Källén-Lehmann の計算

有限温度系での Green 関数は次のように表せる.

$$G_{\alpha\beta}(\boldsymbol{x}_1,t_1;\boldsymbol{x}_2,t_2) = -i\mathrm{Tr}\left(\hat{\rho}\mathcal{T}\psi_\alpha(\boldsymbol{x}_1,t_1)\psi^\dagger_\beta(\boldsymbol{x}_2,t_2)\right) \tag{3.10}$$

非対角であると仮定したので,

$$G(\boldsymbol{x}_1,t_1;\boldsymbol{x}_2,t_2) = -i\mathrm{Tr}\left(\hat{\rho}\mathcal{T}\psi(\boldsymbol{x}_1,t_1)\psi^\dagger(\boldsymbol{x}_2,t_2)\right) \tag{3.10'}$$

を評価すればよい.
ここで, 式 (3.13) や完備関係式を挟むと

$$\begin{aligned}
G(\boldsymbol{x}_1,t_1;\boldsymbol{x}_2,t_2) =& -i\mathrm{Tr}\left(\sum_n \rho_n |n\rangle\langle n|\mathcal{T}\psi(\boldsymbol{x}_1,t_1)\sum_m |m\rangle\langle m|\psi^\dagger(\boldsymbol{x}_2,t_2)\right) & (1)\\
=& -i\sum_l \langle l|\left(\sum_n \rho_n |n\rangle\langle n|\mathcal{T}\psi(\boldsymbol{x}_1,t_1)\sum_m |m\rangle\langle m|\psi^\dagger(\boldsymbol{x}_2,t_2)\right)|l\rangle & (2)\\
=& -i\sum_{l,m,n}\left(\mathcal{T}\langle l|\rho_n|n\rangle\langle n|\psi(\boldsymbol{x}_1,t_1)|m\rangle\langle m|\psi^\dagger(\boldsymbol{x}_2,t_2)|l\rangle\right) & (3)\\
=& -i\sum_{m,n}\left(\mathcal{T}\rho_n\langle n|\psi(\boldsymbol{x}_1,t_1)|m\rangle\langle m|\psi^\dagger(\boldsymbol{x}_2,t_2)|n\rangle\right) & (4)
\end{aligned}$$

式 (2.21) によると, Heisenberg 表示での ψ_α をもちいて

$$\psi_\alpha(\boldsymbol{x},t) = e^{-i(\mathcal{P}\boldsymbol{x}-\mathcal{H}t)}\psi_\alpha e^{i(\mathcal{P}\boldsymbol{x}-\mathcal{H}t)} \tag{2.21}$$

と書き表せるので,

$$\begin{aligned}
G(\boldsymbol{x}_1,t_1;\boldsymbol{x}_2,t_2) =& -i\sum_{m,n}\left(\mathcal{T}\rho_n\langle n|e^{-i(\mathcal{P}\boldsymbol{x}_1-\mathcal{H}t_1)}\psi e^{i(\mathcal{P}\boldsymbol{x}_1-\mathcal{H}t_1)}|m\rangle\langle m|e^{-i(\mathcal{P}\boldsymbol{x}_2-\mathcal{H}t_2)}\psi^\dagger e^{i(\mathcal{P}\boldsymbol{x}_2-\mathcal{H}t_2)}|n\rangle\right) & (5)\\
=& -i\sum_{m,n}\left(\mathcal{T}\rho_n\langle n|e^{-i(\boldsymbol{P}_n\boldsymbol{x}_1-(E_n-\mu N_n)t_1)}\psi e^{i(\boldsymbol{P}_m\boldsymbol{x}_1-(E_m-\mu N_m)t_1)}|m\rangle\langle m|e^{-i(\boldsymbol{P}_m\boldsymbol{x}_2-(E_m-\mu N_m)t_2)}\psi^\dagger e^{i(\boldsymbol{P}_n\boldsymbol{x}_2-(E_n-\mu N_n)t_2)}|n\rangle\right) & (6)\\
=& -i\sum_{m,n}\left(\mathcal{T}\rho_n\langle n|e^{-i((\boldsymbol{P}_n-\boldsymbol{P}_m)\boldsymbol{x}_1-(E_n-\mu_n-E_m+\mu_m)t_1)}\psi|m\rangle\langle m|\psi^\dagger e^{i((\boldsymbol{P}_n-\boldsymbol{P}_m)\boldsymbol{x}_2-(E_n-\mu N_n-E_m+\mu N_m)t_2)}|n\rangle\right) & (7)\\
& & (8)
\end{aligned}$$

ここで,

$$\begin{aligned}
\boldsymbol{P}_{mn} =& \boldsymbol{P}_m - \boldsymbol{P}_n\\
\omega_{mn} =& E_m - \mu N_m - (E_n - \mu N_n)
\end{aligned} \tag{3.16}$$

として書き換えると

$$\begin{aligned}
G(\boldsymbol{x}_1,t_1;\boldsymbol{x}_2,t_2) =& -i\sum_{m,n}\left(\mathcal{T}e^{-i(-\boldsymbol{P}_{mn}\boldsymbol{x}_1+\omega_{mn}t_1)}\rho_n\langle n|\psi|m\rangle\langle m|\psi^\dagger|n\rangle e^{i(-\boldsymbol{P}_{mn}\boldsymbol{x}_2+\omega_{mn}t_2)}\right) & (9)\\
=& -i\theta(t_1-t_2)\sum_{m,n}\left(e^{-i(-\boldsymbol{P}_{mn}\boldsymbol{x}_1+\omega_{mn}t_1)}\rho_n\langle n|\psi|m\rangle\langle m|\psi^\dagger|n\rangle e^{i(-\boldsymbol{P}_{mn}\boldsymbol{x}_2+\omega_{mn}t_2)}\right)\\
& -i(\mp\theta(t_2-t_1))\sum_{m,n}\left(e^{-i(-\boldsymbol{P}_{mn}\boldsymbol{x}_2+\omega_{mn}t_2)}\rho_n\langle n|\psi^\dagger|m\rangle\langle m|\psi|n\rangle e^{i(-\boldsymbol{P}_{mn}\boldsymbol{x}_1+\omega_{mn}t_1)}\right) & (10)\\
=& -i\theta(t_1-t_2)\sum_{m,n}\left(e^{-i(-\boldsymbol{P}_{mn}(\boldsymbol{x}_1-\boldsymbol{x}_2)+\omega_{mn}(t_1-t_2))}\rho_n A_{mn}\right)\\
& -i(\mp\theta(t_2-t_1))\sum_{m,n}\left(e^{-i(\boldsymbol{P}_{mn}(\boldsymbol{x}_1-\boldsymbol{x}_2)-\omega_{mn}(t_1-t_2))}\rho_n A_{nm}\right) & (11)
\end{aligned}$$

$$\begin{aligned}
G(\boldsymbol{p},t_1;t_2) =& -i(2\pi)^3\theta(t_1-t_2)\sum_{m,n}\left(e^{-i\omega_{mn}(t_1-t_2)}\rho_n A_{mn}\right)\delta(\boldsymbol{p}-\boldsymbol{P}_{mn})\\
& \pm i(2\pi)^3\theta(t_2-t_1)\sum_{m,n}\left(e^{i\omega_{mn}(t_1-t_2)}\rho_n A_{nm}\right)\delta(\boldsymbol{p}+\boldsymbol{P}_{mn}) & (12)
\end{aligned}$$

$$G(\boldsymbol{p},\omega) =(2\pi)^3\sum_{m,n}\left(\frac{\rho_n A_{mn}\delta(\boldsymbol{p}-\boldsymbol{P}_{mn})}{\omega-\omega_{mn}+i0} \pm \frac{\rho_n A_{nm}\delta(\boldsymbol{p}+\boldsymbol{P}_{mn})}{\omega+\omega_{mn}-i0}\right) \tag{13}$$

第二項目について, n,m を逆転させると

$$\begin{aligned}
G(\boldsymbol{p},\omega) =& (2\pi)^3\sum_{m,n}\left(\frac{\rho_n A_{mn}\delta(\boldsymbol{p}-\boldsymbol{P}_{mn})}{\omega-\omega_{mn}+i0} \pm \frac{\rho_m A_{mn}\delta(\boldsymbol{p}+\boldsymbol{P}_{nm})}{\omega+\omega_{nm}-i0}\right) & (14)\\
=& (2\pi)^3\sum_{m,n}\left(\frac{\rho_n A_{mn}\delta(\boldsymbol{p}-\boldsymbol{P}_{mn})}{\omega-\omega_{mn}+i0} \pm \frac{\rho_m A_{mn}\delta(\boldsymbol{p}-\boldsymbol{P}_{mn})}{\omega-\omega_{mn}-i0}\right) & (15)\\
=& (2\pi)^3\sum_{m,n}A_{mn}\delta(\boldsymbol{p}-\boldsymbol{P}_{mn})\left(\frac{\rho_n}{\omega-\omega_{mn}+i0} \pm \frac{\rho_m}{\omega-\omega_{mn}-i0}\right) & (16)
\end{aligned}$$

後半の (括弧) について

$$\left(\frac{\rho_n}{\omega-\omega_{mn}+i0}\pm\frac{\rho_m}{\omega+\omega_{mn}-i0}\right)=\left(\frac{e^{-\beta(E_n-\mu N_n-\Omega)}}{\omega-\omega_{mn}+i0}\pm\frac{e^{-\beta(E_m-\mu N_m-\Omega)}}{\omega-\omega_{mn}-i0}\right) \tag{17}$$

$$=e^{-\beta(E_n-\mu N_n-\Omega)}\left(\frac{1}{\omega-\omega_{mn}+i0}\pm\frac{e^{-\beta((E_m-E_n)-\mu(N_m-N_n))}}{\omega-\omega_{mn}-i0}\right) \tag{18}$$

$$=\rho_n\left(\frac{1}{\omega-\omega_{mn}+i0}\pm\frac{e^{-\beta\omega_{mn}}}{\omega-\omega_{mn}-i0}\right) \tag{19}$$

となるので, 最終的に

$$G(\boldsymbol{p},\omega)=(2\pi)^3\sum_{m,n}\rho_n A_{mn}\delta(\boldsymbol{p}-\boldsymbol{P}_{mn})\left(\frac{1}{\omega-\omega_{mn}+i0}\pm\frac{e^{-\beta\omega_{mn}}}{\omega-\omega_{mn}-i0}\right) \tag{20}$$