Gamma = W\beta + \eta$$

where

$$\underbrace{\Gamma}_{d^2 p(T-p) \times 1} = \begin{pmatrix} -\text{vec}(\Phi_p) \\ \mathbf{0}_{d^2 p} \\ \vdots \\ \mathbf{0}_{d^2 p} \end{pmatrix}, \quad \underbrace{\eta}_{d^2 p(T-p) \times 1} = \begin{pmatrix} \eta_{p+1} \\ \vdots \\ \eta_T \end{pmatrix},$$

and

$$\underbrace{\mathbf{W}}_{d^2 p(T-p) \times (d+d^2 p(T-p))} = \begin{pmatrix} \mathbf{0}_{d^2 p} & \cdots & \mathbf{0}_{d^2 p} & -\mathbf{I}_{d^2 p} & & \\ \mathbf{0}_{d^2 p} & \cdots & \mathbf{0}_{d^2 p} & \mathbf{I}_{d^2 p} & -\mathbf{I}_{d^2 p} & \\ \vdots & \cdots & \vdots & & \ddots & \ddots \\ \mathbf{0}_{d^2 p} & \cdots & \mathbf{0}_{d^2 p} & & & \mathbf{I}_{d^2 p} & -\mathbf{I}_{d^2 p} \end{pmatrix}.$$

Therefore, with all things considered, our model can be notated as

$$\boldsymbol{\psi} = \boldsymbol{\zeta} \boldsymbol{\beta} + \boldsymbol{\xi},$$

where

$$\underbrace{\boldsymbol{\psi}}_{(d(T-p)+d^2 p(T-p)) \times 1} = \begin{pmatrix} \mathbf{Y} \\ \boldsymbol{\Gamma} \end{pmatrix}, \quad \underbrace{\boldsymbol{\zeta}}_{(d(T-p)+d^2 p(T-p)) \times (d+d^2 p(T-p))} = \begin{pmatrix} \mathbf{Z} \\ \mathbf{W} \end{pmatrix}, \quad \underbrace{\boldsymbol{\xi}}_{(d(T-p)+d^2 p(T-p)) \times 1} = \begin{pmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\eta} \end{pmatrix}.$$

2 Prediction

$$\hat{\mathbf{y}}_t = E[\mathbf{y}_t | \mathcal{F}_{t-1}] = E[\mathbf{y}_t | \mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-p}]$$

$$\hat{\mathbf{y}}_t = \hat{\boldsymbol{\nu}} + \hat{\boldsymbol{\Phi}}_t^{(1)} \mathbf{y}_{t-1} + \dots + \hat{\boldsymbol{\Phi}}_t^{(p)} \mathbf{y}_{t-p},$$

where

$$\underbrace{\hat{\mathbf{y}}_t}_{d \times 1} = \begin{pmatrix} \hat{y}_{1,t} \\ \vdots \\ \hat{y}_{d,t} \end{pmatrix}, \quad \underbrace{\hat{\boldsymbol{\nu}}}_{d \times 1} = \begin{pmatrix} \hat{\nu}_1 \\ \vdots \\ \hat{\nu}_d \end{pmatrix}, \quad \underbrace{\hat{\boldsymbol{\Phi}}_t^{(l)}}_{d \times d} = \begin{pmatrix} \hat{\phi}_{11,t}^{(l)} & \cdots & \hat{\phi}_{1d,t}^{(l)} \\ \vdots & \ddots & \vdots \\ \hat{\phi}_{d1,t}^{(l)} & \cdots & \hat{\phi}_{dd,t}^{(l)} \end{pmatrix}, \quad l = 1, \dots, p.$$

We can notate as

$$\begin{aligned} \begin{pmatrix} \hat{y}_{1,t} \\ \vdots \\ \hat{y}_{d,t} \end{pmatrix} &= \begin{pmatrix} \hat{\nu}_1 \\ \vdots \\ \hat{\nu}_d \end{pmatrix} + \begin{pmatrix} \hat{\phi}_{11,t}^{(1)} & \cdots & \hat{\phi}_{1d,t}^{(1)} \\ \vdots & \ddots & \vdots \\ \hat{\phi}_{d1,t}^{(1)} & \cdots & \hat{\phi}_{dd,t}^{(1)} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ \vdots \\ y_{d,t-1} \end{pmatrix} + \dots + \begin{pmatrix} \hat{\phi}_{11,t}^{(p)} & \cdots & \hat{\phi}_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots \\ \hat{\phi}_{d1,t}^{(p)} & \cdots & \hat{\phi}_{dd,t}^{(p)} \end{pmatrix} \begin{pmatrix} y_{1,t-p} \\ \vdots \\ y_{d,t-p} \end{pmatrix} \\ &= \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix} + \begin{pmatrix} \phi_{11,t}^{(1)} & \cdots & \phi_{1d,t}^{(1)} & \cdots & \phi_{11,t}^{(p)} & \cdots & \phi_{1d,t}^{(p)} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} & \cdots & \phi_{d1,t}^{(1)} & \cdots & \phi_{dd,t}^{(1)} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ \vdots \\ y_{d,t-p} \end{pmatrix}. \end{aligned}$$

Thus we can write

$$\hat{\mathbf{y}}_t = \hat{\boldsymbol{\nu}} + \hat{\boldsymbol{\Phi}}_t \mathbf{Z}_t.$$