The Standard Incomplete Markets (SIM) Model

Matthew Rognlie

NBER Heterogeneous-Agent Macro Workshop, 2025

Individual household problem

The "standard incomplete markets" model (steady state)

- Individual household i optimizes

$$\max_{\{a_{it}, c_{it}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

subject to period-by-period budget constraint and borrowing constraint

$$a_{it} + c_{it} = (1+r)a_{i,t-1} + Ze_{it} \qquad\qquad a_{it} \geq \underline{a}$$

- Exogenous income state e_{it} follows Markov chain, which we'll usually normalize to 1, Z scales aggregate after-tax income

- Initial assets $a_{i,-1}$ taken as given, standard assumptions on u (CRRA)

Can convert sequential form to Bellman equation

$$\max_{\{a_{it}, c_{it}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

$$a_{it} + c_{it} = (1+r)a_{i,t-1} + Ze_{it}$$

$$a_{it} \geq \underline{a}$$

$$V(e,a) = \max_{c,a'} u(c) + \beta \mathbb{E}[V(e',a') \,|\, e]$$

$$s.t.\ a' + c = (1+r)a + Ze$$

$$a' \geq \underline{a}$$

Solved by **policies** $a'(e,a)$ and $c(e,a)$ in two state variables, **exogenous** income state e and **endogenous** asset state a

Solving the Bellman equation

- Policies $a'(e,a)$ and $c(e,a)$ satisfy standard **first-order condition**

$$u'(c) \geq \beta \mathbb{E}[V_a(e',a') \mid e]$$

with equality unless borrowing constraint binds, and **envelope condition**

$$V_a(e,a) = (1+r)u'(c)$$

- Combined, same as sequential **Euler equation** $u'(c_{it}) \geq \beta(1+r)\mathbb{E}_t[u'(c_{i,t+1})]$
- Can use first-order and envelope conditions to iterate backward on V_a and policies
    - Best way: interpolation and "endogenous gridpoints" (Carroll 2006)
    - Iterating until convergence gives V_a and steady-state policies on a grid

See computation supplement (on GitHub) for more!

```python
def backward_iteration(Va, Pi, a_grid, y, r, beta, eis):
    # step 1: discounting and expectations
    Wa = (beta * Pi) @ Va

    # step 2: solving for asset policy using the first-order condition
    c_endog = Wa**(-eis)
    coh = y[:, np.newaxis] + (1+r)*a_grid

    a = np.empty_like(coh)
    for e in range(len(y)):
        a[e, :] = np.interp(coh[e, :], c_endog[e, :] + a_grid, a_grid)

    # step 3: enforcing the borrowing constraint and backing out consumption
    a = np.maximum(a, a_grid[0])
    c = coh - a

    # step 4: using the envelope condition to recover the derivative of the value function
    Va = (1+r) * c**(-1/eis)

    return Va, a, c
```

Basic backward iteration takes just 9 lines of standard Python code

Consolidated in **sim_steady_state.py**, GitHub links to supplementary notebook, video lectures, and also 2x sped-up version

Distribution of households

Solved household problem, now aggregate

- We've solved problem facing individual household
- Now aggregate into economies with a **continuum** of such households
  - Soon will put in **general equilibrium** …
  - But for now interested in properties of **"partial equilibrium"** model, i.e. taking return r as given

- This is a **heterogeneous-agent economy**
  - Has a **distribution** of households across the two states, e and a

What is distribution of households?

- In principle, it's a **measure** μ
- If finitely many e, then can define $\mu(e, \mathbb{A})$ separately for each e, as measure on subsets $\mathbb{A}$ of the asset space
- Law of motion

$$\mu_{t+1}(e', \mathbb{A}) = \sum_e \mu_t \left(e, (a')^{-1}(e, \mathbb{A})\right) \cdot P(e, e')$$

where $P(e, e')$ is transition probability, $(a')^{-1}(e, \cdot\,)$ is inverse of policy $a'(e, \cdot\,)$

- Measure of $\mathbb{A}$ today is sum of measures yesterday that send you there today

Why measure?

- You might want some nice density function …
  - But if $\beta(1+r) < 1$, there will be a **positive mass at borrowing constraint**
  - (If $\beta(1+r) \geq 1$, can show everyone's assets will diverge to ∞, so we generally don't consider that case…)
  - If finitely many e, this implies **discrete distribution** with only mass points!
    - Countably many histories of e since last time hitting constraint.

- So for generality, we assume an arbitrary measure over assets
  - Will revisit much later when we build a "smoother" model

Calculating distribution in practice

```
def get_lottery(a, a_grid):
    # step 1: find the i such that a' lies between gridpoints a_i and a_(i+1)
    a_i = np.searchsorted(a_grid, a) - 1

    # step 2: obtain lottery probabilities pi
    a_pi = (a_grid[a_i+1] - a)/(a_grid[a_i+1] - a_grid[a_i])

    return a_i, a_pi
```

```
@numba.njit
def forward_policy(D, a_i, a_pi):
    Dend = np.zeros_like(D)
    for e in range(a_i.shape[0]):
        for a in range(a_i.shape[1]):
            # send pi(e,a) of the mass to gridpoint i(e,a)
            Dend[e, a_i[e,a]] += a_pi[e,a]*D[e,a]

            # send 1-pi(e,a) of the mass to gridpoint i(e,a)+1
            Dend[e, a_i[e,a]+1] += (1-a_pi[e,a])*D[e,a]

    return Dend
```

Approximate distribution by point masses on finite grid; when asset policy $a'(e,a)$ lies between two gridpoints, convert it to **"lottery"** between gridpoints with same expectation

Also in **sim_steady_state.py**, notebook, and videos.

Steady state of aggregate model

What is a steady state of model?

- Consists of:
  - **policy functions** $a'(e,a)$ and $c'(e,a)$ that solve Bellman equation
  - **distribution** $\mu(e,\mathbb{A})$ that satisfies steady-state law of motion

$$\mu(e',\mathbb{A}) = \sum_s \mu\left(e,(a')^{-1}(e,\mathbb{A})\right)\cdot P(e,e')$$

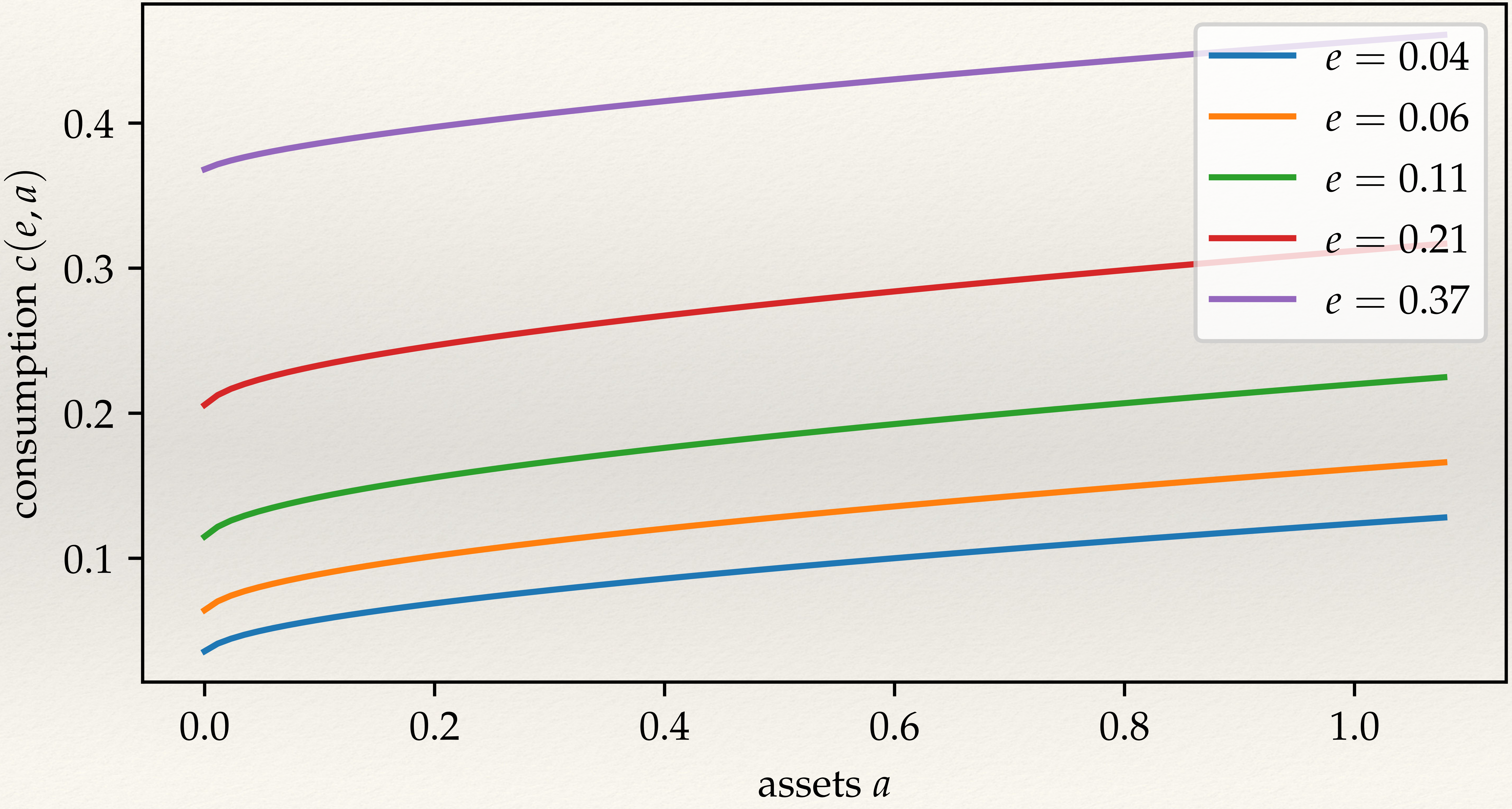
- Can show such a "stationary distribution" exists and is unique if $\beta(1+r)<1$
- **Aggregate assets** and **consumption**:

$$A = \int a d\mu = \int a'(e,a)d\mu \qquad C = \int c d\mu$$

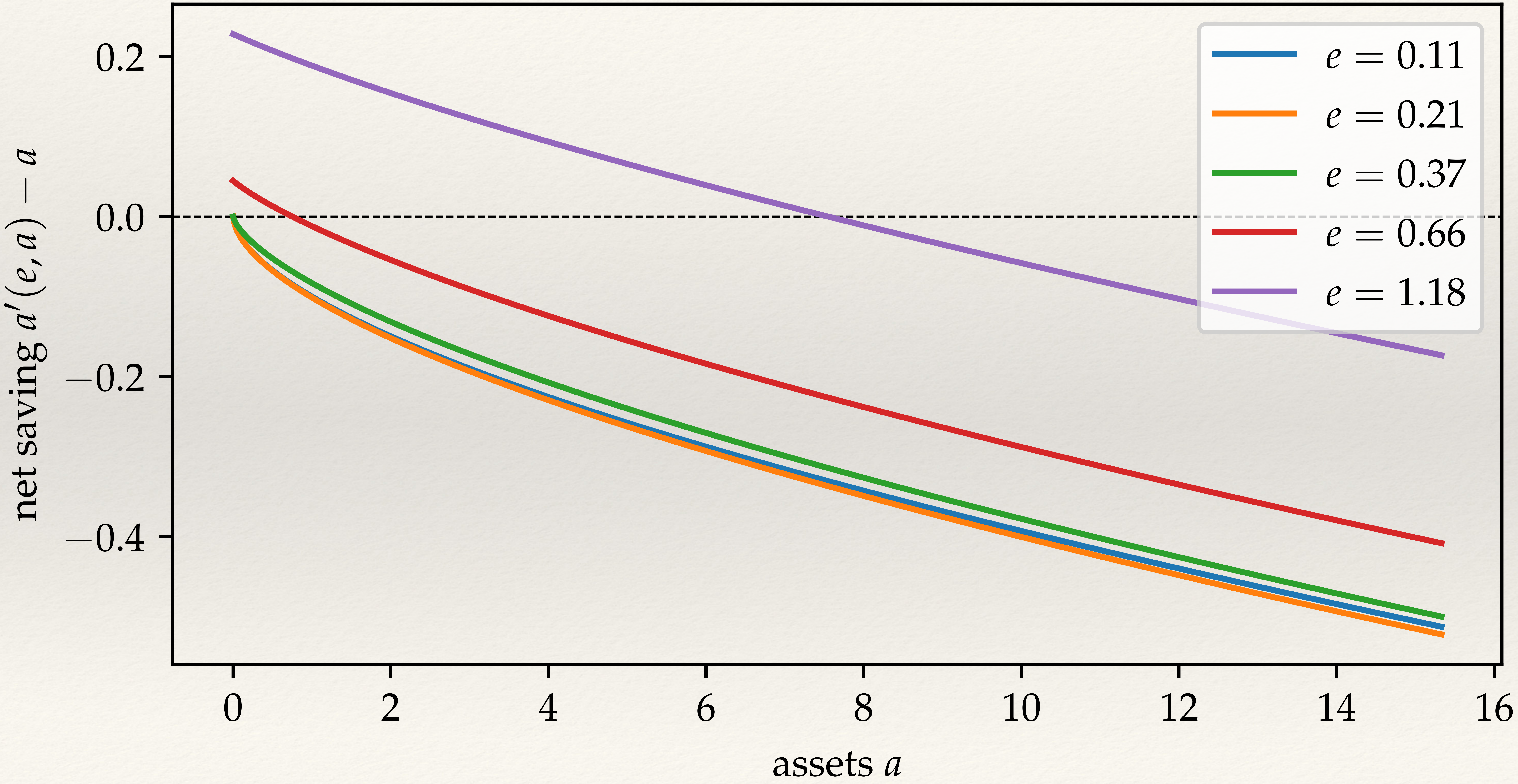
Nice features of model

- Captures key features of **consumption-saving problem** with risk
  - income smoothing, precautionary savings, etc.
- Endogenously generates **wealth distribution** (of assets a)
- Consistent with high **marginal propensities to consume (MPCs)** out of cash on hand, here $mpc(e, a) \equiv (\partial c(e, a)/\partial a)/(1 + r)$
- Unlike representative-agent model, steady-state **asset demand not infinitely elastic** in r, so r can be endogenous
- Easy to extend: other shocks, preference heterogeneity, endogenous labor, life-cycle structure, other assets…

Consumption functions: increasing, concave

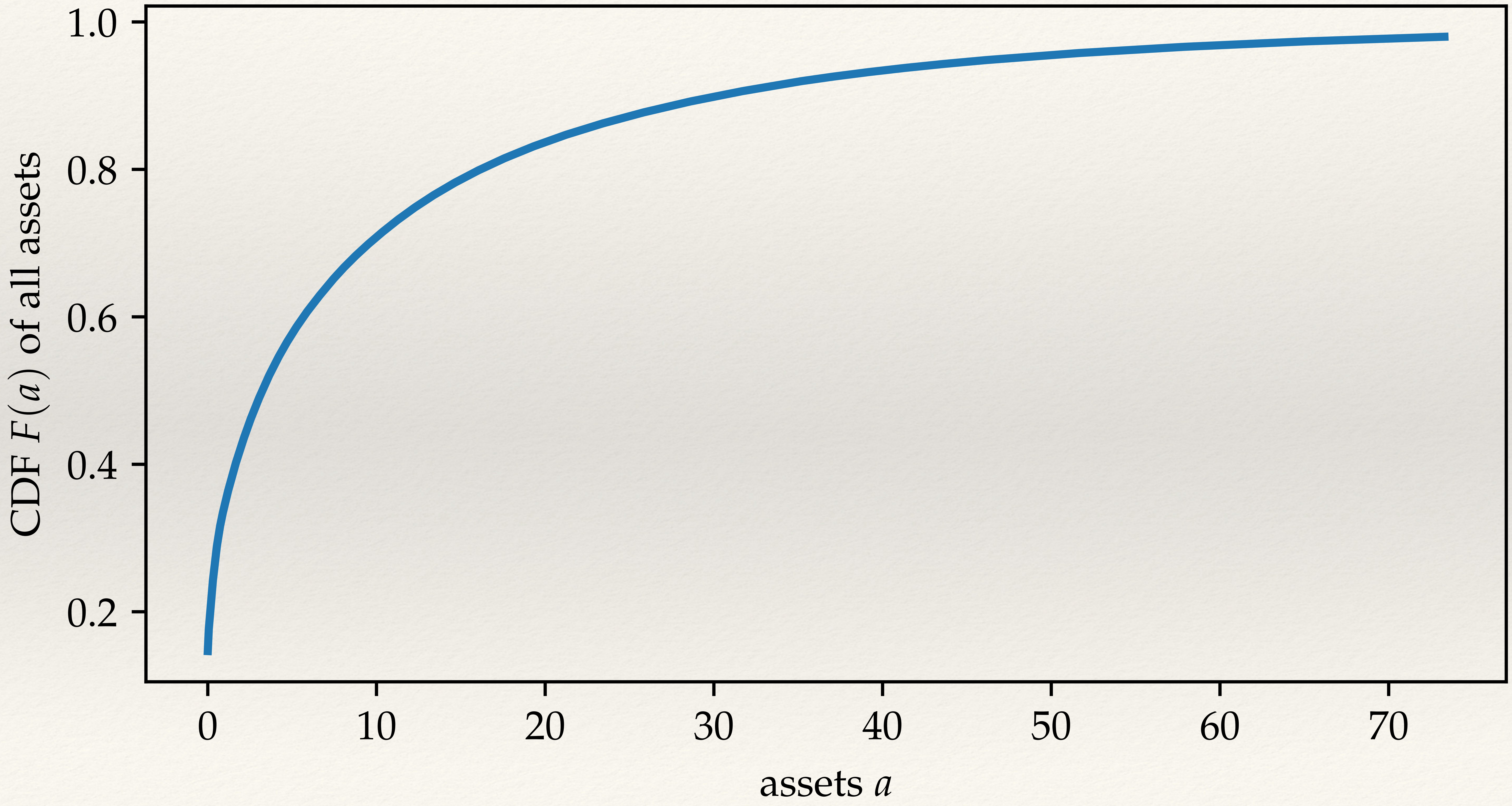


Buffer-stock behavior for each household



"Target" asset levels increase dramatically as we go to higher e, leading to inequality in stationary wealth distribution

Rich asset distribution, endogenous wealth inequality



Calibration of model

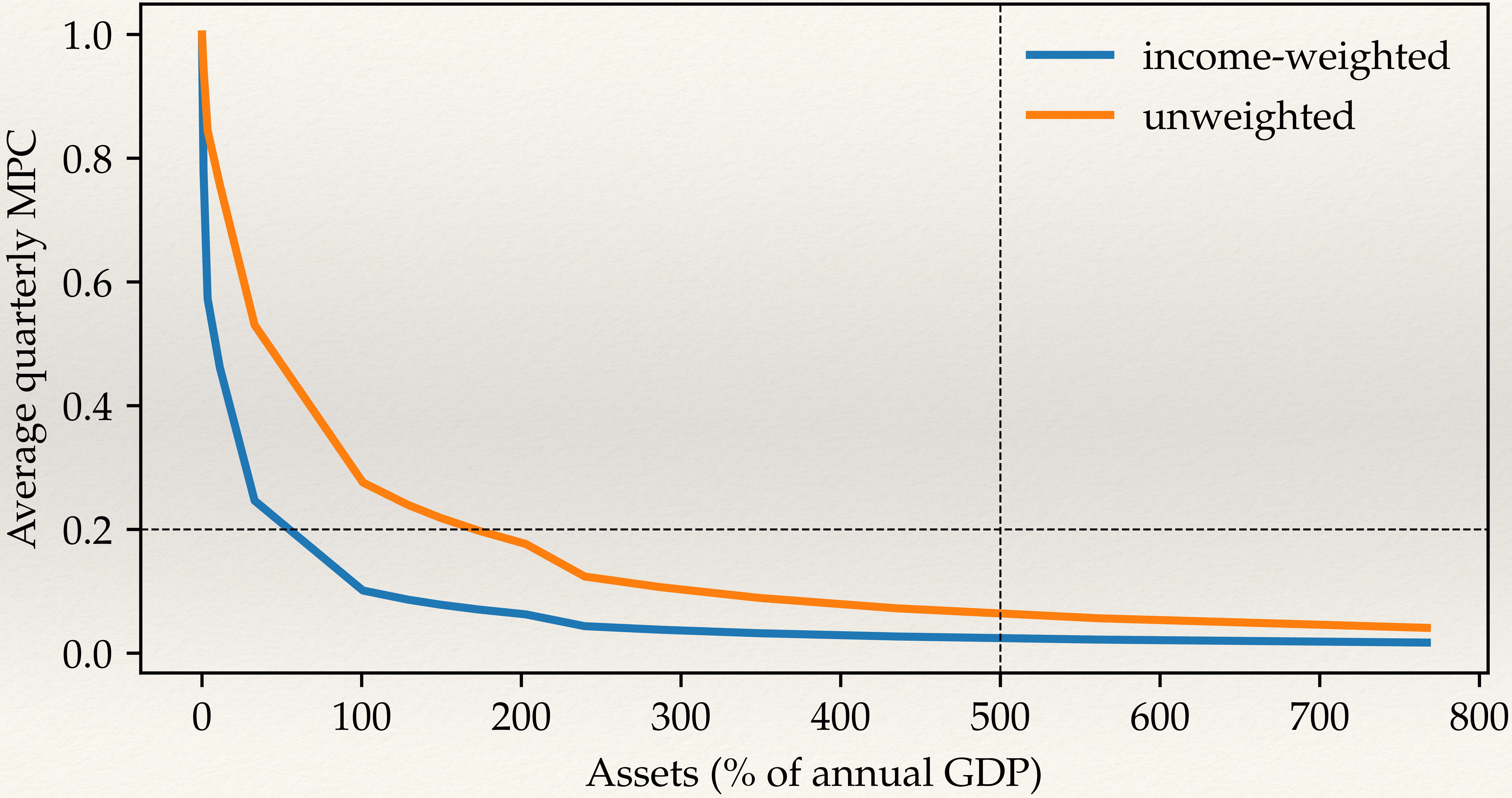
What parameters do we need to calibrate?

- Calibrate to quarterly frequency
- **Income process** e: [usually normalize average e to 1, entire ss scales in Z]
  - calibrate as discrete approximation of lognormal AR(1)
  - annual persistence $\rho = 0.91$, cross-sectional sd $\sigma = 0.92$ (IKC paper), rough approximation of pretax income process in US
  - 11-point Rouwenhorst approximation (see supplement for details)
- **Real rate** r to 2% annually, borrowing **constraint** $\underline{a}$ to 0, **utility** to $u(c) = \log c$
- One parameter remains: **discount factor** β

Two common strategies for calibrating β

- Calibrate to hit target for **aggregate assets**, taken from data
  - Our Ann Rev calibration: assets A at 500% of GDP, given after-tax labor income Z of 70% of GDP, following US
  - (Some others target lower A, interpreted as some notion of "liquid" assets)
- Calibrate to average **marginal propensity to consume**, also taken from data
  - Our Ann Rev calibration: average income-weighted quarterly MPC of 0.2
- Problem: **tradeoff** between two, β that matches one fails other

Asset-MPC tradeoff as we vary β

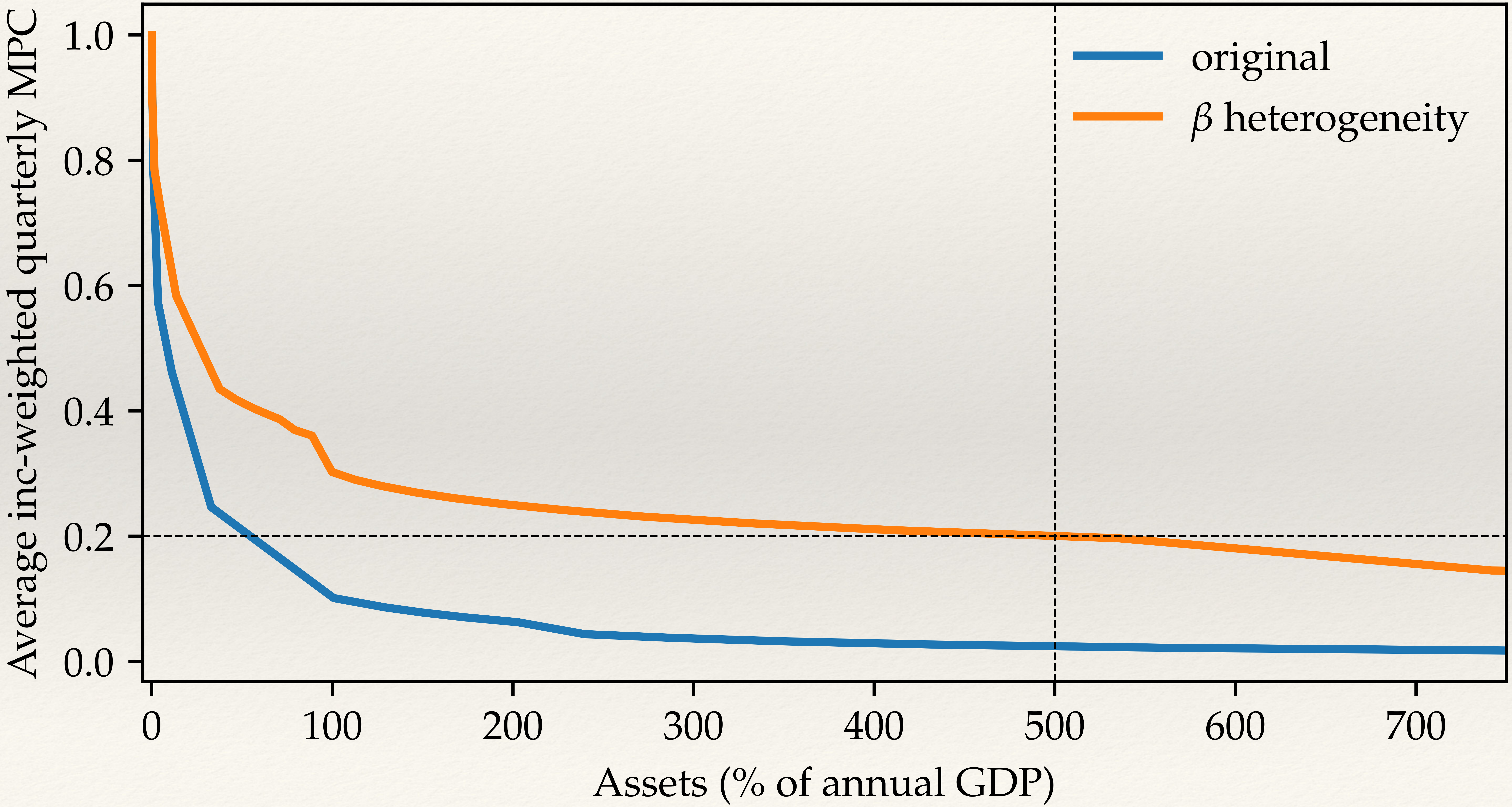


Our solution: β heterogeneity

- By **mixing different** β, we can target both aggregate assets and MPCs
- Exogenous state is now (β, e), can make β either permanent or stochastic
  - Stochastic limits polarization into "spenders" vs. "savers"

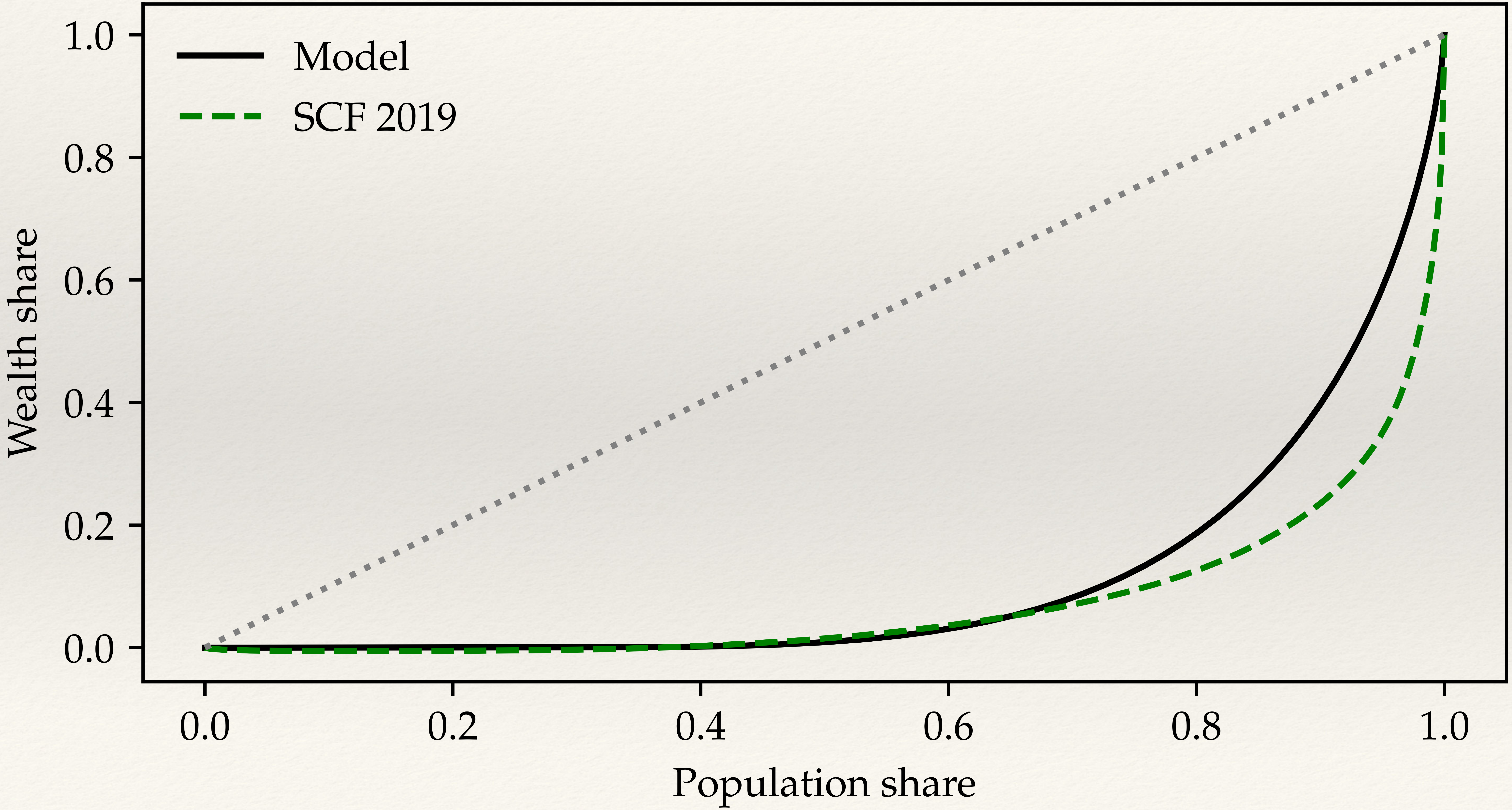
- We'll use simplified version of Annual Review stochastic β process (independent of e):
  - Two βs with 50% shares, each household gets fresh β draw with prob 1%
  - Loosely interpret as new draw of preferences every "generation" (25 years)
  - Calibrate to **hit both asset (500% of GDP) and income-weighted MPC (0.2) targets**
  - Calibrated quarterly βs: approximately 0.955 (impatient) and 0.998 (patient)

New vs. old asset-MPC tradeoff



(In β heterogeneity line, we keep gap between high and low β fixed as we vary the mean.)

Untargeted moment: Lorenz curve vs. US data



Model not bad, but **misses wealth concentration at upper end** (hard to fit without other features)

Annual Review calibration has richer income process and explicitly targets Lorenz curve, gets better fit

Partial equilibrium dynamics

Time-varying aggregate inputs to household problem

- Revisit individual household problem [ignoring β process for notational simplicity]

$$\max_{\{a_{it}, c_{it}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

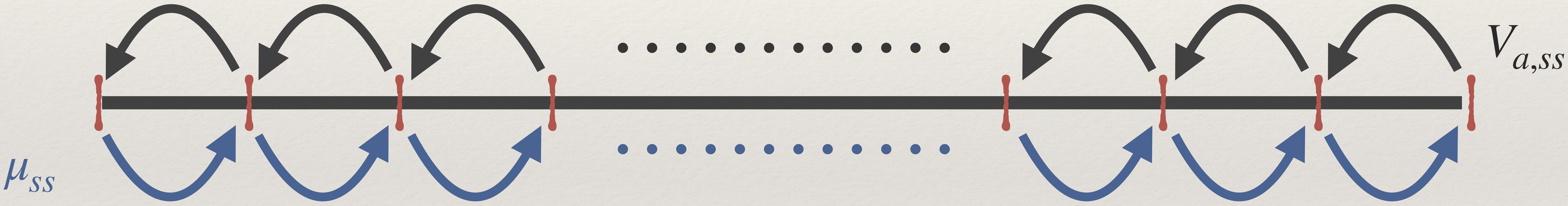
allowing returns r_t^p and aggregate after-tax income Z_t to vary over time

$$a_{it} + c_{it} = (1 + r_t^p) a_{i,t-1} + Z_t e_{it} \qquad a_{it} \geq \underline{a}$$

- Add p to emphasize that r_t^p is **ex-post** return from $t-1$ to t, determined at date t
- Assume distribution of $a_{i,-1}$ is steady state, perfect foresight over $\{r_t^p, Z_t\}_{t=0}^{\infty}$ from date 0 onward ("MIT shock")

Solution uses similar iterations to steady state

1. Start with $V_{aT} = V_{a,ss}$ and iterate backward T times, using time-varying r_t^p and Z_t to obtain policies $a_t'(e,a)$, $c_t(e,a)$ at $t = 0,\ldots,T-1$



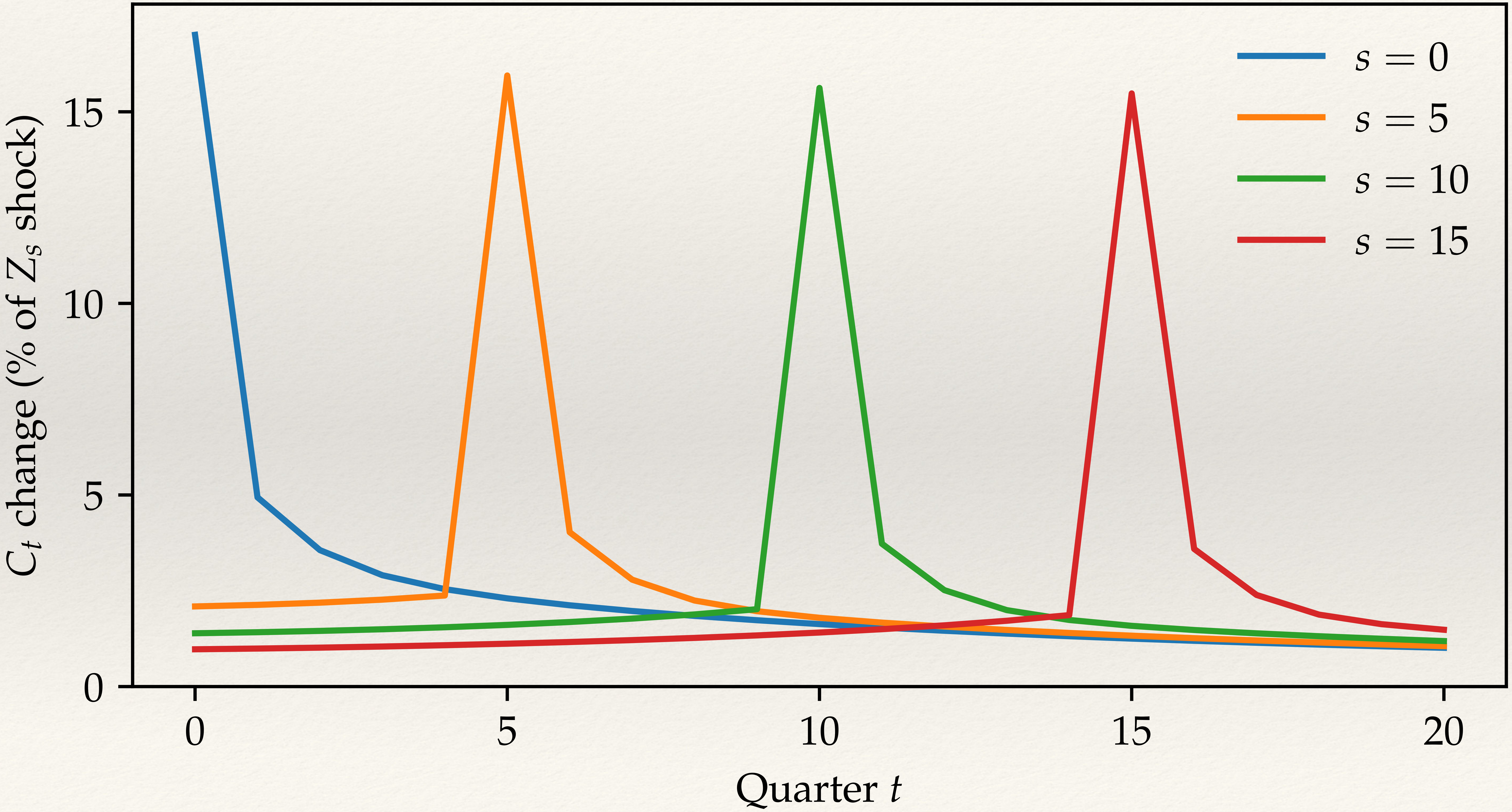
2. Start with distribution $\mu_0 = \mu_{ss}$ and iterate forward $T-1$ times, using time-varying policy function $a_t'(e,a)$ to get $\mu_t(e,\cdot)$ at each t

3. Aggregate policies $a_t'(e,a)$, $c_t(e,a)$ against $\mu_t(e,\cdot)$ at each t to get A_t, C_t

Key observation: r_t^p and Z_t determine everything

- Given sequences of ex-post returns r_t^p and aggregate after-tax income Z_t:
  - We can solve for time-varying policy functions $a_t'(e,a)$ and $c_t(e,a)$
  - These imply a time-varying distribution $\mu_t(e,\cdot\,)$
  - And together these imply time-varying A_t and C_t
- Hence, we can think of A_t and C_t as being **functions** of $\{r_s^p\}_{s=0}^{\infty}$ and $\{Z_s\}_{s=0}^{\infty}$
  - These are **sequence-space functions** $\mathscr{A}_t(\{r_s^p, Z_s\})$ and $\mathscr{C}_t(\{r_s^p, Z_s\})$
  - Plot around steady-state; later, compute derivatives ("sequence-space Jacobians")

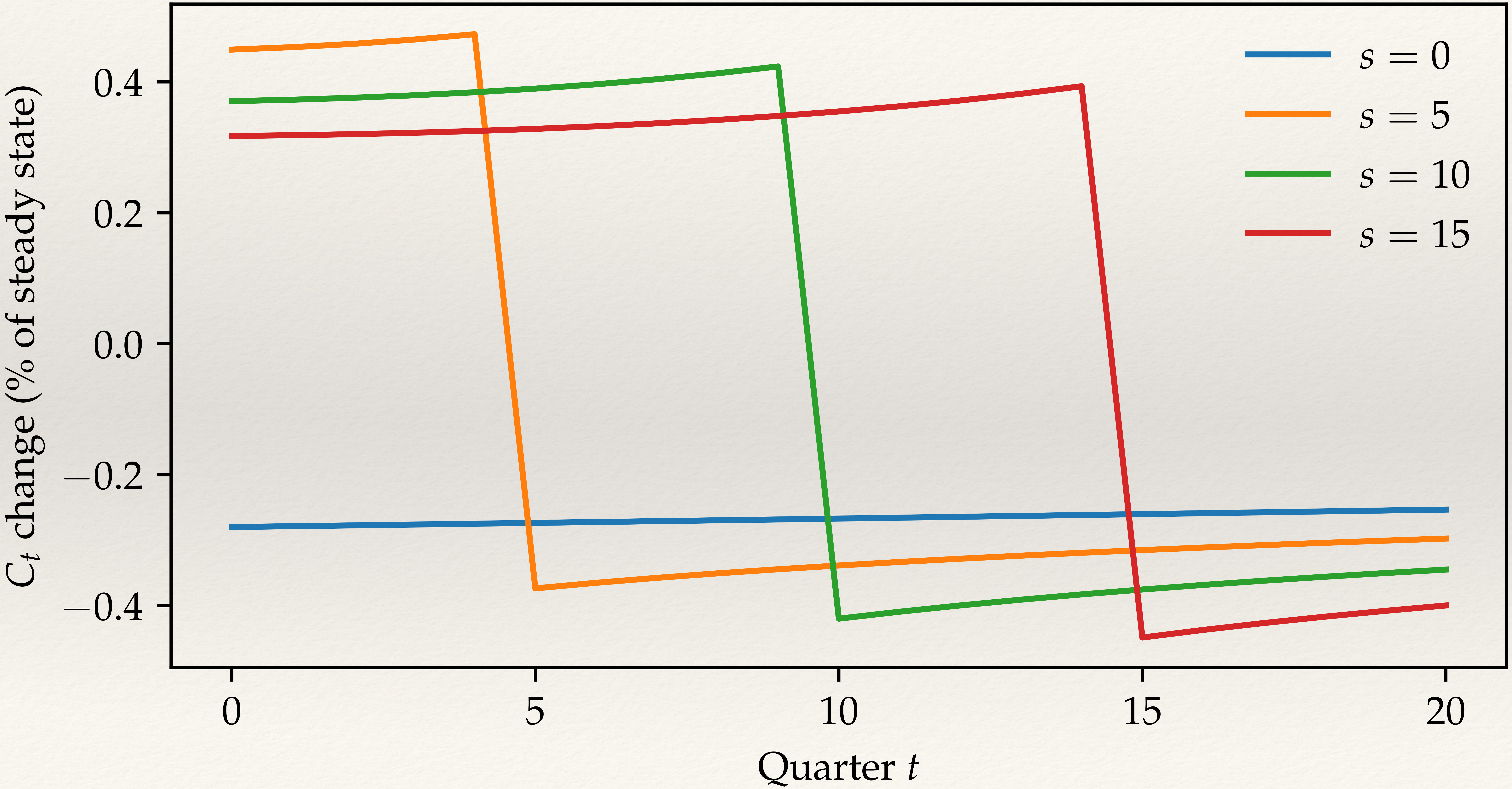
Response to 1% Z_s shocks at different dates s



Response to surprise at quarter 0 slightly less than 20% due to nonlinearity

Elevated spending out of income at other dates as well (will relate to "intertemporal MPCs")

Response to -1 pp r_s^p shocks at different dates s



Consumption increases in anticipation of falling returns, but less than standard Euler equation

If $s = 0$ then it's a **surprise negative return** ("capital loss"). Implied MPC out of loss is only 0.011, **very low** but persistent

Conclusion

Conclusion

- Introduced **standard incomplete markets model**
- Nice features: concave consumption functions, buffer-stock behavior, endogenous wealth distribution
- Resolve steady-state “asset-MPC tradeoff" by introducing β heterogeneity

- Aggregate dynamics a function of $\{r_s^p, Z_s\}$ path
- Elevated consumption in period of income shock, some before and after too
- Consumption response to r smaller than rep agent; MPC out of cap gains still low